Fluid Flow, Nonsmooth Domains, and Heterogeneous Catalysis

Inaugural-Dissertation

zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Heinrich-Heine-Universität Düsseldorf

vorgelegt von

Siegfried Maier
aus Singen am Hohentwiel

Düsseldorf, Dezember 2015

aus dem Institut für Mathematik
der Heinrich-Heine-Universität Düsseldorf

Gedruckt mit der Genehmigung der
Mathematisch-Naturwissenschaftlichen Fakultät der
Heinrich-Heine-Universität Düsseldorf

Referent: Prof. Dr. Jürgen Saal

Korreferent: Prof. Dr. Dieter Bothe

Tag der mündlichen Prüfung: 10.03.2016

Acknowledgments

First of all I would like to express my gratitude to my supervisor Prof. Dr. Jürgen Saal for his support and his guidance. He always had time and answered my questions with patience. I have highly appreciated the nice and professional working atmosphere in his group. I would like to thank him for many advices, hints and numerous fruitful discussions during the last years, and also for the great time in his group.
I also like to express my gratitude to Prof. Dr. Dieter Bothe for the great time in the *Mathematical Modeling and Analysis* group at the *Center of Smart Interfaces*. Also many thanks to him for several invitations to Darmstadt and his great hospitality. He has always been ready for answering my questions and giving support. I benefited a lot from the interesting discussions with him. Especially, thanks for introducing heterogeneous catalysis to me. Also thanks to him for acting as co-referee.
I am also grateful to PD. Dr. Axel Grünrock for being my mentor.
I am very grateful for my time in Darmstadt. There I enjoyed the rich scientific possibilities at the Technische Univeristät Darmstadt, including the *Center of Smart Interfaces*, the *Mathematical Modeling and Analysis* group and the *International Research Training Group 1529*. Especially as an associated member of the international research training group I benefited a lot from special lectures, weekly seminars and workshops. There I had the opportunity to get into contact with the leading experts in the field.
Many thanks to my former colleagues and friends at the Technische Universität Darmstadt, especially Dr. André Fischer, Dr. Christoph Albert and Dr. Stefan Fleckenstein. I could always count on them. I am also highly grateful for the time with my colleagues and friends at the Universität Düsseldorf. The atmosphere here has been extraodinarily pleasant. Many thanks to my colleague and friend Dr. Florian Zanger, and also to Laura Westermann and Fabian Dicken for proofreading this thesis. Special thanks to my colleague and friend Dr. Matthias Köhne who accompanied me in Darmstadt and in Düsseldorf and helped me with guidance, a lot of hints and fruitful discussions. Last but not least, also thanks to him for proofreading.
Above all, I thank my parents Josef and Gabriele Maier and my brother Uwe Maier for constant support, understanding and encouragement during my whole life. This thesis would not have been possible without them.

Siegfried Maier

Contents

Introduction

In this thesis we consider two nonlinear, time and space dependent systems of partial differential equations stemming from fluid dynamics and from chemical engineering on nonsmooth domains. While the first part is devoted to the Navier-Stokes equations on a wedge domain, in the second part a model of heterogeneous catalysis on a finite cylinder is considered. The aim of this thesis is to give an analytic approach for both systems under consideration. We show local-in-time well-posedness results for both systems and a global-in-time result for the catalysis model. In each case the analysis is done in the strong L^p- respectively L^2-setting.

Fluid Flow in a Wedge Domain

The fundamental equations of fluid mechanics are the Navier-Stokes equations, which describe the motion of a viscous incompressible fluid. In the continuum mechanical derivation conservation of momentum and mass are postulated. Only the isothermal case is considered. Therefore the fluid's incompressibility yields that the density is constant. In particular the mass balance equation reduces to the divergence condition of the velocity. Hence only the balance of momentum equation exhibits a time evolution. For an introduction to fluid dynamics and more information on the continuum mechanical derivation see [Bat00, TR00] and [TM05]. In many situations one single fluid may be described entirely satisfactory. However, in many problems several phases are involved. In particular contact line phenomena constitute three-phase problems with usually one solid and two fluid phases. As a typical example one may think of a water drop running down glass. There the glass is the solid, the water and the surrounding air constitute the two fluids. The contact line occurs where glass, water and air meet. At this line a certain *contact angle* between the solid and the interface of the two fluids can be measured. Usually it is not prescribed and it can be observed that it behaves dynamically. Contact line problems occur in a vast variety of situations like ink-jet printing, liquid coating, spin coating, lubrication, painting, condensation and generally in wetting and de-wetting phenomena. An analytic treatment of such problems leads to a wedge domain as a prototype geometry. This is due to a local transformation of the three-phase (liquid/gas/solid) contact line to a wedge domain by employing a suitable Hanzawa transformation, see e.g. [DGH$^+$11] or Section 4 in [PSSS12]. Due to the free boundary of the fluid/gas

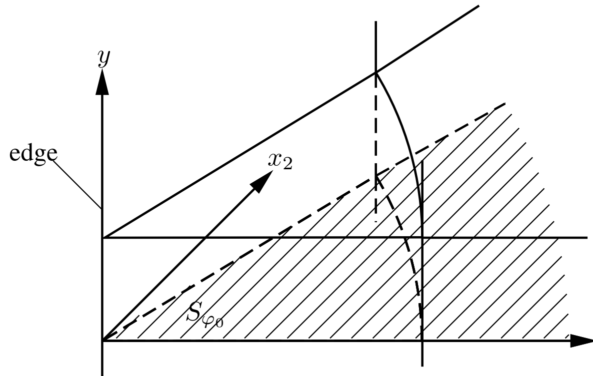

Figure 0.1: three dimensional wedge type domain $G = S_{\varphi_0} \times \mathbb{R}$

interface, this however, usually leads to intricate quasilinear problems with dynamic boundary conditions in wedge domains. An analytical treatment of these problems appears very hard. In fact, it seems only the special values of contact angles, that is $\varphi_0 = 0, \pi/2, \pi$, could be handled so far, cf. [Sol95, FV97, Sch01]. A major objective of this thesis is to show that at least for a specific set of boundary conditions the Stokes equations are well-posed on a three-dimensional wedge for arbitrary angles $\varphi_0 \in (0, \pi)$. Note that contact line problems for the case of Darcy's instead of Stokes flow have been considered in [KM15] for a prescribed contact angle. The underlying wedge problem was content of [KM13].

Let us consider the wedge problem we intend to study in this thesis. Let $T > 0$ be a fixed time and let $G := S_{\varphi_0} \times \mathbb{R}$ denote a three dimensional wedge of angle $\varphi_0 \in (0, \pi)$ with

$$S_{\varphi_0} := \left\{ (x_1, x_2) = (r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2 : \; r > 0, \; 0 < \varphi < \varphi_0 \right\}.$$

We consider the non-stationary Navier-Stokes equations. Here we impose perfect slip boundary conditions on the wedge G, such that the complete initial boundary value problem reads

$$\left\{ \begin{array}{rclll} \partial_t u - \Delta u + \nabla p + (u \cdot \nabla) u & = & f & \text{in} & (0,T) \times G, \\ \operatorname{div} \mathrm{u} & = & 0 & \text{in} & (0,T) \times G, \\ \nu \times \operatorname{curl} u = 0, \quad u \cdot \nu & = & 0 & \text{on} & (0,T) \times \partial G, \\ u(0) & = & u_0 & \text{in} & G. \end{array} \right. \tag{0.1}$$

Here ν denotes the outer normal vector at ∂G. Let $t \in (0,T)$ and $(x, y) = (x_1, x_2, y) \in G$. For given data, here an external force $f = f(t, x)$ and initial velocity field $u_0 = u_0(x)$ we aim to find a unique velocity field $u = u(t, x)$ and a corresponding pressure $p = p(t, x)$ – which is uniquely determined up to a constant – which solve (0.1). Note that (0.1) is semilinear as the convection $(u \cdot \nabla)u$ is the only nonlinearity. Also note that the Stokes equations formally arise from (0.1) just by omitting the convection $(u \cdot \nabla)u$.

The aim of our analysis is to prove the strong well-posedness of the Stokes equations associated to (0.1) in a weighted L^p-setting and the strong local-in-time

L^p-well-posedness of (0.1) in the unweighted setting and for at least some angles and small p-intervals close to 1. The main results of Part I are given by Theorem 3.1, Corollary 3.2 and Theorem 3.3 and may be summarized as follows:
Let the Stokes operator $\mathcal{A}_S$ associated to (0.1) be defined as in Theorem 3.1. Suppose that the integrability index $1 < p < \infty$, the weight exponent $\gamma \in \mathbb{R}$, and the angle $\varphi_0 \in (0, \pi)$ satisfy a certain condition which stems from the spectral properties of $\mathcal{A}_S$. Then $\mathcal{A}_S$ admits a bounded $\mathcal{H}^\infty$-calculus on the γ-weighted L^p-space of solenoidal fields $L^p_{\sigma,\gamma}(G)$ with angle $\phi^\infty_{\mathcal{A}_S} < \pi/2$. Here the domain of $\mathcal{A}_S$ contains all derivatives up to order two, such that we obtain full Sobolev regularity. Consequently, under the same condition the Stokes equations corresponding to (0.1) admit maximal L^p_γ-regularity, again in the sense of full Sobolev regularity. Moreover, let

$$\gamma = 0, \qquad \varphi_0 \in \left(0, \frac{5}{9}\pi\right), \qquad p \in \left(\frac{5}{3}, \frac{2}{3 - \pi/\varphi_0}\right).$$

Then on a given finite time interval the Navier-Stokes equations (0.1) admit a unique strong L^p-solution for sufficiently small initial data.

The strategy we pursue is as follows. In a first step we consider the resolvent problem

$$\begin{cases} (\lambda - \Delta)u &= f \quad \text{in} \quad G, \\ \nu \times \operatorname{curl} u &= 0 \quad \text{on} \quad \partial G, \\ u \cdot \nu &= 0 \quad \text{on} \quad \partial G, \end{cases} \tag{0.2}$$

in the Kondrat'ev space

$$L^p_\gamma(G, \mathbb{R}^3) := L^p(G, |x|^\gamma d(x_1, x_2, y), \mathbb{R}^3)$$

for appropriate $\gamma \in \mathbb{R}$. (Actually in a certain subspace of $L^p_\gamma(G)$, see Chapter 5.) A common approach, which is also utilized here, is to transform this system to a layer by introducing polar coordinates and applying the Euler transformation. The resulting transformed system then can be handled by abstract results on operator sums, cf. [PG75, DV87]. In our situation we apply suitable Kalton-Weis type theorems, cf. [KW01]. In fact, the corresponding transformed linear operator consists of a sum in which every summand admits a bounded $\mathcal{H}^\infty$-calculus. A specific feature here is that some of the operators are non-commuting in the resolvent sense. Here we apply [PS07, Theorem 3.1] which represents a Kalton-Weis type theorem for the non-commuting case based on the Labbas-Terreni commutator condition, which was introduced in [LT87]. Hence the $\mathcal{H}^\infty$-calculus transfers to the full sum. This, in turn, yields this property to be valid for the Laplacian related to (0.2) as well.
Prüss and Simonett already successfully applied this method in [PS07] to the scalar time dependent version of problem (0.2) in case of Dirichlet boundary conditions on ∂G. In fact, Prüss and Simonett precisely recovered the results on maximal regularity for the Dirichlet-Laplacian on wedge type domains obtained

before in [Naz01] by Nazarov via direct methods based on Green's function. The outcome in [Naz01] also covers the case of Neumann boundary conditions.
Having the $\mathcal{H}^\infty$-calculus for the Laplacian corresponding to system (0.2) at hand we turn to the Stokes equations. This is the point where the perfect slip conditions become essential. In fact, for this type of boundary conditions it can be proved that the Helmholtz projection and the Laplacian commute. Thus the Stokes operator can be regarded as the part of the Laplacian in the solenoidal subspace $L^p_{\sigma,\gamma}(G)$, see Chapter 5. This immediately yields the $\mathcal{H}^\infty$-calculus also to hold for the Stokes operator $\mathcal{A}_S$. This in particular implies that $\mathcal{A}_S$ generates a bounded analytic C_0-semigroup on $L^p_{\sigma,\gamma}(G)$ and that it has maximal regularity.
Note that the fact that Helmholtz projection and Laplacian commute in the perfect slip setting has already been utilized by Mitrea and Monniaux in [MM09a] and [MM09b]. Indeed, in [MM09b] well-posedness for the Navier-Stokes system is studied in the context of bounded (graph) Lipschitz domains. For the linear (Hodge-) Stokes operator it is proved that it is the generator of an analytic C_0-semigroup on L^p provided p is within the usual range $((3+\varepsilon)', 3+\varepsilon)$, cf. [MM09a]. Although it is the same set of equations, the outcomes of [MM09a, MM09b] and the underlying note are in some sense not comparable. The roughness of the boundary forces the authors in [MM09a, MM09b] to work in Hodge spaces (i.e. curl u, div $u \in L^p$ instead $\nabla u \in L^p$) which in that case do not coincide with corresponding Sobolev spaces.
Finally, we exploit maximal regularity and apply the contraction mapping principle as it is commonly done to prove the stong local-in-time well-posedness result.

Heterogeneous Catalysis on a Cylindrical Domain

There is a vast variety of examples for heterogeneous catalysis occuring in industrial application of chemical engineering. For instance in contact processes to produce sulfuric acid, the Harber-Bosch process for ammonia synthesis or the production of high-octane gasoline. Another example are exhaust gas converters, in particular the so called *three-way catalytic converter* in automobiles. In the last examples the reduction of nitrogen oxides to nitrogen and oxygen, the oxidation of carbon monoxide to carbon dioxide and the oxidation of unburnt hydrocarbons to carbon dioxide and water take place.
The use of catalysis is twofold: It is employed to increase the speed of chemical reactions and it may be used to change the selectivity in favor of a desired product. In a heterogeneous catalysis the *catalyst*, does not appear in the same phase as the substrate. While the substrate is typically present in a gas or a liquid, the catalyst is usually given on a solid wall, where the catalysis takes place. Therefore it is often referred to as *catalytic wall* or *active surface*. Note that heterogeneous catalysis requires a high area-to-volume ratio. This is the case in porous structures, where the smallest unit is a single pore. For more information on such reactors see [Ari75, Lev99, Whi90].

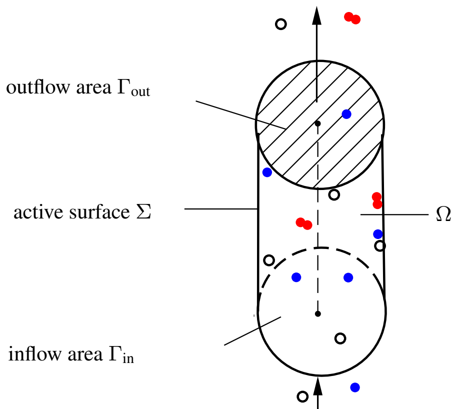

Figure 0.2: catalytic process in a micro pore

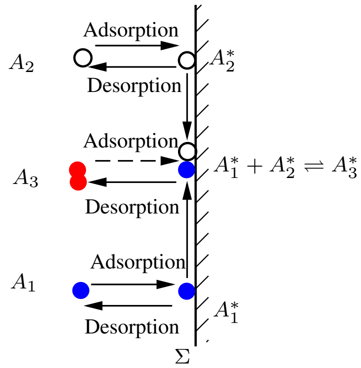

Figure 0.3: scheme of sorption and catalytic reaction on Σ

To get a first impression what happens during a heterogeneous catalysis we may think about the following process: The given advection field pushes chemical species – here the educts – into a micro pore, which we assume to be of cylinder-like shape with circular intersection. There they are driven by diffusive fluxes and adsorbed onto the catalytic wall, i.e. the lateral surface of the cylinder. Here the heterogeneous catalysis, that is a chemical reaction itself takes place. The chemical products are desorbed into the bulk phase again where they are driven outside by the advection field. So there are several mechanisms involved: advection and diffusion in the bulk phase, and sorption, reaction and surface diffusion on the active surface. Note that these processes underlie two different time scales: The advection, diffusion and sorption kinetics take place far more slowly than the chemical reaction. Also note that the sorption as well as the reaction, which may be seen as the most interesting parts of the heterogeneous catalytic process, take place on the surface which is in a mathematical view of interest itself. For more information on catalysis in general see e.g. the textbooks [Mas96] or [Rot08] and the references cited therein.

Let us briefly comment on the model of the heterogeneous catalytic process we study in this thesis. The modeling is based on continuum thermodynamics with several restrictions. In particular we only consider the isothermal case. After the derivation of partial mass balances for bulk and surface molar mass concentrations, constitutive laws for the material dependent quantities are introduced. Especially, we assume that the bulk and surface diffusive fluxes are governed by Fickian diffusion with constant coefficients. An outline of the modeling is given in Chapter 9. A related, but more complicated model is derived in the upcoming work [BMOS]. A note on the complexity of heterogeneous catalysis modeling is [Kei13].

We turn to the full system we intend to study. Due to the geometry of certain mirco-pores which are similar to a cylinder with circular intersection, this seems to be a possible choice for the underlying domain. Therefore, let us choose the domain $\Omega := B_R(0) \times (0, h) \subset \mathbb{R}^3$ with an open two dimensional ball $B_R(0)$ around 0 for a radius $R > 0$ and a height $h > 0$. Throughout the thesis let

$\Gamma_{\text{in}} := B_R(0) \times \{0\}$ denote the bottom of the cylinder, $\Sigma := \partial B_R(0) \times (0, h)$ the lateral surface of the cylinder and $\Gamma_{\text{out}} = B_R(0) \times \{h\}$ the top of the cylinder. In terms of the heterogeneous catalytic process described above the boundary parts Γ_{in}, Σ and Γ_{out} correspond to inflow area through which the species are driven inside, the active surface and the outflow area through which the chemical products are driven outside, cf. Figure 0.2.
For $N \in \mathbb{N}$ chemical species $A_1, ..., A_N$ respectively their adsorbed counterparts $A_1^*, ..., A_N^*$ being involved into the reaction we formulate the complete catalysis system in terms of corresponding concentrations $c_1, ..., c_N$ respectively surface concentrations $c_1^\Sigma, ..., c_N^\Sigma$. It is given by

$$\left\{\begin{array}{rcll} \partial_t c_i + (u \cdot \nabla) c_i - d_i \Delta c_i &=& 0 & \text{in} \quad (0,T) \times \Omega, \\ \partial_t c_i^\Sigma - d_i^\Sigma \Delta_\Sigma c_i^\Sigma &=& r_i^{\text{sorp}}(c_i, c_i^\Sigma) + r_i^{\text{ch}}(c^\Sigma) & \text{on} \quad (0,T) \times \Sigma, \\ (u \cdot \nu) c_i - d_i \partial_\nu c_i &=& g_i^{\text{in}} & \text{on} \quad (0,T) \times \Gamma_{\text{in}}, \\ -d_i \partial_\nu c_i &=& r_i^{\text{sorp}}(c_i, c_i^\Sigma) & \text{on} \quad (0,T) \times \Sigma, \\ -d_i \partial_\nu c_i &=& 0 & \text{on} \quad (0,T) \times \Gamma_{\text{out}}, \\ -d_i^\Sigma \partial_{\nu_\Sigma} c_i^\Sigma &=& 0 & \text{on} \quad (0,T) \times \partial\Sigma, \\ c_i(0) &=& c_{0,i} & \text{in} \quad \Omega, \\ c_i^\Sigma(0) &=& c_{0,i}^\Sigma & \text{on} \quad \Sigma, \end{array}\right. \tag{0.3}$$

where the index i runs over $i = 1, ..., N$ as always in Part II of the thesis. Here r_i^{sorp} is a given sorption rate function and r_i^{ch} stands for the rate of molar mass production due to chemical reactions. System (0.3) consists of $2N$ partial differential equations acting in Ω, respectivley, on Σ. So this system is built up of a diffusion-advection equation for c_i in Ω in the first line, a diffusion-sorption-reaction equation for c_i^Σ on Σ in the second line, complemented by initial conditions, and boundary conditions on each of the boundary components Γ_{in}, Σ and Γ_{out}. On Γ_{in} we impose a inhomogeneous *Danckwerts boundary condition* , which describes the rate of mass feed through the inflow area for some given data g_i^{in}. On Σ, respectively, Γ_{out} we impose nonlinear inhomogeneous, respectively, homogeneous Neumann boundary conditions, where the nonlinear boundary condition on Σ stems from the continuum mechanical derivation of the model. There we may derive bulk and surface balance equations at once, as well as just look at the surface mass balance. This way we obtain an identification of the sorption rate which gives rise to the boundary condition on Σ. Last but not least we ask c_i^Σ to respect a homogeneous Neumann boundary condition on $\partial\Sigma$. Note that (0.3) is a semilinear problem with nonlinearities r_i^{ch} and r_i^{sorp}, where r^{ch} is the only coupling term, which mixes up the different components $c_1^\Sigma, ..., c_N^\Sigma$ and lines. As a constitutive law we choose Fickian diffusion with constant diffusivities $d_i, d_i^\Sigma > 0$. Note that although this special choice of diffusion on the surface does not seem to be physically accurate enough for large surface concentrations, however finite time blow-ups may occur in general in case $d_i^\Sigma \neq d_j^\Sigma$ for some $i, j \in \{1, ..., N\}$, cf. [PS00]. Among others we suppose that the given velocity

field u is solenoidal.
We also make several assumptions on the nonlinearities r_i^{ch} and r_i^{sorp}: Besides some regularity assumption, the sorption r_i^{sorp} is supposed to be monotonically increasing in c_i and monotonically decreasing in c_i^Σ. In view of $r_i^{\text{sorp}} = r_i^{\text{ad}} - r_i^{\text{de}}$ this reflects the following phenomelogical observation: The higher the concentration of a constituent in the bulk – the more is absorbed onto the active surface – the higher the total sorption. Analogously the higher a concentration of an adsorbed species on the surface – the more is desorbed in the bulk – the lower the total sorption. Moreover, the sorption admits linear bounds, i.e. we assume there exist adsorption and desorption constants $k_i^{\text{ad}}, k_i^{\text{de}} > 0$ such that

$$-k_i^{\text{de}} c_i^\Sigma \leq r_i^{\text{sorp}}(c_i, c_i^\Sigma) \leq k_i^{\text{ad}} c_i, \qquad c_i, c_i^\Sigma \geq 0,$$

which seems to be quite restrictive but in particular allows for showing nonnegativity of concentrations without getting too technical. This estimate applied pointwise on Σ implies that $r_i^{\text{sorp}}(0,\cdot) \leq 0$ and $0 \leq r_i^{\text{sorp}}(\cdot, 0)$. In view of $r_i^{\text{sorp}} = r_i^{\text{ad}} - r_i^{\text{de}}$ with positive ad- and desorption rates this may be understood as follows: If there are no bulk species A_i at the surface, there is no adsorption possible. If there are no adsorbed species A_i^*, there is no desorption possible. In particular a pointwise application of the linear bounds implies $r_i^{\text{sorp}}(0,0) = 0$.
We also assume the reaction rate to admit several crucial properties which allow for a systematic treatment. It is well-known that a necessary and sufficient condition providing nonnegativity of concentrations is quasi-positivity of the production rate, see [Pie10]. In our case this assumption has to be combined with the afore stated properties of the sorption rate in order to prove the nonnegativity of c_i and c_i^Σ. Another essential assumption is that r^{ch} admits polynomial growth, which will be employed in the proof of the global existence theorem. Furthermore, we make use of a structural condition on r^{ch}: We assume r^{ch} to admit a triangular structure, which means that there exists an invertible matrix $Q \in \mathbb{R}^{N \times N}$ with strictly positive diagonal entries and $q_{ij} \geq 0$ for all $i, j = 1, ..., N$ such that

$$Qr^{\text{ch}}(y) \leq C \left(1 + \sum_{j=1}^N y_j\right) v, \qquad y \in [0,\infty)^N \tag{0.4}$$

for some constant $C > 0$ and $v = (1, \ldots, 1)$. Note that (0.4) is meant componentwise. This triangular condition has been widely studied by several authors, e.g. [Mor89, BS98, MP92, Pie10, BFPR] and allows for a treatment of the reaction term by a summation of different lines and an iteration over i. Roughly speaking, this condition allows for linear estimates of r_1^{ch}, $r_1^{\text{ch}} + r_2^{\text{ch}}$ and $r_1^{\text{ch}} + r_2^{\text{ch}} + r_3^{\text{ch}}$ and so on. For r^{ch} which is subject to (0.4), quasi-positive, and admits polynomial growth and suitable data, a pure reaction-diffusion system of the form

$$\partial_t c_i - d_i \Delta c_i = r_i^{\text{ch}}(c_1, ..., c_N) \qquad i = 1, ..., N$$

on some suitable domain in $\mathbb{R}^n$ subject to homogeneous Neumann boundary conditions admits a unique global classical solution as has been shown in [Mor89].

A typical example of a reaction rate admiting this triangular structure is a three component system studied by Rothe [Rot84]. There the reaction rate for concentrations $c^\Sigma = (c_1^\Sigma, c_2^\Sigma, c_3^\Sigma)$ reads $r_i^{\text{ch}}(c^\Sigma) = \sigma_i k^{\text{re}}(c_1^\Sigma c_2^\Sigma - c_3^\Sigma)$ with sign vector $\sigma = (-1, -1, 1)$ and $k^{\text{re}} > 0$ denoting a reaction constant, see example (R1) in Chapter 8.

We aim to show strong L^p-local-in-time and also strong L^2-global-in-time well-posedness of (0.3). More precisely, the main results on our heterogeneous catalysis model are given by Proposition 10.1, Theorem 8.1, Theorem 8.3 and may be summarized as follows: The fully inhomogeneous linear catalyst equations corresponding to (0.3) admit maximal L^p-regularity for all $p \in (5/3, \infty)$ and $p \neq 3$. Moreover, system (0.3) admits a unique local strong L^p-solution for $5/3 < p < \infty$ and $p \neq 3$ on small time intervals. Eventually, we show that system (0.3) admits a unique global strong L^2-solution for arbitrarily large data. The major objective is to extend known results from reaction-diffusion equations to the more complicated case of heterogeneous catalysis.

Let us outline the plan of the proofs of these results: In a first step we treat the inhomogeneous linear problem associated to (0.3), where we employ cylindrical L^p-theory [Nau13] and solve the diffusion equations subject to Neumann boundary conditions. At best of the author's knowledge, no reference for the surjectivity of the occuring Neumann trace operator on the cylinder with respect to maximal regularity classes seems to be available. Therefore, we show the surjectivity – here by a reflection in axial direction. As a consequence, we obtain the solvability of the inhomogeneous diffusion equations – but only without the terms $(u \cdot \nu)c_i$ and $(u \cdot \nabla)c_i$. In the next step these are treated as perturbation terms.
The nonlinear system (0.3) is then solved via maximal regularity of the linear system and the contraction mapping principle. Nonnegativity of concentrations c_i and surface concentrations c_i^Σ follows from the quasi-positivity of r^{ch}, the monotonicity and the linear bounds of r_i^{sorp} as well as a suitable sign of the data and $u \cdot \nu$ on the different boundary parts Γ_{in}, Σ, Γ_{out}.
In order to show the global well-posedness of (0.3) we make use of estimates of weak type, which we state in three lemmas: a linear comparison principle to estimate solutions against each other corresponding to their data, L^p-estimates for $2 \leq p \leq \infty$ as well as an estimate which is proven by duality techniques and maximal regularity, cf. [Pie10]. With these three lemmas at hand we are able to prove the global existence theorem: To this end we assume $T^* < \infty$, i.e. that the maximal time interval of existence of the local L^2-solution is finite, and show that the solution stays bounded in the phase space $H^1(\Omega) \times H^1(\Sigma)$ as $T \to T^*$. For this purpose it is sufficient to show a priori L^∞-bounds for the solution on Ω_T respectively Σ_T. To this end we proceed as follows: In a first step we derive L^q-estimates for c_i and $c_i|_\Sigma$ via the comparison principle and the weak-type L^q-estimates. In a second step we treat the surface concentrations. Here we employ the linear bounds of r^{sorp}, the triangular structure of r^{ch}, as well as the nonnegativity of concentrations. From the comparison principle we infer L^p-L^q-estimates

for c_i^Σ against c_i^Σ and $c_i|_\Sigma$. At this point we may exploit the triangular structure of the reaction rate by an iteration over $i = 1, ..., N$. We combine the estimates and infer the L^p-boundedness of concentrations for sufficiently large p, such that an argument as in [BR, Theorem 4] yields the boundedness of c_i, c_i^Σ in $L^\infty(\Omega_T)$ respectively $L^\infty(\Sigma_T)$ for $T < T^*$. Note in passing that for the application of this last argument we employ the linear bounds of r_i^{sorp} and the polynomial growth of r_i^{ch}.

Let us comment on some technical difficulties and compare this thesis to other works. In the last decades there has been a vast variety of results for parabolic equations and reaction-diffusion equations. Usually, a lot of authors work in a weak setting and assume a sufficiently smooth boundary of the underlying domain. We emphasize the special challenge which occurs in the present work when dealing with nonsmooth boundaries and constructing strong L^p-solutions. Since, in general, a direct treatment of such problems is analytically hard, up to now there only seem to be few works in such directions, see [NS11], [NS12], [Nau13] for an operator theoretical approach, [Ama15] for parabolic equations, [Köh13, Chapter 8] for Stokes equations, [Seg13] for coupled elliptic-parabolic systems.
In the proof of the global existence theorem the treatment of the trace of c_i on the surface requires a special effort, since an estimation needs some extra regularity. This is also the point, where a pure L^2-energy approach, without having L^∞- a priori bounds at hand, fails. Also in this view, the surface-bulk-coupling of the system is mathematically a special challenge.
The maximal regularity classes we employ for the surface concentrations and for the corresponding data are mainly used due to the 'weak' coupling which vanishes completely in the discussion of the linear equations. There we are free to choose the same regularity for c_i^Σ on the boundary as in the domain for c_i. Note that in general for stronger coupled systems, admitting an evolution on the boundary, other regularity classes are used [DPZ08].
Note that the treatment of the triangular structure is similar as for instance in the proof of Theorem 3.5 in [Pie10] or in [BFPR]. However, in contrast to [Pie10, BFPR], the problem considered here is more involved since reaction and sorption have to be treated simultanuously and the full system does not decompose into a pure c_i and a pure c_i^Σ system in any step of the derivation of a priori L^∞-bounds.
Overall there seem to be only a few notes considering related models. In [KO00], the existence of a unique weak L^1-solution is proved. In [MS06], the case of fast sorption is considered. See [HJ91] for scale-reduced models which are in particular interesting in porous media. Another work on the existence of time-periodic solutions of heterogeneous catalysis itself is [Bot01].

Structure of the Thesis

This thesis consists of two major parts, Part I (Chapter 3 - 7) contains the analysis of the Stokes- and Navier-Stokes equations subject to perfect slip boundary conditions on three-dimensional wedges and Part II (Chapter 8 - 12) is devoted to the catalytic model from above and its analysis. Part I is based on the joint work with Jürgen Saal [MS14], while the content of Part II is based on the joint work with Dieter Bothe, Matthias Köhne und Jürgen Saal [BKMS]. Before starting with Part I we provide the reader with necessary and useful preliminaries in Chapter 1 and as a preperation we take a close look on the relevant operator theory in Chapter 2.

In Chapter 1 we fix the basic notation and collect some general results on fundamental theorems and auxiliary statements which we use throughout this thesis. Chapter 2 is devoted to operator classes, maximal regularity and operator sums. There we give an introduction to the functional analytic framework which we employ in the analysis of the linear equations in Part I and Part II. We state some important results which we apply in Chapter 5 and Chapter 10 and also have a look on the relations of the individual introduced notions. In particular we give a short motivation of the concept of operator sums.

Part I: In Chapter 3 we give the main results of the analysis of the Stokes- and also the Navier-Stokes equations subject to perfect slip boundary conditions on wedge domains. In Chapter 4 the transformation of the wedge problem in weighted spaces to the layer problem in unweighted spaces is carried out completely. In particular, we compute the resulting operator originating from the Laplacian. In Chapter 5 we decompose it ‘cylindrically’ with respect to the coordinates of the obtained layer. These obtained ‘intersection operators’ are analyzed separately and then are summed up again. By the results stated in Chapter 2 we then derive a bounded $\mathcal{H}^\infty$-calculus of the transformed Laplacian. This result for the Laplacian carries over to the Stokes operator thanks to perfect slip boundary conditions as is shown in Chapter 6. Chapter 7 gives a local-in-time result for small data on arbitrary given finite time intervals for an admissible combination of angles φ_0, weights γ and a corresponding range of p.

Part II: Chapter 8 collects the main results of the analysis of the heterogeneous catalysis model considered in this thesis. In Chapter 9 an outline of the derivation of the underlying catalysis model is given. We show that this model arises from two kinds of mass balances – one for concentrations in the bulk phase and one for adsorbed surface concentrations of the involved chemical species. Chapter 10 is devoted to the study of the linear version of the catalyst equations. We give a rigorous analysis by starting with the homogeneous diffusion equations subject to Neumann boundary conditions, where again the collection of functional analytic theory presented in Chapter 2 is applied. After having proved the surjectivity of the Neumann trace operator, a perturbation argument yields maximal regularity of the fully inhomogeneous linear catalyst equations. Based on this linear result, we obtain the local-in-time well-posedness of the catalyst equations for small times and arbitrary data by the contraction mapping principle in Chapter 11. Besides this we also prove nonnegativity of solutions in Chapter 11. Finally Chapter 12 contains the proof of the strong L^2-well-posedness globally in time. In the first half of Chapter 12 we prove three main auxiliary results: comparison principles, L^p-estimates, and estimates based on a duality argument. Ultimately the proof of the global existence theorem is based on L^∞-estimates and given in the second half of Chapter 12.

We close this thesis by summaries in English and in German.

Chapter 1

Preliminaries

1.1. Basic Notation

Let us fix the basic notation. By $\mathbb{N} = \{1, 2, 3, ...\}$ we denote the natural numbers and denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The symbols $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ stand for integers, rational, real and complex numbers respectively.
Throughout this thesis we employ the following standard symbols for partial derivatives and differential operators: Let $T > 0$ be given and let $\Omega \subset \mathbb{R}^n$ be a domain. For a suitable function $f : (0, T) \times \Omega \to \mathbb{R}$ we denote the time derivative of f by $\partial_t f$ and the derivative with respect to the space coordinate x_k by $\partial_{x_k} f$ for $k = 1, ..., n$. We employ multi-index notation, i.e.

$$\partial^\alpha f := \partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n} f, \qquad \alpha \in \mathbb{N}_0^n.$$

The gradient of f is given by $\nabla f = (\partial_{x_1} f, ..., \partial_{x_n} f)$, while the divergence for a suitable vector field $F : (0, T) \times \Omega \to \mathbb{R}^n$ is defined via

$$\operatorname{div} F := \sum_{k=1}^{n} \partial_{x_k} F_k.$$

Finally the Laplacian of f is defined by

$$\Delta f = \sum_{k=1}^{n} \partial_{x_k}^2 f.$$

In the same manner but understood componentwise we define ΔF. For $n = 3$ and $F : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ we also employ the three-dimensional curl which we define as

$$\operatorname{curl} F := \begin{pmatrix} \partial_{x_2} F_3 - \partial_{x_3} F_2 \\ \partial_{x_3} F_1 - \partial_{x_1} F_3 \\ \partial_{x_1} F_2 - \partial_{x_2} F_1 \end{pmatrix}.$$

Similarly we employ the vector product $a \times b$ for $a, b \in \mathbb{R}^3$. The normal derivative of a function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ shall be denoted by $\partial_\nu u$. Moreover, when dealing with Cauchy problems the time derivative of $u = u(t)$ is also denoted by $\dot{u}$.

Let X be a Banach space. Then the norm of X will be denoted by $\| \cdot \|_X$. For $1 \leq p \leq \infty$ and a measure space (S, Σ, μ), we write $L^p(S, \mu, X)$ for the usual

Bochner-Lebesgue space. If $\Omega \subset \mathbb{R}^n$ and μ is the (Borel-) Lebesgue measure, we also write $L^p(\Omega, X)$. The symbol $W^{k,p}(\Omega, X)$ denotes the X-valued Sobolev space of order $k \in \mathbb{N}_0$, where $W^{0,p} := L^p$. It is defined through

$$W^{k,p}(\Omega, X) := \{u \in L^p(\Omega, X) : \ \partial^\alpha u \in L^p(\Omega, X) \ (\alpha \in \mathbb{N}_0^n, \ |\alpha| \le k)\},$$

where its norm is given by

$$\|u\|_{W^{k,p}(\Omega,X)} := \begin{cases} \left(\sum_{|\alpha|\le k} \|\partial^\alpha u\|^p_{L^p(\Omega,X)}\right)^{1/p} & : p < \infty, \\ \max_{|\alpha|\le k} \|\partial^\alpha u\|_{L^\infty(\Omega,X)} & : p = \infty. \end{cases}$$

For sake of convenience we also employ $H^k := W^{k,2}$. For $s \in (0,\infty) \setminus \mathbb{N}$ let $W^s_p(\Omega, X)$ denote the X-valued Sobolev-Slobodeckij space, where

$$W^s_p(\Omega, X) := \{u \in L^p(\Omega, X) : \ \|u\|_{W^s_p(\Omega,X)} < \infty\},$$

and for $s = k + \lambda$ with $k \in \mathbb{N}_0$ and $\lambda \in (0, 1)$ the norm reads

$$\|u\|_{W^s_p(\Omega,X)} = \|u\|_{W^{k,p}(\Omega,X)} + \left(\sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{\|\partial^\alpha u(x) - \partial^\alpha u(y)\|^p_X}{\|x-y\|^{n+\lambda p}} \, dxdy\right)^{1/p}$$

provided $p < \infty$. We also introduce corresponding spaces for negative k and s respectively as the dual spaces. In the same manner we employ C^m for m-times continuously differentiable functions, and BC^m for those with bounded derivatives up to order $m \in \mathbb{N}_0$. The space of smooth X-valued functions is given through $C^\infty(\Omega, X) := \bigcap_{k=1}^\infty C^k(\Omega, X)$. In order to introduce some auxiliary results in this chapter we also use the symbol $C^{k,\lambda}$ for the usual Hölder space with $k \in \mathbb{N}_0$ and $\lambda \in (0, 1)$ and analogously $BC^{k,\lambda}$. They are defined in the same manner as Sobolev-Slobodeckij space introduced above. Moreover, we need zero time trace spaces which we define in the same manner as all other spaces occured so far. The X-valued Sobolev-Slobodeckij space with zero time trace for instance is given as

$${}_0W^s_p((0,T), X) := \{c_i \in W^s_p((0,T), X) : \ c_i|_{t=0} = 0\} \tag{1.1}$$

with $s \in (0, 1)$ and $sp > 1$. We employ this notation for all appearing spaces whenever time traces make sense.

We work with the duality product in several spaces as e.g. with respect to L^p and $L^{p'}$ with $1/p + 1/p' = 1$. It will be denoted by $(\cdot,\cdot)_{p,p'}$. If no confusion is possible we leave away the index.

For given Banach spaces X, Y the space of bounded linear operators from X to Y shall be denoted by $\mathscr{L}(X, Y)$, where $\mathscr{L}(X) := \mathscr{L}(X, X)$. The subclass of isomorphisms is denoted by $\mathscr{L}_{is}(X, Y)$ or $\mathscr{L}_{is}(X)$, respectively. The space of all real analytic functions from X to Y shall be denoted by $C^\omega(X, Y)$. If A is a linear operator in X then $D(A)$, $R(A)$ and $N(A)$ stand for its domain, range and kernel respectively, where $\sigma(A)$, $\sigma_p(A)$ and $\rho(A)$ mean its spectrum, point spectrum and its resolvent set. For a closed subspace X_0 of X we define the *part* A_0 *of* A *in* X_0 by

$$D(A_0) := \{x \in X_0 \cap D(A) : \ Ax \in X_0\}, \qquad A_0 x := Ax.$$

We write $X = Y$ if the spaces X and Y are isomorphic, i.e. there exists a linear and bijective map $\iota : X \to Y$ which is continuous with respect to the norms of X and Y and admits a continuous inverse. Note that in case X, Y are Banach spaces the continuity of ι^{-1} follows by the continuous inverse theorem.

Suppose $\{X, Y\}$ is an interpolation couple. By $(X, Y)_{\theta,p}$ we denote the real interpolation space with parameters $\theta \in (0, 1)$ and $p \in [1, \infty]$. By $[X, Y]_\theta$ we denote the complex interpolation space with parameter $\theta \in (0, 1)$. An introduction to interpolation theory can be found in [BL76].
Note in passing that the Sobolev-Slobodeckij space introduced above via integral norms coincides with a real interpolation space. That is for $s \in (0, \infty) \setminus \mathbb{N}$ and $s = k + \lambda$ with $k \in \mathbb{N}$ and $\lambda \in (0, 1)$ the representation

$$W_p^s(\Omega, X) = \big(W_p^k(\Omega, X), W_p^{k+1}(\Omega, X)\big)_{\lambda,p}$$

holds true.

All appearing constants, e.g. $C, M > 0$ denote generic constants which may vary from line to line, as long as it is not pointed out otherwise.

Note that further notation which is only employed in Part I respectively Part II is introduced in Section 3.1 respectively Section 8.1. Occasionally, we introduce some extra notation where it is needed. For convenience a list of symbols is added at the end of this thesis. In particular, it contains all maximal regularity spaces and operators we use throughout this thesis.

1.2. Useful Results

In this section we collect some useful results. Among others we give important embedding and also trace theorems.

We frequently employ embedding theorems, in particular Sobolev embedding theorems also of anisotropic type. The following propositions are given in [AF03, Theorem 4.12] and [Ama09, Chapter 3]. Let us first give the definition of class $\mathcal{HT}$.

Definition 1.1 *Let X be a Banach space and let $f \in \mathscr{S}(\mathbb{R}, X)$, where*

$$\mathscr{S}(\mathbb{R}, X) := \left\{ u \in C^\infty(\mathbb{R}, X) : \max_{\alpha,\beta \in \{0,\dots,N\}} \sup_{x \in \mathbb{R}} \|x^\alpha u^{(\beta)}(x)\|_X < \infty \ (N \in \mathbb{N}) \right\}.$$

Set

$$Hf(t) := \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \int\limits_{|s|>\varepsilon} \frac{f(t-s)}{s} ds.$$

Then X is of class $\mathcal{HT}$, if there exists a $p \in (1, \infty)$ such that $H \in \mathscr{L}(L^p(\mathbb{R}, X))$.

Note that all L^p-, $W^{k,p}$- and W^s_p-spaces for $1 < p < \infty$ are of class $\mathcal{HT}$, [KW04]. Now we are able to state

Proposition 1.2 (Sobolev embedding theorem) *Let $\mathcal{F}^s_p = W^{s,p}$ if s is an integer and $\mathcal{F}^s_p = W^s_p$ otherwise.*

(i) Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the cone condition. Let $0 \le t \le s < \infty$ and $1 < p \le q < \infty$ such that

$$s - \frac{n}{p} \ge t - \frac{n}{q}. \tag{1.2}$$

Then the embedding $\mathcal{F}^s_p(\Omega) \hookrightarrow \mathcal{F}^t_q(\Omega)$ holds true.
If t is a nonnegative integer, i.e. $t = k + \lambda$ with $k \in \mathbb{N}_0$ and $\lambda \in (0,1)$, and

$$s - \frac{n}{p} > t \tag{1.3}$$

then $\mathcal{F}^s_p(\Omega) \hookrightarrow BC^{k,\lambda}(\overline{\Omega})$ holds true.

(ii) Let X be a Banach space of class $\mathcal{HT}$ and suppose $T > 0$. Moreover, let $1 \le t \le s < \infty$ and $1 < p \le q < \infty$ satisfy (1.2) for $n = 1$, then the embedding $\mathcal{F}^s_p((0,T), X) \hookrightarrow \mathcal{F}^t_q((0,T), X)$ holds true.
If t is a nonnegative integer, i.e. $t = k + \lambda$ with $k \in \mathbb{N}_0$ and $\lambda \in (0,1)$, and (1.3) is satisfied for $n = 1$ then
$\mathcal{F}^s_p((0,T), X) \hookrightarrow BC^{k,\lambda}([0,T], X)$ holds true.

We continue with anisotropic embeddings which are content of [Ama09, Chapter 3] by H. Amann, in particular of Theorem 3.3.2, Theorem 3.7.5 and Theorem 3.9.1 therein. Note that we do not use the fact that Amann considered the vector valued case.

Proposition 1.3 (Anisotropic embedding theorems) *Let $\Omega_1 \subset \mathbb{R}^{n_1}$, $\Omega_2 \subset \mathbb{R}^{n_2}$ be domains satisfying the cone condition. For the parabolic weight vector $\omega = (2,1)$ let $\omega \cdot n := 2n_1 + n_2$. Let $\mathcal{F}^s_p = W^{s,p}$ if s is an integer and $\mathcal{F}^s_p = W^s_p$ otherwise.*

(i) Suppose $1 < p \le q < \infty$ and $0 \le t < s < \infty$ and that

$$s - \frac{\omega \cdot n}{p} \ge t - \frac{\omega \cdot n}{q}. \tag{1.4}$$

Then the embedding

$$\mathcal{F}_p^{s/2}(\Omega_1, L^p(\Omega_2)) \cap L^p(\Omega_1, \mathcal{F}_p^s(\Omega_2)) \hookrightarrow \mathcal{F}_q^{t/2}(\Omega_1, L^q(\Omega_2)) \cap L^q(\Omega_1, \mathcal{F}_q^t(\Omega_2))$$

holds true.

(ii) Suppose $1 < p < \infty$ *and* $0 \leq s, t < \infty$*, such that* t *is a natural even number with*

$$s > t + \frac{\omega \cdot n}{p}.$$

Then the embedding

$$\mathcal{F}_p^{s/2}(\Omega_1, L^p(\Omega_2)) \cap L^p(\Omega_1, \mathcal{F}_p^s(\Omega_2)) \hookrightarrow C^{t/2}(\Omega_1, C(\Omega_2)) \cap C(\Omega_1, C^t(\Omega_2))$$

holds true.

Occasionally we require some embeddings respectively corresponding estimates with constants being independent of T. In particular such estimates are employed in Part II in the proof of the local existence theorem. The following two lemmas provide necessary results.

Lemma 1.4 *Let* $1 < p < \infty$, $0 < s < 1$, $0 < T < T' < \infty$ *and* X *be a Banach space. Set*

$$E(T) \in \left\{ L^q((0,T), X),\, W_q^t((0,T), X) \right\},$$

such that there is a continuous embedding $W_p^s((0,T'), X) \hookrightarrow E(T')$ *for a certain* $q \in (1, \infty)$*. Let us denote the zero time trace space of* $E(T)$ *by* ${}_0E(T)$*, i.e. we have* ${}_0E(T) = L^q((0,T), X)$ *or* ${}_0E(T) = {}_0W_q^t((0,T), X)$*. Then for all* $T \in (0, T')$ *there is a continuous embedding*

$${}_0W_p^s((0,T), X) \hookrightarrow {}_0E(T)$$

whose embedding constant does not depend on T*.*

Proof. For a fixed $T \in (0, T')$ let ${}_0\mathcal{E}_T$ denote the extension operator from [PSS07, Proposition 6.1]

$${}_0\mathcal{E}_T : {}_0W_p^s((0,T), X) \to {}_0W_p^s((0,T'), X),$$

where $s \in (0,1)$. Its norm does not depend on T. Moreover, let $\mathcal{R}_T$ denote the restriction operator

$$\mathcal{R}_T : {}_0W_p^s((0,T'), X) \to {}_0W_p^s((0,T), X).$$

Then the commuting diagram

$$\begin{tikzcd}
{}_0W_p^s((0,T), X) \arrow[r, "{}_0\mathcal{E}_T"] \arrow[d] & {}_0W_p^s((0,T'), X) \arrow[d] \\
{}_0E(T) & {}_0E(T') \arrow[l, "\mathcal{R}_T"']
\end{tikzcd}$$

implies the assertion. Because of the embedding ${}_0W_p^s((0,T'),X) \hookrightarrow {}_0E(T')$ the embedding constant of

$${}_0W_p^s((0,T),X) \hookrightarrow {}_0E(T)$$

in general depends on T' but not on T. $\square$

Lemma 1.5 *Let X be a Banach space and $0 < T < \infty$. For $s > 1/p - 1/q$ not an integer and $q > p$ the following statements hold true.*

(i) We have

$${}_0W_p^s((0,T),X) \hookrightarrow L^q((0,T),X) \hookrightarrow L^p((0,T),X).$$

Moreover, for all $u \in {}_0W_p^s((0,T),X)$ the estimate

$$\|u\|_{L^p((0,T),X)} \leq CT^\eta \|u\|_{{}_0W_p^s((0,T),X)}$$

is valid for a constant $C > 0$ and an exponent $\eta > 0$ both being independent of T.

(ii) Let $s \in (0,1)$. For a $q \in (1,\infty)$ chosen sufficiently close to p let $\varepsilon > 0$ be a number satisfying $2/p - 2/q < \varepsilon < s$. Set $t := s + 1/p - 1/q - \varepsilon \in (0,1)$. Then we obtain the embedding chain

$${}_0W_p^s((0,T),X) \hookrightarrow {}_0W_q^t((0,T),X) \hookrightarrow {}_0W_p^{s-\varepsilon}((0,T),X).$$

In addition to above, suppose $T \in (0,1)$. Then for all $u \in {}_0W_p^s((0,T),X)$ the estimate

$$\|u\|_{{}_0W_p^{s-\varepsilon}((0,T),X)} \leq CT^\eta \|u\|_{{}_0W_p^s((0,T),X)}$$

is valid for a constant $C > 0$ and an exponent $\eta > 0$ both being independent of T.

Proof. (i): Firstly, for $q > p$ and $1/p = 1/q + 1/q'$ Hölder's inequality yields

$$\left(\int_0^T \|u\|_X^p \, dt\right)^{1/p} \leq \left(\int_0^T 1^{q'} \, dt\right)^{1/q'} \left(\int_0^T \|u\|_X^q \, dt\right)^{1/q} = T^{1/q'} \|u\|_{L^q((0,T),X)}.$$

Therefore we may estimate

$$\|u\|_{L^p((0,T),X)} \leq T^\eta \|u\|_{L^q((0,T),X)}, \tag{1.5}$$

for $\eta := 1/p - 1/q > 0$.
Secondly, Sobolev's embedding theorem yields for $s > 1/p - 1/q$

$$W_p^s((0,T'),X) \hookrightarrow L^q((0,T'),X)$$

for $T' > T$. An application of Lemma 1.4 yields

$${}_0W_p^s((0,T),X) \hookrightarrow L^q((0,T),X),$$

where the embedding constant $C > 0$ is independent of T, but may depend on T'. Therefore

$$\|u\|_{L^p((0,T),X)} \leq CT^\eta \|u\|_{{}_0W_p^s((0,T),X)}.$$

(ii): We proceed similarly to the proof of (i). Let $u \in {}_0W_p^s((0,T),X)$. Firstly, we estimate the two summands in

$$\|u\|_{{}_0W_p^{s-\varepsilon}((0,T),X)} = \|u\|_{L^p((0,T),X)} + \left(\int_0^T \int_0^T \frac{\|u(\tau)-u(\sigma)\|_X^p}{|\tau-\sigma|^{1+(s-\varepsilon)p}} \, d\tau d\sigma \right)^{1/p}$$

separately. The first summand has already been estimated in (1.5). For the second one, we also employ Hölder's inequality with $1/p = 1/q + 1/q'$. We have

$$\begin{aligned} \left(\int_0^T \int_0^T \frac{\|u(\tau)-u(\sigma)\|_X^p}{|\tau-\sigma|^{1+(s-\varepsilon)p}} \, d\tau d\sigma \right)^{1/p} &= \left(\int_0^T \int_0^T \left(\frac{\|u(\tau)-u(\sigma)\|_X}{|\tau-\sigma|^{1/p+(s-\varepsilon)}} \right)^p d\tau d\sigma \right)^{1/p} \\ \leq \left(\int_0^T \int_0^T 1^{q'} \, d\tau d\sigma \right)^{1/q'} &\cdot \left(\int_0^T \int_0^T \frac{\|u(\tau)-u(\sigma)\|_X^q}{|\tau-\sigma|^{q/p+(s-\varepsilon)q}} \, d\tau d\sigma \right)^{1/q} \\ = T^{2/q'} \left(\int_0^T \int_0^T \frac{\|u(\tau)-u(\sigma)\|_X^q}{|\tau-\sigma|^{1+tq}} \, d\tau d\sigma \right)^{1/q} & \end{aligned}$$

with $t = s + 1/p - 1/q - \varepsilon$ due to $q/p = 1 + q/q'$. We may add the esimtates of both terms and conclude

$$\|u\|_{{}_0W_p^{s-\varepsilon}((0,T),X)} \leq T^\eta \|u\|_{{}_0W_q^t((0,T),X)} \tag{1.6}$$

with $\eta := 1/p - 1/q > 0$ since $T \in (0,1)$. This is the fact that

$$W_q^t((0,T),X) \hookrightarrow W_p^{s-\varepsilon}((0,T),X) \tag{1.7}$$

holds. Note that although $q > p$ by assumption, estimate (1.6) and embedding (1.7) remain valid since $(0,T)$ is of finite length, cf. [AF03, Remark 4.13]. Due to the choice of t the Sobolev indices satisfy $s - 1/p \geq t - 1/q$ and hence

$$W_p^s((0,T'),X) \hookrightarrow W_q^t((0,T'),X)$$

holds for $T' \in (T,\infty)$. Thanks to Lemma 1.4 the embedding constant of

$${}_0W_p^s((0,T),X) \hookrightarrow {}_0W_q^t((0,T),X)$$

does not depend on $T < T'$. In view of (1.6) we infer

$$\|u\|_{{}_0W_p^{s-\varepsilon}((0,T),X)} \leq CT^\eta \|u\|_{{}_0W_p^s((0,T),X)}$$

with a constant $C > 0$ and an exponent $\eta > 0$ both being independent of T. □

A useful result for Sobolev spaces on domains $\Omega_1 \times \Omega_2$ is given by the next Lemma.

Lemma 1.6 *Let* $1 < p < \infty$ *and* $0 \leq s < \infty$. *Moreover, let* $\Omega_1 \subset \mathbb{R}^{n_1}$, $\Omega_2 \subset \mathbb{R}^{n_2}$ *be domains. Let* $\mathcal{F}_p^s = W^{s,p}$ *if* s *is an integer and* $\mathcal{F}_p^s = W_p^s$ *otherwise. Then*

$$\mathcal{F}_p^s(\Omega_1 \times \Omega_2) = \mathcal{F}_p^s(\Omega_1, L^p(\Omega_2)) \cap L^p(\Omega_1, \mathcal{F}_p^s(\Omega_2))$$

holds true in the sense of equivalent norms.

Proof. If s is an integer the claim follows by writing down the norms. The case $s \in (0,\infty) \setminus \mathbb{N}$ is contained as a special case in [Ama09, Theorem 3.6.3]. □

In the discussion of inhomogeneous boundary values in Part II suitable trace operators for maximal regularity classes need to be studied. To this end, we employ the following result, which is content of [Mar87, Theorem 2].

