# Dimension results for operator semistable Lévy processes

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# Abstract

Operator semistable Lévy processes are stochastic processes with a selfsimilarity property on a discrete scale. They generalize the better known class of (operator) stable Lévy processes, which have a continuous scaling property. Stochastic processes with certain scaling and selfsimilarity properties have applications in different scientific areas. In particular, they prove to be a useful tool for developing adequate mathematical models. They are applied, for instance, in order to describe natural dynamic processes in physics or pricing formulas in financial mathematics. In the latter case, selfsimilar processes can be seen as an improvement compared to well established models such as the Black-Scholes model, as they allow taking large jumps and long-term dependencies into account. In literature one can find numerous examples for the determination of the Hausdorff and other fractal dimensions for deterministic selfsimilar sets (on a discrete scale), e.g. for Cantor sets or Sierpinski gaskets. However, there do not exist many results on dimension properties for Lévy processes with a scaling or selfsimilarity property on a discrete scale yet.

This cumulative thesis examines dimension properties of operator semistable Lévy processes $X = \{X(t) : t \geq 0\}$ in $\mathbb{R}^d$ with exponent $E$, where $E$ is an invertible linear operator on $\mathbb{R}^d$. The thesis consists of three manuscripts on the subject. In the first manuscript, the Hausdorff dimension of the range and the graph of a stochastic process generated by the limit distribution of the cumulative gains in a series of St. Petersburg games is calculated over the time interval $[\frac{1}{2}, 1]$. This distribution can be defined as a continuous transformation of a non-strictly, semistable distribution. Furthermore, the Hausdorff dimension $\dim_H Gr_X(B)$ for the graph of an arbitrary operator semistable Lévy process $X$ in $\mathbb{R}^d$ and any Borel set $B \subseteq \mathbb{R}_+$ is calculated in the second manuscript by interpreting the graph $\mathrm{Gr}_X(B) = \{(t, X(t)) : t \in B\}$ as a semi-selfsimilar process in $\mathbb{R}^{d+1}$, whose distribution is not full. The Hausdorff dimension is expressed in terms of the real parts of the eigenvalues of the exponent $E$ and the Hausdorff dimension of $B$. In the third manuscript, the results on the path behavior of certain operator semistable Lévy processes are refined by the investigation of exact Hausdorff measure functions. In particular, for the range of certain operator semistable Lévy processes with a partially diagonal exponent exact Hausdorff measure functions are calculated over the time interval $[0, 1]$.

# Zusammenfassung


Operator-semistabile Lévy-Prozesse sind stochastische Prozesse mit einer Selbstähnlichkeitseigenschaft auf einer diskreten Skala. Sie stellen eine Verallgemeinerung der besser bekannten Klasse der operator-stabilen Lévy-Prozesse dar. Stochastische Prozesse mit gewissen Skalierungs- und Selbstähnlichkeitseigenschaften finden Anwendung in verschiedenen wissenschaftlichen Bereichen. Insbesondere haben sie sich als ein nützliches Werkzeug bei der Entwicklung von adäquaten mathematischen Modellen erwiesen. Sie werden beispielsweise verwendet, um natürliche dynamische Prozesse in der Physik oder Preismodelle in der Finanzmathematik zu beschreiben. Im letzteren Fall kann die Verwendung selbstähnlicher Prozesse als eine Verbesserung etablierter Methoden wie des Black-Scholes-Modells gesehen werden, da sie es ermöglichen, langfristige Abhängigkeiten und große Sprünge zu berücksichtigen. In der Literatur finden sich zahlreiche Beispiele für die Bestimmung der Hausdorff- und anderer frakataler Dimensionen von deterministischen selbstähnlichen Mengen, wie beispielsweise die der Cantor-Menge oder des Sierpinski-Dreiecks. Bis heute existieren jedoch wenige Resultate zu Dimensionseigenschaften von Lévy-Prozessen mit einer Skalierungs- oder Selbstähnlichkeitseigenschaft auf einer diskreten Skala.

Die vorliegende kumulative Dissertation untersucht Dimensionseigenschaften von operator-semistabilen Lévy-Prozessen $X = \{X(t) : t \geq 0\}$ in $\mathbb{R}^d$ mit Exponent $E$, wobei es sich bei $E$ um einen invertierbaren linearen Operator auf $\mathbb{R}^d$ handelt. Die Dissertation enthält drei Manuskripte zu diesem Thema. Im ersten Manuskript wird die Hausdorff-Dimension des Bildes und des Graphen eines stochastischen Prozesses über dem Zeitinterval $[\frac{1}{2}, 1]$ berechnet, der von der Grenzverteilung der kumulierten Gewinne in einer Reihe von St. Petersburg-Spielen generiert wird. Diese Verteilung kann als kontinuierliche Transformation einer nicht-strikten, semistabilen Verteilung definiert werden. Zudem wird im zweiten Manuskript eine allgemeine Formel für die Hausdorff-Dimension $\dim_H Gr_X(B)$ des Graphen eines operator-semistabilen Lévy-Prozesses für eine beliebige Borel-Menge $B \subseteq \mathbb{R}_+$ aufgestellt. Dies wird erreicht, indem der Graph $\mathrm{Gr}_X(B) = \{(t, X(t)) : t \in B\}$ als semi-selbstähnlicher Prozess definiert wird, dessen Verteilung jedoch nicht voll ist. Die Hausdorff-Dimension wird in Abhängigkeit der Realteile der Eigenwerte des Exponenten $E$ und der Hausdorff-Dimension von $B$ ausgedrückt. Im dritten Manuskript werden exakte Hausdorff-Maß-Funktionen für gewisse operator-semistabile Lévy-Prozesse mit teilweise diagonalem Exponenten $E$ über dem Zeitintervall $[0, 1]$ ermittelt.

# Contents

# Chapter 1

# Introduction

## 1.1 General information

Stochastic processes with certain scaling and selfsimilarity properties have applications in different scientific fields. In particular, they prove to be a useful tool for developing adequate mathematical models. They are applied, for instance, in order to describe natural dynamic processes in physics or pricing formulas in financial mathematics. In the latter case, selfsimilar processes can be seen as an improvement compared to well established models such as the Black-Scholes model, as they allow taking large jumps and long-term dependencies into account.

In this thesis, I examine a certain class of selfsimilar stochastic processes, the so called operator semistable Lévy processes, in terms of their fractal properties. More precisely, the stochastic processes under consideration are selfsimilar on a discrete scale. In the literature, one can find numerous examples for the determination of the Hausdorff and other fractal dimensions for deterministic selfsimilar sets (on a discrete scale), e.g. for Cantor sets or Sierpinski gaskets. However, until now, there do not exist many results on dimension properties for Lévy processes with a scaling or selfsimilarity property on a discrete scale. For this reason, the aim of the present thesis is to close some of these gaps and generalize already existing results for other classes of Lévy processes on operator semistable processes. In particular, this thesis focuses on the determination of the Hausdorff dimension for the graph of an arbitrary operator semistable Lévy process as well as the calculation of formulas for exact Hausdorff measure functions for the range of a certain class of operator semistable Lévy processes.

A Lévy process $X$ in $\mathbb{R}^d$ is a stochastically continuous process with cádlág paths and stationary and independent increments which starts in $X(0) = 0$ almost surely. The distribution of $X$ is then uniquely determined by the distribution of $X(1)$. The Lévy process $X$ is called $(c^E, c)$-operator semistable if the distribution of $X(1)$ is full, i.e. not supported on any lower dimensional hyperplane, and there exists a linear operator $E$ on $\mathbb{R}^d$ and some $c > 1$ such that

$$\{X(ct)\}_{t\geq 0} \stackrel{\text{fd}}{=} \left\{c^E X(t)\right\}_{t\geq 0}, \tag{1.1}$$

where $\stackrel{\text{fd}}{=}$ denotes the equality of all finite-dimensional distributions and

$$c^E := \sum_{n=0}^{\infty} \frac{(\log c)^n}{n!} E^n.$$

The linear operator $E$ is referred to as the exponent of the operator semistable Lévy process $X$. In case the exponent $E$ is a multiple of the identity, i.e. $E = \frac{1}{\alpha} \cdot I$ for some $\alpha \in (0, 2]$, the process $X$ is called $(c^{1/\alpha}, c)$-semistable. If (1.1) holds for all $c > 0$, the Lévy process is called operator stable. For a comprehensive overview on operator semistable distributions, I refer to the monographs [45] and [50].
This thesis is structured as follows: Section 1.2 contains general definitions and known results that will be useful throughout this thesis. Namely, it states definitions for the Hausdorff and box-counting dimension and introduces exact Hausdorff measure functions for arbitrary Borel sets $F \subseteq \mathbb{R}^d$. I also recall spectral decomposition results from [45] which make it possible to decompose an operator semistable Lévy process according to the distinct real parts of the eigenvalues of the exponent $E$. The section further contains certain uniformity results for the density functions of an operator semistable Lévy process and gives a definition for the expected sojourn times. Section 1.3 contains an overview on existing dimension results for operator stable and semistable Lévy processes and summarizes the results of this thesis. The next chapters contain these results in the form of three enclosed manuscripts titled "Dimension results related to the St. Petersburg game", "Hausdorff dimension of the graph of an operator semistable Lévy process" and "On exact Hausdorff measure functions of operator semistable Lévy processes", respectively. Finally, Chapter 5 concludes this thesis and gives an outlook on possible future lines of research.

## 1.2 Preliminaries

### 1.2.1 Fractal dimensions

Of the many existing fractal dimensions the Hausdorff dimension is probably one of the oldest and most important ones. For an arbitrary subset $F$ of $\mathbb{R}^d$ the $s$-dimensional Hausdorff measure $\mathcal{H}^s(F)$ is defined as

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} |F|_i^s : |F_i| \leq \delta \text{ and } F \subseteq \bigcup_{i=1}^{\infty} F_i \right\},$$

where $|F| = \sup\{\|x - y\| : x, y \in F\}$ denotes the diameter of a subset $F \subseteq \mathbb{R}^d$ and $\|\cdot\|$ is the Euclidean norm. It can be shown that the value $\dim_H F = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}$ exists and is unique for all $F \subseteq \mathbb{R}^d$. The critical value $\dim_H F$ is

called the Hausdorff dimension of $F$. For more details on the Hausdorff dimension compare [20] and [43].
Furthermore, a function $\phi$ belongs to the class $\Phi$ if there exists a constant $\delta > 0$ such that $\phi$ is right continuous and increasing on the open interval $(0, \delta)$, $\phi(0+) = 0$ and fulfills the doubling property. More precisely, there exists a constant $K > 0$ such that

$$\frac{\phi(2s)}{\phi(s)} \leq K \quad \text{for all } 0 < s < \frac{1}{2}\delta. \tag{1.2}$$

For an arbitrary Borel set $F \subseteq \mathbb{R}^d$ and a function $\phi \in \Phi$ the $\phi$-Hausdorff measure is then defined as

$$\phi - m(F) = \liminf_{\epsilon \to \infty} \left\{ \sum_{i=1}^{\infty} \phi(|F_i|) : F \subseteq \bigcup_{i=1}^{\infty} F_i, |F_i| < \epsilon \right\}. \tag{1.3}$$

The function $\phi \in \Phi$ is called an exact Hausdorff measure function for $F \subseteq \mathbb{R}^d$ if $0 < \phi - m(F) < \infty$.
As mentioned before, the Hausdorff dimension is just one of the many fractal dimensions which are frequently used. An alternative is the so called box-counting dimension (for more details, see [20]). For an arbitrary subset $F \subseteq \mathbb{R}^d$ denote by $N_\delta(F)$ the smallest number of closed balls of radius $\delta > 0$ that cover $F$. Then, the lower and the upper box-counting dimension of $F$ are given by

$$\underline{\dim}_{\mathrm{B}} F = \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \quad \text{and} \quad \overline{\dim}_{\mathrm{B}} F = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

and the box-counting dimension is defined as

$$\dim_{\mathrm{B}} F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

provided that this limit exists. The fractal dimensions defined in this section are related as follows:

$$\dim_{\mathrm{H}} F \leqslant \underline{\dim}_{\mathrm{B}} F \leqslant \overline{\dim}_{\mathrm{B}} F \leqslant d.$$

Note that there are many known examples for which the above inequalities are strict.

### 1.2.2 Spectral decomposition

Let $X$ be a $(c^E, c)$-operator semistable Lévy process. You can now factor the minimal polynomial of $E$ into $q_1(x) \cdot \ldots \cdot q_p(x)$ where all roots of $q_i$ have real parts equal to $a_i$

and $a_i \neq a_j$ for $i \neq j$. Without loss of generality, one can additionally assume that $a_i < a_j$ for $i < j$. Note that by Theorem 7.1.10 in [45] $a_j \geq \frac{1}{2}$ for all $j \in \{1, \ldots p\}$. Define $V_j = \mathrm{Ker}(q_j(E))$. According to Theorem 2.1.14 in [45] $V_1 \oplus \cdots \oplus V_p$ is a direct sum decomposition of $\mathbb{R}^d$ into $E$ invariant subspaces. In an appropriate basis, $E$ is then block-diagonal and we may write $E = E_1 \oplus \cdots \oplus E_p$ where $E_j : V_j \to V_j$ and every eigenvalue of $E_j$ has real part equal to $a_j$. Additionally, every $V_j$ is an $E_j$-invariant subspace of dimension $d_j = \dim V_j$ and $d = d_1 + \ldots + d_p$. Write $X(t) = X^{(1)}(t) + \ldots + X^{(p)}(t)$ with respect to this direct sum decomposition, where by Lemma 7.1.17 in [45], $\{X^{(j)}(t), t \geq 0\}$ is a $(c^{E_j}, c)$-operator semistable Lévy process in $V_j$. Additionally, we can now choose an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^d$ such that the $V_j, j \in \{1, \ldots, p\}$, are mutually orthogonal.
Throughout this thesis, we will denote by $\alpha_j = 1/a_j$ the reciprocals of the real parts of the eigenvalues of the exponent $E$ with $0 < \alpha_p < \ldots < \alpha_1 < 2$.

### 1.2.3 Properties of the density functions

In the calculations of this thesis, certain uniformity results for the density functions of a $(c^E, c)$-operator semistable Lévy process $X = \{X(t) : t \geq 0\}$ are needed. For this purpose, let $g_t$ denote the corresponding density function of $X(t)$ for $t > 0$. According to Proposition 28.1 in [50], the random vector $X(t)$ has a continuous and bounded Lebesgue density for every $t > 0$. In their article, Kern and Wedrich [36] refined this result and showed that the mapping $(t, x) \mapsto g_t(x)$ is continuous on $(0, \infty) \times \mathbb{R}^d$ and that

$$\sup_{t \in [1,c)} \sup_{x \in \mathbb{R}^d} |g_t(x)| < \infty. \tag{1.4}$$

Inequality (1.4) directly leads to the following uniformity result for the existence of negative moments (compare Lemma 2.3 in [36]): For an operator semistable Lévy process $X = \{X(t) : t \geq 0\}$ in $\mathbb{R}^d$ and any $\delta > 0$ we have

$$\sup_{t \in [1,c)} \mathbb{E}\left[\|X(t)\|^{-\delta}\right] < \infty.$$

Furthermore, the following lemma states a positivity result for the density functions:

**Lemma 1.1** (cf. Lemma 2.4 in [36])
*Let $\{X(t)\}_{t \geq 0}$ be an operator semistable Lévy process with $\alpha_1 > 1$, $d_1 = 1$ and with density*

*$g_t$ as above. Then there exist constants $K > 0$, $r > 0$ and uniformly bounded Borel sets $J_t \subseteq \mathbb{R}^{d-1} \cong V_2 \oplus \cdots \oplus V_p$ for $t \in [1, c)$ such that*

$$g_t(x_1, \ldots, x_p) \geq K > 0 \quad \textit{for all } (x_1, \ldots, x_p) \in [-r, r] \times J_t.$$

*Further, we can choose $\{J_t\}_{t \in [1,c)}$ such that $\lambda^{d-1}(J_t) \geq R > 0$ for every $t \in [1, c)$. Note that the constants $K, r$ and $R$ do not depend on $t \in [1, c)$.*

Note that for an operator stable Lévy process it is sufficient to validate the above properties for the case that $t = 1$ as the self-similarity property of the process then ensures the transferability of the result to all $t > 0$.

### 1.2.4 Expected sojourn times

For a Lévy process $X = \{X(t) : t \geq 0\}$ let

$$T(a, s) = \int_0^s 1_{B(0,a)}(X(t))dt$$

be the sojourn time up to time $s > 0$ in the closed ball $B(0, a)$ with radius $a > 0$. Here, $1_F(x)$ denotes the indicator function that equals 1 if $x \in F$ and 0 if $x \notin F$. Furthermore, let $K_1 > 0$ be a fixed constant. A family $\Lambda(a)$ of cubes of side $a$ in $\mathbb{R}^d$ is called $K_1$-nested if no ball of radius $a$ in $\mathbb{R}^d$ can intersect more than $K_1$ cubes of $\Lambda(a)$. Below, I choose $\Lambda(a)$ to be the family of cubes in $\mathbb{R}^d$ of the form $[k_1 a, (k_1 + 1)a] \times \ldots \times [k_d a, (k_d + 1)a]$ where $(k_1, \ldots, k_d) \in \mathbb{Z}^d$. One can easily see that the above defined family $\Lambda(a)$ is $3^d$-nested.
The definition given in Section 1.2.1 of this thesis makes it clear that in order to calculate the Hausdorff dimension it is essential to find a suitable sequence of coverings. For this purpose, Pruitt and Taylor [48] analyzed the connection to the expected sojourn times and proved the following remarkable covering lemma (see Lemma 6.1 in [48]):

**Lemma 1.2**
*Let $X = \{X(t)\}_{t \geq 0}$ be a Lévy process in $\mathbb{R}^d$ and let $\Lambda(a)$ be a fixed $K_1$-nested family of cubes in $\mathbb{R}^d$ of side $a$ with $0 < a \leq 1$. For any $u \geq 0$ let $M_u(a, s)$ be the number of cubes in $\Lambda(a)$ hit by $X(t)$ at some time $t \in [u, u + s]$. Then*

$$\mathbb{E}\left[M_u(a, s)\right] \leq 2\, K_3 s \cdot \left(\mathbb{E}\left[T\left(\tfrac{a}{3}, s\right)\right]\right)^{-1}.$$

Generating results for the Hausdorff dimension and exact Hausdorff measure functions for operator semistable Lévy processes can now in parts be transferred to analyzing the asymptotic behavior of the expected sojourn times $\mathbb{E}[T(a,s)]$.

## 1.3 Overview

In the past, efforts have been made to generate dimension results for Lévy processes with certain self-similarity properties. An overview on existing results can be found in [38] and [59]. For an $\alpha$-stable Lévy process $X$ in $\mathbb{R}^d$, i.e. an operator stable Lévy process with exponent $E = \frac{1}{\alpha} \cdot I$, and $\alpha \in (0,2]$, Blumenthal and Getoor [3] examined the range $X([0,1]) = \{X(t) : t \in [0,1]\}$ over the time interval $[0,1]$ and showed that in this case the Hausdorff dimension is $\dim_H X([0,1]) = \min(\alpha, d)$ almost surely. Subsequently, Pruitt and Taylor [48] calculated $\dim_H X([0,1])$ in case that $X$ is a Lévy process in $\mathbb{R}^d$ with independent stable marginals of index $\alpha_1 \geq \ldots \geq \alpha_d$. In 2005, Meerschaert and Xiao [46] generated a formula for the Hausdorff dimension $\dim_H X(B)$ of an operator stable Lévy process $X$ in $\mathbb{R}^d$ and an arbitrary Borel set $B \subseteq \mathbb{R}_+$. Their result is based on the work of Becker-Kern, Meerschaert and Scheffler [2] who calculated the Hausdorff dimension $\dim_H X([0,1])$ for an operator stable Lévy process in $\mathbb{R}^d$ under the additional assumption that for $\alpha_1 > \min(1,d)$ the density of $X(1)$ is positive at the origin.
Starting point of the calculations of this thesis is the work from Kern and Wedrich [36] who generalized the dimension result in [46] and calculated the Hausdorff dimension $\dim_H X(B)$ for the range of an operator semistable Lévy process $X$ in $\mathbb{R}^d$ over an arbitrary Borel set $B \subseteq \mathbb{R}_+$. For $d \geq 2$ they showed that

$$\dim_{\mathrm{H}} X(B) = \begin{cases} \alpha_1 \dim_{\mathrm{H}} B & \text{if } \alpha_1 \dim_{\mathrm{H}} B \leq d_1, \\ 1 + \alpha_2 \left( \dim_{\mathrm{H}} B - \frac{1}{\alpha_1} \right) & \text{if } \alpha_1 \dim_{\mathrm{H}} B > d_1 \end{cases}$$

almost surely (see Theorem 3.1 in [36]). In case that the process $X$ is one-dimensional the dimension formula reads as follows (compare Theorem 3.3 in [36]):

$$\dim_{\mathrm{H}} X(B) = \min(\alpha \dim_{\mathrm{H}} B, 1) \quad \text{almost surely.}$$

Note that the Hausdorff dimension only depends on the real parts of the eigenvalues of the exponent $E$ of the process $X$ and the Hausdorff dimension $\dim_H B$ of the Borel set

$B \subseteq \mathbb{R}_+$. The proofs of the results above are split into two parts validating $\dim_H X(B) \geq \gamma$ and $\dim_H X(B) \leq \gamma$ for some $\gamma \geq 0$, respectively. To obtain the upper bound a suitable sequence of coverings for $X(B)$ is chosen. This method goes back to Hendricks [28] and Pruitt and Taylor [48]. The proof of the lower bound uses standard capacity arguments applying Frostman's lemma and utilizing the relationship between the Hausdorff dimension and the capacitary dimension as stated in Frostman's theorem [34, 43]. The methods described here will, in parts, be useful in the proofs of the results in this thesis, where they are generalized, where needed, and adapted to the specific requirements.

The first manuscript in Chapter 2, which was published in Probability and Mathematical Statistics [37], contains an examination of the fractal properties of a process with a specific semistable and non-stable distribution. More specifically, the manuscript deals with the limit distribution of the cumulative gains in a series of St. Petersburg games. By [11], the corresponding process $(Y(t))_{t\in[\frac{1}{2},1]}$ can be defined as a continuous transformation of a non-strictly semistable Lévy process $(X(t))_{t\geq 0}$ which fulfills

$$X(2^k t) \stackrel{\mathrm{d}}{=} 2^k (X(t) + kt)$$

for every $k \in \mathbb{Z}$ and $t \geq 0$. Note that, due to the shift term, the so defined process is not operator semistable in the sense of the definition in Section 1.1. In Chapter 2, the Hausdorff and the box-counting dimension of the range and the graph of this particular semistable process are calculated over the time interval $[\frac{1}{2}, 1]$. Furthermore, the results are compared to the fractal dimension of the corresponding limiting objects when the gains are given by a deterministic sequence initiated by Hugo Steinhaus [51].

In the manuscript in Chapter 3, which is accepted for publication in the Journal of Fractal Geometry [56], the methods applied to the particular semistable Lévy process in the foregoing chapter are generalized in order to calculate the Hausdorff dimension $\dim_H \mathrm{Gr}_X(B)$ of the graph of an arbitrary operator semistable Lévy process $X$ in $\mathbb{R}^d$ and any Borel set $B \subseteq \mathbb{R}_+$. To do so, the process $Z = \{Z(t) : t \geq 0\}$, defined by $Z(t) := (t, X(t))$ for all $t \geq 0$, is introduced. Consequently, this gives us $\dim_H \mathrm{Gr}_X(B) = \dim_H Z(B)$. The process $Z$ is again a Lévy process which fulfills the scaling property (1.1) of an operator semistable process but is itself not operator semistable in the sense of the definition given in the introduction as the distribution of $Z(1)$ is not full. Nevertheless, for the reasons mentioned above, one is now able to use the parts of the results and the corresponding proofs in the paper of Kern and Wedrich [36] where fullness of the process was not required. All other parts, however, have to be calculated with enhanced methods. The method of generating dimension results for a class of Lévy processes in $\mathbb{R}^d$ by interpreting the graph as a $(d+1)$-dimensional Lévy process can also be found in Manstavičius [41].

The manuscript in Chapter 4 was submitted for publication to the Electronic Journal of Probability in April 2016 and offers a refinement of the results on the fractal properties of an operator semistable Lévy process as stated in Chapter 3 by dealing with the subject of exact Hausdorff measure functions. For the range of an $\alpha$-stable Lévy process an exact Hausdorff measure function was formulated by Taylor in [53]. Furthermore, Pruitt and Taylor [48] studied sample path properties of Lévy processes with independent stable components. Based on their work, Hou and Ying [30] determined exact Hausdorff measure functions for the range of an operator stable Lévy process with diagonal exponent $E$ over the time interval $[0,1]$. In the third manuscript, exact Hausdorff measure functions for certain operator semistable Lévy processes with $\alpha_1 < d_1$ and diagonal principal exponent, i.e. $E_1 = \alpha_1^{-1} \cdot I^{d_1}$, where $I^{d_1}$ denotes the identity operator on the $d_1$-dimensional subspace $V_1$ (compare Section 1.2.2), are calculated. In the proofs a distinction must be made whether the process is of Taylor type $A$ or $B$ (see [53] for details). A Lévy process is said to be of type $A$ if the continuous density function $g_1$ of $X(1)$ is strictly positive in zero and of type $B$, otherwise. The distinction between the two different types is necessary, because the respective Lévy processes display different asymptotic behavior concerning their sojourn times as defined in Section 1.2.4. Furthermore, in order to determine exact Hausdorff measure functions, sharp upper and lower bounds for the expected sojourn times are needed. The proofs follow the outline given in [30]. Nevertheless, the applied methods in the third manuscript go beyond simple adjustments of the arguments given there.