Proposition 1.7 (Trace theorem) *Let* Ω *be a bounded or unbounded Lipschitz domain. Suppose* $1 < p < \infty$ *and* $s > l - 1/p$. *In case* $s - 1/p \geq 1$ *suppose in addition that* Ω *is a* $C^{k,\lambda}$*-domain and* $k + \lambda > s - 1/p$. *Moreover, suppose* $s - 1/p$ *is not an integer. Then the mapping*

$$Tr_l : \mathcal{F}_p^s(\Omega) \to \prod_{j=0}^{l-1} W_p^{s-1/p-j}(\partial\Omega)$$

is a surjection and admits a bounded linear right inverse, where $\mathcal{F}_p^s = W^{s,p}$ *if s is an integer and* $\mathcal{F}_p^s = W_p^s$ *otherwise.*

In the proof of the local existence theorem in Part II of this thesis we employ mapping properties for Nemytskij operators acting on Sobolev-Slobodekij spaces. See Section 3.1 in [Sic96] for the following result:

Proposition 1.8 *Let* $0 < s < 1$ *and* $1 < p < \infty$. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be sufficiently smooth.*
(i) Suppose $W_p^s \hookrightarrow L^\infty$. *Then the following assertions are equivalent:*

- $g : W_p^s \to W_p^s$.
- *There exists a constant* $C > 0$, *such that*

$$\|g \circ \phi\|_{W_p^s} \leq C \|g'\|_{L^\infty(\phi(\mathbb{R}^n))} \|\phi\|_{W_p^s}$$

  holds for all $\phi \in W_p^s(\mathbb{R}^n)$.
- $g(0) = 0$ *and* $g' \in L^\infty_{loc}(\mathbb{R})$.

(ii) Suppose $W_p^s \not\hookrightarrow L^\infty$. *Then the following assertions are equivalent:*

- $g : W_p^s \to W_p^s$.

- *There exists a constant $C > 0$, such that*

$$\|g \circ \phi\|_{W_p^s} \leq C\|g'\|_{L^\infty}\|\phi\|_{W_p^s}$$

holds for all $\phi \in W_p^s(\mathbb{R}^n)$.

- $g(0) = 0$ *and* $g' \in L^\infty$.

In order to prove the local existence result in Part II we also employ

Lemma 1.9 (Mean value theorem – integral representation) *Let X, Y be Banach spaces and let $K \subset X$ be convex and $f \in C^1(K, Y)$. Then*

$$f(u) - f(v) = \left[\int_0^1 Df[tu + (1-t)v]dt\right](u - v) \qquad (u, v \in K).$$

Moreover, the estimate

$$\|f(u) - f(v)\|_Y \leq \sup_{w \in K} \|Df[w]\|_{\mathscr{L}(X,Y)}\|u - v\|_X \qquad (u, v \in K)$$

holds true, where Df denotes the Fréchet derivative of f.

For the proof of the local-in-time results in Chapter 7 and Chapter 11 we employ the contraction mapping principle.

Proposition 1.10 (Contraction mapping principle) *Let (M, d) be a complete metric space and suppose the map $\Phi : M \to M$ is a contraction, i.e. there exists a $\theta \in (0, 1)$, such that*

$$d(\Phi(x), \Phi(y)) \leq \theta d(x, y) \qquad (x, y \in M).$$

Then Φ admits a unique fixed point $x_0 \in M$, i.e. $\Phi(x_0) = x_0$.

The global-in-time existence result in Part II makes use of L^p-estimates, where Gronwall's inequality is required. It reads

Lemma 1.11 (Gronwall inequality) *Let $T > 0$ be given and let $\varphi \in L^1(0, T)$ be nonnegative and satisfy a.e. the integral inequality*

$$\varphi(t) \leq C_1 \int_0^t \varphi(s)ds + C_2$$

for constants $C_1, C_2 > 0$. Then

$$\varphi(t) \leq C_2(1 + C_1 te^{C_2 t})$$

holds for a.e. $t \in (0, T)$.

Chapter 2

Operator Classes, Maximal Regularity and Closedness of Operator Sums

In this chapter we give basic definitions and important results on operator classes used throughout this work. In particular, we introduce the notions of *sectoriality*, *bounded $\mathcal{H}^\infty$-calculus* and *bounded imaginary powers*. For a comprehensive introduction to these concepts we refer to [DHP03, KW04] and [Haa06]. We give some results on the relationships of these operator properties and in particular illuminate the link of the corresponding angles.
We introduce *maximal regularity* in the setting of Cauchy problems and concentrate on the connection to the operator properties introduced before.
Having the standard results at hand we put the focus on the *operator sum* method which we employ in Chapter 5. For more information on this topic, see [DV87, KW01, PSS07].

2.1. Operator Classes

Let us start with the notion of sectoriality for which we refer to [DHP03]. Let $\phi \in (0, \pi)$ be fixed. We denote the complex sector of angle ϕ by

$$\Sigma_\phi := \{z \in \mathbb{C} : \ z \neq 0, \ |\arg(z)| < \phi\}.$$

Definition 2.1 *A closed linear operator A in a Banach space X is called sectorial, if*

(i) $\overline{D(A)} = X$, $\overline{R(A)} = X$,

(ii) $(-\infty, 0) \subset \rho(A)$ *and there is a* $c > 0$ *such that* $\|t(t + A)^{-1}\|_{\mathscr{L}(X)} \leq c$ *for all* $t > 0$.

In this case it is well-known (Taylor expansion), that there exists a $\phi \in [0, \pi)$ such that the uniform estimate in (ii) extends to all $\lambda \in \Sigma_{\pi-\phi}$. We call

$$\phi_A := \inf\{\phi : \ \rho(-A) \supset \Sigma_{\pi-\phi}, \ \sup_{\lambda\in\Sigma_{\pi-\phi}} \|\lambda(\lambda + A)^{-1}\|_{\mathscr{L}(X)} < \infty\}$$

the spectral angle of A. The class of sectorial operators is denoted by $\mathcal{S}(X)$.

It is well-known that in case $A \in \mathcal{S}(X)$ with $\phi_A < \pi/2$ then $-A$ generates a bounded holomorphic C_0-semigroup on X. Note that the density of $R(A)$ in X in Definition 2.1 implies that A is injective. The following statements can be found in [Haa06, Proposition 2.2.1].

Proposition 2.2 *Let A be a sectorial operator on a Banach space X.*

(a) $N(A) \cap \overline{R(A)} = \{0\}$, i.e. density of $R(A)$ implies $N(A) = \{0\}$.

(b) If X is reflexive it follows that $X = N(A) \oplus \overline{R(A)}$.

We turn to the notion $\mathcal{H}^\infty$-calculus. For a comprehensive introduction to this concept we refer to [DHP03, KW01, KW04, Haa06].

For $\sigma \in (0, \pi)$ we define

$$\mathcal{H}^\infty(\Sigma_\sigma) := \{f : \Sigma_\sigma \to \mathbb{C} : \ f \text{ holomorphic}, \ \|f\|_\infty < \infty\}$$

where

$$\|f\|_\infty := \sup\{|f(z)| : \ z \in \Sigma_\sigma\}.$$

For $\rho(z) := z/(1+z)^2$ we define the subalgebra

$$\mathcal{H}_0(\Sigma_\sigma) := \{f \in \mathcal{H}^\infty(\Sigma_\sigma) : \ \exists C, \varepsilon > 0 \, \forall z \in \Sigma_\sigma : |f(z)| \leq C|\rho(z)|^\varepsilon\}.$$

Let A be a sectorial operator in X with spectral angle ϕ_A. Let $\sigma \in (\phi_A, \pi)$ and $\theta \in (\phi_A, \sigma)$. The path

$$\Gamma = \{te^{i\theta} : \ \infty > t \geq 0\} \cup \{te^{-i\theta} : \ 0 \leq t < \infty\} \tag{2.1}$$

stays with the only possible exception at zero in the resolvent set of A. Note that Γ is oriented counterclockwise. It may be shown that due to Cauchy's integral formula and the sectoriality of the operator A

$$f(A) := \frac{1}{2\pi i} \int\limits_\Gamma f(\mu)(\mu - A)^{-1} d\mu \tag{2.2}$$

is a well-defined element of $\mathscr{L}(X)$ for every $f \in \mathcal{H}_0(\Sigma_\sigma)$. Formula (2.2) gives rise to the algebra homomorphism

$$\Phi_A : \mathcal{H}_0(\Sigma_\sigma) \to \mathscr{L}(X), \quad f \mapsto f(A), \tag{2.3}$$

called *Dunford calculus*. We extend Φ_A to $f \in \mathcal{H}^\infty(\Sigma_\sigma)$ by

$$f(A) := \rho(A)^{-1}(\rho f)(A), \qquad D(f(A)) := \{x \in X : (\rho f)(A)x \in D(A) \cap R(A)\},$$

which gives rise to a closed and densely defined operator in X which may be unbounded in general. This definition is compatible to the one above in case $f \in \mathcal{H}_0(\Sigma_\sigma)$ (Cauchy's Theorem, resolvent identity). Let us introduce the notion of a *bounded $\mathcal{H}^\infty$-calculus*, which goes back to McIntosh [McI86].

Definition 2.3 *Let $A \in \mathcal{S}(X)$. The operator A is said to admit a bounded $\mathcal{H}^\infty$-calculus on X, if there exists $\sigma > \phi_A$ such that Φ_A given in (2.3) is bounded (with respect to the topologies on $\mathcal{H}^\infty(\Sigma_\sigma)$ and $\mathscr{L}(X)$), that is there exists a $C_\sigma > 0$, such that*

$$\|f(A)\|_{\mathscr{L}(X)} \leq C_\sigma \|f\|_\infty \qquad (f \in \mathcal{H}_0(\Sigma_\sigma)). \tag{2.4}$$

We denote by $\mathcal{H}^\infty(X)$ the class of operators admitting a bounded $\mathcal{H}^\infty$-calculus on X. The number

$$\phi_A^\infty := \inf\{\sigma \in (\phi_A, \pi) : \ (2.4) \text{ holds}\}$$

is called $\mathcal{H}^\infty$-angle of A.

Remark 2.4 *The boundedness as it is given in (2.4) is equivalent to require $\|f(A)\|_{\mathscr{L}(X)} \leq C_\sigma$ for all $f \in \mathcal{H}_0(\Sigma_\sigma)$ with $\|f\|_\infty \leq 1$.*

Let us give a reformulation of the integral representation given in (2.2).

Lemma 2.5 *Let X be a Banach space, $T \in \mathcal{H}^\infty(X)$ and let $\phi \in (\phi_T, \pi)$. Then*

$$h(T) = \frac{1}{2\pi i} \int_\Gamma \frac{h(\lambda)}{\lambda} T(\lambda - T)^{-1} d\lambda \qquad (h \in \mathcal{H}_0(\Sigma_\phi)), \tag{2.5}$$

where Γ denotes the path in (2.1).

Proof. Let $\phi \in (\phi_T, \pi)$ and let $\theta_1, \theta_2 \in (\phi_T, \phi)$ with $\theta_1 < \theta_2$ and $0 < r_2 < r_1$. For $k = 1, 2$ set

$$\Gamma_k := \{te^{i\theta_k} : \ \infty > t \geq r_k\} \cup \{r_k e^{i\varphi} : \ \theta_k \geq \varphi \geq -\theta_k\} \cup \{te^{-i\theta_k} : \ r_k \leq t < \infty\},$$

such that Γ_1 stays on the right hand side of Γ_2. Let I_0 denote the right-hand side of (2.5). Then

$$I_0 = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{h(\lambda)}{\lambda} \left(\frac{1}{2\pi i} \int_{\Gamma_2} \mu(\mu - T)^{-1} d\mu \right) (\lambda - T)^{-1} d\lambda$$

and due to the resolvent identity we have

$$(\mu - \lambda)(\lambda - T)^{-1}(\mu - T)^{-1} = (\lambda - T)^{-1} - (\mu - T)^{-1} \qquad (\lambda \in \Gamma_1, \mu \in \Gamma_2).$$

Therefore

$$
\begin{aligned}
I_0 =& \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{h(\lambda)}{\lambda} \mu (\lambda - T)^{-1} (\mu - T)^{-1} d\mu d\lambda \\
=& \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{h(\lambda)\mu}{\lambda(\mu - \lambda)} (\lambda - T)^{-1} d\mu d\lambda \\
& \qquad - \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{h(\lambda)\mu}{\lambda(\mu - \lambda)} \mu (\mu - T)^{-1} d\mu d\lambda \\
=& \frac{1}{2\pi i} \int_{\Gamma_1} \frac{h(\lambda)}{\lambda} \underbrace{\left(\frac{1}{2\pi i} \int_{\Gamma_2} \frac{\mu}{\mu - \lambda} d\mu \right)}_{=:I(\lambda)} (\lambda - T)^{-1} d\lambda \\
& + \frac{1}{2\pi i} \int_{\Gamma_2} \mu \underbrace{\left(\frac{1}{2\pi i} \int_{\Gamma_1} \frac{h(\lambda)/\lambda}{\lambda - \mu} d\lambda \right)}_{=:J(\mu)} (\mu - T)^{-1} d\mu.
\end{aligned}
$$

By Cauchy's theorem $\mathrm{ind}_{\Gamma_2}(\lambda) = 1$ and $\mathrm{ind}_{\Gamma_1}(\mu) = 0$ yield $I(\lambda) = \lambda$ and $J(\mu) = 0$. Hence by taking the limit $r_1, r_2 \to 0$ the assertion follows. □

It is well-known that, if $A \in \mathcal{H}^\infty(X)$, then Φ_A extends to a bounded algebra homomorphism from $\mathcal{H}^\infty(\Sigma_\sigma)$ to $\mathscr{L}(X)$ for $\sigma > \phi_A^\infty$, cf. [KW04]. That is (2.4) holds true for all $f \in \mathcal{H}^\infty(\Sigma_\sigma)$. This is due to the so called Convergence lemma, see e.g. [Haa06, Proposition 5.1.4].

Lemma 2.6 *Let $A \in \mathcal{S}(X)$ and $\sigma \in (\phi_A, \pi)$. Suppose $(h_n)_n \subset \mathcal{H}^\infty(\Sigma_\sigma)$ be a sequence with $\sup_{n\in\mathbb{N}} \|h_n\|_\infty < \infty$ and that $h(z) := \lim_{n\to\infty} h_n(z)$ exists pointwise on Σ_σ.*
If $\sup_{n\in\mathbb{N}} \|h_n(A)\|_{\mathscr{L}(S)} < \infty$, then $h(A) \in \mathscr{L}(X)$ and $h_n(A)x \to h(A)x$ for all $x \in X$.

We turn to another important class: operators admitting bounded imaginary powers. Again we refer to [DHP03, KW04, Haa06].

Definition 2.7 *Let $A \in \mathcal{S}(X)$. Then A is said to admit bounded imaginary powers if*
$A^{is} \in \mathscr{L}(X)$ for all $s \in \mathbb{R}$ and if there exists a constant $C > 0$, such that

$$\|A^{is}\|_{\mathscr{L}(X)} \le C \qquad (|s| \le 1).$$

The class of such operators shall be denoted by $\mathcal{BIP}(X)$. The number

$$\theta_A := \limsup_{|s|\to\infty} \frac{1}{|s|} \log \|A^{is}\|_{\mathscr{L}(X)}$$

is called power angle of $A \in \mathcal{BIP}(X)$.

The relationships between $\mathcal{H}^\infty$-calculus, bounded imaginary powers and sectoriality

$$\mathcal{H}^\infty(X) \subset \mathcal{BIP}(X) \subset \mathcal{S}(X), \qquad \phi_A \le \theta_A \le \phi_A^\infty \tag{2.6}$$

hold true. To see this we make use of the following characterization of bounded imaginary powers. An operator A admits bounded imaginary powers if and only if there exist a constant $M \ge 1$ and a number θ, such that

$$\|A^{is}\|_{\mathscr{L}(X)} \le M e^{\theta|s|} \qquad (s \in \mathbb{R}). \tag{2.7}$$

The power angle may be characterized through $\theta_A = \inf\{\theta : (2.7) \text{ holds}\}$, see [DHP03]. Then for $s \in \mathbb{R}$ and $\sigma \in (0,\pi)$ we consider the imaginary power function $f : \Sigma_\sigma \to \mathbb{C}$, $f(z) := z^{is}$. From

$$|f(z)| = |z^{is}| = |e^{is\log z}| = e^{|s|arg(z)} \le e^{\sigma|s|} \qquad (z \in \Sigma_\sigma)$$

we infer $\|f\|_\infty \le e^{\sigma|s|}$ and consequently $\theta_A \le \phi_A^\infty$. Hence (2.6) follows.

By a well-known result the $\mathcal{BIP}(X)$ property of A allows for a description of the domain of fractional powers $D(A^\alpha)$ for $0 < \alpha < 1$ by means of complex interpolation, see e.g. [Tri78]. Let

$$X_\alpha := (D(A^\alpha), \|\cdot\|_\alpha), \qquad \|x\|_\alpha := \|x\|_X + \|A^\alpha x\|_X \qquad (0 < \alpha < 1).$$

Then by [DHP03, Section 2.3]

$$D(A) \subset X_\alpha \subset X \qquad (0 < \alpha < 1)$$

and we have

Proposition 2.8 *Suppose $A \in \mathcal{BIP}(X)$ and let $0 < \alpha < 1$. Then X_α is isomorphic to the complex interpolation space $[X, D(A)]_\alpha$.*

A proof of Proposition 2.8 can be found in [Tri78] or [Yag84]. See also [See71].

Remark 2.9 *The statement of Proposition 2.8 is false in general if A is 'only' sectorial even in case X is a Hilbert space. This follows e.g. from [Are04, 4.4.10 Fractional powers and BIP] in combination with [Ven93], see also [MS00].*

We turn to the $\mathcal{R}$-bounded versions of sectoriality and holomorphic functional calculus. For an introduction of $\mathcal{R}$-boundedness see [DHP03, KW04]. The $\mathcal{R}$-boundedness of a family of operators is stronger then the uniform boundedness with respect to the operator norm and allows for a characterization of maximal regularity of a Cauchy problem due to celebrated results by Weis, [Wei01a, Wei01b] (see next section).

Definition 2.10 ($\mathcal{R}$-boundedness) *Let X and Y be Banach spaces. Then a family of operators*

$\mathcal{T} \subset \mathscr{L}(X,Y)$ is called $\mathcal{R}$-bounded, if there are a constant $C > 0$ and a $p \in [1,\infty)$, such that for all $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$, and all independent symmetric $\{-1,1\}$-valued random variables ε_j on a probability space $(\Omega, \mathcal{M}, P)$ for $j = 1, ..., N$ we have that

$$\|\sum_{j=1}^{N} \varepsilon_j T_j x_j\|_{L^p(\Omega,Y)} \leq C\|\sum_{j=1}^{N} \varepsilon_j x_j\|_{L^p(\Omega,X)}. \tag{2.8}$$

The smallest $C > 0$ such that (2.8) is satisfied is called $\mathcal{R}$-bound of $\mathcal{T}$ and denoted by $\mathcal{R}(\mathcal{T})$.

Based on the notion of $\mathcal{R}$-boundedness we introduce $\mathcal{R}$-sectoriality.

Definition 2.11 *A sectorial operator $A \in \mathcal{S}(X)$ is called $\mathcal{R}$-sectorial if there exists an angle $\phi \in (0,\pi)$ and a constant $C_\phi > 0$, such that*

$$\mathcal{R}\left(\{\lambda(\lambda + A)^{-1} :\ \lambda \in \Sigma_{\pi-\phi}\}\right) \leq C_\phi. \tag{2.9}$$

The class of $\mathcal{R}$-sectorial operators is denoted by $\mathcal{RS}(X)$. The number

$$\phi_A^{\mathcal{R}} := \inf\{\phi \in (0,\pi) :\ (2.9) \text{ holds}\} \tag{2.10}$$

is called $\mathcal{R}$-angle of A.

We have $\phi_A \leq \phi_A^{\mathcal{R}}$. Similarly, we introduce the notion of $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus:

Definition 2.12 *Let $A \in \mathcal{S}(X)$. We say that A admits an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus, if there exists a $\sigma > \phi_A$, such that $f(A) \in \mathscr{L}(X)$ for all $f \in \mathcal{H}_0(\Sigma_\sigma)$ and if there exists a constant $C_\sigma > 0$, such that*

$$\mathcal{R}\left(\{f(A) :\ f \in \mathcal{H}_0(\Sigma_\sigma),\ \|f\|_\infty \leq 1\}\right) \leq C_\sigma. \tag{2.11}$$

The number

$$\phi_A^{\mathcal{R}\infty} := \inf\{\sigma \in (\phi_A, \pi) :\ (2.11) \text{ holds}\}$$

is called $\mathcal{RH}^\infty$-angle of A.

A wide range of operators are known to admit a bounded $\mathcal{H}^\infty$-calculus or even an $\mathcal{R}$-bounded one. Let us have a look at some examples:

Example 2.13 *(a) In Hilbert spaces linear m-accretive operators admit an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus.*

(b) A classical example is given by the Laplacian $A_L := -\Delta$ in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ with domain $D(A_L) = W^{2,p}(\mathbb{R}^n)$. It is well-known that $A_L \in \mathcal{RH}^\infty(L^p(\mathbb{R}^n))$ with $\phi_{A_L}^{\mathcal{R}\infty} = 0$. This follows by Fourier transform and application of the Mikhlin multiplier theorem to estimate the Dunford integral.

(c) Another more general example is given by parameter-elliptic operators in $L^p(\mathbb{R}^n, X)$ for a Banach space X of class $\mathcal{HT}$ and $1 < p < \infty$, see [DHP03].

(d) *In some situations also the Stokes operator A_S as e.g. in halfspace subject to Robin boundary conditions is known to admit a bounded $\mathcal{H}^\infty$-calculus with $\phi^\infty_{A_S} = 0$, cf. [Saa03].*

In the abstract results applied below the notion of of property (α) for Banach spaces appears. Let us give its definition. For more information we refer again to [DHP03, KW01, KW04]. Here we only remark that reflexive L^p spaces and their closed subspaces, hence all crucial spaces used in this thesis, enjoy this property.

Definition 2.14 *(Property (α)) For a probability space $\mathcal{P}$ let $\mathcal{E}_\mathcal{P}$ denote the set of all independent symmetric $\{-1, 1\}$-valued random variables on $\mathcal{P}$. A Banach space X admits property (α) if there exist probability spaces $\mathcal{P} = (\Omega, \mathcal{M}, P)$, $\mathcal{P}' = (\Omega', \mathcal{M}', P')$, $p \in [1, \infty)$ and a constant $\alpha > 0$, such that for all $N \in \mathbb{N}$, $x_{jk} \in X$, $a_{jk} \in \mathbb{C}$, $|a_{jk}| \leq 1$, and $(\varepsilon_j)_{j=1,\dots,N} \subset \mathcal{E}_\mathcal{P}$, $(\varepsilon'_k)_{k=1,\dots,N} \subset \mathcal{E}_{\mathcal{P}'}$ the estimate*

$$\|\sum_{j,k=1}^{N} \varepsilon_j \varepsilon'_k a_{jk} x_{jk}\|_{L^p(\Omega\times\Omega',X)} \leq \alpha \|\sum_{j,k=1}^{N} \varepsilon_j \varepsilon'_k x_{jk}\|_{L^p(\Omega\times\Omega',X)}$$

holds.

Let us give some known results in case X is of class $\mathcal{HT}$ or admits property (α). The following proposition can be found in [CP01].

Proposition 2.15 *Let X be a Banach space of class $\mathcal{HT}$ and suppose $A \in \mathcal{BIP}(X)$. Then $A \in \mathcal{RS}(X)$ and $\phi^\mathcal{R}_A \leq \theta_A$.*

Altogether, we obtain the following inclusion chain for a given Banach space X of class $\mathcal{HT}$:

$$\mathcal{RH}^\infty(X) \subset \mathcal{H}^\infty(X) \subset \mathcal{BIP}(X) \subset \mathcal{RS}(X) \subset \mathcal{S}(X). \tag{2.12}$$

The corresponding angles satisfy

$$\phi_A \leq \phi^\mathcal{R}_A \leq \theta_A \leq \phi^\infty_A \leq \phi^{\mathcal{R}\infty}_A. \tag{2.13}$$

A result on the equality of angles is given in [KW01, Proposition 5.1]. Here no restriction on the underlying Banach space is required:

Proposition 2.16 *Let X be a Banach space. If $A \in \mathcal{H}^\infty(X) \cap \mathcal{RS}(X)$ then we have $\phi^\infty_A = \phi^\mathcal{R}_A$.*

In [KW04, Remark 12.10] the equivalences of $\mathcal{H}^\infty$-calculus and $\mathcal{RH}^\infty$-calculus is given in case we additionally assume that X admits property (α):

Proposition 2.17 *Let the Banach space X enjoy property (α). Then $A \in \mathcal{H}^\infty(X)$ if and only if $A \in \mathcal{RH}^\infty(X)$. In this case $\phi^\infty_A = \phi^{\mathcal{R}\infty}_A$.*

2.2. Maximal Regularity

In this section we define the notion of maximal regularity of a Cauchy problem, that is of the corresponding operator A, respectively. For more information on this topic see again e.g. [DHP03] and [KW04]. We give sufficient criteria for maximal regularity in terms of the operator properties introduced above and also give a famous characterization by Weis. Maximal regularity needs to be defined rigorously since several different notions appear even in closely related literature, such that this term may differ from author to author. We consider the Cauchy problem

$$\begin{cases} \dot{u} + Au &= f \quad \text{in} \quad J = (0,T), \\ u(0) &= u_0, \end{cases} \tag{2.14}$$

with $T \in (0, \infty]$ for a closed and densely defined operator $A : D(A) \to X$ and data f and u_0.

Definition 2.18 *Let $1 < p < \infty$, let X be a Banach space and $J = (0,T)$ for $T \in (0,\infty]$. Suppose A is closed and densely defined. Then A is said to admit maximal L^p-regularity on X for J if for each $f \in L^p(J,X)$ and each initial value $u_0 \in I_p(A) := (X, D(A))_{1-1/p,p}$ there exists a unique solution $u : J \to D(A)$ of (2.14) satisfying*

$$\|u\|_{W^{1,p}(J,X)} + \|Au\|_{L^p(J,X)} \leq C \left(\|f\|_{L^p(J,X)} + \|u_0\|_{I_p(A)} \right) \tag{2.15}$$

with a constant $C > 0$ independent of f, u_0. We denote the class of operators which admit maximal L^p-regularity for J by $\mathcal{MR}_T(X)$.

Note that due to [Sob64] the class $\mathcal{MR}_T(X)$ does not depend on p. If A admits maximal L^p-regularity for a $p \in (1,\infty)$ then for all $p \in (1,\infty)$. Therefore it remains meaningful to speak of 'maximal regularity' only instead of 'maximal L^p-regularity'.

Remark 2.19 *Maximal regularity of A in the sense from above means that*

$$\begin{pmatrix} \partial_t + A \\ \gamma_0 \end{pmatrix} : W^{1,p}(J,X) \cap L^p(J, D(A)) \to L^p(J,X) \times I_p(A)$$

is an isomorphism. However, there are different definitions in literature. In particular some authors ask only for $\|\dot{u}\|_{L^p(J,X)}$ in (2.15) and not $\|u\|_{W^{1,p}(J,X)}$. In case of $J = \mathbb{R}_+$ and $0 \notin \rho(A)$ these two requirements differ, since then (2.15) with $\|\dot{u}\|_{L^p(J,X)}$ would be too weak to guarantee the invertibility of $\partial_t + A$. However, in case of finite $T < \infty$ or $0 \in \rho(A)$, we may replace $\|u\|_{W^{1,p}(J,X)}$ by $\|\dot{u}\|_{L^p(J,X)}$ and the two definitions coincide, see [KW04]. Note that this is the case in this thesis, since we only work with maximal regularity on finite time intervals.
More generally, following the spirit of Definition 2.18 we will introduce the notion of maximal regularity with respect to an initial boundary value problem via the isomorphism property, see Definition 8.4.

Let us briefly comment on the maximal regularity of shifted operators: When $A + \delta \in \mathcal{MR}_T(X)$ for a shift $\delta > 0$ we have to distinguish in general the cases $T < \infty$ and $T = \infty$ again. In case $T < \infty$ this still yields $A \in \mathcal{MR}_T(X)$ by replacing f by $e^{\delta \cdot} f \in L^p(J, X)$ and u by $e^{\delta \cdot} u \in W^{1,p}(J, X)$. Apparently, in case $T = \infty$ this does not work.

Due to a celebrated result by Weis [Wei01b, Theorem 4.2] a characterization of maximal regularity may be given in terms of $\mathcal{R}$-sectoriality provided the underlying Banach space is of class $\mathcal{HT}$. We state this result for the special case of $0 \in \rho(A)$.

Proposition 2.20 *Let X be a Banach space of class $\mathcal{HT}$ and let $A \in \mathcal{S}(X)$ with $\phi_A < \pi/2$. Suppose $0 \in \rho(A)$. Then A admits maximal regularity on X for $J = (0, \infty)$ if and only if $A \in \mathcal{RS}(X)$ with $\phi_A^{\mathcal{R}} < \pi/2$.*

Let us state a corollary to the famous Dore-Venni Theorem of 1987 [DV87]. In view of Proposition 2.15 and Proposition 2.20 it yields a sufficient condition for A having maximal regularity. See Section 2.3 below for the precise statement of the Dore-Venni theorem.

Proposition 2.21 *Let X be a Banach space of class $\mathcal{HT}$ and let $J = (0, T)$ for $T \in (0, \infty]$. Suppose $A \in \mathcal{BIP}(X)$ with $\theta_A < \pi/2$. Then $A \in \mathcal{MR}_T(X)$.*

Remark 2.22 *Note that this in particular implies that $A \in \mathcal{H}^\infty(X)$, with angle $\phi_A^\infty < \pi/2$ yields $A \in \mathcal{MR}_T(X)$, by (2.12) and (2.13). In applications this is more convenient than showing $\mathcal{BIP}$.*

2.3. Closedness of Operator Sums

Before coming to the basic ideas of studying operator sums we consider the *canonical extension* of an operator.

Definition 2.23 *Let X be a Banach space and let $A : D(A) \to X$ be an operator in X. For $1 < p < \infty$ and a domain $\Omega \subset \mathbb{R}^n$ we define the* canonical extension *of A to $L^p(\Omega, X)$ by*

$$\tilde{A}u := Au, \qquad u \in D(\tilde{A}) := L^p(\Omega, D(A)).$$

All properties which we employ carry over from A to its canonical extension. This is given by

Lemma 2.24 *Let $\Omega \subset \mathbb{R}^n$ be a domain. Suppose X is a Banach space and $A : D(A) \to X$ is an operator in X, closed and densely defined. Let $1 < p < \infty$ and let $\tilde{A}$ denote its canonical extension to $L^p(\Omega, X)$. Then*

(a) $\rho(\tilde{A}) = \rho(A)$, $\sigma_p(\tilde{A}) = \sigma_p(A)$ *and so on.*

(b) $A \in \mathcal{S}(X)$ *with spectral angle* ϕ_A *implies* $\tilde{A} \in \mathcal{S}(L^p(\Omega, X))$ *with* $\phi_{\tilde{A}} = \phi_A$.

(c) *$A \in \mathcal{H}^\infty(X)$ with $\mathcal{H}^\infty$-angle ϕ_A^∞ implies $\tilde{A} \in \mathcal{H}^\infty(L^p(\Omega, X))$ with $\phi_{\tilde{A}}^\infty = \phi_A^\infty$.*

(d) *Let $\Omega' \subset \mathbb{R}^m$ be a domain, $X = L^p(\Omega')$ and let A be accretive in X. Then $\tilde{A}$ is accretive in $L^p(\Omega \times \Omega') = L^p(\Omega, L^p(\Omega'))$.*

In the following chapters, we will denote $\tilde{A}$ again by A in slight abuse of notation.

Let us briefly give a motivation from an application point of view why studying the closedness of sums of operators. Indeed, it gives another equivalent formulation of maximal regularity of a Cauchy problem, i.e. an operator sum of the form '$d/dt + A$' needs to be studied. We may read this as the sum of two closed operators. The question of maximal regularity of the Cauchy problem (2.14) – for convenience with $u_0 = 0$ – is the question if $d/dt + A$ is invertible. More precisely, suppose X is a Banach space and $A \in \mathcal{S}(X)$ with $\phi_A < \pi/2$, such that A generates a holomorphic C_0-semigroup on X. For a finite time interval $J = (0, T)$ we consider

$$B : D(B) \to L^p(J, X), \quad Bu := \frac{d}{dt}u, \qquad D(B) = {}_0W^{1,p}(J, X)$$

and define the canonical extension $\tilde{A}$ of A to $E := L^p(J, X)$ as in Defnition 2.23 for $\Omega = J$, i.e.

$$\tilde{A}u := Au, \qquad u \in D(\tilde{A}) := L^p(J, D(A)).$$

Then $\tilde{A} \in \mathcal{S}(E)$ and $\phi_{\tilde{A}} = \phi_A$, by Lemma 2.24. We have

$$D(\tilde{A}) \cap D(B) = L^p(J, D(A)) \cap {}_0W^{1,p}(J, X)$$

and we may work with the sum

$$\tilde{A} + B : D(\tilde{A}) \cap D(B) \to E.$$

From $A \in \mathcal{S}(X)$ with $\phi_A < \pi/2$ it follows that $N(\tilde{A} + B) = \{0\}$ and also that $\overline{R(\tilde{A} + B)} = E$. Now it becomes clear, that the following three statements are equivalent:

(i) The Cauchy problem (2.14) admits maximal regularity.

(ii) The operator sum $\tilde{A} + B$ is invertible in E.

(iii) The operator sum $\tilde{A} + B$ is closed in E.

In particular this motivates for an investigation of the closedness of the sum of closed operators. A lot results on this topic are known by now, e.g. the celebrated Theorem by Dore and Venni from 1987, [DV87]:

Proposition 2.25 *Let E be a Banach space of class $\mathcal{HT}$ and let $A, B \in \mathcal{S}(E)$ with $0 \in \rho(A) \cap \rho(B)$. Suppose*

(i) *$(\lambda - A)^{-1}(\mu - B)^{-1} = (\mu - B)^{-1}(\lambda - A)^{-1}$ for all $\lambda \in \rho(A)$, $\mu \in \rho(B)$,* *(ii)* $A, B \in \mathcal{BIP}(E)$ *with power angles* θ_A, θ_B*,*

(iii) $\theta_A + \theta_B < \pi$

are valid. Then the sum $A + B : D(A) \cap D(B) \to E$ *is closed and* $0 \in \rho(A + B)$*.*

The restriction $0 \in \rho(A) \cap \rho(B)$ in Proposition 2.25 is not necessary. Moreover, even
$A + B \in \mathcal{BIP}(E)$ may be obtained as has been shown in [PS90, Theorem 4].

The next Proposition is crucial in our approach, since it gives a sufficient condition for the invertibility of an operator sum. It is due to Prüß, cf. [Prü93, Theorem 8.5]. By (2.12) and (2.13) the class $\mathcal{BIP}(X)$ appearing in the statement of the proposition below contains the class $\mathcal{H}^\infty(X)$. Hence it applies to our situation in Chapter 5.

Proposition 2.26 *Suppose the Banach space* E *belongs to the class* $\mathcal{HT}$ *and assume*

(1) $\omega_A + A, \omega_B + B \in \mathcal{BIP}(E)$ *for some* $\omega_A, \omega_B \in \mathbb{R}$*,*

(2) A *and* B *are resolvent commuting,*

(3) $\theta_{A+\omega_A} + \theta_{B+\omega_B} < \pi$*.*

Then $A + B$ *with its natural domain* $D(A + B) = D(A) \cap D(B)$ *is closed and* $\sigma(A + B) \subset \sigma(A) + \sigma(B)$*. In particular, if* $\sigma(A) \cap \sigma(-B) = \emptyset$ *then* $A + B$ *is invertible.*

Besides this approach to the closedness question with bounded imaginary powers, there are also results on an approach with $\mathcal{R}$-sectoriality and $\mathcal{H}^\infty$-calculus. Here Kalton and Weis gave an 'asymmetrical' result in [KW01, Theorem 6.3] in the sense that the closedness of $A + B$ follows if one operator is $\mathcal{R}$-sectorial and the other one admits a bounded $\mathcal{H}^\infty$-calculus. More precisely, they proved

Proposition 2.27 *Let* X *be a Banach space and let* $A \in \mathcal{RS}(X)$ *and* $B \in \mathcal{H}^\infty(X)$ *be two resolvent commuting operators such that* $\phi_A^{\mathcal{R}} + \phi_B^\infty < \pi$*. Then the operator sum* $A + B$ *with domain* $D(A + B) := D(A) \cap D(B)$ *is closed and sectorial. Moreover, there exists* $C > 0$ *such that*

$$\|Ax\|_X + \|Bx\|_X \leq C\|(A + B)x\|_X \qquad (x \in D(A + B))$$

is satisfied and $0 \in \rho(A) \cup \rho(B)$ *implies* $0 \in \rho(A + B)$*. If* X *enjoys property* (α)*, then* $A + B \in \mathcal{RS}(X)$ *and* $\phi_{A+B}^{\mathcal{R}} \leq \max\{\phi_A^{\mathcal{R}}, \phi_B^\infty\}$*.*

Not that this is of particular interest when Cauchy problems are considered, since the property $A \in \mathcal{BIP}(X)$ is weakened to $A \in \mathcal{RS}(X)$.
Let us have a look at two more results, which we will directly apply in Chapter 5. The first one gives a "symmetric" statement on the sum $A+B$ and the composition AB provided A and B are resolvent commuting, see [KW01, Theorem 4.4] or [NS12, Proposition 3.5]:

Proposition 2.28 *Let X be a Banach space of class $\mathcal{HT}$ having property (α). Suppose*
$A, B \in \mathcal{H}^\infty(X)$ with $\phi_A^\infty + \phi_B^\infty < \pi$ be two resolvent commuting operators.

(a) Then $A+B$ admits an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus with $\phi_{A+B}^{\mathcal{R}\infty} \le \max\{\phi_A^\infty, \phi_B^\infty\}$.

(b) Let further $0 \in \rho(A)$. Then AB admits an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus with $\phi_{A+B}^{\mathcal{R}\infty} \le \phi_A^\infty + \phi_B^\infty$.

The second result deals with the non-commuting case and is crucial for this work. Let us give the analogous statement to Proposition 2.28 by Prüss and Simonett in the case A and B are non-commuting, see [PS07, Theorem 3.1]. To this end we state a commutator condition by Labbas and Terreni:

(**LT**) *Commutator condition by Labbas-Terreni.* Let $0 \in \rho(A)$. Further, assume that there are constants $C > 0$, $0 \le \alpha < \beta < 1$, $\psi_A > \phi_A^\infty$, $\psi_B > \phi_B^\infty$ satisfying $\psi_A + \psi_B < \pi$ and such that for all $\lambda \in \Sigma_{\pi-\psi_A}$ and all $\mu \in \Sigma_{\pi-\psi_B}$,

$$\|A(\lambda + A)^{-1}[A^{-1}, (\mu + B)^{-1}]\|_{\mathscr{L}(X)} \le \frac{C}{(1 + |\lambda|)^{1-\alpha}|\mu|^{1+\beta}}, \tag{2.16}$$

where $[A, B] = AB - BA$ denotes the commutator.

Having the Labbas-Terreni commutator condition at hand the statment on non-commuting operators reads as

Proposition 2.29 *Let E be a Banach space having property (α). Moreover, let $A, B \in \mathcal{H}^\infty(E)$ and suppose* (**LT**) *is valid. Then there exists a shift $\nu > 0$ such that $\nu + A + B \in \mathcal{H}^\infty(E)$ with $\phi_{\nu+A+B}^\infty \le \max\{\psi_A, \psi_B\}$.*

Remark 2.30 *Notice that in [PS07] instead of property (α) for E the stronger property of an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus for B is assumed. However, in spaces having property (α) this is equivalent to having merely a bounded $\mathcal{H}^\infty$-calculus, see Proposition 2.17.*

Part I.

Stokes- and Navier-Stokes Equations with Perfect Slip on Wedge Type Domains

Chapter 3

Main Results

The content of Part I is based on the joint work with Jürgen Saal [MS14]. Both authors contributed equally to [MS14]. The author of this thesis checked if the underlying approach is applicable. He computed the transformation and did a major part of the analysis of the arisen operators. The nonlinear result in this thesis is not included in [MS14].

We study the well-posedness of the Stokes equations subject to perfect slip boundary conditions on wedge domains. We employ the operator sum method to show that the Stokes operator admits a bounded $\mathcal{H}^\infty$-calculus in weighted L^p_γ-spaces. As a consequence the linear Stokes equations admit maximal regularity. This in turn implies strong local-in-time well-posedness of the Navier-Stokes equations on wedges with suitable angles for small data and a p-interval close to 1 in the unweighted L^p-setting.

We consider the Navier-Stokes equations subject to perfect slip boundary conditions given as

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u + \nabla p + (u\cdot\nabla)u &=& f & \text{in} \quad (0,T)\times G, \\ \operatorname{div} \mathrm{u} &=& 0 & \text{in} \quad (0,T)\times G, \\ \nu\times\operatorname{curl} u = 0, \quad u\cdot\nu &=& 0 & \text{on} \quad (0,T)\times \partial G, \\ u(0) &=& u_0 & \text{in} \quad G. \end{array}\right. \tag{3.1}$$

Here

$$G = S_{\varphi_0}\times\mathbb{R}, \quad S_{\varphi_0} := \left\{(x_1,x_2) = (r\cos\varphi, r\sin\varphi)\in\mathbb{R}^2 :\ r>0,\ 0<\varphi<\varphi_0\right\}$$

represents a domain of wedge type and ν denotes the outer normal vector at ∂G. As a standard linearization we consider the Stokes equations

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u + \nabla p &=& f & \text{in} \quad (0,T)\times G, \\ \operatorname{div} \mathrm{u} &=& 0 & \text{in} \quad (0,T)\times G, \\ \nu\times\operatorname{curl} u = 0, \quad u\cdot\nu &=& 0 & \text{on} \quad (0,T)\times \partial G, \\ u(0) &=& u_0 & \text{in} \quad G. \end{array}\right. \tag{3.2}$$

Our main result on the Stokes system reads as follows.

Theorem 3.1 *Assume that* $1 < p < \infty$, $\gamma \in \mathbb{R}$, *and* $\varphi_0 \in (0, \pi)$ *satisfy*

$$\min\left\{1, \left(\frac{\pi}{\varphi_0} - 1\right)^2\right\} > \left(2 - \frac{2+\gamma}{p}\right)^2. \tag{3.3}$$

Then the Stokes operator

$$\begin{aligned} &\mathcal{A}_S u := -\Delta u, \\ u \in D(\mathcal{A}_S) = \Big\{ &u \in L^p_{\sigma,\gamma}(G) : \ \nu \times \textit{curl}\, u = 0, \ \nu \cdot u = 0 \text{ on } \partial G, \\ &u/|(x_1, x_2)|^2, \ \partial^\alpha u \in L^p_\gamma(G, \mathbb{R}^3) \ (\alpha \in \mathbb{N}_0^3, \ |\alpha| \le 2) \Big\} \end{aligned}$$

associated to system (3.2) admits a bounded $\mathcal{H}^\infty$*-calculus on* $L^p_{\sigma,\gamma}(G)$ *with* $\mathcal{H}^\infty$*-angle*
$\phi^\infty_{\mathcal{A}_S} < \pi/2$.

Theorem 3.1 in particular implies that $\mathcal{A}_S$ generates a bounded analytic C_0-semigroup on
$L^p_{\sigma,\gamma}(G)$ and that it has maximal regularity. Hence we also have

Corollary 3.2 *Suppose the assumptions of Theorem 3.1 hold and let* $J = (0, T)$ *be a time interval with* $T \in (0, \infty)$. *Then for each* $f \in L^p(J, L^p_{\sigma,\gamma}(G))$ *and* $u_0 \in (L^p_{\sigma,\gamma}(G), D(\mathcal{A}_S))_{1-1/p,p}$ *there exists a unique solution* $u \in L^p(J, L^p_{\sigma,\gamma}(G))$ *of (3.2) possessing the regularity*

$$u, u/|\cdot|^2, \partial_t u, \partial^\alpha u \in L^p(J, L^p_\gamma(G, \mathbb{R}^3)) \qquad (|\alpha| \le 2).$$

In particular, the map $[u \mapsto f]$ *defines an isomorphism between the corresponding spaces.*

Corollary 3.2 yields the following result concerning the nonlinear system (3.1).

Theorem 3.3 *(Local-in-time existence, small data). Let* $\gamma = 0$, $\varphi_0 \in (0, \frac{5}{9}\pi)$, $p \in (\frac{5}{3}, \frac{2}{3-\pi/\varphi_0})$. *Suppose* $T \in (0, \infty)$ *is given. Let the spaces* $\mathbb{E}^G_{p,\sigma}(T)$, $\mathbb{I}^G_{p,\sigma}$ *be given as in the definition below in Section 3.1. Then there exists a* $\kappa > 0$, *such that for all* $u_0 \in \mathbb{I}^G_{p,\sigma}$ *with* $\|u_0\|_{\mathbb{I}^G_{p,\sigma}} < \kappa$ *the nonlinear problem (7.1) admits a unique solution* $u \in \mathbb{E}^G_{p,\sigma}(T)$.

Remark 3.4 *We remark that by obvious modifications of the proofs our main results remain valid in case that the underlying domain is a two-dimensional wedge. Then we have*
$G = S_{\varphi_0} \subset \mathbb{R}^2$ *and the boundary conditions take the form*

$$\textit{curl}\, u = 0, \quad u \cdot \nu = 0,$$

where curl $u = \partial_{x_1} u_2 - \partial_{x_2} u_1$ *for a two dimensional vector field* u.

We proceed as follows. In Chapter 4 we transform the resolvent problem associated to (3.2) via polar coordinates and Euler transformation to a degenerate problem on a layer. In Chapter 5 we prove an $\mathcal{H}^\infty$-calculus for the related linear operator of the transformed system. In Chapter 6 it is demonstrated how this result transfers to the Stokes operator associated to (3.2), i.e., we prove Theorem 3.1. Finally in Chapter 7 we show well-posedness of (3.1), i.e. we prove Theorem 3.3.

3.1. Notation in Part I

Let us fix the notation which we use in Part I. For a domain $\Omega \subset \mathbb{R}^n$ let $C_c^\infty(\Omega, X)$ denote the space of smooth and compactly supported X-valued functions defined on Ω and $C_{c,\sigma}^\infty(\Omega, \mathbb{R}^n) := \{\varphi \in C_c^\infty(\Omega, \mathbb{R}^n) : \operatorname{div} \varphi = 0\}$. For $\gamma \in \mathbb{R}$ set

$$\mu_\gamma(U) := \int\limits_U |(x_1, x_2)|^\gamma \, d(x_1, x_2, y) \quad (U \in \mathscr{B}(\mathbb{R}^3)),$$

where $\mathscr{B}(\mathbb{R}^3)$ denotes the Borel σ-algebra. On the wedge $G = S_{\varphi_0} \times \mathbb{R}$ we define weighted Bochner-Lebesgue and Sobolev spaces via

$$\begin{aligned} L^p_\gamma(G, X) &:= L^p(G, \mu_\gamma, X), \\ W^{k,p}_\gamma(G, X) &:= \{u \in L^p_\gamma(G, X) | \, \partial^\alpha u \in L^p_\gamma(G, X) \; (\alpha \in \mathbb{N}_0^3, \; |\alpha| \le k)\} \end{aligned}$$

for $k \in \mathbb{N}$, $1 \le p \le \infty$.

For $X = \mathbb{R}$ we define the weighted version of the homogeneous Sobolev space of first order via

$$\widehat{W}^{1,p}_\gamma(G) := \{\varphi \in L^1_{loc}(G) : \; \nabla \varphi \in L^p_\gamma(G, \mathbb{R}^3)\} \tag{3.4}$$

and where $u \in L^p_{loc}(G)$ means that u is p-integrable on every compact $K \subset G$. For the local-in-time existence result proved in Chapter 7 we employ the maximal regularity spaces

$$\begin{aligned} &\mathbb{E}^G_p(T) := W^{1,p}((0,T), L^p(G, \mathbb{R}^3)) \cap L^p((0,T), W^{2,p}(G, \mathbb{R}^3)), \\ &\mathbb{F}^G_p(T) := L^p((0,T) \times G, \mathbb{R}^3), \end{aligned}$$

and also set

$$\begin{aligned} \mathbb{E}^G_{p,\sigma}(T) &:= W^{1,p}((0,T), L^p_\sigma(G)) \cap L^p((0,T), D(\mathcal{A}_S)), \\ \mathbb{F}^G_{p,\sigma}(T) &:= L^p((0,T), L^p_\sigma(G)), \\ \mathbb{I}^G_{p,\sigma} &:= (L^p_\sigma(G), D(\mathcal{A}_S))_{1-1/p,p}\,. \end{aligned}$$

Chapter 4

Transformation of the Linear Resolvent Problem

In the present and the next chapter we consider the resolvent problem

$$\begin{cases} (\lambda - \Delta)u &= f \quad \text{in} \quad G, \\ \nu \times \operatorname{curl} u &= 0 \quad \text{on} \quad \partial G, \\ u \cdot \nu &= 0 \quad \text{on} \quad \partial G, \end{cases} \tag{4.1}$$

on a three-dimensional wedge as it is given above. We aim to prove that the associated Laplacian admits a bounded $\mathcal{H}^\infty$-calculus. To this end we proceed as follows. In the first step we introduce cylinder coordinates, while in a second step we apply the Euler transformation. In a third step we rescale the appearing terms such that in the transformed setting we can work in unweighted L^p-spaces on a layer. This chapter is devoted to the transformation of (4.1).

Let again $\varphi_0 \in (0, \pi)$ denote the angle of the wedge and set $I := (0, \varphi_0)$. The transformation to polar coordinates, respectively cylinder coordinates is given by

$$\psi_P : \mathbb{R}_+ \times I \times \mathbb{R} \to G, \quad (r, \varphi, y) \mapsto (r\cos\varphi, r\sin\varphi, y) = (x_1, x_2, y).$$

Since we deal with vector fields, we also employ the standard orthogonal basis for cylinder coordinates in $\mathbb{R}^3$ given by

$$e_r = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}, \qquad e_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \qquad e_y = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The orthogonal transformation matrix $\mathcal{O}$ for a vector field then reads

$$\mathcal{O} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In radial direction we apply the Euler transformation $r = e^x$, where in slight abuse of notation $x \in \mathbb{R}$ denotes the new variable. We set $\Omega := \mathbb{R} \times I \times \mathbb{R}$ and

$$\psi_E : \Omega \to \mathbb{R}_+ \times I \times \mathbb{R}, \quad (x, \varphi, y) \mapsto (e^x, \varphi, y) =: (r, \varphi, y).$$

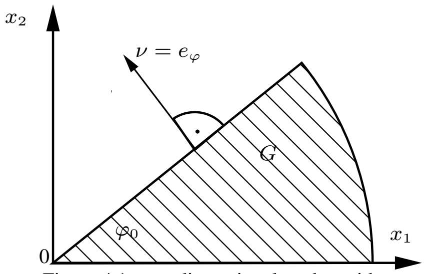

Figure 4.1: two-dimensional wedge with vertex in 0 and angle φ_0

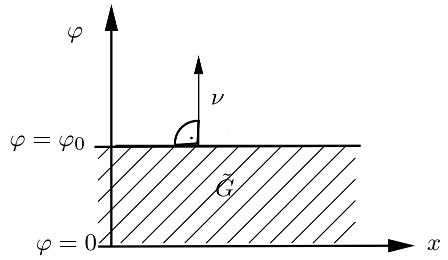

Figure 4.2: strip in polar coordinates after applying polar coordinate and Euler transformation

It is then clear that

$$\psi := \psi_P \circ \psi_E : \Omega \to G$$

is a diffeomorphism.

Having introduced all required transformations we define the induced pull-back mapping for a parameter $\beta \in \mathbb{R}$ which we determine below.

Definition 4.1 *Let $\beta \in \mathbb{R}$ be given. Suppose u is the solution of (4.1). We define the pull-back Θ^* through*

$$v := \Theta^* u := e^{-\beta x} \mathcal{O}^{-1} u \circ \psi.$$

We call the inverse of Θ^* push-forward and denote it by $\Theta_* = (\Theta^*)^{-1}$.

By Definition 4.1 it is clear that the push-forward is given by

$$u = \Theta_* v = \mathcal{O}(e^{\beta x} v) \circ \psi^{-1}. \tag{4.2}$$

In the following we also employ $\tilde{\Theta}^*$ which is given through

$$g = (g_x, g_\varphi, g_y) := \tilde{\Theta}^* f := e^{2x} \Theta^* f. \tag{4.3}$$

Then we have

Lemma 4.2 *Let $\tilde{\Theta}^*$ be given through (4.3). Then*

$$\tilde{\Theta}^* : L^p_\gamma(G, \mathbb{R}^3) \to L^p(\Omega, \mathbb{R}^3)$$

is an isomorphism.

Proof. $\tilde{\Theta}^*$ is well-defined: To see this let $f \in L^p_\gamma(G, \mathbb{R}^3)$. Then

$$\begin{aligned} \|\tilde{\Theta}^* f\|^p_{L^p(\Omega,\mathbb{R}^3)} &= \int\limits_{\mathbb{R}_y} \left(\int\limits_{\mathbb{R}_x \times I_\varphi} |e^{(2-\beta)x} \mathcal{O}^{-1} f(\psi(x,\varphi,y))|^p d(x,\varphi) \right) dy \\ &\le \underbrace{\|\mathcal{O}^{-1}\|^p_{L^\infty(I,\mathbb{R}^{3\times 3})}}_{\le 1} \int\limits_{\mathbb{R}_y} \left(\int\limits_{\mathbb{R}_x \times I_\varphi} |e^{(2-\beta)x} f(\psi(x,\varphi,y))|^p d(x,\varphi) \right) dy. \end{aligned} \tag{4.4}$$

We employ the diffeomorphism $\psi : \Omega \to G$ from above with $\det \psi' = e^{2x}$ which yields the transform

$$\begin{aligned}
&\int\limits_{\mathbb{R}_y} \left(\int\limits_{\mathbb{R}_x \times I_\varphi} |e^{(2-\beta)x} f(\psi(x,\varphi,y))|^p d(x,\varphi) \right) dy \\
&= \int\limits_{\mathbb{R}_y} \left(\int\limits_{S_{\varphi 0}} ||(x_1,x_2,0)|^{2-\beta} f(x_1,x_2,y)|^p \cdot |(x_1,x_2,0)|^{-2} d(x_1,x_2) \right) dy \\
&= \int\limits_G |f(x_1,x_2,y)|^p |(x_1,x_2,0)|^{p(2-\beta)-2} d(x_1,x_2,y) \\
&= \|f\|^p_{L^p_\gamma(G,\mathbb{R}^3)},
\end{aligned} \tag{4.5}$$

where in the last step we used $p(2-\beta) = \gamma + 2$. Since $\tilde{\Theta}^*$ is linear (4.4) and (4.5) also imply that

$$\tilde{\Theta}^* : L^p_\gamma(G,\mathbb{R}^3) \to L^p(\Omega,\mathbb{R}^3)$$

is continuous. Its inverse is given by $\tilde{\Theta}_* u = \mathcal{O} e^{(\beta-2)x} v \circ \psi^{-1}$. The open mapping principle yields that $\tilde{\Theta}_* : L^p(\Omega,\mathbb{R}^3) \to L^p_\gamma(G,\mathbb{R}^3)$ is continuous, too. $\square$

Therefore g given in (4.3) in particular fullfils $g \in L^p(\Omega,\mathbb{R}^3)$ and the choice $p(2-\beta) = \gamma+2$, i.e. $\beta = 2 - (\gamma+2)/p$ allows for a treatment of the transformed system in unweighted spaces. Note that this choice of β has already been utilized in [PS07]. In order to transform system (4.1) we compute the transformed differential operators div, curl, Δ and the transformed boundary conditions. These formal computations are given in the following. Employing $u = \Theta_* v$ we have:

$$\begin{aligned}
\partial_{x_1} u_1 \circ \psi = e^{(\beta-1)x} \big\{ &(\beta \cos^2\varphi + \sin^2\varphi) v_x + (\cos\varphi \sin\varphi - \beta \cos\varphi \sin\varphi) v_\varphi \\
&+ \cos^2\varphi (\partial_x v_x) - \cos\varphi \sin\varphi (\partial_\varphi v_x) - \cos\varphi \sin\varphi (\partial_x v_\varphi) \\
&+ \sin^2\varphi (\partial_\varphi v_\varphi) \big\},
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
\partial_{x_2} u_1 \circ \psi = e^{(\beta-1)x} \big\{ &(\beta \cos\varphi \sin\varphi - \cos\varphi \sin\varphi) v_x + (-\beta \sin^2\varphi - \cos^2\varphi) v_\varphi \\
&+ \cos\varphi \sin\varphi (\partial_x v_x) + \cos^2\varphi (\partial_\varphi v_x) - \sin^2\varphi (\partial_x v_\varphi) \\
&- \cos\varphi \sin\varphi (\partial_\varphi v_\varphi) \},
\end{aligned} \tag{4.7}$$

$$\partial_y u_1 \circ \psi = e^{\beta x} \{ \cos\varphi (\partial_y v_x) - \sin\varphi (\partial_y v_\varphi) \}, \tag{4.8}$$

$$\begin{aligned}
\partial_{x_1} u_2 \circ \psi = e^{(\beta-1)x} \big\{ &(\beta \cos\varphi \sin\varphi - \cos\varphi \sin\varphi) v_x + (\beta \cos^2\varphi + \sin^2\varphi) v_\varphi \\
&+ \cos\varphi \sin\varphi (\partial_x v_x) - \sin^2\varphi (\partial_\varphi v_x) + \cos^2\varphi (\partial_x v_\varphi) \\
&- \cos\varphi \sin\varphi (\partial_\varphi v_\varphi) \},
\end{aligned} \tag{4.9}$$

$$\begin{aligned}\partial_{x_2}u_2\circ\psi = e^{(\beta-1)x}&\left\{(\beta\sin^2\varphi+\cos^2\varphi)v_x+(\beta\cos\varphi\sin\varphi-\cos\varphi\sin\varphi)v_\varphi\right.\\&+\sin^2\varphi(\partial_x v_x)+\cos\varphi\sin\varphi(\partial_\varphi v_x)+\cos\varphi\sin\varphi(\partial_x v_\varphi)\\&\left.+\cos^2\varphi(\partial_\varphi v_\varphi)\right\},\end{aligned}\tag{4.10}$$

$$\partial_y u_2\circ\psi = e^{\beta x}\left\{\sin\varphi(\partial_y v_x)+\cos\varphi(\partial_y v_\varphi)\right\},\tag{4.11}$$

$$\partial_{x_1}u_3\circ\psi = e^{(\beta-1)x}\left\{\beta\cos\varphi v_y+\cos\varphi(\partial_x v_y)-\sin\varphi(\partial_\varphi v_y)\right\},\tag{4.12}$$

$$\partial_{x_2}u_3\circ\psi = e^{(\beta-1)x}\left\{\beta\sin\varphi v_y+\sin\varphi(\partial_x v_y)+\cos\varphi(\partial_\varphi v_y)\right\},\tag{4.13}$$

$$\partial_y u_3\circ\psi = e^{\beta x}\partial_y v_y.\tag{4.14}$$

Consequently, we obtain

$$\begin{aligned}(\operatorname{div} u)\circ\psi &= (\operatorname{div}\Theta_* v)\circ\psi\\&= e^{(\beta-1)x}(\beta v_x+v_x+\partial_x v_x+\partial_\varphi v_\varphi)+e^{\beta x}\partial_y v_y,\end{aligned}\tag{4.15}$$

and

$$\begin{aligned}\tilde\Theta^*(\operatorname{curl} u) &= e^{2x}\left(e^{-x}\partial_\varphi v_y-\partial_y v_\varphi\right)e_r+e^{2x}\left(\partial_y v_x-e^{-x}(\beta v_y+\partial_x v_y)\right)e_\varphi\\&\quad+e^{x}\left((\beta+1)v_\varphi+\partial_x v_\varphi-\partial_\varphi v_x\right)e_y.\end{aligned}\tag{4.16}$$

Note that we use $\tilde\Theta^*(\operatorname{curl} u)$ in order to transform the boundary conditions, while $(\operatorname{div} u)\circ\psi$ is employed in Chapter 6 in the proof of Lemma 6.1 for the investigation of the Stokes operator. We turn to the transformation of the second order derivatives appearing in the Laplacian.

$$\begin{aligned}(\partial_{x_1}^2 u_1)\circ\psi = e^{(\beta-2)x}&\left\{\left(\beta^2\cos^3\varphi-\beta\cos^3\varphi+3\beta\cos\varphi\sin^2\varphi-3\cos\varphi\sin^2\varphi\right)v_x\right.\\&+(\beta-1)(-\beta\cos^2\varphi\sin\varphi+2\cos^2\varphi\sin\varphi-\sin^3\varphi)v_\varphi\\&+(2\beta\cos^3\varphi-\cos^3\varphi+3\cos\varphi\sin^2\varphi)(\partial_x v_x)\\&+(-2\beta\cos^2\varphi\sin\varphi+2\cos^2\varphi\sin\varphi-2\sin^3\varphi)(\partial_\varphi v_x)\\&+(-2\beta\cos^2\varphi\sin\varphi+3\cos^2\varphi\sin\varphi-\sin^3\varphi)(\partial_x v_\varphi)\\&+(2\beta\cos\varphi\sin^2\varphi-4\cos\varphi\sin^2\varphi)(\partial_\varphi v_\varphi)\\&+\cos^3\varphi(\partial_x^2 v_x)-2\cos^2\varphi\sin\varphi(\partial_x\partial_\varphi v_x)+\cos\varphi\sin^2\varphi(\partial_\varphi^2 v_x)\\&\left.-\cos^2\varphi\sin\varphi(\partial_x^2 v_\varphi)+2\cos\varphi\sin^2\varphi(\partial_\varphi\partial_x v_\varphi)-\sin^3\varphi(\partial_\varphi^2 v_\varphi)\right\},\end{aligned}\tag{4.17}$$

$$
\begin{aligned}
(\partial_{x_2}^2 u_1)\circ\psi =& e^{(\beta-2)x}\Big\{(\beta^2\cos\varphi\sin^2\varphi - 3\beta\cos\varphi\sin^2\varphi + 2\cos\varphi\sin^2\varphi + \beta\cos^3\varphi \\
& - \cos^3\varphi)v_x + (-\beta^2\sin^3\varphi + \beta\sin^3\varphi - 3\beta\cos^2\varphi\sin\varphi \\
& + 3\cos^2\varphi\sin\varphi)v_\varphi + (2\beta\cos\varphi\sin^2\varphi - 3\cos\varphi\sin^2\varphi + \cos^3\varphi)(\partial_x v_x) \\
& + (2\beta\cos^2\varphi\sin\varphi - 4\cos^2\varphi\sin\varphi)(\partial_\varphi v_x) \\
& + (-2\beta\sin^3\varphi + \sin^3\varphi - 3\cos^2\varphi\sin\varphi)(\partial_x v_\varphi) \\
& + (-2\beta\cos\varphi\sin^2\varphi + 2\cos\varphi\sin^2\varphi - 2\cos^3\varphi)(\partial_\varphi v_\varphi) \\
& + \cos\varphi\sin^2\varphi(\partial_x^2 v_x) + 2\cos^2\varphi\sin\varphi(\partial_x\partial_\varphi v_x) + \cos^3\varphi(\partial_\varphi^2 v_x) \\
& - \sin^3\varphi(\partial_x^2 v_\varphi) - 2\cos\varphi\sin^2\varphi(\partial_x\partial_\varphi v_\varphi) - \cos^2\varphi\sin\varphi(\partial_\varphi^2 v_\varphi)\Big\},
\end{aligned}
\tag{4.18}
$$

$$
(\partial_y^2 u_1)\circ\psi = e^{\beta x}(\cos\varphi(\partial_y^2 v_x) - \sin\varphi(\partial_y^2 v_\varphi)),
\tag{4.19}
$$

$$
\begin{aligned}
(\partial_{x_1}^2 u_2)\circ\psi =& e^{(\beta-2)x}\Big\{(\beta^2\cos^2\varphi\sin\varphi - 3\beta\cos^2\varphi\sin\varphi + 2\cos^2\varphi\sin\varphi + \beta\sin^3\varphi \\
& - \sin^3\varphi)v_x + (\beta^2\cos^3\varphi - \beta\cos^3\varphi + 3\beta\cos\varphi\sin^2\varphi \\
& - 3\cos\varphi\sin^2\varphi)v_\varphi + (2\beta\cos^2\varphi\sin\varphi - 3\cos^2\varphi\sin\varphi + \sin^3\varphi)(\partial_x v_x) \\
& + (-2\beta\cos\varphi\sin^2\varphi + 4\cos\varphi\sin^2\varphi)(\partial_\varphi v_x) \\
& + (2\beta\cos^3\varphi - \cos^3\varphi + 3\cos\varphi\sin^2\varphi)(\partial_x v_\varphi) \\
& + (-2\beta\cos^2\varphi\sin\varphi + 2\cos^2\varphi\sin\varphi - 2\sin^3\varphi)(\partial_\varphi v_\varphi) \\
& + \cos^2\varphi\sin\varphi(\partial_x^2 v_x) - 2\cos\varphi\sin^2\varphi(\partial_x\partial_\varphi v_x) + \sin^3\varphi(\partial_\varphi^2 v_x) \\
& + \cos^3\varphi(\partial_x^2 v_\varphi) - 2\cos^2\varphi\sin\varphi(\partial_x\partial_\varphi v_\varphi) + \cos\varphi\sin^2\varphi(\partial_\varphi^2 v_\varphi)\Big\},
\end{aligned}
\tag{4.20}
$$

$$
\begin{aligned}
(\partial_{x_2}^2 u_2)\circ\psi =& e^{(\beta-2)x}\Big\{(\beta^2\sin^3\varphi - \beta\sin^3\varphi + 3\beta\cos^2\varphi\sin\varphi - 3\cos^2\varphi\sin\varphi)v_x \\
& + (\beta^2\cos\varphi\sin^2\varphi - 3\beta\cos\varphi\sin^2\varphi + 2\cos\varphi\sin^2\varphi + \beta\cos^3\varphi \\
& - \cos^3\varphi)v_\varphi + (2\beta\sin^3\varphi - \sin^3\varphi + 3\cos^2\varphi\sin\varphi)(\partial_x v_x) \\
& + (2\beta\cos\varphi\sin^2\varphi - 2\cos\varphi\sin^2\varphi + 2\cos^3\varphi)(\partial_\varphi v_x) \\
& + (2\beta\cos\varphi\sin^2\varphi - 3\cos\varphi\sin^2\varphi + \cos^3\varphi)(\partial_x v_\varphi) \\
& + (2\beta\cos^2\varphi\sin\varphi - 4\cos^2\varphi\sin\varphi)(\partial_\varphi v_\varphi) \\
& + \sin^3\varphi(\partial_x^2 v_x) + 2\cos\varphi\sin^2\varphi(\partial_x\partial_\varphi v_x) + \cos^2\varphi\sin\varphi(\partial_\varphi^2 v_x) \\
& + \cos\varphi\sin^2\varphi(\partial_x^2 v_\varphi) + 2\cos^2\varphi\sin\varphi(\partial_x\partial_\varphi v_\varphi) + \cos^3\varphi(\partial_\varphi^2 v_\varphi)\Big\}
\end{aligned}
\tag{4.21}
$$

$$
(\partial_y^2 u_2)\circ\psi = e^{\beta x}(\sin\varphi(\partial_y^2 v_x) + \cos\varphi(\partial_y^2 v_\varphi)),
\tag{4.22}
$$

$$(\partial_{x_1}^2 u_3)\circ\psi = e^{(\beta-2)x}\left\{\beta^2\cos^2\varphi - \beta\cos^2\varphi + \beta\sin^2\varphi)v_y + (2\beta\cos^2\varphi - \cos^2\varphi\right.$$
$$+\sin^2\varphi)(\partial_x v_y) + (-2\beta\cos\varphi\sin\varphi + 2\cos\varphi\sin\varphi)(\partial_\varphi v_y)$$
$$\left.+\cos^2\varphi(\partial_x^2 v_y) - 2\cos\varphi\sin\varphi(\partial_x\partial_\varphi v_y) + \sin^2\varphi(\partial_\varphi^2 v_y)\right\}, \tag{4.23}$$

$$(\partial_{x_2}^2 u_3)\circ\psi = e^{(\beta-2)x}\left\{(\beta^2\sin^2\varphi - \beta\sin^2\varphi + \beta\cos^2\varphi)v_y + (2\beta\sin^2\varphi - \sin^2\varphi\right.$$
$$+\cos^2\varphi)(\partial_x v_y) + (2\beta\cos\varphi\sin\varphi - 2\cos\varphi\sin\varphi)(\partial_\varphi v_y)$$
$$\left.+\sin^2\varphi(\partial_x^2 v_y) + 2\cos\varphi\sin\varphi(\partial_x\partial_\varphi v_y) + \cos^2\varphi(\partial_\varphi^2 v_y)\right\}, \tag{4.24}$$

$$(\partial_y^2 u_3)\circ\psi = e^{\beta x}(\partial_y^2 v_y). \tag{4.25}$$

We sum up the second order derivatives componentwise and obtain the scalar Laplacian of each component:

$$\begin{aligned}\Delta u_1\circ\psi =&(\partial_{x_1}^2 u_1)\circ\psi + (\partial_{x_2}^2 u_1)\circ\psi + (\partial_y^2 u_1)\circ\psi\\ =&e^{(\beta-2)x}\left\{(\beta^2\cos\varphi - \cos\varphi)v_x + (-\beta^2\sin\varphi + \cos^2\varphi\sin\varphi + \sin^3\varphi)v_\varphi\right.\\ &+2\beta\cos\varphi(\partial_x v_x) - 2\sin\varphi(\partial_\varphi v_x) - 2\beta\sin\varphi(\partial_x v_\varphi) - 2\cos\varphi(\partial_\varphi v_\varphi)\\ &+\cos\varphi(\partial_x^2 v_x) + \cos\varphi(\partial_\varphi^2 v_x) - \sin\varphi(\partial_x^2 v_\varphi) - \sin\varphi(\partial_\varphi^2 v_\varphi)\\ &\left.+e^{2x}\cos\varphi(\partial_y^2 v_x) - e^{2x}\sin\varphi(\partial_y^2 v_\varphi)\right\},\end{aligned}$$

$$\begin{aligned}\Delta u_2\circ\psi =&(\partial_{x_1}^2 u_2)\circ\psi + (\partial_{x_2}^2 u_2)\circ\psi + (\partial_y^2 u_2)\circ\psi\\ =&e^{(\beta-2)x}\left\{(\beta^2\sin\varphi - \cos^2\varphi\sin\varphi - \sin^3\varphi)v_x + (\beta^2\cos\varphi - \cos\varphi)v_\varphi\right.\\ &+2\beta\sin\varphi(\partial_x v_x) + 2\cos\varphi(\partial_\varphi v_x) + 2\beta\cos\varphi(\partial_x v_\varphi) - 2\sin\varphi(\partial_\varphi v_\varphi)\\ &+\sin\varphi(\partial_x^2 v_x) + \sin\varphi(\partial_\varphi^2 v_x) + \cos\varphi(\partial_x^2 v_\varphi) + \cos\varphi(\partial_\varphi^2 v_\varphi)\\ &\left.+e^{2x}\sin\varphi(\partial_y^2 v_x) + e^{2x}\cos\varphi(\partial_y^2 v_\varphi)\right\},\end{aligned}$$

$$\begin{aligned}\Delta u_3\circ\psi =&(\partial_{x_1}^2 u_3)\circ\psi + (\partial_{x_2}^2 u_3)\circ\psi + (\partial_y^2 u_3)\circ\psi\\ =&e^{(\beta-2)x}\left\{e^{2x}(\partial_y^2 v_y) + (\partial_x^2 v_y) + (\partial_\varphi^2 v_y) + 2\beta(\partial_x v_y) + \beta^2 v_y\right\}.\end{aligned}$$

Having the transformed second order derivatives at hand we may compute the full transformation of the Laplacian via

$$\tilde{\Theta}^*(\Delta u) = e^{(2-\beta)x}\mathcal{O}^{-1}\begin{pmatrix}\Delta u_1\circ\psi\\ \Delta u_2\circ\psi\\ \Delta u_3\circ\psi\end{pmatrix},$$

and obtain

$$\tilde{\Theta}^*(\Delta u) = \begin{pmatrix} e^{2x}(\partial_y^2 v_x) + (\partial_x^2 v_x) + 2\beta(\partial_x v_x) + \beta^2 v_x + (\partial_\varphi^2 v_x) - v_x - 2(\partial_\varphi v_\varphi) \\ e^{2x}(\partial_y^2 v_\varphi) + (\partial_x^2 v_\varphi) + 2\beta(\partial_x v_\varphi) + \beta^2 v_\varphi + (\partial_\varphi^2 v_\varphi) - v_\varphi + 2(\partial_\varphi v_x) \\ e^{2x}(\partial_y^2 v_y) + (\partial_x^2 v_y) + 2\beta(\partial_x v_y) + \beta^2 v_y + (\partial_\varphi^2 v_y) \end{pmatrix}.$$

We introduce the polynomial P given through

$$P(\partial_x) := -(\partial_x^2 + (2\beta)\partial_x + \beta^2). \tag{4.26}$$

Then we end up with

$$\tilde{\Theta}^*(\Delta u) = \begin{pmatrix} e^{2x}\partial_y^2 v_x - P(\partial_x)v_x + \partial_\varphi^2 v_x - v_x - 2\partial_\varphi v_\varphi \\ e^{2x}\partial_y^2 v_\varphi - P(\partial_x)v_\varphi + \partial_\varphi^2 v_\varphi - v_\varphi + 2\partial_\varphi v_x \\ e^{2x}\partial_y^2 v_y - P(\partial_x)v_y + \partial_\varphi^2 v_y \end{pmatrix}.$$

It remains to transform the boundary conditions

$$u \cdot \nu = 0, \qquad \nu \times \operatorname{curl} u = 0 \quad \text{on } \partial G.$$

They are equivalent to

$$u \cdot \nu = 0, \quad (\operatorname{curl} u) \cdot \tau_1 = 0, \quad (\operatorname{curl} u) \cdot \tau_2 = 0 \quad \text{on } \partial G$$

for two linearly independent tangential vectors τ_1, τ_2. Besides using $\nu = \pm e_\varphi$ it is nearby to choose

$$\tau_1 = e_r, \quad \tau_2 = e_y$$

at $\varphi = 0$ and $\varphi = \varphi_0$ respectively. Together with (4.16) this yields

$$\partial_\varphi v_x = 0, \quad v_\varphi = 0, \quad \partial_\varphi v_y = 0 \quad \text{on } \partial\Omega = \mathbb{R} \times \{0, \varphi_0\} \times \mathbb{R}.$$

Remark 4.3 *We employ two different transformations for u and for f, namely $v = \Theta^* u$ and $g = \tilde{\Theta}^* f$. This is due to the factor e^{2x} arising from second order derivatives with respect to the variables x_1 and x_2. Employing Θ^* and $\tilde{\Theta}^*$ we may absorb this factor such that we end up with $g \in L^p(\Omega, \mathbb{R}^3)$ in (4.3) and have*

$$\tilde{\Theta}^*((\lambda - \Delta)u) = \tilde{\Theta}^*((\lambda - \Delta)\Theta_* v) = g.$$

In particular we obtain

$$\tilde{\Theta}^*(\lambda u) = \tilde{\Theta}^*(\lambda \Theta_* v) = e^{2x}\lambda v.$$

As a consequence for the analysis the domain of the Laplacian is transformed via Θ^* and the ground space via $\tilde{\Theta}^*$.

Summarizing our computations, the transformed system (4.1) reads as

$$\begin{cases} e^{2x}\lambda v_x - e^{2x}\partial_y^2 v_x + P(\partial_x)v_x - \partial_\varphi^2 v_x + v_x + 2\partial_\varphi v_\varphi &= g_x \quad \text{in } \Omega, \\ e^{2x}\lambda v_\varphi - e^{2x}\partial_y^2 v_\varphi + P(\partial_x)v_\varphi - \partial_\varphi^2 v_\varphi + v_\varphi - 2\partial_\varphi v_x &= g_\varphi \quad \text{in } \Omega, \\ e^{2x}\lambda v_y - e^{2x}\partial_y^2 v_y + P(\partial_x)v_y - \partial_\varphi^2 v_y &= g_y \quad \text{in } \Omega, \\ \partial_\varphi v_x = 0, \ \ v_\varphi = 0, \ \ \partial_\varphi v_y &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.27}$$

In the next chapter we prove strong well-posedness for system (4.27). As in [PS07] one difficulty here is to handle the non-standard differential operator $e^{2x}(\lambda - \partial_y^2)$ and the fact that this operator and $P(\partial_x)$ do not commute.

Chapter 5

Holomorphic Functional Calculus and Maximal Regularity of the Laplacian

The aim of this chapter is to prove an $\mathcal{H}^\infty$-calculus for the linear operator corresponding to problem (4.1). This will be derived by building up the full operator by its single parts via the operator sum method introduced in Chapter 2. In fact, we will prove that each single part admits a bounded $\mathcal{H}^\infty$-calculus. Based on commutative (Proposition 2.28, i.e. [KW01]) and non-commutative (Proposition 2.29, i.e. [PS07]) Kalton-Weis theorems this property transfers to the full linear operator, cf. Chapter 2.

For $1 < p < \infty$ and $\Omega = \mathbb{R}^2 \times I$ we set

$$X := L^p\left(\Omega, \mathbb{R}^3\right).$$

Note that for the sake of convenience from now on we write the space variables in the order $(x, y, \varphi) \in \mathbb{R}^2 \times I$, although we keep the order of components as $v = (v_x, v_\varphi, v_y)$. Occasionally we also write $\mathbb{R}_x$, $\mathbb{R}_y$, I_φ to indicate the relation between domain and the corresponding variable. Note that we may interchange the order of Sobolev spaces in the sense that for all domains $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^m$, integers $k, l \in \mathbb{N}_0$ and a Banach space E Fubini's theorem yields

$$W^{k,p}(\Omega_1, W^{l,p}(\Omega_2, E)) = W^{l,p}(\Omega_2, W^{k,p}(\Omega_1, E))$$

and the norms are equal.

The full operator given in (4.27) consists of the following single parts:

(1) Let P be given by (4.26). We define B in $L^p(\mathbb{R})$ by means of

$$Bu(x) = P(\partial_x)u(x), \quad x \in \mathbb{R}, \quad u \in D(B) = W^{2,p}(\mathbb{R}).$$

Its spectrum is given by the parabola $P(i\mathbb{R})$, which is symmetric about the real axis, open to the right, and has its vertex in $a_0 := -\beta^2 \in \mathbb{R}$. It is known that $\omega + B \in \mathcal{H}^\infty(L^p(\mathbb{R}))$ for $\omega > -a_0$, with $\phi^\infty_{\omega+B} < \pi/2$, see [PS07]. Additionally, $B - a_0$ is accretive in X which can be seen as follows: The shifted operator

$B - a_0 = -\partial_x^2 - 2\beta\partial_x$ admits the same domain as B. Hence we have to show that $((B-a_0)u, u|u|^{p-2})_{p,p'} \geq 0$ for arbitrary $u \in D(B)$. Indeed, from

$$\begin{aligned}\partial_x \left(u(x)|u(x)|^{p-2}\right) &= u'(x)|u(x)|^{p-2} + u(x)(p-2)|u(x)|^{p-3} \cdot \mathrm{sign}(u(x))u'(x) \\ &= (p-1)u'(x)|u(x)|^{p-2} \qquad\qquad (x \neq 0)\end{aligned}$$

and

$$-2\beta \int_{\mathbb{R}} u'(x)(u(x)|u(x)|^{p-2})dx = 2\beta(p-1)\int_{\mathbb{R}} u'(x)u(x)|u(x)|^{p-2}dx,$$

that is

$$\int_{\mathbb{R}} u'(x)(u(x)|u(x)|^{p-2})dx = 0,$$

we infer

$$\begin{aligned}\left((B-a_0)u, u|u|^{p-2}\right)_{p,p'} &= -(u'', u|u|^{p-2})_{p,p'} - 2\beta(u', u|u|^{p-2})_{p,p'} \\ &= (p-1)\int_{\mathbb{R}} |u'(x)|^2|u(x)|^{p-2}dx \geq 0 \qquad (u \in D(B)).\end{aligned} \tag{5.1}$$

Therefore $B - a_0$ is accretive. By Lemma 2.24 all listed properties also hold true for the canonical extension of B to X which we again denote by B.

(2) We denote by L_y the Laplacian in $L^p(\mathbb{R})$ in the y-variable:

$$L_y u(y) = -\partial_y^2 u(y), \qquad y \in \mathbb{R},\ u \in D(L_y) = W^{2,p}(\mathbb{R}).$$

The operator L_y admits a bounded $\mathcal{H}^\infty$-calculus in $L^p(\mathbb{R})$ with $\phi_{L_y}^\infty = 0$, see Example 2.13 (b). The spectrum is $\sigma(L_y) = [0,\infty)$. Furthermore L_y is accretive. As for B, by Lemma 2.24 the same holds true for the canonical extension to X which we again denote by L_y.

(3) We also have to deal with the multiplication operator M in $L^p(\mathbb{R})$ defined by

$$\begin{aligned}Mu(x) &= e^{2x}u(x), \quad x \in \mathbb{R}, \\ D(M) &= \{u \in L^p(\mathbb{R}) : [x \mapsto e^{2x}u(x)] \in L^p(\mathbb{R})\}.\end{aligned}$$

Its spectrum is given by $\sigma(M) = [0,\infty)$. Moreover, this operator admits a bounded $\mathcal{H}^\infty$-calculus with $\phi_M^\infty = 0$. To see this let $\sigma \in (0,\pi)$ and $\theta \in (0,\sigma)$ and let Γ denote the path in (2.1). Note that for all $\lambda \in \rho(M)$ the resolvent $(\lambda - M)^{-1}$ is realized through the multiplication with $(\lambda - e^{2x})^{-1}$. Then due to Cauchy's formula

$$\|f(M)g\|_{L^p(\mathbb{R})}^p = \int_{\mathbb{R}} \left| \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - e^{2x})^{-1}d\lambda\, g(x)\right|^p dx = \int_{\mathbb{R}} |f(e^{2x})|^p|g(x)|^p dx,$$

and hence

$$\|f(M)g\|_{L^p(\mathbb{R})} \leq \|f\|_\infty \|g\|_{L^p(\mathbb{R})} \qquad (f \in \mathcal{H}_0(\Sigma_\sigma),\ g \in L^p(\mathbb{R})).$$

Hence $\phi_M^\infty = 0$. Likewise the canonical extension of M to X enjoys the same properties by Lemma 2.24 and will again be denoted by M.

(4) We define $L_{N,D}$ in $L^p(I, \mathbb{R}^2)$ and L_N in $L^p(I)$ by

$$L_{N,D}v' := \begin{pmatrix} 1-\partial_\varphi^2 & 2\partial_\varphi \\ -2\partial_\varphi & 1-\partial_\varphi^2 \end{pmatrix} v', \quad L_N v_y := -\partial_\varphi^2 v_y$$

on

$$\begin{aligned} D(L_{N,D}) &:= \{v' = (v_x, v_\varphi) \in W^{2,p}(I, \mathbb{R}^2) :\ \partial_\varphi v_x = 0,\ v_\varphi = 0 \text{ on } \partial I\}, \\ D(L_N) &:= \{v_y \in W^{2,p}(I) :\ \partial_\varphi v_y = 0 \text{ on } \partial I\}, \end{aligned}$$

respectively. So, $L_{N,D}$ is subject to the Neumann conditions of v_x and the Dirichlet conditions of v_φ and L_N to the Neumann conditions of v_y in (4.27). Furthermore, we set

$$\begin{gathered} Lv := \begin{pmatrix} L_{N,D} & 0 \\ 0 & L_N \end{pmatrix} v, \quad v \in D(L), \\ D(L) := \{v \in W^{2,p}(I, \mathbb{R}^3) : \partial_\varphi v_x = 0,\ v_\varphi = 0,\ \partial_\varphi v_y = 0 \text{ on } \partial I\} \end{gathered}$$

in $L^p(I, \mathbb{R}^3)$. The spectrum of these operators can be determined explicitly. In fact, it is straight forward to verify that

$$\sigma_p(L_{N,D}) = \left\{ \left(\frac{\pi k}{\varphi_0} \pm 1 \right)^2 :\ k \in \mathbb{N} \right\} \cup \{1\}$$

with corresponding eigenfunctions

$$\begin{pmatrix} v_x^k \\ v_\varphi^k \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\pi k}{\varphi_0}\varphi\right) \\ \pm\sin\left(\frac{\pi k}{\varphi_0}\varphi\right) \end{pmatrix}, \quad k \in \mathbb{N}_0,\ \varphi \in I.$$

Next, by well-known results on eigenvalues of the Neumann-Laplacian we obtain

$$\sigma(L_N) = \sigma_p(L_N) = \{0\} \cup \left\{ \frac{\pi^2}{\varphi_0^2} k^2 :\ k \in \mathbb{N} \right\}.$$

Consequently, $\sigma(L) = \sigma_p(L) = \sigma_p(L_{N,D}) \cup \sigma_p(L_N)$, that is

$$\sigma(L) = \{0\} \cup \{1\} \cup \left\{ \left(\frac{\pi k}{\varphi_0} \pm 1 \right)^2 :\ k \in \mathbb{N} \right\} \cup \left\{ \frac{\pi^2}{\varphi_0^2} k^2 :\ k \in \mathbb{N} \right\}. \tag{5.2}$$

Furthermore, L admits a bounded $\mathcal{H}^\infty$-calculus on $L^p(I, \mathbb{R}^3)$ with $\phi_L^\infty = 0$, see e.g. [Duo90]. Therein the author constructs $\mathcal{H}^\infty$-estimates for elliptic operators on smooth domains in L^p for $1 < p < \infty$. Again, by Lemma 2.24 the canonical extension of L to X enjoys the same properties and will again be denoted by L.

As it is shown later on (see Lemma 6.1), eigenfunctions to the eigenvalue 0 will play no further rôle when dealing with the Stokes equations. Thus, we may exclude it. Note that this is even essential for the applicability of Proposition 2.26 below. To exclude the corresponding eigenspace we set

$$L_0^p(I) := \left\{ u \in L^p(I) : \int_I u(\varphi)d\varphi = 0 \right\}.$$

The projection onto this subspace is given as

$$\pi_0 : L^p(I) \longrightarrow L_0^p(I), u \mapsto u - \frac{1}{|I|} \int_I u(\varphi)d\varphi.$$

Then

$$\Pi_0 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi_0 \end{pmatrix} \tag{5.3}$$

is the projection onto the new ground space

$$X_0 := L^p(\mathbb{R}^2, L^p(I) \times L^p(I) \times L_0^p(I)). \tag{5.4}$$

We obviously have $(1 - \Pi_0)X = L^p(\mathbb{R}^2, E_0)$ with

$$E_0 = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle, \tag{5.5}$$

hence the decomposition

$$X = X_0 \oplus L^p(\mathbb{R}^2, E_0). \tag{5.6}$$

In particular we have $\Pi_0 L = L\Pi_0$. Thus $L_0 := L|_{X_0}$ is well-defined from its natural domain $D(L_0) := \Pi_0 D(L) = D(L) \cap X_0$ to X_0 and we have

$$\sigma(L_0) = \{1\} \cup \left\{ \left(\frac{\pi k}{\varphi_0} \pm 1 \right)^2 : k \in \mathbb{N} \right\} \cup \left\{ \frac{\pi^2}{\varphi_0^2} k^2 : k \in \mathbb{N} \right\}. \tag{5.7}$$

Remark 5.1 *By similar arguments we also could exclude the eigenspace corresponding to the eigenvalue 1. Observe that then the span of the two excluded spaces contains solenoidal fields, e.g. $\phi \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$ given through*

$$\phi(x, y) := \phi_\rho(x, y) \cdot e^{-(\beta+1)x} \begin{pmatrix} -y \\ 0 \\ xe^{-x} \end{pmatrix},$$

$$\phi_\rho(x, y) := \begin{cases} e^{-\frac{\rho^2}{\rho^2 - (x^2+y^2)}} & : |(x, y)| < \rho \\ 0 & : |(x, y)| \geq \rho \end{cases}, \quad x, y \in \mathbb{R},$$

where $\rho > 0$ is a given radius, such that $\phi_\rho \in C_c^\infty(\mathbb{R}^2)$ (with $\operatorname{supp}\phi_\rho \subset B_\rho(0)$) implies $\phi \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$. To check the divergence free condition we apply the transformed divergence (4.15) to the function $\phi = (\phi_x, 0, \phi_y)$ and have

$$\mathit{div}\,\phi = e^{(\beta-1)x}\left((\beta+1)\phi_x + \partial_x\phi_x\right) + e^{\beta x}\partial_y\phi_y,$$

and

$$\begin{aligned}(\beta+1)\phi_x + \partial_x\phi_x + e^x\partial_y\phi_y = &-(\beta+1)e^{-\frac{\rho^2}{\rho^2-(x^2+y^2)}-(\beta+1)x}y \\ &+ e^{-\frac{\rho^2}{\rho^2-(x^2+y^2)}-(\beta+1)x}\left(\frac{2\rho^2xy}{(\rho^2-(x^2+y^2))^2} + (\beta+1)y\right) \\ &- e^{-\frac{\rho^2}{\rho^2-(x^2+y^2)}-(\beta+1)x}\frac{2\rho^2xy}{(\rho^2-(x^2+y^2))^2} = 0.\end{aligned}$$

However, we want such fields to be included in the approach to the Stokes equations in Chapter 6.

By permanence properties of the $\mathcal{H}^\infty$-calculus this property remains valid for L_0, i.e. we have

$$L_0 \in \mathcal{H}^\infty(X_0), \quad \phi_{L_0}^\infty = 0, \tag{5.8}$$

cf. [DHP03, Proposition 2.11].

The full linear operator related to (4.27) is now build up by an operator sum. We start by considering the operator $A := (\kappa + L_y)M$ in X_0 for fixed $\kappa > 0$ and with natural domain

$$D(A) := \{u \in D(M) : \ Mu \in D(L_y)\}.$$

Lemma 5.2 *The operator A defined above admits a bounded $\mathcal{H}^\infty$-calculus on X_0 with $\phi_A^\infty = 0$.*

Proof. Since L_y has a bounded $\mathcal{H}^\infty$-calculus on X_0 with $\phi_{L_y}^\infty = 0$ this remains true for the shifted operator $\kappa + L_y$. By the fact that X_0 has property (α), $M \in \mathcal{H}^\infty(X_0)$ with $\phi_M^\infty = 0$, and since $0 \in \rho(\kappa + L_y)$ for $\kappa > 0$, we may apply Proposition 2.28, (that is [NS12, Proposition 3.5]) which yields the result. □

Next, we consider $A + B$ with natural domain $D(A) \cap D(B)$. Note that this domain reads as

$$\begin{aligned}D(A) \cap D(B) =&\{v \in D(M) : \ Mv \in D(L_y)\} \cap D(B) \\ =&\{v \in L^p(\mathbb{R}^2, \mathbb{R}^3) : \ [(x,y) \mapsto e^{2x}v(x,y)] \in L^p(\mathbb{R}_x, W^{2,p}(\mathbb{R}_y, \mathbb{R}^3))\} \\ &\cap W^{2,p}(\mathbb{R}_x, L^p(\mathbb{R}_y, \mathbb{R}^3)).\end{aligned} \tag{5.9}$$

Since A and B are non-commuting (in the resolvent sense), we employ Proposition 2.29 (that is [PS07, Theorem 3.1]) which is based on the Labbas Terreni commutator condition (LT), see (2.16). Having Proposition 2.29 at hand, we can prove

Lemma 5.3 *There is a $\nu > 0$ such that*

$$\nu + A + B \in \mathcal{H}^\infty(X_0), \quad \phi^\infty_{\nu+A+B} < \frac{\pi}{2}.$$

Proof. We compute the commutator in the Labbas Terreni condition (2.16) for A and B. To this end, note that on the one hand it is clear that B and $L_y + \kappa$ are resolvent commuting, while on the other hand M and B do not commute. Instead, we have the relation

$$MBf = e^{2x}P(\partial_x)f = P(\partial_x - 2)e^{2x}f =: B_{-2}Mf \tag{5.10}$$

for all $f \in D(MB) := \{v \in D(B) : \; Bv \in D(M)\}$, satisfied in the sense of distributions, where at this point M and B are regarded as operators in $L^p(\mathbb{R})$. Note that $D(B_{-2}) = D(B)$. Now, fix $\eta > 0$ such that

$$\sigma(\eta + B) \cup \sigma(\eta + B_{-2}) \subset \{z \in \mathbb{C} : \; \mathrm{Re} z > 0\}.$$

For $\varphi \in C_c^\infty(\mathbb{R})$ we therefore obtain

$$\begin{aligned} \big((\eta + B_{-2})^{-1}M(\eta + B)f, (\eta + B'_{-2})\varphi\big) &= (M(\eta + B)f, \varphi) \\ &= ((\eta + B_{-2})Mf, \varphi) \\ &= (Mf, (\eta + B'_{-2})\varphi). \end{aligned}$$

Since $(\eta + B'_{-2})(C_c^\infty(\mathbb{R}))$ lies dense in $L^{p'}(\mathbb{R})$, where $1/p + 1/p' = 1$, this yields that

$$M(D(MB)) \subset D(B),$$

hence $D(MB) \subset D(BM)$. Setting $f = (\mu + \eta + B)^{-1}g$ we arrive at

$$(\mu + \eta + B_{-2})^{-1}Mg = M(\mu + \eta + B)^{-1}g \quad (g \in D(M), \; \mu \in \Sigma_{\pi/2+\varepsilon}).$$

Regarded as operators in X_0 again, we further compute for $g \in D(M)$ and

$$f = (\mu + \eta + B)^{-1}g \in D(MB) \subset D(BM) :$$

$$\begin{aligned} & (\mu + \eta + B)Mf - M(\mu + \eta + B)f = BMf - \underbrace{MB}_{=B_{-2}M} f \\ \Leftrightarrow \quad & Mf - (\mu + \eta + B)^{-1}M(\mu + \eta + B)f = (\mu + \eta + B)^{-1}(B - B_{-2})Mf \\ \Leftrightarrow \quad & M(\mu + \eta + B)^{-1}g - (\mu + \eta + B)^{-1}Mg \\ & \qquad = (\mu + \eta + B)^{-1}(B - B_{-2}) \underbrace{M(\mu + \eta + B)^{-1}}_{=(\mu+\eta+B_{-2})^{-1}M} g \\ \Leftrightarrow \quad & [M, (\mu + \eta + B)^{-1}]g = (\mu + \eta + B)^{-1}(B - B_{-2})(\mu + \eta + B_{-2})^{-1}Mg. \end{aligned}$$

For $A = (\kappa + L_y)M$ this results in

$$[(\omega + A), (\mu + \eta + B)^{-1}] = (\mu + \eta + B)^{-1}Q(\mu + \eta + B_{-2})^{-1}(\omega + A) \tag{5.11}$$

on $D(A)$ for all $\mu \in \Sigma_{\pi/2+\varepsilon}$, $\omega > 0$, and with the first order differential operator

$$Q = B - B_{-2} = P(\partial_x) - P(\partial_x - 2) = Q(\partial_x) = -4\partial_x - 4\beta + 4. \tag{5.12}$$

Note that we have $D(AB) \subset D(BA)$, since

$$\begin{aligned} D(AB) = D(L_y MB) &= \{v \in D(MB) : \; MBv \in D(L_y)\} \\ &\subset \{v \in D(BM) : \; B_{-2}Mv \in D(L_y)\} \\ &= \{v \in D(M) : \; Mv \in D(L_y B)\} \\ &\overset{L_yB=BL_y}{=} \{v \in D(M) : \; Mv \in D(BL_y)\} = D(BA). \end{aligned}$$

Therefore for all $g \in X_0$ and $f := (\omega + A)^{-1}g \in D(A)$ it follows

$$(\omega + A) \underbrace{(\mu + \eta + B)^{-1} f}_{\in D(AB) \subset D(BA)} \in D(B).$$

Hence for such $g \in X_0$ and $f \in D(A)$ we infer from (5.11)

$$\begin{aligned} & [(\omega + A), (\mu + \eta + B)^{-1}]f = (\mu + \eta + B)^{-1} Q (\mu + \eta + B_{-2})^{-1} (\omega + A) f \\ \Leftrightarrow \quad & (\mu + \eta + B)^{-1}(\omega + A)f - (\omega + A)(\mu + \eta + B)^{-1} f \\ & = -(\mu + \eta + B)^{-1} Q (\mu + \eta + B_{-2})^{-1}(\omega + A) f \\ \Leftrightarrow \quad & (\mu + \eta + B)^{-1} g - (\omega + A)(\mu + \eta + B)^{-1}(\omega + A)^{-1} g \\ & = -(\mu + \eta + B)^{-1} Q (\mu + \eta + B_{-2})^{-1} g \\ \Leftrightarrow \quad & (\lambda + \omega + A)^{-1}(\mu + \eta + B)^{-1} g \\ & \qquad - (\lambda + \eta + A)^{-1}(\omega + A)(\mu + \eta + B)^{-1}(\omega + A)^{-1} g \\ & = -(\lambda + \omega + A)^{-1}(\mu + \eta + B)^{-1} Q (\mu + \eta + B_{-2})^{-1} g \\ \Leftrightarrow \quad & (\omega + A)(\lambda + \omega + A)^{-1}[(\omega + A)^{-1}, (\mu + \eta + B)^{-1}] g \\ & = -(\lambda + \omega + A)^{-1}(\mu + \eta + B)^{-1} Q (\mu + \eta + B_{-2})^{-1} g. \end{aligned}$$

Thus we obtain

$$\begin{aligned} & (\omega + A)(\lambda + \omega + A)^{-1}[(\omega + A)^{-1}, (\mu + \eta + B)^{-1}] \\ & = -(\lambda + \omega + A)^{-1}(\mu + \eta + B)^{-1} Q (\mu + \eta + B_{-2})^{-1} \end{aligned}$$

which implies by the sectoriality estimates and interpolation

$$\|(\omega + A)(\lambda + \omega + A)^{-1}[(\omega + A)^{-1}, (\mu + \eta + B)^{-1}]\|_{\mathscr{L}(X_0)} \leq \frac{C}{(1 + |\lambda|)|\mu|^{3/2}}$$

for all $\lambda \in \Sigma_{\pi-\delta}$ and $\mu \in \Sigma_{\pi/2+\varepsilon}$ for fixed $\varepsilon \in (0, \pi/2 - \phi_B^\infty)$ and $\delta \in (0, \pi/2 - \varepsilon)$. Hence, the Labbas Terreni condition (2.16) holds true with $\alpha = 0$, $\beta = 1/2$, $\psi_A = \delta$ and $\psi_B = \pi/2 - \varepsilon$. Applying Proposition 2.29 in combination with Lemma 5.2 the assertion follows. □

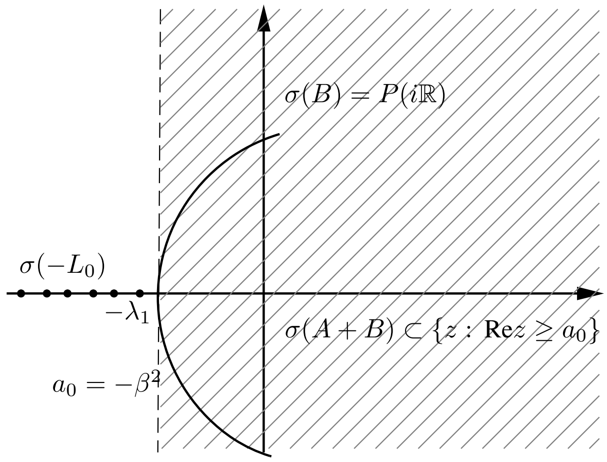

Figure 5.1: spectral condition $\sigma(-L_0) \cap \sigma(A+B) = \emptyset$

Applying Proposition 2.26 to $A+B$ and L_0 leads to

Lemma 5.4 *Let the operator* $A+B+L_0$ *in* X_0 *with domain* $D(A+B) \cap D(L_0)$ *be defined as above. Furthermore, let* $\lambda_1 > 0$, *being the first eigenvalue of* L_0 *(see (5.7)), satisfy*

$$\lambda_1 > \beta^2 = \left(2 - \frac{2}{p} - \frac{\gamma}{p}\right)^2. \tag{5.13}$$

Then $A+B+L_0$ *is invertible.*

Proof. By Lemma 5.3 and (5.8) conditions (1), (2) and (3) of Proposition 2.26 are readily fulfilled. Hence, $A+B+L_0$ is closed. Next, $A+B+\beta^2$ with domain $D(A) \cap D(B)$ is accretive: We have

$$\begin{aligned}&\big(((\kappa + L_y)M + B + \beta^2)u, u|u|^{p-2}\big)_{p,p',\mathbb{R}^2} = (\kappa Mu, u|u|^{p-2})_{p,p',\mathbb{R}^2} \\ &+ (L_y Mu, u|u|^{p-2})_{p,p',\mathbb{R}^2} + ((B+\beta^2)u, u|u|^{p-2})_{p,p',\mathbb{R}^2} \qquad (u \in D(A+B)).\end{aligned}$$

The third term is greater or equal zero by the accretivity of $B+\beta^2$ by (5.1). From

$$\begin{aligned}\partial_y(u|u|^{p-2}) &= (\partial_y u)|u|^{p-2} + u(p-2)|u|^{p-3}\text{sign}(u)(\partial_y u) \\ &= (\partial_y u)|u|^{p-2}(p-1) \qquad\qquad (x \neq 0)\end{aligned}$$

and

$$\begin{aligned}(\kappa Mu, u|u|^{p-2})_{p,p',\mathbb{R}^2} &= \kappa \int\limits_{\mathbb{R}^2} e^{2x}|u|^p d(x,y) \geq 0, \\ (L_y Mu, u|u|^{p-2})_{p,p',\mathbb{R}^2} &= -\int\limits_{\mathbb{R}^2} e^{2x}(\partial_y^2 u)(u|u|^{p-2}) d(x,y) \\ &= (p-1)\int\limits_{\mathbb{R}^2} e^{2x}|\partial_y u|^2|u|^{p-2} d(x,y) \geq 0 \qquad (u \in D(A+B))\end{aligned}$$

the accretivity of $A+B+\beta^2$ on $D(A+B)$ follows. As a consequence we obtain

$$\sigma(A+B) \subset \{z \in \mathbb{C} : \ \mathrm{Re} z \geq -\beta^2\}.$$

Thus, if condition (5.13) is satisfied, we have $\sigma(A+B) \cap \sigma(-L_0) = \emptyset$ and Proposition 2.26 yields the assertion. □

Note that $A+B+L_0$ represents the full linear operator of the transformed problem (4.27). Combining Lemma 5.3 and Lemma 5.4 leads to

Proposition 5.5 *Let condition (5.13) be satisfied. Then we have*

$$A+B+L_0 \in \mathcal{H}^\infty(X_0), \quad \phi^\infty_{A+B+L_0} < \frac{\pi}{2}.$$

Proof. For simplicity set $T = A+B+L_0$. In view of Lemma 5.3 we know that $\phi^\infty_{\nu+A+B} < \pi/2$ and by the discussion before also that $\phi^\infty_{L_0} = 0$. Due to the fact that $\nu+A+B$ and L_0 are resolvent commuting, Proposition 2.28, that is the standard Kalton-Weis theorem, therefore implies $\nu+T \in \mathcal{H}^\infty(X_0)$ and $\phi^\infty_{\nu+T} < \pi/2$. Now, fix $\phi \in (\phi^\infty_{\nu+T}, \pi)$ and let for $\theta \in (\phi^\infty_{\nu+T}, \phi)$ the path Γ be given as in (2.1). Then for $h \in \mathcal{H}_0(\Sigma_\phi)$ we have to estimate the Dunford integral

$$h(T) = \frac{1}{2\pi i} \int_\Gamma h(\lambda)(\lambda - T)^{-1} d\lambda. \tag{5.14}$$

If we split this integral into two parts corresponding either to $|\lambda| \leq 1$ or to $|\lambda| > 1$, then the desired estimate for small λ easily follows from $0 \in \rho(T)$ which has been proved in Lemma 5.4. On the other hand, the part corresponding to $|\lambda| > 1$ easily reduces to $\nu + T \in \mathcal{H}^\infty(X_0)$ which has been derived above. Hence the assertion is proved. □

Now we are in position to rigorously prove the equivalence of problems (4.1) and (4.27). To this end, recall that the domain of $A+B+L_0$ in view of (5.9) is given as

$$\begin{aligned}
&D(A+B+L_0) \\
&= \{v \in X : \ e^{2x} v \in L^p(\mathbb{R}_x \times I_\varphi, W^{2,p}(\mathbb{R}_y, \mathbb{R}^3))\} \cap W^{2,p}(\mathbb{R}_x, L^p(I_\varphi \times \mathbb{R}_y, \mathbb{R}^3)) \\
&\cap \{v \in X_0 : \ v \in L^p(\mathbb{R}^2, W^{2,p}(I_\varphi, \mathbb{R}^3)), \ \partial_\varphi v_x = v_\varphi = \partial_\varphi v_y = 0 \text{ on } \partial\Omega\}.
\end{aligned}$$

Implicitely all mixed derivatives with respect to the variables x and φ are contained in $D(A+B+L_0)$. This follows by Lemma 1.6, due to

$$L^p(\Omega_1, W^{2,p}(\Omega_2)) \cap W^{2,p}(\Omega_1, L^p(\Omega_2)) = W^{2,p}(\Omega_1 \times \Omega_2) \tag{5.15}$$

(in the sense of equivalent norms) for all domains $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^m$. Applying (5.15) to $\Omega_1 = \mathbb{R}_x$, $\Omega_2 = I_\varphi$ we obtain a more explicit representation of the domain $D(A+B+L_0)$, that is

$$D(A+B+L_0) = \Big\{ v = (v_x, v_\varphi, v_y) \in X_0 : \; e^{2x} v \in L^p(\mathbb{R}_x \times I_\varphi, W^{2,p}(\mathbb{R}_y, \mathbb{R}^3)), \\ v \in L^p(\mathbb{R}_y, W^{2,p}(\mathbb{R}_x \times I_\varphi, \mathbb{R}^3)), \quad \partial_\varphi v_x = v_\varphi = \partial_\varphi v_y = 0 \text{ on } \partial\Omega \Big\}. \tag{5.16}$$

Hence we obtain the following lemma on the equivalence of the wedge and the layer problem.

Lemma 5.6 *Let $1 < p < \infty$, $\gamma \in \mathbb{R}$, and $\varphi_0 \in (0, \pi)$ be given such that condition (5.13) is satisfied. Assume that $f \in \tilde{\Theta}_* X_0$ and $g = \tilde{\Theta}^* f$. Then $v \in D(A+B+L_0)$ (which is given through (5.16)) is the unique solution of (4.27) if and only if $u = \Theta_* v \in D(\mathcal{A}_\kappa)$ (given through (5.21)) is the unique solution of (4.1). In particular, $\tilde{\Theta}^* \in \mathscr{L}_{is}(\tilde{\Theta}_* X_0, X_0)$ and*

$$\Theta^* \in \mathscr{L}_{is}(D(\mathcal{A}_\kappa), D(A+B+L_0)), \tag{5.17}$$

where here the domains are equipped with the modified graph norms

$$\|u\|_{D(\mathcal{A}_\kappa)} = \|u\|_{L^p_\gamma(G,\mathbb{R}^3)} + \|\mathcal{A}_\kappa u\|_{L^p_\gamma(G,\mathbb{R}^3)} + \|u/|(\cdot,\cdot,0)|^2\|_{L^p_\gamma(G,\mathbb{R}^3)},$$

$$\|v\|_{D(A+B+L_0)} = \|v\|_{L^p(\Omega,\mathbb{R}^3)} + \|(A+B+L_0)v\|_{L^p(\Omega,\mathbb{R}^3)} + \|e^{2x} v\|_{L^p(\Omega,\mathbb{R}^3)},$$

respectively.

Proof. Let $1 < p < \infty$, $\gamma \in \mathbb{R}$, and $\varphi_0 \in (0, \pi)$ be given such that condition (5.13) is satisfied. Let Θ^* be the pull-back defined in (4.2) and $\tilde{\Theta}^*$ be the transformation given in (4.3). By Lemma 4.2

$$\tilde{\Theta}^* : L^p_\gamma(G, \mathbb{R}^3) \to L^p(\Omega, \mathbb{R}^3)$$

is an isomorphism with inverse $\tilde{\Theta}_* = \Theta_* e^{-2x}$. Utilizing decomposition (5.6) we see that

$$L^p_\gamma(G, \mathbb{R}^3) = \tilde{\Theta}_* X_0 \oplus \tilde{\Theta}_* L^p(\mathbb{R}^2, E_0), \tag{5.18}$$

hence that also

$$\tilde{\Theta}^* : \tilde{\Theta}_* X_0 \to X_0$$

is an isomorphism with X_0 defined in (5.4).