# Chapter 2

# Manuscript 1
# Dimension results related to the St. Petersburg game

*Joint work with Peter Kern*

## Declaration

### General information:

| | |
|---|---|
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### Contributions:

In total, I contributed roughly 50 % of the content of this manuscript. More specifically, my contributions are as follows:

- I have contributed to the Introduction.
- I have written major parts of Section 2.2 - *Hausdorff dimension of the St. Petersburg game*.

## 2.1 Introduction

The famous St. Petersburg game is easily formulated as a simple coin tossing game. The player's gain $Y = 2^T$ in a single game can be expressed by means of the stopping time $T = \inf\{n \in \mathbb{N} : X_n = 1\}$ of repeated independent tosses $(X_n)_{n\in\mathbb{N}}$ of a fair coin until it first lands heads. For a sequence of gains $(Y_n)_{n\in\mathbb{N}}$ in independent St. Petersburg games the partial sum $S_n = \sum_{k=1}^n Y_k$ denotes the total gain in the first $n$ games. To find a fair entrance fee for playing the game is commonly called the St. Petersburg problem, frequently raised to the status of a paradox. Since the expectation $\mathbb{E}[Y] = \infty$ is infinite, a fair premium cannot be constructed by the help of the usual law of large numbers. We refer to Jorland [33] and Dutka [16] for the history of the St. Petersburg game and for early solutions of the 300 year old problem.

The first step towards a mathematically satisfactory solution has been achieved by Feller [24, 25] who showed that a premium depending on the number $n$ of games can fulfill a certain weak law of large numbers

$$\frac{S_n}{n \log_2 n} \to 1 \quad \text{in probability,}$$

where $\log_2$ denotes logarithm to the base 2. However, Feller's result does not tell if the game is dis- or advantageous for the player, i.e. if $S_n - n\log_2 n$ is likely to be negative or positive. This question can only be answered by a weak limit theorem and the first theorem of this kind has been shown by Martin-Löf [42] for the subsequence $k(n) = 2^n$

$$\frac{S_{k(n)} - k(n)\log_2 k(n)}{k(n)} \to X \quad \text{in distribution.} \tag{2.1}$$

The limit $X$ is infinitely divisible with characteristic function $\exp(\psi(y))$, where

$$\psi(y) = \int_{0+}^{\infty} e^{iyx} - 1 - iyx \cdot 1_{\{x\leqslant 1\}}\, d\phi(x)$$

and the Lévy measure $\phi$ is concentrated on $2^{\mathbb{Z}}$ with $\phi(\{2^k\}) = 2^{-k}$ for $k \in \mathbb{Z}$. Hence $X$ is a semistable random variable and the corresponding Lévy process $\{X(t)\}_{t\geqslant 0}$ with $X(1) \stackrel{\mathrm{d}}{=} X$ is a (non-strictly) semistable Lévy process fulfilling the semi-selfsimilarity condition

$$X(2^k t) \stackrel{\mathrm{d}}{=} 2^k (X(t) + kt) \quad \text{for every } k \in \mathbb{Z} \text{ and } t \geqslant 0. \tag{2.2}$$

For details on semistable random variables and Lévy processes we refer to the monographs [45, 50]. For a semistable setup in general there exists a continuum of possible limit distributions by variation of the subsequence $k(n)\to\infty$ in (2.1). The possible limit distributions for the St. Petersburg game have been characterized by Csörgő and Dodunekova [11] as follows. For $n\in\mathbb{N}$ let us introduce the quantity

$$\gamma_n = n\cdot 2^{-\lceil \log_2 n\rceil}\in(\tfrac12,1], \tag{2.3}$$

which determines the relative position of $n\in\mathbb{N}$ between its nearest consecutive powers of 2. If $k(n)\to\infty$ is a subsequence such that $\gamma_{k(n)}\to t\in[\frac12,1]$ Csörgő and Dodunekova [11] have shown that

$$\frac{S_{k(n)}-k(n)\log_2 k(n)}{k(n)}\to Y(t):=t^{-1}(X(t)-t\log_2 t) \tag{2.4}$$

in distribution, where $Y(\frac12)\stackrel{\mathrm{d}}{=}Y(1)\stackrel{\mathrm{d}}{=}X$ due to (2.2); cf. also [54]. In fact in Theorem 2.2 of [11] the necessary and sufficient condition for convergence in distribution of the normalized and centralized sums $S_n$ along the subsequence $k(n)\to\infty$ should be stated in terms of the so-called circular convergence of $\gamma_{k(n)}$; for details we refer to page 301 in [9] or page 241 in [10]. It is also possible to interpret (2.4) as convergence in distribution of stochastic processes on the space $D[\frac12,1]$ of càdlàg functions $\varphi:[\frac12,1]\to\mathbb{R}$ equipped with the Skorohod $J_1$-topology as follows. First, a direct application of Theorem 14.14 in [35] shows that

$$\left\{\frac{S_{\lfloor 2^n t\rfloor}-\lfloor 2^n t\rfloor n}{2^n}\right\}_{t\in[\frac12,1]}\to\{X(t)\}_{t\in[\frac12,1]} \tag{2.5}$$

in distribution on $D[\frac12,1]$. Alternatively, one may deduce (2.5) from Theorem 2.1 of Fazekas [23]. Secondly, observe that

$$\frac{S_{\lfloor 2^n t\rfloor}-\lfloor 2^n t\rfloor\log_2\lfloor 2^n t\rfloor}{\lfloor 2^n t\rfloor}=\frac{2^n}{\lfloor 2^n t\rfloor}\left(\frac{S_{\lfloor 2^n t\rfloor}-\lfloor 2^n t\rfloor n}{2^n}-\frac{\lfloor 2^n t\rfloor}{2^n}\log_2\frac{\lfloor 2^n t\rfloor}{2^n}\right),$$

where convergence of the deterministic centerings

$$x_n(t)=-\frac{\lfloor 2^n t\rfloor}{2^n}\log_2\frac{\lfloor 2^n t\rfloor}{2^n}\to -t\log_2 t=x(t)$$

and normalizations

$$y_n(t)=\frac{2^n}{\lfloor 2^n t\rfloor}\to t^{-1}=y(t)$$

can also be interpreted as convergence in $D[\frac{1}{2},1]$ with continuous limit functions $x$ and $y$. Since addition and multiplication are continuous in every element of $D[\frac{1}{2},1]\times D[\frac{1}{2},1]$ for which the coordinate-functions have no common discontinuities (see Theorem 4.1 in [57], respectively Theorem 13.3.2 in [58]), an application of the continuous mapping theorem yields

$$\left\{\frac{S_{\lfloor 2^n t\rfloor}-\lfloor 2^n t\rfloor\log_2\lfloor 2^n t\rfloor}{\lfloor 2^n t\rfloor}\right\}_{t\in[\frac{1}{2},1]}\to\left\{t^{-1}(X(t)-t\log_2 t)=Y(t)\right\}_{t\in[\frac{1}{2},1]}$$

in distribution on $D[\frac{1}{2},1]$. Hence we have convergence in distribution of stochastic processes in (2.4) for $k(n)=\lfloor 2^n t\rfloor$ for which circular convergence of $\gamma_{k(n)}$ towards $t\in[\frac{1}{2},1]$ holds.
The object of our study are local fluctuations of the sample paths of the stochastic process $Y=\{Y(t)\}_{t\in[\frac{1}{2},1]}$. Figure 2.1 shows typical (approximative) sample paths of $\{Y(t)\}_{t\in[\frac{1}{2},1]}$ generated by $n=2^{16}$ simulated St. Petersburg games. Note that the sample paths do only have upward jumps due to the fact that the Lévy measure $\phi$ is concentrated on $2^{\mathbb{Z}}$.
The main goal of our paper is to determine the Hausdorff dimension of the range $Y([\frac{1}{2},1])=\{Y(t):t\in[\frac{1}{2},1]\}$ and the graph $G_Y([\frac{1}{2},1])=\{(t,Y(t)):t\in[\frac{1}{2},1]\}$ of the stochastic process $Y$ encoding all the possible distributional limits of St. Petersburg games. For an arbitrary subset $F\subseteq\mathbb{R}^d$ the $s$-dimensional Hausdorff measure is defined as

$$\mathcal{H}^s(F)=\lim_{\delta\to 0}\inf\left\{\sum_{i=1}^{\infty}|F_i|^s:|F_i|\leqslant\delta\text{ and }F\subseteq\bigcup_{i=1}^{\infty}F_i\right\},$$

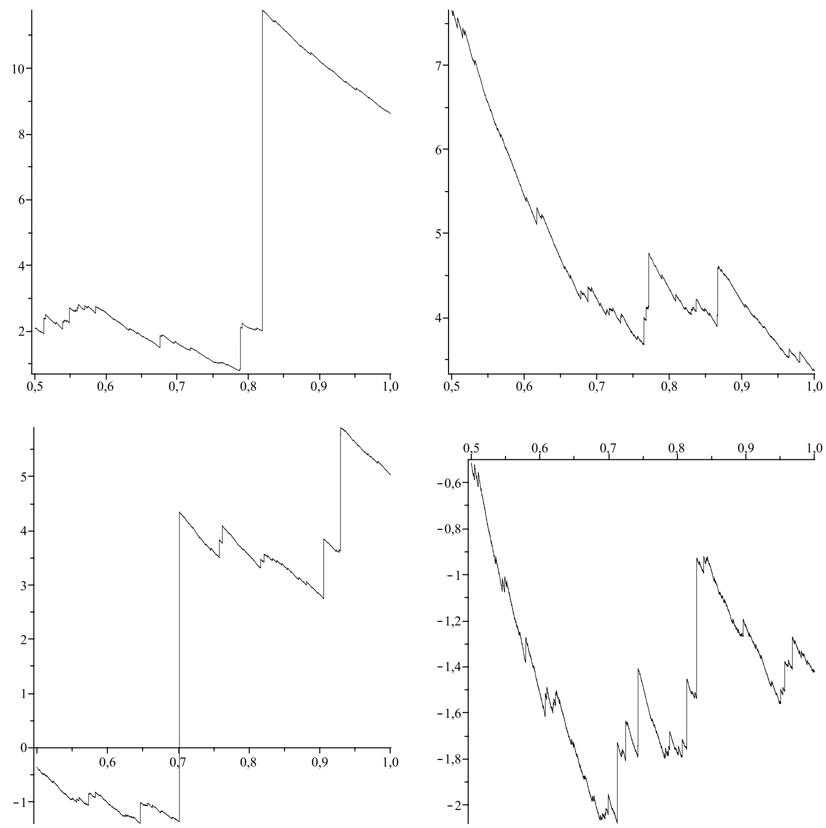
where $|F|=\sup\{\|x-y\|:x,y\in F\}$ denotes the diameter of a set $F\subseteq\mathbb{R}^d$ and $\|\cdot\|$ is the Euclidean norm. It can now be shown that there exists a unique value $\dim_{\mathrm{H}}F\geqslant 0$ so that $\mathcal{H}^s(F)=\infty$ for all $0\leqslant s<\dim_{\mathrm{H}}F$ and $\mathcal{H}^s(F)=0$ for all $s>\dim_{\mathrm{H}}F$. This critical value is called the Hausdorff dimension of $F$. Specifically, we have

$$\dim_{\mathrm{H}}F=\inf\{s:\mathcal{H}^s(F)=0\}=\sup\{s:\mathcal{H}^s(F)=\infty\}.$$

For details on the Hausdorff dimension we refer to [19, 43].
Now let $F\subseteq\mathbb{R}^d$ be a Borel set and denote by $\mathcal{M}^1(F)$ the set of probability measures on $F$. For $s>0$ the $s$-energy of $\mu\in\mathcal{M}^1(F)$ is defined by

$$I_s(\mu)=\int_F\int_F\frac{\mu(dx)\mu(dy)}{\|x-y\|^s}.$$

Figure 2.1: Simulation of four approximations to the sample paths of $Y$. For better visibility the jumps are shown as vertical lines.




By Frostman's lemma, e.g., see [34, 43], there exists a probability measure $\mu \in \mathcal{M}^1(F)$ with $I_s(\mu) < \infty$ if $\dim_{\mathrm{H}} F > s$. In this case $F$ is said to have positive $s$-capacity $C_s(F)$ given by

$$C_s(F) = \sup\{I_s(\mu)^{-1} : \mu \in \mathcal{M}^1(F)\}$$

and the capacitary dimension of $F$ is defined by

$$\dim_{\mathrm{C}} F = \sup\{s > 0 : C_s(F) > 0\} = \inf\{s > 0 : C_s(F) = 0\}.$$

A consequence of Frostman's theorem, e.g., see [34, 43], is that for Borel sets $F \subseteq \mathbb{R}^d$ the Hausdorff and capacitary dimension coincide. Therefore, one can prove lower bounds for the Hausdorff dimension with a simple capacity argument: if $I_s(\mu) < \infty$ for some $\mu \in \mathcal{M}^1(F)$ then $\dim_{\mathrm{H}} F = \dim_{\mathrm{C}} F \geqslant s$.

An alternative fractal dimension is the so called box-counting dimension (see, e.g., [19]). For this purpose let $N_\delta(F)$ be the smallest number of closed balls of radius $\delta$ that cover the set $F \subseteq \mathbb{R}^d$. The lower and the upper box-counting dimensions of an arbitrary set $F \subseteq \mathbb{R}^d$ are now defined as

$$\underline{\dim}_{\mathrm{B}} F = \liminf_{\delta\to 0} \frac{\log N_\delta(F)}{-\log \delta} \quad \text{and} \quad \overline{\dim}_{\mathrm{B}} F = \limsup_{\delta\to 0} \frac{\log N_\delta(F)}{-\log \delta} \tag{2.6}$$

and the box-counting dimension of $F$ is given by

$$\dim_{\mathrm{B}} F = \lim_{\delta\to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

provided that this limit exists. The different fractal dimensions are related as follows:

$$\dim_{\mathrm{H}} F \leqslant \underline{\dim}_{\mathrm{B}} F \leqslant \overline{\dim}_{\mathrm{B}} F \leqslant d. \tag{2.7}$$

Note that there are plenty of sets $F \subseteq \mathbb{R}^d$ where these inequalities are strict.
In Section 2 we will determine the Hausdorff and box-counting dimension of the range $Y([\frac{1}{2}, 1])$ and the graph $G_Y([\frac{1}{2}, 1])$ for almost all sample paths of the stochastic process $Y$. Additionally, in Section 3 we will also consider a deterministic sequence introduced by Steinhaus [51] which is called the "*Steinhaus sequence*" according to [13]. The Steinhaus sequence $(x_n)_{n\in\mathbb{N}}$ is defined by $x_n = 2^k$ if $n = 2^{k-1} + m \cdot 2^k$ for some $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Alternatively, as in Vardi [55], one can define $x_n$ to be twice the highest power of 2 dividing $n$. The Steinhaus sequence is explicitly given by

$$2, 4, 2, 8, 2, 4, 2, 16, 2, 4, 2, 8, 2, 4, 2, 32, 2, 4, 2, 8, 2, 4, 2, 16, 2, 4, 2, 8, 2, 4, 2, 64, ...$$

and has relative frequencies $\lim_{n\to\infty} n^{-1} \operatorname{card}\{1 \leqslant j \leqslant n : x_j = 2^k\} = 2^{-k}$ for $k \in \mathbb{N}$. The sequence $(x_n)_{n\in\mathbb{N}}$ has been considered as entrance fees for repeated St. Petersburg games in [51, 13] and has been proven to be a sequence of nearly asymptotically fair premiums in a certain sense. For details we refer to [13]. In contrast to [51, 13] we will consider the Steinhaus sequence as a sequence of possible gains in repeated St. Petersburg games. Again, we will determine the Hausdorff and box-counting dimension of the range and the graph of the corresponding limiting object of the centralized and normalized Steinhaus sequence. To do so, we will employ results for iterated function systems as presented in [22].

## 2.2 Hausdorff dimension of the St. Petersburg game

### 2.2.1 Hausdorff dimension of the range

In this section we evaluate the Hausdorff dimension of the range of the stochastic process $Y = \{Y(t)\}_{t\in[\frac{1}{2},1]}$. We employ common techniques used to calculate Hausdorff dimensions of selfsimilar Lévy processes (see [59, 46, 36]) and adapt them to our situation. Note that the given process $Y$ is neither a Lévy process nor does it have the selfsimilarity property of a semistable process. The result is stated in the theorem below.

**Theorem 2.1**
*We have* $\dim_{\mathrm{H}} Y([\frac{1}{2},1]) = 1$ *almost surely.*

Note that Theorem 2.1 together with (2.7) yields

$$\dim_{\mathrm{H}} Y([\tfrac{1}{2},1]) = \dim_{\mathrm{B}} Y([\tfrac{1}{2},1]) = 1$$

almost surely. Since $Y$ is a process on $\mathbb{R}$ it is obvious that $\dim_{\mathrm{H}} Y([\frac{1}{2},1]) \leqslant 1$ almost surely. For the proof of Theorem 2.1 it is hence sufficient to prove the following lemma.

**Lemma 2.2**
*We have* $\dim_{\mathrm{H}} Y([\frac{1}{2},1]) \geqslant 1$ *almost surely.*

*Proof.* As mentioned above we can write

$$Y(t) = t^{-1}\left(X(t) - t\log_2 t\right),$$

where $X = \{X(t)\}_{t\geqslant 0}$ is a semistable Lévy process. To prove the assertion we will apply Frostman's theorem [34, 43] with the probability measure $\sigma = 2\lambda|_{[\frac{1}{2},1]}$, where $\lambda$ denotes Lebesgue measure. For this purpose let $0 < \gamma < 1$ and note that $\sigma$ is an admissible measure for Frostman's lemma, i.e.

$$\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} \frac{\sigma(ds)\,\sigma(dt)}{|s-t|^{\gamma}} < \infty.$$

By Frostman's theorem it is now sufficient to show that for all $\gamma \in (0,1)$

$$\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} \mathbb{E}\left[|Y(s)-Y(t)|^{-\gamma}\right] ds\,dt < \infty.$$

For $r \in (0,1]$ let $g_r$ be the Lebesgue density of $X(r)$ chosen from the class $C^\infty(\mathbb{R})$ by Proposition 2.8.1 in [50]. Then we have $M := \sup_{r\in(0,1]} \sup_{x\in\mathbb{R}} |g_r(x)| < \infty$ as in Lemma 3 of [8]; see also Lemma 2.2 in [36]. By symmetry of the integrand we get

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} \mathbb{E}\left[|Y(s)-Y(t)|^{-\gamma}\right] ds\,dt \\
&= 2\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{t} \mathbb{E}\left[\left|s^{-1}X(s)-\log_2 s - t^{-1}\left(X(s)+(X(t)-X(s))\right)+\log_2 t\right|^{-\gamma}\right] ds\,dt \\
&= 2\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{t}\int_{\mathbb{R}^2} \left|s^{-1}x-\log_2 s - t^{-1}(x+y)+\log_2 t\right|^{-\gamma} g_s(x)\, g_{t-s}(y)\, d\lambda^2(x,y)\, ds\,dt \\
&= 2\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{t}\int_{\mathbb{R}^2} \left|\frac{t-s}{st}\,x+\log_2\left(\frac{t}{s}\right)-\frac{y}{t}\right|^{-\gamma} g_s(x)\, g_{t-s}(y)\, d\lambda^2(x,y)\, ds\,dt \\
&= 2\int_{\frac{1}{2}}^{1}\int_{0}^{t-\frac{1}{2}}\int_{\mathbb{R}^2} \left|\frac{w}{t(t-w)}\,x+\log_2\left(\frac{t}{t-w}\right)-\frac{y}{t}\right|^{-\gamma} g_{t-w}(x)\, g_{w}(y)\, d\lambda^2(x,y)\, dw\,dt,
\end{aligned}$$

where in the last equality we substituted $w = t-s$. Now we write $w \in (0,\frac{1}{2}]$ as $w = 2^{-m}r$ with $m = m(w) \in \mathbb{N}$ and $r \in (\frac{1}{2},1]$. This leads us to

$$\begin{aligned}
g_w(y) &= \frac{d}{dy}\mathbb{P}\left(X(w)\leqslant y\right) \\
&= \frac{d}{dy}\mathbb{P}\left(X(2^{-m}r)\leqslant y\right) = \frac{d}{dy}\mathbb{P}\left(2^{-m}(X(r)-mr)\leqslant y\right) \\
&= \frac{d}{dy}\mathbb{P}\left(X(r)\leqslant 2^m y+mr\right) = 2^m g_r\left(2^m y+mr\right).
\end{aligned}$$

Using the substitutions $v = 2^m y + mr$ and $u = \frac{t}{2^{-m}}\left(\frac{w}{t(t-w)}x+\log_2\left(\frac{t}{t-w}\right)+\frac{mw}{t}\right)$ we get

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left|\frac{w}{t(t-w)}\,x+\log_2\left(\frac{t}{t-w}\right)-\frac{y}{t}\right|^{-\gamma} g_{t-w}(x)\, g_{w}(y)\, d\lambda^2(x,y) \\
&= 2^m\int_{\mathbb{R}^2} \left|\frac{w}{t(t-w)}\,x+\log_2\left(\frac{t}{t-w}\right)-\frac{y}{t}\right|^{-\gamma} g_{t-w}(x)\, g_{r}(2^m y+mr)\, d\lambda^2(x,y) \\
&= \int_{\mathbb{R}^2} \left|\frac{w}{t(t-w)}\,x+\log_2\left(\frac{t}{t-w}\right)-\frac{2^{-m}}{t}\,v+\frac{mw}{t}\right|^{-\gamma} g_{t-w}(x) g_{r}(v)\, d\lambda^2(x,v) \\
&= \frac{t-w}{r}\int_{\mathbb{R}^2} \left|\frac{2^{-m}}{t}(u-v)\right|^{-\gamma} g_{t-w}(x(u))\, g_{r}(v)\, d\lambda^2(u,v)
\end{aligned}$$

$$= \frac{t^\gamma (t-w) 2^{m\gamma}}{r} \left( \int_A + \int_{A^\complement} \right) |u-v|^{-\gamma}\, g_{t-w}(x(u))\, g_r(v)\, d\lambda^2(u,v),$$

where $A$ denotes the set $A = \{(u,v) \in \mathbb{R}^2 : |u-v| \leqslant 1\}$. We now estimate the two integrals separately. First,

$$\begin{aligned} &\int_A |u-v|^{-\gamma}\, g_{t-w}(x(u))\, g_r(v)\, d\lambda^2(u,v) \\ &\quad \leqslant M \int_{\mathbb{R}} \left( \int_{v-1}^{v} (v-u)^{-\gamma} du + \int_{v}^{v+1} (u-v)^{-\gamma} du \right) g_r(v)\, dv \\ &\quad = M \int_{\mathbb{R}} \frac{2}{1-\gamma} g_r(v)\, dv = \frac{2M}{1-\gamma} \end{aligned}$$

and secondly,

$$\begin{aligned} \int_{A^\complement} |u-v|^{-\gamma}\, g_{t-w}(x(u))\, g_r(v)\, d\lambda^2(u,v) &\leqslant \int_{A^\complement} g_{t-w}(x(u))\, g_r(v)\, d\lambda^2(u,v) \\ \leqslant \frac{r}{t-w} \int_{\mathbb{R}^2} g_{t-w}(x)\, g_r(v)\, d\lambda^2(x,v) &= \frac{r}{t-w}. \end{aligned}$$

This leads us to

$$\begin{aligned} &\int_{\mathbb{R}^2} \left| \frac{w}{t(t-w)} x + \log_2\left(\frac{t}{t-w}\right) - \frac{y}{t} \right|^{-\gamma} g_{t-w}(x)\, g_w(y)\, d\lambda^2(x,y) \\ &\quad \leqslant t^\gamma 2^{m\gamma} \left( \frac{2M(t-w)}{r(1-\gamma)} + 1 \right) \leqslant t^\gamma 2^{m\gamma} \left( \frac{4M}{1-\gamma} + 1 \right) =: K t^\gamma 2^{m\gamma}. \end{aligned}$$

Taken all together, we obtain

$$\begin{aligned} \int_{\frac12}^1 \int_{\frac12}^1 \mathbb{E}\left[|Y(s)-Y(t)|^{-\gamma}\right] ds\, dt &\leqslant 2K \int_{\frac12}^1 \int_0^{t-\frac12} t^\gamma 2^{m(w)\gamma}\, dw\, dt \\ = 2K \int_0^{\frac12} \int_{\frac12 + w}^1 t^\gamma 2^{m(w)\gamma}\, dt\, dw &= 2K \sum_{m\in\mathbb{N}} \int_{2^{-(m+1)}}^{2^{-m}} \int_{\frac12 + w}^1 t^\gamma 2^{m\gamma}\, dt\, dw \qquad (2.8) \\ \leqslant 2K \sum_{m\in\mathbb{N}} \int_{\frac12}^1 \int_{\frac12}^1 t^\gamma 2^{m\gamma}\, dt\, 2^{-m}\, dr &= K \sum_{m\in\mathbb{N}} \left(2^{\gamma-1}\right)^m \int_{\frac12}^1 t^\gamma\, dt < \infty, \end{aligned}$$

since $\gamma - 1 < 0$. This concludes our proof. □

### 2.2.2 Hausdorff dimension of the graph

In this section we show that the dimension result for the range of the stochastic process $Y$ also holds for its graph $G_Y([\frac{1}{2},1])$. We will split the proof into two parts, first verifying $\alpha = 1$ as an upper bound and secondly as a lower bound for the Hausdorff dimension of the graph.

We first calculate the upper bound for the Hausdorff dimension of the graph of the semistable Lévy process $X$ and later on transfer the result to the process $Y$. As $X$ is not strictly semistable we cannot use the dimension results of [36], without modifying it according to our situation. This parallels investigations for stable Lévy processes, where dimension results for the strictly stable (symmetric) Cauchy process and for an asymmetric (non-strictly) stable Cauchy process have been addressed separately. Nevertheless, the Hausdorff dimension of the range, respectively the graph, coincides for both Cauchy processes; see [3, 27, 53].