Observe that by the discussion in Chapter 4 – in particular Remark 4.3, we also have

$$\tilde{\Theta}^*(\kappa - \Delta) u = \tilde{\Theta}^* f = g = (A+B+L_0)\Theta^* u. \tag{5.19}$$

Thus, we can define

$$\mathcal{A}_\kappa u := (\kappa - \Delta) u, \quad u \in D(\mathcal{A}_\kappa) := \Theta_* D(A+B+L_0), \tag{5.20}$$

which is an operator in $\tilde{\Theta}_* X_0$. By the transforms calculated in Chapter 4 it is straight forward to show that $D(\mathcal{A}_\kappa)$ is explicitly given as

$$
\begin{aligned}
D(\mathcal{A}_\kappa) = \Big\{ & u \in \tilde{\Theta}_* X_0 : \ u/|(\cdot,\cdot,0)|^2, \ \partial^\alpha u \in L^p_\gamma(G,\mathbb{R}^3) \ (|\alpha| \le 2), \\
& \nu \times \operatorname{curl} u = 0, \ \nu \cdot u = 0 \text{ on } \partial G \Big\}.
\end{aligned} \tag{5.21}
$$

To this end we employ again the isomorphism $\tilde{\Theta}^*$ between the L^p-ground spaces: For
$v \in D(A + B + L_0)$ let $u = \Theta_* v$. First we use

$$
\begin{aligned}
\|u\|_{L^p_\gamma(G,\mathbb{R}^3)} &= \|\Theta_* v\|_{L^p_\gamma(G,\mathbb{R}^3)} = \|\Theta_* e^{-2x} e^{2x} v\|_{L^p_\gamma(G,\mathbb{R}^3)} \\
&= \|\tilde{\Theta}_*(e^{2x} v)\|_{L^p_\gamma(G,\mathbb{R}^3)} \le C \|e^{2x} v\|_{L^p(\Omega,\mathbb{R}^3)},
\end{aligned}
$$

$$
\|u/|(\cdot,\cdot,0)|^2\|_{L^p_\gamma(G,\mathbb{R}^3)} = \|\Theta_* v/|(\cdot,\cdot,0)|^2\|_{L^p_\gamma(G,\mathbb{R}^3)} = \|\tilde{\Theta}_* v\|_{L^p_\gamma(G,\mathbb{R}^3)} \le C \|v\|_{L^p(\Omega,\mathbb{R}^3)},
$$

for a constant $C > 0$. In a second step we consider the second order derivatives with exception $\partial_{x_1}\partial_y u$ and $\partial_{x_2}\partial_y u$. We directly see that

$$
\partial^2_{x_1} u, \ \partial^2_{x_2} u, \ \partial^2_y u, \ \partial_{x_1}\partial_{x_2} u \in L^p_\gamma(G,\mathbb{R}^3)
$$

by (4.17)-(4.25) and since

$$
v, \ e^{2x} v, \ \partial_y e^{2x} v, \ \partial^2_y e^{2x} v, \ \partial_x v, \ \partial_\varphi v, \ \partial_x \partial_\varphi v, \ \partial^2_x v, \ \partial^2_\varphi v \in L^p(\Omega,\mathbb{R}^3)
$$

due to the assumption $v \in D(A + B + L_0)$. Note that the transforms of the second order mixed partial derivative $\partial_{x_1}\partial_{x_2} u$ admits the same structure as $\partial^2_{x_1} u$ and $\partial^2_{x_2} u$ concerning the weight. In a third step we take care of first order derivatives by suitable interpolation inequalities. Indeed, by explicitly considering the L^p-norms of $e^x v$, $e^x \partial_\varphi v$ and $e^x \partial_x v$, performing partial integrations (in case of $e^x \partial_\varphi v$ and $e^x \partial_x v$), employing Hölder's inequality and Young's inequality we obtain

$$
\|e^x v\|_{L^p(\Omega,\mathbb{R}^3)} \le \|v\|_{L^p(\Omega,\mathbb{R}^3)} + \|e^{2x} v\|_{L^p(\Omega,\mathbb{R}^3)}, \tag{5.22}
$$

$$
\|e^x \partial_\varphi v\|_{L^p(\Omega,\mathbb{R}^3)} \le C \|e^{2x} v\|^{1/2}_{L^p(\Omega,\mathbb{R}^3)} \|\partial^2_\varphi v\|^{1/2}_{L^p(\Omega,\mathbb{R}^3)}, \tag{5.23}
$$

$$
\|e^x \partial_x v\|_{L^p(\Omega,\mathbb{R}^3)} \le C \|e^{2x} v\|^{1/2}_{L^p(\Omega,\mathbb{R}^3)} \|\partial^2_x v\|^{1/2}_{L^p(\Omega,\mathbb{R}^3)} + C \|v\|^{1/2}_{L^p(\Omega,\mathbb{R}^3)} \tag{5.24}
$$

with a constant $C > 0$. Estimate (5.22) follows by splitting up the integral into two parts, one for $x \in (-\infty, 0)$ where $e^x \le 1$ and the other one for $x \in (0,\infty)$ where $e^x \le e^{2x}$. Estimate (5.23) for instance can be seen by employing

$$
\begin{aligned}
\int_{\mathbb{R}\times I} |e^x \partial_\varphi v|^p d(x,\varphi) &= \int_{\mathbb{R}\times I} |e^x \partial_\varphi v|^2 \, |e^x \partial_\varphi v|^{p-2} d(x,\varphi) \\
&= - \int_{\mathbb{R}\times I} (e^{2x} v)(\partial^2_\varphi v) |e^x \partial_\varphi v|^{p-2} (p-1) d(x,\varphi).
\end{aligned}
$$

Then after the application of Hölder's inequality and Young's inequality the factor $|e^x \partial_\varphi v|^{p-2}$ may be absorbed into the left-hand side. Thus we infer $e^x v$, $e^x \partial_\varphi v$, $e^x \partial_x v \in L^p(\Omega, \mathbb{R}^3)$ and hence by considering the first order transformations (4.6)-(4.14), that

$$\partial_{x_1} u, \partial_{x_2} u, \partial_y u \in L^p_\gamma(G, \mathbb{R}^3).$$

Because of

$$W^{2,p}_\gamma(S_{\varphi_0} \times \mathbb{R}_y) = L^p(S_{\varphi_0}, W^{2,p}(\mathbb{R}_y)) \cap W^{2,p}_\gamma(S_{\varphi_0}, L^p(\mathbb{R}_y))$$

we also see that $\partial_{x_1}\partial_y u, \partial_{x_2}\partial_y u \in L^p_\gamma(G, \mathbb{R}^3)$. This yields the representation (5.21) of the domain of $\mathcal{A}_\kappa$. Employing $\Theta^* = e^{-2x}\tilde{\Theta}^*$ and the isomorphism $\tilde{\Theta}^*$ with respect to L^p yields

$$\|\underbrace{\Theta^* u}_{=e^{-2x}\tilde{\Theta}^* u}\|_{L^p(\Omega,\mathbb{R}^3)} + \|\underbrace{(A+B+L_0)\Theta^* u}_{=\tilde{\Theta}^*\mathcal{A}_\kappa u}\|_{L^p(\Omega,\mathbb{R}^3)} + \|\underbrace{e^{2x}\Theta^* u}_{=\tilde{\Theta}^* u}\|_{L^p(\Omega,\mathbb{R}^3)}$$

$$\leq C\left(\|u/|(\cdot,\cdot,0)|^2\|_{L^p_\gamma(G,\mathbb{R}^3)} + \|\mathcal{A}_\kappa u\|_{L^p_\gamma(G,\mathbb{R}^3)} + \|u\|_{L^p_\gamma(G,\mathbb{R}^3)}\right) \qquad (u \in D(\mathcal{A}_\kappa))$$

for a constant $C > 0$. Likewise, the converse estimate can be shown. Hence (5.17) follows. $\square$

By the fact that the property of having a bounded $\mathcal{H}^\infty$-calculus is invariant under conjugation with isomorphisms the bounded $\mathcal{H}^\infty$-calculus carries over from $A + B + L_0$ to $\mathcal{A}_\kappa$, see the statement on this permanence property in [DHP03, Proposition 2.11 (vi)]. This result is given by

Proposition 5.7 *For $\kappa > 0$ let $\mathcal{A}_\kappa$ be defined as above. Let $1 < p < \infty$, $\gamma \in \mathbb{R}$, and $\varphi_0 \in (0, \pi)$ be given such that condition (5.13) is satisfied. Then we have*

$$\mathcal{A}_\kappa \in \mathcal{H}^\infty(\tilde{\Theta}_* X_0), \quad \phi^\infty_{\mathcal{A}_\kappa} < \frac{\pi}{2}.$$

Proof. Due to Lemma 2.5 the Dunford integral (2.2) can be written as

$$h(T) = \frac{1}{2\pi i}\int_\Gamma \frac{h(\lambda)}{\lambda} T(\lambda - T)^{-1} d\lambda \tag{5.25}$$

for $h \in \mathcal{H}_0(\Sigma_\phi)$, $T \in \mathcal{S}(X)$, and $\phi \in (\phi_T, \pi)$. The fact that $T \in \mathcal{H}^\infty(X)$ then means that

$$\|h(T)\|_{\mathscr{L}(X)} \leq C\|h\|_\infty \quad (h \in \mathcal{H}_0(\Sigma_\phi)) \tag{5.26}$$

for $h(T)$ given through (5.25). By Proposition 5.5 this is true for $T = A + B + L_0$ and $\phi \in (\phi^\infty_{A+B+L_0}, \pi)$.

Now, observe that in view of (5.19) we have

$$\mathcal{A}_\kappa u = \tilde{\Theta}_*(A + B + L_0)\Theta^* u \quad (u \in D(\mathcal{A}_\kappa)).$$

Thanks to Lemma 5.6 this yields

$$\mathcal{A}_\kappa(\lambda-\mathcal{A}_\kappa)^{-1}=\tilde{\Theta}_*(A+B+L_0)(\lambda-(A+B+L_0))^{-1}\tilde{\Theta}^*$$

for $\lambda\in\rho(\mathcal{A}_\kappa)=\rho(A+B+L_0)$. By this representation and formula (5.25) we easily achieve that (5.26) remains valid for $T=\mathcal{A}_\kappa$ and $\phi\in(\phi_{A+B+L_0},\pi)$. Hence the assertion is proved. □

Proposition 5.5 in combination with Remark 2.22 yields that $A+B+L_0$ admits maximal regularity. Then by Lemma 5.6 we infer

Corollary 5.8 *Suppose the assumptions of Proposition 5.7 are satisfied and let the time interval $J=(0,T)$ with $T\in(0,\infty)$. Then for each $f\in L^p(J,\tilde{\Theta}_*X_0)$ there exists a unique solution $u\in L^p(J,\tilde{\Theta}_*X_0)$ of (4.1) such that*

$$u/|(x_1,x_2)|^2,\partial_t u,\partial^\alpha u\in L^p(J,L^p_\gamma(G,\mathbb{R}^3))\qquad(\alpha\in\mathbb{N}_0^3,\ |\alpha|\le 2).$$

In particular, the map $[u\mapsto f]$ defines an isomorphism between the corresponding spaces.

Before turning to the Stokes equations let us have a closer look at the essential condition (5.13). Especially we are interested when it is allowed to choose $\gamma=0$, that is when we can work in the unweighted setting. The relationship on the first eigenvalue λ_1 of L_0 can be written as

$$(2-\sqrt{\lambda_1})p-2<\gamma<(2+\sqrt{\lambda_1})p-2,$$

Since $\lambda_1=\min\{1,(\frac{\pi}{\varphi_0}-1)^2\}$, we have a closer look at the condition

$$\left(3-\frac{\pi}{\varphi_0}\right)p-2<\gamma<\left(1+\frac{\pi}{\varphi_0}\right)p-2\tag{5.27}$$

in terms of $\gamma\in\mathbb{R}$, $p\in(1,\infty)$ and the angle $\varphi_0\in(0,\pi)$. The following tabular displays γ-intervals for some characteristic angles φ_0.

φ_0	$\gamma\in$	$\gamma=0:\ p\in$
$\varphi_0\le\frac{\pi}{2}$	$(p-2,3p-2)$	$(1,2)$
$\varphi_0=\frac{3}{4}\pi$	$\left(\frac{5}{3}p-2,\frac{7}{3}p-2\right)$	$\left(1,\frac{6}{5}\right)$
$\varphi_0=(1-\varepsilon)\pi$	$((3-1/(1-\varepsilon))p-2,(1+1/(1-\varepsilon))p-2)$	$\left(1,\frac{2(1-\varepsilon)}{2-3\varepsilon}\right)$

In terms of condition (5.13) the answer to the above question is illustrated in the last column of the table.

Remark 5.9 *Let us compare the situation here to some known conditions on the weight γ for the heat equation in a wedge. Nazarov discussed the case of Dirichlet and Neumann boundary conditions in [Naz01]. In the special case of a three-dimensional wedge Nazarovs' conditions take the form*

$$2-\frac{2}{p}-\lambda_D<\frac{\gamma}{p}<2-\frac{2}{p}+\lambda_D$$

for a Dirichlet boundary condition and

$$2 - \frac{2}{p} - \min\{\lambda_N, 2\} < \frac{\gamma}{p} < 2 - \frac{2}{p}$$

for a Neumann boundary condition. Here $\lambda_D = \lambda_N = \pi/\varphi_0$ denote the square roots of the first nonnegative eigenvalues of the related azimuthal operators which corresponding to L in this work.

Thus, in the situations considered in [Naz01] the admissible range for γ is larger than the range for perfect slip obtained by condition (5.13). We remark, however, that for the problem considered in this work the form of the first eigenvalue $\lambda_1 = \min\{1, (\pi/\varphi_0 - 1)^2\}$ in (5.13) is due to the fact that we have to transform a system including vector fields. We also remark that by excluding the eigenspace corresponding to the eigenvalue 1 of L (see (5.2)) our condition would improve in case that $\varphi_0 < \pi/2$. Then, however, we miss some solenoidal functions, see also Remark 5.1. On the other hand, including the eigenspace corresponding to the eigenvalue 0 would cause our approach to fail, since then the condition $\sigma(A + B) \cap \sigma(-L) = \emptyset$ (see proof of Lemma 5.4) cannot be satisfied anymore.

Chapter 6

The Stokes Equations on a Wedge

We turn to the Stokes equations (3.2). To this end, we first have to fix a suitable space of solenoidal vector fields. Let $1 < p < \infty$ and $1/p + 1/p' = 1$. In our setting it seems appropriate to choose

$$L^p_{\sigma,\gamma}(G) := \left\{ u \in L^p_\gamma(G, \mathbb{R}^3) : \int_G u \cdot \nabla\varphi d(x_1, x_2, y) = 0 \ \left(\varphi \in \widehat{W}^{1,p'}_{-\gamma'}(G)\right) \right\},$$

where $\gamma' = \gamma p'/p$ and $\widehat{W}^{1,p'}_{-\gamma'}(G)$ is given as in (3.4). Note that by

$$\begin{aligned} \left| \int_G u \cdot \nabla\varphi d(x_1, x_2, y) \right| &\le \int_G |u||(x_1, x_2)|^{\gamma/p} |\nabla\varphi||(x_1, x_2)|^{-\gamma/p} d(x_1, x_2, y) \\ &\le \|u\|_{L^p_\gamma(G,\mathbb{R}^3)} \|\nabla\varphi\|_{L^{p'}_{-\gamma p'/p}(G,\mathbb{R}^3)} \qquad (u \in L^p_{\sigma,\gamma}(G),\ \varphi \in \widehat{W}^{1,p'}_{-\gamma'}(G)) \end{aligned}$$

Hölder's inequality guarantees well-definedness of $L^p_{\sigma,\gamma}(G)$. Note that because of $C_c^\infty(G) \subset \widehat{W}^{1,p}_{-\gamma}(G)$, it is obvious that $u \in L^p_{\sigma,\gamma}(G)$ satisfies $\operatorname{div} u = 0$ in the sense of distributions. Moreover, by the generalized Gauß theorem, cf. [Gal11, Theorem III.2.2], the trace $\nu \cdot u$ is well-defined in the trace space (Slobodeckij space) $W_p^{-1/p}(\mathcal{O})$ for every bounded domain $\mathcal{O}$ such that

$$\overline{\mathcal{O}} \subset \partial G \setminus (\{(0,0)\} \times \mathbb{R}).$$

Hence $u \cdot \nu = 0$ on ∂G is fulfilled at least in a local sense away from $\{(0,0)\} \times \mathbb{R}$. Our intention is to regard the Stokes operator as the part of the Laplacian in the space $\tilde{\Theta}_* X_0$. For this purpose, we first need to show

Lemma 6.1 *There is a canonical embedding*

$$L^p_{\sigma,\gamma}(G) \hookrightarrow \tilde{\Theta}_* X_0, \tag{6.1}$$

that is, $L^p_{\sigma,\gamma}(G)$ *can be regarded as a closed subspace of* $\tilde{\Theta}_* X_0$.

Proof. Consider the factor space

$$Y := L^p_\gamma(G, \mathbb{R}^3) / \tilde{\Theta}_* L^p(\mathbb{R}^2, E_0)$$

with E_0 defined in (5.5). Recall that an element of $L^p(\mathbb{R}^2, E_0)$ is represented by $(0, 0, w)$ with $w \in L^p(\mathbb{R}^2)$. Applying the transformed divergence operator (see (4.15)) to $(0, 0, w)$ yields $\partial_y w = 0$. Thus w is constant in y which results in $w = 0$. This implies

$$L^p_{\sigma,\gamma}(G) \cap \tilde{\Theta}_* L^p(\mathbb{R}^2, E_0) = \{0\},$$

hence that

$$L^p_{\sigma,\gamma}(G) \hookrightarrow Y.$$

From decomposition (5.18) we infer that Y is isomorphic to $\tilde{\Theta}_* X_0$ (with respect to the L^p_γ-norm), hence embedding (6.1) is well-defined in a canonical way. Since $L^p_{\sigma,\gamma}(G)$ and $\tilde{\Theta}_* X_0$ are obviously closed with respect to the norm in $L^p_\gamma(G, \mathbb{R}^3)$, the claim is proved. □

Remark 6.2 *Observe that the embedding operator which maps $L^p_{\sigma,\gamma}(G)$ isomorphically onto a closed subspace of $\tilde{\Theta}_* X_0$ is represented by $\tilde{\Theta}_* \Pi_0 \tilde{\Theta}^*$ with Π_0 defined in (5.3). Hence we identify $L^p_{\sigma,\gamma}(G)$ actually with $\tilde{\Theta}_* \Pi_0 \tilde{\Theta}^*(L^p_{\sigma,\gamma}(G))$. However, the fact that $\tilde{\Theta}^*(\kappa - \Delta)\Theta_* = A + B + L$ commutes with Π_0 justifies it to work directly with $L^p_{\sigma,\gamma}(G)$ in the set up of the Stokes operator, as it is presented below.*

Let $\mathcal{A}_\kappa : D(\mathcal{A}_\kappa) \subset \tilde{\Theta}_* X_0 \to \tilde{\Theta}_* X_0$ be the Laplacian as defined in (5.20) with domain $D(\mathcal{A}_\kappa)$ as given in (5.21). We also set $\mathcal{A} := \mathcal{A}_0$, i.e. for $\kappa = 0$. Thanks to Lemma 6.1 (and Remark 6.2) we can define the Stokes operator as the part of $\mathcal{A}$ in $L^p_{\sigma,\gamma}(G)$. We set

$$\begin{aligned} \mathcal{A}_S u &:= \mathcal{A}|_{L^p_{\sigma,\gamma}(G)} u, \quad u \in D(\mathcal{A}_S), \\ D(\mathcal{A}_S) &:= \{u \in D(\mathcal{A}) \cap L^p_{\sigma,\gamma}(G) : \ \mathcal{A}u \in L^p_{\sigma,\gamma}(G)\}. \end{aligned}$$

Note that then (3.2) is equivalent to the Cauchy problem

$$\begin{cases} \dot{u} + \mathcal{A}_S\, u = f & \text{in } (0, T), \\ \qquad u(0) = 0, \end{cases} \tag{6.2}$$

with $f \in L^p((0, T), L^p_{\sigma,\gamma}(G))$. The following lemma justifies the above definition of the Stokes operator.

Lemma 6.3 *We have*

$$D(\mathcal{A}_S) = D(\mathcal{A}) \cap L^p_{\sigma,\gamma}(G).$$

Proof. We only have to show, that the right hand side is a subset of $D(\mathcal{A}_S)$. To this end, let $u \in D(\mathcal{A}) \cap L^p_{\sigma,\gamma}(G)$. We have to show that $f := \mathcal{A}u = \text{curl}^2 u \in L^p_{\sigma,\gamma}(G)$. The divergence theorem yields

$$\begin{aligned} \int_G f \cdot \nabla\varphi \, d(x_1, x_2, y) &= \int_G (\text{curl}\,\text{curl}\, u) \cdot \nabla\varphi d(x_1, x_2, y) \\ &= \int_{\partial G} (\nu \times \text{curl}\, u) \cdot \nabla\varphi d\sigma = 0 \end{aligned}$$

for all $\varphi \in C^\infty\left(\overline{G} \setminus (\{(0,0)\} \times \mathbb{R})\right) \cap \widehat{W}^{1,p'}_{-\gamma'}(G)$. This in turn implies $f \in L^p_{\sigma,\gamma}(G)$ and hence $u \in D(\mathcal{A}_S)$, provided $C^\infty(\overline{G} \setminus (\{(0,0)\} \times \mathbb{R})) \cap \widehat{W}^{1,p'}_{-\gamma'}(G)$ lies dense in $\widehat{W}^{1,p'}_{-\gamma'}(G)$. This can be seen by a mollifier argument. □

Remark 6.4 *For $\gamma = 0$ we can work with the Helmholtz projection $\mathbb{P}$ as usually. It is given by*

$$\mathbb{P} : L^p(G, \mathbb{R}^3) \to L^p_\sigma(G), \quad u \mapsto u - \nabla p,$$

where p is the solution of the weak Neumann problem

$$(\nabla p, \nabla \varphi) = (u, \nabla \varphi) \quad (\varphi \in \widehat{W}^{1,p'}(G)),$$

for $u \in L^p(G, \mathbb{R}^3)$ and $1/p + 1/p' = 1$. We refer to [BM86] for the existence of the Helmholtz decomposition of $L^p(G, \mathbb{R}^3)$, $1 < p < \infty$. Note also that in this case we have

$$L^p_\sigma(G) = \overline{C^\infty_{c,\sigma}(G)}^{L^p}.$$

With this projection at hand the Stokes operator takes the form

$$\mathcal{A}_S u = \mathbb{P}\mathcal{A}u = -\mathbb{P}\Delta u, \quad u \in D(\mathcal{A}_S). \tag{6.3}$$

This representation will be utilized in the next section.

By Lemma 6.3 we have the relation

$$(\lambda - \mathcal{A}_S)^{-1} = (\lambda - \mathcal{A})^{-1}|_{L^p_{\sigma,\gamma}(G)}, \tag{6.4}$$

for $\lambda \in \rho(\mathcal{A}) = \rho(\mathcal{A}_S)$. Proposition 5.7 therefore immediately implies

Proposition 6.5 *For $\kappa > 0$ let $\mathcal{A}_{S,\kappa} := \kappa + \mathcal{A}_S$ with $\mathcal{A}_S$ the Stokes operator as defined above. Let $1 < p < \infty$, $\gamma \in \mathbb{R}$, and $\varphi_0 \in (0, \pi)$ be given such that condition (5.13) is satisfied. Then we have*

$$\mathcal{A}_{S,\kappa} \in \mathcal{H}^\infty(L^p_{\sigma,\gamma}(G)), \quad \phi^\infty_{\mathcal{A}_{S,\kappa}} < \frac{\pi}{2}.$$

Remark 6.6 *Applying a scaling argument like in [McC81, BM88, NS03, Saa03] to the $\mathcal{H}^\infty$-estimate for $\mathcal{A}_{S,\kappa}$ yields that Proposition 6.5 also holds for $\kappa = 0$. This, of course, is also true for Proposition 5.7 and essentially relies on the fact that a wedge is scaling invariant. More precisely, for $r > 0$ we consider*

$$(J_r u)(x_1, x_2, y) := u(rx_1, rx_2, ry), \qquad (x_1, x_2, y) \in G.$$

As in [Saa03] we set

$$(\mathcal{A}_S)_r := r^2 J_r \mathcal{A}_S J_{1/r}$$

and have

$$\begin{aligned} (\mathcal{A}_S)_{1/r} &= \frac{1}{r^2} J_{1/r} \mathcal{A}_S J_r = \mathcal{A}_S, \\ (\lambda - \kappa - r^2 \mathcal{A}_S)^{-1} &= J_{1/r}(\lambda - \kappa - \mathcal{A}_S)^{-1} J_r \qquad (\lambda - \kappa \in \rho(\mathcal{A}_S)). \end{aligned}$$

Hence we obtain for $f \in L^p_\gamma(G,\mathbb{R}^3)$, $h \in \mathcal{H}_0(\Sigma_\phi)$ *with angles* $\phi \in (\phi^\infty_{\mathcal{A}_{S,\kappa}},\pi)$, $\theta \in (\phi^\infty_{\mathcal{A}_{S,\kappa}},\phi)$ *and the path* Γ *as in (2.1) that*

$$\begin{aligned} h\left(\frac{\kappa}{r^2}+\mathcal{A}_S\right)f &= \frac{1}{2\pi i}\int\limits_\Gamma h(\mu)\left(\mu-\frac{\kappa}{r^2}-\mathcal{A}_S\right)^{-1} f d\mu \\ &= \frac{1}{2\pi i}\int\limits_\Gamma h(\mu) r^2 (r^2\mu-\kappa-r^2\mathcal{A}_S)^{-1} f d\mu \\ &= \frac{1}{2\pi i}\int\limits_\Gamma h\left(\frac{\lambda}{r^2}\right)(\lambda-\kappa-r^2\mathcal{A}_S)^{-1} f d\lambda \\ &= J_{1/r}\left(\frac{1}{2\pi i}\int\limits_\Gamma h\left(\frac{\lambda}{r^2}\right)(\lambda-\kappa-\mathcal{A}_S)^{-1} d\lambda\right) J_r f \\ &= J_{1/r}(J_{1/r^2}h)(\kappa+\mathcal{A}_S)J_r f. \end{aligned} \tag{6.5}$$

Note that $J_{1/r^2}h$ *is well-defined and uniformly bounded. Thus by the integral transformation* $(z_1,z_2,z_3)=(rx_1,rx_2,ry)$ *and Proposition 6.5 we infer*

$$\begin{aligned} \left\|h\left(\frac{\kappa}{r^2}+\mathcal{A}_S\right)f\right\|_{L^p_\gamma(G,\mathbb{R}^3)} &= \|J_{1/r}(J_{1/r^2}h)(\kappa+\mathcal{A}_S)J_r f\|_{L^p_\gamma(G,\mathbb{R}^3)} \\ &= r^{(3+\gamma)/p}\|(J_{1/r^2}h)(\kappa+\mathcal{A}_S)J_r f\|_{L^p_\gamma(G,\mathbb{R}^3)} \\ &\le r^{(3+\gamma)/p}C\|(J_{1/r^2}h)\|_\infty\|J_r f\|_{L^p_\gamma(G,\mathbb{R}^3)} \\ &= C\|h\|_\infty\|f\|_{L^p_\gamma(G,\mathbb{R}^3)} \end{aligned}$$

for a constant $C>0$ *which is independent of* r. *Passing to the limit* $r\to\infty$ *yields the assertion.*

Note that Proposition 6.5 and Remark 6.6 imply Theorem 3.1, our main result for the linearized situation.

Let us briefly comment on open questions. Among others one question is whether the $\mathcal{H}^\infty$-angle of the Stokes operator which is shown to be less than $\pi/2$ can be improved to 0. Note that here this is only due to the polynomial operator arising from the radial direction.
Up to now it also seems to be open if Theorem 3.1 and Corollary 3.2 remain valid for other boundary conditions than perfect slip.

Chapter 7

The Navier-Stokes Equations

We consider the nonlinear Navier-Stokes equations (3.1) on the three-dimensional wedge G. For simplicity we restrict to the case $\gamma = 0$, i.e. the unweighted setting. We prove Theorem 3.3, i.e. we derive a local-in-time strong L^p-solution for small initial data, angles $\varphi_0 \in (0, 5\pi/9)$ and a p-interval contained in $(5/3, \infty)$. By Remark 6.4 we may reformulate (3.1) by applying the Helmholtz projection $\mathbb{P}$ to the first line in (3.1), such that (3.1) is equivalent to

$$\begin{cases} \dot{u}(t) + \mathcal{A}_S u(t) &= F(u(t)), \quad t \in (0, T), \\ \qquad\qquad u(0) &= u_0, \end{cases} \tag{7.1}$$

where F contains the nonlinearity $F(u) = -\mathbb{P}((u \cdot \nabla)u)$. Let

$$\mathcal{L}_T u := \begin{pmatrix} \partial_t + \mathcal{A}_S \\ \gamma_0 \end{pmatrix} u, \qquad \mathcal{R}_T(u) := \begin{pmatrix} F(u) \\ u_0 \end{pmatrix}.$$

Here we employ the spaces

$$\mathbb{E}_p^G(T),\ \mathbb{F}_p^G(T),\ \mathbb{E}_{p,\sigma}^G(T),\ \mathbb{F}_{p,\sigma}^G(T),\ \mathbb{I}_{p,\sigma}^G,$$

introduced in Section 3.1. By Corollary 3.2 in case $\gamma = 0$

$$\mathcal{L}_T : \mathbb{E}_{p,\sigma}^G(T) \to \mathbb{F}_{p,\sigma}^G(T) \times \mathbb{I}_{p,\sigma}^G$$

constitutes an isomorphism. Hence we may reformulate (7.1) as the fixed point equation

$$u = \mathcal{L}_T^{-1} \mathcal{R}_T(u) =: \Phi_T(u). \tag{7.2}$$

Lemma 7.1 *Let $T \in (0, \infty)$ be given and let $p \in (5/3, \infty)$. Then*

(a) $F : \mathbb{E}_p^G(T) \to \mathbb{F}_p^G(T)$ is real analytic.

(b) The Fréchet derivative of F is given through

$$DF[v]w = -\mathbb{P}\left((v \cdot \nabla)w + (w \cdot \nabla)v\right) \qquad (v, w \in \mathbb{E}_p^G(T)). \tag{7.3}$$

Proof. (a): We show that F is well-defined: To this end let $a \geq 2$ and $p > 3/a+1$. Let $u \in \mathbb{E}_p^G(T)$. We apply Hölder's inequality in time and space for indices

$$\frac{1}{p} = \frac{1}{2p} + \frac{1}{2p}, \qquad \frac{1}{p} = \frac{1}{ap} + \frac{a-1}{ap}$$

and the Sobolev embedding theorem in space, where we assume that the conditions

$$ap > \frac{ap}{a-1}, \qquad p > 3\frac{a-2}{a}$$

are fulfilled. Thus we infer

$$\begin{aligned} \|(u\cdot\nabla)u\|_{\mathbb{F}_p^G(T)} &\leq \|u\|_{L^{2p}((0,T),L^{ap}(G,\mathbb{R}^3))}\|\nabla u\|_{L^{2p}((0,T),L^{ap/(a-1)}(G,\mathbb{R}^{3\times 3}))} \\ &\leq C\|u\|^2_{L^{2p}((0,T),W^{1,ap/(a-1)}(G,\mathbb{R}^3))} \end{aligned} \tag{7.4}$$

for some constant $C > 0$. Because of $p > 3/a+1$ there is an $\varepsilon > 0$, such that $p > 3/a + 1 + \varepsilon$. Setting

$$s := \frac{1+\varepsilon}{2p} \in (0,1)$$

the Sobolev embedding theorem in space with

$$\frac{ap}{a-1} > p, \qquad p > \frac{3}{a} + 1 + \varepsilon$$

and in time with

$$s - \frac{1}{p} > -\frac{1}{2p} \qquad \Leftrightarrow \qquad \frac{1+\varepsilon}{2p} > \frac{1}{2p}$$

yields

$$\begin{aligned} \|u\|_{L^{2p}((0,T),W^{1,a/(a-1)}(G,\mathbb{R}^3))} &\leq C\|u\|_{L^{2p}((0,T),W_p^{2(1-s)}(G,\mathbb{R}^3))} \\ &\leq C\|u\|_{W_p^s((0,T),W_p^{2(1-s)}(G,\mathbb{R}^3))}. \end{aligned} \tag{7.5}$$

By the mixed derivative theorem the embedding

$$W^{1,p}((0,T),L^p(G)) \cap L^p((0,T),W^{2,p}(G)) \hookrightarrow W_p^s((0,T),W_p^{2(1-s)}(G))$$

holds true and hence

$$\|u\|_{W_p^s((0,T),W_p^{2(1-s)}(G,\mathbb{R}^3))} \leq C\|u\|_{\mathbb{E}_p^G(T)}. \tag{7.6}$$

Combining (7.4), (7.5) and (7.6) we infer

$$\|(u\cdot\nabla)u\|_{\mathbb{F}_p^G(T)} \leq C\|u\|^2_{\mathbb{E}_p^G(T)} \qquad (u \in \mathbb{E}_p^G(T)) \tag{7.7}$$

if the conditions $p > 3/a+1$, $p > 3(a-2)/a$ are satisfied for a constant $C > 0$ depending on G, T, p. The best possible value of $a \geq$ is the graphs' intersection of

$$p = \frac{3}{a} + 1, \qquad p = 3\frac{a-2}{a}$$

which is given by $(a_0, p_0) = (9/2, 5/3)$. Hence F is a well-defined mapping for all $p \in (5/3, \infty)$. $F \in C^\omega(\mathbb{E}_p^G(T), \mathbb{F}_p^G(T))$ follows from the fact that F is a product of linear functions.

(b): We show that

$$\frac{\|F(u) - F(v) - \mathbb{P}\left((v \cdot \nabla)(v-u) + (v-u) \cdot \nabla v\right)\|_{\mathbb{F}_p^G(T)}}{\|u-v\|_{\mathbb{E}_p^G(T)}} \to 0,$$

as

$$\|u-v\|_{\mathbb{E}_p^G(T)} \to 0. \tag{7.8}$$

We have

$$\begin{aligned}
&F(u) - F(v) - \mathbb{P}\left((v \cdot \nabla)(u-v) + ((u-v) \cdot \nabla)v\right) \\
&\quad = -\mathbb{P}\left((u \cdot \nabla)u - (v \cdot \nabla)v - (v \cdot \nabla)u + (v \cdot \nabla)v - (u \cdot \nabla)v + (v \cdot \nabla)v\right) \\
&\quad = -\mathbb{P}\left((u \cdot \nabla)(u-v) + (v \cdot \nabla)(v-u)\right) \\
&\quad = -\mathbb{P}\left(((u-v) \cdot \nabla)(u-v)\right),
\end{aligned}$$

and hence by (7.7)

$$\begin{aligned}
&\|F(u) - F(v) - \mathbb{P}\left((v \cdot \nabla)(u-v) + ((u-v) \cdot \nabla)v\right)\|_{\mathbb{F}_p^G(T)} \\
&\qquad\qquad \leq C\|u-v\|^2_{\mathbb{E}_p^G(T)} \qquad (u, v \in \mathbb{E}_p^G(T)).
\end{aligned}$$

This yields the assertion. □

In the proof of Lemma 7.1 we made use of several Sobolev embeddings and employed the *mixed derivative theorem* instead of anisotropic embeddings. It goes back to Sobolevskii [Sob75], see also [DSS08, Lemma 4.1] or the appendix of [PSS07]. The only reason why employing mixed derivatives here is due to the application of Hölder's inequality where we obtained different integrability exponents. Proposition 1.3 on the other hand is formulated for common 'p' only. However, more general anisotropic embeddings with different 'p' are feasable, cf. for instance [Tri78], but for simplicity not employed in this thesis.

Remark 7.2 *Lemma 7.1 holds for the p-interval $(5/3, \infty)$. This is due to the convection term in the Navier-Stokes equations and was already shown in [Saa07] for a domain in $\mathbb{R}^n$ with arbitrary $n \geq 2$. There the p-interval turns out to be $((n+2)/3, \infty)$.*

We have all ingredients at hand to prove Theorem 3.3.

Proof of Theorem 3.3. The nonlinear problem (7.1) admits a unique solution $u \in \mathbb{E}_{p,\sigma}^G(T)$ if u is the unique fixed point of Φ_T given in (7.2). We apply the contraction mapping principle to Φ_T. Let $r > 0$. We set

$$M := \|\mathcal{L}_T^{-1}\|_{\mathscr{L}(\mathbb{F}_{p,\sigma}^G(T) \times \mathbb{I}_{p,\sigma}^G, \mathbb{E}_{p,\sigma}^G(T))}$$

and choose

$$\varepsilon := \frac{1}{2M}, \qquad \|u_0\|_{\mathbb{I}^G_{p,\sigma}} < \frac{r}{2M} =: \kappa.$$

Let $\bar{B}_r(0)$ denote the closed ball in $\mathbb{E}^G_{p,\sigma}(T)$ around zero with radius r.

Contraction property: For $u, v \in \bar{B}_r(0)$ we obtain by Lemma 7.1 in combination with the mean value theorem

$$\begin{aligned}\|\Phi_T(u) - \Phi_T(v)\|_{\mathbb{E}^G_{p,\sigma}(T)} &\leq M\|F(u) - F(v)\|_{\mathbb{F}^G_{p,\sigma}(T)} \\ &\leq M \sup_{\|w\|_{\mathbb{E}^G_{p,\sigma}(T)} \leq r} \|DF[w]\|_{\mathscr{L}(\mathbb{E}^G_{p,\sigma}(T)),\mathbb{F}^G_{p,\sigma}(T))} \|u - v\|_{\mathbb{E}^G_{p,\sigma}(T)}.\end{aligned}$$

By Lemma 7.1 $DF[0] = 0$ and DF is continuous. Hence for ε given above we may choose $r > 0$, such that

$$\sup_{\|w\|_{\mathbb{E}^G_{p,\sigma}(T) \leq r}} \|DF[w]\|_{\mathscr{L}(\mathbb{E}^G_{p,\sigma}(T)),\mathbb{F}^G_{p,\sigma}(T))} < \varepsilon,$$

and we infer

$$\|\Phi_T(u) - \Phi_T(v)\|_{\mathbb{E}^G_{p,\sigma}(T)} \leq \frac{1}{2}\|u - v\|_{\mathbb{E}^G_{p,\sigma}(T)} \qquad (u, v \in \bar{B}_r(0)).$$

Self mapping property : Let $u \in \bar{B}_r(0)$. We proceed analogously to the contraction estimate:

$$\begin{aligned}\|\Phi_T(u)\|_{\mathbb{E}^G_{p,\sigma}(T)} &\leq M \left(\|\mathcal{R}_T(u) - \mathcal{R}_T(0)\|_{\mathbb{F}^G_{p,\sigma}(T)\times\mathbb{I}^G_{p,\sigma}} + \|\mathcal{R}_T(0)\|_{\mathbb{F}^G_{p,\sigma}(T)\times\mathbb{I}^G_{p,\sigma}}\right) \\ &\leq M \left(\|F(u) - F(0)\|_{\mathbb{F}^G_{p,\sigma}(T)} + \|u_0\|_{\mathbb{I}^G_{p,\sigma}}\right) \\ &\leq M \left(\varepsilon\|u\|_{\mathbb{E}^G_{p,\sigma}(T)} + \|u_0\|_{\mathbb{I}^G_{p,\sigma}}\right) \\ &\leq M \left(\frac{1}{2M}r + \frac{r}{2M}\right) = r \qquad (u \in \bar{B}_r(0)).\end{aligned}$$

From the contraction mapping principle we infer that Φ_T admits a unique fixed point $u \in \bar{B}_r(0)$. This completes the proof. □

Remark 7.3 *In the two-dimensional case, i.e. on S_{φ_0} the result improves in terms of φ_0 and p: For $\gamma = 0$, $\varphi_0 \in (0, \frac{2}{3}\pi)$ and $p \in (\frac{4}{3}, \frac{2}{3-\pi/\varphi_0})$ Theorem 3.3 holds true.*

Remark 7.4 *Note that at least for angles small enough and a corresponding range of p Corollary 3.2 is still strong enough to show that there exists a unique local-in-time strong L^p-solution. On the one hand Lemma 7.1 needs $p > 5/3$. On the other hand (5.27) for $\gamma = 0$ yields*

$$p \in \left(\frac{2}{1 + \frac{\pi}{\varphi_0}}, \frac{2}{3 - \frac{\pi}{\varphi_0}}\right),$$

such that we eventually obtain a well-posedness result for

$$p \in \left(\frac{5}{3}, \frac{2}{3 - \frac{\pi}{\varphi_0}} \right).$$

Note that $\varphi_0 < 5/9\,\pi$ *guarantees that this* p*-interval is not empty. In particular Theorem 3.3 covers the case of* $\varphi_0 \leq \pi/2$ *and* $p \in (5/3, 2)$.

Part II.

Global Solutions for a Class of Heterogeneous Catalysis Models

Chapter 8

Main Results

The content of Part II is based on the joint work with Dieter Bothe, Matthias Köhne and Jürgen Saal [BKMS]. The authors contributed equally to [BKMS]. The application of cylindrical L^p-theory was done by the author of this thesis. The remainder linear theory arose in mutual work with the other authors. As well the local-in-time existence result, the nonnegativity of concentrations, as weak-type estimates are due to him. For the proof of the final global-in-time existence result he did a significant contribution.

A heterogeneous catalysis in a finite three-dimensional cylinder is studied. The system under consideration is built-up of a diffusion-advection system acting in the bulk phase and a reaction-diffusion-sorption system acting on the catalytic wall, i.e. here the lateral surface of the cylinder. We assume Fickian diffusion with constant coefficients. Moreover, sorption kinetics is assumed to be general while the catalysis is assumed to be a reversible chemical reaction with triangular structure. Our main result includes the existence of a unique global strong L^2-solution to this model.

Let $\Omega := B_R(0) \times (0,h) \subset \mathbb{R}^3$ denote a finite three-dimensional cylinder of height $h > 0$. Its boundary decomposes into bottom Γ_{in}, top Γ_{out} and lateral surface Σ, standing for inflow area, outflow area and active surface. For given constant diffusivities $d_i, d_i^\Sigma > 0$ we aim to solve

$$\left\{\begin{array}{rcll}
\partial_t c_i + (u\cdot\nabla)c_i - d_i\Delta c_i &=& 0 & \text{in} \quad (0,T)\times\Omega,\\
\partial_t c_i^\Sigma - d_i^\Sigma \Delta_\Sigma c_i^\Sigma &=& r_i^{\text{sorp}}(c_i, c_i^\Sigma) + r_i^{\text{ch}}(c^\Sigma) & \text{on} \quad (0,T)\times\Sigma,\\
(u\cdot\nu)c_i - d_i\partial_\nu c_i &=& g_i^{\text{in}} & \text{on} \quad (0,T)\times\Gamma_{\text{in}},\\
-d_i\partial_\nu c_i &=& r_i^{\text{sorp}}(c_i, c_i^\Sigma) & \text{on} \quad (0,T)\times\Sigma,\\
-d_i\partial_\nu c_i &=& 0 & \text{on} \quad (0,T)\times\Gamma_{\text{out}},\\
-d_i^\Sigma\partial_{\nu_\Sigma} c_i^\Sigma &=& 0 & \text{on} \quad (0,T)\times\partial\Sigma,\\
c_i(0) &=& c_{0,i} & \text{in} \quad \Omega,\\
c_i^\Sigma(0) &=& c_{0,i}^\Sigma & \text{on} \quad \Sigma,
\end{array}\right. \tag{8.1}$$

for the unknown concentrations (c_i, c_i^Σ) with $i = 1, ..., N$.

($\mathbf{A}^{\text{vel}}$) Throughout this part we assume that the velocity field satisfies

$$u \in \mathbb{U}_p^\Omega(T) := W^{1,p}((0,T), L^p(\Omega,\mathbb{R}^3)) \cap L^p((0,T), W^{2,p}(\Omega,\mathbb{R}^3)), \tag{8.2}$$

for given time $T > 0$. Moreover, let

$$u \cdot \nu \leq 0 \quad \text{on } \Gamma_{\text{in}}, \qquad u \cdot \nu = 0 \quad \text{on } \Sigma, \qquad u \cdot \nu \geq 0 \quad \text{on } \Gamma_{\text{out}},$$

and $\operatorname{div} u = 0$ in the distributional sense.

Examples for sorption and reaction rates

We give a few examples for sorption and reaction rate functions.

(S1) Let $k_i^{\text{ad}}, k_i^{\text{de}} > 0$ denote adsorption and desorption rate constants. The simplest sorption rate is given by the linear *Henry law*, i.e.

$$r_{H,i}^{\text{sorp}}(c_i, c_i^\Sigma) = k_i^{\text{ad}} c_i - k_i^{\text{de}} c_i^\Sigma.$$

This law models dilute systems.

(S2) For moderate concentrations, *Langmuir's law* given by

$$r_{L,i}^{\text{sorp}}(c_i, c_i^\Sigma) = k_i^{\text{ad}} c_i \left(1 - \frac{c_i^\Sigma}{c_{\infty,i}^\Sigma}\right) - k_i^{\text{de}} c_i^\Sigma$$

may be employed. Here $c_{\infty,i}^\Sigma > 0$ denotes the maximum capacity constant for adsorption of species i. In an application of our main results, we actually consider a modified version; see Remark 12.7, which satisfies all of our assumptions on the sorption rate stated in Chapter 11.

(R1) A standard example for a reaction rate function considers a reversible chemical reaction of type $A + B \rightleftharpoons P$ with $N = 3$ components. If mass action kinetics is employed, the mass productions are governed by the rate function

$$r_R^{\text{ch}}(c^\Sigma) = \begin{pmatrix} -k^{\text{re}}(c_1^\Sigma c_2^\Sigma - \kappa c_3^\Sigma) \\ -k^{\text{re}}(c_1^\Sigma c_2^\Sigma - \kappa c_3^\Sigma) \\ +k^{\text{re}}(c_1^\Sigma c_2^\Sigma - \kappa c_3^\Sigma) \end{pmatrix}.$$

Here $k^{\text{re}} > 0$ denotes the rate constant of the forward reaction, while κ is the equilibrium constant for this reaction, determined as the ratio between forward and backward reaction rates. For more information on r_R^{ch} and a global analysis of the corresponding reaction-diffusion system, including equilibria see [Rot84].

Main Results

The main results of this part are the local-in-time existence of a unique nonnegative strong L^p-solution and the global-in-time existence of a unique nonnegative strong L^2-solution. In all chapters, $(0, T)$ denotes a finite time interval. The local existence result reads as follows. For the assumptions imposed on the sorption and reaction rates see the beginning of Chapter 11 and Chapter 12.

Theorem 8.1 *(Local existence) Let $T' > 0$ be given and $5/3 < p < \infty$ with $p \neq 3$. Suppose u satisfies (A^{vel}), r^{sorp} satisfies (A_F^{sorp}), (A_M^{sorp}), (A_B^{sorp}) and r^{ch} fulfills (A_F^{ch}), (A_N^{ch}), (A_P^{ch}). Then for every set of data*

$$\begin{aligned}
&g_i^{in} \in W_p^{1/2-1/2p}((0,T'), L^p(\Gamma_{in})) \cap L^p((0,T'), W_p^{1-1/p}(\Gamma_{in})),\\
&c_{0,i} \in W_p^{2-2/p}(\Omega),\\
&c_{0,i}^{\Sigma} \in W_p^{2-2/p}(\Sigma),
\end{aligned}$$

which, if $p > 3$, satisfy the compatibility conditions

$$\begin{array}{rcll}
(u|_{t=0} \cdot \nu)c_{0,i} - d_i \partial_\nu c_{0,i} &=& g_i^{in}|_{t=0} & \text{on } \Gamma_{in},\\
-d_i \partial_\nu c_{0,i} &=& r_i^{sorp}(c_{0,i}, c_{0,i}^{\Sigma}) & \text{on } \Sigma,\\
-d_i \partial_\nu c_{0,i} &=& 0 & \text{on } \Gamma_{out},\\
-d_i^{\Sigma} \partial_{\nu_\Sigma} c_{0,i}^{\Sigma} &=& 0 & \text{on } \partial\Sigma,
\end{array}$$

there exists a $T^* \in (0, T')$ and a unique strong solution (c_i, c_i^{Σ}) of (8.1) satisfying

$$\begin{aligned}
c_i &\in W^{1,p}((0,T), L^p(\Omega)) \cap L^p((0,T), W^{2,p}(\Omega)),\\
c_i^{\Sigma} &\in W^{1,p}((0,T), L^p(\Sigma)) \cap L^p((0,T), W^{2,p}(\Sigma)),
\end{aligned}$$

for all $T \in (0, T^*)$. If additionally $g_i^{in} \leq 0$ on Γ_{in}, $c_{0,i} \geq 0$, in Ω, $c_{0,i}^{\Sigma} \geq 0$ on Σ, then c_i and c_i^{Σ} are nonnegative a.e., either.

Remark 8.2 *If c_i, c_i^{Σ} are not continuous functions, then $c_i \geq 0$ has to be understood in the a.e. sense with respect to Lebesgue measure on Ω, and $c_i^{\Sigma} \geq 0$ with respect to the surface measure on Σ.*

The global existence is given by

Theorem 8.3 *(Global existence) Let the assumptions of Theorem 8.1 be satisfied for $p = 2$ and $T' = \infty$. Additionally, assume that r^{ch} fulfills (A_S^{ch}), see (12.2) and that $c_{0,i} \in BC(\Omega)$, $c_{0,i}^{\Sigma} \in BC(\Sigma)$, and that $-g_i^{in}$ $c_{0,i}$, and $c_{0,i}^{\Sigma}$ are nonnegative. Then the local solution (c_i, c_i^{Σ}) extends to a global solution of (8.1), i.e., for $p = 2$ the assertions of Theorem 8.1 hold for every $T > 0$.*

8.1. Notation in Part II

When working in time-space domains we use the notation $\Omega_T := (0,T) \times \Omega$ and analogously $\Sigma_T := (0,T) \times \Sigma$ for a finite $T > 0$. We set $H^k := W^{k,2}$ for $k \in \mathbb{N}$.

We denote

$$[0,\infty)^N := \{x \in \mathbb{R}^N : \ x_i \geq 0 \text{ for all } i = 1, ..., N\}$$

and call it the *closed cone* in $\mathbb{R}^N$ for $N \in \mathbb{N}$. For the sake of convenience we employ $x \leq y$ for $x, y \in \mathbb{R}^N$ and mean $x_k \leq y_k$ for all $k = 1, ..., N$.

We denote by f^+, f^- the positive and negative part of a function f, i.e., we set $f^+ := \max\{0, f\}$, $f^- := -\min\{0, f\}$. Moreover, we use sets of functions, whose elements are nonnegative or nonpositive, e.g. we write $L^\infty(\Omega)^+$ for functions which admit a bounded essential supremum on Ω and which are nonnegative a.e. in Ω. With the corresponding meaning we employ e.g. $L^\infty(\Omega)^-$.

Throughout this work let $\nabla_\Sigma u := (\nabla u)|_{\partial\Omega} - \nu(\nu \cdot (\nabla u)|_{\partial\Omega})$ denote the surface gradient and let moreover $\Delta_\Sigma u = \nabla_\Sigma \cdot \nabla_\Sigma u$ denote the Laplace-Beltrami operator on Σ. The normal derivative of c_i^Σ with respect to $\partial\Sigma$ shall be denoted by $\partial_{\nu_\Sigma} c_i^\Sigma$.

8.1.1. Maximal Regularity Spaces

For $1 < p < \infty$ let us introduce the following maximal regularity spaces. The solution spaces for the unknown functions c_i, c_i^Σ are given by

$$
\begin{aligned}
\mathbb{E}_p^\Omega(T) &:= W^{1,p}((0,T), L^p(\Omega)) \cap L^p((0,T), W^{2,p}(\Omega)), \\
\mathbb{E}_p^\Sigma(T) &:= W^{1,p}((0,T), L^p(\Sigma)) \cap L^p((0,T), W^{2,p}(\Sigma)).
\end{aligned}
$$

For the data spaces we first establish regularity classes. We set

$$
\begin{aligned}
\mathbb{F}_p^\Omega(T) &:= L^p((0,T) \times \Omega), \\
\mathbb{F}_p^\Sigma(T) &:= L^p((0,T) \times \Sigma), \\
\mathbb{G}_p^{\text{in}}(T) &:= W_p^{1/2-1/2p}((0,T), L^p(\Gamma_{\text{in}})) \cap L^p((0,T), W_p^{1-1/p}(\Gamma_{\text{in}})), \\
\mathbb{G}_p^\Sigma(T) &:= W_p^{1/2-1/2p}((0,T), L^p(\Sigma)) \cap L^p((0,T), W_p^{1-1/p}(\Sigma)), \\
\mathbb{G}_p^{\text{out}}(T) &:= W_p^{1/2-1/2p}((0,T), L^p(\Gamma_{\text{out}})) \cap L^p((0,T), W_p^{1-1/p}(\Gamma_{\text{out}})), \\
\mathbb{I}_p^\Omega &:= W_p^{2-2/p}(\Omega), \\
\mathbb{I}_p^\Sigma &:= W_p^{2-2/p}(\Sigma).
\end{aligned}
$$

We define the tupel data space for the catalyst equations without initial data through

$$
\mathbb{F}_p^{\Omega,\Sigma}(T) := \mathbb{F}_p^\Omega(T) \times \mathbb{F}_p^\Sigma(T) \times \mathbb{G}_p^{\text{in}}(T) \times \mathbb{G}_p^\Sigma(T) \times \mathbb{G}_p^{\text{out}}(T) \times \{0\}
$$

and the tupel data space with initial spaces through

$$
\mathbb{F}_{p,I}^{\Omega,\Sigma}(T) := \mathbb{F}_p^{\Omega,\Sigma}(T) \times \mathbb{I}_p^\Omega \times \mathbb{I}_p^\Sigma.
$$

In some statements we also employ the Dirichlet trace space on Σ, which is given through

$$
\mathbb{H}_p^\Sigma(T) := W_p^{1-1/2p}((0,T), L^p(\Sigma)) \cap L^p((0,T), W_p^{2-1/p}(\Sigma)).
$$

In Part II of this thesis we employ zero time trace spaces also in case of the maximal regularity classes introduced above. We write e.g. ${}_0\mathbb{E}_p^{\Omega}(T)$, ${}_0\mathbb{E}_p^{\Sigma}(T)$ etc.

In Part II we make use of a generalized notion of maximal regularity suitable for inhomogeneous initial boundary value problems.

Definition 8.4 *If $\mathcal{L}$ is a linear operator from the Banach space $\mathbb{E}$ into the Banach space $\mathbb{F}$, we say that $\mathcal{L}$ has* maximal regularity *if $\mathcal{L} : \mathbb{E} \to \mathbb{F}$ is an isomorphism.*

For instance, $\mathcal{L}$ can represent the full left hand side of (8.1), where we choose $\mathbb{E} = \mathbb{E}_p^{\Omega}(T)^N \times \mathbb{E}_p^{\Sigma}(T)^N$ as the space of solutions and $\mathbb{F} = \mathbb{F}_{p,I}^{\Omega,\Sigma}(T)^N$ as the data class. Note that Definition 8.4 is consistent with Definition 2.18.

Chapter 9

Outline of the Modeling

Let us briefly comment on the fundamental modeling of the heterogeneous catalysis. For a more general model see the upcoming work [BMOS]. The modeling is based on continuum mechanics in consistency with the second law of thermodynamics. We restrict ourselves to the isothermal case and assume that the bulk phase is occupied by a dilute mixture, i.e. one of the given constituents $A_1, ..., A_{N+1}$ is the solvent – without loss of generality A_{N+1} – and $A_1, ..., A_N$ are solutes. The corresponding molar concentrations and mass densities are denoted by $c_1, ..., c_{N+1}$ and $\rho_1, ..., \rho_{N+1}$. Similarly, we write $A_1^*, ..., A_N^*$ for the corresponding adsorbed species and so on. We assume the solvent to be an incompressible fluid, such that the total molar mass and the total mass density are constant. We derive partial mass balances (continuity equations) by employing a fixed controll volume and applying the divergence theorem in its bulk and surface version to the integral balances for A_i. This way we obtain the differential form of the balance equations in the underlying domain $\Omega \subset \mathbb{R}^3$ with active surface $\Sigma \subset \partial\Omega$. Let u denote the velocity field describing the motion of the solvent A_N. In absence of chemical reactions in the bulk phase and in view of the conditions div $u = 0$ and $u \cdot \nu = 0$ on Σ the balance equations read

$$\left\{ \begin{array}{rcll} \partial_t \rho_i + \operatorname{div}(\rho_i u + j_i) & = & 0 & \text{in} \quad (0,T) \times \Omega, \\ \partial_t \rho_i^\Sigma + \operatorname{div}_\Sigma (j_i^\Sigma) & = & M_i r_i^{\text{ch}} + M_i r_i^{\text{sorp}} & \text{on} \quad (0,T) \times \Sigma. \end{array} \right.$$

Here M_i denotes the molar mass, j_i the diffusive bulk flux for A_i, j_i^Σ the diffusive surface flux for A_i^*, r_i^{ch} the molar mass production rate due to chemical reactions and r_i^{sorp} the one due to sorption. Here we also made use of a pure surface mass balance in order to identify r_i^{sorp}. In terms of molar mass concentrations given through $c_i = \rho_i / M_i$ and $c_i^\Sigma = \rho_i^\Sigma / M_i$ and molar diffusive fluxes $J_i := j_i / M_i$, $J_i^\Sigma := j_i^\Sigma / M_i$ we may formulate molar mass balance relations

$$\left\{ \begin{array}{rcll} \partial_t c_i + \operatorname{div}(c_i u + J_i) & = & 0 & \text{in} \quad (0,T) \times \Omega, \\ \partial_t c_i^\Sigma + \operatorname{div}_\Sigma (J_i^\Sigma) & = & r_i^{\text{ch}} + r_i^{\text{sorp}} & \text{on} \quad (0,T) \times \Sigma, \end{array} \right.$$

with $r_i^{\text{sorp}} = r_i^{\text{ad}} - r_i^{\text{de}}$ and r_i^{ad} standing for the pure adsorption and r_i^{de} for the pure desorption rate.

In order to close the model we have to introduce consitutive laws for the material dependent quantaties J_i, J_i^Σ, r_i^{ch} and r_i^{sorp}. This is done in consistency with the entropy inequality, i.e. the total free energy is non-increasing in case of a closed system.

- The simplest possible and meaningful choice of the diffusive fluxes J_i is Fickian diffusion wich constant diffusivities $d_i > 0$ which goes back to [Fic55] and read
$$J_i = -d_i \nabla c_i, \qquad i = 1, ..., N.$$
Note that due to the diluteness assumption in the bulk this is a possible natural choice.

- The surface fluxes are more involved since it is not clear why A^*_{N+1} should be a major absored species. In general the diluteness assumption is only satisfied in the bulk and does not carry over to the active surface. However, for the mathematical treatment we choose the same type of fluxes on the surface as in the bulk and constitute
$$J_i^\Sigma = -d_i^\Sigma \nabla_\Sigma c_i^\Sigma, \qquad i = 1, ..., N$$
with surface diffusion coefficients $d_i^\Sigma > 0$.

- Following the spirit of [BD15] we choose a suitable ratio of forward and backward reaction rate. Let us note that we intend to restrict ourselves to reversible reactions. For $R = R(N) \in \mathbb{N}$ of such reactions of the adsorbed species $A_1^*, ..., A_N^*$ taking place simultaneously we formally write
$$\alpha_1^j A_1^* + ... + \alpha_N^j A_N^* \rightleftharpoons \beta_1^j A_1^* + ... + \beta_N^j A_N^*, \qquad 1 \leq j \leq R,$$
where $\alpha^j, \beta^j \in \mathbb{N}_0^N$ denote stoichiometric coefficients of the j^{th} reaction. By mass action kinetics the production rate is the difference of forward and backward rates, where – up to constants – each of them are products of powers of concentrations, see [SFH89, Esp95].

- The sorption rate is chosen analogously. We insert the quotient $r_i^{\mathrm{ad}}/r_i^{\mathrm{de}}$. Then by choosing – without loss of generality r_i^{de} – the remaining rate r_i^{ad} is given implicitely.

Closing this model with these listed constitutive laws we end up with
$$\left\{ \begin{array}{rcll} \partial_t c_i + (u \cdot \nabla) c_i - d_i \Delta c_i & = & 0 & \text{in} \quad (0,T) \times \Omega, \\ \partial_t c_i^\Sigma - d_i^\Sigma \Delta_\Sigma c_i^\Sigma & = & r_i^{\mathrm{ch}} + r_i^{\mathrm{sorp}} & \text{on} \quad (0,T) \times \Sigma. \end{array} \right.$$

This system of partial differential equations has to be complemented by suitable boundary conditions on $\partial\Omega$ and – since describing an evolution of concentrations c_i and c_i^Σ – also by initial conditions.

Let us briefly comment on the model. In this work we assume sorption kinetics with no maximal capacity on the catalytic wall. Another possible approach to heterogeneous catalysis – different from the one we choose here – is a model with a maximal capacity. Indeed, it seems to lead to a more realistic model if *free sites* are introduced on the active surface, see the upcoming paper [BMOS]. Such a model would better fit to the original Langmuir-Hinshelwood sorption kinetics

which goes back to the works of Langmuir and can be found in the pioneering work [Lan18].
Another modification could be of further interest: Replacing Fickian diffusion by Maxwell-Stefan type diffusion for fluxes in the bulk as well as on the surface. It is not clear at all why diluteness should hold on the surface. Therefore Fickian diffusion seems to be less appropriate than Maxwell-Stefan diffusion, which recently attracted great attention by several authors in different fields including fluid dynamics [BD15, Bot11, HMPW].

Chapter 10

Linear Equations

We discuss a suitable linearization of the catalyst equations as it is given below in (10.1) and show maximal regularity for $5/3 < p < \infty$ and $p \neq 3$ by means of cylindrical L^p-theory, the surjectivity of the Neumann trace operator and a perturbation argument. For homogeneous cylindrical problems, such as arising as an auxiliary problem in this work, a rich literature concerning $\mathcal{H}^\infty$-calculus respectively $\mathcal{R}$-sectoriality is available. Therefore, only an application of existing theory is required here. Unfortunately, trace operators for our purposes have not been studied so far to the best of the author's knowledge. Hence the proof of the surjectivity of the trace operator is carried out in Section 10.2. Finally, the velocity terms are treated as lower order perturbation terms. Here we employ suitable results on pointwise multiplication and the Neumann series lemma.

For given data

$$(f_i, f_i^\Sigma, g_i^{\text{in}}, g_i^\Sigma, g_i^{\text{out}}, 0, c_{0,i}, c_{0,i}^\Sigma) \in \mathbb{F}_{p,I}^{\Omega,\Sigma}(T)$$

we consider the linear system:

$$\left\{\begin{array}{rcll}
\partial_t c_i + (u\cdot\nabla)c_i - d_i\Delta c_i &=& f_i & \text{in} \quad (0,T)\times\Omega,\\
\partial_t c_i^\Sigma - d_i^\Sigma \Delta_\Sigma c_i^\Sigma &=& f_i^\Sigma & \text{on} \quad (0,T)\times\Sigma,\\
(u\cdot\nu)c_i - d_i\partial_\nu c_i &=& g_i^{\text{in}} & \text{on} \quad (0,T)\times\Gamma_{\text{in}},\\
-d_i\partial_\nu c_i &=& g_i^\Sigma & \text{on} \quad (0,T)\times\Sigma,\\
-d_i\partial_\nu c_i &=& g_i^{\text{out}} & \text{on} \quad (0,T)\times\Gamma_{\text{out}},\\
-d_i^\Sigma\partial_{\nu_\Sigma} c_i^\Sigma &=& 0 & \text{on} \quad (0,T)\times\partial\Sigma,\\
c_i(0) &=& c_{0,i} & \text{in} \quad \Omega,\\
c_i^\Sigma(0) &=& c_{0,i}^\Sigma & \text{in} \quad \Sigma.
\end{array}\right. \tag{10.1}$$

Our purpose is to solve (10.1) for the unknown concentrations c_i, c_i^Σ. The main result of this chapter is given by

Proposition 10.1 (Maximal regularity) *Let $5/3 < p < \infty$ with $p \neq 3$ and let $T > 0$ be finite. Suppose the velocity field u satisfies assumption (A^{vel}). Then (10.1) admits a unique solution*

$$(c_i, c_i^\Sigma) \in \mathbb{E}_p^\Omega(T) \times \mathbb{E}_p^\Sigma(T)$$

if and only if the data satisfy the regularity condition

$$(f_i, f_i^\Sigma, g_i^{\text{in}}, g_i^\Sigma, g_i^{\text{out}}, 0, c_{0,i}, c_{0,i}^\Sigma) \in \mathbb{F}_{p,I}^{\Omega,\Sigma}(T),$$

and in case of $p > 3$ the compatibility conditions

$$\begin{array}{rcll} (u|_{t=0} \cdot \nu)c_{0,i} - d_i \partial_\nu c_{0,i} &=& g_i^{in}|_{t=0} & \text{on } \Gamma_{in}, \\ -d_i \partial_\nu c_{0,i} &=& g_i^{\Sigma}|_{t=0} & \text{on } \Sigma, \\ -d_i \partial_\nu c_{0,i} &=& g_i^{out}|_{t=0} & \text{on } \Gamma_{out}, \\ -d_i \partial_\nu c_{0,i}^{\Sigma} &=& 0 & \text{on } \partial\Sigma. \end{array}$$

Additionally, the corresponding solution operator ${}_0\mathcal{S}_T$ with respect to zero time trace spaces satisfies

$$\|{}_0\mathcal{S}_\tau\|_{\mathscr{L}\left({}_0\mathbb{F}_p^{\Omega,\Sigma}(\tau)^N,\, {}_0\mathbb{E}_p^{\Omega}(\tau)^N \times {}_0\mathbb{E}_p^{\Sigma}(\tau)^N\right)} \leq M \qquad (0 < \tau < T)$$

for a constant $M > 0$ independent of τ.

Plan of the proof: System (10.1) decomposes completely into two systems: One for the concentrations c_i in Ω and one for the boundary concentrations c_i^Σ on Σ. In the first step we neglect the velocity terms $(u \cdot \nabla)c_i$ and $(u \cdot \nu)c_i$ – playing the rôle of perturbation terms – and consider only homogeneous boundary data, i.e. we start with

$$\left\{ \begin{array}{rcll} \partial_t c_i - d_i \Delta c_i &=& f_i & \text{in} \quad (0,T) \times \Omega, \\ -d_i \partial_\nu c_i &=& 0 & \text{on} \quad (0,T) \times \partial\Omega, \qquad (i = 1, ..., N) \\ c_i(0) &=& c_{0,i} & \text{in} \quad \Omega, \end{array} \right. \tag{10.2}$$

and

$$\left\{ \begin{array}{rcll} \partial_t c_i^\Sigma - d_i^\Sigma \Delta_\Sigma c_i^\Sigma &=& f_i^\Sigma & \text{on} \quad (0,T) \times \Sigma, \\ -d_i^\Sigma \partial_{\nu_\Sigma} c_i^\Sigma &=& 0 & \text{on} \quad (0,T) \times \partial\Sigma, \qquad (i = 1, ..., N) \\ c_i^\Sigma(0) &=& c_{0,i}^\Sigma & \text{on} \quad \Sigma. \end{array} \right. \tag{10.3}$$

We proceed as follows: We solve (10.2) and (10.3) separately via cylindrical L^p-theory. For more information on this topic see [NS11, NS12, Nau12, Nau13]. Then a symmetric extension in axial direction of Ω yields the surjectivity of the Neumann trace operator and consequently the solvability of the inhomogeneous initial boundary value problem. By perturbation arguments the obtained result carries over to (10.1). Note that in the perturbation step the condition $p > 5/3$ is employed in order to apply multiplication results.

Remark 10.2 (Compatibility conditions) *Note that in this thesis we only have to take care of compatibility conditions between boundary data and initial data in case of $p > 3$. This is due to the special choice and combination of boundary conditions on each boundary component Γ_{in}, Σ and Γ_{out}. In general a* weakly singular domain *such as a finite cylinder considered here, leads to a second set of compatibility conditions which naturally arise between the boundary data at edges of adjacent smooth boundary components. These conditions occur for all $p > p_0$ for a certain $p_0 \in (1, \infty)$ depending on the order of the involved boundary operators. For more information on the compatibility conditions arising from singularities at the boundary of the underlying domain see, e.g., [Köh13, Chapter 8] or [Seg13] and the references cited therein, in particular [BDM03].*

10.1. $\mathcal{H}^\infty$-Calculus for the Laplacian

We employ the theory on operator sums introduced in Chapter 2 to solve (10.2) and (10.3). More precisely, we directly apply results by Nau and Saal who proved results on homogeneous cylindrical problems like (10.2) by employing Kalton-Weis type theorems, cf. Chapter 2. To work in the functional analytic setting given therein we define

$$A_i := -d_i\Delta, \quad A_i : D(A_i) \subset L^p(\Omega) \to L^p(\Omega),$$
$$D(A_i) := \{c_i \in W^{2,p}(\Omega) : \ -d_i\partial_\nu c_i = 0 \text{ on } \partial\Omega\}, \quad (i = 1, ..., N)$$

and analogously

$$A_i^\Sigma := -d_i^\Sigma \Delta_\Sigma, \quad A_i^\Sigma : D(A_i^\Sigma) \subset L^p(\Sigma) \to L^p(\Sigma),$$
$$D(A_i^\Sigma) := \{c_i^\Sigma \in W^{2,p}(\Sigma) : \ -d_i^\Sigma \partial_{\nu_\Sigma} c_i^\Sigma = 0 \text{ on } \partial\Sigma\} \quad (i = 1, ..., N).$$

In this section we aim to show that A_i, A_i^Σ admit a bounded $\mathcal{H}^\infty$-calculus which implies the desired maximal regularity, cf. [DHP03], [KW04]. For A_i we directly apply [Nau13, Theorem 4.1]. To this end, we employ the following cylindrical decomposition. We set
$V_1 := B_R(0) \subset \mathbb{R}^2$ and $V_2 := (0, h) \subset \mathbb{R}$ for the radius $R > 0$ and height $h > 0$ of the cylinder Ω.

- Let us start with intersection given through the ball $V_1 = B_R(0)$:

$$A_{i,1} : D(A_{i,1}) \subset L^p(V_1) \to L^p(V_1), \quad A_{i,1}c_i := -d_i(\partial_{x_1}^2 + \partial_{x_2}^2)c_i,$$
$$D(A_{i,1}) := \{c_i \in W^{2,p}(V_1) : \ B_{i,1}c_i := -d_i\partial_\nu c_i = 0 \text{ on } \partial V_1\}.$$

- The remaining dimension of the cylinder leads to an operator on an interval:

$$A_{i,2} : D(A_{i,2}) \subset L^p(V_2) \to L^p(V_2), \quad A_{i,2}c_i := -d_i\partial_{x_3}^2 c_i,$$
$$D(A_{i,2}) := \{c_i \in W^{2,p}(V_2) : \ B_{i,2}c_i := -d_i\partial_\nu c_i = 0 \text{ on } \partial V_2\}.$$

Obviously we are in the setting of [Nau13] in the case of the strong Neumann Laplacian given on both intersections V_1, V_2. Therefore [Nau13, Theorem 4.1 a)] yields that $A_i + \delta$ for some $\delta > 0$ admits a bounded $\mathcal{H}^\infty$-calculus on $L^p(\Omega)$ with $\mathcal{H}^\infty$-angle $\phi_{A_i+\delta}^\infty < \frac{\pi}{2}$. This implies maximal regularity for (10.2) on finite intervals $(0, T)$ by Remark 2.22.

We continue with the discussion of A_i^Σ. Here we first employ polar coordinates and apply afterwards [Nau13, Theorem 4.1]. To this end, we introduce

$$\psi_R^\Sigma : (0, 2\pi) \to \mathbb{R}^2, \quad \varphi \mapsto \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \end{pmatrix}$$

and the induced pull-back

$$\Psi^\Sigma : L^p(\partial V_1) \to L^p((0, 2\pi)), \quad \Psi^\Sigma c_i^\Sigma := c_i^\Sigma \circ \psi_R^\Sigma.$$

With Ψ^Σ at hand we are able to describe the occurring Laplace-Beltrami operator $\Delta_{\Sigma,1}$ on ∂V_1 concretely. Indeed, we have

$$\Psi^\Sigma(-d_i^\Sigma \Delta_{\Sigma,1} c_i^\Sigma) = -\frac{d_i^\Sigma}{R^2}\partial_\varphi^2(c_i^\Sigma \circ \psi_R^\Sigma) \qquad (c_i^\Sigma \in W^{2,p}(\partial V_1)). \tag{10.4}$$

This allows to consider the Ψ^Σ-transformed operator $-\partial_\varphi^2$ on $(0,2\pi)$ subject to periodic boundary conditions instead of $\Delta_{\Sigma,1}$ on ∂V_1 directly. The domain of $-\partial_\varphi^2$ is given as

$$D(-\partial_\varphi^2) = \{c_i^\Sigma \in W^{2,p}((0,2\pi)) : \ \partial_\varphi^j c_i^\Sigma|_{\varphi=0} = \partial_\varphi^j c_i^\Sigma|_{\varphi=2\pi} \ \ (j=0,1)\}.$$

We have

Lemma 10.3 *Let the pull-back Ψ^Σ be given as above, then*

(a) $\Psi^\Sigma : L^p(\partial V_1) \to L^p((0,2\pi))$ *is an isomorphism.*

(b) $\Psi^\Sigma : D(\Delta_{\Sigma,1}) \to D(-\partial_\varphi^2)$ *is an isomorphism, i.e. with respect to the graph norms.*

Proof. (a): First of all we show that

$$\Psi^\Sigma : L^p(\partial V_1) \to L^p((0,2\pi))$$

is well-defined and continuous. Let us compute Gram's determinant of ψ_R^Σ:

$$\det\begin{pmatrix} \partial_R\psi_R^\Sigma \cdot \partial_R\psi_R^\Sigma & \partial_R\psi_R^\Sigma \cdot \partial_\varphi\psi_R^\Sigma \\ \partial_\varphi\psi_R^\Sigma \cdot \partial_R\psi_R^\Sigma & \partial_\varphi\psi_R^\Sigma \cdot \partial_\varphi\psi_R^\Sigma \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix} = R^2.$$

Hence by definition of the surface integral we have

$$\int\limits_{\partial V_1} |c_i^\Sigma(x_1,x_2)|^p d\sigma(x_1,x_2) = \int\limits_0^{2\pi} |c_i^\Sigma(\psi_R^\Sigma(\varphi))|^p R d\varphi, \qquad (c_i^\Sigma \in L^p(\partial V_1))$$

and therefore

$$\begin{aligned} \|\Psi^\Sigma c_i^\Sigma\|_{L^p((0,2\pi))}^p = \int\limits_0^{2\pi} |(c_i^\Sigma \circ \psi_R^\Sigma)(\varphi)|^p d\varphi &= \frac{1}{R}\int\limits_{\partial V_1} |c_i^\Sigma(x_1,x_2)|^p d\sigma(x_1,x_2) \\ &= \frac{1}{R}\|c_i^\Sigma\|_{L^p(\partial V_1)}^p, \end{aligned}$$

and equivalently

$$\|\Psi^\Sigma c_i^\Sigma\|_{L^p((0,2\pi))} = R^{-1/p}\|c_i^\Sigma\|_{L^p(\partial V_1)} \qquad (c_i^\Sigma \in L^p(\partial V_1)). \tag{10.5}$$

This yields that $\Psi^\Sigma : L^p(\partial V_1) \to L^p((0,2\pi))$ is well-defined and continuous. Since the function ψ_R^Σ is bijective, the same holds true for Ψ^Σ. The continuity of $(\Psi^\Sigma)^{-1}$

also follows by (10.5).
(b): Let $c_i^\Sigma \in D(\Delta_{\Sigma,1})$. We show that

$$\| -\frac{d_i^\Sigma}{R^2}\partial_\varphi^2(\Psi^\Sigma c_i^\Sigma)\|_{L^p((0,2\pi))} = R^{-1/p}\|d_i^\Sigma \Delta_{\Sigma,1} c_i^\Sigma\|_{L^p(\partial V_1)},$$

then the isomorphism property of Ψ^Σ with respect to the graph norms on the domains readily follows. By (10.4) we have

$$\begin{aligned}
\| -\frac{d_i^\Sigma}{R^2}\partial_\varphi^2(\Psi^\Sigma c_i^\Sigma)\|^p_{L^p((0,2\pi))} &= \int_0^{2\pi} \left| -\frac{d_i^\Sigma}{R^2}\partial_\varphi^2 c_i^\Sigma(\psi_R^\Sigma(\varphi))\right|^p d\varphi \\
&= \frac{1}{R}\int_0^{2\pi} |\Psi^\Sigma(-d_i^\Sigma \Delta_{\Sigma,1} c_i^\Sigma)|^p R d\varphi \\
&= \frac{1}{R}\int_0^{2\pi} |((-d_i^\Sigma \Delta_{\Sigma,1} c_i^\Sigma)\circ \psi_R^\Sigma)(\varphi)|^p R d\varphi \\
&= \frac{1}{R}\int_{\partial V_1} |-d_i^\Sigma \Delta_{\Sigma,1} c_i^\Sigma(x_1,x_2)|^p d\sigma(x_1,x_1) \\
&= \frac{1}{R}\| -d_i^\Sigma \Delta_{\Sigma,1} c_i^\Sigma\|^p_{L^p(\partial V_1)}, \qquad (c_i^\Sigma \in D(\Delta_{\Sigma,1})),
\end{aligned}$$

where R^{-1} is due to Gram's determinant. Due to (a) we obtain

$$\|\Psi^\Sigma c_i^\Sigma\|_{D(\partial_\varphi^2)} = R^{-1/p}\|c_i^\Sigma\|_{D(\Delta_{\Sigma,1})}.$$

Hence the graph norms are equivalent and $\Psi^\Sigma : D(\Delta_{\Sigma,1}) \to D(\partial_\varphi^2)$ is an isomorphism. □

Having Lemma 10.3 at hand we go on with the cylindrical decomposition of (10.3) respectively A_i^Σ.

Analogously to A_i we resolve the transformation of A_i^Σ into two cylindrical parts:

- We set

$$\begin{aligned}
&A_{i,1}^\Sigma : D(A_{i,1}^\Sigma) \subset L^p((0,2\pi)) \to L^p((0,2\pi)), \quad A_{i,1}^\Sigma c_i^\Sigma := -\frac{d_i}{R^2}\partial_\varphi^2 c_i^\Sigma, \\
&D(A_{i,1}^\Sigma) := \{c_i^\Sigma \in W^{2,p}((0,2\pi)) : \ \partial_\varphi^j c_i^\Sigma|_{\varphi=0} = \partial_\varphi^j c_i^\Sigma|_{\varphi=2\pi} \ \ (j=0,1)\}.
\end{aligned}$$

- The axial direction of Ω corresponds to

$$\begin{aligned}
&A_{i,2}^\Sigma : D(A_{i,2}^\Sigma) \subset L^p(V_2) \to L^p(V_2), \quad A_{i,2}^\Sigma c_i^\Sigma := -d_i^\Sigma \partial_{x_3}^2 c_i^\Sigma, \\
&D(A_{i,2}^\Sigma) := \{c_i^\Sigma \in W^{2,p}(V_2) : \ -d_i^\Sigma \partial_{x_3} c_i^\Sigma = 0 \,\text{on}\, \partial V_2\}.
\end{aligned}$$

Note that Lemma 10.3 justifies to directly work with $A_{i,1}^{\Sigma}$ instead of $\Delta_{\Sigma,1}$. Applying again [Nau13, Theorem 4.1 a)] we infer that there is a shift $\delta > 0$ such that $\delta + A_i^{\Sigma}$ for some $\delta > 0$ admits a bounded $\mathcal{H}^{\infty}$-calculus on $L^p(\Sigma)$ with $\mathcal{H}^{\infty}$-angle $\phi^{\infty}_{A_i^{\Sigma}+\delta} < \frac{\pi}{2}$. This implies maximal regularity for (10.3) on finite intervals $(0,T)$ by Remark 2.22. We summarize the results of subsection 10.1 in

Lemma 10.4 *Let $1 < p < \infty$ with $p \neq 3$ and $T > 0$ be given.*

(a) *System (10.2) admits a unique solution $c_i \in \mathbb{E}_p^{\Omega}(T)$ if and only if the data satisfies the regularity condition $f_i \in \mathbb{F}_p^{\Omega}(T)$, $c_{0,i} \in \mathbb{I}_p^{\Omega}$ and in case of $p > 3$ the compatibility condition $-d_i \partial_\nu c_{0,i} = 0$ on $\partial\Omega$.*

(b) *System (10.3) admits a unique solution $c_i^{\Sigma} \in \mathbb{E}_p^{\Sigma}(T)$ if and only if the data satisfies the regularity condition $f_i^{\Sigma} \in \mathbb{F}_p^{\Sigma}(T)$, $c_{0,i}^{\Sigma} \in \mathbb{I}_p^{\Sigma}$ and in case of $p > 3$ the compatibility condition $-d_i^{\Sigma} \partial_{\nu_\Sigma} c_{0,i} = 0$ on $\partial\Sigma$.*

10.2. Inhomogeneous Neumann Boundary Conditions

We turn to the discussion of inhomogeneous boundary values. We show surjectivity of the trace operator which leads to the solvability of the corresponding inhomogeneous boundary value problem.

Lemma 10.5 *Let $1 < p < \infty$ with $p \neq 3$ and let $T > 0$. Then the Neumann trace operator*

$$\begin{aligned} \gamma_1 : \mathbb{E}_p^{\Omega}(T) &\to \mathbb{G}_p^{in}(T) \times \mathbb{G}_p^{\Sigma}(T) \times \mathbb{G}_p^{out}(T) \\ c &\mapsto (\partial_\nu c|_{\Gamma_{in}}, \partial_\nu c|_{\Sigma}, \partial_\nu c|_{\Gamma_{out}}) \end{aligned}$$

is a retraction.

Proof. Assume that

$$g^{\text{in}} \in \mathbb{G}_p^{\text{in}}(T),\ g^{\Sigma} \in \mathbb{G}_p^{\Sigma}(T),\ g^{\text{out}} \in \mathbb{G}_p^{\text{out}}(T)$$

are given boundary data. We show that there exists a $c \in \mathbb{E}_p^{\Omega}(T)$ such that

$$\partial_\nu c = g^{\text{in}} \quad \text{on } \Gamma_{\text{in}}, \tag{10.6}$$
$$\partial_\nu c = g^{\Sigma} \quad \text{on } \Sigma, \tag{10.7}$$
$$\partial_\nu c = g^{\text{out}} \quad \text{on } \Gamma_{\text{out}}. \tag{10.8}$$

Note that for the sake of readability we skip the index i and we work with the domain $\Omega = B_R(0) \times (-h,h)$, such that $\Gamma_{\text{in}} = B_R(0) \times \{-h\}$ and $\Gamma_{\text{out}} = B_R(0) \times \{h\}$ in this section. Let us define the halfspaces

$$H_{-h} := \mathbb{R}^2 \times (-h, \infty), \quad H_h := \mathbb{R}^2 \times (-\infty, h).$$

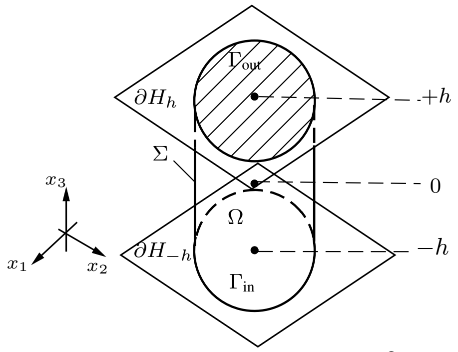

Figure 10.1: Finite cylinder bounded by $\partial H_{\pm h}$

We proceed in two steps.

Step 1. Due to [AF03, Chapter 5] there is an extension of g^{in} to

$$\tilde{g}^{\text{in}} \in \mathbb{G}_p^{\partial H_{-h}}(T),$$

and of g^{out} to

$$\tilde{g}^{\text{out}} \in \mathbb{G}_p^{\partial H_h}(T).$$

We have

$$\tilde{g}^{\text{in}}(0) \in W_p^{1-3/p}(\partial H_{-h}), \quad \tilde{g}^{\text{out}}(0) \in W_p^{1-3/p}(\partial H_h).$$

In case that $p > 3$ we may choose a $\tilde{c}_0^{\text{in}} \in W_p^{2-2/p}(H_{-h})$ and $\tilde{c}_0^{\text{out}} \in W_p^{2-2/p}(H_h)$ such that

$$\partial_\nu \tilde{c}_0^{\text{in}} = \tilde{g}^{\text{in}}(0) \quad \text{on } \partial H_{-h},$$
$$\partial_\nu \tilde{c}_0^{\text{out}} = \tilde{g}^{\text{out}}(0) \quad \text{on } \partial H_h,$$

due to Proposition 1.7, that is the known trace result by Marschall with respect to mere space variables and without time. In case that $p < 3$ let $\tilde{c}_0^{\text{in}} \in W_p^{2-2/p}(H_{-h})$ and
$\tilde{c}_0^{\text{out}} \in W_p^{2-2/p}(H_h)$ be arbitrary. Due to [DHP07] we may solve the parabolic problem

$$\begin{cases} \partial_t v^{\text{in}} - \Delta v^{\text{in}} &= 0 \quad \text{in} \quad (0,T) \times H_{-h}, \\ \partial_\nu v^{\text{in}} &= \tilde{g}^{\text{in}} \quad \text{on} \quad (0,T) \times \partial H_{-h}, \\ v^{\text{in}}(0) &= \tilde{c}_0^{\text{in}} \quad \text{in} \quad H_{-h}, \end{cases}$$

for a unique solution

$$v^{\text{in}} \in \mathbb{E}_p^{H_{-h}}(T)$$

and analogously we solve

$$\begin{cases} \partial_t v^{\text{out}} - \Delta v^{\text{out}} &= 0 \quad \text{in} \quad (0,T) \times H_h, \\ \partial_\nu v^{\text{out}} &= \tilde{g}^{\text{out}} \quad \text{on} \quad (0,T) \times \partial H_h, \\ v^{\text{out}}(0) &= \tilde{c}_0^{\text{out}} \quad \text{in} \quad H_h, \end{cases}$$

for a unique solution

$$v^{\text{out}} \in \mathbb{E}_p^{H_h}(T).$$

Let the cut-off function $\zeta \in C^\infty(\mathbb{R}, [0,1])$ satisfy

$$\zeta|_{(-\infty,-h/3)} = 1, \quad \zeta|_{(h/3,\infty)} = 0. \tag{10.9}$$

Then the convex combination

$$v := \zeta v^{\text{in}}|_\Omega + (1-\zeta) v^{\text{out}}|_\Omega \in \mathbb{E}_p^\Omega(T)$$

fulfills the boundary conditions on top and bottom of Ω by construction. Note that this construction in particular respects all appearing compatibility conditions between boundary and initial data.

Step 2. It remains to show that there exists a $w \in \mathbb{E}_p^\Omega(T)$ such that

$$\begin{aligned}
\partial_\nu w &= 0 && \text{on } \Gamma_{\text{in}},\\
\partial_\nu w &= g^\Sigma - \partial_\nu v && \text{on } \Sigma,\\
\partial_\nu w &= 0 && \text{on } \Gamma_{\text{out}}.
\end{aligned}$$

To this end we reduce this problem to an equation on a bounded C^2-domain, which works as follows. To this end, first define Ω_{-h} as the domain resulting from extending Ω in some way boundedly and smoothly (at least in the C^2-sense) on the top. For instance, we connect half of a ball to Ω at Γ_{out}. We also set $\Sigma_{-h} := \partial\Omega_{-h} \setminus \overline{\Gamma}_{\text{in}}$. In a similar manner we define Ω_{+h} and Σ_{+h} by extending Ω suitably at the bottom. Then, let $G_\pm$ denote the domains resulting from reflecting $\Omega_{\pm h}$ at $h_\pm$ and set $\Gamma_\pm := \partial G_\pm$. Then $G_\pm$ has e.g. the form of a 'pill'. It is clear that this way we always can find a suitable extension such that $G_\pm$ is of class C^2.

Let ζ be the cut-off function with the properties given in (10.9). We extend the function $\zeta(g^\Sigma - \partial_\nu v)$ to

$$\tilde{g}_-^\Sigma \in \mathbb{G}_p^{\Sigma_{-h}}(T) = W_p^{1/2-1/2p}((0,T), L^p(\Sigma_{-h}) \cap L^p((0,T), W_p^{1-1/p}(\Sigma_{-h}))$$

by 0 and $\tilde{g}_-^\Sigma$ to

$$\hat{g}_-^\Sigma \in \mathbb{G}_p^{\Gamma_-}(T)$$

by even reflection. Note that the extension by even reflection conserves the regularity $W_p^{1-1/p}$ in axial direction. Next, let $\hat{c}_{\text{in}}^\Sigma \in W_p^{2-2/p}(G_-)$ solve a suitable Neumann problem on G_- with boundary data

$$\partial_\nu \hat{c}_{\text{in}}^\Sigma = \hat{g}_-^\Sigma(0) \quad \text{on } \Gamma_-,$$

if $p > 3$. Observe that then $\hat{c}_{\text{in}}^\Sigma$ is even with respect to Γ_{in}. We solve the problem

$$\left\{\begin{aligned}
\partial_t w^{\text{in}} - \Delta w^{\text{in}} &= 0 && \text{in} \quad (0,T)\times G_-,\\
\partial_\nu w^{\text{in}} &= \hat{g}_-^\Sigma && \text{on} \quad (0,T)\times \Gamma_-,\\
w^{\text{in}}(0) &= \hat{c}_{\text{in}}^\Sigma && \text{in} \quad G_-,
\end{aligned}\right. \tag{10.10}$$

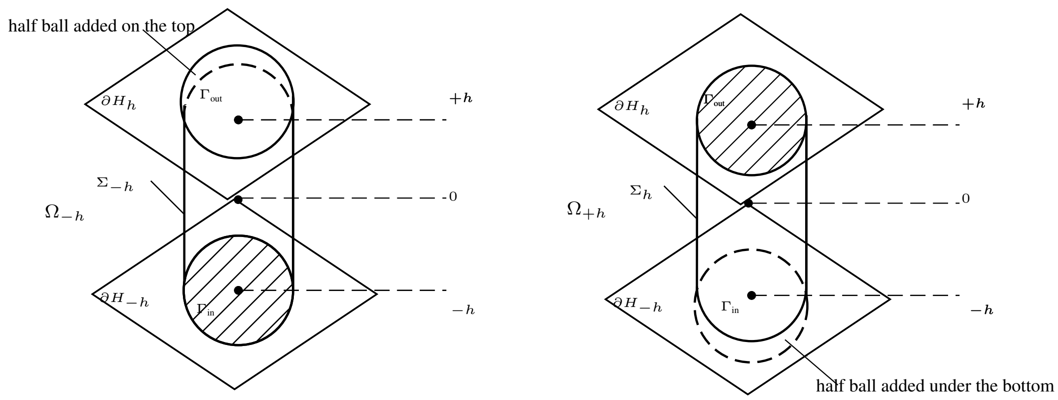

Figure 10.2: Left hand side: half 'pill' Ω_{-h}, right hand side: half 'pill' Ω_{+h}

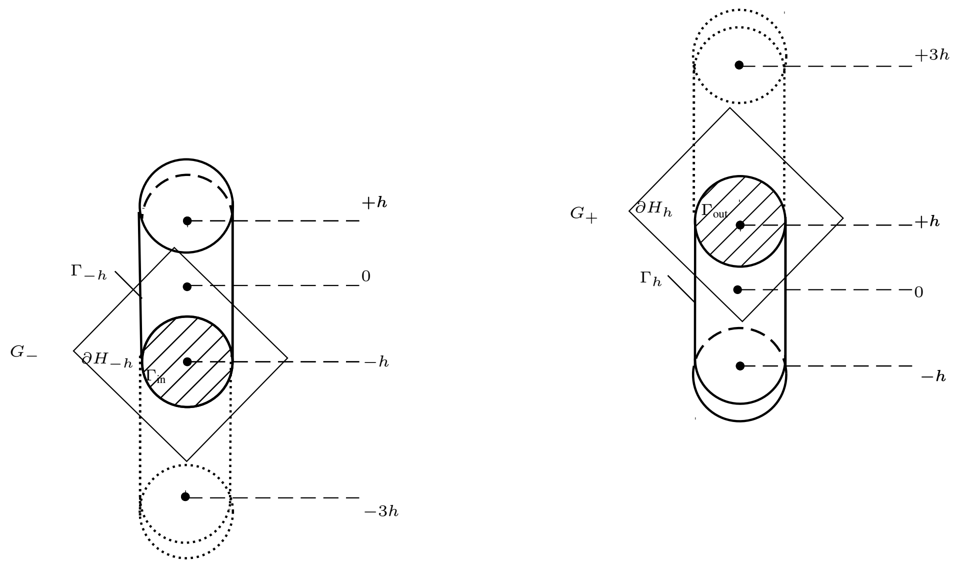

Figure 10.3: Left hand side: 'pill' G_-, right hand side: 'pill' G_+

by [DHP07, Theorem 2.1] to obtain

$$w^{\text{in}} \in \mathbb{E}_p^{G_-}(T).$$

Defining

$$\tilde{w}^{\text{in}} := w^{\text{in}}|_{\Omega_{-h}} \in \mathbb{E}_p^{\Omega_{-h}}(T),$$

we have $\partial_\nu \tilde{w}^{\text{in}} = \tilde{g}_-^\Sigma$ on Σ_{-h}. Since $\hat{g}_-^\Sigma$ and $\hat{c}_{\text{in}}^\Sigma$ are even in axial direction we have $\partial_\nu \tilde{w}^{\text{in}} = 0$ on Γ_{in}. In order to see $\partial_\nu \tilde{w}^{\text{in}} = 0$ let us set

$$\bar{w}^{\text{in}}(x', x_3) = \begin{cases} w^{\text{in}}(x', x_3) & : x_3 \geq -h, \\ w^{\text{in}}(x', -2h - x_3) & : x_3 < -h, \end{cases} \qquad (x' \in B_R(0) \subset \mathbb{R}^2)$$

such that $\bar{w}^{\text{in}}$ is even with respect to $x_3 = -h$. One easily verifies that $\bar{w}^{\text{in}}$ is a solution of (10.10). Since (10.10) is uniquely solvable we infer $\bar{w}^{\text{in}} = w^{\text{in}}$. Moreover, Sobolev embedding theorem applied in axial direction yields

$$
\begin{aligned}
\partial_{x_3} w^{\text{in}} &\in W^{1,p}((-2h,h)\times B_R(0)) \\
&= W^{1,p}((-2h,h), L^p(B_R(0))) \cap L^p((-2h,h), W^{1,p}(B_R(0))) \\
&\hookrightarrow C^0((-2h,h), L^p(B_R(0)))
\end{aligned}
$$

for all $p > 1$. Therefore we may employ the continuity of $\partial_{x_3} w^{\text{in}}$ in the x_3-direction. Indeed, taking the limit from above for $x_3 > -h$ we have

$$
\lim_{\varepsilon \downarrow 0} \partial_{x_3} w^{\text{in}}(\,\cdot\,, -h+\varepsilon) = \partial_{x_3} w^{\text{in}}(\,\cdot\,, -h),
$$

while taking the limit from below for $x_3 < -h$ we have

$$
\begin{aligned}
\lim_{\varepsilon \downarrow 0} \partial_{x_3} w^{\text{in}}(\,\cdot\,, -h-\varepsilon) &= \lim_{\varepsilon \downarrow 0} \partial_{x_3} \bar{w}^{\text{in}}(\,\cdot\,, -h-\varepsilon) \\
&= \lim_{\varepsilon \downarrow 0} \left(\partial_{x_3} w^{\text{in}}(\,\cdot\,, -2h - x_3)\right)|_{x_3=-h-\varepsilon} \\
&= -\lim_{\varepsilon \downarrow 0} \partial_{x_3} w^{\text{in}}(\,\cdot\,, -h+\varepsilon) \\
&= -\partial_{x_3} w^{\text{in}}(\,\cdot\,, -h).
\end{aligned}
$$

Therefore $\partial_{x_3} w^{\text{in}}(\cdot, -h) = 0$ and hence $\partial_\nu \tilde{w}^{\text{in}} = 0$ on Γ_{in} holds true.

Analogously we proceed with Γ_{out}. Here we extend $(1-\zeta)(g^\Sigma - \partial_\nu v)$ to obtain a

$$
\tilde{w}^{\text{out}} \in \mathbb{E}_p^{\Omega_{+h}}(T)
$$

with $\partial_\nu \tilde{w}^{\text{out}} = (1-\zeta)(g^\Sigma - \partial_\nu v)$ on Σ and $\partial_\nu \tilde{w}^{\text{out}} = 0$ on Γ_{out}. Let the cut-off functions $\tilde{\zeta}_1, \tilde{\zeta}_2 \in C^\infty(\mathbb{R}, [0,1])$ satisfy

$$
\tilde{\zeta}_1 = \begin{cases} 1 & : \ (-\infty, h/2) \\ 0 & : \ (2h/3, \infty) \end{cases}, \qquad \tilde{\zeta}_2 = \begin{cases} 0 & : \ (-\infty, -2h/3) \\ 1 & : \ (-h/2, \infty) \end{cases}.
$$

Then the combination

$$
w := \tilde{\zeta}_1 \tilde{w}^{\text{in}}|_\Omega + \tilde{\zeta}_2 \tilde{w}^{\text{out}}|_\Omega \in \mathbb{E}_p^\Omega(T)
$$

satisfies by construction $\partial_\nu w = 0$ on $(0,T) \times \Gamma_{\text{in}}$ and $\partial_\nu w = 0$ on $(0,T)\times\Gamma_{\text{out}}$. The remaining inhomogeneous boundary condition on Σ is satisfied either, since

$$
\begin{aligned}
\partial_\nu w &= \tilde{\zeta}_1 \partial_\nu \tilde{w}^{\text{in}} + \tilde{\zeta}_2 \partial_\nu \tilde{w}^{\text{out}} \\
&= \tilde{\zeta}_1 \zeta (g^\Sigma - \partial_\nu v) + \tilde{\zeta}_2 (1-\zeta)(g^\Sigma - \partial_\nu v) \\
&= \zeta(g^\Sigma - \partial_\nu v) + (1-\zeta)(g^\Sigma - \partial_\nu v) = g^\Sigma - \partial_\nu v \qquad \text{on } (0,T)\times\Sigma.
\end{aligned}
$$

Putting together Step 1 and Step 2 we define $c := v + w \in \mathbb{E}_p^\Omega(T)$ and obtain that c satisfies (10.6)-(10.8). Thus we have proved that there exists a bounded linear right-inverse to γ_1 which yields that the trace operator

$$
\gamma_1 : \mathbb{E}_p^\Omega(T) \to \mathbb{G}_p^{\text{in}}(T) \times \mathbb{G}_p^\Sigma(T) \times \mathbb{G}_p^{\text{out}}(T)
$$

in fact is a retraction. $\square$

Having Lemma 10.5 on the surjectivity of the Neumann trace operator at hand, we turn to the fully inhomogeneous Neumann system which is given through

$$\left\{\begin{array}{rcll} \partial_t c_i - d_i \Delta c_i &=& f_i & \text{in } (0,T)\times\Omega, \\ \partial_t c_i^\Sigma - d_i^\Sigma \Delta_\Sigma c_i^\Sigma &=& f_i^\Sigma & \text{on } (0,T)\times\Sigma, \\ -d_i\partial_\nu c_i &=& g_i^{\text{in}} & \text{on } (0,T)\times\Gamma_{\text{in}}, \\ -d_i\partial_\nu c_i &=& g_i^{\Sigma} & \text{on } (0,T)\times\Sigma, \\ -d_i\partial_\nu c_i &=& g_i^{\text{out}} & \text{on } (0,T)\times\Gamma_{\text{out}}, \\ -d_i^\Sigma\partial_{\nu_\Sigma} c_i^\Sigma &=& 0 & \text{on } (0,T)\times\partial\Sigma, \\ c_i(0) &=& c_{0,i} & \text{in } \Omega, \\ c_i^\Sigma(0) &=& c_{0,i}^\Sigma & \text{in } \Sigma. \end{array}\right. \qquad (i=1,...,N) \qquad (10.11)$$

The result of this section is given by

Lemma 10.6 *Let $1<p<\infty$ with $p\neq 3$ and let $T>0$ be given. Then (10.11) admits a unique solution*

$$(c_i, c_i^\Sigma) \in \mathbb{E}_p^\Omega(T)\times\mathbb{E}_p^\Sigma(T)$$

if and only if the data satisfies the regularity conditions

$$(f_i, f_i^\Sigma, g_i^{in}, g_i^\Sigma, g_i^{out}, 0, c_{0,i}, c_{0,i}^\Sigma) \in \mathbb{F}_{p,I}^{\Omega,\Sigma}(T)$$

and in case of $p>3$ the compatibility conditions

$$\begin{array}{rcll} -d_i\partial_\nu c_{0,i} &=& g_i^{in}|_{t=0} & \text{on } \Gamma_{in}, \\ -d_i\partial_\nu c_{0,i} &=& g_i^{\Sigma}|_{t=0} & \text{on } \Sigma, \\ -d_i\partial_\nu c_{0,i} &=& g_i^{out}|_{t=0} & \text{on } \Gamma_{out}, \\ -d_i\partial_\nu c_{0,i}^\Sigma &=& 0 & \text{on } \partial\Sigma. \end{array}$$

Additionally, the corresponding solution operator ${}_0\mathcal{S}_T$ with respect to homogeneous initial values satisfies

$$\|{}_0\mathcal{S}_\tau\|_{\mathscr{L}({}_0\mathbb{F}_p^{\Omega,\Sigma}(\tau)^N, {}_0\mathbb{E}_p^\Omega(\tau)^N\times{}_0\mathbb{E}_p^\Sigma(\tau)^N)} \le M \qquad (0<\tau<T) \qquad (10.12)$$

for a constant $M>0$ independent of τ.

Proof. By Lemma 10.5 for given $g_i^{\text{in}}\in\mathbb{G}_p^{\text{in}}(T)$, $g_i^\Sigma\in\mathbb{G}_p^\Sigma(T)$, $g_i^{\text{out}}\in\mathbb{G}_p^{\text{out}}(T)$ there exists a $c_i^1\in\mathbb{E}_p^\Omega(T)$ with

$$\begin{array}{ll} -d_i\partial_\nu c_i^1 = g_i^{\text{in}} & \text{on } (0,T)\times\Gamma_{\text{in}}, \\ -d_i\partial_\nu c_i^1 = g_i^{\Sigma} & \text{on } (0,T)\times\Sigma, \\ -d_i\partial_\nu c_i^1 = g_i^{\text{out}} & \text{on } (0,T)\times\Gamma_{\text{out}}. \end{array}$$

Secondly, due to Lemma 10.4 for $f_i \in \mathbb{F}_p^\Omega(T)$, $c_{0,i} \in \mathbb{I}_p^\Omega(T)$ we find a unique $c_i^2 \in \mathbb{E}_p^\Omega(T)$ such that

$$\begin{cases} \partial_t c_i^2 - d_i \Delta c_i^2 &= f_i - (\partial_t - d_i\Delta) c_i^1 \quad \text{in} \quad (0,T) \times \Omega, \\ -d_i \partial_\nu c_i^2 &= 0 \quad \text{on} \quad (0,T) \times \Gamma_{\text{in}}, \\ -d_i \partial_\nu c_i^2 &= 0 \quad \text{on} \quad (0,T) \times \Sigma, \\ -d_i \partial_\nu c_i^2 &= 0 \quad \text{on} \quad (0,T) \times \Gamma_{\text{out}}, \\ c_i^2(0) &= c_{0,i} - c_i^1(0) \quad \text{in} \quad \Omega. \end{cases}$$

By construction $c_i := c_i^1 + c_i^2 \in \mathbb{E}_p^\Omega(T)$ satisfies (10.11). We employ the extension operator in zero time trace spaces from [PSS07, Proposition 6.1] to get

$$\|(f_i, f_i^\Sigma, g_i^{\text{in}}, g_i^\Sigma, g_i^{\text{out}}, 0)\|_{{}_0\mathbb{F}_p^{\Omega,\Sigma}(T)} \leq K \|(f_i, f_i^\Sigma, g_i^{\text{in}}, g_i^\Sigma, g_i^{\text{out}}, 0)\|_{{}_0\mathbb{F}_p^{\Omega,\Sigma}(\tau)},$$

where $K = K(T) > 0$ denotes its norm and is independent of $\tau < T$ due to the zero time trace. Hence we infer

$$\begin{aligned} &\|{}_0\mathcal{S}_{\tau,i}(f_i, f_i^\Sigma, g_i^{\text{in}}, g_i^\Sigma, g_i^{\text{out}}, 0)\|_{{}_0\mathbb{E}_p^\Omega(\tau)\times{}_0\mathbb{E}_p^\Sigma(\tau)} \\ &\qquad \leq \|{}_0\mathcal{S}_{T,i}(f_i, f_i^\Sigma, g_i^{\text{in}}, g_i^\Sigma, g_i^{\text{out}}, 0)\|_{{}_0\mathbb{E}_p^\Omega(T)\times{}_0\mathbb{E}_p^\Sigma(T)} \\ &\qquad \leq M' \|(f_i, f_i^\Sigma, g_i^{\text{in}}, g_i^\Sigma, g_i^{\text{out}}, 0)\|_{{}_0\mathbb{F}_p^{\Omega,\Sigma}(T)} \\ &\qquad \leq M' K \|(f_i, f_i^\Sigma, g_i^{\text{in}}, g_i^\Sigma, g_i^{\text{out}}, 0)\|_{{}_0\mathbb{F}_p^{\Omega,\Sigma}(\tau)}, \end{aligned}$$

which yields the estimate (10.12) of the solution operator ${}_0\mathcal{S}_\tau$ with a constant $M := M'(T)K(T) > 0$ being independent of $\tau < T$. □

10.3. Perturbation

Assume that u satisfies (A^{vel}). We prove Proposition 10.1 by perturbation. To this end, we have to show that the results obtained in Lemma 10.6 carry over when adding the two perturbation terms $(u \cdot \nabla)c_i$ and $(u \cdot \nu)c_i$. Here we employ $\mathbb{U}_p^\Omega(T)$ as given in (8.2) and the Dirichlet trace space

$$\mathbb{U}_p^{\text{in}}(T) := W_p^{1-1/2p}((0,T), L^p(\Gamma_{\text{in}}, \mathbb{R}^3)) \cap L^p((0,T), W_p^{2-1/p}(\Gamma_{\text{in}}, \mathbb{R}^3)).$$

In the following we restrict to $5/3 < p < \infty$ with $p \neq 3$ to apply pointwise multiplication results.

Proof of Proposition 10.1. The proof is carried out in three steps.

Step 1. We estimate both perturbation terms occuring in (10.1). Let $\varepsilon \in (0,1)$ be sufficiently small. Then the following algebra properties hold for $5/3 < p < \infty$, cf. [KS]:

$$\begin{aligned} &\mathbb{U}_p^\Omega(T) \cdot W_p^{1/2-\varepsilon/2}((0,T), L^p(\Omega, \mathbb{R}^3)) \cap L^p((0,T), W_p^{1-\varepsilon}(\Omega, \mathbb{R}^3)) \hookrightarrow \mathbb{F}_p^\Omega(T), \\ &\mathbb{U}_p^{\text{in}}(T) \cdot W_p^{1-1/2p-\varepsilon/2}((0,T), L^p(\Gamma_{\text{in}})) \cap L^p((0,T), W_p^{2-1/p-\varepsilon}(\Gamma_{\text{in}})) \hookrightarrow \mathbb{G}_p^{\text{in}}(T). \end{aligned}$$

The first embedding follows by a direct calculation and the second by taking trace of the following embedding

$$\begin{aligned}\mathbb{U}_p^\Omega(T)\cdot W_p^{1-\varepsilon/2}((0,T),L^p(\Omega,\mathbb{R}^3))\cap L^p((0,T),W_p^{2-\varepsilon}(\Omega,\mathbb{R}^3))\\ \hookrightarrow W_p^{1/2}((0,T),L^p(\Omega))\cap L^p((0,T),W^{1,p}(\Omega)),\end{aligned}$$

which follows by a straight forward calculation, too. For $0<\tau<T$ we infer the following estimates:

$$\begin{aligned}\|(u\cdot\nabla)c_i\|_{{}_0\mathbb{F}_p^\Omega(\tau)} &\le C\|u\|_{\mathbb{U}_p^\Omega(T)}\|\nabla c_i\|_{{}_0W_p^{1/2-\varepsilon/2}((0,\tau),L^p(\Omega,\mathbb{R}^3))\cap L^p((0,\tau),W_p^{1-\varepsilon}(\Omega,\mathbb{R}^3))}\\ &\le C\|u\|_{\mathbb{U}_p^\Omega(T)}\|c_i\|_{{}_0W_p^{1-\varepsilon/2}((0,\tau),L^p(\Omega))\cap L^p((0,\tau),W_p^{2-\varepsilon}(\Omega))}\\ &\le C\tau^\eta\|c_i\|_{{}_0\mathbb{E}_p^\Omega(\tau)} \qquad (c_i\in{}_0\mathbb{E}_p^\Omega(\tau))\end{aligned}$$

with a constant $C>0$ and an exponent $\eta>0$ both being independent of τ thanks to Lemma 1.5. Analogously

$$\begin{aligned}\|(u\cdot\nu)c_i\|_{{}_0\mathbb{G}_p^{\text{in}}(\tau)} &\le C\|u\|_{\mathbb{U}_p^{\text{in}}(T)}\|c_i\|_{{}_0W_p^{1-1/2p-\varepsilon/2}((0,\tau),L^p(\Gamma_{\text{in}}))\cap L^p((0,\tau),W_p^{2-1/p-\varepsilon}(\Gamma_{\text{in}}))}\\ &\le C\|u\|_{\mathbb{U}_p^\Omega(T)}\|c_i\|_{{}_0W_p^{1-\varepsilon/2}((0,\tau),L^p(\Omega))\cap L^p((0,\tau),W_p^{2-\varepsilon}(\Omega))}\\ &\le C\tau^\eta\|c_i\|_{{}_0\mathbb{E}_p^\Omega(\tau)} \qquad (c_i\in{}_0\mathbb{E}_p^\Omega(\tau))\end{aligned}$$

with a constant $C>0$ and an exponent $\eta>0$ both being independent of τ thanks to Lemma 1.5. It follows that the linear operator

$$\begin{aligned}\mathcal{B}:{}_0\mathbb{E}_p^\Omega(\tau)^N\times{}_0\mathbb{E}_p^\Sigma(\tau)^N &\to {}_0\mathbb{F}_p^{\Omega,\Sigma}(\tau)^N\\ \mathcal{B}(c,c^\Sigma) &= ((u\cdot\nabla)c_i,\,0,\,(u\cdot\nu)c_i,\,0,\,0,\,0)_{i=1,\dots,N}\end{aligned}$$

may be estimated by

$$\|\mathcal{B}(c,c^\Sigma)\|_{{}_0\mathbb{F}_p^{\Omega,\Sigma}(\tau)^N}\le C\tau^\eta\|(c,c^\Sigma)\|_{{}_0\mathbb{E}_p^\Omega(\tau)^N\times{}_0\mathbb{E}_p^\Sigma(\tau)^N},\quad ((c,c^\Sigma)\in{}_0\mathbb{E}_p^\Omega(\tau)^N\times{}_0\mathbb{E}_p^\Sigma(\tau)^N) \tag{10.13}$$

with a constant $C>0$ and an exponent $\eta>0$ both being independent of $\tau<T$.