**Proposition 2.3**
*Let $\{Z(t) := (t, X(t))\}_{t\geqslant 0}$. Then almost surely*

$$\dim_{\mathrm{H}} Z([\tfrac{1}{2},1]) \leqslant 1.$$

Let $T(a,s) = \int_0^s 1_{B(0,a)}(Z(t))\,dt$ denote the sojourn time of the Lévy process $Z$ up to time $s$ in the closed ball $B(0,a) \subseteq \mathbb{R}^2$ with radius $a$ centered at the origin. To prove Proposition 2.3 we need the following lemma.

**Lemma 2.4**
*Let $Z$ be the stochastic process defined in Proposition 2.3. There exists a positive and finite constant $K$ such that for all $0 < a \leqslant 1$ and $\frac{a}{\sqrt{2}} \leqslant s \leqslant 1$ we have*

$$\mathbb{E}\left[T(a,s)\right] \geqslant Ka.$$

*Proof.* Fix $0 < a \leqslant 1$ and let $0 < \delta \leqslant \frac{1}{\sqrt{2}}$, so that

$$2^{-(i_0+1)} < a\delta \leqslant 2^{-i_0} < \frac{a}{\sqrt{2}} < s,$$

for some $i_0 \in \mathbb{N}_0$. Furthermore we choose $0 < \delta \leqslant \frac{1}{\sqrt{2}}$ small enough (i.e. $i_0 \in \mathbb{N}_0$ big enough) so that

$$2i \leqslant \frac{a}{2\sqrt{2}} 2^i \quad \text{for all } i > i_0 \tag{2.9}$$

and that additionally

$$\mathbb{P}\left(\sup_{r\in[1,2)} X(r) < \frac{1}{\delta 2\sqrt{2}}\right) - \mathbb{P}\left(\inf_{r\in[1,2)} X(r) \leqslant -\frac{1}{\delta 2\sqrt{2}}\right) \geqslant \frac{1}{2}. \tag{2.10}$$

Inequality (2.10) holds for $\delta > 0$ small enough since $X$ is a Lévy process and it can be assumed that it has càdlàg paths. Thus both $\sup_{r\in[1,2)} X(r)$ and $\inf_{r\in[1,2)} X(r)$ are random variables. We have

$$\begin{aligned}
&\mathbb{E}\left[T(a,s)\right] = \int_0^s \mathbb{P}\left(\|Z(t)\| < a\right) dt \geqslant \int_0^s \mathbb{P}\left(|X(t)| < \frac{a}{\sqrt{2}}, t < \frac{a}{\sqrt{2}}\right) dt \\
&= \int_0^{\frac{a}{\sqrt{2}}} \mathbb{P}\left(|X(t)| < \frac{a}{\sqrt{2}}\right) dt \geqslant \int_0^{2^{-i_0}} \mathbb{P}\left(|X(t)| < \frac{a}{\sqrt{2}}\right) dt \\
&= \sum_{i=i_0+1}^{\infty} \int_{2^{-i}}^{2^{-i+1}} \mathbb{P}\left(|X(t)| < \frac{a}{\sqrt{2}}\right) dt = \sum_{i=i_0+1}^{\infty} 2^{-i} \int_1^2 \mathbb{P}\left(|X(2^{-i}r)| < \frac{a}{\sqrt{2}}\right) dr \\
&= \sum_{i=i_0+1}^{\infty} 2^{-i} \int_1^2 \mathbb{P}\left(|2^{-i}(X(r) - ir)| < \frac{a}{\sqrt{2}}\right) dr \\
&= \sum_{i=i_0+1}^{\infty} 2^{-i} \int_1^2 \mathbb{P}\left(|X(r) - ir| < \frac{2^i a}{\sqrt{2}}\right) dr.
\end{aligned}$$

By (2.9) and (2.10) the probability above can be estimated from below by

$$\begin{aligned}
&\mathbb{P}\left(|X(r) - ir| < \frac{2^i a}{\sqrt{2}}\right) = \mathbb{P}\left(-\frac{a}{\sqrt{2}}2^i + ir < X(r) < \frac{a}{\sqrt{2}}2^i + ir\right) \\
&\geqslant \mathbb{P}\left(-\frac{a}{\sqrt{2}}2^i + 2i < X(r) < \frac{a}{\sqrt{2}}2^i\right) \\
&= \mathbb{P}\left(X(r) < \frac{a}{\sqrt{2}}2^i\right) - \mathbb{P}\left(X(r) \leqslant -\frac{a}{\sqrt{2}}2^i + 2i\right) \\
&\geqslant \mathbb{P}\left(\sup_{r\in[1,2)} X(r) < \frac{a}{\sqrt{2}}2^i\right) - \mathbb{P}\left(\inf_{r\in[1,2)} X(r) \leqslant -\frac{a}{\sqrt{2}}2^i + 2i\right)
\end{aligned}$$

$$\geqslant \mathbb{P}\left(\sup_{r\in[1,2)} X(r) < \frac{a}{2\sqrt{2}}2^{i_0+1}\right) - \mathbb{P}\left(\inf_{r\in[1,2)} X(r) \leqslant -\frac{a}{2\sqrt{2}}2^{i_0+1}\right)$$

$$\geqslant \mathbb{P}\left(\sup_{r\in[1,2)} X(r) < \frac{1}{\delta 2\sqrt{2}}\right) - \mathbb{P}\left(\inf_{r\in[1,2)} X(r) \leqslant -\frac{1}{\delta 2\sqrt{2}}\right) \geqslant \frac{1}{2}.$$

Note that $\delta$ does not depend on $a$. It follows that

$$\mathbb{E}\left[T(a,s)\right] \geqslant \sum_{i=i_0+1}^{\infty} 2^{-i}\int_1^2 \frac{1}{2}\,dr = \frac{1}{2}\sum_{i=i_0+1}^{\infty} 2^{-i} = \frac{1}{2}2^{-i_0} \geqslant \frac{1}{2}\delta a =: Ka,$$

which concludes the proof. □

*Proof.* Proposition 2.3 Let $K_1 > 0$ be a fixed constant. A family $\Lambda(a)$ of cubes of side $a \in (0,1]$ in $\mathbb{R}^2$ is called $K_1$-nested if no ball of radius $a$ in $\mathbb{R}^2$ can intersect more than $K_1$ cubes of $\Lambda(a)$. For any $u \geqslant 0$ let $M_u(a,s)$ be the number of these cubes hit by the Lévy process $Z$ at some time $t \in [u, u+s]$. Then a famous covering lemma of Pruitt and Taylor [48, Lemma 6.1] states that

$$\mathbb{E}[M_u(a,s)] \leqslant 2K_1 s \cdot (\mathbb{E}[T(\tfrac{a}{3},s)])^{-1}.$$

Lemma 2.4 now enables us to construct a covering of $Z([\frac{1}{2},1])$ whose expected $s$-dimensional Hausdorff measure is finite for every $s > 1$. The arguments are in complete analogy to the proof of part (i) of Lemma 3.4 in [36] and thus omitted. □

In order to transfer the result of Proposition 2.3 to the process $Y$ we introduce the continuous function $\tau : [\frac{1}{2},1] \times \mathcal{K}_1 \to [\frac{1}{2},1] \times \mathcal{K}_2$ with

$$\begin{pmatrix} t \\ x \end{pmatrix} \mapsto \tau(t,x) = \begin{pmatrix} t \\ t^{-1}x - \log_2 t \end{pmatrix}, \tag{2.11}$$

where $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathbb{R}$ are not further specified compact intervals that can vary throughout the paper. It can easily be shown, that for a fixed compact interval $\mathcal{K}_1 \subseteq \mathbb{R}$ the function $\tau$ is bi-Lipschitz when choosing $\mathcal{K}_2$ such that $[\frac{1}{2},1] \times \mathcal{K}_2 = \mathrm{Im}(\tau)$. We can now write all elements $(t, Y(t))^\top \in G_Y([\frac{1}{2},1])$ as

$$\begin{pmatrix} t \\ Y(t) \end{pmatrix} = \begin{pmatrix} t \\ t^{-1}X(t) - \log_2 t \end{pmatrix} = \tau\left(t, X(t)\right).$$

Since $X$ is a Lévy process, it can be assumed that all paths are cádlág and hence that for all fixed $\omega \in \Omega$ there exists a compact interval $\mathcal{K}_1 \subseteq \mathbb{R}$ such that

$$X(t)(\omega) \in \mathcal{K}_1 \quad \text{for all } t \in [\tfrac{1}{2}, 1].$$

This means that for $Z = (Z(t) = (t, X(t)))_{t \in [\frac{1}{2},1]}$ and all $\omega \in \Omega$ we have

$$\dim_{\mathrm{H}} Z([\tfrac{1}{2}, 1])(\omega) = \dim_{\mathrm{H}} \tau(Z([\tfrac{1}{2}, 1]))(\omega) = \dim_{\mathrm{H}} G_Y([\tfrac{1}{2}, 1])(\omega)$$

by Lemma 1.8 in [17]. Since we have shown in Proposition 2.3 that $\dim_{\mathrm{H}} Z([\frac{1}{2}, 1]) \leqslant 1$ almost surely, we have thus proven the following upper bound.

**Theorem 2.5**
*We have* $\dim_{\mathrm{H}} G_Y([\frac{1}{2}, 1]) \leqslant 1$ *almost surely.*

To prove the lower bound for the Hausdorff dimension of the graph we can use the same technique as for the lower bound in case of the range of $Y$.

**Theorem 2.6**
*We have* $\dim_{\mathrm{H}} G_Y([\frac{1}{2}, 1]) \geqslant 1$ *almost surely.*

*Proof.* Let $0 < \gamma < 1$. By (2.8) we get

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} \mathbb{E}\left[\|(s, Y(s))^{\top} - (t, Y(t))^{\top}\|^{-\gamma}\right] ds\, dt \\
&= \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} \mathbb{E}\left[\left((s-t)^2 + (Y(s) - Y(t))^2\right)^{-\frac{\gamma}{2}}\right] ds\, dt \\
&\leqslant \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} \mathbb{E}\left[|Y(s) - Y(t)|^{-\gamma}\right] ds\, dt < \infty.
\end{aligned}$$

As in Lemma 2.2 the assertion follows by Frostman's theorem. □

With similar techniques it is also possible to proof the following dimension result for the box-counting dimension of the graph of the St. Petersburg process $Y$.

**Theorem 2.7**
*We have* $\dim_{\mathrm{B}} G_Y([\frac{1}{2}, 1]) = 1$ *almost surely.*

*Proof.* The lower bound follows directly from the almost sure inequalities

$$1 \leqslant \dim_{\mathrm{H}} G_Y([\tfrac{1}{2},1]) \leqslant \underline{\dim}_{\mathrm{B}} G_Y([\tfrac{1}{2},1]) \leqslant \overline{\dim}_{\mathrm{B}} G_Y([\tfrac{1}{2},1]).$$

For the upper bound it is now sufficient to verify $\overline{\dim}_{\mathrm{B}} G_Y([\frac{1}{2},1]) \leqslant 1$ almost surely. Due to the nature of the upper box-counting dimension (see (2.6)) we can again calculate the upper bound for $\overline{\dim}_{\mathrm{B}} Z([\frac{1}{2},1]) \leqslant 1$ with the same covering arguments as in the proof of part (i) of Lemma 3.4 in [36]. With the bi-Lipschitz invariance of the upper box-counting dimension (see Section 3.2 in [19]) the proof concludes. □

**Remark 2.8**

*If one prefers to flip an unfair coin this naturally leads to so called generalized St. Petersburg games as treated in [8, 12, 26, 47]. Let $p \in (0,1)$ be the probability of the coin falling heads and let $q = 1-p$. The single gain in a generalized St. Petersburg game is given by $q^{-T/\alpha}$ for some $\alpha > 0$. We focus on the classical situation $\alpha = 1$ and slightly modify the gain to $q^{1-T}p^{-1}$ for ease of notation, which results in the limit theorem*

$$\frac{S_{k(n)} - k(n)\log_{1/q} k(n)}{k(n)} \to Y(t) = t^{-1}(X(t) - t\log_{1/q} t)$$

*in distribution, whenever*

$$q^{\lceil \log_{1/q} k(n)\rceil} k(n) \to t \in [q,1],$$

*where $\{X(t)\}_{t\geqslant 0}$ is a semistable Lévy process with the semi-selfsimilarity property*

$$X(q^{-k}t) \stackrel{\mathrm{d}}{=} q^{-k}(X(t)+kt) \quad \text{for every } k\in\mathbb{Z} \text{ and } t \geqslant 0.$$

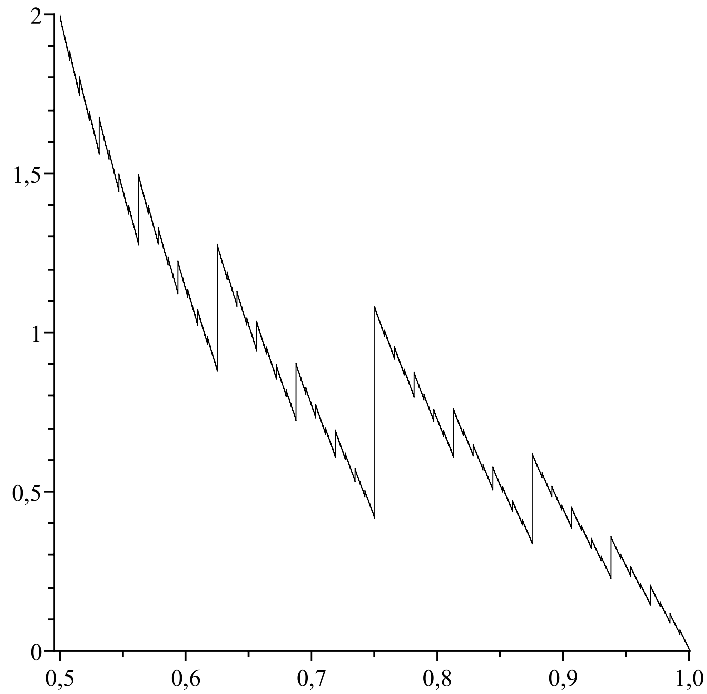
*We emphasize that with the above techniques our Theorems 2.1–2.7 also hold for the process $\{Y(t)\}_{t\in[q,1]}$ in this generalized situation when replacing the interval by $[q,1]$. Presumably, similar results can be shown for general $\alpha > 0$.*

## 2.3 Hausdorff dimension of the Steinhaus sequence

Recall the definition of the Steinhaus sequence $(x_n)_{n\in\mathbb{N}}$ given in the Introduction. The asymptotic properties of $(x_n)_{n\in\mathbb{N}}$ have been analyzed in full detail by Csörgő and Simons [13]. Let $s(n) = x_1 + \cdots + x_n$ and $\gamma_n = n\cdot 2^{-\lceil \log_2 n\rceil} \in (\frac{1}{2},1]$ as in (2.3). Then by Theorem 3.3 in [13] we have for any $n\in\mathbb{N}$

$$\frac{s(n) - n\log_2 n}{n} = \xi(\gamma_n), \tag{2.12}$$

Figure 2.2: Graph of $\xi$ on the interval $[\frac{1}{2}, 1)$. For better visibility the jumps of $\xi$ are shown as vertical lines.




where the function $\xi : [\frac{1}{2}, 1] \to [0, 2]$ is defined by

$$\xi(\gamma) = 2 - \log_2 \gamma - \frac{1}{\gamma} \sum_{k=1}^{\infty} \frac{k\varepsilon_k}{2^k}$$

and the sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subseteq \{0, 1\}$ is given by the dyadic expansion $\gamma = \sum_{k=0}^{\infty} \frac{\varepsilon_k}{2^k}$ of $\gamma \in [\frac{1}{2}, 1]$ with the convention that $\varepsilon_k = 0$ for infinitely many $k \in \mathbb{N}$. By Theorem 3.1 in [13] the function $\xi$ is càdlàg with $\xi(\frac{1}{2}) = 2 = \xi(1)$ and has jumps precisely at the dyadic rationals in $(\frac{1}{2}, 1]$. All these jumps are upward and the largest jump occurs from $\xi(1-) = 0$ to $\xi(1) = 2$. The graph of $\xi$ seems to inhere fractal properties as can be seen in Figure 2.2 below, a replication of Figure 1 in [13]. It follows directly from (2.12) that the sequence $(s(n))_{n\in\mathbb{N}}$ of total gains satisfies the asymptotic property of Feller

$$\frac{s(n)}{n \log_2 n} \to 1 \tag{2.13}$$

as $n \to \infty$; see [13]. Note that Feller's law of large numbers does not hold in an almost sure sense. According to classical results in [6, 1, 14] it is known that

$$\limsup_{n\to\infty} \frac{S_n}{n\log_2 n} = \infty \quad \text{and} \quad \liminf_{n\to\infty} \frac{S_n}{n\log_2 n} = 1 \quad \text{almost surely.} \tag{2.14}$$

More precisely, we have $\mathrm{LIM}\{S_n/(n\log_2 n) : n \in \mathbb{N}\} = [1,\infty]$ almost surely by Corollary 1 in [55], where LIM denotes the set of accumulation points. But there is a version of the strong law of large numbers by [15] when neglecting the largest gain

$$\frac{S_n - \max_{1\leqslant k\leqslant n} X_k}{n\log_2 n} \to 1 \quad \text{almost surely.}$$

A comparison of (2.13) and (2.14) shows that the Steinhaus sequence belongs to an exceptional nullset with respect to (2.14) concerning the almost sure limit behavior of the total gain in repeated St. Petersburg games.

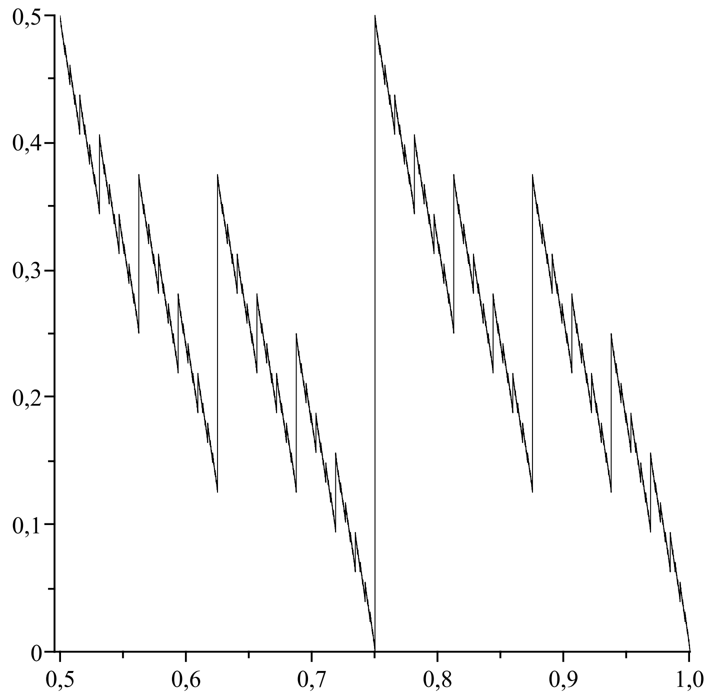
Moreover, for any sequence $k_n \to \infty$ with $\gamma_{k_n} = k_n \cdot 2^{-\lceil \log_2 k_n \rceil} \to \gamma \in [\frac{1}{2}, 1]$ we get from (2.12)

$$\emptyset \neq \mathrm{LIM}\left\{ \frac{s(k_n) - k_n \log_2 k_n}{k_n} : n \in \mathbb{N} \right\} \subseteq \{\xi(\gamma), \xi(\gamma-)\}.$$

Hence we may consider the function $\xi$ as the corresponding limiting object when using the same centering and normalization sequences as in (2.4). We will now show that the Steinhaus sequence is not exceptional concerning the local fluctuations of the limit measured by the Hausdorff or box-counting dimension.

It follows directly from the above stated properties of $\xi$ given in Theorem 3.1 of [13] that the range $\xi([\frac{1}{2}, 1])$ is equal to the interval $(0, 2]$ and hence $\dim_{\mathrm{H}} \xi([\frac{1}{2}, 1]) = 1$ by Theorem 1.12 in [17]. This shows that $\dim_{\mathrm{H}} \xi([\frac{1}{2}, 1])$ coincides with the Hausdorff dimension of the range of a typical sample path of $\{Y(t)\}_{t\in[\frac{1}{2},1]}$. Clearly, by (2.7) we also have $\dim_{\mathrm{H}} \xi([\frac{1}{2}, 1]) = \dim_{\mathrm{B}} \xi([\frac{1}{2}, 1]) = 1$. A look at Figure 2.2 suggests that it is merely the graph and not the range of $\xi$ that should inhere fractal properties. In the sequel we will argue that also the graph $G_\xi([\frac{1}{2}, 1])$ is typical concerning the almost sure dimension properties of the sample graph of $\{Y(t)\}_{t\in[\frac{1}{2},1]}$. To this aim we will apply the inverse $\tau^{-1}$ of the bi-Lipschitz function $\tau$ defined in (2.11) with $\mathcal{K}_2 = [0, 2] \subseteq \mathbb{R}$. I.e., we now consider the function $\tau^{-1} : [\frac{1}{2}, 1] \times [0, 2] \to [\frac{1}{2}, 1] \times \mathcal{K}_1$ with $\tau^{-1}(t, x) = (t, t(x + \log_2 t))^\top$, where the compact interval $\mathcal{K}_1 \subseteq \mathbb{R}$ is chosen such that $[\frac{1}{2}, 1] \times \mathcal{K}_1 = \operatorname{Im} \tau^{-1}$. Applied to the graph

Figure 2.3: Image of $\tau^{-1}(G_\xi[\frac{1}{2},1))$. For better visibility the jumps are shown as vertical lines.




of $\xi$ we get for any $\gamma \in [\frac{1}{2},1]$

$$\tau^{-1}(\gamma,\xi(\gamma)) = \begin{pmatrix} \gamma \\ \gamma(\xi(\gamma)+\log_2\gamma) \end{pmatrix} = \begin{pmatrix} \gamma \\ 2\gamma - \sum_{k=1}^{\infty}\frac{k\varepsilon_k}{2^k} \end{pmatrix}$$

and by bi-Lipschitz invariance we have

$$\dim_{\mathrm{H}} G_\xi([\tfrac{1}{2},1]) = \dim_{\mathrm{H}} \tau^{-1}(G_\xi([\tfrac{1}{2},1])). \tag{2.15}$$

The same equality holds for upper and lower box-counting dimensions; e.g., see [19]. The image $\tau^{-1}(G_\xi([\frac{1}{2},1)))$ is illustrated in Figure 2.3 and shows perfect selfsimilarity. To see this, we may write $\tau^{-1}(\gamma,\xi(\gamma)) = (\gamma, f(\gamma))^\top$ with

$$f(\gamma) = 2\gamma - \sum_{k=1}^{\infty}\frac{k\varepsilon_k}{2^k}.$$

**Lemma 2.9**

*For any* $\gamma \in [\frac{1}{2}, 1)$ *we have*

$$f(\tfrac{1}{2}\gamma + \tfrac{1}{2}) = \tfrac{1}{2}(1 - \gamma + f(\gamma)) = f(\tfrac{1}{2}\gamma + \tfrac{1}{4}).$$

*Proof.* For the dyadic expansion $\gamma = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^k}$ of $\gamma \in [\frac{1}{2}, 1)$ we necessarily have $\varepsilon_1 = 1$. Consequently,

$$\tfrac{1}{2}\gamma + \tfrac{1}{2} = \tfrac{1}{2} + \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^{k+1}} = \sum_{k=1}^{\infty} \frac{\varepsilon'_k}{2^k} \quad \text{with} \quad \varepsilon'_k = \begin{cases} 1 & k = 1, \\ \varepsilon_{k-1} & k \geqslant 2 \end{cases}$$

and

$$\tfrac{1}{2}\gamma + \tfrac{1}{4} = \tfrac{1}{4} + \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^{k+1}} = \sum_{k=1}^{\infty} \frac{\varepsilon''_k}{2^k} \quad \text{with} \quad \varepsilon''_k = \begin{cases} 1 & k = 1, \\ 0 & k = 2, \\ \varepsilon_{k-1} & k \geqslant 3. \end{cases}$$

It follows that

$$f(\tfrac{1}{2}\gamma + \tfrac{1}{2}) = 2(\tfrac{1}{2}\gamma + \tfrac{1}{2}) - \sum_{k=1}^{\infty} \frac{k\varepsilon'_k}{2^k} = \gamma + \tfrac{1}{2} - \sum_{k=2}^{\infty} \frac{k\varepsilon_{k-1}}{2^k} \quad \text{and}$$

$$f(\tfrac{1}{2}\gamma + \tfrac{1}{4}) = 2(\tfrac{1}{2}\gamma + \tfrac{1}{4}) - \sum_{k=1}^{\infty} \frac{k\varepsilon''_k}{2^k} = \gamma - \sum_{k=3}^{\infty} \frac{k\varepsilon_{k-1}}{2^k} = \gamma + \tfrac{1}{2} - \sum_{k=2}^{\infty} \frac{k\varepsilon_{k-1}}{2^k}.$$

This shows $f(\frac{1}{2}\gamma + \frac{1}{2}) = f(\frac{1}{2}\gamma + \frac{1}{4}) = \gamma + \frac{1}{2} - \sum_{k=2}^{\infty} \frac{k\varepsilon_{k-1}}{2^k}$ and furthermore we get