Step 2. We give the construction of the solution of (10.1) as a sum $c_i=\hat c_i+\bar c_i$, $c_i^\Sigma=\hat c_i^\Sigma+\bar c_i^\Sigma$. Let $(\hat c_i,\hat c_i^\Sigma)\in\mathbb{E}_p^\Omega(\tau)\times\mathbb{E}_p^\Sigma(\tau)$ be the solution to (10.11) with g_i^{in} replaced by some $\hat g_i^{\text{in}}$ satisfying the compatibility condition $-d_i\partial_\nu\hat c_{0,i}=\hat g_i^{\text{in}}(0)$ in case $p>3$ and which exists according to Lemma 10.6. Next, we set

$$\bar f_i=-(u\cdot\nabla)\hat c_i,\qquad \bar g_i^{\text{in}}=g_i^{\text{in}}-\hat g_i^{\text{in}}-(u\cdot\nu)\hat c_i.$$

Note that then for $p>3$ the compatibility condition

$$\bar g_i^{\text{in}}|_{t=0}=g_i^{\text{in}}|_{t=0}-\hat g_i^{\text{in}}|_{t=0}-(u|_{t=0}\cdot\nu)c_{0,i}=0$$

is satisfied by construction. Thus, the task is reduced to prove that for $0 < \tau \le T$ there exists a unique solution $(\bar{c}_i, \bar{c}_i^\Sigma) \in {}_0\mathbb{E}_p^\Omega(\tau) \times {}_0\mathbb{E}_p^\Sigma(\tau)$ of

$$\left\{\begin{aligned} \partial_t\bar{c}_i + (u\cdot\nabla)\bar{c}_i - d_i\Delta\bar{c}_i &= \bar{f}_i && \text{in} && (0,\tau)\times\Omega,\\ \partial_t\bar{c}_i^\Sigma - d_i^\Sigma\Delta_\Sigma\bar{c}_i^\Sigma &= 0 && \text{on} && (0,\tau)\times\Sigma,\\ (u\cdot\nu)\bar{c}_i - d_i\partial_\nu\bar{c}_i &= \bar{g}_i^{\text{in}} && \text{on} && (0,\tau)\times\Gamma_{\text{in}},\\ -d_i\partial_\nu\bar{c}_i &= 0 && \text{on} && (0,\tau)\times\Sigma,\\ -d_i\partial_\nu\bar{c}_i &= 0 && \text{on} && (0,\tau)\times\Gamma_{\text{out}},\\ -d_i^\Sigma\partial_{\nu_\Sigma}\bar{c}_i^\Sigma &= 0 && \text{on} && (0,\tau)\times\partial\Sigma,\\ \bar{c}_i(0) &= 0 && \text{in} && \Omega,\\ \bar{c}_i^\Sigma(0) &= 0 && \text{in} && \Sigma. \end{aligned}\right. \tag{10.14}$$

This will be done in the final step.

Step 3. We show the unique solvability of (10.14) on some interval $(0,\tau)$. The proof will show that τ is independent of the data $f_i, f_i^\Sigma, g_i^{\text{in}}, g_i^\Sigma, g_i^{\text{out}}, c_{0,i}, c_{0,i}^\Sigma$. Due to the linearity of the system solvability then carries over to the whole time interval $(0,T)$.

We apply a Neumann series argument to (10.14). To this end let us reformulate (10.14) by means of the operators ${}_0\mathcal{L}_\tau$ induced by the left-hand side of (10.14) and $\mathcal{B}$. Let

$$\bar{F}_i = (\bar{f}_i, 0, \bar{g}_i^{\text{in}}, 0, 0, 0) \in {}_0\mathbb{F}_p^{\Omega,\Sigma}(\tau)^N, \qquad (i = 1, ..., N)$$

such that (10.14) is equivalent to

$${}_0\mathcal{L}_\tau(\bar{c}, \bar{c}^\Sigma) + \mathcal{B}(\bar{c}, \bar{c}^\Sigma) = \bar{F} \qquad ((\bar{c}, \bar{c}^\Sigma) \in {}_0\mathbb{E}_p^\Omega(\tau)^N \times {}_0\mathbb{E}_p^\Sigma(\tau)^N).$$

Due to

$${}_0\mathcal{L}_\tau + \mathcal{B} = (I + \mathcal{B}\,{}_0\mathcal{S}_\tau)\,{}_0\mathcal{L}_\tau$$

with the solution operator ${}_0\mathcal{S}_\tau = {}_0\mathcal{L}_\tau^{-1}$ from Lemma 10.6, the invertibility of $(I + \mathcal{B}\,{}_0\mathcal{S}_\tau)$ from ${}_0\mathbb{F}_p^{\Omega,\Sigma}(\tau)^N$ to ${}_0\mathbb{E}_p^\Omega(\tau)^N \times {}_0\mathbb{E}_p^\Sigma(\tau)^N$ and, in turn, of ${}_0\mathcal{L}_\tau + \mathcal{B}$ from ${}_0\mathbb{E}_p^\Omega(\tau)^N \times {}_0\mathbb{E}_p^\Sigma(\tau)^N$ to ${}_0\mathbb{F}_p^{\Omega,\Sigma}(\tau)^N$ readily follows from (10.13) if we choose τ so small that $C\tau^\eta M < 1$ with M from (10.12). Note that this is possible since M is independent of $\tau < T$. $\square$

Chapter 11

Local Well-Posedness

In this chapter we derive unique solvability of the nonlinear catalyst equations. After stating the precise assumptions on the nonlinearities, i.e. r_i^{sorp} and r_i^{ch} we show nonnegativity of concentrations and surface concentrations for data admiting the 'right' sign in Section 11.1. Finally, in Section 11.2 we prove the local-in-time well-posedness result given through Theorem 8.1. We consider

$$\left\{\begin{array}{rcll} \partial_t c_i + (u\cdot\nabla)c_i - d_i\Delta c_i &=& f_i & \text{in} \quad (0,T)\times\Omega, \\ \partial_t c_i^\Sigma - d_i^\Sigma \Delta_\Sigma c_i^\Sigma &=& r_i^{\text{sorp}}(c_i, c_i^\Sigma) + r_i^{\text{ch}}(c^\Sigma) & \text{on} \quad (0,T)\times\Sigma, \\ (u\cdot\nu)c_i - d_i\partial_\nu c_i &=& g_i^{\text{in}} & \text{on} \quad (0,T)\times\Gamma_{\text{in}}, \\ -d_i\partial_\nu c_i &=& r_i^{\text{sorp}}(c_i, c_i^\Sigma) & \text{on} \quad (0,T)\times\Sigma, \\ -d_i\partial_\nu c_i &=& 0 & \text{on} \quad (0,T)\times\Gamma_{\text{out}}, \\ -d_i^\Sigma \partial_{\nu_\Sigma} c_i^\Sigma &=& 0 & \text{on} \quad (0,T)\times\partial\Sigma, \\ c_i(0) &=& c_{0,i} & \text{in} \quad \Omega, \\ c_i^\Sigma(0) &=& c_{0,i}^\Sigma & \text{on} \quad \Sigma, \end{array}\right. \tag{11.1}$$

in the strong L^p-sense, locally in time. Recall when the index i is used we mean $i = 1, ..., N$, and have e.g. $c = (c_i)_{i=1,...,N}$ and $c^\Sigma = (c_i^\Sigma)_{i=1,...,N}$.

Let us state the assumptions on the sorption and reaction terms. They are required to satisfy the following assumptions:

($\mathbf{A_F^{sorp}}$) The sorption rate acts as a function satisfying

$$r_i^{\text{sorp}} = r_i^{\text{sorp}}(c_i, c_i^\Sigma), \quad r_i^{\text{sorp}} \in C^2(\mathbb{R}^2), \quad \nabla r_i^{\text{sorp}} \in BC^1(\mathbb{R}^2, \mathbb{R}^2).$$

($\mathbf{A_M^{sorp}}$) The sorption rate increases monotonically in c_i and decreases monotonically in c_i^Σ.

($\mathbf{A_B^{sorp}}$) The sorption rate admits linear bounds

$$-k_i^{\text{de}} c_i^\Sigma \leq r_i^{\text{sorp}}(c_i, c_i^\Sigma) \leq k_i^{\text{ad}} c_i \qquad (c_i, c_i^\Sigma \geq 0)$$

for given adsorption and desorption constants $k_i^{\text{ad}}, k_i^{\text{de}} > 0$.

($\mathbf{A_F^{ch}}$) We assume that the chemical reactions fulfill

$$r_i^{\text{ch}} = r_i^{\text{ch}}(c^\Sigma), \quad r^{\text{ch}} \in C^1([0,\infty)^N, \mathbb{R}^N).$$

($\mathbf{A_N^{ch}}$) The reaction is supposed to be quasi-positive, i.e.

$$r_i^{\mathrm{ch}}(y) \geq 0, \qquad (y \in [0,\infty)^N,\ y_i = 0).$$

($\mathbf{A_P^{ch}}$) The reaction admits polynomial growth, i.e. there exist a constant $M > 0$ and an exponent $\gamma \in [1,\infty)$ if $p \in [2,\infty)$ and $\gamma = \frac{1}{1-p/2} \in [1,6]$ if $p \in (5/3,2)$, such that

$$|r^{\mathrm{ch}}(y)| \leq M\,(1+|y|^{\gamma}) \qquad (y \in [0,\infty)^N).$$

Additionally, suppose the Jacobian fulfills

$$|(r^{\mathrm{ch}})'(y)| \leq M\left(1+|y|^{\gamma-1}\right) \qquad (y \in [0,\infty)^N).$$

Remark 11.1 *Let us comment on the polynomial growth conditions of r^{ch} and $r^{ch\prime}$.*

a) *In (A_P^{ch}) we restrict to $\gamma \in [1,6]$ if $p \in (5/3,2)$. This is due to the embedding*

$$_0\mathbb{E}_p^{\Sigma}(T) \hookrightarrow L^{p\gamma}(\Sigma_T)$$

for $p > 5/3$, cf. Proposition 1.3, which we employ in the proof of the local existence result. In case $p \geq 2$ only an arbitrary polynomial growth is required.

b) *The growth rate γ of r^{ch} in (A_P^{ch}) yields that r^{ch} acts as a Nemytskij operator*

$$r^{ch} : L^{p\gamma}(\Sigma_T)^N \to L^p(\Sigma_T)^N,$$

cf. [AZ90, Theorem 3.1]. Analogously the growth rate $\gamma - 1$ of $(r^{ch})'$ in (A_P^{ch}) yields

$$(r^{ch})' : L^{p\gamma}(\Sigma_T)^N \to L^{p\gamma/(\gamma-1)}(\Sigma_T)^{N\times N}.$$

In particular $(r^{ch})'$ maps a ball $\bar{B}_\delta$ of radius $\delta > 0$ in $L^{p\gamma}(\Sigma_T)^N$ into a ball of radius
$k(\delta) > 0$ in $L^{p\gamma/(\gamma-1)}(\Sigma_T)^{N\times N}$. Hence a pointwise application of the mean value theorem to the real function r^{ch} gives us

$$r^{ch}(c^{\Sigma}(t,x)) - r^{ch}(z^{\Sigma}(t,x)) = \int_0^1 r^{ch\prime}\left(z^{\Sigma}(t,x) + \tau(c^{\Sigma}(t,x) - z^{\Sigma}(t,x))\right)\,d\tau \cdot (c^{\Sigma}(t,x) - z^{\Sigma}(t,x)), \quad ((t,x) \in \Sigma_T)$$

and Hölder's inequality with

$$\frac{1}{p} = \frac{1}{p\gamma/(\gamma-1)} + \frac{1}{p\gamma}$$

yields

$$\begin{aligned}
&\|r^{ch}(c^\Sigma) - r^{ch}(z^\Sigma)\|_{L^p(\Sigma_T)^N} \\
&\qquad \le \sup_{v^\Sigma \in \bar{B}_\delta} \|(r^{ch})'(v^\Sigma)\|_{L^{p\gamma/(\gamma-1)}(\Sigma_T)^{N\times N}} \|c^\Sigma - z^\Sigma\|_{L^{p\gamma}(\Sigma_T)^N} \\
&\qquad \le k(\delta)\|c^\Sigma - z^\Sigma\|_{L^{p\gamma}(\Sigma_T)^N} \qquad (c^\Sigma, z^\Sigma \in \bar{B}_\delta),
\end{aligned}$$

i.e. r^{ch} *acts as a locally Lipschitz continuous Nemytskij operator, cf. [AZ90, Theorem 3.10].*

For $y \in \mathbb{R}^N$ let us denote $y^+ = (y_i^+)_{i=1,\dots,N}$ where as before $y_i^+ = \max\{0, y_i\}$. Since we do not know a priori whether a corresponding solution (c, c^Σ) is non-negative, we extend r^{ch} as

$$r_{i,+}^{\mathrm{ch}} : \mathbb{R}^N \to \mathbb{R}, \qquad r_{i,+}^{\mathrm{ch}}(y) := r_i^{\mathrm{ch}}(y^+) \qquad (y \in \mathbb{R}^N).$$

Then (11.1) remains meaningful even if c or c^Σ take negative values.

The main result of this chapter is given by Theorem 8.1. The proof is based on maximal regularity of the linear system (10.1) proved in Chapter 10 and the contraction mapping principle. We start by proving the nonnegativity of c_i and of c_i^Σ.

11.1. Nonnegativity of Concentrations

For data admitting the right sign we have the following result.

Lemma 11.2 *(Nonnegativity) Let* $5/3 < p < \infty$ *with* $p \neq 3$. *Let* $T > 0$ *and let* $c_{0,i} \in \mathbb{I}_p^{\Omega,+}$, $c_{0,i}^\Sigma \in \mathbb{I}_p^{\Sigma,+}$ *and* $g_i^{in} \in \mathbb{G}_p^{in}(T)^-$ *be given. Suppose* u *satisfies (*A^{vel}*),* r^{sorp} *satisfies (*A_F^{sorp}*), (*A_M^{sorp}*), (*A_B^{sorp}*) and* r^{ch} *fulfills (*A_F^{ch}*), (*A_N^{ch}*), (*A_P^{ch}*). Moreover, suppose* $(c_i, c_i^\Sigma) \in \mathbb{E}_p^\Omega(T) \times \mathbb{E}_p^\Sigma(T)$ *is a strong* L^p*-solution of (11.1). Then*

$$c_i \ge 0 \quad \text{a.e. in } \Omega_T, \qquad c_i^\Sigma \ge 0 \quad \text{a.e. on } \Sigma_T$$

hold true.

Proof. Let $\phi_\varepsilon \in C^\infty(\mathbb{R})$ be a pointwise approximation of

$$\phi(r) = \begin{cases} -r & : r \le 0 \\ 0 & : r > 0 \end{cases},$$

as $\varepsilon \to 0$, which satisfies $\phi_\varepsilon \geq 0$, $\phi'_\varepsilon \leq 0$ and $\phi''_\varepsilon \geq 0$, e.g.

$$\phi_\varepsilon(r) := \begin{cases} -re^{\varepsilon/r} & : r \leq 0 \\ 0 & : r > 0 \end{cases}.$$

Then we have for $c_i^- = \max\{0, -c_i\}$ that

$$\phi_\varepsilon(c_i) \to c_i^-, \qquad c_i\phi'_\varepsilon(c_i) \to c_i \left\{ \begin{array}{ll} -1 & : c_i < 0 \\ 0 & : c_i \geq 0 \end{array} \right\} = c_i^- \tag{11.2}$$

as $\varepsilon \to 0$. We show

$$\lim_{\varepsilon \to 0} \left[\int_\Omega \phi_\varepsilon(c_i)dx + \int_\Sigma \phi_\varepsilon(c_i^\Sigma)d\sigma \right] \leq 0.$$

Applying ϕ_ε to c_i we obtain by partial integration

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \phi_\varepsilon(c_i)dx &= \int_\Omega \phi'_\varepsilon(c_i)\partial_t c_i dx = \int_\Omega \phi'_\varepsilon(c_i)\text{div}\,(d_i\nabla c_i - uc_i)dx \\
&= \int_{\partial\Omega} \phi'_\varepsilon(c_i)(d_i\partial_\nu c_i - (u\cdot\nu)c_i)d\sigma - \int_\Omega \phi''_\varepsilon(c_i)\nabla c_i \cdot (d_i\nabla c_i - uc_i)dx \\
&= -\int_{\Gamma_{\text{in}}} \phi'_\varepsilon(c_i)g_i^{\text{in}}d\sigma - \int_\Sigma \phi'_\varepsilon(c_i)r_i^{\text{sorp}}(c_i, c_i^\Sigma)d\sigma - \int_{\Gamma_{\text{out}}} \phi'_\varepsilon(c_i)(u\cdot\nu)c_i d\sigma \\
&\quad - d_i\int_\Omega \phi''_\varepsilon(c_i)|\nabla c_i|^2dx + \int_\Omega \phi''_\varepsilon(c_i)\nabla c_i uc_i dx
\end{aligned} \tag{11.3}$$

due to the boundary conditions. In the same way we have

$$\begin{aligned}
\frac{d}{dt}\int_\Sigma \phi_\varepsilon(c_i^\Sigma)d\sigma &= \int_\Sigma \phi'_\varepsilon(c_i^\Sigma)\partial_t c_i^\Sigma d\sigma = \int_\Sigma \phi'_\varepsilon(c_i^\Sigma)(d_i^\Sigma\Delta_\Sigma c_i^\Sigma + r_i^{\text{sorp}}(c_i, c_i^\Sigma) + r_{i,+}^{\text{ch}}(c^\Sigma))d\sigma \\
&= -d_i^\Sigma\int_\Sigma \phi''_\varepsilon(c_i^\Sigma)|\nabla_\Sigma c_i^\Sigma|^2d\sigma + \int_\Sigma \phi'_\varepsilon(c_i^\Sigma)(r_i^{\text{sorp}}(c_i, c_i^\Sigma) + r_{i,+}^{\text{ch}}(c^\Sigma))d\sigma.
\end{aligned} \tag{11.4}$$

Let us go through all the integrals appearing on the right-hand side of (11.3) and (11.4). The first and the fourth integrals on the right-hand side of (11.3) and the first integral on the right-hand side of (11.4) are negative or zero such that we may drop them. The remaining four integrals are treated as follows: We combine the sorption boundary integrals to

$$\int_\Sigma (\phi'_\varepsilon(c_i^\Sigma) - \phi'_\varepsilon(c_i))r_i^{\text{sorp}}(c_i, c_i^\Sigma)d\sigma$$

and split up this integral into three integrals on

$$\{\text{sign}(c_i) = \text{sign}(c_i^\Sigma)\}, \qquad \{c_i \le 0 \le c_i^\Sigma\}, \qquad \{c_i^\Sigma \le 0 \le c_i\}.$$

When c_i and c_i^Σ have the same sign this integral tends to 0 as $\varepsilon \to 0$. Whereas on $\{c_i \le 0 \le c_i^\Sigma\}$ we have

$$\begin{aligned}\int_\Sigma & \mathbb{1}_{\{c_i \le 0 \le c_i^\Sigma\}} (\phi'_\varepsilon(c_i^\Sigma) - \phi'_\varepsilon(c_i)) r_i^{\text{sorp}}(c_i, c_i^\Sigma) d\sigma \\ &\to \int_\Sigma r_i^{\text{sorp}}(c_i, c_i^\Sigma) d\sigma \le \int_\Sigma r_i^{\text{sorp}}(0, c_i^\Sigma) d\sigma \le 0\end{aligned}$$

as $\varepsilon \to 0$ by monotonicity of r_i^{sorp} and $r_i^{\text{sorp}}(0, c_i^\Sigma) \le 0$. In the same way on $\{c_i^\Sigma \le 0 \le c_i\}$ we obtain

$$\begin{aligned}\int_\Sigma & \mathbb{1}_{\{c_i^\Sigma \le 0 \le c_i\}} (\phi'_\varepsilon(c_i^\Sigma) - \phi'_\varepsilon(c_i)) r_i^{\text{sorp}}(c_i, c_i^\Sigma) d\sigma \\ &\to -\int_\Sigma r_i^{\text{sorp}}(c_i, c_i^\Sigma) d\sigma \le -\int_\Sigma r_i^{\text{sorp}}(c_i, 0) d\sigma \le 0\end{aligned}$$

as $\varepsilon \to 0$ by monotonicity of r_i^{sorp} and $r_i^{\text{sorp}}(c_i, 0) \ge 0$. Because of $u \cdot \nu \ge 0$ on Γ_{out} we see that

$$-\int_{\Gamma_{\text{out}}} \phi'_\varepsilon(c_i)(u \cdot \nu) c_i d\sigma \to -\int_{\Gamma_{\text{out}}} c_i^-(u \cdot \nu) d\sigma \le 0.$$

We treat the boundary integral with the chemical reaction term by the quasi-positivity of r^{ch} as follows. We show

$$\int_\Sigma \phi'_\varepsilon(c_i^\Sigma) r_{i,+}^{\text{ch}}(c^\Sigma) d\sigma \le 0$$

through

$$\begin{aligned}\int_\Sigma \phi'_\varepsilon(c_i^\Sigma) r_{i,+}^{\text{ch}}(c^\Sigma) d\sigma &= \int_\Sigma \phi'_\varepsilon(c_i^\Sigma) \mathbb{1}_{\{c_i^\Sigma > 0\}} r_{i,+}^{\text{ch}}(c^\Sigma) d\sigma \\ &+ \int_\Sigma \phi'_\varepsilon(c_i^\Sigma) \mathbb{1}_{\{c_i^\Sigma = 0\}} r_{i,+}^{\text{ch}}(c^\Sigma) d\sigma + \int_\Sigma \phi'_\varepsilon(c_i^\Sigma) \mathbb{1}_{\{c_i^\Sigma < 0\}} r_{i,+}^{\text{ch}}(c^\Sigma) d\sigma.\end{aligned}$$

The first integral vanishes by the properties of ϕ'_ϵ, the second one is less or equal zero by quasi-positivity and $\phi'_\varepsilon(0) \le 0$ as $\varepsilon \to 0$ and the third one is less or equal zero by definition of the extension of r^{ch} to $\mathbb{R}^N$, i.e. because of $c_i^\Sigma < 0$ implies $c_i^{\Sigma+} = 0$ and $r_i^{\text{ch}}(c^{\Sigma+}) \ge 0$ by quasi-positivity. We turn to the remaining integral $\int_\Omega \phi''_\varepsilon(c_i) \nabla c_i u c_i dx$. For its treatment we make use of $\nabla(\phi'_\varepsilon(c_i)) = \phi''_\varepsilon(c_i) \nabla c_i$ and in the same manner of $\nabla(\phi_\varepsilon(c_i)) = \phi'_\varepsilon(c_i)\nabla c_i$ and integrate by parts twice, such that

$$\begin{aligned}\int_\Omega \phi''_\varepsilon(c_i)\nabla c_i \cdot u c_i dx &= \int_\Omega \nabla(\phi'_\varepsilon(c_i)) \cdot u c_i dx \\ &= \int_{\partial\Omega} \phi'_\varepsilon(c_i)(u\cdot\nu)c_i d\sigma - \int_\Omega \phi'_\varepsilon(c_i) \underbrace{\operatorname{div}(uc_i)}_{=u\cdot\nabla c_i} dx \\ &= \int_{\partial\Omega} \phi'_\varepsilon(c_i)(u\cdot\nu)c_i d\sigma - \int_\Omega \nabla(\phi_\varepsilon(c_i))u dx \\ &= \int_{\partial\Omega} \phi'_\varepsilon(c_i)(u\cdot\nu)c_i d\sigma - \int_{\partial\Omega} \phi_\varepsilon(c_i)(u\cdot\nu) d\sigma,\end{aligned}$$

where in the second and in the last step we made use of $\operatorname{div} u = 0$. Employing (11.2) we see

$$\int_\Omega \phi''_\varepsilon(c_i)\nabla c_i \cdot u c_i dx = \int_{\partial\Omega} (\phi'_\varepsilon(c_i)c_i - \phi_\varepsilon(c_i))(u\cdot\nu)d\sigma \to 0$$

as $\varepsilon \to 0$. Therefore summing up (11.3) and (11.4), integration in time over $[0, t]$ and taking the limit $\varepsilon \to 0$ yields

$$\begin{aligned}&\int_\Omega c_i^-(t)dx + \int_\Sigma c_i^{\Sigma-}(t)d\sigma = \int_\Omega \phi(c_i(t))dx + \int_\Sigma \phi(c_i^\Sigma(t))d\sigma \\ &\le \int_\Omega \phi(c_{0,i})dx + \int_\Sigma \phi(c_{0,i}^\Sigma)d\sigma = \int_\Omega (c_{0,i})^- dx + \int_\Sigma (c_{0,i}^\Sigma)^- d\sigma = 0\end{aligned}$$

which in turn gives us $c_i^- = 0$ a.e. in Ω_T, $c_i^{\Sigma-} = 0$ a.e. on Σ_T and therefore $c_i \ge 0$ a.e. in Ω_T, $c_i^\Sigma \ge 0$ a.e. on Σ_T. Note in passing that for $\varepsilon \to 0$ we make use of Lebesgue's theorem on dominated convergence. □

11.2. Existence of Solutions

Let $T' > 0$ be given and $T \le T'$. Assume a set of fixed data

$$(f_i, 0, g_i^{\text{in}}, 0, 0, 0, c_{0,i}, c_{0,i}^\Sigma) \in \mathbb{F}_{p,I}^{\Omega,\Sigma}(T')$$

is given. We denote by

$$\mathcal{L}_{T,i} : \mathbb{E}_p^\Omega(T) \times \mathbb{E}_p^\Sigma(T) \to \mathbb{F}_p^{\Omega,\Sigma}(T)$$

the isomorphism induced by maximal regularity of (10.1) (Proposition 10.1), that is, $\mathcal{L}_{T,i}$ is the full linear operator on the left hand side of (10.1) except for the time traces. The full nonlinear problem (8.1) then is reformulated as

$$\begin{aligned} \mathcal{L}_{T,i}(c_i, c_i^\Sigma) &= (f_i, 0, g_i^{\text{in}}, 0, 0, 0) + \mathcal{N}_{T,i}(c, c^\Sigma), \\ c_i(0) = c_{0,i}, \quad & c_i^\Sigma(0) = c_{i,0}^\Sigma, \quad i = 1, \dots, N, \end{aligned} \tag{11.5}$$

where $\mathcal{N}_{T,i}$ includes the nonlinear sorption and reaction terms, i.e.,

$$\mathcal{N}_{T,i}(c, c^\Sigma) := \big(0, r_i^{\text{sorp}}(c_i, c_i^\Sigma) + r_{i,+}^{\text{ch}}(c^\Sigma), 0, r_i^{\text{sorp}}(c_i, c_i^\Sigma), 0, 0\big).$$

In order keep the constants resulting from the estimates below independent of T, we employ a suitable zero time trace splitting as described in the following.

First we take care of the compatibility condition arising from the nonlinear boundary condition on Σ. Taking time trace results in $r_i^{\text{sorp}}(c_{0,i}, c_{0,i}^\Sigma) \in W_p^{1-3/p}(\Sigma)$, which will be extended to $\mathbb{G}_p^\Sigma(T)$ by setting

$$r_i^* := \mathcal{R}_\Sigma e^{t\Delta_{\Sigma(-\infty,\infty)}} \mathcal{E}_{\Sigma_{(-\infty,\infty)}} r_i^{\text{sorp}}(c_{0,i}, c_{0,i}^\Sigma).$$

Here $\mathcal{E}_{\Sigma_{(-\infty,\infty)}}$ denotes the extension operator from the lateral surface Σ to the infinite cylinder $\Sigma_{(-\infty,\infty)}$ and $\mathcal{R}_\Sigma$ the corresponding restriction operator (note that both act as bounded operators on the function classes considered here, cf. [AF03]). Since $e^{t\Delta_{\Sigma(-\infty,\infty)}}$ has the same regularizing properties as the Laplacian on the whole space $\mathbb{R}^n$, for which the desired regularity is well known [PSS07], we see that $r_i^* \in \mathbb{G}_p^\Sigma(T)$.

Now we define the reference solution $(c_i^*, c_i^{\Sigma *}) \in \mathbb{E}_p^\Omega(T') \times \mathbb{E}_p^\Sigma(T')$, existing according to Proposition 10.1, via

$$\mathcal{L}_{T,i}(c_i^*, c_i^{\Sigma *}) = (f_i, 0, g_i^{\text{in}}, r_i^*, 0, 0), \quad c_i(0) = c_{0,i} \quad \text{in } \Omega, \quad c_i^\Sigma(0) = c_{0,i}^\Sigma \quad \text{on } \Sigma. \tag{11.6}$$

Decomposing (c_i, c_i^Σ) as

$$c_i = \bar{c}_i + c_i^*, \quad c_i^\Sigma = \bar{c}_i^\Sigma + c_i^{\Sigma *}$$

and subtracting (11.6) from (11.5), we end up with the reduced and equivalent zero time trace problem

$${}_0\mathcal{L}_{T,i}(\bar{c}, \bar{c}^\Sigma) = {}_0\mathcal{N}_{T,i}(\bar{c}, \bar{c}^\Sigma) \qquad (i = 1, ..., N).$$

Here ${}_0\mathcal{L}_{T,i}$ denotes the restriction of $\mathcal{L}_{T,i}$ to ${}_0\mathbb{E}_p^\Omega(T) \times {}_0\mathbb{E}_p^\Sigma(T)$ and

$${}_0\mathcal{N}_{T,i}(\bar{c}, \bar{c}^\Sigma) := \mathcal{N}_{T,i}(\bar{c}_i + c_i^*, \bar{c}_i^\Sigma + c_i^{\Sigma *}) - (0, 0, 0, r_i^*, 0, 0).$$

Next, we define ${}_0\Phi_T := ({}_0\Phi_{T,i})_{i=1,\dots,N}$ through

$$\begin{aligned} {}_0\Phi_{T,i} :\ & {}_0\mathbb{E}_p^\Omega(T)^N \times {}_0\mathbb{E}_p^\Sigma(T)^N \to {}_0\mathbb{E}_p^\Omega(T)^N \times {}_0\mathbb{E}_p^\Sigma(T)^N, \\ {}_0\Phi_{T,i}(\bar{c}, \bar{c}^\Sigma) &:= {}_0\mathcal{S}_{T,i}\, {}_0\mathcal{N}_{T,i}(\bar{c}, \bar{c}^\Sigma), \qquad (i = 1, ..., N), \end{aligned}$$

with the bounded linear inverse ${}_0\mathcal{S}_{T,i}$ of ${}_0\mathcal{L}_{T,i}$ given in Proposition 10.1.

Proof of Theorem 8.1. We apply the contraction mapping principle to ${}_0\Phi_T$, i.e. we show that there exists a $\delta > 0$, such that the mapping ${}_0\Phi_T$ constitutes a contraction on the closed ball $\bar{B}_\delta(0) \subset {}_0\mathbb{E}_p^\Omega(T)^N \times {}_0\mathbb{E}_p^\Sigma(T)^N$ and fulfills ${}_0\Phi_T : \bar{B}_\delta(0) \to \bar{B}_\delta(0)$.

(i) Contraction property: Let $(\bar{c}, \bar{c}^\Sigma), (\bar{z}, \bar{z}^\Sigma) \in \bar{B}_\delta(0)$. Then we have

$$\begin{aligned} \|{}_0\Phi_T(\bar{c}, \bar{c}^\Sigma) - {}_0\Phi_T(\bar{z}, \bar{z}^\Sigma)\|_{{}_0\mathbb{E}_p^\Omega(T)^N \times {}_0\mathbb{E}_p^\Sigma(T)^N} &\leq C\|{}_0\mathcal{N}_T(\bar{c}, \bar{c}^\Sigma) - {}_0\mathcal{N}_T(\bar{z}, \bar{z}^\Sigma)\|_{{}_0\mathbb{F}_p^{\Omega,\Sigma}(T)^N} \\ &= C\|r^{\text{sorp}}(\bar{c} + c^*, \bar{c}^\Sigma + c^{\Sigma*}) - r^{\text{sorp}}(\bar{z} + c^*, \bar{z}^\Sigma + c^{\Sigma*})\|_{(\mathbb{F}_p^\Sigma(T) \cap {}_0\mathbb{G}_p^\Sigma(T))^N} \\ &\quad + C\|r^{\text{ch}}((\bar{c}^\Sigma + c^{\Sigma*})^+) - r^{\text{ch}}((\bar{z}^\Sigma + c^{\Sigma*})^+)\|_{\mathbb{F}_p^\Sigma(T)^N} \end{aligned} \tag{11.7}$$

with

$$C := \sup\left\{\|{}_0\mathcal{S}_T\|_{\mathscr{L}({}_0\mathbb{F}_p^{\Omega,\Sigma}(T)^N,\, {}_0\mathbb{E}_p^\Omega(T)^N \times {}_0\mathbb{E}_p^\Sigma(T)^N)} : \, T \in (0, T']\right\}$$

independent of T, cf. Proposition 10.1. From Remark 11.1 we infer that

$$\|r^{\text{ch}}((\bar{c}^\Sigma + c^{\Sigma*})^+) - r^{\text{ch}}((\bar{z}^\Sigma + c^{\Sigma*})^+)\|_{L^p(\Sigma_T)^N} \leq L\|\bar{c}^\Sigma - \bar{z}^\Sigma\|_{L^{p\gamma}(\Sigma_T)^N}$$

for a constant $L > 0$ depending on δ but not on T and $\gamma \in [1, 6]$. Note in passing that we also used that $h \mapsto h^+$ is globally Lipschitz continuous from $L^{p\gamma}(\Sigma_T)$ to $L^{p\gamma}(\Sigma_T)$ with Lipschitz constant 1. By the fact that $p > 5/3$ we can estimate as

$$\|\bar{c}^\Sigma - \bar{z}^\Sigma\|_{L^{p\gamma}(\Sigma_T)^N} \leq KT^\eta\|\bar{c}^\Sigma - \bar{z}^\Sigma\|_{{}_0\mathbb{E}_p^\Sigma(T)^N}$$

with a constant $K > 0$ and an exponent $\eta > 0$ independent of T, see Remark 11.1 (a). For the T-independence of K see Lemma 1.5. We arrive at

$$\|r^{\text{ch}}((\bar{c}^\Sigma + c^{\Sigma*})^+) - r^{\text{ch}}((\bar{z}^\Sigma + c^{\Sigma*})^+)\|_{\mathbb{F}_p^\Sigma(T)^N} \leq LKT^\eta\|\bar{c}^\Sigma - \bar{z}^\Sigma\|_{{}_0\mathbb{E}_p^\Sigma(T)^N}. \tag{11.8}$$

We turn to the estimate of the sorption rate. By ($\mathrm{A}_\mathrm{F}^{\text{sorp}}$), ($\mathrm{A}_\mathrm{B}^{\text{sorp}}$) the mapping r_i^{sorp} acts as a Nemytskij operator from $W_p^s(\Sigma_T) \times W_p^s(\Sigma_T)$ to $W_p^s(\Sigma_T)$ for $s \in (0,1)$, cf. Proposition 1.8. Note that ($\mathrm{A}_\mathrm{B}^{\text{sorp}}$) implies $r_i^{\text{sorp}}(0,0) = 0$ which is needed therein. A pointwise application of the mean value theorem to the real function r_i^{sorp} gives us

$$\begin{aligned} &r_i^{\text{sorp}}(z_i(t,x), z_i^\Sigma(t,x)) - r_i^{\text{sorp}}(c_i(t,x), c_i^\Sigma(t,x)) \\ &= \int_0^1 \nabla r_i^{\text{sorp}}(c_i + \tau(z_i - c_i), c_i^\Sigma + \tau(z_i^\Sigma - c_i^\Sigma))(t,x)\, d\tau \begin{pmatrix} z_i(t,x) - c_i(t,x) \\ z_i^\Sigma(t,x) - c_i^\Sigma(t,x) \end{pmatrix}. \end{aligned}$$

for $(t,x) \in \Sigma_T$. Utilizing $\nabla r_i^{\text{sorp}} \in BC^1(\mathbb{R}^2, \mathbb{R}^2)$ yields

$$\begin{aligned} &\|r_i^{\text{sorp}}(z_i, z_i^\Sigma) - r_i^{\text{sorp}}(c_i, c_i^\Sigma)\|_{W_p^s(\Sigma_T)} \\ &\qquad \leq \|\nabla r_i^{\text{sorp}}\|_{BC^1(\mathbb{R}^2,\mathbb{R}^2)} \left(\|z_i - c_i\|_{W_p^s(\Sigma_T)} + \|z_i^\Sigma - c_i^\Sigma\|_{W_p^s(\Sigma_T)}\right), \end{aligned}$$

i.e. the global Lipschitz continuity of the induced Nemytskij operator of r_i^{sorp}. Hence we may employ

$$W_p^s(\Sigma_T) = W_p^s((0,T), L^p(\Sigma)) \cap L^p((0,T), W_p^s(\Sigma))$$

due to Lemma 1.6 for $s \in (0,1)$ and

$$\begin{aligned}
{}_0\mathbb{H}_p^\Sigma(T) \times {}_0\mathbb{E}_p^\Sigma(T) &\hookrightarrow ({}_0W_p^{1-1/p+\epsilon}((0,T), L^p(\Sigma)) \cap L^p((0,T), W_p^{1-1/p+\epsilon}(\Sigma)))^2 \\
&\xrightarrow{r_i^{\text{sorp}}} {}_0W_p^{1-1/p+\epsilon}((0,T), L^p(\Sigma)) \cap L^p((0,T), W_p^{1-1/p+\epsilon}(\Sigma)) \hookrightarrow {}_0\mathbb{G}_p^\Sigma(T),
\end{aligned}$$

for sufficiently small $\epsilon > 0$, such that we obtain similarly as for the estimate of the reaction term that

$$\begin{aligned}
&\|r_i^{\text{sorp}}(\bar{c} + c^*, \bar{c}^\Sigma + c^{\Sigma *}) - r_i^{\text{sorp}}(\bar{z} + c^*, \bar{z}^\Sigma + c^{\Sigma *})\|_{{}_0\mathbb{G}_p^\Sigma(T)^N} \\
&\qquad \le L'K'T^\eta \left(\|\bar{c}_i - \bar{z}_i\|_{{}_0\mathbb{E}_p^\Omega(T)} + \|\bar{c}_i^\Sigma - \bar{z}_i^\Sigma\|_{{}_0\mathbb{E}_p^\Sigma(T)} \right) \qquad (11.9)
\end{aligned}$$

with constants $L', K' > 0$ and an exponent $\eta > 0$ independent of $T < T'$, see Lemma 1.5. Combining (11.8) and (11.9) yields

$$\begin{aligned}
&\|{}_0\Phi_T(\bar{c}, \bar{c}^\Sigma) - {}_0\Phi_T(\bar{z}, \bar{z}^\Sigma)\|_{{}_0\mathbb{E}_p^\Omega(T)^N \times {}_0\mathbb{E}_p^\Sigma(T)^N} \\
&\le C(LK + L'K')T^\eta \|(\bar{c}, \bar{c}^\Sigma) - (\bar{z}, \bar{z}^\Sigma)\|_{{}_0\mathbb{E}_p^\Omega(T)^N \times {}_0\mathbb{E}_p^\Sigma(T)^N}
\end{aligned}$$

for $(\bar{c}, \bar{c}^\Sigma), (\bar{z}, \bar{z}^\Sigma) \in \bar{B}_\delta(0)$. We choose T so small that

$$C(LK + L'K')T^\eta \le \frac{1}{2}, \qquad (11.10)$$

which is possible since all other constants appearing in (11.10) are independent of $T < T'$. Hence ${}_0\Phi_T$ is a contraction on $\bar{B}_\delta(0)$.

(ii) Self mapping property: Let $(\bar{c}, \bar{c}^\Sigma) \in \bar{B}_\delta(0)$. Then we have

$$\begin{aligned}
&\|{}_0\Phi_T(\bar{c}, \bar{c}^\Sigma)\|_{{}_0\mathbb{E}_p^\Omega(T)^N \times {}_0\mathbb{E}_p^\Sigma(T)^N} \le C\|{}_0\mathcal{N}_T(\bar{c} + c^*, \bar{c}^\Sigma + c^{\Sigma *})\|_{{}_0\mathbb{F}_p^{\Omega,\Sigma}(T)^N} \\
&\le C\|r^{\text{sorp}}(\bar{c} + c^*, \bar{c}^\Sigma + c^{\Sigma *}) - r^*\|_{{}_0\mathbb{G}_p^\Sigma(T)^N} + C\|r^{\text{ch}}((\bar{c}^\Sigma + c^{\Sigma *})^+)\|_{\mathbb{F}_p^\Sigma(T)^N}.
\end{aligned}$$

Analogously to (i) we estimate the reaction term by

$$\begin{aligned}
&\|r^{\text{ch}}((\bar{c}^\Sigma + c^{\Sigma *})^+)\|_{L^p(\Sigma_T)^N} \\
&\le \|r^{\text{ch}}((\bar{c}^\Sigma + c^{\Sigma *})^+) - r^{\text{ch}}((c^{\Sigma *})^+)\|_{L^p(\Sigma_T)^N} + \|r^{\text{ch}}((c^{\Sigma *})^+)\|_{L^p(\Sigma_T)^N} \\
&\le L\|\bar{c}^\Sigma\|_{L^{p\gamma}(\Sigma_T)^N} + \|r^{\text{ch}}((c^{\Sigma *})^+)\|_{L^p(\Sigma_T)^N} \\
&\le LKT^\eta \|\bar{c}^\Sigma\|_{{}_0\mathbb{E}_p^\Sigma(T)^N} + \|r^{\text{ch}}((c^{\Sigma *})^+)\|_{L^p(\Sigma_T)^N} \qquad (11.11)
\end{aligned}$$

with constants $L, K > 0$ being independent of T. For the sorption term we obtain in the same manner as in (i) the estimate

$$\begin{aligned}
&\|r_i^{\text{sorp}}(\bar{c}_i + c_i^*, \bar{c}_i^\Sigma + c_i^{\Sigma *}) - r_i^*\|_{{}_0\mathbb{G}_p^\Sigma(T)} \\
&\le \|r_i^{\text{sorp}}(\bar{c}_i + c_i^*, \bar{c}_i^\Sigma + c_i^{\Sigma *}) - r_i^{\text{sorp}}(c_i^*, c_i^{\Sigma *})\|_{{}_0\mathbb{G}_p^\Sigma(T)} \\
&\qquad + \|r_i^{\text{sorp}}(c_i^*, c_i^{\Sigma *})\|_{\mathbb{G}_p^\Sigma(T)} + \|r_i^*\|_{{}_0\mathbb{G}_p^\Sigma(T)} \\
&\le L'K'T^\eta \|(\bar{c}_i, \bar{c}_i^\Sigma)\|_{{}_0\mathbb{E}_p^\Omega(T)^N \times {}_0\mathbb{E}_p^\Sigma(T)^N} + \|r_i^{\text{sorp}}(c_i^*, c_i^{\Sigma *})\|_{\mathbb{G}_p^\Sigma(T)} + \|r_i^*\|_{{}_0\mathbb{G}_p^\Sigma(T)} \quad (11.12)
\end{aligned}$$

with constants $L', K' > 0$ and an exponent $\eta > 0$ being all independent of T. Putting together (11.11) and (11.12) yields

$$\begin{aligned}
\|{}_0\Phi_T(\bar{c}, \bar{c}^\Sigma)\|_{{}_0\mathbb{E}_p^\Omega(T)^N \times {}_0\mathbb{E}_p^\Sigma(T)^N} &\le C(LK + L'K')T^\eta \delta + \|r^{\text{ch}}((c^{\Sigma *})^+)\|_{L^p(\Sigma_T)^N} \\
&\qquad + \|r_i^{\text{sorp}}(c_i^*, c_i^{\Sigma *})\|_{\mathbb{G}_p^\Sigma(T)} + \|r_i^*\|_{{}_0\mathbb{G}_p^\Sigma(T)}.
\end{aligned}$$

Since c_i^*, $c_i^{\Sigma *}$, and r_i^* are fixed functions, notice that the latter three terms can be made small by choosing $T > 0$ small. Thus, by choosing T so small that the sum of those three terms is less that $\delta/2$ and such that (11.10) is satisfied, we arrive at

$$\|{}_0\Phi_T(\bar{c}, \bar{c}^\Sigma)\|_{{}_0\mathbb{E}_p^\Omega(T)^N \times {}_0\mathbb{E}_p^\Sigma(T)^N} \le \delta \qquad ((\bar{c}, \bar{c}^\Sigma) \in \bar{B}_\delta(0)).$$

The proof of Theorem 8.1 is now complete. $\square$

Chapter 12

Global Well-Posedness

In this chapter we show that the local-in-time strong solutions obtained via Theorem 8.1 in fact exist globally, provided the reaction rates satisfy some structural condition that allows for the derivation of the necessary a-priori estimates. To be precise, we will consider the nonlinear problem

$$\left\{\begin{array}{rcllr} \partial_t c_i + (u\cdot\nabla)c_i - d_i\Delta c_i &=& f_i & \text{in} & \mathbb{R}_+\times\Omega,\\ \partial_t c_i^\Sigma - d_i^\Sigma\Delta_\Sigma c_i^\Sigma &=& r_i^{\text{sorp}}(c_i,\, c_i^\Sigma) + r_i^{\text{ch}}(c^\Sigma) & \text{in} & \mathbb{R}_+\times\Sigma,\\ (u\cdot\nu)c_i - d_i\partial_\nu c_i &=& g_i^{\text{in}} & \text{on} & \mathbb{R}_+\times\Gamma_{\text{in}},\\ -d_i\partial_\nu c_i &=& r_i^{\text{sorp}}(c_i,\, c_i^\Sigma) & \text{on} & \mathbb{R}_+\times\Sigma,\\ -d_i\partial_\nu c_i &=& 0 & \text{on} & \mathbb{R}_+\times\Gamma_{\text{out}},\\ -d_i^\Sigma\partial_{\nu_\Sigma} c_i^\Sigma &=& 0 & \text{on} & \mathbb{R}_+\times\partial\Sigma,\\ c_i(0) &=& c_{0,i} & \text{in} & \Omega,\\ c_i^\Sigma(0) &=& c_{0,i}^\Sigma & \text{on} & \Sigma, \end{array}\right. \tag{12.1}$$

where we assume ($\mathrm{A}_\mathrm{F}^{\text{sorp}}$), ($\mathrm{A}_\mathrm{M}^{\text{sorp}}$), ($\mathrm{A}_\mathrm{B}^{\text{sorp}}$). In addition to ($\mathrm{A}_\mathrm{F}^{\text{ch}}$), ($\mathrm{A}_\mathrm{N}^{\text{ch}}$) and ($\mathrm{A}_\mathrm{P}^{\text{ch}}$) we make the following assumption on the structure of the reaction rates.

($\mathbf{A}_\mathbf{S}^{\textbf{ch}}$) The structure of the reaction rates is such that there exists an invertible lower triangular matrix $Q = (q_{ij})_{1\le i,j\le N} \in \mathbb{R}^{N\times N}$ with strictly positive diagonal entries such that

$$Qr^{\text{ch}}(y) \le C\left(1 + \sum_{j=1}^{N} y_j\right)v, \qquad y\in[0,\infty)^N \tag{12.2}$$

for some constant $C > 0$ and $v = (1,\,\ldots,\,1)$.

Reaction-diffusion systems with the triangular condition given through ($\mathrm{A}_\mathrm{S}^{\text{ch}}$) have been widely studied by several authors; see for instance [Pie10] and the references cited therein. When proving global existence results, condition (12.2) allows for an iteration scheme which has been applied successfully in many situations. A major objective of Chapter 12 is to generalize this iteration scheme for standard reaction-diffusion systems subject to ($\mathrm{A}_\mathrm{S}^{\text{ch}}$) to heterogeneous catalysis. The main difference compared to standard systems lies in the fact that the reaction takes place on the boundary instead inside the bulk and that we also have to deal with terms arising from sorption processes.

In this setting, we obtain the global-in-time well-posedness result given by Theorem 8.3. The proof of Theorem 8.3 is given in Section 12.2. Besides the maximal regularity estimates obtained in Chapter 10 it also requires some comparison principles and some weak-type estimates, which are provided by the following results in Section 12.1.

12.1. Comparison Principle and Weak-type Estimates

Lemma 12.1 *Let $T > 0$ and let $1 < p < \infty$. Let $\alpha,\, \beta > 0$.*

(a) *Assume $f \in \mathbb{F}_p^{\Omega}(T)$, $g^{in} \in \mathbb{G}_p^{in}(T)$, $g^{\Sigma} \in \mathbb{G}_p^{\Sigma}(T)$ and $v_0 \in \mathbb{I}_p^{\Omega}$ with*

$$(u \cdot \nu)v_0 - \beta\partial_\nu v_0 = g^{in}(0) \quad \text{on } \Gamma_{in}, \qquad -\beta\partial_\nu v_0 = g^{\Sigma}(0) \quad \text{on } \Sigma,$$

if $p > 3$. Let $v \in \mathbb{E}_p^{\Omega}(T)$ be a strong solution to

$$\left\{\begin{array}{rcll} \alpha\partial_t v + (u\cdot\nabla)v - \beta\Delta v &=& f & \text{in } \ (0,\, T)\times\Omega, \\ (u\cdot\nu)v - \beta\partial_\nu v &=& g^{in} & \text{on } \ (0,\, T)\times\Gamma_{in}, \\ -\beta\partial_\nu v &=& g^{\Sigma} & \text{on } \ (0,\, T)\times\Sigma, \\ -\beta\partial_\nu v &=& 0 & \text{on } \ (0,\, T)\times\Gamma_{out}, \\ v(0) &=& v_0 & \text{in } \ \Omega. \end{array}\right.$$

If $f,\, v_0 \geq 0$ and $g^{in},\, g^{\Sigma} \leq 0$, then $v \geq 0$.

(b) *Assume $f \in \mathbb{F}_p^{\Sigma}(T)$ and $v_0 \in \mathbb{I}_p^{\Sigma}$ with $-\beta\partial_{\nu_\Sigma} v_0 = 0$ on $\partial\Sigma$, if $p > 3$. Let $v \in \mathbb{E}_p^{\Sigma}(T)$ be a strong solution to*

$$\left\{\begin{array}{rcll} \alpha\partial_t v - \beta\Delta_\Sigma v &=& f & \text{on } \ (0,\, T)\times\Sigma, \\ -\beta\partial_{\nu_\Sigma} v &=& 0 & \text{on } \ (0,\, T)\times\partial\Sigma, \\ v(0) &=& v_0 & \text{on } \ \Sigma. \end{array}\right.$$

If $f,\, v_0 \geq 0$, then $v \geq 0$.

Proof. The proof follows the lines of the proof of Lemma 11.2, except that here we deal with a linear problem, only. □

We turn to the first kind of weak-type estimates, i.e. L^p-estimates. First we state and prove two lemmas which give us estimates for (10.1) with respect to the L^p-norm. This will be carried out in detail since the treatment of the velocity term $(u \cdot \nabla)c_i$ requires some effort and is worth a closer look. This yields the desired estimates only in case of $p < \infty$, see Lemma 12.3. As a second step we obtain the remaining case $p = \infty$ for c_i by employing the comparison principle (Lemma

12.1). This is content of Lemma 12.5. Let us start by stating an auxiliary lemma, which together with its proof can be found in [Fis13].

Lemma 12.2 *Let $G \subset \mathbb{R}^n$ be a domain, $0 \le a < b \le T < \infty$ and suppose $\alpha, \beta \ge 2$ satisfy*

$$\frac{1}{\alpha} + \frac{n}{2\beta} \begin{cases} > \frac{n}{4}, & \text{if } n = 2 \text{ and } \alpha = 2, \text{ or } n = 1 \text{ and } \alpha = 4, \\ \ge \frac{n}{4}, & \text{else.} \end{cases} \tag{12.3}$$

Then it holds true that

$$X_{a,b} := L^\infty((a,b), L^2(G)) \cap L^2((a,b), W^{1,2}(G)) \hookrightarrow L^\alpha((a,b), L^\beta(G))$$

and there is a constant $C > 0$ depending on G, α, β, T but not on a, b such that

$$\|v\|_{L^\alpha((a,b),L^\beta(G))} \le C \left(\text{ess sup}_{t\in[a,b]} \|v(t)\|^2_{L^2(G)} + \int_a^b \|\nabla v(s)\|^2_{L^2(G)} ds \right)^{1/2} \tag{12.4}$$

for all $v \in X_{a,b}$.

We make use of Lemma 12.2 to treat the advection term in the proof of Lemma 12.3. More precisely, we estimate $\int_{\Omega_T} u \cdot \nabla(c_i^{p/2}) c_i^{p/2} d(t,x)$ by an application of Hölder's inequality for suitable exponents and Young's inequality. Consequently, we have to deal with $\|c_i^{p/2}\|_{L^\alpha(\Omega_T)}$ for a suitbale $\alpha \in (2,\infty)$. At this point Lemma 12.2 comes into play for $\alpha = \beta$. Let us go on with

Lemma 12.3 *Let $T^* > 0$ and $0 < T < T^*$ and let $5/3 < p < \infty$. Assume (A^{vel}) is fulfilled and that the data satisfies the regularity condition*

$$(f_i, f_i^\Sigma, g_i^{in}, g_i^\Sigma, 0, 0, c_{0,i}, c_{0,i}^\Sigma) \in \mathbb{F}^{\Omega,\Sigma}_{p,I}(T)$$

and in case of $p > 3$ the compatibility conditions

$$(u \cdot \nu)c_{0,i} - d_i \partial_\nu c_{0,i} = g_i^{in}(0) \quad \text{on } \Gamma_{in}, \qquad -d_i \partial_\nu c_{0,i} = g_i^\Sigma(0) \quad \text{on } \Sigma.$$

Let $(c_i, c_i^\Sigma) \in \mathbb{E}^\Omega_p(T) \times \mathbb{E}^\Sigma_p(T)$ be the strong solution of (10.1). Moreover, suppose that

$$f_i, f_i^\Sigma, -g_i^{in}, -g_i^\Sigma, c_{0,i}, c_{0,i}^\Sigma \ge 0.$$

Then there exists a constant $M > 0$ depening only on

$$\|u\|_{\mathbb{U}^\Omega_p(T)}, d_i, d_i^\Sigma, p, \Omega \tag{12.5}$$

but which is independent of T, such that

$$\begin{aligned} \|c_i(t)\|^p_{L^p(\Omega)} + \|c_i^\Sigma(t)\|^p_{L^p(\Sigma)} &+ \int_0^t \left(\|c_i^{p/2}(s)\|^2_{H^1(\Omega)} + \|(c_i^\Sigma)^{p/2}\|^2_{H^1(\Sigma)} \right) ds \\ &\le M \Big(\|c_{0,i}\|^p_{L^p(\Omega)} + \|c_{0,i}^\Sigma\|^p_{L^p(\Sigma)} + \|f_i\|^p_{L^p(\Omega_T)} + \|f_i^\Sigma\|^p_{L^p(\Sigma_T)} \\ &\quad + \|g_i^{in}\|^p_{L^p(\Gamma_{in,T})} + \|g_i^\Sigma\|^p_{L^p(\Sigma_T)} \Big) \end{aligned} \tag{12.6}$$

holds for all $t \in (0,T)$.