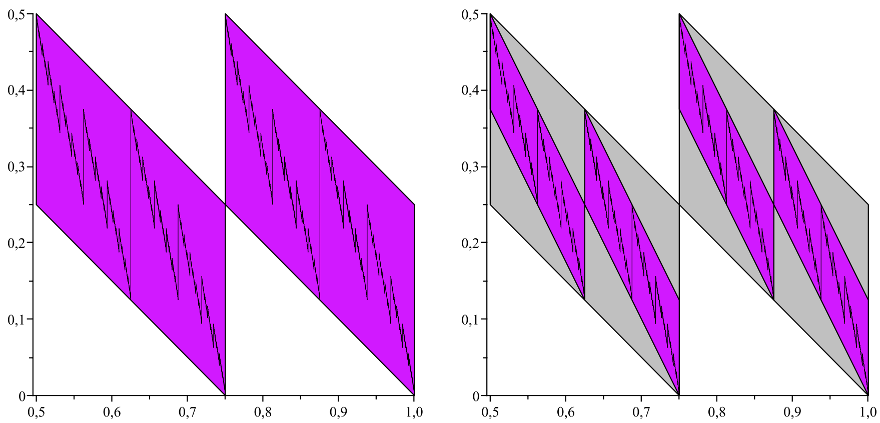
$$\begin{aligned} \gamma + \tfrac{1}{2} - \sum_{k=2}^{\infty} \frac{k\varepsilon_{k-1}}{2^k} &= \gamma + \tfrac{1}{2} - \tfrac{1}{2}\sum_{k=1}^{\infty} \frac{(k+1)\varepsilon_k}{2^k} \\ &= \gamma + \tfrac{1}{2} - \tfrac{1}{2}\sum_{k=1}^{\infty} \frac{k\varepsilon_k}{2^k} - \tfrac{1}{2}\gamma = \tfrac{1}{2}(1 - \gamma + f(\gamma)) \end{aligned}$$

concluding the proof. □

Let $\tau_0, \tau_1 : [\frac{1}{2}, 1] \times [0, \frac{1}{2}] \to [\frac{1}{2}, 1] \times [0, \frac{1}{2}]$ be the affine contractions given by

$$\tau_0(x, y) = \begin{pmatrix} \frac{1}{2}x + \frac{1}{4} \\ \frac{1}{2}(1 - x + y) \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix},$$

Figure 2.4: Contractions generating the image. The highlighted parallelograms are $\tau_0(D)$, $\tau_1(D)$ with $D = [\frac{1}{2}, 1] \times [0, \frac{1}{2}]$ (left) and their first iterates $\tau_0(\tau_0(D))$, $\tau_1(\tau_0(D))$, $\tau_0(\tau_1(D))$, $\tau_1(\tau_1(D))$ (right).




$$\tau_1(x,y) = \begin{pmatrix} \frac{1}{2}x + \frac{1}{2} \\ \frac{1}{2}(1 - x + y) \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Then $\tau_0$ maps $\tau^{-1}(G_\xi([\frac{1}{2}, 1)))$ onto its left half and $\tau_1$ onto its right half, i.e. for any $\gamma \in [\frac{1}{2}, 1)$ we have

$$\tau^{-1}(\tfrac{1}{2}\gamma + \tfrac{1}{4}, \xi(\tfrac{1}{2}\gamma + \tfrac{1}{4})) = \tau_0(\gamma, f(\gamma)) = \tau_0\big(\tau^{-1}(\gamma, \xi(\gamma))\big) \quad \text{and}$$

$$\tau^{-1}(\tfrac{1}{2}\gamma + \tfrac{1}{2}, \xi(\tfrac{1}{2}\gamma + \tfrac{1}{2})) = \tau_1(\gamma, f(\gamma)) = \tau_1\big(\tau^{-1}(\gamma, \xi(\gamma))\big),$$

which follows directly from Lemma 2.9. These contraction properties are illustrated in Figure 2.4 and show that the image $\tau^{-1}(G_\xi([\frac{1}{2}, 1)))$ can be generated by an iterated function system. By Hutchinson [31] there exists a unique non-empty compact set $F \subseteq [\frac{1}{2}, 1] \times [0, \frac{1}{2}]$, called the attractor, such that $F = \tau_0(F) \cup \tau_1(F)$ which fulfills

$$F = \bigcap_{r=1}^{\infty} \bigcup_{(i_1,\ldots,i_r)\in\{0,1\}^r} \tau_{i_r} \circ \cdots \circ \tau_{i_1}([\tfrac{1}{2}, 1] \times [0, \tfrac{1}{2}]).$$

In fact for any $(i_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ the iterated contractions $\tau_{i_r} \circ \cdots \circ \tau_{i_1}$ applied to the square $[\frac{1}{2}, 1] \times [0, \frac{1}{2}]$ converge to a single point of $F$ as $r \to \infty$ and every element of $F$ can be obtained in this way. More precisely, our construction shows that for $\gamma \in [\frac{1}{2}, 1)$ with dyadic expansion $\gamma = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^k}$ we have $\varepsilon_1 = 1$ and

$$d\big(\tau_{\varepsilon_r} \circ \cdots \circ \tau_{\varepsilon_2}([\tfrac{1}{2}, 1] \times [0, \tfrac{1}{2}]), \tau^{-1}(\gamma, \xi(\gamma))\big) \to 0$$

as $r \to \infty$, where $d(A, x) = \inf\{\|y - x\| : y \in A\}$ for $A \subseteq \mathbb{R}^2$ and $x \in \mathbb{R}^2$. Since we required $\varepsilon_k = 0$ for infinitely many $k \in \mathbb{N}$, the only possible limit points of $F$ missing are those with $\tau_{\varepsilon_k} = \tau_1$ for all but finitely many $k \geqslant 2$. For these necessarily $\gamma = \frac{1}{2} + \sum_{k=2}^{\infty} \frac{\varepsilon_k}{2^k} \in (\frac{1}{2}, 1]$ is a dyadic rational and we have

$$d\big(\tau_{\varepsilon_r} \circ \cdots \circ \tau_{\varepsilon_2}([\tfrac{1}{2}, 1] \times [0, \tfrac{1}{2}]), \tau^{-1}(\gamma, \xi(\gamma-))\big) \to 0$$

as $r \to \infty$. The above arguments show that $F$ is the closure of $\tau^{-1}(G_\xi([\frac{1}{2}, 1)))$ and since the dyadic rationals are countable, by elementary properties of the Hausdorff dimension and (2.15) we get

$$\dim_{\mathrm{H}} F = \dim_{\mathrm{H}} \tau^{-1}(G_\xi([\tfrac{1}{2}, 1))) = \dim_{\mathrm{H}} G_\xi([\tfrac{1}{2}, 1]). \tag{2.16}$$

The same equality holds for upper and lower box-counting dimensions; e.g., see page 44 in [19].

A common way to calculate the fractal dimension of the self-affine invariant set $F$ is by means of the singular value function. For an overview of such methods we refer to [22]. The linear part of both affine mappings $\tau_0$ and $\tau_1$ is equal to the linear contraction with associated matrix

$$L = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1/2 \end{pmatrix}.$$

By induction one easily calculates for $r \in \mathbb{N}$

$$L^r = \begin{pmatrix} 1/2^r & 0 \\ -r/2^r & 1/2^r \end{pmatrix}$$

and the singular values of $L^r$ are the positive roots of the eigenvalues of $(L^r)^\top L^r$ which calculate as

$$\alpha_1^{(r)} = \frac{1}{2^r}\sqrt{\frac{r^2 + 2 + \sqrt{r^4 + 4r^2}}{2}} \quad \text{and} \quad \alpha_2^{(r)} = \frac{1}{2^r}\sqrt{\frac{r^2 + 2 - \sqrt{r^4 + 4r^2}}{2}}.$$

These determine the singular value function of $L^r$ for $r \in \mathbb{N}$ given by

$$\varphi^s(L^r) = \begin{cases} (\alpha_1^{(r)})^s & \text{for } 0 < s \leqslant 1, \\ \alpha_1^{(r)}(\alpha_2^{(r)})^{s-1} & \text{for } 1 < s \leqslant 2. \end{cases} \tag{2.17}$$

Now the affinity dimension of $F$ is defined by

$$\dim_{\mathrm{A}} F = \inf\Big\{s>0 : \sum_{r=1}^{\infty} \sum_{(i_1,\dots,i_r)\in\{0,1\}^r} \varphi^s(L_{i_r}\circ\cdots\circ L_{i_1}) < \infty\Big\}$$
$$= \inf\Big\{s>0 : \sum_{r=1}^{\infty} 2^r \varphi^s(L^r) < \infty\Big\},$$

where $L_0$, $L_1$ are the linear parts of the affine contractions $\tau_0$, respectively $\tau_1$, and the last equality holds since $L_0 = L = L_1$ in our situation. The special form of the singular values $\alpha_1^{(r)}$, $\alpha_2^{(r)}$ of $L^r$ together with (2.17) shows that $\dim_{\mathrm{A}} F = 1$.
Since the union $F = \tau_0(F)\cup\tau_1(F)$ is disjoint, by Proposition 2 in [21] we get a lower bound for the Hausdorff dimension of $F$

$$\dim_{\mathrm{H}} F \geqslant \inf\Big\{s>0 : \sum_{r=1}^{\infty} \sum_{(i_1,\dots,i_r)\in\{0,1\}^r} \big(\varphi^s((L_{i_r}\circ\cdots\circ L_{i_1})^{-1})\big)^{-1} < \infty\Big\}$$
$$= \inf\Big\{s>0 : \sum_{r=1}^{\infty} 2^r \big(\varphi^s(L^{-r})\big)^{-1} < \infty\Big\}.$$

Again, by induction one easily calculates for $r\in\mathbb{N}$

$$L^{-r} = \begin{pmatrix} 2^r & 0 \\ r\,2^r & 2^r \end{pmatrix}$$

and the singular values of $L^{-r}$ are

$$\beta_1^{(r)} = 2^r\sqrt{\frac{r^2+2+\sqrt{r^4+4r^2}}{2}} \quad \text{and} \quad \beta_2^{(r)} = 2^r\sqrt{\frac{r^2+2-\sqrt{r^4+4r^2}}{2}},$$

which shows that $\dim_{\mathrm{H}} F \geqslant 1$. Since by Proposition 1 in [22] we have

$$\dim_{\mathrm{H}} F \leqslant \underline{\dim}_{\mathrm{B}} F \leqslant \overline{\dim}_{\mathrm{B}} F \leqslant \dim_{\mathrm{A}} F,$$

altogether the above calculations show:

**Theorem 2.10**
*We have* $\dim_{\mathrm{H}} G_\xi([\frac12,1]) = 1 = \dim_{\mathrm{B}} G_\xi([\frac12,1])$.

This shows that the graph of $\xi$, being the limiting object of the Steinhaus sequence (considered as a possible sequence of total gains in repeated St. Petersburg games), is not exceptional concerning the Hausdorff or box-counting dimension of the sample graph $G_Y([\frac12,1])$ calculated in Section 2.

## Chapter 3

# Manuscript 2
# Hausdorff dimension of the graph of an operator semistable Lévy process

# Declaration

## General information:

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## 3.1 Introduction

Let $X = (X(t))_{t\geq 0}$ be a Lévy process in $\mathbb{R}^d$. Namely, $X$ is a stochastically continuous process with càdlàg paths that has stationary and independent increments and starts in $X(0) = 0$ almost surely. The distribution of $X$ is uniquely determined by the distribution of $X(1)$ which can be an arbitrary infinitely divisible distribution. The process $X$ is called $(c^E, c)$-operator semistable, if the distribution of $X(1)$ is full, i.e. not supported on any lower dimensional hyperplane, and there exists a linear operator $E$ on $\mathbb{R}^d$ such that

$$\{X(ct)\}_{t\geq 0} \stackrel{\text{fd}}{=} \left\{c^E X(t)\right\}_{t\geq 0} \qquad \text{for some } c > 1. \tag{3.1}$$

Here $\stackrel{\text{fd}}{=}$ denotes equality of all finite dimensional distributions and

$$c^E := \sum_{n=0}^{\infty} \frac{(\log c)^n}{n!} E^n.$$

If for some $\alpha \in (0, 2]$ the exponent $E$ is a multiple of the identity, i.e. $E = \alpha \cdot I$, we call the process $(c^{1/\alpha}, c)$-semistable. The Lévy process is called operator stable if (3.1) holds for all $c > 0$.

The aim of this paper is to calculate the Hausdorff dimension $\dim_H \mathrm{Gr}_X(B)$ of the graph $\mathrm{Gr}_X(B) = \{(t, X(t)) : t \in B\}$ of an operator semistable Lévy process $X = (X(t))_{t\geq 0}$ for an arbitrary Borel set $B \subseteq \mathbb{R}_+$.

For an arbitrary subset $F$ of $\mathbb{R}^d$ the $s$-dimensional Hausdorff measure $\mathcal{H}^s(F)$ is defined as

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \inf \left\{ \sum_{i=1}^{\infty} |F|_i^s : |F_i| \leq \delta \text{ and } F \subseteq \bigcup_{i=1}^{\infty} F_i \right\},$$

where $|F| = \sup\{\|x - y\| : x, y \in F\}$ denotes the diameter of a set $F \subseteq \mathbb{R}^d$ and $\|\cdot\|$ is the Euclidean norm. It can be shown that the value $\dim_H F = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}$ exists and is unique for all subsets $F \subseteq \mathbb{R}^d$. The critical value $\dim_H F$ is called the Hausdorff dimension of $F$. Further details on the Hausdorff dimension can be found in [20] and [43].

In the past efforts have been made to generate dimension results for Lévy processes, which fulfill certain scaling properties. An overview can for example be found in [38] or [59]. For an operator semistable Lévy process $X$ and an arbitrary Borel set $B \subseteq \mathbb{R}_+$ Kern and Wedrich [36] calculated the Hausdorff dimension of the range $\dim_H X(B)$ in terms of the real parts of the eigenvalues of the exponent $E$ and the Hausdorff dimension of $B$. The result is a generalization of the one stated in Meerschaert and Xiao [46], who calculated the Hausdorff dimension $\dim_H X(B)$ for an operator stable Lévy process.

For an arbitrary operator semistable Lévy process $X$ our aim is to adapt the methods used to prove the results above by interpreting the graph $\mathrm{Gr}_X(B) = \{(t, X(t)) : t \in B\}$ as a process on $\mathbb{R}^{d+1}$, which fulfills the scaling property (3.1) for a certain exponent but whose distribution is not full. The method of generating dimension results for a class of Lévy processes by interpreting the graph as a $(d+1)$-dimensional Lévy process has also been employed by Manstavičius in [41].

The most prominent example of a semistable, non-stable distribution is perhaps the limit distribution of the cumulative gains in a series of St. Petersburg games. In this particular case, Kern and Wedrich [37] already calculated the Hausdorff dimension $\dim_H \mathrm{Gr}_X([0,1])$ of the corresponding graph over the interval $[0,1]$ employing the method described above. Dimension results for the graph of a stable Lévy process can be found in [4] and [32]. Furthermore, in the case that $X$ is a dilation stable Lévy process on $\mathbb{R}^d$, i.e. an operator stable Lévy process with a diagonal exponent, Xiao and Lin [60] calculated the Hausdorff dimension $\dim_H \mathrm{Gr}_X(B)$ for an arbitrary Borel set $B \subseteq \mathbb{R}_+$ and Hou [29] determined an exact Hausdorff measure function for $\mathrm{Gr}_X([0,1])$.

This paper is structured as follows: In Section 2.1 we recall spectral decomposition results from [45], which enable us to decompose the exponent $E$ and thereby the operator semistable Lévy process $X$ according to the distinct real parts of the eigenvalues of $E$. Section 2.2 contains certain uniformity and positivity results from [36] for the density functions of the process $X$, which will be helpful in the proofs of our main results. The main results on the Hausdorff dimension of the graph of an operator semistable Lévy process are stated and proven in Section 3.

Throughout this paper $K$ denotes an unspecified positive and finite constant that can vary in each occurrence. Fixed constants will be denoted by $K_1, K_2$, etc.

## 3.2 Preliminaries

### 3.2.1 Spectral decomposition

Let $X$ be a $(c^E, c)$-operator semistable Lévy process. Factor the minimal polynomial of $E$ into $q_1(x) \cdot \ldots \cdot q_p(x)$ where all roots of $q_i$ have real parts equal to $a_i$ and $a_i < a_j$ for $i < j$. Let $\alpha_j = a_j^{-1}$ so that $\alpha_1 > \ldots > \alpha_p$, and note that $0 < \alpha_j \leq 2$ by Theorem 7.1.10 in [45]. Define $V_j = \operatorname{Ker}(q_j(E))$. According to Theorem 2.1.14 in [45] $V_1 \oplus \cdots \oplus V_p$ is then a direct sum decomposition of $\mathbb{R}^d$ into $E$ invariant subspaces. In an appropriate basis, $E$ is then block-diagonal and we may write $E = E_1 \oplus \cdots \oplus E_p$ where $E_j : V_j \to V_j$ and every eigenvalue of $E_j$ has real part equal to $a_j$. Especially, every $V_j$ is an $E_j$-invariant subspace of dimension $d_j = \dim V_j$ and $d = d_1 + \ldots + d_p$. Write $X(t) = X^{(1)}(t) + \ldots + X^{(p)}(t)$ with respect to this direct sum decomposition, where by Lemma 7.1.17 in [45], $\{X^{(j)}(t), t \geq 0\}$ is a $(c^{E_j}, c)$-operator semistable Lévy process on $V_j$. We can now choose an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^d$ such that the $V_j, j \in \{1, \ldots, p\}$, are mutually orthogonal and throughout this paper we will let $\|x\| = \sqrt{\langle x, x \rangle}$ be the associated Euclidean norm. In particular we have for $t = c^r m > 0$ that

$$\|X(t)\|^2 \stackrel{\mathrm{d}}{=} \|c^{rE} X(m)\|^2 = \|c^{rE_1} X^{(1)}(m)\|^2 + \ldots + \|c^{rE_p} X^{(p)}(m)\|^2, \tag{3.2}$$

with $r \in \mathbb{Z}$ and $m \in [1, c)$.

The following lemma states a result on the growth behavior of the exponential operators $t^{E_j}$ near the origin $t = 0$. It is a variation of Lemma 2.1 in [46] and a direct consequence of Corollary 2.2.5 in [45].

**Lemma 3.1**

*For every $j \in \{1, \ldots, p\}$ and every $\epsilon > 0$ there exists a finite constant $K \geq 1$ such that for all $0 < t \leq 1$ we have*

$$K^{-1} t^{a_j + \epsilon} \leq \|t^{E_j}\| \leq K t^{a_j - \epsilon} \tag{3.3}$$

*and*

$$K^{-1} t^{-(a_j - \epsilon)} \leq \|t^{-E_j}\| \leq K t^{-(a_j + \epsilon)}. \tag{3.4}$$

### 3.2.2 Properties of the density function

The following three lemmas state uniformity results of operator semistable Lévy processes. They will be very helpful in the proofs of our main theorems. The lemmas are taken from Kern and Wedrich [36]. Let $X = \{X(t)\}_{t\geq 0}$ be a full dimensional operator semistable Lévy process on $\mathbb{R}^d$ and $g_t, t > 0$, the corresponding continuous density functions. Lemma 2.2 in [36] states the following:

**Lemma 3.2**
*The mapping $(t, x) \mapsto g_t(x)$ is continuous on $(0, \infty) \times \mathbb{R}^d$ and we have*

$$\sup_{t\in[1,c)} \sup_{x\in\mathbb{R}^d} |g_t(x)| < \infty. \tag{3.5}$$

As a consequence we get a result on the existence of negative moments of an operator semistable Lévy process $X = \{X(t)\}_{t\geq 0}$ on $\mathbb{R}^d$ given in Lemma 2.3 of [36].

**Lemma 3.3**
*For any $\delta \in (0, d)$ we have*

$$\sup_{t\in[1,c)} \mathbb{E}[\|X(t)\|^{-\delta}] < \infty. \tag{3.6}$$

Furthermore, we will need a uniform positivity result for the density functions taken from Lemma 2.4 of [36].

**Lemma 3.4**
*Let $\{X(t)\}_{t\geq 0}$ be an operator semistable Lévy process with maximal index $\alpha_1 > 1$, $d_1 = 1$ and with density $g_t$ as above. Then there exist constants $K > 0$, $r > 0$ and uniformly bounded Borel sets $J_t \subseteq \mathbb{R}^{d-1} \cong V_2 \oplus \cdots \oplus V_p$ for $t \in [1, c)$ such that*

$$g_t(x_1, \dots, x_p) \geq K > 0 \text{ for all } (x_1, \dots, x_p) \in [-r, r] \times J_t. \tag{3.7}$$

*Further, we can choose $\{J_t\}_{t\in[1,c)}$ such that $\lambda^{d-1}(J_t) \geq R$ for every $t \in [1, c)$. Note that the constants $K$, $r$ and $R$ do not depend on $t \in [1, c)$.*

**Remark 3.5**
*Note that $\alpha_1 > 1$ is a necessary condition in Lemma 3.4. To see that, take for example the $\alpha_1$-stable subordinator with $0 < \alpha_1 < 1$. Here the support of the density function is a subset of $\mathbb{R}_+$, so that (3.7) does not hold for any $r > 0$.*

## 3.3 Main results

The following two Theorems are the main results of this paper. The constants $\alpha_1, \alpha_2$ and $d_1$ are defined as in Section 2.1 by means of the spectral decompostition.

**Theorem 3.6**
*Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator semistable Lévy process on $\mathbb{R}^d$ with $d \geq 2$. Then for any Borel set $B \subseteq \mathbb{R}_+$ we have almost surely*

$$\dim_H \mathrm{Gr}_X(B) = \begin{cases} \dim_H B \cdot \max(\alpha_1, 1), & \textit{if } \alpha_1 \dim_H B \leq d_1, \\ 1 + \max(\alpha_2, 1) \cdot (\dim_H B - \frac{1}{\alpha_1}), & \textit{if } \alpha_1 \dim_H B > d_1. \end{cases}$$

The dimension result for the one-dimensional case reads as follows:

**Theorem 3.7**
*Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a $(c^{1/\alpha}, c)$-semistable Lévy process on $\mathbb{R}$. Then for any Borel set $B \subseteq \mathbb{R}_+$ we have almost surely*

$$\dim_H \mathrm{Gr}_X(B) = \begin{cases} \dim_H B \cdot \max(\alpha, 1), & \textit{if } \alpha \dim_H B \leq 1, \\ 1 + \dim_H B - \frac{1}{\alpha}, & \textit{if } \alpha \dim_H B > 1. \end{cases}$$

Let $X = (X(t))_{t \geq 0}$ be a $(c^E, c)$-operator semistable Lévy process on $\mathbb{R}^d$ and let $\alpha_1 > \ldots > \alpha_p$ denote the reciprocals of the real parts of the eigenvalues of $E$ as defined in Section 2.1. We want to calculate the Hausdorff dimension of the graph $\mathrm{Gr}_X(B)$ of $X$ for an arbitrary Borel set $B \subseteq \mathbb{R}_+$. Therefore, we define the process $Z = (Z(t))_{t \geq 0}$ as $Z(t) = (t, X(t))$ for all $t \geq 0$. This gives us $\dim_H Z(B) = \dim_H \mathrm{Gr}_X(B)$. One can easily see that $Z$ is also a Lévy process and fulfills the scaling property of a $(c^F, c)$-operator semistable process where

$$F = \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix}.$$

Nevertheless, the process $Z$ itself is not operator semistable in the sense of the definition given in the Introduction as the distribution of $Z(1)$ is obviously not full.
As mentioned in the Introduction, the Hausdorff dimension $\dim_H X(B)$ of the range of an operator semistable Lévy process $X$ has already been calculated in [36] as

$$\dim_{\mathrm{H}} X(B) = \begin{cases} \alpha_1 \dim_{\mathrm{H}} B & \text{if } \alpha_1 \dim_{\mathrm{H}} B \leq d_1, \\ 1 + \alpha_2 \left( \dim_{\mathrm{H}} B - \frac{1}{\alpha_1} \right) & \text{if } \alpha_1 \dim_{\mathrm{H}} B > d_1, \end{cases} \tag{3.8}$$

almost surely for $d \geq 2$. Hence, for the reasons mentioned above, we are now able to use the parts of the result (3.8) and the corresponding proofs where fullness of the process was not required. All other parts, however, have to be calculated anew.
The proof of Theorem 3.1 is split into two parts. First we will obtain the upper bounds for $\dim_H \mathrm{Gr}_X(B)$ by choosing a suitable sequence of coverings. This method goes back to Pruitt and Taylor [48] and Hendricks [28]. Afterwards we will use standard capacity arguments in order to prove the lower bounds.

### 3.3.1 Upper Bounds

For a Lévy process $\{X(t)\}_{t\geq 0}$ let

$$T_X(a,s) = \int_0^s 1_{B(0,a)}(X(t))dt \tag{3.9}$$

be the sojourn time in the closed ball $B(0,a)$ with radius $a$ centered at the origin up to time $s > 0$.
The following covering lemma is due to Pruitt and Taylor (see Lemma 6.1 in [48]):

**Lemma 3.8**
*Let $Z = \{Z(t)\}_{t\geq 0}$ be a Lévy process in $\mathbb{R}^{d+1}$ and let $\Lambda(a)$ be a fixed $K_1$-nested family of cubes in $\mathbb{R}^{d+1}$ of side $a$ with $0 < a \leq 1$. For any $u \geq 0$ let $M_u(a,s)$ be the number of cubes in $\Lambda(a)$ hit by $Z(t)$ at some time $t \in [u, u+s]$. Then*

$$\mathbb{E}\left[M_u(a,s)\right] \leq 2\,K_1 s \cdot \left(\mathbb{E}\left[T_Z\left(\tfrac{a}{3}, s\right)\right]\right)^{-1}.$$

In order to prove the upper bounds of Theorem 3.6 we now need to calculate sharp lower bounds of the expected sojourn times $\mathbb{E}[T_Z(a,s)]$ of the graph $Z = \{(t, X(t)), t \geq 0\}$ of an operator semistable Lévy process on $\mathbb{R}^d$.
In their paper Kern and Wedrich calculated in Theorem 2.6 in [36] upper and lower bounds for the expected sojourn times $\mathbb{E}[T_X(a,s)]$ of an operator semistable Lévy process:

**Theorem 3.9**
*Let $X = \{X(t)\}_{t\geq 0}$ be as in Theorem 3.1. For any $0 < \alpha_2'' < \alpha_2 < \alpha_2' < \alpha_1'' < \alpha_1 < \alpha_1'$ there exist positive and finite constants $K_6, \ldots, K_9$ such that*

*(i) if $\alpha_1 \leq d_1$, then for all $0 < a \leq 1$ and $a^{\alpha_1} \leq s \leq 1$ we have*

$$K_6 a^{\alpha_1'} \leq \mathbb{E}[T_X(a,s)] \leq K_7 a^{\alpha_1''}.$$

*(ii)* *if* $\alpha_1 > d_1 = 1$, *for all* $0 < a \le a_0$ *with* $a_0 > 0$ *sufficiently small, and all* $a^{\alpha_2} \le s \le 1$ *we have*

$$K_8 a^{\rho'} \le \mathbb{E}[T_X(a,s)] \le K_9 a^{\rho''},$$

*where* $\rho'' = 1 + \alpha_2''(1 - \frac{1}{\alpha_1})$ *and* $\rho' = 1 + \alpha_2'(1 - \frac{1}{\alpha_1})$.

Looking at the proof of the lower bounds of Theorem 3.9 (i) (i.e. Theorem 2.6 (i) in [36]), we find that the condition $\alpha_1 \le d_1$ is not needed here. Hence, the same proof additionally gives us the following corollary:

**Corollary 3.10**

*Let* $X = \{X(t)\}_{t\ge 0}$ *be as in Theorem 3.1. For any* $0 < \alpha_1 < \alpha_1'$ *there exists a positive and finite constant* $\tilde{K}_6$ *such that for all* $0 < a \le 1$ *and* $a^{\alpha_1} \le s \le 1$ *we have*

$$\tilde{K}_6 a^{\alpha_1'} \le \mathbb{E}[T_X(a,s)].$$

Similarly to the results above we will now calculate lower bounds for the expected sojourn times $\mathbb{E}[T_Z(a,s)]$ of the graph $Z = \{(t, X(t)), t \ge 0\}$ of an operator semistable Lévy process on $\mathbb{R}^d$. The upper bounds can also be calculated but are not stated here as they are not needed to determine the Hausdorff dimension.

**Theorem 3.11**

*Let* $Z = \{(t, X(t)), t \ge 0\}$, *where* $X = \{X(t), t \ge 0\}$ *is as in Theorem 3.1.*

*(i)* *If* $\alpha_1 \ge 1$, *there exists a positive and finite constant* $K_2$ *such that for all* $0 < a \le 1$ *and* $a^{\alpha_1} \le s \le 1$ *and any* $\alpha_1 < \alpha_1'$

$$\mathbb{E}[T_Z(a,s)] \ge K_2 a^{\alpha_1'}.$$

*(ii)* *If* $\alpha_1 < 1$, *there exists a positive and finite constant* $K_3$ *such that for all* $0 < a \le 1$ *and* $a \le s \le 1$ *and any* $\epsilon > 0$

$$\mathbb{E}[T_Z(a,s)] \ge K_3 a^{1+\epsilon}.$$

*(iii)* *If* $\alpha_1 > d_1 = 1$ *and* $\alpha_2 \ge 1$, *there exists a positive and finite constant* $K_4$ *such that for any* $0 < \alpha_2 < \alpha_2' < \alpha_1$ *and all* $a > 0$ *small enough, say* $0 < a \le a_0$, *and all* $a^{\alpha_2} \le s \le 1$

$$\mathbb{E}[T_Z(a,s)] \ge K_4 a^{1+\alpha_2'(1-\frac{1}{\alpha_1})}.$$

*(iv) If $\alpha_1 > d_1 = 1$ and $\alpha_2 < 1$, there exists a positive and finite constant $K_5$ such that for all $a > 0$ small enough, say $0 < a \le a_0$, and all $\frac{a}{\sqrt{p+1}} \le s \le 1$*

$$\mathbb{E}[T_Z(a,s)] \ge K_5 a^{2-\frac{1}{\alpha_1}}.$$

*Proof.* (i) & (ii) Let $\alpha_1' > \alpha_1$. Looking at the proof of Corollary 3.10 (i.e. Theorem 2.6 part (i) in [36]) one realizes that the fullness is not needed there. Hence we can use this result to prove part (i) and (ii) of the present theorem. In order to do so we need to further examine the exponent

$$F = \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix}$$

of the process $Z$. Analogously to Section 2.1 denote by $\tilde{\alpha}_1 > \ldots > \tilde{\alpha}_q$ the reciprocals of the real parts of the eigenvalues of $F$ and by $\tilde{d}_1$ the dimension of the $F_1$ invariant subspace of $\mathbb{R}^{d+1}$, where $F_1$ is (analagously to $E_1$) the blockmatrix, whose eigenvalues have real part equal to $\tilde{\alpha}_1^{-1}$. Furthermore, let $\tilde{\alpha}_1'$ be such that $\tilde{\alpha}_1' = \tilde{\alpha}_1 + \alpha_1' - \alpha_1$.
In part (i) we have that $\alpha_1 \ge 1$. Then $\tilde{\alpha}_1 = \alpha_1$ and $\tilde{d}_1 \ge d_1$. By Corollary 3.10 there now exists a positive constant $K_2$ such that

$$\mathbb{E}[T_Z(a,s)] \ge K_2 a^{\tilde{\alpha}_1'} = K_2 a^{\alpha_1'}$$

for all $0 < a \le 1$ and $a^{\alpha_1} \le s \le 1$.
On the other hand in part (ii) we have $\alpha_1 < 1$. Then $\tilde{\alpha}_1 = 1$ and $\tilde{d}_1 = 1$. For any $\epsilon > 0$, by Corollary 3.10 there now exists a postive constant $K_3$ such that

$$\mathbb{E}[T_Z(a,s)] \ge K_3 a^{\tilde{\alpha}_1+\epsilon} = K_3 a^{1+\epsilon}$$

for all $0 < a \le 1$ and $a \le s \le 1$.
(iii) Let $0 < \alpha_j < \alpha_j' < \alpha_{j-1}$ for all $j = 2, \ldots, p$. Choose $i_0, i_1 \in \mathbb{N}_0$ such that $c^{-i_0} < a \le c^{-i_0+1}$ and $c^{-i_1} < c^{-i_0\alpha_2'} \le c^{-i_1+1}$. For $t \in (0,1]$ we can write $t = mc^{-i}$ with $m \in [1,c)$ and $i \in \mathbb{N}_0$. By Lemma 3.1 we then have

$$\|X^{(j)}(t)\| \stackrel{\mathrm{d}}{=} \|c^{-iE_j} X^{(j)}(m)\| \le \|c^{-iE_j}\| \, \|X^{(j)}(m)\| \le K\, c^{-i/\alpha_j'} \|X^{(j)}(c^i t)\| \tag{3.10}$$

for all $j = 1, \ldots, p$. Note that, since $d_1 = 1$, for $j = 1$ in (3.10) we can choose $K = 1$ and $\alpha_1' = \alpha_1$. Furthermore, since $\alpha_2' > 1$ there exists a constant $a_0 > 0$ such that for all

$0 < a \le a_0$ we have $a^{\alpha_2'} \le \frac{a}{\sqrt{p+1}}$. Altogether, for $0 < a \le a_0$ this gives us

$$\begin{aligned}
\mathbb{E}[T_Z(a,s)] &= \int_0^s \mathbb{P}\left(\|Z(t)\| < a\right) dt = \int_0^s \mathbb{P}\left(\|(t, X(t))\| < a\right) dt \\
&\ge \int_0^s \mathbb{P}\left(|X^{(1)}(t)| < \frac{a}{\sqrt{p+1}}, \|X^{(j)}(t)\| < \frac{a}{\sqrt{p+1}}, 2 \le j \le p, |t| < \frac{a}{\sqrt{p+1}}\right) dt \\
&\ge \int_0^{a^{\alpha_2'}} \mathbb{P}\left(|X^{(1)}(t)| < \frac{a}{\sqrt{p+1}}, \|X^{(j)}(t)\| < \frac{a}{\sqrt{p+1}}, 2 \le j \le p\right) dt \\
&\ge \int_0^{c^{-i_1}} \mathbb{P}\left(|X^{(1)}(t)| < \frac{a}{\sqrt{p+1}}, \|X^{(j)}(t)\| < \frac{a}{\sqrt{p+1}}, 2 \le j \le p\right) dt \\
&= \sum_{i=i_1+1}^{\infty} \int_{c^{-i}}^{c^{-i+1}} \mathbb{P}\left(|X^{(1)}(t)| < \frac{a}{\sqrt{p+1}}, \|X^{(j)}(t)\| < \frac{a}{\sqrt{p+1}}, 2 \le j \le p\right) dt \\
&\ge \sum_{i=i_1+1}^{\infty} \int_{c^{-i}}^{c^{-i+1}} \mathbb{P}\left(|X^{(1)}(c^i t)| < \frac{c^{\frac{i}{\alpha_1} - i_0}}{\sqrt{p+1}}, \|X^{(j)}(c^i t)\| < K^{-1} \frac{c^{\frac{i}{\alpha_j'} - i_0}}{\sqrt{p+1}}, 2 \le j \le p\right) dt \\
&\ge \sum_{i=i_1+1}^{\infty} c^{-i} \int_1^c \mathbb{P}\begin{pmatrix} |X^{(1)}(m)| < \frac{c^{\frac{i}{\alpha_1} - i_0}}{\sqrt{p+1}} \text{ and} \\ \|X^{(j)}(m)\| < K^{-1} \frac{c^{\frac{i}{\alpha_j'} - i_0}}{\sqrt{p+1}}, 2 \le j \le p \end{pmatrix} dm,
\end{aligned}$$

where the penultimate inequality follows from (3.10). By Lemma 3.4 choose $K_{10} > 0$, $r > 0$ and uniformly bounded Borel sets $J_m \subseteq \mathbb{R}^{d-1}$ with Lebesgue measure $0 < K_9 \le \lambda^{d-1}(J_m) < \infty$ for every $m \in [1, c)$ such that the bounded continuous density $g_m(x_1, \ldots, x_p)$ of $X(m) = X^{(1)}(m) + \ldots + X^{(p)}(m)$ fulfills

$$g_m(x_1, \ldots, x_p) \ge K_{10} > 0 \quad \text{for all} \quad (x_1, \ldots, x_p) \in [-r, r] \times J_m$$

and for every $m \in [1, c)$. Since $\{J_m\}_{m \in [1,c)}$ is uniformly bounded by Lemma 2.4 we are able to choose $0 < \delta \le c^{-3} < 1$ such that

$$\bigcup_{m \in [1,c)} J_m \subseteq \left\{ \|x_j\| \le \frac{K^{-1} c^{\frac{-\alpha_1}{\alpha_p}}}{\delta \sqrt{p+1}}, 2 \le j \le p \right\}.$$

Let $\eta = c^{\frac{2}{\alpha_p}} / \left(r\sqrt{p+1}\right)$.

Since $\alpha_1 > \alpha_2' > 1$ there exists a constant $a_0 \in (0, 1]$ such that $(\eta a)^{\alpha_1} < (\delta a)^{\alpha_2'}$ for all $0 < a \le a_0$. Now, choose $i_2, i_3 \in \mathbb{N}_0$ such that $c^{-i_2} < \left(\delta c^{-i_0+1}\right)^{\alpha_2'} \le c^{-i_2+1}$ and

$c^{-i_3} < \left(\eta c^{-i_0}\right)^{\alpha_1} \leq c^{-i_3+1}$. Note that

$$c^{-i_3} < \left(\eta c^{-i_0}\right)^{\alpha_1} < (\delta a)^{\alpha_2'} \leq \left(\delta c^{-i_0+1}\right)^{\alpha_2'} \leq c^{-i_2+1}$$

and

$$c^{-(i_1+1)} \geq c^{-2} \cdot c^{-i_0\alpha_2'} \geq \left(c^{-2} \cdot c^{-i_0}\right)^{\alpha_2'} = \left(c^{-3} \cdot c^{-i_0+1}\right)^{\alpha_2'} \geq \left(\delta c^{-i_0+1}\right)^{\alpha_2'} > c^{-i_2},$$

hence $i_3 \geq i_2 - 1$ and $i_1 + 1 \leq i_2$. We further have for all $i = i_2, \ldots, i_3 + 1$ and every $j = 2, \ldots, p$

$$\frac{c^{i/\alpha_1 - i_0}}{\sqrt{p+1}} \leq \frac{c^{(i_3+1)/\alpha_1 - i_0}}{\sqrt{p+1}} \leq \frac{c^{2/\alpha_1}(\eta c^{-i_0})^{-1} c^{-i_0}}{\sqrt{p+1}} = \frac{c^{2/\alpha_1}}{\eta\sqrt{p+1}} < r \tag{3.11}$$

and, since $\alpha_2' \geq \alpha_j'$ for $j = 2, \ldots, p$,

$$\begin{aligned} \frac{c^{i/\alpha_j' - i_0}}{\sqrt{p+1}} &\geq \frac{c^{i_2/\alpha_j' - i_0}}{\sqrt{p+1}} \geq \frac{(\delta c^{-i_0+1})^{-\alpha_2'/\alpha_j'} c^{-i_0}}{\sqrt{p+1}} \\ &= \frac{(\delta^{-1} c^{i_0-1})^{\alpha_2'/\alpha_j'} c^{-i_0}}{\sqrt{p+1}} \geq \frac{c^{-\alpha_2'/\alpha_j'}}{\delta\sqrt{p+1}} \geq \frac{c^{-\alpha_1/\alpha_p}}{\delta\sqrt{p+1}}. \end{aligned} \tag{3.12}$$

Let $I_m = (-\frac{c^{i/\alpha_1 - i_0}}{\sqrt{p+1}}, \frac{c^{i/\alpha_1 - i_0}}{\sqrt{p+1}}) \times J_m$. Then together with the calculations above, we get using (3.11) and (3.12)

$$\begin{aligned} \mathbb{E}[T(a,s)] &\geq \sum_{i=i_2}^{i_3+1} c^{-i} \int_1^c P\left( \begin{array}{c} |X^{(1)}(m)| < \frac{c^{i/\alpha_1 - i_0}}{\sqrt{p+1}} \text{ and} \\ \|X^{(j)}(m)\| \leq K^{-1} \frac{c^{i/\alpha_j' - i_0}}{\sqrt{p+1}}, 2 \leq j \leq p \end{array} \right) dm \\ &\geq \sum_{i=i_2}^{i_3+1} c^{-i} \int_1^c \int_{I_m} g_m(x)\, dx\, dm \geq \sum_{i=i_2}^{i_3+1} c^{-i} (c-1)\, 2\, \frac{c^{i/\alpha_1 - i_0}}{\sqrt{p+1}} \cdot K_{10} \cdot K_9 \\ &= K c^{-i_0} \sum_{i=i_2}^{i_3+1} \left(c^{-i}\right)^{1-\frac{1}{\alpha_1}} = K c^{-i_0} \left( \frac{1 - \left(c^{-(i_3+2)}\right)^{1-\frac{1}{\alpha_1}}}{1 - c^{\frac{1}{\alpha_1} - 1}} - \frac{1 - \left(c^{-i_2}\right)^{1-\frac{1}{\alpha_1}}}{1 - c^{\frac{1}{\alpha_1} - 1}} \right) \\ &= K c^{-i_0} \left( \left(c^{-i_2}\right)^{1-\frac{1}{\alpha_1}} - \left(c^{-(i_3+2)}\right)^{1-\frac{1}{\alpha_1}} \right) \\ &\geq K_{41} \left(c^{-i_0}\right)^{1+\alpha_2'\left(1-\frac{1}{\alpha_1}\right)} - K_{42} \left(c^{-i_0}\right)^{\alpha_1}. \end{aligned} \tag{3.13}$$

Since $1+\alpha_2'(1-\frac{1}{\alpha_1})<1+\alpha_1(1-\frac{1}{\alpha_1})=\alpha_1$ we have $\left(c^{-i_0}\right)^{\alpha_1-\left(1+\alpha_2'\left(1-\frac{1}{\alpha_1}\right)\right)}\to 0$ if $a\to 0$, i.e. $i_0\to\infty$. Hence we can further choose $a_0$ sufficiently small, such that

$$\mathbb{E}[T_Z(a,s)]\geq K_4 a^{1+\alpha_2'(1-\frac{1}{\alpha_1})}$$

for all $0<a\leq a_0$.

(iv) Let $0<\alpha_j<\alpha_j'<\alpha_{j-1}$ for all $j=2,\dots,p$, and additionally, let $\alpha_2<\alpha_2'<1$. Now choose $i_0,i_1\in\mathbb{N}_0$ such that $c^{-i_0}<a\leq c^{-i_0+1}$ and $c^{-i_1}<\frac{a}{\sqrt{p+1}}\leq c^{-i_1+1}$. For $t\in(0,1]$ we can write $t=mc^{-i}$ with $m\in[1,c)$ and $i\in\mathbb{N}_0$. By (3.10) we then have

$$\|X^{(j)}(t)\|\overset{\mathrm{d}}{=}\|c^{-iE_j}X^{(j)}(m)\|\leq K\,c^{-i/\alpha_j'}\|X^{(j)}(c^it)\| \tag{3.14}$$

for all $j=1,\dots,p$. Note that, since $d_1=1$, for $j=1$ in (3.14) we can choose $K=1$ and $\alpha_1'=\alpha_1$. Similarly to the proof of part (iii), this gives us

$$\begin{aligned}
\mathbb{E}[T_Z(a,s)] &\geq \int_0^{\frac{a}{\sqrt{p+1}}}\mathbb{P}\left(|X^{(1)}(t)|<\frac{a}{\sqrt{p+1}},\|X^{(j)}(t)\|<\frac{a}{\sqrt{p+1}},2\leq j\leq p\right)dt\\
&\geq \int_0^{c^{-i_1}}\mathbb{P}\left(|X^{(1)}(t)|<\frac{a}{\sqrt{p+1}},\|X^{(j)}(t)\|<\frac{a}{\sqrt{p+1}},2\leq j\leq p\right)dt\\
&\geq \sum_{i=i_1+1}^{\infty}\int_{c^{-i}}^{c^{-i+1}}\mathbb{P}\left(|X^{(1)}(c^it)|<\frac{c^{\frac{i}{\alpha_1}-i_0}}{\sqrt{p+1}},\|X^{(j)}(c^it)\|<K^{-1}\frac{c^{\frac{i}{\alpha_j'}-i_0}}{\sqrt{p+1}},2\leq j\leq p\right)dt\\
&\geq \sum_{i=i_1+1}^{\infty}c^{-i}\int_1^c\mathbb{P}\begin{pmatrix}|X^{(1)}(m)|<\frac{c^{\frac{i}{\alpha_1}-i_0}}{\sqrt{p+1}}\text{ and}\\ \|X^{(j)}(m)\|<K^{-1}\frac{c^{\frac{i}{\alpha_j'}-i_0}}{\sqrt{p+1}},2\leq j\leq p\end{pmatrix}dm,
\end{aligned}$$

where the penultimate inequality follows from (3.14). As in the proof of part (iii), by Lemma 3.4 choose $K_{10}>0$, $r>0$ and uniformly bounded Borel sets $J_m\subseteq\mathbb{R}^{d-1}$ with Lebesgue measure $0<K_9\leq\lambda^{d-1}(J_m)<\infty$ for every $m\in[1,c)$ such that the bounded continuous density $g_m(x_1,\dots,x_p)$ of $X(m)=X^{(1)}(m)+\ldots+X^{(p)}(m)$ fulfills

$$g_m(x_1,\dots,x_p)\geq K_{10}>0\quad\text{for all}\quad(x_1,\dots,x_p)\in[-r,r]\times J_m$$

and for every $m\in[1,c)$. Since $\{J_m\}_{m\in[1,c)}$ is uniformly bounded by Lemma 2.4 we are now able to choose $0<\delta\leq(\sqrt{p+1}\cdot c^3)^{-1}<1$ such that

$$\bigcup_{m\in[1,c)}J_m\subseteq\left\{\|x_j\|\leq\frac{K^{-1}c^{\frac{-\alpha_1}{\alpha_p}}}{\delta\sqrt{p+1}},2\leq j\leq p\right\}.$$

Let $\eta = c^{\frac{2}{\alpha_p}} / \big(r\sqrt{p+1}\big)$.

Since $\alpha_1 > 1$ there exists a constant $0 < a_0 \le 1$ such that we have $(\eta a)^{\alpha_1} < \delta a$ for all $0 < a \le a_0$. Now, choose $i_2, i_3 \in \mathbb{N}_0$ such that $c^{-i_2} < \delta c^{-i_0+1} \le c^{-i_2+1}$ and $c^{-i_3} < \big(\eta c^{-i_0}\big)^{\alpha_1} \le c^{-i_3+1}$. Note that

$$c^{-i_3} < \big(\eta c^{-i_0}\big)^{\alpha_1} < (\eta a)^{\alpha_1} < \delta a \le \delta c^{-i_0+1} \le c^{-i_2+1}$$

and, since $\delta \le \frac{1}{\sqrt{p+1}} \cdot c^{-3}$,

$$c^{-(i_1+1)} \ge c^{-2} \cdot \frac{a}{\sqrt{p+1}} > \frac{c^{-3}}{\sqrt{p+1}} \cdot c^{-i_0+1} \ge \delta c^{-i_0+1} > c^{-i_2}.$$

Hence, we also get $i_2 - 1 \le i_3$ and $i_1 + 1 \le i_2$. As in (3.11), we further have for all $i = i_2, \ldots, i_3 + 1$ that

$$\frac{c^{i/\alpha_1 - i_0}}{\sqrt{p+1}} \le r \tag{3.15}$$

and, since $\alpha'_j < 1$ for all $j = 2, \ldots, p$,

$$\begin{aligned} \frac{c^{i/\alpha'_j - i_0}}{\sqrt{p+1}} &\ge \frac{c^{i_2/\alpha'_j - i_0}}{\sqrt{p+1}} \ge \frac{\big(\delta c^{-i_0+1}\big)^{-1/\alpha'_j} c^{-i_0}}{\sqrt{p+1}} = \frac{\big(\delta^{-1} c^{i_0-1}\big)^{1/\alpha'_j} c^{-i_0}}{\sqrt{p+1}} \\ &\ge \frac{c^{-1/\alpha'_j}}{\delta\sqrt{p+1}} \ge \frac{c^{-1/\alpha_p}}{\delta\sqrt{p+1}} \ge \frac{c^{-\alpha_1/\alpha_p}}{\delta\sqrt{p+1}}. \end{aligned} \tag{3.16}$$

Define the subsets $\{I_m : m \in [1, c)\} \subseteq \mathbb{R}^d$ as above. Similarly to the calculations in (3.13), using (3.15) and (3.16) we arrive at

$$\mathbb{E}[T_Z(a,s)] \ge K c^{-i_0} \left( \big(c^{-i_2}\big)^{1-\frac{1}{\alpha_1}} - \Big(c^{-(i_3+2)}\Big)^{1-\frac{1}{\alpha_1}} \right) \tag{3.17}$$

Altogether, we get

$$\begin{aligned} \mathbb{E}[T_Z(a,s)] &\ge K_{51} c^{-i_0} \left(c^{-i_0}\right)^{1-\frac{1}{\alpha_1}} - K_{52} c^{-i_0} \left(c^{-i_0}\right)^{\alpha_1 - 1} \\ &= K_{51} \left(c^{-i_0}\right)^{2-\frac{1}{\alpha_1}} - K_{52} \left(c^{-i_0}\right)^{\alpha_1}. \end{aligned}$$

Since $\alpha_1 > 1$ and therefore $2 - 1/\alpha_1 = 1 + (1 - 1/\alpha_1) < 1 + \alpha_1(1 - 1/\alpha_1) = \alpha_1$, we can choose $a_0$ sufficiently small, such that

$$\mathbb{E}[T_Z(a,s)] \ge K_5 a^{2-\frac{1}{\alpha_1}}.$$

for all $0 < a \le a_0$. □

Similarly to the proof of Lemma 3.4 in [36], we can now find a suitable covering of $Z(B)$ and prove the desired upper bounds.