Proof. We proceed in two steps – In a first step we treat the c_i-equations, in a second step we come to the c_i^Σ-equations.
Step 1. Let us first treat the equation in Ω. Note that by Lemma 12.1 we have $c_i \geq 0$. Up to a change of c_i into $e^{-t}c_i$, g_i^{in} into $e^{-t}g_i^{\text{in}}$, g_i^Σ into $e^{-t}g_i^\Sigma$, f_i into $e^{-t}f_i$, and $c_{0,i}$ into $e^{-t}c_{0,i}$ we obtain

$$\partial_t c_i + c_i + u\cdot\nabla c_i - d_i\Delta c_i = \partial_t c_i + c_i + \operatorname{div}(uc_i - d_i\nabla c_i) = f_i \qquad \text{in } \Omega_T. \tag{12.7}$$

Note that this is possible due to the linearity of (10.1). Multiplying (12.7) by pc_i^{p-1} and integrating over Ω yields

$$\frac{d}{dt}\int_\Omega c_i^p dx + p\int_\Omega c_i^p dx + p\int_\Omega (\operatorname{div}(uc_i - d_i\nabla c_i))c_i^{p-1}dx = p\int_\Omega f_i c_i^{p-1}dx.$$

Partial integration yields

$$\begin{aligned}
&p\int_\Omega (\operatorname{div}(uc_i - d_i\nabla c_i))c_i^{p-1}dx\\
&= p\int_{\partial\Omega}((u\cdot\nu)c_i - d_i\partial_\nu c_i)c_i^{p-1}d\sigma - p\int_\Omega (uc_i - d_i\nabla c_i)(p-1)c_i^{p-2}(\nabla c_i)dx\\
&= p\int_{\Gamma_{\text{in}}} g_i^{\text{in}}c_i^{p-1}d\sigma + p\int_\Sigma g_i^\Sigma c_i^{p-1}d\sigma + p\int_{\Gamma_{\text{out}}}(u\cdot\nu)c_i^p d\sigma\\
&\quad - p\int_\Omega (uc_i - d_i\nabla c_i)(p-1)c_i^{p-2}(\nabla c_i)dx.
\end{aligned}$$

Note that

$$\nabla(c_i^{p/2}) = \tfrac{p}{2}c_i^{p/2-1}\nabla c_i = \tfrac{p}{2}c_i^{(p-2)/2}\nabla c_i$$

yields

$$4\tfrac{p-1}{p}d_i|\nabla c_i^{p/2}|^2 = d_i p(p-1)|\nabla c_i|^2 c_i^{p-2},$$

from which we infer

$$d_i p(p-1)\int_\Omega |\nabla c_i|^2 c_i^{p-2}dx = 4\tfrac{p-1}{p}d_i\int_\Omega |\nabla(c_i^{p/2})|^2 dx.$$

Hence we obtain

$$\begin{aligned}
&\frac{d}{dt}\|c_i\|^p_{L^p(\Omega)} + p\|c_i\|^p_{L^p(\Omega)} + 4\tfrac{p-1}{p}d_i\int_\Omega |\nabla(c_i^{p/2})|^2 dx\\
&\leq p\int_\Omega f_i c_i^{p-1}dx - p\int_{\Gamma_{\text{in}}} g_i^{\text{in}}c_i^{p-1}d\sigma - p\int_\Sigma g_i^\Sigma c_i^{p-1}d\sigma - p\int_{\Gamma_{\text{out}}}(u\cdot\nu)c_i^p d\sigma\\
&\quad + p(p-1)\int_\Omega c_i^{p-1}u\cdot\nabla c_i dx
\end{aligned}$$

and because of $\|c_i\|^p_{L^p(\Omega)} = \|c_i^{p/2}\|^2_{L^2(\Omega)}$ we may rewrite this inequality in the form

$$\begin{aligned}\frac{d}{dt}\|c_i^{p/2}\|^2_{L^2(\Omega)} + p\|c_i^{p/2}\|^2_{L^2(\Omega)} + 4\tfrac{p-1}{p}d_i \int_\Omega |\nabla(c_i^{p/2})|^2 dx \\ \le p\int_\Omega f_i c_i^{p-1}dx - p\int_{\Gamma_{\text{in}}} g_i^{\text{in}} c_i^{p-1}d\sigma - p\int_\Sigma g_i^\Sigma c_i^{p-1}d\sigma - p\int_{\Gamma_{\text{out}}}(u\cdot\nu)c_i^p d\sigma \\ + p(p-1)\int_\Omega c_i^{p-1} u\cdot\nabla c_i dx.\end{aligned}$$

Employing

$$2(p-1)\nabla(c_i^{p/2})c_i^{p/2} = 2(p-1)(\tfrac{p}{2}c_i^{p/2-1}\nabla c_i)c_i^{p/2} = p(p-1)c_i^{p-1}\nabla c_i$$

we obtain

$$p(p-1)\int_\Omega u c_i^{p-1}\nabla c_i dx = 2(p-1)\int_\Omega u\cdot\nabla(c_i^{p/2})c_i^{p/2}dx,$$

such that

$$\begin{aligned}\frac{d}{dt}\|c_i^{p/2}\|^2_{L^2(\Omega)} + p\|c_i^{p/2}\|^2_{L^2(\Omega)} + 4\tfrac{p-1}{p}d_i \int_\Omega |\nabla(c_i^{p/2})|^2 dx \\ \le p\int_\Omega f_i c_i^{p-1}dx - p\int_{\Gamma_{\text{in}}} g_i^{\text{in}} c_i^{p-1}d\sigma - p\int_\Sigma g_i^\Sigma c_i^{p-1}d\sigma - p\int_{\Gamma_{\text{out}}}(u\cdot\nu)c_i^p d\sigma \\ + 2(p-1)\int_\Omega u\cdot\nabla(c_i^{p/2})c_i^{p/2}dx.\end{aligned}$$

We turn to the three boundary integrals. By Lemma 11.2 we may neglect the third one, since it is negative or zero. Let $1 = 1/p + 1/p'$. Because of $p' = p/(p-1)$ we may use

$$\|c_i^{p-1}\|^{p'}_{L^{p'}(\Omega)} = \int_\Omega c_i^{(p-1)p'}dx = \int_\Omega c_i^p dx = \|c_i\|^p_{L^p(\Omega)}.$$

This in combination with Hölder's inequality and Young's inequality yields

$$\begin{aligned}\left| p\int_\Omega f_i c_i^{p-1}dx\right| \le p\|f_i\|_{L^p(\Omega)}\|c_i^{p-1}\|_{L^{p'}(\Omega)} \le M(\varepsilon)\|f_i\|^p_{L^p(\Omega)} + \varepsilon\frac{p}{p'}\|c_i^{p-1}\|^{p'}_{L^{p'}(\Omega)} \\ = M(\varepsilon)\|f_i\|^p_{L^p(\Omega)} + \varepsilon(p-1)\|c_i\|^p_{L^p(\Omega)},\end{aligned}$$

and

$$\begin{aligned}\left| p\int_{\Gamma_{\text{in}}} g_i^{\text{in}} c_i^{p-1}d\sigma\right| \le p\|g_i^{\text{in}}\|_{L^p(\Gamma_{\text{in}})}\|c_i^{p-1}\|_{L^{p'}(\Gamma_{\text{in}})} \le M(\varepsilon)\|g_i^{\text{in}}\|^p_{L^p(\Gamma_{\text{in}})} + \varepsilon\tfrac{p}{p'}\|c_i^{p-1}\|^{p'}_{L^{p'}(\partial\Omega)} \\ = M(\varepsilon)\|g_i^{\text{in}}\|^p_{L^p(\Gamma_{\text{in}})} + \varepsilon(p-1)\|c_i\|^p_{L^p(\partial\Omega)}\end{aligned}$$

with a constant $M > 0$ depending on p and $\varepsilon > 0$, which in turn is arbitrary. Again, we reformulate the L^p-norm of c_i in terms of the L^2-norm of $c_i^{p/2}$. Then we employ a standard trace theorem, such that

$$\|c_i\|_{L^p(\partial\Omega)}^p = \|c_i^{p/2}\|_{L^2(\partial\Omega)}^2 \leq C\|c_i^{p/2}\|_{H^1(\Omega)}^2$$

for a constant $C > 0$ yields

$$\left| p \int_\Omega f_i c_i^{p-1} dx \right| \leq M(\varepsilon)\|f_i\|_{L^p(\Omega)}^p + \varepsilon(p-1)\|c_i^{p/2}\|_{L^2(\Omega)}^2$$

and

$$\left| p \int_{\Gamma_{\text{in}}} g_i^{\text{in}} c_i^{p-1} d\sigma \right| \leq M(\varepsilon)\|g_i^{\text{in}}\|_{L^p(\Gamma_{\text{in}})}^p + \varepsilon'(p-1)\|c_i^{p/2}\|_{H^1(\Omega)}^2$$

for a constant $M > 0$ depending on $\varepsilon' = \varepsilon C$. We treat g_i^Σ in the same manner as g_i^{in}, and we arrive at

$$-p \int_\Sigma g_i^\Sigma c_i^{p-1} d\sigma \leq M(\varepsilon')\|g_i^\Sigma\|_{L^p(\Sigma)}^p + \varepsilon'(p-1)\|c_i^{p/2}\|_{H^1(\Omega)}^2.$$

Combining these estimates we infer for an $\varepsilon > 0$

$$\begin{aligned} \frac{d}{dt}\|c_i^{p/2}\|_{L^2(\Omega)}^2 &+ p\|c_i^{p/2}\|_{L^2(\Omega)}^2 + 4\tfrac{p-1}{p} d_i \|\nabla(c_i^{p/2})\|_{L^2(\Omega,\mathbb{R}^3)}^2 \\ &\leq M(\varepsilon)\|f_i\|_{L^p(\Omega)}^p + M(\varepsilon)\|g_i^{\text{in}}\|_{L^p(\Gamma_{\text{in}})}^p + M(\varepsilon)\|g_i^\Sigma\|_{L^p(\Sigma)}^p \\ &\quad + 3(p-1)\varepsilon\|c_i^{p/2}\|_{H^1(\Omega)}^2 + 2(p-1)\int_\Omega u \cdot \nabla(c_i^{p/2}) c_i^{p/2} dx. \end{aligned}$$

Choosing

$$\varepsilon = \tfrac{1}{6(p-1)} \min\left\{p, 4\tfrac{p-1}{p} d_i\right\} > 0$$

we have

$$3(p-1)\varepsilon\|c_i^{p/2}\|_{H^1(\Omega)}^2 = \frac{1}{2}\min\left\{p, 4\tfrac{p-1}{p} d_i\right\}\|c_i^{p/2}\|_{H^1(\Omega)}^2$$

and we may absorb the ε-term on the right-hand side such that

$$\begin{aligned} &\frac{d}{dt}\|c_i^{p/2}\|_{L^2(\Omega)}^2 + \frac{p}{2}\|c_i^{p/2}\|_{L^2(\Omega)}^2 + 2\tfrac{p-1}{p} d_i \|\nabla(c_i^{p/2})\|_{L^2(\Omega,\mathbb{R}^3)}^2 \\ &\leq M\|f_i\|_{L^p(\Omega)}^p + M\|g_i^{\text{in}}\|_{L^p(\Gamma_{\text{in}})}^p + M\|g_i^\Sigma\|_{L^p(\Sigma)}^p + 2(p-1)\int_\Omega u \cdot \nabla(c_i^{p/2}) c_i^{p/2} dx \end{aligned} \tag{12.8}$$

holds with a constant $M > 0$.
We turn to the treatment of $\int_\Omega |u \cdot \nabla(c_i^{p/2}) c_i^{p/2}| dx$, which we will absorb into the left-hand side. Integration of (12.8) over $[0, t']$ with $t' \leq t \leq T$ yields

$$\|c_i^{p/2}(t')\|_{L^2(\Omega)}^2 + \delta_i \int_0^{t'} \|c_i^{p/2}\|_{H^1(\Omega)}^2 ds \leq \|c_{0,i}^{p/2}\|_{L^2(\Omega)}^2 + M\|f_i\|_{L^p(\Omega_{t'})}^p$$
$$+ M\|g_i^{\text{in}}\|_{L^p(\Gamma_{\text{in},t'})}^p + M\|g_i^\Sigma\|_{L^p(\Sigma_{t'})}^p + 2(p-1) \int_{\Omega_{t'}} u \cdot (\nabla(c_i^{p/2})) c_i^{p/2} d(t,x) \quad (12.9)$$

for

$$\delta_i := \min\left\{\frac{p}{2}, 2\frac{p-1}{p} d_i\right\} > 0.$$

We choose $\alpha := \beta := 10/3$. Then $1/\alpha + 3/2\alpha = 3/4$ and Lemma 12.2 is applicable. For arbitrary $\varepsilon > 0$ Hölder's inequality for the relation $\frac{1}{2} + \frac{1}{2} = 1$ and afterwards for $\frac{1}{5} + \frac{1}{10/3} = \frac{1}{2}$ and Young's inequality yield

$$\int_{\Omega_{t'}} |u \cdot \nabla(c_i^{p/2}) c_i^{p/2}| d(s,x) \leq \|u\|_{L^5(\Omega_{t'}, \mathbb{R}^3)} \|\nabla(c_i^{p/2})\|_{L^2(\Omega_{t'}, \mathbb{R}^3)} \|c_i^{p/2}\|_{L^\alpha(\Omega_{t'})}$$
$$\leq \varepsilon \|\nabla(c_i^{p/2})\|_{L^2(\Omega_{t'}, \mathbb{R}^3)}^2 + C(\varepsilon)\|u\|_{L^5(\Omega_{t'}, \mathbb{R}^3)}^2 \|c_i^{p/2}\|_{L^\alpha(\Omega_{t'})}^2.$$

Note that for all $p > 5/3$

$$\mathbb{U}_p^\Omega(T) \hookrightarrow L^5(\Omega_{t'}, \mathbb{R}^3)$$

due to the relation

$$2 - \frac{5}{p} > -1$$

of the anisotropic indices of the spaces $\mathbb{U}_p^\Omega(T)$ and $L^5(\Omega_{t'}, \mathbb{R}^3)$ thanks to Proposition 1.3. By estimate (12.9) we infer

$$\|c_i^{p/2}(t')\|_{L^2(\Omega)}^2 + \delta_i' \int_0^{t'} \|c_i^{p/2}\|_{H^1(\Omega)}^2 ds \leq \|c_{0,i}^{p/2}\|_{L^2(\Omega)}^2 + M\|f_i\|_{L^p(\Omega_{t'})}^p + M\|g_i^{\text{in}}\|_{L^p(\Gamma_{\text{in}})}^p$$
$$+ M\|g_i^\Sigma\|_{L^p(\Sigma_{t'})}^p + C\|u\|_{L^5(\Omega_{t'}, \mathbb{R}^3)}^2 \|c_i^{p/2}\|_{L^\alpha(\Omega_t')}^2,$$

where $\delta_i' := \delta_i - \varepsilon > 0$. Taking the supremum over all $t' \leq t$ on both sides and using Lemma 12.2 with $\alpha = \beta$ we obtain

$$\begin{aligned}\sup_{t'\in[0,t]} \|c_i^{p/2}(t')\|^2_{L^2(\Omega)} + \int_0^t \|(c_i^{p/2})\|^2_{H^1(\Omega)} ds \leq C\Big[&\|c_{0,i}^{p/2}\|^2_{L^2(\Omega)} + M\|f_i\|^p_{L^p(\Omega_t)} \\ &+M\|g_i^{\text{in}}\|^p_{L^p(\Gamma_{\text{in},t})} + M\|g_i^\Sigma\|^p_{L^p(\Sigma_t)} \\ +\|u\|^2_{L^5(\Omega_t,\mathbb{R}^3)} \cdot \Big(\sup_{t'\in[0,t]} \|c_i^{p/2}(t')\|^2_{L^2(\Omega)} &+ \int_0^t \|\nabla c_i^{p/2}(s)\|^2_{L^2(\Omega,\mathbb{R}^3)} ds\Big)\Big]\end{aligned} \tag{12.10}$$

where $C > 0$ denotes a constant being independent of t by Lemma 12.2. Now we choose $m \in \mathbb{N}$ sufficiently large for

$$\|u\|^2_{L^5((\frac{h-1}{m}T,\frac{h}{m}T),L^5(\Omega,\mathbb{R}^3))} \leq \frac{1}{8C} \qquad h \in \{1, ..., m\},$$

to hold and obtain

$$\|u\|^2_{L^5((\frac{h-1}{m}t,\frac{h}{m}t),L^5(\Omega,\mathbb{R}^3))} \leq 4\|u\|^2_{L^5((\frac{h-1}{m}T,\frac{h}{m}T),L^5(\Omega,\mathbb{R}^3))} \leq \frac{1}{2C}.$$

By (12.10) it follows for all $h \in \{1, ..., m\}$ that

$$\begin{aligned}&\|(c_i(\tfrac{h}{m}t))^{p/2}\|^2_{L^2(\Omega)} + \int_{\frac{h-1}{m}t}^{\frac{h}{m}t} \|(c_i^{p/2})^{p/2}\|^2_{H^1(\Omega)} ds \\ &\leq C\Big[\|(c_i(\tfrac{h-1}{m}t))^{p/2}\|^2_{L^2(\Omega)} + M\int_{\frac{h-1}{m}t}^{\frac{h}{m}t} (\|f_i\|^p_{L^p(\Omega_{t'})} + \|g_i^{\text{in}}\|^p_{L^p(\Gamma_{\text{in}})} + \|g_i^\Sigma\|^p_{L^p(\Sigma)}) ds\Big].\end{aligned}$$

Iterating this computation for $h \in \{1, ..., m\}$ yields

$$\begin{aligned}&\|c_i(t)\|^p_{L^p(\Omega)} + \int_0^t \|c_i^{p/2}(s)\|_{H^1(\Omega)} ds \\ &\qquad\leq M\Big(\|c_{0,i}\|^p_{L^p(\Omega)} + \|f_i\|^p_{L^p(\Omega_T)} + \|g_i^{\text{in}}\|^p_{L^p(\Gamma_{\text{in},T})} + \|g_i^\Sigma\|^p_{L^p(\Sigma_T)}\Big)\end{aligned} \tag{12.11}$$

with a constant $M > 0$ depending merely on the parameters given in (12.5).

Step 2. We consider

$$\partial_t c_i^\Sigma - d_i^\Sigma \Delta_\Sigma c_i^\Sigma = f_i^\Sigma \qquad \text{on } \Sigma_T,$$

and as already seen in Step I we rescale this equation by replacing c_i^Σ by $e^{-t}c_i^\Sigma$, $c_{0,i}^\Sigma$ by $e^{-t}c_{0,i}^\Sigma$, c_i by $e^{-t}c_i$, and f_i^Σ by $e^{-t}f_i^\Sigma$, such that we deal with

$$\partial_t c_i^\Sigma + c_i^\Sigma - d_i^\Sigma \Delta_\Sigma c_i^\Sigma = f_i^\Sigma \qquad \text{on } \Sigma_T.$$

Analogously to Step I we multiply by $p(c_i^\Sigma)^{p-1}$ and obtain

$$\frac{d}{dt}\|c_i^\Sigma\|_{L^p(\Sigma)}^p + p\|c_i^\Sigma\|_{L^p(\Sigma)}^p + d_i^\Sigma p(p-1)\int_\Sigma |\nabla_\Sigma c_i^\Sigma|^2 (c_i^\Sigma)^{p-2} d\sigma = p\int_\Sigma f_i^\Sigma (c_i^\Sigma)^{p-1} d\sigma.$$

We infer

$$\frac{d}{dt}\|c_i^\Sigma\|_{L^p(\Sigma)}^p + p\|c_i^\Sigma\|_{L^p(\Sigma)}^p + d_i^\Sigma p(p-1)\int_\Sigma |\nabla_\Sigma c_i^\Sigma|^2 (c_i^\Sigma)^{p-2} d\sigma \le p\int_\Sigma f_i^\Sigma (c_i^\Sigma)^{p-1} d\sigma.$$

By the same techniques as above employed in Step I we have

$$p\int_\Sigma f_i^\Sigma (c_i^\Sigma)^{p-1} d\sigma \le M(\varepsilon)\|f_i^\Sigma\|_{L^p(\Sigma)}^p + (p-1)\varepsilon\|c_i^\Sigma\|_{L^p(\Sigma)}^p$$

for a constant $M > 0$ which depends on $\varepsilon > 0$. Note that analogously to the treatment of the gradient of c_i in Step I

$$d_i^\Sigma p(p-1)\int_\Sigma |\nabla_\Sigma c_i^\Sigma|^2 (c_i^\Sigma)^{p-2} d\sigma = 4\tfrac{p-1}{p} d_i^\Sigma \int_\Sigma |\nabla_\Sigma((c_i^\Sigma)^{p/2})|^2 d\sigma.$$

We arrive at

$$\begin{aligned} \frac{d}{dt}\|c_i^\Sigma\|_{L^p(\Sigma)}^p + \|c_i^\Sigma\|_{L^p(\Sigma)}^p + 4\tfrac{p-1}{p} d_i^\Sigma \|\nabla_\Sigma((c_i^\Sigma)^{p/2})\|_{L^2(\Sigma,\mathbb{R}^3)}^2 \\ \le M(\varepsilon)\|f_i^\Sigma\|_{L^p(\Sigma)}^p + (p-1)\varepsilon\|c_i^\Sigma\|_{L^p(\Sigma)}^p. \end{aligned} \tag{12.12}$$

Choosing $\varepsilon := \frac{1}{2(p-1)} > 0$ we may absorb the last term on the right hand side of (12.12) into $\|c_i^\Sigma\|_{L^p(\Sigma)}^p$ and obtain

$$\frac{d}{dt}\|(c_i^\Sigma)^{p/2}\|_{L^2(\Sigma)}^2 + \frac{1}{2}\|(c_i^\Sigma)^{p/2}\|_{L^2(\Sigma)}^2 + 4\frac{p-1}{p} d_i^\Sigma \int_\Sigma |\nabla_\Sigma (c_i^\Sigma)^{p/2}|^2 d\sigma \le M\|f_i^\Sigma\|_{L^p(\Sigma)}^p$$

with $M = M(\varepsilon)$. By choosing

$$\delta'' := \min\left\{\frac{1}{2}, 4\frac{p-1}{p} d_i^\Sigma\right\} > 0$$

and integrating over $[0,t]$ it follows

$$\begin{aligned} \|(c_i^\Sigma)^{p/2}(t)\|_{L^2(\Sigma)}^2 + \delta_i'' \int_0^t \|(c_i^\Sigma)^{p/2}\|_{H^1(\Sigma)}^2 ds \\ \le \|c_{0,i}^\Sigma\|_{L^p(\Sigma)}^p + M\|f_i^\Sigma(s)\|_{L^p(\Sigma_t)}^p \quad (t < T). \end{aligned} \tag{12.13}$$

Summing up the estimates (12.11) and (12.13) the assertion given through (12.6) follows with a constant $M > 0$ depending merely on the parameters in (12.5). □

Remark 12.4 *Note that the full strength of Lemma 12.3 is not exploited in the following. We only make use of $p = 2$ for the c_i-estimates in order to prove Lemma 12.5, see Case 1, $q = 2$ therein. Indeed it would have been sufficient to give L^2-estimtates of c_i only. Since L^2-estimates for system (10.1) nearly take the same effort as L^p-estimates, we preferred to directly give L^p-estimates of the full linear system (10.1).*

Lemma 12.5 *Let $T^* > 0$ and $0 < T < T^*$ and let $2 \leq p < \infty$ and $2 \leq q \leq \infty$. Let $\alpha,\ \beta > 0$. Assume $f \in \mathbb{F}_p^\Omega(T)$, $g^{in} \in \mathbb{G}_p^{in}(T)$, $g^\Sigma \in \mathbb{G}_p^\Sigma(T)$ and $v_0 \in \mathbb{I}_p^\Omega \cap BC(\Omega)$ with*

$$(u \cdot \nu)v_0 - \beta\partial_\nu v_0 = g^{in}(0) \quad \text{on } \Gamma_{in}, \qquad -\beta\partial_\nu v_0 = g^\Sigma(0) \quad \text{on } \Sigma,$$

if $p > 3$. Let $v \in \mathbb{E}_p^\Omega(T)$ be a strong solution to

$$\left\{ \begin{array}{rcllr} \alpha\partial_t v + (u \cdot \nabla)v - \beta\Delta v &=& f & \text{in} & (0,\,T) \times \Omega, \\ (u \cdot \nu)v - \beta\partial_\nu v &=& g^{in} & \text{on} & (0,\,T) \times \Gamma_{in}, \\ -\beta\partial_\nu v &=& g^\Sigma & \text{on} & (0,\,T) \times \Sigma, \\ -\beta\partial_\nu v &=& 0 & \text{on} & (0,\,T) \times \Gamma_{out}, \\ v(0) &=& v_0 & \text{in} & \Omega. \end{array} \right. \tag{12.14}$$

Then

$$\|v\|_{L^q(\Omega_T)} + \|v\|_{L^q(\Sigma_T)} \leq C\Big(\|f\|_{L^q(\Omega_T)} + \|g^{in}\|_{L^q(\Gamma_{in,T})} + \|g^\Sigma\|_{L^q(\Sigma_T)} + \|v_0\|_{L^q(\Omega)}\Big)$$

for some constant $C = C(q) > 0$ that is independent of p and $0 < T < T^*$.

Proof. If the right hand side is infinite, nothing has to be shown. So, we may now assume that the data admits finite L^q-norm. We distinguish two cases:
Case 1: $p > 5/2$, such that $\mathbb{E}_p^\Omega(T) \hookrightarrow BC(\overline{\Omega_T})$, see Proposition 1.3.
$q = 2$*:* This follows from Lemma 12.3.
$q = \infty$*:* Let $\mathcal{L}_T$ denote the operator given by the left-hand side of system (12.14), such that $\mathcal{L}_T v = (f, g^{\text{in}}, g^\Sigma, 0, v_0)$. For fixed $5/2 < r < 3$ let $\phi \in \mathbb{E}_r^\Omega(T)$ denote the solution of

$$\mathcal{L}_T\phi = (1, -1, -1, 0, 1) =: F.$$

Note that $\mathbb{E}_r^\Omega(T) \hookrightarrow BC(\overline{\Omega_T})$ by Proposition 1.3 and that no compatibility conditions for the ϕ-data occur. Let

$$\delta := \|f\|_{L^\infty(\Omega_T)} + \|g^{\text{in}}\|_{L^\infty(\Gamma_{\text{in},T})} + \|g^\Sigma\|_{L^\infty(\Sigma_T)} + \|v_0\|_{L^\infty(\Omega)}$$

and let $\bar{v} \in \mathbb{E}_p^\Omega(T)$ be given by $\mathcal{L}_T\bar{v} = \delta F$. Then the comparison principle (Lemma 12.1 (a)) applied to $\bar{v} - v$ yields $v \leq \bar{v}$ on $\overline{\Omega_T}$. Since $\bar{v} = \delta\phi$ and $\|\bar{v}\|_{BC} = \delta\|\phi\|_{BC}$ it follows that

$$v \leq \delta\left(\|\phi\|_{L^\infty(\Omega_T)} + \|\phi\|_{L^\infty(\Sigma_T)}\right) =: \delta C$$

on $\overline{\Omega_T}$. Analogously $-\delta C \leq v$, such that $\|v\|_{L^\infty(\Omega_T)} \leq C\delta$. Since in this case $v \in BC(\overline{\Omega_T})$ we trivially have

$$\|v\|_{L^\infty(\Sigma_T)} \leq \|v\|_{L^\infty(\Omega_T)} \leq C\delta,$$

and as a consequence

$$\|v\|_{L^\infty(\Sigma_T)} + \|v\|_{L^\infty(\Omega_T)} \leq C\delta.$$

For $2 \leq q \leq \infty$ we set

$$X_q := L^q(\Omega_T) \times L^q(\Sigma_T), \quad Y_q := L^q(\Omega_T) \times L^q(\Gamma_{\text{in},T}) \times L^q(\Sigma_T) \times \{0\} \times L^q(\Omega).$$

Let $\mathcal{S}_T$ denote the system's solution operator. By the L^2- and the L^∞-estimates obtained above we have

$$\mathcal{S}_T \in \mathscr{L}(Y_2, X_2) \cap \mathscr{L}(Y_\infty, X_\infty).$$

By the Riesz-Thorin interpolation theorem $\mathcal{S}_T \in \mathscr{L}(Y_q, X_q)$ which yields the assertion for Case 1.

Case 2: $2 \leq p \leq 5/2$.
For convenience set $F := (f, 0, g^{\text{in}}, g^\Sigma, 0, 0, v_0, 0) \in \mathbb{F}^{\Omega,\Sigma}_{p,I}(T)$. Let $r > 5/2$ and assume

$$F_n := (f_n, 0, g_n^{\text{in}}, g_n^\Sigma, 0, 0, v_{0,n}, 0) \in \mathbb{Y}_{r,q} :=$$
$$\mathbb{F}^{\Omega,\Sigma}_{r,I}(T) \cap (L^q(\Omega_T) \times \{0\} \times L^q(\Gamma_{\text{in},T}) \times L^q(\Sigma_T) \times \{0\} \times \{0\} \times L^q(\Omega) \times \{0\})$$

for $n \in \mathbb{N}$, such that $F_n \to F$ in $\mathbb{Y}_{p,q}$ (e.g. by extension of F to $\mathbb{R} \times \mathbb{R}^3$ respectively $\mathbb{R} \times (\partial B_R(0) \times \mathbb{R})$ and mollification). Let $v_n \in \mathbb{E}^\Omega_r(T)$ denote the corresponding solution. Then Case 1 applies to F_n, v_n and there exists a $C > 0$ independent of $n \in \mathbb{N}$, such that

$$\|v_n\|_{L^q(\Omega_T)} + \|v_n\|_{L^q(\Sigma_T)} \leq C\|F_n\|_{L^q}. \tag{12.15}$$

Obviously $(v_n)_n$ is a Cauchy sequence in

$$\mathbb{X}_{p,q} := \mathbb{E}^\Omega_p(T) \cap (L^q(\Omega_T) \times L^q(\Sigma_T)),$$

such that we may pass to the limit $n \to \infty$ in (12.15). Hence we obtain $v_n \to v$ in $\mathbb{X}_{p,q}$ with v being the solution to F and

$$\|v\|_{L^q(\Omega_T)} + \|v\|_{L^q(\Sigma_T)} \leq C\|F\|_{L^q}.$$

$\square$

The next Lemma is standard for equations on standard domains Ω, cf. [Pie10, Lemma 3.4] or [BFPR]. Here we give a proof since we employ it on Σ:

Lemma 12.6 *Let $T^* > 0$ and let $1 < p,\, q < \infty$. Let $\mu > 0$ and let arbitrary coefficients $\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N \in \mathbb{R}$ be given. Assume $f, g_1, \ldots, g_N \in \mathbb{F}_p^\Sigma(T)$ and suppose that $u_0, v_0^1, \ldots, v_0^N \in \mathbb{I}_p^\Sigma \cap BC(\Sigma)$ with $\partial_{\nu_\Sigma} u_0 = \partial_{\nu_\Sigma} v_0^j = 0$ on $\partial\Sigma$, if $p > 3$. Let $u, v_1, \ldots, v_N \in \mathbb{E}_p^\Sigma(T)$ be strong solutions to*

$$\begin{cases} \partial_t u - \mu\Delta_\Sigma u &= f, & \alpha_j\partial_t v_j - \beta_j\Delta_\Sigma v_j &= g_j & \text{on} \quad (0,\,T)\times\Sigma, \\ -\mu\partial_{\nu_\Sigma} u &= 0, & -\beta_j\partial_{\nu_\Sigma} v_j &= 0 & \text{on} \quad (0,\,T)\times\partial\Sigma, \\ u(0) &= u_0, & v_j(0) &= v_0^j & \text{on} \quad \Sigma, \end{cases} \tag{12.16}$$

for some $0 < T < T^*$. If $f \leq \sum_{j=1}^N g_j$, then

$$\|u^+\|_{L^q(\Sigma_T)} \leq C\left(1 + \sum_{j=1}^N \|v_j\|_{L^q(\Sigma_T)}\right)$$

for some constant $C > 0$ depending on

$$\Sigma, T^*, q, \mu, \alpha_j, \beta_j, \|u_0\|_{BC(\Sigma)},\, \|v_0^1\|_{BC(\Sigma)},\, \ldots,\, \|v_0^N\|_{BC(\Sigma)} \tag{12.17}$$

but which is independent of $0 < T < T^*$.

Proof. Let $\theta \in L^{q'}(\Sigma_T)^+$ for $1/q + 1/q' = 1$ and let $0 < \tau < T$. By the transformation $t \mapsto \tau - t$ applied to the equation for u in (12.16) with data $f = \theta$, $u_0 = 0$ we obtain the backward heat equation

$$\begin{cases} -[\partial_t\phi + \mu\Delta_\Sigma\phi] &= \theta(\tau - t) & \text{on } (0,\tau)\times\Sigma, \\ -\mu\partial_{\nu_\Sigma}\phi &= 0 & \text{on } (0,\tau)\times\partial\Sigma, \\ \phi(\tau) &= 0 & \text{on } \Sigma, \end{cases} \tag{12.18}$$

which admits maximal $L^{q'}$-regularity, cf. Chapter 10. Hence (12.18) admits a unique solution $\phi \in \mathbb{E}_{q'}^\Sigma(\tau)$, which is nonnegative by the same arguments employed in the proof of Lemma 11.2. This solution satisfies

$$\|\phi\|_{W^{1,q'}((0,\tau),L^{q'}(\Sigma))} + \|\phi\|_{L^{q'}((0,\tau),W^{2,q'}(\Sigma))} \leq C\|\theta\|_{L^{q'}(\Sigma_\tau)}. \tag{12.19}$$

Consequently we obtain $\phi \in BUC([0,\tau], L^{q'}(\Sigma))$ and

$$\|\phi(s)\|_{L^{q'}(\Sigma)} = \|\int_s^\tau \partial_t\phi dt\|_{L^{q'}(\Sigma)} \leq \int_0^\tau \|\partial_t\phi\|_{L^{q'}(\Sigma)} dt \leq T^{1/q}\|\partial_t\phi\|_{L^{q'}(\Sigma_\tau)} \tag{12.20}$$

for $s \in (0,\tau)$. Combining (12.19) and (12.20) we obtain

$$\sup_{s\in[0,\tau]} \|\phi(s)\|_{L^{q'}(\Sigma)} + \|\partial_t\phi\|_{L^{q'}(\Sigma_\tau)} + \mu\|\Delta_\Sigma\phi\|_{L^{q'}(\Sigma_\tau)} \leq C\|\theta\|_{L^{q'}(\Sigma_\tau)} \tag{12.21}$$

for a constant $C > 0$ depending on Σ, T, q, μ but not on $\tau < T$. Hence by plugging in the first line of (12.18) and multiple partial integrations – once in time and twice in space – we have

$$\begin{aligned}\int\limits_{\Sigma_\tau} u\theta d(t,\sigma) &= -\int\limits_{\Sigma_\tau} u[\partial_t\phi + \mu\Delta_\Sigma\phi]d(t,\sigma)\\ &= \int\limits_{\Sigma} u_0\phi(0)d\sigma + \int\limits_{\Sigma_\tau} \phi\partial_t u d(t,\sigma) + \mu\int\limits_0^\tau\int\limits_\Sigma \nabla_\Sigma\cdot\nabla_\Sigma\phi d\sigma dt\\ &= \int\limits_{\Sigma} u_0\phi(0)d\sigma + \int\limits_{\Sigma_\tau} \phi(\partial_t u - \mu\Delta_\Sigma u)d(t,\sigma).\end{aligned}$$

Due to the nonnegativity of ϕ and $f \le \sum_{j=1}^N g_j$ we may estimate as follows and perform partial integrations backwards:

$$\begin{aligned}\int\limits_{\Sigma_\tau} u\theta d(t,\sigma) &\le \int\limits_{\Sigma} u_0\phi(0)d\sigma + \sum_{j=1}^N \int\limits_{\Sigma_\tau} \phi(\alpha_j\partial_t v_j - \beta_j\Delta_\Sigma v_j)d(t,\sigma)\\ &= \int\limits_{\Sigma} (u_0 - \sum_{j=1}^N \alpha_j v_0^j)\phi(0)d\sigma + \sum_{j=1}^N \int\limits_{\Sigma_\tau} (-\alpha_j\partial_t\phi - \beta_j\Delta_\Sigma\phi)\, v_j d(t,\sigma).\end{aligned}$$

Therefore, by (12.21) and Hölder's inequality for $1/q + 1/q' = 1$ we obtain

$$\int\limits_{\Sigma_\tau} u\theta d(t,\sigma) \le C\|\theta\|_{L^{q'}(\Sigma_\tau)} \left(1 + \sum_{j=1}^N \|v_j\|_{L^q(\Sigma_\tau)}\right)$$

with a constant $C > 0$ depending merely on the constants occuring in (12.17). Employing

$$\|u^+\|_{L^q(\Sigma_T)} = \sup\left\{ \int\limits_0^T\int\limits_\Sigma u\theta\,\mathrm{d}\sigma(x)\,\mathrm{d}t \;:\; \theta \in L^{q'}(\Sigma_T)^+,\ \|\theta\|_{L^{q'}(\Sigma_T)} < 1 \right\}$$

the assertion follows. □

12.2. Proof of Theorem 8.3

Having the comparison principle and the weak-type estimates at hand, we are in a position to prove Theorem 8.3.

Proof of Theorem 8.3. Since according to Theorem 8.1 the (local-in-time) solutions to (12.1) generate a local semi-flow in the phase space which coincides with $\mathbb{I}_2^{\Omega} \times \mathbb{I}_2^{\Sigma} = H^1(\Omega) \times H^1(\Sigma)$, we may assume $T^* < \infty$ and show that the solution stays bounded in $H^1(\Omega) \times H^1(\Sigma)$ on $(0, T^*)$ in order to obtain a contradiction. It is sufficient to establish L^∞-bounds for the solution in order to obtain boundedness in the phase space. To this end we employ [BR, Theorem 4], which yields that solutions stay bounded in L^∞ if the nonlinearities are bounded in a certain L^q-norm for sufficiently large q. This is where the linear and polynomial growth of r^{sorp} and r^{ch} enter the game, due to which it is sufficient to show that solutions are bounded in an L^r-norm with sufficiently large r. Note in passing that this argument had been widely used for reaction-diffusion equations, cf. e.g. [BFPR], and goes back to [LSU68, Theorem III 7.1].

Then the H^1-boundedness of solutions follows as is shown in the last part of this proof. The uniqueness of global solutions is a consequence of uniqueness of local solutions.

We will now derive L^∞-bounds, which requires several steps. Note, however, that we may use the fact that $c_i,\, c_i^\Sigma \geq 0$ on $(0, T^*)$ thanks to Lemma 11.2.

Step 1. We have $c_i \in \mathbb{E}_2^{\Omega}(T) = H^1((0, T), L^2(\Omega)) \cap L^2((0, T), H^2(\Omega))$ and

$$\left\{\begin{array}{rcllr} \partial_t c_i + (u \cdot \nabla) c_i - d_i \Delta c_i &=& f_i & \text{in} \quad (0, T) \times \Omega, & \\ (u \cdot \nu) c_i - d_i \partial_\nu c_i &=& g_i^{\text{in}} & \text{on} \quad (0, T) \times \Gamma_{\text{in}}, & \\ -d_i \partial_\nu c_i &=& r_i^{\text{sorp}}(c_i,\, c_i^\Sigma) & \text{on} \quad (0, T) \times \Sigma, & (12.22) \\ -d_i \partial_\nu c_i &=& 0 & \text{on} \quad (0, T) \times \Gamma_{\text{out}}, & \\ c_i(0) &=& c_{0,i} & \text{in} \quad \Omega, & \end{array}\right.$$

for $i = 1, \ldots, N$ and all $0 < T < T^*$. Now, $r_i^{\text{sorp}}(c_i,\, c_i^\Sigma) \geq -k_i^{\text{de}} c_i^\Sigma$. Thus, by Lemma 12.1 (a) we have $0 \leq c_i \leq z_i$ for the unique maximal regular solution z_i to

$$\left\{\begin{array}{rcll} \partial_t z_i + (u \cdot \nabla) z_i - d_i \Delta z_i &=& f_i & \text{in} \quad (0, T) \times \Omega, \\ (u \cdot \nu) z_i - d_i \partial_\nu z_i &=& g_i^{\text{in}} & \text{on} \quad (0, T) \times \Gamma_{\text{in}}, \\ -d_i \partial_\nu z_i &=& -C c_i^\Sigma & \text{on} \quad (0, T) \times \Sigma, \\ -d_i \partial_\nu z_i &=& 0 & \text{on} \quad (0, T) \times \Gamma_{\text{out}}, \\ z_i(0) &=& c_{0,i} & \text{in} \quad \Omega, \end{array}\right.$$

with some appropriate constant $C = C((k_j^{\text{de}})_{j=1,\ldots,N}) > 0$. Note that this problem allows for a strong solution in the L^2-setting without any compatibility conditions between the right hand sides of the boundary conditions and the initial value. Since z_i is a solution to a linear problem, we may write $z_i = u_i + v_i$, where u_i

solves

$$\begin{cases} \partial_t u_i + (u \cdot \nabla) u_i - d_i \Delta u_i &= 0 && \text{in} \quad (0,\,T) \times \Omega, \\ (u \cdot \nu) u_i - d_i \partial_\nu u_i &= 0 && \text{on} \quad (0,\,T) \times \Gamma_{\text{in}}, \\ -d_i \partial_\nu u_i &= -C c_i^\Sigma && \text{on} \quad (0,\,T) \times \Sigma, \\ -d_i \partial_\nu u_i &= 0 && \text{on} \quad (0,\,T) \times \Gamma_{\text{out}}, \\ u_i(0) &= 0 && \text{in} \quad \Omega, \end{cases}$$

and v_i solves

$$\begin{cases} \partial_t v_i + (u \cdot \nabla) v_i - d_i \Delta v_i &= f_i && \text{in} \quad (0,\,T) \times \Omega, \\ (u \cdot \nu) v_i - d_i \partial_\nu v_i &= g_i^{\text{in}} && \text{on} \quad (0,\,T) \times \Gamma_{\text{in}}, \\ -d_i \partial_\nu v_i &= 0 && \text{on} \quad (0,\,T) \times \Sigma, \\ -d_i \partial_\nu v_i &= 0 && \text{on} \quad (0,\,T) \times \Gamma_{\text{out}}, \\ v_i(0) &= c_{0,i} && \text{in} \quad \Omega. \end{cases}$$

For these solutions we have

$$\|u_i\|_{L^q(\Omega_T)} + \|u_i\|_{L^q(\Sigma_T)} \le C' \|c_i^\Sigma\|_{L^q(\Sigma_T)}$$

as well as

$$\|v_i\|_{L^q(\Omega_T)} + \|v_i\|_{L^q(\Sigma_T)} \le A_i$$

provided that $2 \le q \le \infty$. Here, we employed Lemma 12.5 to obtain postitive constants $C' = C'(C, q)$ and $A_i = A_i(\|f\|_{L^q(\Omega_T)},\, \|g^{\text{in}}\|_{L^q(\Gamma_{\text{in},T})},\, \|c_{0,i}\|_{L^q(\Omega)},\, q)$ that are independent of $0 < T < T^*$. Since $\|c_i\|_{L^q} \le \|z_i\|_{L^q} \le \|u_i\|_{L^q} + \|v_i\|_{L^q}$ we may sum up the above estimates to obtain

$$\sum_{i=1}^N \|c_i\|_{L^q(\Omega_T)} + \sum_{i=1}^N \|c_i\|_{L^q(\Sigma_T)} \le C^* \left(1 + \sum_{j=1}^N \|c_j^\Sigma\|_{L^q(\Sigma_T)} \right), \tag{12.23}$$

where $C^* = C^*(C', (A_j)_{j=1,\dots,N}) > 0$ denotes a constant that is independent of $0 < T < T^*$. Note that this estimate is available for all $2 \le q \le \infty$.

Step 2. We have $c_i^\Sigma \in \mathbb{E}_2^\Sigma(T) = H^1((0,\,T),\, L^2(\Sigma)) \cap L^2((0,\,T),\, H^2(\Sigma))$ and

$$\begin{cases} \partial_t c_i^\Sigma - d_i^\Sigma \Delta_\Sigma c_i^\Sigma &= r_i^{\text{sorp}}(c_i,\, c_i^\Sigma) + r_i^{\text{ch}}(c^\Sigma) && \text{on} \quad (0,\,T) \times \Sigma, \\ -d_i^\Sigma \partial_{\nu_\Sigma} c_i^\Sigma &= 0 && \text{on} \quad (0,\,T) \times \partial\Sigma, \\ c_i^\Sigma(0) &= c_{0,i}^\Sigma && \text{on} \quad \Sigma, \end{cases} \tag{12.24}$$

for $i = 1,\, \dots,\, N$ and all $0 < T < T^*$. Now we use the triangular structure of the reaction rates that is guaranteed by ($\mathrm{A}_\mathrm{S}^{\mathrm{ch}}$) to treat the cases $i = 1$ and $i = 2,\, \dots,\, N$ separately.

Step 2.1. According to assumption ($\mathrm{A}_{\mathrm{B}}^{\mathrm{sorp}}$) we have $r_1^{\mathrm{sorp}}(c_1, c_1^\Sigma) \leq k_1^{\mathrm{ad}} c_1$ and according to ($\mathrm{A}_{\mathrm{S}}^{\mathrm{ch}}$) we have $q_{11} r_1^{\mathrm{ch}}(c^\Sigma) \leq C(1 + \sum_{j=1}^N c_j^\Sigma)$ for some fixed constant $C > 0$. Thus, by Lemma 12.1 (b) we have $0 \leq c_1^\Sigma \leq z_1^\Sigma$ for the unique solution z_1^Σ to

$$\left\{\begin{array}{rcll} \partial_t z_1^\Sigma - d_1^\Sigma \Delta_\Sigma z_1^\Sigma &=& C'\Big(1 + c_1 + \sum_{j=1}^N c_j^\Sigma\Big) & \text{on } (0, T) \times \Sigma, \\ -d_1^\Sigma \partial_{\nu_\Sigma} z_1^\Sigma &=& 0 & \text{on } (0, T) \times \partial\Sigma, \\ z_1^\Sigma(0) &=& c_{0,1}^\Sigma & \text{on } \Sigma, \end{array}\right.$$

with some appropriate constant $C' = C'(C, k_1^{\mathrm{ad}}, q_{11}) > 0$. Since z_1^Σ is a solution to a linear problem, we may write $z_1^\Sigma = u_1^\Sigma + v_1^\Sigma$, where u_1^Σ solves

$$\left\{\begin{array}{rcll} \partial_t u_1^\Sigma - d_1^\Sigma \Delta_\Sigma u_1^\Sigma &=& C'\Big(1 + c_1 + \sum_{j=1}^N c_j^\Sigma\Big) & \text{on } (0, T) \times \Sigma, \\ -d_1^\Sigma \partial_{\nu_\Sigma} u_1^\Sigma &=& 0 & \text{on } (0, T) \times \partial\Sigma, \\ u_1^\Sigma(0) &=& 0 & \text{on } \Sigma, \end{array}\right.$$

and v_1^Σ solves

$$\left\{\begin{array}{rcll} \partial_t v_1^\Sigma - d_1^\Sigma \Delta_\Sigma v_1^\Sigma &=& 0 & \text{on } (0, T) \times \Sigma, \\ -d_1^\Sigma \partial_{\nu_\Sigma} v_1^\Sigma &=& 0 & \text{on } (0, T) \times \partial\Sigma, \\ v_1^\Sigma(0) &=& c_{0,1}^\Sigma & \text{on } \Sigma. \end{array}\right.$$

For these solutions we have

$$\|u_1^\Sigma\|_{L^q(\Sigma_T)} \leq L \|u_1^\Sigma\|_{{}_0\mathbb{E}_p^\Sigma(T)} \leq MLC' \left(\Theta + \|c_1\|_{L^p(\Sigma_T)} + \sum_{j=1}^N \|c_j^\Sigma\|_{L^p(\Sigma_T)}\right)$$

with $\Theta = \Theta(p) = \|1\|_{L^p(\Sigma_{T^*})}$ as well as

$$\|(v_1^\Sigma)^+\|_{L^q(\Sigma_T)} \leq A_1,$$

provided that $2 \leq p,\, q < \infty$. Here, $L = L(p, q) > 0$ denotes the norm of the embedding ${}_0\mathbb{E}_p^\Sigma(T) \hookrightarrow L^q(\Sigma_T)$ (see Proposition 1.3) and $M = M(p) > 0$ denotes the norm of the solution operator in the L^p-setting for the time interval $(0, T)$, which are both independent of $0 < T < T^*$ thanks to the homogeneous initial condition, see Lemma 1.5. Moreover $A_1 = A_1(\|c_{0,1}^\Sigma\|_{BC(\Sigma)}, q) > 0$ denotes the constant delivered by Lemma 12.6, which is also independent of $0 < T < T^*$. Note that $0 \leq z_1^\Sigma \leq u_1^\Sigma + (v_1^\Sigma)^+$. Therefore,

$$\begin{aligned} \|c_1^\Sigma\|_{L^q(\Sigma_T)} &\leq \|z_1^\Sigma\|_{L^q(\Sigma_T)} \leq \|u_1^\Sigma\|_{L^q(\Sigma_T)} + \|(v_1^\Sigma)^+\|_{L^q(\Sigma_T)} \\ &\leq C''\left(1 + \|c_1\|_{L^p(\Sigma_T)} + \sum_{j=1}^N \|c_j^\Sigma\|_{L^p(\Sigma_T)}\right), \end{aligned}$$

provided that $2 \leq p,\, q < \infty$. Here $C'' = C''(MLC',\, \Theta,\, A_1)$ denotes a positive constant, which is independent of $0 < T < T^*$.

Step 2.2. Now fix $i \in \{\, 2,\, \ldots,\, N\,\}$. By (12.24) we obtain

$$q_{ii}(\partial_t c_i^\Sigma - d_i^\Sigma \Delta_\Sigma c_i^\Sigma) + \sum_{j<i} q_{ij}\left(\partial_t c_j^\Sigma - d_j^\Sigma \Delta_\Sigma c_j^\Sigma\right) = [Qr^{\mathrm{sorp}}(c,\, c^\Sigma)]_i + [Qr^{\mathrm{ch}}(\, c^\Sigma)]_i$$

on $(0,\, T) \times \Sigma$. According to assumption ($\mathrm{A}_\mathrm{B}^{\mathrm{sorp}}$) we have

$$[Qr^{\mathrm{sorp}}(c,\, c^\Sigma)]_i = \sum_{j\leq i} q_{ij} r_j^{\mathrm{sorp}}(c_j,\, c_j^\Sigma) \leq \sum_{j\leq i, q_{ij}>0} q_{ij} k_j^{\mathrm{ad}} c_j - \sum_{j\leq i, q_{ij}<0} q_{ij} k_j^{\mathrm{de}} c_j^\Sigma$$

and according to ($\mathrm{A}_\mathrm{S}^{\mathrm{ch}}$) we have $[Qr^{\mathrm{ch}}(\, c^\Sigma)]_i \leq C(1 + \sum_{j=1}^N c_j^\Sigma)$ for some fixed constant $C > 0$. Thus, by Lemma 12.1 (b) we have $0 \leq c_i^\Sigma \leq z_i^\Sigma$ for the unique maximal regular solution z_i^Σ to

$$\begin{cases} \partial_t z_i^\Sigma - d_i^\Sigma \Delta_\Sigma z_i^\Sigma &=& C'\Big(1 + \sum_{j\leq i} c_j + \sum_{j=1}^N c_j^\Sigma\Big) & & \\ & & \quad - \sum_{j<i} r_{ij}\left(\partial_t c_j^\Sigma - d_j^\Sigma \Delta_\Sigma c_j^\Sigma\right) & \text{in} & (0,\, T) \times \Sigma, \\ -d_i^\Sigma \partial_{\nu_\Sigma} z_i^\Sigma &=& 0 & \text{on} & (0,\, T) \times \partial\Sigma, \\ z_i^\Sigma(0) &=& c_{0,i}^\Sigma & \text{on} & \Sigma, \end{cases}$$

with some appropriate constant $C' = C'(C,\, k_j^{\mathrm{ad}},\, k_j^{\mathrm{de}},\, q_{ij}) > 0$ and $r_{ij} = q_{ij}/q_{ii}$. Since z_i^Σ is a solution to a linear problem, we may write $z_i^\Sigma = u_i^\Sigma + v_i^\Sigma$, where u_i^Σ solves

$$\begin{cases} \partial_t u_i^\Sigma - d_i^\Sigma \Delta_\Sigma u_i^\Sigma &=& C'\Big(1 + \sum_{j\leq i} c_j + \sum_{j=1}^N c_j^\Sigma\Big) & \text{in} & (0,\, T) \times \Sigma, \\ -d_i^\Sigma \partial_{\nu_\Sigma} u_i^\Sigma &=& 0 & \text{on} & (0,\, T) \times \partial\Sigma, \\ u_i^\Sigma(0) &=& 0 & \text{on} & \Sigma, \end{cases}$$

and v_i^Σ solves

$$\begin{cases} \partial_t v_i^\Sigma - d_i^\Sigma \Delta_\Sigma v_i^\Sigma &=& - \sum_{j<i} r_{ij}\left(\partial_t c_j^\Sigma - d_j^\Sigma \Delta_\Sigma c_j^\Sigma\right) & \text{in} & (0,\, T) \times \Sigma, \\ -d_i^\Sigma \partial_{\nu_\Sigma} v_i^\Sigma &=& 0 & \text{on} & (0,\, T) \times \partial\Sigma, \\ v_i^\Sigma(0) &=& c_{0,i}^\Sigma & \text{on} & \Sigma. \end{cases}$$

For these solutions we have

$$\|u_i^\Sigma\|_{L^q(\Sigma_T)} \leq L \|u_i^\Sigma\|_{{}_0\mathbb{E}_p^\Sigma(T)} \leq MLC' \left(\Theta + \sum_{j\leq i} \|c_j\|_{L^p(\Sigma_T)} + \sum_{j=1}^N \|c_j^\Sigma\|_{L^p(\Sigma_T)} \right)$$

with $\Theta = \Theta(p) = \|1\|_{L^p(\Sigma_{T^*})}$ as well as

$$\|(v_i^\Sigma)^+\|_{L^q(\Sigma_T)} \leq A_i \left(1 + \sum_{j<i} \|c_j^\Sigma\|_{L^q(\Sigma_T)} \right),$$

provided that $2 \le p, q < \infty$. Here, $L = L(p, q) > 0$ denotes the norm of the embedding ${}_0\mathbb{E}_p^\Sigma(T) \hookrightarrow L^q(\Sigma_T)$ and $M = M(p) > 0$ denotes the norm of the solution operator in the L^p-setting for the time interval $(0, T)$, which are both independent of $0 < T < T^*$ thanks to the homogeneous initial condition, see Lemma 1.5. Moreover

$$A_i = A_i((\|c_{0,j}^\Sigma\|_{BC(\Sigma)})_{j=1,\dots,i}, q) > 0$$

denotes the constant delivered by Lemma 12.6, which is also independent of $0 < T < T^*$. We again have $0 \le z_i^\Sigma \le u_i^\Sigma + (v_i^\Sigma)^+$. Therefore,

$$\begin{aligned}\|c_i^\Sigma\|_{L^q(\Sigma_T)} &\le \|z_i^\Sigma\|_{L^q(\Sigma_T)} \le \|u_i^\Sigma\|_{L^q(\Sigma_T)} + \|(v_i^\Sigma)^+\|_{L^q(\Sigma_T)} \\ &\le C''\left(1 + \sum_{j\le i}\|c_j\|_{L^p(\Sigma_T)} + \sum_{j=1}^N \|c_j^\Sigma\|_{L^p(\Sigma_T)}\right) + A_i \sum_{j<i}\|c_j^\Sigma\|_{L^q(\Sigma_T)},\end{aligned}$$

provided that $2 \le p, q < \infty$. Here $C'' = C''(MLC', \Theta, A_i)$ denotes a positive constant, which is independent of $0 < T < T^*$.