**Lemma 3.12**
*Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator semistable Lévy process on $\mathbb{R}^d$ with $d \geq 2$. Then for any Borel set $B \subseteq \mathbb{R}_+$ we have almost surely*

$$\dim_H \mathrm{Gr}_X(B) \leq \begin{cases} \alpha_1 \dim_H B, & \text{if } \alpha_1 \dim_H B \leq d_1, \alpha_1 \geq 1, \quad (i) \\ \dim_H B, & \text{if } \alpha_1 \dim_H B \leq d_1, \alpha_1 < 1, \quad (ii) \\ 1 + \alpha_2(\dim_H B - \frac{1}{\alpha_1}), & \text{if } \alpha_1 \dim_H B > d_1, \alpha_1 > \alpha_2 \geq 1, (iii) \\ 1 + \dim_H B - \frac{1}{\alpha_1}, & \text{if } \alpha_1 \dim_H B > d_1, \alpha_1 > 1 > \alpha_2. \ (iv) \end{cases}$$

*Proof.* (i) Assume $\alpha_1 \dim_H B \leq d_1$ and $\alpha_1 \geq 1$. Analogously to the proof of Lemma 3.4 in [36] for the case $\alpha_1 \dim_H B \leq 1$, it follows by Lemma 3.8 and Theorem 3.11 (i) that $\dim_H Z(B) \leq \alpha_1 \dim_H B$ almost surely.
(ii) Assume $\alpha_1 \dim_H B \leq d_1$ and $\alpha_1 < 1 \leq d_1$. For $\gamma > \dim_H B$, choose $\beta > 1$ such that $\gamma' = 1 - \beta + \gamma > \dim_H B$. For $\varepsilon \in (0,1]$, by definition of the Hausdorff dimension, there exists a sequence $\{I_i\}_{i\in\mathbb{N}}$ of intervals in $\mathbb{R}_+$ of length $|I_i| < \varepsilon$ such that

$$B \subseteq \bigcup_{i=1}^{\infty} I_i \quad \text{and} \quad \sum_{i=1}^{\infty} |I_i|^{\gamma'} < 1.$$

Let $s_i = b_i := |I_i|$; then $b_i/3 < s_i$. It follows by Lemma 3.8 and Theorem 3.11 (ii) that $Z(I_i)$ can be covered by $M_i$ cubes $C_{ij} \in \Lambda(b_i)$ of side $b_i$ such that for every $i \in \mathbb{N}$ we have

$$\mathbb{E}[M_i] \leq 2K_1 s_i \left(\mathbb{E}\left[T_Z\left(\tfrac{b_i}{3}, s_i\right)\right]\right)^{-1} \leq 2K_1 s_i K_3^{-1} \left(\tfrac{b_i}{3}\right)^{-\beta} = K\, s_i b_i^{-\beta} = K\, |I_i|^{1-\beta}.$$

Note that $Z(B) \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{M_i} C_{ij}$, where $b_i\sqrt{d+1}$ is the diameter of $C_{ij}$. In other words, $\{C_{ij}\}$ is a $(\varepsilon\sqrt{d+1})$-covering of $X(B)$. By monotone convergence we have

$$\mathbb{E}\left[\sum_{i=1}^{\infty} M_i b_i^{\gamma}\right] = \sum_{i=1}^{\infty} \mathbb{E}\left[M_i b_i^{\gamma}\right] \leq \sum_{i=1}^{\infty} K\, |I_i|^{1-\beta}\, |I_i|^{\gamma} = K \sum_{i=1}^{\infty} |I_i|^{\gamma'} \leq K.$$

Letting $\varepsilon \to 0$, i.e $b_i \to 0$ and applying Fatou's lemma, we get

$$\mathbb{E}\left[\mathcal{H}^{\gamma}(X(B))\right] \leq \mathbb{E}\left[\liminf_{\varepsilon\to 0} \sum_{i=1}^{\infty} \sum_{j=1}^{M_i} \left(b_i\sqrt{d+1}\right)^{\gamma}\right]$$

$$\leq \liminf_{\varepsilon\to 0} \sqrt{d+1}^{\,\gamma}\,\mathbb{E}\left[\sum_{i=1}^{\infty} M_i b_i^{\gamma}\right] \leq \sqrt{d+1}^{\,\gamma} K < \infty,$$

which shows that $\dim_H Z(B) \leq \gamma$ almost surely. And since $\gamma > \dim_H B$ is arbitrary, we get $\dim_H Z(B) \leq \dim_H B$ almost surely.

(iii) Assume $\alpha_1 \dim_H B > d_1$ and $\alpha_2 \geq 1$. Since $\dim_H B \leq 1$, we have $\alpha_1 > d_1 = 1$. For $\gamma > \dim_H B$ choose $\alpha_2' > \alpha_2$ such that $\gamma' = 1 - \frac{\alpha_2'}{\alpha_2} + \frac{\alpha_2'}{\alpha_2}\gamma > \dim_H B$. For $\varepsilon \in (0,1]$ define $\{I_i\}_{i\in\mathbb{N}}$ as in part (ii) and let $s_i := |I_i|$ and $b_i := |I_i|^{\frac{1}{\alpha_2}}$. Then $(b_i/3)^{\alpha_2} < s_i$. Again, by Lemma 3.8 and Theorem 3.11 (iii) it follows that $Z(I_i)$ can be covered by $M_i$ cubes $C_{ij} \in \Lambda(b_i)$ of side $b_i$ such that for every $i \in \mathbb{N}$ we have

$$\begin{aligned}\mathbb{E}[M_i] &\leq 2K_1 s_i \left(\mathbb{E}\left[T_Z\left(\tfrac{b_i}{3}, s_i\right)\right]\right)^{-1} \leq 2K_1 s_i K_4^{-1}\left(\tfrac{b_i}{3}\right)^{-1-\alpha_2'(1-\frac{1}{\alpha_1})}\\ &= K\, s_i b_i^{-1-\alpha_2'(1-\frac{1}{\alpha_1})} = K\,|I_i|^{1-\frac{1}{\alpha_2}-\frac{\alpha_2'}{\alpha_2}\cdot(1-\frac{1}{\alpha_1})}.\end{aligned}$$

By monotone convergence we have

$$\mathbb{E}\left[\sum_{i=1}^{\infty} M_i b_i^{1+\alpha_2'(\gamma-\frac{1}{\alpha_1})}\right] \leq \sum_{i=1}^{\infty} K\,|I_i|^{1-\frac{1}{\alpha_2}-\frac{\alpha_2'}{\alpha_2}\cdot(1-\frac{1}{\alpha_1})}\,|I_i|^{\frac{1}{\alpha_2}+\frac{\alpha_2'}{\alpha_2}(\gamma-\frac{1}{\alpha_1})} = K\sum_{i=1}^{\infty}|I_i|^{\gamma'} \leq K.$$

Since $\gamma > \dim_H B$ and $\alpha_2' > \alpha_2$ are arbitrary, with the same arguments as in part (ii) we get $\dim_H Z(B) \leq 1 + \alpha_2(\dim_H B - \frac{1}{\alpha_1})$ almost surely.

(iv) Assume $\alpha_1 \dim_H B > d_1$ and $\alpha_2 < 1$. Since $\dim_H B \leq 1$, we have $\alpha_1 > d_1 = 1$. Let $\gamma = \gamma' > \dim_H B$. For $\varepsilon \in (0,1]$ define $\{I_i\}_{i\in\mathbb{N}}$ as in part (ii) and let $s_i := |I_i|$ and $b_i := |I_i|$. Then $b_i/(3\sqrt{p+1}) < s_i$. Again, by Lemma 3.8 and Theorem 3.11 (iv) it follows that $Z(I_i)$ can be covered by $M_i$ cubes $C_{ij} \in \Lambda(b_i)$ of side $b_i$ such that for every $i \in \mathbb{N}$ we have

$$\mathbb{E}[M_i] \leq 2K_1 s_i \left(\mathbb{E}\left[T_Z\left(\tfrac{b_i}{3}, s_i\right)\right]\right)^{-1} \leq 2K_1 s_i K_5^{-1}\left(\tfrac{b_i}{3}\right)^{-2+\frac{1}{\alpha_1}} = K\, s_i b_i^{-2+\frac{1}{\alpha_1}} = K\,|I_i|^{-1+\frac{1}{\alpha_1}}.$$

By monotone convergence we have

$$\mathbb{E}\left[\sum_{i=1}^{\infty} M_i b_i^{1+\gamma-\frac{1}{\alpha_1}}\right] \leq \sum_{i=1}^{\infty} K\,|I_i|^{-1+\frac{1}{\alpha_1}}\,|I_i|^{1+\gamma-\frac{1}{\alpha_1}} = K\sum_{i=1}^{\infty}|I_i|^{\gamma} = K\sum_{i=1}^{\infty}|I_i|^{\gamma'} \leq K.$$

Since $\gamma > \dim_H B$ is arbitrary, we get $\dim_H Z(B) \leq 1 + \dim_H B - \frac{1}{\alpha_1}$ almost surely. □

### 3.3.2 Lower Bounds

In order to obtain the lower bounds of $\dim_H \mathrm{Gr}_X(B)$ we apply Frostman's Lemma and Theorem and use the relationship between the Hausdorff dimension and the capacitary dimension (see [20, 43] for details).

**Lemma 3.13**
*Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator semistable Lévy process on $\mathbb{R}^d$ with $d \geq 2$. Then for any Borel set $B \subseteq \mathbb{R}_+$ we have almost surely*

$$\dim_H \mathrm{Gr}_X(B) \geq \begin{cases} \alpha_1 \dim_H B, & \text{if } \alpha_1 \dim_H B \leq d_1, \alpha_1 \geq 1, \quad (i) \\ \dim_H B, & \text{if } \alpha_1 \dim_H B \leq d_1, \alpha_1 < 1, \quad (ii) \\ 1 + \alpha_2(\dim_H B - \frac{1}{\alpha_1}), & \text{if } \alpha_1 \dim_H B > d_1, \alpha_1 > \alpha_2 \geq 1, (iii) \\ 1 + \dim_H B - \frac{1}{\alpha_1}, & \text{if } \alpha_1 \dim_H B > d_1, \alpha_1 > 1 > \alpha_2. \ (iv) \end{cases}$$

*Proof.* (i)+(iii) Since projections are Lipschitz continuous, we have $\dim_H \mathrm{Gr}_X(B) \geq \dim_H X(B)$. Hence, the desired lower bounds in these two parts can be deduced from the dimension result (3.8) for the range of an operator semistable process.
(ii) Choose $0 < \gamma < \dim_H B \leq 1$. Then by Frostman's lemma there exists a probability measure $\sigma$ on $B$ such that

$$\int_B \int_B \frac{\sigma(ds)\sigma(dt)}{|s-t|^\gamma} < \infty. \tag{3.18}$$

In order to prove $\dim_H Gr_X(B) = \dim_H Z(B) \geq \gamma$ almost surely, by Frostman's theorem [34, 43] it suffices to show that

$$\int_B \int_B \mathbb{E}\left[\|Z(s) - Z(t)\|^{-\gamma}\right] \sigma(ds)\,\sigma(dt) < \infty. \tag{3.19}$$

Let $s, t \in B \subseteq \mathbb{R}_+$. Then

$$\mathbb{E}\left[\left\|\begin{pmatrix} t \\ X(t) \end{pmatrix} - \begin{pmatrix} s \\ X(s) \end{pmatrix}\right\|^{-\gamma}\right] \leq \mathbb{E}\left[|s-t|^{-\gamma}\right] = |s-t|^{-\gamma}.$$

Hence, (3.19) follows directly from (3.18).
(iv) Assume $\alpha_1 \dim_H B > d_1$ then $\alpha_1 > d_1 = 1$. Choose $1 < \gamma < 1 + \dim_H B - \frac{1}{\alpha_1}$, then $\rho = \gamma - 1 + \frac{1}{\alpha_1} < \dim_H B$. By Frostman's lemma, there exists again a probability measure $\sigma$ on $B$ such that

$$\int_B \int_B \frac{\sigma(ds)\sigma(dt)}{|s-t|^\rho} < \infty.$$

Again, in order to verify (3.19) we split the domain of integration into two parts. First assume that $|s-t| = mc^{-i} \leq 1$ with $m \in [1, c)$ and $i \in \mathbb{N}_0$. Since $d_1 = 1$ we get

$$\begin{aligned}
\mathbb{E}\left[\left\|\begin{pmatrix} t \\ X(t)\end{pmatrix} - \begin{pmatrix} s \\ X(s)\end{pmatrix}\right\|^{-\gamma}\right] &\leq \mathbb{E}\left[\left(c^{-i\frac{2}{\alpha_1}} \cdot |X^{(1)}(m)|^2 + |s-t|^2\right)^{-\frac{\gamma}{2}}\right] \\
&\leq K \int_{\mathbb{R}} \frac{1}{c^{-i\frac{\gamma}{\alpha_1}} \cdot |x_1|^\gamma + |s-t|^\gamma} \cdot g_m(x_1) dx_1 \\
&= K \int_{\mathbb{R}} \frac{1}{m^{-\frac{\gamma}{\alpha_1}} (mc^{-i})^{\frac{\gamma}{\alpha_1}} \cdot |x_1|^\gamma + |s-t|^\gamma} \cdot g_m(x_1) dx_1 \\
&\leq K \int_{\mathbb{R}} \frac{1}{c^{-\frac{\gamma}{\alpha_1}} \cdot |s-t|^{\frac{\gamma}{\alpha_1}} |x_1|^\gamma + |s-t|^\gamma} \cdot g_m(x_1) dx_1 \\
&\leq K \int_{\mathbb{R}} \frac{1}{|s-t|^{\frac{\gamma}{\alpha_1}} |x_1|^\gamma + |s-t|^\gamma} \cdot g_m(x_1) dx_1 \\
&= K \cdot |s-t|^{-\frac{\gamma}{\alpha_1}} \int_{\mathbb{R}} \frac{1}{|x_1|^\gamma + |s-t|^{\gamma(1-\frac{1}{\alpha_1})}} \cdot g_m(x_1) dx_1 =: K \cdot |s-t|^{-\frac{\gamma}{\alpha_1}} \cdot I_m,
\end{aligned}$$

where $g_m(x_1)$ is the density function of $X^{(1)}(m)$. Let

$$F_m(r_1) = \mathbb{P}\left(|X^{(1)}(m)| \leq r_1\right) = \int_{|x_1| \leq r_1} g_m(x_1) dx_1$$

and note that by Lemma 3.2

$$\sup_{m \in [1,c)} \sup_{x_1 \in \mathbb{R}} |g_m(x_1)| \leq K_8 < \infty.$$

This leads to

$$F_m(r_1) \leq 1 \wedge 2K_8 \cdot r_1 \quad \forall r_1 \geq 0 \text{ and } \forall m \in [1, c).$$

We denote $z = |s-t|^{1-\frac{1}{\alpha_1}}$. By using integration by parts, we deduce

$$\begin{aligned}
I_m &= \int_0^\infty \frac{1}{r_1^\gamma + z^\gamma} F_m(dr_1) \\
&= \left[\frac{1}{r_1^\gamma + z^\gamma} F_m(r_1)\right]_0^\infty + \int_0^\infty \frac{\gamma r_1^{\gamma-1}}{(r_1^\gamma + z^\gamma)^2} F_m(r_1) dr_1 \\
&\leq K \int_0^\infty \frac{\gamma r_1^{\gamma-1}}{(r_1^\gamma + z^\gamma)^2} r_1 dr_1 = K \int_0^\infty \frac{\gamma r_1^\gamma}{(r_1^\gamma + z^\gamma)^2} dr_1
\end{aligned}$$

$$= K \int_0^\infty \frac{z\gamma \cdot (zs_1)^\gamma}{((zs_1)^\gamma + z^\gamma)^2}\, ds_1$$

$$= K z^{-(\gamma-1)} \cdot \int_0^\infty \frac{\gamma s_1^\gamma}{(s_1^\gamma + 1)^2}\, ds_1$$

$$\leq K z^{-(\gamma-1)} = K\, |s-t|^{-(\gamma-1)(1-\frac{1}{\alpha_1})},$$

where the last integral is finite since $\gamma > 1$. Together we get for $|s-t| \leq 1$

$$\mathbb{E}\left[\left\| \begin{pmatrix} t \\ X(t) \end{pmatrix} - \begin{pmatrix} s \\ X(s) \end{pmatrix} \right\|^{-\gamma}\right] \leq K\, |s-t|^{-\gamma+1-\frac{1}{\alpha_1}} = K\, |s-t|^{-\rho}.$$

For $|s-t| \geq 1$ we have

$$\sup_{|s-t|\geq 1} \mathbb{E}\left[\left\| \begin{pmatrix} t \\ X(t) \end{pmatrix} - \begin{pmatrix} s \\ X(s) \end{pmatrix} \right\|^{-\gamma}\right] \leq \sup_{|s-t|\geq 1} \mathbb{E}\left[|s-t|^{-\gamma}\right] = \sup_{|s-t|\geq 1} |s-t|^{-\gamma} \leq 1.$$

Therefore it follows from the calculations above that

$$\int_B \int_B \mathbb{E}\left[\left\| \begin{pmatrix} t \\ X(t) \end{pmatrix} - \begin{pmatrix} s \\ X(s) \end{pmatrix} \right\|^{-\gamma}\right] \sigma(ds)\sigma(dt) < \infty.$$

Using Frostman's theorem we get

$$\dim_H Gr_X(E) \geq \gamma.$$

Since $\gamma < 1 + \dim_H B - \frac{1}{\alpha_1}$ was arbitrary this concludes the proof. □

### 3.3.3 Proof of Main Results

Theorem 3.6 now follows directly from Lemma 3.12 and Lemma 3.13. It remains to prove the corresponding dimension result for the one-dimensional case as stated in Theorem 3.7. For $\alpha \dim_H B \leq 1$ Lemma 3.12 and 3.13 are still valid for $d = 1$ with $\alpha_1 := \alpha$. In case $\alpha \dim_H B > 1 = d$ the proof runs analogously to Lemma 3.12 part (iv) and Lemma 3.13 part (iv).

**Remark 3.14**
*For $B = [0,1]$, an alternative way to calculate $\dim_H \mathrm{Gr}_X(B)$ can be to examine the index*

*introduced by Khoshnevisan et al. in [39], which depends on the asymptotic behavior of the Lévy exponent of the process $X$. As this is subject of current research, it is not addressed in the present paper.*

# Chapter 4

# Manuscript 3
# On exact Hausdorff measure functions of operator semistable Lévy processes

*Joint work with Peter Kern*

# Declaration

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## Contributions:

In total, I contributed roughly 80 % of the content of this manuscript. More specifically, my contributions are as follows:

- I have written major parts of the Introduction.
- I have written major parts of the Preliminaries.
- I have developed the Main Results.
- I have written major parts of the Proofs.

## 4.1 Introduction

Let $X = \{X(t) : t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$. More precisely, $X$ is a stochastically continuous process with cádlág paths and stationary and independent increments that starts in $X(0) = 0$ almost surely. Then the distribution of $X$ is uniquely determined by the distribution of $X(1)$ which can be an arbitrary infinitely divisible distribution. For $c > 1$ and a linear operator $E$ on $\mathbb{R}^d$ we call the Lévy process $X$ $(c^E, c)$-operator semistable if the distribution of $X(1)$ is full, i.e. not supported on any lower dimensional hyperplane, and

$$\{X(ct)\}_{t\geq 0} \stackrel{\text{fd}}{=} \left\{c^E X(t)\right\}_{t\geq 0}, \tag{4.1}$$

where $\stackrel{\text{fd}}{=}$ denotes equality of all finite-dimensional distributions and

$$c^E := \sum_{n=0}^{\infty} \frac{(\log c)^n}{n!} E^n.$$

The linear operator $E$ is referred to as the exponent of the operator semistable Lévy process $X$. If (4.1) holds for all $c > 1$, the Lévy process is called operator stable. If the exponent $E$ is a multiple of the identity, i.e. $E = 1/\alpha \cdot I$, where necessarily $\alpha \in (0, 2]$, the process $X$ is simply called $(c^{1/\alpha}, c)$-semistable. In case (4.1) holds for all $c > 0$, the Lévy process is called operator stable with exponent $E$, or $\alpha$-stable in case $E = 1/\alpha \cdot I$, where $\alpha = 2$ refers to the Brownian motion case.

In the past, efforts have been made to generate results on exact Hausdorff measure functions for the range of stable Lévy processes. The case of Brownian motion was studied by Ciesielski and Taylor [7, 52]. An exact Hausdorff measure function for the range of an $\alpha$-stable Lévy process was formulated by Taylor [53]. It turned out that the gauge function depends on whether the Lebesgue density of $X(1)$ is positive or zero in the origin, which by Taylor were called stable processes of type $A$ or type $B$, respectively. Furthermore, Pruitt and Taylor [48] studied sample path properties of Lévy processes with independent stable components, including exact Hausdorff measures. Based on their work, Hou and Ying [30] determined an exact Hausdorff measure function for the range of certain operator stable Lévy process of type $A$ with diagonal exponent $E$. They emphasize without proof that similar methods also lead to an exact Hausdorff measure function for type $B$, see Remark 1 in [30]. For an overview on general dimension results for Lévy processes see [38] and [59].

Our aim is to generalize the results of Hou and Ying in two respects. Firstly, we consider the more general class of operator semistable Lévy process with the weaker discrete scaling (4.1) and, secondly, we will relax the assumption that the exponent $E$ should be diagonal and show that it suffices to require diagonality for a principal component $E_1$; see Section 2.2 for details. The Hausdorff dimension for the range and the graph of operator semistable Lévy processes have recently been determined in [36] and [56], respectively. The special case of the limit process in subsequent coin-tossing games of the famous St. Petersburg paradox has been studied in [37] in detail.

The methods applied in this paper are similar to the ones used in [53] and [30]. The paper is structured as follows. Section 2.1 gives the definition of an exact Hausdorff measure function for an arbitrary Borel set $F \subseteq \mathbb{R}^d$. In Section 2.2 we recall spectral decomposition results as stated in [45] which enable us to decompose the operator semistable Lévy process $X$ according to the distinct real parts of the eigenvalues of the exponent $E$. Then, Section 2.3 contains first results for the expected sojourn times of operator semistable Lévy processes. The main results are stated and proven in Section 3 and 4, respectively.

Throughout this paper, $K$ denotes an unspecified positive and finite constant that can vary in each occurrence, whereas fixed constants will be denoted by $K_1, C_1, K_2, C_2$, etc.

## 4.2 Preliminaries

### 4.2.1 Exact Hausdorff measure functions

A function $\phi$ is said to belong to the class $\Phi$ if there exists a constant $\delta > 0$ such that $\phi$ is right-continuous and increasing on the open interval $(0, \delta)$, $\phi(0+) = 0$ and fulfills the doubling property, i.e. there exists a constant $K_1 > 0$ such that

$$\frac{\phi(2s)}{\phi(s)} \leq K_1 \quad \text{for all } 0 < s < \frac{1}{2}\delta. \tag{4.2}$$

For a function $\phi \in \Phi$ the $\phi$-Hausdorff measure of an arbitrary Borel set $F \subseteq \mathbb{R}^d$ is then defined as

$$\phi - m(F) = \liminf_{\epsilon \to \infty} \left\{ \sum_{i=1}^{\infty} \phi(|F_i|) : F \subseteq \bigcup_{i=1}^{\infty} F_i, |F_i| < \epsilon \right\}, \tag{4.3}$$

where $|F| = \sup\{\|x - y\| : x, y \in F\}$ denotes the diameter of a set $F \subseteq \mathbb{R}^d$ and $\|\cdot\|$ is the Euclidean norm. The function $\phi \in \Phi$ is called an exact Hausdorff measure function for

$F \subseteq \mathbb{R}^d$ if $0 < \phi - m(F) < \infty$. We emphasize that all the gauge functions $\phi$ appearing in this paper belong to the class $\Phi$.

For an arbitrary Borel measure $\mu$ on $\mathbb{R}^d$ and a function $\phi \in \Phi$, the upper $\phi$-density of $\mu$ at $x \in \mathbb{R}^d$ is defined as

$$\overline{D}_{\mu}^{\phi} = \limsup_{r\to 0} \frac{\mu(B(x,r))}{\phi(2r)}, \tag{4.4}$$

where $B(x,r)$ denotes the closed ball with radius $r$ centered at $x$. The following lemma is similar to Lemma 2.1 in [30] and is a direct consequence of the results in [49].

**Lemma 4.1**

*For a given $\phi \in \Phi$, there exists a positive constant $K_2$ such that for any Borel measure $\mu$ on $\mathbb{R}^d$ and every Borel set $F \subseteq \mathbb{R}^d$, we have*

$$\phi - m(F) \geq K_2\, \mu(F) \inf_{x\in F} \frac{1}{\overline{D}_{\mu}^{\phi}(x)}. \tag{4.5}$$

### 4.2.2 Spectral decomposition

Let $X$ be a $(c^E, c)$-operator semistable Lévy process. Factor the minimal polynomial of $E$ into $q_1(x) \cdot \ldots \cdot q_p(x)$ where all roots of $q_i$ have real parts equal to $a_i$ and $a_i \neq a_j$ for $i \neq j$. Without loss of generality, one can additionally assume that $a_i < a_j$ for $i < j$. Note that $a_j \geq \frac{1}{2}$ for all $j \in \{1, \ldots p\}$ by Theorem 7.1.10 in [45]. Define $V_j = \operatorname{Ker}(q_j(E))$. According to Theorem 2.1.14 in [45] $V_1 \oplus \cdots \oplus V_p$ is then a direct sum decomposition of $\mathbb{R}^d$ into $E$ invariant subspaces. In an appropriate basis, $E$ is then block-diagonal and we may write $E = E_1 \oplus \cdots \oplus E_p$ where $E_j : V_j \to V_j$ and every eigenvalue of $E_j$ has real part equal to $a_j$. Especially, every $V_j$ is an $E_j$-invariant subspace of dimension $d_j = \dim V_j$ and $d = d_1 + \ldots + d_p$. Write $X(t) = X^{(1)}(t) + \ldots + X^{(p)}(t)$ with respect to this direct sum decomposition, where by Lemma 7.1.17 in [45], $X^{(j)} = \{X^{(j)}(t), t \geq 0\}$ is a $(c^{E_j}, c)$-operator semistable Lévy process on $V_j$. We can now choose an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^d$ such that the $V_j, j \in \{1, \ldots, p\}$, are mutually orthogonal and throughout this paper we will let $\|x\| = \sqrt{\langle x, x\rangle}$ be the associated Euclidean norm. In particular we have for $t = c^r m > 0$ that

$$\|X(t)\|^2 \stackrel{\mathrm{d}}{=} \|c^{rE} X(m)\|^2 = \|c^{rE_1} X^{(1)}(m)\|^2 + \ldots + \|c^{rE_p} X^{(p)}(m)\|^2, \tag{4.6}$$

with $r \in \mathbb{Z}$ and $m \in [1, c)$.

Throughout this paper, we will denote by $\alpha_j = 1/a_j$ the reciprocals of the real parts of the eigenvalues of the exponent $E$. We assume that the process $X$ has no Gaussian component in which case $0 < \alpha_p < \ldots < \alpha_1 < 2$. Note that in this paper, we will only consider operator semistable Lévy processes with diagonal principal exponent, i.e. $E_1 = \alpha_1^{-1} \cdot I^{d_1}$, where $I^{d_1}$ denotes the identity operator on the $d_1$-dimensional subspace $V_1$. Since $V_1 \cong \mathbb{R}^{d_1}$ we may consider $X^{(1)}$ as an operator semistable Lévy process on $\mathbb{R}^{d_1}$ with diagonal exponent $E_1 = \alpha_1^{-1} \cdot I^{d_1}$ and identity matrix $I^{d_1} \in \mathbb{R}^{d_1 \times d_1}$. Unless otherwise stated, throughout this paper there will be no restriction on all the other spectral components $j = 2, \ldots, p$, i.e. $X^{(j)}$ is an arbitrary $(c^{E_j}, c)$-operator semistable Lévy process on $V_j \cong \mathbb{R}^{d_j}$, where the real part of any eigenvalue of the exponent $E_j$ is equal to $a_j = \alpha_j^{-1} \in (0, 2)$, but in general we do not assume that $E_j$ is diagonal for $j = 2, \ldots, p$.