Step 2.3. Now we may combine the estimates obtained in Step 2.1 and Step 2.2 recursively and infer that

$$\sum_{i=1}^N \|c_i^\Sigma\|_{L^q(\Sigma_T)} \le C^*\left(1 + \sum_{j=1}^N \|c_j\|_{L^p(\Sigma_T)} + \sum_{j=1}^N \|c_j^\Sigma\|_{L^p(\Sigma_T)}\right) \tag{12.25}$$

provided that $2 \le p, q < \infty$. Here, $C^* = C^*(C'', (A_j)_{j=1,\dots,N}) > 0$ is independent of $0 < T < T^*$.

Step 3. We combine estimates (12.23) and (12.25) to obtain

$$\begin{aligned}&\sum_{i=1}^N \|c_i\|_{L^q(\Omega_T)} + \sum_{i=1}^N \|c_i\|_{L^q(\Sigma_T)} + \sum_{i=1}^N \|c_i^\Sigma\|_{L^q(\Sigma_T)} \\ &\qquad \le C^*\left(1 + \sum_{j=1}^N \|c_j\|_{L^p(\Omega_T)} + \sum_{j=1}^N \|c_j\|_{L^p(\Sigma_T)} + \sum_{j=1}^N \|c_j^\Sigma\|_{L^p(\Sigma_T)}\right),\end{aligned} \tag{12.26}$$

provided that $2 \le p, q < \infty$. Here, $C^* > 0$ is independent of $0 < T < T^*$. Using this inequality for $p = 2$ we may obtain L^q-L^2-estimates for arbitrary $2 \le q < \infty$.

Step 4.1. The surface concentrations satisfy

$$\begin{cases} \partial_t c_i^\Sigma - d_i^\Sigma \Delta_\Sigma c_i^\Sigma &= f_i^\Sigma \quad \text{on} \quad (0,T)\times\Sigma, \\ -d_i^\Sigma \partial_{\nu_\Sigma} c_i^\Sigma &= 0 \quad \text{on} \quad (0,T)\times\partial\Sigma, \\ c_i^\Sigma|_{t=0} &= c_{0,i}^\Sigma \quad \text{on} \quad \Sigma, \end{cases}$$

with $f_i^\Sigma := r_i^{\text{sorp}}(c_i, c_i^\Sigma) + r_i^{\text{ch}}(c^\Sigma)$. Due to the polynomial growth of the nonlinearities we may estimate f^Σ in terms of c and c^Σ and employ (12.26) to obtain

$$\sum_{i=1}^N \|f_i^\Sigma\|_{L^r(\Sigma_T)} \le C^* \left(1 + \sum_{j=1}^N \|c_j\|_{L^p(\Omega_T)} + \sum_{j=1}^N \|c_j\|_{L^p(\Sigma_T)} + \sum_{j=1}^N \|c_j^\Sigma\|_{L^p(\Sigma_T)}\right),$$

where $2 \le p$, $r < \infty$ and $C^* > 0$ is independent of $0 < T < T^*$. Thus, for given $2 \le q \le \infty$ we may use this estimate for sufficiently large $2 \le r < \infty$ together with a classical result from [LSU68], which yields the estimate

$$\sum_{i=1}^N \|c_i^\Sigma\|_{L^q(\Sigma_T)} \le C^* \left(1 + \sum_{j=1}^N \|c_j\|_{L^p(\Omega_T)} + \sum_{j=1}^N \|c_j\|_{L^p(\Sigma_T)} + \sum_{j=1}^N \|c_j^\Sigma\|_{L^p(\Sigma_T)}\right). \tag{12.27}$$

Let us note that [LSU68, Theorem III.7.1] is stated for Dirichlet boundary conditions, but the result remains true in the Neumann case; see [BR, Theorem 4], whose proof carries over to smooth manifolds such as Σ. Note that in contrast to (12.25) obtained in the second step, the estimate (12.27) is available for all $2 \le p < \infty$ and all $2 \le q \le \infty$, while $C^* > 0$ is still independent of $0 < T < T^*$.

Step 4.2. Now we combine estimates (12.23) and (12.27) to obtain

$$\begin{aligned}\sum_{i=1}^N \|c_i\|_{L^q(\Omega_T)} &+ \sum_{i=1}^N \|c_i\|_{L^q(\Sigma_T)} + \sum_{i=1}^N \|c_i^\Sigma\|_{L^q(\Sigma_T)} \\ &\le C^* \left(1 + \sum_{j=1}^N \|c_j\|_{L^p(\Omega_T)} + \sum_{j=1}^N \|c_j\|_{L^p(\Sigma_T)} + \sum_{j=1}^N \|c_j^\Sigma\|_{L^p(\Sigma_T)}\right),\end{aligned} \tag{12.28}$$

provided that $2 \le p < \infty$ and $2 \le q \le \infty$. Here, $C^* > 0$ is independent of $0 < T < T^*$. Using this inequality for $p = 2$ we may in particular obtain L^q-L^2-estimates for arbitrary $2 \le q \le \infty$.

Step 5.1. The estimate (12.28) applied for $p = 2$ and $q = \infty$ yields

$$\begin{aligned}\sum_{i=1}^N \|c_i(t)\|_{L^2(\Omega)}^2 &+ \sum_{i=1}^N \|c_i(t)\|_{L^2(\Sigma)}^2 + \sum_{i=1}^N \|c_i^\Sigma(t)\|_{L^2(\Sigma)}^2 \\ &\le C^* \Bigg(1 + \sum_{j=1}^N \int_0^t \|c_j(s)\|_{L^2(\Omega)}^2 \, ds + \sum_{j=1}^N \int_0^t \|c_j(s)\|_{L^2(\Sigma)}^2 \, ds \\ &\qquad + \sum_{j=1}^N \int_0^t \|c_j^\Sigma(s)\|_{L^2(\Sigma)}^2 \, ds\Bigg)\end{aligned}$$

for all $0 < t < T < T^*$, where $C^* > 0$ is independent of $0 < T < T^*$. Thus, a standard Gronwall argument (Lemma 1.11) implies

$$\|c_i\|_{L^2(\Omega_T)},\ \|c_i\|_{L^2(\Sigma_T)},\ \|c_i^\Sigma\|_{L^2(\Sigma_T)} \le M e^{\omega T}, \qquad 0 < T < T^*, \tag{12.29}$$

with some constants M, $\omega > 0$, which are independent of $0 < T < T^*$.

Step 5.2. The estimate (12.28) again applied for $p = 2$ and $q = \infty$ together with (12.29) yields

$$\|c_i\|_{L^\infty(\Omega_T)},\ \|c_i\|_{L^\infty(\Sigma_T)},\ \|c_i^\Sigma\|_{L^\infty(\Sigma_T)} \leq Me^{\omega T}, \qquad 0 < T < T^*, \quad (12.30)$$

with some constants M, $\omega > 0$, which are independent of $0 < T < T^*$.

Step 6. Now the obtained a priori estimates (12.30) carry over from L^∞ to $H^1(\Omega)$ and $H^1(\Sigma)$. This may be seen by the following argument: Due to the L^∞-estimates, the L^2-solution of (8.1) is contained in $\mathbb{E}_p^\Omega(T) \times \mathbb{E}_p^\Sigma(T)$ for each $1 < p < \infty$ with $p \neq 3$ by bootstrapping. Here the crucial estimate is

$$\|r_i^{\text{sorp}}(c_i, c_i^\Sigma)\|_{\mathbb{G}_p^\Sigma(T)} \leq C\|(c_i, c_i^\Sigma)\|_{\mathbb{E}_q^\Omega(T)\times\mathbb{E}_q^\Sigma(T)}$$

with suitable $q < p$, see the proof of Theorem 8.1. This, in turn, yields

$$c_i \in BC(\overline{\Omega_T}), \qquad c_i^\Sigma \in BC(\overline{\Sigma_T}), \qquad (T < T^*)$$

due to Proposition 1.3. Hence by plugging in c, c^Σ into r^{sorp}, r^{ch}, we may consider (8.1) again, as a linear problem, this time for data being continuous in time. More precisely, we consider (10.1) for data

$$\begin{aligned}
f_i &\in BC([0,T^*), L^2(\Omega)), & f_i^\Sigma &\in BC([0,T^*), L^2(\Sigma)),\\
g_i^{\text{in}} &\in BC([0,T^*), L^2(\Gamma_{\text{in}})), & g_i^\Sigma &\in BC([0,T^*), L^2(\Sigma)),\\
g_i^{\text{out}} &\in BC([0,T^*), L^2(\Gamma_{\text{out}})),
\end{aligned}$$

and

$$c_{0,i} \in H^1(\Omega), \qquad c_{0,i}^\Sigma \in H^1(\Sigma).$$

Following the strategy of the proof of our linear result in Chapter 10, in particular by transferring Lemma 10.5, in a very similar manner we obtain that the unique solution of (8.1) satisfies

$$c_i \in BC([0,T^*), H^1(\Omega)), \qquad c_i^\Sigma \in BC([0,T^*), H^1(\Sigma)),$$

and the corresponding a priori estimates

$$\begin{aligned}
\sup_{t\in[0,T]} (\|c_i(t)\|_{H^1(\Omega)} + \|c_i^\Sigma(t)\|_{H^1(\Sigma)}) \leq Me^{\omega T} \sup_{t\in[0,T]} \big(&\|f_i(t)\|_{L^2(\Omega)} + \|f_i^\Sigma(t)\|_{L^2(\Sigma)}\\
&+\|g_i^{\text{in}}(t)\|_{L^2(\Gamma_{\text{in}})} + \|g_i^\Sigma(t)\|_{L^2(\Sigma)} + \|g_i^{\text{out}}(t)\|_{L^2(\Gamma_{\text{out}})}\big), \qquad (T < T^*)
\end{aligned} \quad (12.31)$$

for constants $M, \omega > 0$ independent of T. Hence, we may pass to the limit $T \to T^*$ and see that both sides of (12.31) stay finite. The proof of Theorem 8.3 is now complete. □

Remark 12.7 (Sorption and reaction examples) *A few remarks on the examples given in the introduction concerning the assumptions (A_F^{sorp}), (A_M^{sorp}), (A_B^{sorp}), (A_F^{ch}), (A_N^{ch}), (A_P^{ch}) and (A_S^{ch}) stated in Chapter 11 and Chapter 12 are in order here. Evidently, Henry's law (S1) satisfies all of our assumptions. However, Langmuir's law (S2) needs to be modified in order to meet all assumptions on the sorption. To this end we introduce ζ^+, a smooth cut-off function, which approximates $(\cdot)^+$ pointwise and ζ^B a smooth and bounded function with bounded derivatives up to order 2, which is monotonically increasing. In addition suppose $\zeta^+(0) = 0$ and $\zeta^B(0) = 0$. Then we consider*

$$\tilde{r}_{L,i}^{sorp}(c_i, c_i^\Sigma) = k_i^{ad}\zeta^B(c_i)\zeta^+\left(1 - \frac{\zeta^+(c_i^\Sigma)}{c_{\infty,i}^\Sigma}\right) - k_i^{de}c_i^\Sigma,$$

which indeed satisfies (A_F^{sorp}), (A_M^{sorp}) and (A_B^{sorp}) and therefore is covered by our main results. This modification is only necessary due to technical reasons, since

- *(A_F^{sorp}) is violated due to $\nabla r_{L,i}^{sorp} \notin BC^1(\mathbb{R}^2, \mathbb{R}^2)$,*
- *(A_M^{sorp}) is not guaranteed since $c_i^\Sigma \leq c_{\infty,i}^\Sigma$ is not postulated and*
- *for the same reason (A_B^{sorp}) is not satisfied since $(1 - c_i^\Sigma/c_{\infty,i}^\Sigma)$ could be negative.*

For the time being it is not clear to the author whether there are still global solutions in case we omit the cut-off functions. Observe that in our model there is no maximal capacity on the active surface, which is required in the original Langmuir law (S2) to gain nonnegativity of concentrations. Nonnegativity in turn is employed in the proof of the global existence result.

The reaction rate r_R^{ch} given in (R1) satisfies (A_F^{ch}), admits quadratic growth (A_P^{ch}), is quasi-positive (A_N^{ch}) and respects the triangular structure (A_S^{ch}) with corresponding matrix

$$Q_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Remark 12.8 *There are some questions left open. To the best of the author's knowledge time asymptotics and equilibria of (12.1) are unknown up to now. We only remark that to this end the stationary equations have to be studied for nonconstant solutions, which seems to be quite involved.*
Another open question arises when studiyng a biquadratic reaction rate – say for $A_1^*, ..., A_4^*$ – rate instead of the one from above, which admits "only" linear backward rates. The biquadratic reaction $A_1^* + A_2^* \rightleftharpoons A_3^* + A_4^*$ admits quadratic forward and backwards rates of the form $r_i^{ch}(c^\Sigma) = \sigma_i(-c_1^\Sigma c_2^\Sigma + c_3^\Sigma c_4^\Sigma)$ with a sign vector $\sigma = (-1, -1, 1, 1)$. Reaction-diffusion systems with such reaction rates are known to admit global weak solutions in any space dimension. In one and two space dimensions solutions stay bounded if the initial data is bounded. However

in three space dimensions blow ups could occur in finite time, [Pie10, Problem 3] and [GV09]. There only an estimate for the Hausdorff dimension of the set of blow-up points is known. Therefore a guess would be that an analogue of our global existence result also holds for the biquadratic case, since the reaction on Σ *takes place on a two-dimensional submanifold of* $\mathbb{R}^3$*.*

Summary

In this thesis we consider two nonlinear time-dependent systems of partial differential equations and investigate their well-posedness. The first system stems from fluid mechanics and is treated in Part I. The second one originates from chemical engineering and is content of Part II. Both systems are studied on three dimensional domains with nonsmooth boundaries for strong solutions. In case of the second system we give a global-in-time analysis.

I. Stokes- and Navier-Stokes Equations with Perfect Slip on Wedge Type Domains

In the first part we consider the Stokes- and Navier-Stokes equations subject to perfect slip boundary conditions on a three dimensional domain of wedge type $G = S_{\varphi_0} \times \mathbb{R}$ with

$$S_{\varphi_0} := \left\{(x_1, x_2) = (r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2 : \ r > 0, \ 0 < \varphi < \varphi_0\right\}.$$

Let $T > 0$ be given and let $t \in (0, T)$ and $(x, y) = (x_1, x_2, y) \in G$. In the following ν denotes the outer normal at the boundary ∂G of the domain G. For a given external force $f = f(t, x)$ and a given initial velocity field $u_0 = u_0(x)$ we aim to find a unique velocity field $u = u(t, x)$ and a corresponding pressure $p = p(t, x)$ – which is unique up to a constant – that satisfy the Stokes equations

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u + \nabla p &=& f & \text{in} \quad (0,T) \times G, \\ \operatorname{div} \mathrm{u} &=& 0 & \text{in} \quad (0,T) \times G, \\ \nu \times \operatorname{curl} u = 0, \quad u \cdot \nu &=& 0 & \text{on} \quad (0,T) \times \partial G, \\ u(0) &=& u_0 & \text{in} \quad G, \end{array}\right. \tag{13.1}$$

respectively the Navier-Stokes equations which formally arise from (13.1) by adding the convection term $(u \cdot \nabla)u$ in the first line. We intend to prove that the Navier-Stokes system corresponding to (13.1) is strongly L^p-well-posed – locally in time – for nontrivial angles $\varphi_0 \in (0, \pi)$ of the wedge. We choose L^p_γ as a ground space, which is standard in this setting. The main results of Part I are given through Theorem 3.1, Corollary 3.2 und Theorem 3.3 and may be summarized as:

- Let $1 < p < \infty$, $\gamma \in \mathbb{R}$ and $\varphi_0 \in (0, \pi)$ satisfy the condition

$$\min \left\{ 1, \left(\frac{\pi}{\varphi_0} - 1 \right)^2 \right\} > \left(2 - \frac{2 + \gamma}{p} \right)^2 . \tag{13.2}$$

  Then the Stokes operator $\mathcal{A}_S$ defined as in Theorem 3.1 admits a bounded $\mathcal{H}^\infty$-calculus on $L^p_{\sigma,\gamma}(G)$ with $\phi^\infty_{\mathcal{A}_S} < \pi/2$.

- Under the assumptions from above the Stokes equations given in (13.1) admit maximal L^p_γ-regularity.

- Let $\gamma = 0$, $\varphi_0 \in (0, \frac{5}{9}\pi)$, $p \in (\frac{5}{3}, \frac{2}{3-\pi/\varphi_0})$. For given $T \in (0, \infty)$ the Navier-Stokes equations associated to (13.1) admit a unique strong L^p-solution on $(0, T)$ for sufficiently small initial data.

Strategy of the proof: We proceed in several steps. In a first step we transform the resolvent problem associated to (13.1) to a three dimensional layer of height φ_0 by employing polar respectively cylinder coordinates and the Euler transformation $r = e^x$. By introducing a parameter β and choosing it appropriately we may manipulate the weight such that the new ground space on the layer $\Omega = \mathbb{R}^2 \times (0, \varphi_0)$ is the unweighted $L^p(\Omega, \mathbb{R}^3)$. Now we apply the operator sum method we introduced in Chapter 2 to this transformed problem. In particular we make use of Proposition 2.29 for the case of non-commuting operators and apply it to e^{2x} and $-(\partial_x^2 + (2\beta)\partial_x + \beta^2)$, see also [PSS07, Theorem 3.1]. For this it is crucial that the Labbas-Terreni commutator condition is satisfied. Altogether the results for the intersection operators carry over to the operator sum, i.e. to the whole transformed Laplacian up to a shift. A result on invertibility by Prüß [Prü93, Theorem 8.5] shows that we get rid of this shift if a certain spectral condition is satisfied. This condition leads to (13.2). Since the employed transformation is an isomorphism the $\mathcal{H}^\infty$-calculus carries over to the Laplacian on G in the weighted setting.
In a third step we make use of the perfect slip boundary conditions, which are curcial for the approach chosen here: The Laplacian subject to these boundary conditions and the Helmholtz projection commute, cf. Mitrea und Monniaux [MM09a, MM09b]. Hence resolvent estimates and consequently the $\mathcal{H}^\infty$-calculus carry over to the Stokes operator, which yields the first main result. Then the maximal regularity of (13.1) follows from a corollary to the Dore-Venni Theorem. Finally, an application of the contraction mapping principle yields the strong L^p-well-posedness of the Navier-Stokes equations locally in time for $\gamma = 0$, corresponding angles φ_0 and a small p-interval.

II. Global Solutions for a Class of Heterogeneous Catalysis Models

For a finite cylinder $\Omega \subset \mathbb{R}^3$ with $\Gamma_{\text{in}}, \Sigma, \Gamma_{\text{out}}$ denoting circular bottom, lateral surface and top, we consider the heterogeneous catalyst equations

$$\begin{cases} \partial_t c_i + (u \cdot \nabla) c_i - d_i \Delta c_i &= 0 & \text{in} \quad (0,T) \times \Omega, \\ \partial_t c_i^\Sigma - d_i^\Sigma \Delta_\Sigma c_i^\Sigma &= r_i^{\text{sorp}}(c_i, c_i^\Sigma) + r_i^{\text{ch}}(c^\Sigma) & \text{on} \quad (0,T) \times \Sigma, \\ (u \cdot \nu) c_i - d_i \partial_\nu c_i &= g_i^{\text{in}} & \text{on} \quad (0,T) \times \Gamma_{\text{in}}, \\ -d_i \partial_\nu c_i &= r_i^{\text{sorp}}(c_i, c_i^\Sigma) & \text{on} \quad (0,T) \times \Sigma, \\ -d_i \partial_\nu c_i &= 0 & \text{on} \quad (0,T) \times \Gamma_{\text{out}}, \\ -d_i^\Sigma \partial_{\nu_\Sigma} c_i^\Sigma &= 0 & \text{on} \quad (0,T) \times \partial\Sigma, \\ c_i(0) &= c_{0,i} & \text{in} \quad \Omega, \\ c_i^\Sigma(0) &= c_{0,i}^\Sigma & \text{on} \quad \Sigma, \end{cases} \tag{13.3}$$

with $i = 1, ..., N$. Concentrations of species $X_1, ..., X_N$ are denoted by $c_1, ..., c_N$, while $c_1^\Sigma, ..., c_N^\Sigma$ stand for their adsorbed surface concentration counter parts. Here r_i^{ch} denotes the reaction rate of the catalysis and r_i^{sorp} the sorption rate, which is given as the difference of adsorption and desorption rate. We study (13.3) for local-in-time strong L^p-well-posedness and global-in-time strong L^2-well-posedness. In particular for a given velocity field $u = u(t,x)$, a given mass feed $g_i^{\text{in}} = g_i^{\text{in}}(t,x)$ and given initial concentrations $c_{0,i} = c_{0,i}(x)$ and $c_{0,i}^\Sigma = c_{0,i}^\Sigma(x)$ we aim to find a vector of concentrations $(c_1, ..., c_N)$ with $c_i = c_i(t,x)$ and a vector of surface concentrations $(c_1^\Sigma, ..., c_N^\Sigma)$ with $c_i^\Sigma = c_i^\Sigma(t,x)$ that satisfy (13.3) in the L^p-respectively L^2-sense.
Among others the nonlinear sorption and reaction rates r_i^{sorp} and r_i^{ch} satisfy the following assumptions: The sorption rate is supposed to be monotonically increasing in the first and monotonically decreasing in the second argument, it admits linear bounds. The reaction rate is quasi-positive, admits polynomial growth and satisfies a triangular structural condition. The main results in Part II are given by Proposition 10.1, Theorem 8.1 and Theorem 8.3 and may be summarized as:

- The fully inhomogeneous linear system admits maximal L^p-regularity provided $p \in (5/3, \infty)$ and $p \neq 3$.
- System (13.3) admits a unique local and strong L^p-solution for $5/3 < p < \infty$ and $p \neq 3$ in case of small times and arbitrary data.
- System (13.3) admits a unique global strong L^2-solution for arbitrary data.

Strategy of the proof: In order to treat the associated inhomogeneous linear problem we employ cylindrical L^p-theory [Nau13] and solve the diffusion equations subject to Neumann boundary conditions. We show the surjectivity of the Neumann trace operator with respect to the maximal regularity classes on the

cylinder by reflection in axial direction. As a consequene we obtain the solvability of the inhomogeneous diffusion equations – but only without the terms $(u \cdot \nu)c_i$ and $(u \cdot \nabla)c_i$. In the next step the latter are treated as perturbation terms. Hence a standard Neumann series argument yields the well-posedness of the fully inhomogeneous linear system including advection.
The nonlinear system (13.3) is then solved for $5/3 < p < \infty$ and $p \neq 3$ via maximal regularity of the linear system and the contraction mapping principle. Nonnegativity of the concentrations c_i and the surface concentrations c_i^Σ follows from the quasi-positivity of r^{ch}, the monotonicity and the linear bounds of r_i^{sorp}, as well as the suitable sign of the data and $u \cdot \nu$ on the different boundary parts Γ_{in}, Σ, Γ_{out}.
In order to show the global well-posedness of (13.3) we make use of three lemmas: a linear comparison principle to estimate solutions against each other in accordance to their data, linear L^p-estimates for $2 \leq p \leq \infty$, as well as an estimate of the nonnegative part of c_i^Σ based on a combination of a duality argument and maximal regularity, cf. [Pie10]. With these three lemmas at hand we are able to prove the global existence theorem: To this end we assume that the maximal time interval of existence of the local L^2-solution is finite, i.e. $T^* < \infty$ and show that the solution stays bounded in the phase space $H^1(\Omega) \times H^1(\Sigma)$ as $T \to T^*$. For this purpose it is sufficient to show a priori L^∞-bounds for the solution on Ω_T respectively Σ_T.

Zusammenfassung

In der vorliegenden Dissertation werden zwei nicht-lineare zeitabhängige Systeme partieller Differentialgleichungen auf Wohlgestelltheit untersucht. Das erste System stammt aus der Strömungsmechanik und wird in Teil I behandelt. Das zweite System entspringt dem Chemieingenieurswesen und ist Gegenstand von Teil II. Beide Systeme werden auf dreidimensionalen nicht-glatten Gebieten auf eindeutige starke Lösungen untersucht – im Falle des zweiten Systems global in der Zeit.

I. Navier-Stokes-Gleichungen mit "perfect slip"-Randbedingungen auf einem Keilgebiet

Im ersten Teil werden die Stokes- und Navier-Stokes-Gleichungen mit "perfect slip"-Randbeding- ungen auf einem drei-dimensionalen Keilgebiet $G = S_{\varphi_0} \times \mathbb{R}$ mit

$$S_{\varphi_0} := \big\{(x_1, x_2) = (r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2 : \; r > 0, \; 0 < \varphi < \varphi_0\big\}$$

betrachtet. Seien $T > 0$ gegeben, $t \in (0, T)$ und $(x, y) = (x_1, x_2, y) \in G$. Im Folgenden bezeichne ν die äußere Normale an den Rand ∂G des Gebiets G. Für eine gegebene äußere Kraft $f = f(t, x)$ und ein gegebenes Anfangsgeschwindigkeitsfeld $u_0 = u_0(x)$ ist ein eindeutiges Geschwindigkeitsfeld $u = u(t, x)$ und ein zugehöriger – bis auf eine Konstante eindeutig bestimmter – Druck $p = p(t, x)$ gesucht, welche die Stokes-Gleichungen

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u + \nabla p &=& f & \text{in} \quad (0,T)\times G, \\ \operatorname{div} \mathrm{u} &=& 0 & \text{in} \quad (0,T)\times G, \\ \nu \times \operatorname{curl} u = 0, \quad u\cdot\nu &=& 0 & \text{auf} \quad (0,T)\times \partial G, \\ u(0) &=& u_0 & \text{in} \quad G, \end{array}\right. \tag{14.1}$$

bzw. die zugehörigen Navier-Stokes-Gleichungen, welche formal aus der ersten Zeile durch Hinzufügen des Konvektionsterms $(u\cdot\nabla)u$ hervorgehen, lösen. Wir möchten beweisen, dass das zu (14.1) gehörende Navier-Stokes-System für "nicht-triviale" Öffnungswinkel $\varphi_0 \in (0, \pi)$ des Keils zeitlich lokal im starken L^p-Sinne wohlgestellt ist. Als Grundraum wählen wir den für solche Probleme üblichen L^p_γ. Die Hauptresultate von Teil I sind gegeben durch Theorem 3.1, Korollar 3.2 und Theorem 3.3 und lauten zusammengefasst:

- Erfüllen $1 < p < \infty$, $\gamma \in \mathbb{R}$ und $\varphi_0 \in (0, \pi)$ die Bedingung

$$\min\left\{1, \left(\frac{\pi}{\varphi_0} - 1\right)^2\right\} > \left(2 - \frac{2+\gamma}{p}\right)^2, \tag{14.2}$$

  so besitzt der wie in Theorem 3.1 definierte Stokes-Operator $\mathcal{A}_S$ einen beschränkten $\mathcal{H}^\infty$-Kalkül auf $L^p_{\sigma,\gamma}(G)$ mit $\phi^\infty_{\mathcal{A}_S} < \pi/2$.

- Unter den Voraussetzungen wie oben besitzen die Stokes-Gleichungen (14.1) maximale L^p_γ-Regularität.

- Seien $\gamma = 0$, $\varphi_0 \in (0, \frac{5}{9}\pi)$ und $p \in (\frac{5}{3}, \frac{2}{3-\pi/\varphi_0})$. Für gegebenes $T \in (0, \infty)$ besitzen die zu (14.1) gehörenden Navier-Stokes-Gleichungen genau eine starke L^p-Lösung auf $(0, T)$ für hinreichend kleine Anfangsdaten.

Vorgehen: Wir gehen in mehreren Schritten vor. In einem ersten Schritt transformieren wir das zu (14.1) gehörige Resolventenproblem mittels Polar- bzw. Zylinderkoordinaten und der Euler-Transformation $r = e^x$ auf eine dreidimensionale Schicht der Höhe φ_0. Durch passende Wahl eines zum Ausgleich des Gewichts eingeführten Parameters β ergibt sich als Grundraum auf der Schicht $\Omega = \mathbb{R}^2 \times (0, \varphi_0)$ der ungewichtete $L^p(\Omega, \mathbb{R}^3)$. Auf das so erhaltene transformierte Problem wird in einem zweiten Schritt die in Kapitel 2 vorgestellte Operatorsummenmethode angewendet. Insbesondere wird Proposition 2.29, siehe auch [PSS07, Theorem 3.1] für den Fall nicht-kommutierender Operatoren auf e^{2x} und $-(\partial_x^2 + (2\beta)\partial_x + \beta^2)$ angewendet. Entscheidend ist hierbei, dass die Labbas-Terreni-Bedingung erfüllt ist. Insgesamt übertragen sich somit bekannte Resultate für die Querschnittsoperatoren auf die Operatorsumme, d.h. also auf den gesamten transformierten Laplace-Operator bis auf eine Verschiebung. Ein Resultat über Invertierbarkeit von Prüß [Prü93, Theorem 8.5] zeigt, dass auf diese Verschiebung verzichtet werden kann, sofern eine gewisse Spektralbedingung erfüllt ist, die auf (14.2) führt. Da die benutzte Transformation ein Isomorphismus ist, überträgt sich der $\mathcal{H}^\infty$-Kalkül auf den Laplace-Operator auf G in den gewichteten Räumen.
In einem dritten Schritt werden die "perfect slip"-Randbedingungen ausgenutzt, welche entscheidend für diesen Zugang sind: Da der Laplace-Operator bezüglich "perfect-slip"- Randbedingungen und die Helmholtz-Projektion kommutieren, vgl. Mitrea und Monniaux [MM09a], [MM09b], übertragen sich die Resolventenabschätzungen und somit auch der $\mathcal{H}^\infty$-Kalkül auf den Stokes-Operator, was somit das erste Hauptresultat liefert. Aus einem Korollar zum Satz von Dore-Venni folgt damit bereits die maximale Regularität von (14.1). Letztlich folgt aus einer Anwendung des Banach'schen Fixpunktsatzes die zeitlich lokale L^p-Wohlgestelltheit der Navier-Stokes-Gleichungen für die Wahl $\gamma = 0$, passende Öffnungswinkel φ_0 und ein kleines zugehöriges p-Intervall.

II. Globale Lösungen für eine Klasse von Modellen zur Heterogenen Katalyse

Für einen endlichen Zylinder $\Omega \subset \mathbb{R}^3$ mit rundem Boden Γ_{in}, Mantel Σ und Deckel Γ_{out} betrachten wir die Gleichungen zur Heterogenen Katalyse

$$\begin{cases} \partial_t c_i + (u \cdot \nabla)c_i - d_i \Delta c_i &= 0 & \text{in} & (0,T) \times \Omega, \\ \partial_t c_i^\Sigma - d_i^\Sigma \Delta_\Sigma c_i^\Sigma &= r_i^{\text{sorp}}(c_i, c_i^\Sigma) + r_i^{\text{ch}}(c^\Sigma) & \text{auf} & (0,T) \times \Sigma, \\ (u \cdot \nu)c_i - d_i \partial_\nu c_i &= g_i^{\text{in}} & \text{auf} & (0,T) \times \Gamma_{\text{in}}, \\ -d_i \partial_\nu c_i &= r_i^{\text{sorp}}(c_i, c_i^\Sigma) & \text{auf} & (0,T) \times \Sigma, \\ -d_i \partial_\nu c_i &= 0 & \text{auf} & (0,T) \times \Gamma_{\text{out}}, \\ -d_i^\Sigma \partial_{\nu_\Sigma} c_i^\Sigma &= 0 & \text{on} & (0,T) \times \partial\Sigma, \\ c_i(0) &= c_{0,i} & \text{in} & \Omega, \\ c_i^\Sigma(0) &= c_{0,i}^\Sigma & \text{auf} & \Sigma, \end{cases} \tag{14.3}$$

wobei $i = 1, ..., N$. Die Konzentrationen der chemischen Spezies $X_1, ..., X_N$ seien mit $c_1, ..., c_N$ bezeichnet, während $c_1^\Sigma, ..., c_N^\Sigma$ für die zugehörigen adsorbierten Oberflächenkonzentrationen stehen. Dabei bezeichnen r_i^{ch} die Reaktionsraten der Katalyse und r_i^{sorp} die Sorptionsraten, die sich als Differenz aus Adsorptions- und Desorptionsraten ergeben. Wir untersuchen (14.3) auf zeitlich lokale starke L^p-Wohlgestelltheit und zeitlich globale starke L^2-Wohlgestelltheit. Insbesondere ist für ein gegebenes Geschwindigkeitsfeld $u = u(t,x)$, eine gegebene Massenzufuhr $g_i^{\text{in}} = g_i^{\text{in}}(t,x)$ und gegebene Anfangskonzentrationen $c_{0,i} = c_{0,i}(x)$ und $c_{0,i}^\Sigma = c_{0,i}^\Sigma(x)$ ein Vektor von Konzentrationen $(c_1, ..., c_N)$ mit $c_i = c_i(t,x)$ und ein Vektor von Oberflächenkonzentrationen $(c_1^\Sigma, ..., c_N^\Sigma)$ mit $c_i^\Sigma = c_i^\Sigma(t,x)$ gesucht, welcher (14.3) im L^p- bzw. L^2-Sinne erfüllt.
Unter anderem sollen die nicht-linearen Sorptions- und Reaktionsraten r_i^{sorp} und r_i^{ch} folgende Annahmen erfüllen: Die Sorptionsrate ist monoton wachsend im ersten und monoton fallend im zweiten Argument. Sie besitzt lineare Schranken. Die Reaktionsrate ist quasi-positiv, besitzt polynomielles Wachstum und genügt einer Dreiecksstruktur-Bedingung. Die Hauptresultate von Teil II sind gegeben durch Proposition 10.1, Theorem 8.1 und Theorem 8.3 und lauten zusammengefasst:

- Das voll-inhomogene lineare Katalysatorsystem besitzt maximale L^p-Regularität für $p \in (5/3, \infty)$ und $p \neq 3$.
- Das System (14.3) besitzt genau eine lokale, starke L^p-Lösung für $p \in (5/3, \infty)$ und $p \neq 3$ im Falle kleiner Zeiten und beliebiger Anfangsdaten.
- Das System (14.3) besitzt genau eine globale starke L^2-Lösung für beliebige Anfangsdaten.

Vorgehen: Um das inhomogene lineare Problem zu behandeln, verwenden wir zunächst zylindrische L^p-Theorie [Nau13] und lösen die homogenen Neumann-Diffusionsprobleme. Wir zei-gen die Surjektivität des Neumann-Spuroperators bzgl. der Maximale-Regularitätsklassen auf dem Zylinder durch eine Spiegelung in axiale Richtung. Dadurch erhalten wir die Lösbarkeit des inhomogenen Diffusionsproblems – ohne die Terme $(u{\cdot}\nu)c_i$ bzw. $(u{\cdot}\nabla)c_i$. Diese werden anschließend als Störterme behandelt. Ein Standard-Neumann-Reihen-Argument liefert somit die Wohlgestelltheit des voll-inhomogenen linearen Systems.
Das nicht-lineare System (14.3) wird für $5/3 < p < \infty$ und $p \neq 3$ mittels maximaler Regulartität den linearen Gleichungen und des Banach'schen Fixpunktsatzes gelöst. Die Nichtnegativität der Konzentrationen c_i und Oberflächenkonzentrationen c_i^Σ folgt aus der Quasi-Positivität von r^{ch}, der Monotonie und der linearen Schranken von r_i^{sorp}, sowie den richtigen Vorzeichen der Daten als auch der Spuren $u \cdot \nu$ auf den Randstücken Γ_{in}, Σ, Γ_{out}.
Für die globale Wohlgestelltheit von (14.3) benötigen wir drei Hilfsaussagen: ein lineares Vergleichsprinzip um Lösungen gegeneinander gemäß ihrer Daten abzuschätzen, lineare L^p- Abschätzungen für $2 \leq p \leq \infty$ sowie eine auf einer Kombination aus einem Dualitätsargument und maximaler Regularität beruhende Abschätzung des nicht-negativen Teils von c_i^Σ, vgl. [Pie10]. Mittels dieser Hilfssätze lässt sich der globale Existenzsatz beweisen: Dazu nehmen wir an, das aus dem lokalen Existenzsatz erhaltene maximale Existenzintervall ist endlich, d.h. $T^* < \infty$, und zeigen, dass die L^2-Lösung in der Norm des Phasenraums $H^1(\Omega) \times H^1(\Sigma)$ beschränkt bleibt für $T \to T^*$. Dazu reicht es, a priori L^∞-Schranken für die Lösung in Ω_T bzw. auf Σ_T nachzuweisen.

Bibliography

[AF03] R. A. Adams and J. J. F. Fournier. *Sobolev Spaces*, volume 140 of *Pure and Applied Mathematics*. Academic Press, 2 edition, 2003.

[Ama09] H. Amann. *Anisotropic Function Spaces and Maximal Regularity for Parabolic Problems. Part 1: Function Spaces.*, volume 6. Jindřich Nečas Center for Mathematical Modeling Lecture Notes Prague, 2009.

[Ama15] H. Amann. Uniformly Regular and Singular Riemannian Manifolds. In *Elliptic and Parabolic Equations*, volume 119 of *Springer Proceedings in Mathematics and Statistics*, pages 1–43. Springer International Publishing, 2015.

[Are04] W. Arendt. Semigroups and Evolution Equations: Functional Calculus, Regularity and Kernel Estimates. In: C.M. Dafermos, et al. (Ed.), Evolutionary Equations, *Handb. Differ. Equ.*, vol. I, Elsevier/North-Holland, Amsterdam, pages 1–85. 2004.

[Ari75] R. Aris. *The Mathematical Theory of Diffusion and Reaction in Permeable Catalysts*, volume I/II. Claredon Press, 1975.

[AZ90] J. Appell and P. P. Zabrejko. *Nonlinear Superposition Operators*, volume 95 of *Camebridge Tracts in Mathematics*. Cambridge University Press, 1990.

[Bat00] G. K. Batchelor. *An Introduction to Fluid Dynamics*. Cambridge Mathematical Library. Cambridge University Press, 2000.

[BD15] D. Bothe and W. Dreyer. Continuum Thermodynamics of Chemically Reacting Fluid Mixtures. *Acta Mechanica*, 226(6):1757–1805, 2015.

[BDM03] C. Bernardi, M. Dauge, and Y. Maday. Polynomials in the Sobolev World. *Preprint of the Laboratoire Jacques-Louis Lions - LJLL , Institut de Recherche Mathématique de Rennes - IRMAR*, R03038, 2003.

[BFPR] D. Bothe, A. Fischer, M. Pierre, and G. Rolland. Global Wellposedness for a Class of Reaction-Advection-Anisotropic-Diffusion Systems. *In preparation, arXiv:1602.02798.*

[BKMS] D. Bothe, M. Köhne, S. Maier, and J. Saal. Global Well-Posedness of a Heterogeneous Catalysis Model. *In preparation, arXiv:1510.06195v1*.

[BL76] J. Bergh and J. Löfström. *Interpolation Spaces: An Introduction*. Grundlehren der mathematischen Wissenschaften. Springer, 1976.

[BM86] M. E. Bogovskiĭ and V. N. Maslennikova. Elliptic Boundary Value Problems in Unbounded Domains with Noncompact and Nonsmooth Boundaries. *Rendiconti del Seminario Matematico e Fisico di Milano*, 56(1):125–138, 1986.

[BM88] W. Borchers and T. Miyakawa. L^2 Decay for the Navier-Stokes Flow in Halfspaces. *Mathematische Annalen*, 282(1):139–155, 1988.

[BMOS] D. Bothe, J. Málek, V. Orava, and O. Souček. Heterogeneous Catalysis – Modeling and Analysis. *In preparation*.

[Bot01] D. Bothe. Periodic Solutions of a Nonlinear Evolution Problem from Heterogeneous Catalysis. *Differential and Integral Equations*, 14:641–670, 2001.

[Bot11] D. Bothe. On the Maxwell-Stefan Approach to Multicomponent Diffusion. In *Parabolic Problems*, volume 80 of *Progress in Nonlinear Differential Equations and Their Applications*, pages 81–93. Springer Basel, 2011.

[BR] D. Bothe and G. Rolland. Global Existence for a Class of Reaction-Diffusion Systems with Mass Action Kinetics and Concentration-Dependent Diffusivities. *Acta Applicandae Mathematicae*, pages 1–33. DOI 10.1007/s10440-014-9968-y, 2014.

[BS98] S. Bonafede and D. Schmitt. Triangular Reaction Diffusion Systems with Integrable Initial Data. *Non-Linear Anal.*, 33(7):785–801, 1998.

[CP01] P. Clément and J. Prüss. An Operator-Valued Transference Principle and Maximal Regularity on Vector-Valued L_p-spaces. *Evolution Equations and Their Applications in Physical and Life Sciences (Bad Herrenalb, 1998)*, 215:67–87, 2001.

[DGH$^+$11] R. Denk, M. Geißert, M. Hieber, J. Saal, and O. Sawada. The Spin-Coating Process: Analysis of the Free Boundary Value Problem. *Commun. Partial Differ. Equations*, 36:1145–1192, 2011.

[DHP03] R. Denk, M. Hieber, and J. Prüss. *$\mathcal{R}$-Boundedness and Problems of Elliptic and Parabolic Type*, volume 166, No. 788. Memoirs of the AMS, 2003.

[DHP07] R. Denk, M. Hieber, and J. Prüss. Optimal Lp-Lq-Estimates for Parabolic Boundary Value Problems with Inhomogeneous Data. *Mathematische Zeitschrift*, 257(1):193–224, 2007.

[DPZ08] R. Denk, J. Prüss, and R. Zacher. Maximal -Regularity of Parabolic Problems with Boundary Dynamics of Relaxation Type. *Journal of Functional Analysis*, 255(11):3149 – 3187, 2008.

[DSS08] R. Denk, J. Saal, and J. Seiler. Inhomogeneous Symbols, the Newton Polygon, and Maximal Lp-Regularity. *Russian J. Math. Phys.*, 15(2):171–192, 2008.

[Duo90] X. T. Duong. H_∞ Functional Calculus of Elliptic Operators with C^∞ Coefficients on L^p Spaces of Smooth Domains. *J. Austral. Math. Soc. Series A*, 48:113–123, 1990.

[DV87] G. Dore and A. Venni. On the Closedness of the Sum of Two Closed Operators. *Mathematische Zeitschrift*, 196:189–202, 1987.

[Esp95] J. H. Espenson. *Chemical Kinetics and Reaction Mechanisms*. Advanced Chemistry Series. McGraw-Hill, 1995.

[Fic55] A. Fick. Ueber Diffusion. *Annalen der Physik*, 170(1):59–86, 1855.

[Fis13] A. Fischer. *Well-Posedness and Asymptotic Behavior in Reactive and Electro-Kinetic Flow Processes* . Berichte aus der Mathematik. Shaker, Aachen, 2013. Dissertation Technische Universität Darmstadt.

[FV97] A. Friedman and J. L. Velázquez. Time-Dependent Coating Flows in a Strip. I: The Linearized Problem. *Trans. Am. Math. Soc.*, 349:2981–3074, 1997.

[Gal11] G. P. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Steady-State Problems*. Monographs in Mathematics. Springer, 2011.

[GV09] T. Goudon and A. Vasseur. Regularity Analysis for Systems of Reaction-Diffusion Equations. *Ann. Sci. École Norm. Sup.*, 43(1), 2009.

[Haa06] M. Haase. *The Functional Calculus for Sectorial Operators*. Operator Theory Advances and Applications. Springer, Basel, 2006.

[HJ91] U. Hornung and W. Jäger. Diffusion, Convection, Adsorption and Reaction of Chemicals in Porous Media. *J. Differential Equations*, 92:199–225, 1991.

[HMPW] M. Herberg, M. Meyries, J. Prüss, and M. Wilke. Reaction-Diffusion Systems of Maxwell-Stefan Type with Reversible Mass-Action Kinetics. *In preparation, arXiv:1310.4723*.

[Kei13] F. Keil. Complexities in Modeling of Heterogeneous Catalytic Reactions. *Computers and Mathematics with Applications*, 65:1674–1697, 2013.

[KM13] H. Knüpfer and N. Masmoudi. Well-Posedness and Uniform Bounds for a Nonlocal Third Order Evolution Operator on an Infinite Wedge. *Communications in Mathematical Physics*, 320(2):395–424, 2013.

[KM15] H. Knüpfer and N. Masmoudi. Darcy's Flow with Prescribed Contact Angle: Well-Posedness and Lubrication Approximation. *Archive for Rational Mechanics and Analysis*, 218(2):589–646, 2015.

[KO00] P. Knabner and F. Otto. Solute Transport in Porous Media with Equilibrium and Non-Equilibrium Multiple-Site Adsorption: Uniquenes of Weak Solutions. *Nonlinear Analysis*, 42:381–403, 2000.

[Köh13] M. Köhne. *L_p-Theory for Incompressible Newtonian Flows: Energy Preserving Boundary Conditions, Weakly Singular Domains*. Springer Spektrum, Wiesbaden, 2013. Dissertation Technische Universität Darmstadt.

[KS] M. Köhne and J. Saal. Multiplication and Analytic Nemytskij Operators in Anisotropic Bessel-Potential and Sobolev Slobodeckij Spaces. *In preparation*.

[KW01] N. J. Kalton and L. Weis. The H^∞-Calculus and Sums of Closed Operators. *Math. Ann.*, 321:319–345, 2001.

[KW04] P. Kunstmann and L. Weis. Maximal L^p-Regularity for Parabolic Equations, Fourier Multiplier Theorems and H^∞-Functional Calculus. In *Functional Analytic Methods for Evolution Equations*, volume 1855 of *Lecture Notes in Mathematics*, pages 65–311. Springer Berlin Heidelberg, 2004.

[Lan18] I. Langmuir. The Adsorption of Gases on Plane Surfaces of Glass, Mica and Platinum. *Journal of the American Chemical Society*, 40(9):1361–1403, 1918.

[Lev99] O. Levenspiel. *Chemical Reaction Engineering (3^{rd} edition)*. Wiley, 1999.

[LSU68] O. A. Ladyzhenskaia, V. A. Solonnikov, and N. N. Ural'tseva. *Linear and Quasi-linear Equations of Parabolic Type*. American Mathematical Society, Translations of Mathematical Monographs. American Mathematical Society, 1968.

[LT87] R. Labbas and B. Terreni. Sommes d'opérateurs linéaires de type parabolique. *Boll. Unione Mat. Ital.*, 7:545–569, 1987.

[Mar87] J. Marschall. The Trace of Sobolev-Slobodeckij Spaces on Lipschitz Domains. *Manuscripta Mathematica*, 58(1-2):47–65, 1987.

[Mas96] R. I. Masel. *Principles of Adsorption and Reaction on Solid Surfaces*. Wiley Series in Chemical Engineering. Wiley, 1996.

[McC81] M. McCracken. The Resolvent Problem for the Stokes Equations on Halfspace in L_p. *SIAM Journal on Mathematical Analysis*, 12(2):201–228, 1981.

[McI86] A. McIntosh. Operators which have an H^∞-Functional Calculus. *Mini-Conference on Operator Theory and Partial Differential equations*, pages 210–231, 1986.

[MM09a] M. Mitrea and S. Monniaux. On the Analyticity of the Semigroup Generated by the Stokes Operator with Neumann-type Boundary Conditions on Lipschitz Subdomains of Riemannian Manifolds. *Transactions of The American Mathematical Society*, 361:3125–3157, 2009.

[MM09b] M. Mitrea and S. Monniaux. The Nonlinear Hodge-Navier-Stokes Equations in Lipschitz Domains. *Differential and Integral Equations*, 22:339–356, 2009.

[Mor89] J. Morgan. Global Existence for Semilinear Parabolic Systems. *SIAM J. Math. Anal.*, 20(5):1128–1144, September 1989.

[MP92] R. H. Martin and M. Pierre. Nonlinear Reaction-Diffusion Systems. *Nonlinear Equations in the Applied Sciences, Math. Sci. Ehg.*, 185:363–398, 1992.

[MS00] C. Martinez and M. Sanz. The Theory of Fractional Powers of Operators. volume 187 of *North-Holland Mathematics Studies*. North-Holland, 2000.

[MS06] F. Marpeau and M. Saad. Mathematical Analysis of Radionuclides Displacement in Porous Media with Nonlinear Adsorption. *J. Differential Equations*, 228:412–439, 2006.

[MS14] S. Maier and J. Saal. Stokes and Navier–Stokes Equations with Perfect Slip on Wedge Type Domains. *Discrete and Continuous Dynamical Systems, Series S*, 7(5):1045–1063, 2014.

[Nau12] T. Nau. *L^p-Theory of Cylindrical Boundary Value Problems. An Operator-Valued Fourier Multiplier and Functional Calculus Approach.* Research. Wiesbaden: Springer Spektrum. xvi, 188 p. , 2012. Dissertation Universität Konstanz.

[Nau13] T. Nau. The Laplacian on Cylindrical Domains. *Integral Equations and Operator Theory*, 75(3):409–431, 2013.

[Naz01] A. I. Nazarov. L_p-Estimates for a Solution to the Dirichlet Problem and to the Neumann Problem for the Heat Equation in a Wedge with Edge of Arbitrary Codimension. *J. Math. Sci.*, 106:2989–3014, 2001.

[NS03] A. Noll and J. Saal. H^∞-Calculus for the Stokes Operator on Lq-Spaces. *Math. Z.*, 244:651–688, 2003.

[NS11] T. Nau and J. Saal. $\mathcal{R}$-sectoriality of Cylindrical Boundary Value Problems. *Parabolic problems. The Herbert Amann Festschrift to the occasion of his 70th birthday. Basel: Birkhäuser. Progress in Nonlinear Differential Equations and Their Applications 80, 479-505.*, 2011.

[NS12] T. Nau and J. Saal. H^∞-Calculus for Cylindrical Boundary Value Problems. *Adv. Differ. Equ.*, 17(7-8):767–800, 2012.

[PG75] G. Da Prato and P. Grisvard. Sommes d'opérateurs linéaires et équations différentielles opérationnelles. *J. Math. Pures Appl.*, 54, no. 3(9):305–387, 1975.

[Pie10] M. Pierre. Global Existence in Reaction-Diffusion Systems with Control of Mass: A Survey. *Milan J. Math.*, 78:417–455, 2010.

[Prü93] J. Prüss. *Evolutionary Integral Equations and Applications*. Monographs in Mathematics, 87. Birkhäuser Verlag, Basel, 1993.

[PS90] J. Prüss and H. Sohr. On Operators with Bounded Imaginary Powers in Banach Spaces. *Math. Z.*, 203:429–452, 1990.

[PS00] M. Pierre and D. Schmitt. Blowup in Reaction-Diffusion Systems with Dissipation of Mass. *SIAM Review*, 42(1):93–106, 2000.

[PS07] J. Prüss and G. Simonett. H^∞-Calculus for the Sum of Non-Commuting Operators. *Trans. Amer. Math. Soc*, 359:3549–3565, 2007.

[PSS07] J. Prüss, J. Saal, and G. Simonett. Existence of Analytic Solutions for the Classical Stefan Problem. *Math. Ann.*, 338:703–755, 2007.

[PSSS12] J. Prüss, S. Shimizu, Y. Shibata, and G. Simonett. On Well-Posedness of Incompressible Two-Phase Flows with Phase Transitions: The Case of Equal Densities. *Evolution Equations and Control Theory*, 1:171–194, 2012.

[Rot84] F. Rothe. *Global Solutions of Reaction-Diffusion Systems*. Lecture Notes in Mathematics. Springer, 1984.

[Rot08] G. Rothenberg. *Catalysis: Concepts and Green Applications*. Wiley, 2008.

[Saa03] J. Saal. *Robin Boundary Conditions and Bounded H^∞-Calculus for the Stokes Operator*. Logos-Verlag, 2003. Dissertation Technische Universität Darmstadt.

[Saa07] J. Saal. Strong Solutions to the Navier-Stokes Equations in Bounded and Unbounded Domains with a Moving Boundary. *Electron. J. Diff. Eqns.,*, (15):365–375, 2007.

[Sch01] B. Schweizer. A Well-Posed Model for Dynamic Contact Angles. *Nonlinear Anal., Theory Methods Appl.*, 43:109–125, 2001.

[See71] R. Seeley. Norms and Domains of the Complex Powers A_B^z. *American Journal of Mathematics*, 93(2):299–309, 1971.

[Seg13] T. Seger. *Elliptic-Parabolic System with Applications to Lithium–Ion Battery Models*. Bibliothek der Universität Konstanz, 2013. Dissertation Universität Konstanz.

[SFH89] J. I. Steinfeld, J. S. Francisco, and W. L. Hase. *Chemical Kinetics and Dynamics*. Prentice Hall, 1989.

[Sic96] W. Sickel. Composition Operators Acting on Sobolev Spaces of Fractional Order - A Survey on Sufficient and Necessary Conditions. *Conf. Proceedings: Function Spaces, Differential Operators and Nonlinear Analysis*, pages 159–182, 1996.

[Sob64] P. E. Sobolevskii. Coerciveness Inequalities for Abstract Parabolic Equations. *Transl.: Soviet. Math. (Doklady)*, 5:894–897, 1964.

[Sob75] P. E. Sobolevskii. Fractional Powers of Coercively Positive Sums of Operators. *Soviet Math. Dokl.*, 6:1638–1641, 1975.

[Sol95] V. A. Solonnikov. On Some Free Boundary Problems for the Navier-Stokes Equations with Moving Contact Points and Lines. *Math. Ann.*, 302:743–772, 1995.

[TM05] R. Temam and A. Miranville. *Mathematical Modeling in Continuum Mechanics*. Cambridge University Press, 2005.

[TR00] C. Truesdell and K. R. Rajagopal. *An Introduction to the Mechanics of Fluids*. Modeling and Simulation in Science, Engineering & Technology. Springer, 2000.

[Tri78] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators*. North-Holland Mathematical Library. North-Holland Publishing Company, 1978.

[Ven93] A. Venni. A Counterexample Concerning Imaginary Powers of Linear Operators. *Functional Analysis and Related Topics*, 1540:381–387, 1993.

[Wei01a] L. Weis. A New Approach to Maximal L_p-Regularity. *Evolution Equations and Their Applications in Physical and Life Sciences (Bad Herrenalb, 1998)*, 215:195–214, 2001.

[Wei01b] L. Weis. Operator-Valued Fourier Multiplier Theorems and Maximal L_p-Regularity. *Math. Ann.*, 319:735–758, 2001.

[Whi90] M. G. White. *Heterogeneous Catalysis*. Prentice-Hall, 1990.

[Yag84] A. Yagi. Coïncidence entre des espaces d'interpolation et des domaines de puissances fractionnaires d'opérateurs. *C.R. Acad. Sci. Paris*, 299:173–176, 1984.

Index

List of Symbols

Maximal Regularity Spaces

Operators and Mappings

Transformations