### 4.2.3 Expected sojourn times

For a Lévy process $X = \{X(t) : t \geq 0\}$ let

$$T(a,s) = \int_0^s 1_{B(0,a)}(X(t))dt,$$

be the sojourn time up to time $s > 0$ in the closed ball $B(0,a)$ with radius $a > 0$ and centered at the origin. We now determine sharp upper and lower bounds for the expected sojourn times $\mathbb{E}[T(a,s)]$ of an operator semistable Lévy process with diagonal exponent $E$. Although, in this paper we only need the result for $\alpha_1 < d_1$, for completeness we also include the result for $\alpha_1 > d_1$.

**Lemma 4.2**
*Let $X$ be a $(c^E, c)$-operator semistable Lévy process on $\mathbb{R}^d$ with diagonal principal exponent $E_1$.*

*(i) If $\alpha_1 < d_1$, there exist constants $K_4$, $K_5 > 0$ such that for all $0 < a \leq 1$ and $a^{\alpha_1} \leq s \leq 1$,*

$$K_4 a^{\alpha_1} \leq \mathbb{E}[T(a,s)] \leq K_5 a^{\alpha_1}.$$

*(ii) If $d \geq 2$ and $\alpha_1 > d_1$ then $d_1 = 1$ and we further assume that $E_2$ is diagonal. Then there exist constants $K_6, K_7 > 0$ such that for all $a > 0$ small enough, say $0 < a \leq a_0$, and all $a^{\alpha_2} \leq s \leq 1$,*

$$K_6 a^{\rho} \leq \mathbb{E}[T(a,s)] \leq K_7 a^{\rho},$$

*where* $\rho = 1 + \alpha_2(1 - 1/\alpha_1)$.

*Proof.* The assertions can be proven by only slightly varying the proof of Theorem 2.6 in [36] and using the fact that for $E_j = \alpha_j^{-1} \cdot I^{d_j}$, where $I^{d_j} \in \mathbb{R}^{d_j \times d_j}$ denotes the identity operator on $V_j$, we have $\|t^{E_j}\| = t^{1/\alpha_j}$ for all $t \geq 0$. □

## 4.3 Main Result

Let $\alpha_1$ and $d_1$ be as defined in Section 4.2.2 by means of the spectral decomposition. As in [30] we were only able to fully solve the question of exact Hausdorff measures for the range of operator semistable Lévy processes in the case $\alpha_1 < d_1$ but also give partial results for the case $\alpha_1 > d_1$. We will consider operator semistable Lévy processes of type $A$ and type $B$, simultaneously. If $\alpha_1 < d_1$ and $X$ is of type $B$ we will need the following assumption on the tail asymptotic of sojourn times.

**Assumption 4.3**
*Let $X$ be a $(c^E, c)$-operator semistable Lévy process of type $B$ on $\mathbb{R}^d$ with diagonal principal exponent $E_1$ and $0 < \alpha_1 < 1$. We suppose that there exist constants $K_8, \lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$*

$$\mathbb{P}\left(T(a,1) > \lambda a^{\alpha_1}\right) \leq \exp\left(-K_8 \lambda^{\frac{1}{1-\alpha_1}}\right).$$

Note that if $X$ is an operator stable Lévy process of type $B$ with $\alpha_1 < d_1$ and diagonal principal exponent $E_1$, then the projection of $X^{(1)}$ onto any coordinate-axis is a stable subordinator and thus necessarily $\alpha_1 < 1$. In this case it is known that Assumption 4.3 holds true by Lemma 6 in [53] or Lemma 5.2 in [48]. In our more general operator semistable case it is an open question whether the same tail asymptotic of the sojourn times holds true. Hence in case of type $B$ we will need to suppose that Assumption 4.3 holds true. The following theorem states the main result of this paper.

**Theorem 4.4**
*Let $X$ be a $(c^E, c)$-operator semistable Lévy process on $\mathbb{R}^d$ with diagonal principal exponent $E_1$.*

*(i) If $X$ is of type $A$ and $0 < \alpha_1 < \min\{2, d_1\}$ then*

$$\phi(a) = a^{\alpha_1} \log\log\frac{1}{a}$$

*is an exact Hausdorff measure function for almost all sample paths of $X$ over the interval* $[0,1]$.

*(ii) If $X$ is of type $B$ and $0 < \alpha_1 < 1$, then, given Assumption 4.3,*

$$\phi(a) = a^{\alpha_1}\left(\log\log\frac{1}{a}\right)^{1-\alpha_1}$$

*is an exact Hausdorff measure function for almost all sample paths of $X$ over the interval* $[0,1]$.

## 4.4 Proof

To prove our main result we will show that the asserted $\phi$-Hausdorff measures of the range of $X$ are both, greater than zero and less than infinity.

### 4.4.1 Greater than zero

The following tail asymptotic of the sojourn times is true for any Lévy process and will be used if $X$ is of type A. The proof can be found in Lemma 3.2 of Hou and Ying [30] and uses the Markov inequality.

**Lemma 4.5**
*Let $X$ be a Lévy process on $\mathbb{R}^d$. Then for all $0 < \delta < 1$, $\lambda > 0$ and $a > 0$, we have that*

$$\mathbb{P}\left(T(a,1) > \lambda\mathbb{E}[T(2a,1)]\right) \le \frac{1}{1-\delta}\cdot\exp\left(-\delta\lambda\right). \tag{4.7}$$

**Remark 4.6**
*Note that* (4.7) *is only meaningful if $\lambda > 1$ and in this case the right-hand side takes it minimum at $\delta = 1 - 1/\lambda$. Thus for $\lambda > 1$ the inequality* (4.7) *becomes strongest in the form*

$$\mathbb{P}\left(T(a,1) > \lambda\mathbb{E}[T(2a,1)]\right) \le \lambda\exp(1-\lambda).$$

*Note further that for an operator semistable Lévy process of type B with diagonal principal exponent $E_1$ and $\alpha_1 < 1$ by Lemma 2.2(i) our Assumption 4.3 is stronger than* (4.7) *for large values of $\lambda$.*

**Lemma 4.7**
*Let $X$ be a $(c^E, c)$-operator semistable Lévy process on $\mathbb{R}^d$ with diagonal principal exponent $E_1$.*

*(i) If $X$ is of type A and $0 < \alpha_1 < \min\{2, d_1\}$ then for*

$$\phi(a) = a^{\alpha_1} \log\log \frac{1}{a}$$

*there exists a positive constant $K_{91}$ such that for all $t_0 \in [0,1]$ we have almost surely*

$$\limsup_{a\to 0} \frac{1}{\phi(a)} \cdot \int_0^1 1_{B(X(t_0),a)}(X(t))\, dt \leq K_{91}. \tag{4.8}$$

*(ii) If $X$ is of type B and $0 < \alpha_1 < 1$ then, given Assumption 4.3, for*

$$\phi(a) = a^{\alpha_1} \left(\log\log \frac{1}{a}\right)^{1-\alpha_1}$$

*there exists a positive constant $K_{92}$ such that for all $t_0 \in [0,1]$ we have almost surely*

$$\limsup_{a\to 0} \frac{1}{\phi(a)} \cdot \int_0^1 1_{B(X(t_0),a)}(X(t))\, dt \leq K_{92}. \tag{4.9}$$

*Proof.* Let $t_0 \in [0,1]$. Define

$$Y(t) = \begin{cases} X(t_0) - X(t_0 - t), & \text{if } 0 \leq t < t_0, \\ \\ X(t), & \text{if } t \geq t_0. \end{cases}$$

Using a change of variable by setting $u := t_0 - t$ and $v := t - t_0$ we get

$$\begin{aligned}
&\int_0^1 1_{B(X(t_0),a)}(X(t))dt \\
&= \int_0^{t_0} 1_{B(X(t_0),a)}(X(t))dt + \int_{t_0}^1 1_{B(X(t_0),a)}(X(t))dt \\
&= -\int_{t_0}^0 1_{B(X(t_0),a)}(X(t_0-u))du + \int_0^{1-t_0} 1_{B(X(t_0),a)}(X(v+t_0))dv \\
&= \int_0^{t_0} 1_{B(0,a)}(Y(u))du + \int_0^{1-t_0} 1_{B(0,a)}(X(v+t_0) - X(t_0))dv
\end{aligned}$$

$$\le \int_0^1 1_{B(0,a)}(Y(u))du + \int_0^1 1_{B(0,a)}(X(v+t_0)-X(t_0))dv.$$

Note that the processes $(X(t))_{t\ge 0}$, $(Y(t))_{t\ge 0}$ and $(X(t+t_0)-X(t_0))_{t\ge 0}$ have the same finite-dimensional distributions. Hence, it is sufficient to show that there exists a constant $K_9 > 0$ such that

$$\mathbb{P}\left(\limsup_{a\to 0} \frac{T(a,1)}{\phi(a)} < \frac{K_9}{2}\right) = 1.$$

For $X$ of type $A$, $0 < \alpha_1 < \min\{2, d_1\}$ and $a > 0$ small enough, we have by Lemma 4.2(i) and Lemma 4.5 that

$$\mathbb{P}\left(T(a,1) > \frac{K_4}{2^{\alpha_1}}\lambda a^{\alpha_1}\right) \le \frac{1}{1-\delta}\cdot \exp\left(-\delta\lambda\right)$$

for all $\delta \in (0,1)$ and all $\lambda > 0$. Now choose $\lambda = \frac{2}{\delta}\log\log\frac{1}{a}$. Then for $a > 0$ small enough

$$\mathbb{P}\left(T(a,1) > \frac{2K_4}{\delta} a^{\alpha_1}\log\log\frac{1}{a}\right) \le \frac{1}{1-\delta}\cdot\left(\log\frac{1}{a}\right)^{-2} \tag{4.10}$$

For $n \in \mathbb{N}$ define $a_n := 2^{-n}$ and $E_n := \{T(a_n,1) > \frac{2K_4}{\delta}\cdot a_n^{\alpha_1}\log\log\frac{1}{a_n}\}$. By (4.10) we get for sufficiently large $N \in \mathbb{N}$

$$\sum_{n=N}^{\infty}\mathbb{P}(E_n) \le \frac{1}{1-\delta}\sum_{n=N}^{\infty}\left(\log\frac{1}{a_n}\right)^{-2} = \frac{(\log 2)^{-2}}{1-\delta}\sum_{n=N}^{\infty}\frac{1}{n^2} < \infty.$$

Applying the Borel-Cantelli lemma, for almost all $\omega$ there exists an integer $N(\omega)$ such that the event $E_n$ does not occur for $n \ge N(\omega)$. For $a > 0$ small enough, we can find $n_0 \ge N(\omega)$ such that $a_{n_0+1} \le a \le a_{n_0}$ which gives us

$$\frac{T(a)}{a^{\alpha_1}\log\log\frac{1}{a}} \le \frac{T(a_{n_0})}{a_{n_0+1}^{\alpha_1}\log\log\frac{1}{a_{n_0}}} \le \frac{2K_4 a_{n_0}^{\alpha_1}\log\log\frac{1}{a_{n_0}}}{\delta a_{n_0+1}^{\alpha_1}\log\log\frac{1}{a_{n_0}}} < \frac{K_4 2^{1+\alpha_1}}{\delta}.$$

For $K_{91} := K_4\, 2^{2+\alpha_1}/\delta$ this concludes the proof of part (i).

Now let $X$ be of type $B$ and $0 < \alpha_1 < 1$. By Assumption 4.3 there exist positive constants $K_8$, $\lambda_0 > 0$ such that for all $\lambda \ge \lambda_0$

$$\mathbb{P}\left(T(a,1) > \lambda a^{\alpha_1}\right) \le \exp\left(-K_8\lambda^{\frac{1}{1-\alpha_1}}\right).$$

Put $\lambda = \left(\frac{2}{K_8}\cdot\log\log\frac{1}{a}\right)^{1-\alpha_1}$. For all $a > 0$ sufficiently small, such that $\lambda \ge \lambda_0$, we then get

$$\mathbb{P}\left(T(a,1) > \left(\frac{2}{K_8}\cdot\log\log\frac{1}{a}\right)^{1-\alpha_1}\cdot a^{\alpha_1}\right) \le \exp\left(-K_8\cdot\left(\frac{2}{K_8}\log\log\frac{1}{a}\right)\right)$$

$$= \exp\left(-2\log\log\frac{1}{a}\right) = \left(\log\frac{1}{a}\right)^{-2}.$$

Let $a_n = 2^{-n}$ and $E_n = \left\{T(a_n,1) > \left(\frac{2}{K_8}\cdot\log\log\frac{1}{a_n}\right)^{1-\alpha_1}\cdot a_n^{\alpha_1}\right\}$. Then for $N \in \mathbb{N}$ sufficiently large

$$\sum_{n=N}^{\infty}\mathbb{P}(E_n) \leq \sum_{n=N}^{\infty}\left(\log\frac{1}{a_n}\right)^{-2} = (\log 2)^{-2}\sum_{n=N}^{\infty}\frac{1}{n^2} < \infty.$$

By Borel Cantelli, for almost all $\omega$ there exists an integer $N(\omega)$ such that $E_n$ does not occur for $n \geq N(\omega)$. If $a_{n+1} \leq a < a_n$ and $n \geq N(\omega)$

$$\frac{T(a)}{a^{\alpha_1}\left(\log\log\frac{1}{a}\right)^{1-\alpha_1}} \leq \frac{T(a_n)}{a_{n+1}^{\alpha_1}\left(\log\log\frac{1}{a_n}\right)^{1-\alpha_1}} \leq 2\cdot K_8^{\alpha_1-1}.$$

Setting $K_{92} := 4\cdot(K_8)^{\alpha_1-1}$ concludes the proof. □

**Theorem 4.8**
*Let $X$ be a $(c^E,c)$-operator semistable Lévy process on $\mathbb{R}^d$ with diagonal principal exponent $E_1$.*

*(i) If $X$ is of type A and $0<\alpha_1<\min\{2,d_1\}$ then for*

$$\phi(a) = a^{\alpha_1}\log\log\frac{1}{a}$$

*we have $\phi-m(X([0,1]))>0$ almost surely.*

*(ii) If $X$ is of type B and $0<\alpha_1<1$ then, given Assumption 4.3, for*

$$\phi(a) = a^{\alpha_1}\left(\log\log\frac{1}{a}\right)^{1-\alpha_1}$$

*we have $\phi-m(X([0,1]))>0$ almost surely.*

*Proof.* For all subsets $A\subseteq\mathbb{R}^d$ define the random Borel measure $\mu$ as

$$\mu(A) = \int_0^1 1_A(X(t))dt.$$

This gives us $\mu(X([0,1]))=1$ for all $\omega\in\Omega$. Let $F=\{X(t_0):t_0\in[0,1]\text{ and (4.8) holds}\}\subseteq X([0,1])$. By Tonelli's theorem we have almost surely

$$\mu(F) = \int_0^1 1_F(X(t))dt = \int_0^1 1_{\{X(t_0)\,:\,t_0\in[0,1]\text{ and (4.8) holds}\}}(X(t))dt$$

$$= \int_0^1 \int_0^1 1_{\{X(t_0)\,:\,(4.8)\text{ holds}\}}(X(t))dt_0\, dt$$
$$= \int_0^1 \int_0^1 1_{\{X(t_0)\,:\,(4.8)\text{ holds}\}}(X(t))dt\, dt_0 = \int_0^1 1dt_0 = 1.$$

Applying Lemma 4.1 and Lemma 4.7 and using the fact that $\phi$ is ultimately increasing, we have that almost surely

$$\phi - m(F) \geq K_2\, \mu(F) \inf_{X(t_0)\in F} \left( \limsup_{a\to\infty} \frac{\mu(B(X(t_0),a))}{\phi(2a)} \right)^{-1}$$
$$\geq K_2 \cdot 1 \cdot \inf_{X(t_0)\in F} \left( \limsup_{a\to\infty} \frac{\mu(B(X(t_0),a))}{\phi(2a)} \right)^{-1} \geq \frac{K_2}{\max\{K_{91}, K_{92}\}} > 0.$$

Since $F \subseteq X([0,1])$ this concludes the proof. □

**Remark 4.9**

*Similarly, if $X$ is of type A, $d \geq 2$ and $\alpha_1 > d_1 = 1$ then for*

$$\phi(a) = a^\rho \log\log \frac{1}{a} \tag{4.11}$$

*with $\rho = 1 + \alpha_2(1 - 1/\alpha_1)$ we have $\phi - m(X([0,1])) > 0$ almost surely. This follows analogously to the proof of Theorem 4.8(i) using Lemma 4.2(ii) instead of part (i) in the proof of Lemma 4.7. Unfortunately, in this case we were not able to show that $\phi - m(X([0,1])) < \infty$.*

### 4.4.2 Less than infinity

**Lemma 4.10**

*Let $X$ be a $(c^E, c)$-operator semistable Lévy process on $\mathbb{R}^d$ with diagonal principal exponent $E_1$.*

*(i) If $X$ is of type A, then there exists a constant $K_{10} > 0$ such that for all $0 < \lambda < 1$ and $0 < \tau < 1$*

$$\mathbb{P}\left( \sup_{0\leq t\leq \tau} \|X(t)\| \leq \tau^{\frac{1}{\alpha_1}} \lambda \right) \geq \exp\left(-K_{10}\lambda^{-\alpha_1}\right). \tag{4.12}$$

*(ii) If $X$ is of type $B$ and $0 < \alpha_1 < 1$, then there exist constants $K_{11}, \lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$ and $0 < \tau < 1$*

$$\mathbb{P}\left(\sup_{0\le t\le \tau} \|X(t)\| \le \tau^{\frac{1}{\alpha_1}}\lambda\right) \ge \exp\left(-K_{11}\lambda^{-\frac{\alpha_1}{1-\alpha_1}}\right).$$

*Proof.* (i) Let $p(t,\cdot)$ be the density function of $X(t)$, $t \ge 0$. Since the process is of type $A$, the density function $p(1,\cdot)$ is bounded and continuous and $p(1,0) > 0$. Hence, we can find $\delta, \eta > 0$ such that for all $x \in \mathbb{R}^d$ with $\|x\| < 2\delta$ we have that $p(1,x) \ge \eta$. Then for $\|x\| < \delta$ this leads to

$$\begin{aligned}
&\mathbb{P}(\|X(1)+x\| < \delta) = \int_{\mathbb{R}^d} 1_{\{\|y+x\|<\delta\}}\, p(1,y)dy \\
&\ge \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} 1_{\{|y_1+x_1|<\frac{\delta}{\sqrt{d}}\}}\cdot\ldots\cdot 1_{\{|y_d+x_d|<\frac{\delta}{\sqrt{d}}\}}\, p(1,y)\, dy_1\cdots dy_d \\
&= \int_{-2\delta}^{2\delta}\cdots\int_{-2\delta}^{2\delta} 1_{\{|y_1+x_1|<\frac{\delta}{\sqrt{d}}\}}\cdot\ldots\cdot 1_{\{|y_d+x_d|<\frac{\delta}{\sqrt{d}}\}}\, p(1,y)\, dy_1\cdots dy_d \\
&\ge\ \eta\int_{-2\delta}^{2\delta}\cdots\int_{-2\delta}^{2\delta} 1_{\{|y_1+x_1|<\frac{\delta}{\sqrt{d}}\}}\cdot\ldots\cdot 1_{\{|y_d+x_d|<\frac{\delta}{\sqrt{d}}\}}\, dy_1\cdots dy_d \\
&=\ \eta\prod_{i=1}^{d}\left(\frac{\delta}{\sqrt{d}} - x_i - \left(-\frac{\delta}{\sqrt{d}} - x_i\right)\right) = \eta\prod_{i=1}^{d}\left(\frac{2\delta}{\sqrt{d}}\right) = \eta\left(\frac{2\delta}{\sqrt{d}}\right)^d =: C_2 > 0.
\end{aligned}$$

Furthermore, since the process $X$ has càdlàg paths, it is almost surely bounded on finite intervals. Hence, by tightness we can find $r > 1$ large enough such that

$$\mathbb{P}\left(\sup_{0\le t\le 1}\|X(t)\| \ge r-\delta\right) < \frac{1}{2}C_2.$$

Altogether, we get for all $\|x\| < \delta$

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le t\le 1}\|X(t)+x\| < r, \|X(1)+x\| < \delta\right) \\
&= \mathbb{P}\left(\|X(1)+x\| < \delta\right) - \mathbb{P}\left(\sup_{0\le t\le 1}\|X(t)+x\| \ge r, \|X(1)+x\| < \delta\right) \\
&\ge C_2 - \mathbb{P}\left(\sup_{0\le t\le 1}\|X(t)+x\| \ge r\right) \\
&\ge C_2 - \mathbb{P}\left(\sup_{0\le t\le 1}\|X(t)\| \ge r-\delta\right) > \frac{1}{2}C_2.
\end{aligned}$$

Let $k \in \mathbb{N}$. By induction, it now follows from the properties of a Lévy process that

$$\mathbb{P}\left(\sup_{0\le t\le k}\|X(t)\|<r\right)\ge \mathbb{P}\left(\sup_{0\le t\le k}\|X(t)\|<r,\|X(k)\|<\delta\right)$$

$$\ge \mathbb{P}\Bigg(\sup_{k-1\le t\le k}\|X(t)\|<r,\|X(k)\|<\delta,\sup_{0\le t\le k-1}\|X(t)\|<r,\|X(k-1)\|<\delta\Bigg)$$

$$\sup_{0\le t\le k-1}\|X(t)\|<r,\|X(k-1)\|<\delta\Bigg)$$

$$=\int_{[0,r)}\int_{\{\|x\|<\delta\}}\mathbb{P}\left(\sup_{k-1\le t\le k}\|X(t)-X(k-1)+x\|<r,\|X(k)-X(k-1)+x\|<\delta\right)$$

$$d\mathbb{P}_{\left(\sup\limits_{0\le t\le k-1}\|X(t)\|,\|X(k-1)\|\right)}(x,y)$$

$$=\int_{[0,r)}\int_{\{\|x\|<\delta\}}\mathbb{P}\left(\sup_{k-1\le t\le k}\|X(t)+x\|<r,\|X(1)+x\|<\delta\right)$$

$$d\mathbb{P}_{\left(\sup\limits_{0\le t\le k-1}\|X(t)\|,\|X(k-1)\|\right)}(x,y)$$

$$\ge\frac{1}{2}C_2\cdot\mathbb{P}\left(\sup_{0\le t\le k-1}\|X(t)\|<r,\|X(k-1)\|<\delta\right)\ge\left(\frac{1}{2}C_2\right)^k=\exp\left(-k\log\left(2C_2^{-1}\right)\right).$$

For $u>1$ choose $k\in\mathbb{N}$ with $k\le u<k+1$. Then for all $r>1$ large enough we have

$$\mathbb{P}\left(\sup_{0\le t\le u}\|X(t)\|<r\right)\ge\mathbb{P}\left(\sup_{0\le t\le k+1}\|X(t)\|<r\right)$$

$$\ge\exp\left(-(k+1)\log\left(2\,C_2^{-1}\right)\right)=\exp\left(-k\cdot\frac{k+1}{k}\cdot\log\left(2\,C_2^{-1}\right)\right)$$

$$\ge\exp\left(-u\cdot 2\log\left(2\,C_2^{-1}\right)\right)=:\exp\left(-C_3\,u\right),$$

where $C_3>0$ is a constant independent from $u$. Now let $0<\tau<1$. Then there exists an $i\in\mathbb{N}_0$ such that $c^{-(i+1)}\le\tau<c^{-i}$ and for $0<\lambda<1$ there exists a $j\in\mathbb{N}$ such that $c^{j-2}\le\lambda^{-\alpha_1}r^{\alpha_1}<c^{j-1}$. Using the fact that for diagonal $E_1$ we have $\|s^E\|\le s^{1/\alpha_1}$ for $0<s<1$, this leads us to

$$\mathbb{P}\left(\sup_{0\le t\le\tau}\|X(t)\|\le\tau^{\frac{1}{\alpha_1}}\lambda\right)\ge\mathbb{P}\left(\sup_{0\le t\le c^{-i}}\|X(t)\|\le c^{-\frac{i+1}{\alpha_1}}\lambda\right)$$

$$= \mathbb{P}\left(\sup_{0\le t\le c^j} \|X(c^{-j-i}t)\| \le c^{-\frac{i+1}{\alpha_1}}\lambda\right) \ge \mathbb{P}\left(\sup_{0\le t\le c^j} c^{-\frac{j+i}{\alpha_1}}\|X(t)\| \le c^{-\frac{i+1}{\alpha_1}}\lambda\right)$$

$$\ge \mathbb{P}\left(\sup_{0\le t\le c^j} \|X(t)\| \le c^{\frac{j-1}{\alpha_1}}\lambda\right) \ge \mathbb{P}\left(\sup_{0\le t\le c^j} \|X(t)\| < r\right)$$

$$\ge \exp\left(-C_3\cdot c^j\right) \ge \exp\left(-K_{10}\lambda^{-\alpha_1}\right),$$

where $K_{10} := C_3 \cdot c^2 \cdot r^{\alpha_1}$.

(ii) Let $0 < \tau < 1$. Then there exists an $i_1 \in \mathbb{N}_0$ with $c^{-(i_1+1)} \le \tau < c^{-i_1}$. We have

$$\mathbb{P}\left(\sup_{0\le t\le \tau} \|X(t)\| \le \tau^{\frac{1}{\alpha_1}}\lambda\right) \ge \mathbb{P}\left(\sup_{0\le t\le c^{-i_1}} \|X(t)\| \le c^{-\frac{i_1+1}{\alpha_1}}\lambda\right)$$

$$= \mathbb{P}\left(\sup_{0\le t\le 1} \|X(c^{-i_1}t)\| \le c^{-\frac{i_1+1}{\alpha_1}}\lambda\right) \ge \mathbb{P}\left(\sup_{0\le t\le 1} \|X(t)\| \le c^{-\frac{1}{\alpha_1}}\lambda\right) =: g(\lambda),$$

where the last inequality follows from the fact that for diagonal $E_1$ we have $\|m\|^E \le m^{1/\alpha_1}$ for all $m \in (0,1]$. Furthermore for all $k \in \mathbb{N}\setminus\{1\}$ we can find a $j \in \mathbb{N}_0$ such that $c^{-(j+1)} \le k^{-1} < c^{-j}$. Since $X$ has independent and stationary increments we get

$$g(\lambda) \ge \mathbb{P}\left(\bigcap_{i=1}^{k}\left\{\sup_{\frac{i-1}{k}\le t\le \frac{i}{k}} \left\|X(t) - X\left(\frac{i-1}{k}\right)\right\| < k^{-1}c^{-\frac{1}{\alpha_1}}\lambda\right\}\right)$$

$$= \prod_{i-1}^{k} \mathbb{P}\left(\sup_{0\le t\le k^{-1}} \|X(t)\| < k^{-1}c^{-\frac{1}{\alpha_1}}\lambda\right)$$

$$\ge \left[\mathbb{P}\left(\sup_{0\le t\le c^{-j}} \|X(t)\| < c^{-(j+1)}c^{-\frac{1}{\alpha_1}}\lambda\right)\right]^k$$

$$= \left[\mathbb{P}\left(\sup_{0\le t\le 1} \|X(c^{-j}t)\| < c^{-j}c^{-\frac{1}{\alpha_1}-1}\lambda\right)\right]^k$$

$$\ge \left[\mathbb{P}\left(\sup_{0\le t\le 1} \|X(t)\| < c^{j(\frac{1}{\alpha_1}-1)}c^{-\frac{1}{\alpha_1}-1}\lambda\right)\right]^k$$

$$= \left[\mathbb{P}\left(\sup_{0\le t\le 1} \|X(t)\| < c^{(j+1)(\frac{1}{\alpha_1}-1)}c^{-\frac{2}{\alpha_1}}\lambda\right)\right]^k$$

$$\ge \left[\mathbb{P}\left(\sup_{0\le t\le 1} \|X(t)\| < k^{\frac{1}{\alpha_1}-1}c^{-\frac{2}{\alpha_1}}\lambda\right)\right]^k = \left[g\left(k^{\frac{1}{\alpha_1}-1}c^{-\frac{1}{\alpha_1}}\lambda\right)\right]^k.$$

Define $h(\lambda) = \log g(\lambda)$. Then

$$h(\lambda) \geq k \cdot h\left(k^{\frac{1}{\alpha_1}-1} c^{-\frac{1}{\alpha_1}} \lambda\right).$$

Furthermore define the sequence $(x_k)_{k\in\mathbb{N}}$ as $x_k = k^{1-\frac{1}{\alpha_1}} c^{\frac{1}{\alpha_1}}$, then $x_{k+1}/x_k \to 1$ and $x_k \to 0$ as $k \to \infty$. We get

$$h(x_k) \geq k \cdot h(1) = c^{\frac{1}{1-\alpha_1}} \cdot x_k^{\frac{\alpha_1}{\alpha_1-1}} \cdot h(1) \geq x_k^{-\frac{\alpha_1}{1-\alpha_1}} \cdot h(1).$$

Since for all $k \in \mathbb{N}$ we have

$$g(1) \geq \left[\mathbb{P}\left(\sup_{0\leq t\leq 1} \|X(t)\| < k^{\frac{1}{\alpha_1}-1} c^{-\frac{2}{\alpha_1}}\right)\right]^k,$$

there exists a $k_0 \in \mathbb{N}$ such that the right-hand-side is strictly positive for all $k \geq k_0$, i.e. $g(1) > 0$. Hence, there exists a finite constant $K > 0$ such that $h(1) = \log g(1) \geq -K$. Since $h$ is non-increasing, there is a $\lambda_0 > 0$ such that for $0 < \lambda \leq \lambda_0$ with $x_{k+1} \leq \lambda < x_k$ and $k \geq k_0$ we have

$$\begin{aligned} h(\lambda) &\geq h(x_{k+1}) = (x_{k+1})^{-\frac{\alpha_1}{1-\alpha_1}} \cdot h(1) \geq -K \cdot (x_{k+1})^{-\frac{\alpha_1}{1-\alpha_1}} \\ &= -K \cdot \left(\frac{x_{k+1}}{x_k}\right)^{-\frac{\alpha_1}{1-\alpha_1}} \cdot (x_k)^{-\frac{\alpha_1}{1-\alpha_1}} \geq -K_{11}\, \lambda^{-\frac{\alpha_1}{1-\alpha_1}}. \end{aligned}$$

Altogether we arrive at

$$g(\lambda) \geq \exp\left(-K_{11}\, \lambda^{-\frac{\alpha_1}{1-\alpha_1}}\right)$$

for all $0 < \lambda < \lambda_0$. □

**Lemma 4.11**

*Let $X$ be a $(c^E, c)$-operator semistable Lévy process on $\mathbb{R}^d$ with diagonal principal exponent $E_1$.*

*(i) For the principal component $j = 1$ there exists a constant $K_{12} > 0$ such that for all $i \in \mathbb{Z}$ and all $a > 0$ we have*

$$\mathbb{P}\left(\|X^{(1)}(c^{-i})\| > ac^{-\frac{i}{\alpha_1}}\right) = \mathbb{P}(\|X^{(1)}(1)\| > a) \leq K_{12}\, a^{-\alpha_1}. \tag{4.13}$$

*(ii) For all other components $j=2,\dots,p$ and arbitrary $\delta'>0$, $\delta_j\in(0,\alpha_j^{-1})$ there exists a constant $K_{j2}>0$ such that for all $i\in\mathbb{Z}$ and all $a\geq a_0\geq 1$ we have*

$$\mathbb{P}\left(\|X^{(j)}(c^{-i})\|>ac^{-i(\frac{1}{\alpha_j}-\delta_j)}\right)\leq K_{j2}\,a^{-(\alpha_j-\delta')}. \tag{4.14}$$

*Proof.* (i) Let $\nu$ be a $(c^{1/\alpha_1},c)$-semistable law on $\mathbb{R}$. One can show (see the Remarks in section 3 of [44]) that for all $t>0$

$$\nu(\{|x|>t\})=t^{-\alpha}f(t), \tag{4.15}$$

where $f$ is a bounded, asymptotically log-periodic function. Let $X_1,\dots,X_{d_1}$ denote the marginals of $X^{(1)}(1)$, i.e. $X_j=\langle X^{(1)}(1),e_j\rangle$ with canonical basis vector $e_j$, and let $X_j(t)=\langle X^{(1)}(t),e_j\rangle$ be the Lévy process generated by $X_j$. Since

$$X_j(ct)=\langle X^{(1)}(ct),e_j\rangle=\langle c^{E_1}X^{(1)}(t),e_j\rangle=c^{1/\alpha_1}\langle X^{(1)}(t),e_j\rangle=c^{1/\alpha_1}X_j(t)$$

the distributions of the marginals $X_j$ are $(c^{1/\alpha_1},c)$-semistable on $\mathbb{R}$. Hence, by (4.15) there exists a finite constant $C_1>0$ such that

$$\mathbb{P}\left(|X_j|>a\right)\leq C_1a^{-\alpha_1}\quad\text{ for all }\quad a>0,\ j=1,\dots,d_1.$$

Since $\|X^{(1)}(1)\|\leq K\cdot\|X^{(1)}(1)\|_1=K\cdot\sum_{j=1}^{d_1}|X_j|$, we further get

$$\begin{aligned}\mathbb{P}\left(\|X^{(1)}(1)\|>a\right)&\leq\mathbb{P}\left(\sum_{j=1}^{d_1}|X_j|>\frac{a}{K}\right)\leq\mathbb{P}\left(\bigcup_{j=1}^{d_1}\left\{|X_j|>\frac{a}{d_1K}\right\}\right)\\&\leq\sum_{j=1}^{d_1}\mathbb{P}\left(|X_j|>\frac{a}{d_1K}\right)\leq C_1\left(\frac{a}{d_1K}\right)^{-\alpha_1}=:K_{12}\,a^{-\alpha_1},\end{aligned}$$

which concludes the proof of (i).
(ii) By Lemma 2.1 in [36] we have for any $r\in[1,c)$

$$\begin{aligned}\|X^{(j)}(rc^{-i})\|&=\|c^{-iE_j}X^{(j)}(r)\|\leq\|c^{-iE_j}\|\|X^{(j)}(r)\|\\&\leq K\cdot c^{-i(a_j-\delta_j)}\|X^{(j)}(r)\|=K\cdot c^{-i(\frac{1}{\alpha_j}-\delta_j)}\|X^{(j)}(r)\|\end{aligned} \tag{4.16}$$

and hence we get

$$\mathbb{P}\left(\|X^{(j)}(c^{-i})\| > ac^{-i(\frac{1}{\alpha_j}-\delta_j)}\right) \leq \mathbb{P}\left(\|X^{(j)}(1)\| > K^{-1}a\right). \tag{4.17}$$

As in part (i), for the marginals $X_1, \ldots, X_{d_j}$ of $X^{(j)}(1)$, i.e. $X_k = \langle X^{(j)}(1), e_k \rangle$ with canonical basis vector $e_k$, we get

$$\mathbb{P}\left(\|X^{(j)}(1)\| > K^{-1}a\right) \leq \sum_{k=1}^{d_j} \mathbb{P}\left(|X_k| > C_2 a\right). \tag{4.18}$$

In view of Theorem 8.2.1 in [45], an application of Theorem 6.3.25(a) in [45] gives

$$\mathbb{P}\left(|X_k| > C_2 a\right) = \mathbb{P}\left(|\langle X^{(j)}(1), e_k \rangle| > C_2 a\right) \leq C_{2k} a^{-\frac{1}{\alpha_j}+\delta'} = C_{2k} a^{-(\alpha_j - \delta')} \tag{4.19}$$

for all $a \geq a_0$ and some $a_0 \geq 1$ independent of $k = 1, \ldots, d_j$. Now, (4.14) follows directly from (4.17)–(4.19). □

**Lemma 4.12**

*Let $X$ be a $(c^E, c)$-operator semistable Lévy process on $\mathbb{R}^d$ with diagonal principal exponent $E_1$. Given $\varepsilon \in (0,1)$, $\delta_1 := 0$ and $\delta_j \in (0, \alpha_j^{-1})$ for $j = 2, \ldots, p$, there exists a constant $a_0 > 0$ such that for all $a \geq a_0$ and all $i \in \mathbb{N}_0$ we have*

$$\sup_{t \in [0, c^{-i}]} \mathbb{P}\left(\|X^{(j)}(t)\| > ac^{-i(\frac{1}{\alpha_j}-\delta_j)}\right) \leq \varepsilon < 1.$$

*Proof.* Using (4.16) in case $j = 2, \ldots, p$ and the semistability in case $j = 1$, we get

$$\begin{aligned}
\sup_{t \in [0, c^{-i}]} \mathbb{P}\left(\|X^{(j)}(t)\| > ac^{-i(\frac{1}{\alpha_j}-\delta_j)}\right) &\leq \sup_{r \in [1,c)} \sup_{k \geq i} \mathbb{P}\left(\|X^{(j)}(rc^{-k})\| > ac^{-i(\frac{1}{\alpha_j}-\delta_j)}\right) \\
&\leq \sup_{r \in [1,c)} \sup_{k \geq i} \mathbb{P}\left(K \cdot c^{-k(\frac{1}{\alpha_j}-\delta_j)} \|X^{(j)}(r)\| > ac^{-i(\frac{1}{\alpha_j}-\delta_j)}\right) \\
&= \sup_{r \in [1,c)} \mathbb{P}\left(\|X^{(j)}(r)\| > \frac{a}{K}\right).
\end{aligned}$$

Since $\left(X^{(j)}(r)\right)_{r \in [1,c)}$ is stochastically continuous and hence weakly relatively compact, it follows by Prohorov's theorem that for $\varepsilon \in (0,1)$ there exists $a_0 > 0$ such that for all $a \geq a_0$ we have

$$\sup_{r \in [1,c)} \mathbb{P}\left(\|X^{(j)}(r)\| > \frac{a}{K}\right) \leq \varepsilon < 1,$$

concluding the proof. □

**Lemma 4.13**
*Let $X$ be a $(c^E, c)$-operator semistable Lévy process on $\mathbb{R}^d$ with diagonal principal exponent $E_1$. Given $\varepsilon \in (0,1)$, $\delta_1 := 0$ and $\delta_j \in (0, \alpha_j^{-1})$ for $j = 2, \ldots, p$, there exists a constant $a_0 > 0$ such that for all $a \geq a_0$ and all $i \in \mathbb{N}_0$ we have*

$$\mathbb{P}\left(\sup_{t\in[0,c^{-i}]} \|X^{(j)}(t)\| > 2ac^{-i(\frac{1}{\alpha_j}-\delta_j)}\right) \leq \frac{1}{1-\varepsilon} \cdot \mathbb{P}\left(\|X^{(j)}(c^{-i})\| > ac^{-i(\frac{1}{\alpha_j}-\delta_j)}\right).$$

*Proof.* For $N \in \mathbb{N}$ and $n = 1, \ldots, N$ define $Y_{n,N} := X^{(j)}(k_n^N) - X^{(j)}(k_{n-1}^N)$, where $k_n^N := \frac{n}{N}\, c^{-i}$. Then $Y_{1,N}, \ldots, Y_{N,N}$ are independent and $\sum_{k=1}^n Y_{k,N} = X^{(j)}(k_n^N)$. By Lemma 4.12 for any $\varepsilon \in (0,1)$ there exists a constant $a_0 > 0$ such that for all $a \geq a_0$ we have

$$\begin{aligned}
&\sup_{0\leq n\leq N} \mathbb{P}\left(\|\sum_{k=1}^N Y_{k,N} - \sum_{k=1}^n Y_{k,N}\| > ac^{-i(\frac{1}{\alpha_j}-\delta_j)}\right)\\
&= \sup_{0\leq n\leq N} \mathbb{P}\left(\|X^{(j)}(k_N^N) - X^{(j)}(k_n^N)\| > ac^{-i(\frac{1}{\alpha_j}-\delta_j)}\right)\\
&= \sup_{0\leq n\leq N} \mathbb{P}\left(\|X^{(j)}(k_N^N - k_n^N)\| > ac^{-i(\frac{1}{\alpha_j}-\delta_j)}\right)\\
&= \sup_{0\leq n\leq N} \mathbb{P}\left(\|X^{(j)}\left(\left(1-\tfrac{n}{N}\right) c^{-i}\right)\| > ac^{-i(\frac{1}{\alpha_j}-\delta_j)}\right)\\
&\leq \sup_{t\in[0,c^{-i}]} \mathbb{P}\left(\|X^{(j)}(t)\| > ac^{-i(\frac{1}{\alpha_j}-\delta_j)}\right) \leq \varepsilon < 1.
\end{aligned}$$

Using the Lévy-Ottaviani inequality (see Lemma 3.21 in [5]) and the fact that $(X^{(j)}(t))_{t\geq 0}$ has right-continuous paths, it follows that

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,c^{-i}]} \|X^{(j)}(t)\| > 2ac^{-i(\frac{1}{\alpha_j}-\delta_j)}\right)\\
&= \lim_{N\to\infty} \mathbb{P}\left(\sup_{0\leq n\leq N} \|X^{(j)}(k_n^N)\| > 2ac^{-i(\frac{1}{\alpha_j}-\delta_j)}\right)\\
&\leq \lim_{N\to\infty} \frac{1}{1-\varepsilon}\, \mathbb{P}\left(\|X^{(j)}(k_N^N)\| > ac^{-i(\frac{1}{\alpha_j}-\delta_j)}\right)\\
&= \frac{1}{1-\varepsilon}\, \mathbb{P}\left(\|X^{(j)}(c^{-i})\| > ac^{-i(\frac{1}{\alpha_j}-\delta_j)}\right),
\end{aligned}$$

concluding the proof. □

For a Lévy process $X = \{X(t) : t \geq 0\}$ define the first exit time from the closed ball $B(0,a)$

$$P(a) = \inf\{t \geq 0 : \|X(t)\| > a\};$$

and the maximum displacement process for $t > 0$ as

$$M(t) = \sup_{0 \leq s \leq t} \|X(s)\|.$$

Note that for $a, r > 0$ the first exit time $P(a)$ and the maximum displacement process $M(r)$ are related by

$$\{P(a) < r\} = \{M(r) > a\}. \tag{4.20}$$

**Lemma 4.14**

*Let $X$ be a $(c^E, c)$-operator semistable Lévy process on $\mathbb{R}^d$ with diagonal principal exponent $E_1$. Then for*

$$\phi(a) = \begin{cases} a^{\alpha_1} \log\log \frac{1}{a}, & \text{if } X \text{ is of type } A \text{ and } 0 < \alpha_1 < d_1 \\ \\ a^{\alpha_1} \left(\log\log \frac{1}{a}\right)^{1-\alpha_1}, & \text{if } X \text{ is of type } B \text{ and } 0 < \alpha_1 < 1 \end{cases} \tag{4.21}$$

*there exist constants $K_{13}, K_{14}, \gamma_0 > 0$ such that*

$$\mathbb{P}\left(\sup_{\gamma \leq a \leq \delta} \frac{P(a)}{\phi(a)} < K_{13}\right) \leq \exp\left(-K_{14} \cdot (-\log\gamma)^{\frac{1}{8}}\right), \tag{4.22}$$

*for all $0 < \gamma \leq \gamma_0$ and $\delta \geq \gamma^{1/6}$.*

*Proof.* First assume that $X$ is of type $A$ and $\alpha_1 \in (0, \min\{d_1, 2\})$, thus $\phi(a) = a^{\alpha_1} \log\log \frac{1}{a}$. By regular variation techniques, it can be shown that for $\alpha_1 < d_1$ the function $\psi$, defined by $\psi(s) = s^{1/\alpha_1}(\log\log 1/s)^{-1/\alpha_1}$, is asymptotically inverse to $\phi$ in the sense that

$$\phi(\psi(s)) \sim s \text{ as } s \to 0+ \quad \text{and} \quad \psi(\phi(a)) \sim a \text{ as } a \to 0+. \tag{4.23}$$

Owing to the fact that $\{M(t) > a\} = \{P(a) < t\}$, instead of estimating the probability that $\frac{P(a)}{\phi(a)}$ remains small, we will now estimate the probability that $\frac{M(t)}{\psi(t)}$ remains large. Therefore, define a sequence $t_k = \exp(-k^2)$, $k \geq 1$. Then for all $k$ there exists an $i_k \in \mathbb{N}_0$ such that $c^{-(i_k+1)} \leq t_k < c^{-i_k}$. Define

$$M'(t_k) = \sup_{t_{k+1} \leq t \leq t_k} \|X(t) - X(t_{k+1})\|$$

and $C_4 := (3K_{10})^{1/\alpha_1}$, where $K_{10}$ is the constant in Lemma 4.10 (a). Furthermore, define

$$D_k := \left\{ \frac{M(t_k)}{\psi(t_k)} > 2 \cdot C_4 \right\}, \ G_k := \left\{ \frac{M'(t_k)}{\psi(t_k)} > C_4 \right\} \text{ and } H_k := \left\{ \frac{M(t_{k+1})}{\psi(t_k)} > C_4 \right\}.$$

Then

$$\begin{aligned} D_k &= \left\{ \sup_{0 \le t \le t_k} \|X(t)\| > 2C_4\psi(t_k) \right\} \\ &= \left\{ \sup_{0 \le t \le t_{k+1}} \|X(t)\| > 2C_4\psi(t_k) \right\} \cup \left\{ \sup_{t_{k+1} \le t \le t_k} \|X(t)\| > 2C_4\psi(t_k) \right\} \\ &\subseteq \left\{ \sup_{0 \le t \le t_{k+1}} \|X(t)\| > 2C_4\psi(t_k) \right\} \\ &\quad \cup \left\{ \sup_{t_{k+1} \le t \le t_k} \|X(t) - X(t_{k+1})\| + \|X(t_{k+1})\| > 2C_4\psi(t_k) \right\} \\ &\subseteq \left\{ \sup_{0 \le t \le t_{k+1}} \|X(t)\| > 2C_4\psi(t_k) \right\} \cup \{\|X(t_{k+1})\| > C_4\psi(t_k)\} \\ &\quad \cup \left\{ \sup_{t_{k+1} \le t \le t_k} \|X(t) - X(t_{k+1})\| > C_4\psi(t_k) \right\} \\ &\subseteq \left\{ \sup_{0 \le t \le t_{k+1}} \|X(t)\| > C_4\psi(t_k) \right\} \cup \left\{ \sup_{t_{k+1} \le t \le t_k} \|X(t) - X(t_{k+1})\| > C_4\psi(t_k) \right\} \\ &= H_k \cup G_k. \end{aligned}$$

And for all $m \in \mathbb{N}$ this gives us

$$\bigcap_{k=m+1}^{2m} D_k \subseteq \left( \bigcap_{k=m+1}^{2m} G_k \right) \cup \left( \bigcup_{k=m+1}^{2m} H_k \right).$$

Note that the sets $(G_k)_{k\in\mathbb{N}}$ are pairwise independent. Set $\mathbb{P}(G_k) = 1 - p_k$ and $\mathbb{P}(H_k) = q_k$. Applying Lemma 4.10 (a) we have for sufficiently large $k$

$$\begin{aligned} p_k &= \mathbb{P}\left(M'(t_k) \le C_4\psi(t_k)\right) \\ &= \mathbb{P}\left( \sup_{t_{k+1} \le t \le t_k} \|X(t) - X(t_{k+1})\| \le C_4\psi(t_k) \right) \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{P}\left( \sup_{0\le t\le t_k - t_{k+1}} \|X(t)\| \le C_4 \psi(t_k) \right) \\
&\ge \mathbb{P}\left( \sup_{0\le t\le t_k} \|X(t)\| \le C_4 \cdot t_k^{\frac{1}{\alpha_1}} \left(\log\log \frac{1}{t_k}\right)^{-\frac{1}{\alpha_1}} \right) \\
&\ge \exp\left( -K_{10} \left( C_4 \left(\log\log(1/t_k)\right)^{-\frac{1}{\alpha_1}} \right)^{-\alpha_1} \right) \\
&= \exp\left( -K_{10} \cdot \frac{1}{3K_{10}} \cdot \log\log \frac{1}{t_k} \right) \\
&= \exp\left( -\frac{1}{3} \log\log \frac{1}{t_k} \right) = \exp\left( -\frac{1}{3} \log k^2 \right) = k^{-\frac{2}{3}}.
\end{aligned}
$$

On the other hand, choosing $\delta' \in (0, \alpha_p)$, $\delta_1 := 0$ and $\delta_j = \frac{1}{\alpha_j} - \frac{1}{\alpha_1} \in (0, \alpha_j^{-1})$ for $j = 2, \ldots, p$, then for sufficiently large $k \in \mathbb{N}$ we get by Lemma 4.13 and Lemma 4.11

$$
\begin{aligned}
q_k &= \mathbb{P}\left( \sup_{0\le t\le t_{k+1}} \|X(t)\| > C_4 \psi(t_k) \right) \\
&\le \sum_{j=1}^{p} \mathbb{P}\left( \sup_{0\le t\le t_{k+1}} \|X^{(j)}(t)\| > \frac{C_4 \cdot t_k^{\frac{1}{\alpha_1}} (\log\log \frac{1}{t_k})^{-\frac{1}{\alpha_1}}}{p} \right) \\
&= \sum_{j=1}^{p} \mathbb{P}\left( \sup_{0\le t\le t_{k+1}} \|X^{(j)}(t)\| > \frac{C_4 \cdot t_k^{\frac{1}{\alpha_1}} (\log\log \frac{1}{t_k})^{-\frac{1}{\alpha_1}} t_{k+1}^{-(\frac{1}{\alpha_j} - \delta_j)} t_{k+1}^{\frac{1}{\alpha_j} - \delta_j}}{p} \right) \\
&\le \sum_{j=1}^{p} \mathbb{P}\left( \sup_{0\le t\le c^{-i_{k+1}}} \|X^{(j)}(t)\| > \frac{C_4 \cdot t_k^{\frac{1}{\alpha_1}} \left(\log\log \frac{1}{t_k}\right)^{-\frac{1}{\alpha_1}} t_{k+1}^{-(\frac{1}{\alpha_j} - \delta_j)} \cdot c^{-(i_{k+1}+1)(\frac{1}{\alpha_j} - \delta_j)}}{p} \right) \\
&\le \sum_{j=1}^{p} K^{(j)} \cdot \mathbb{P}\left( \|X^{(j)}(c^{-i_{k+1}})\| > \frac{C_4 \cdot t_k^{\frac{1}{\alpha_1}} \left(\log\log \frac{1}{t_k}\right)^{-\frac{1}{\alpha_1}} t_{k+1}^{-\frac{1}{\alpha_1}} \cdot c^{-\frac{1}{\alpha_1}} \cdot c^{-i_{k+1}(\frac{1}{\alpha_j} - \delta_j)}}{2p} \right) \\
&\le \sum_{j=1}^{p} \widetilde{K}^{(j)} \cdot \left( t_k^{\frac{1}{\alpha_1}} \left(\log\log \frac{1}{t_k}\right)^{-\frac{1}{\alpha_1}} t_{k+1}^{-\frac{1}{\alpha_1}} \right)^{-(\alpha_j - \delta')}
\end{aligned}
$$

$$\leq \sum_{j=1}^{p} \widetilde{K}^{(j)} \left(t_k^{-1} t_{k+1} \log\log \tfrac{1}{t_k}\right)^{\frac{\alpha_j - \delta'}{\alpha_1}} \leq K \cdot \left(t_k^{-1} t_{k+1} \log\log \tfrac{1}{t_k}\right)^{\frac{\alpha_p - \delta'}{\alpha_1}}$$

$$= K \cdot \left(\exp\left(k^2 - (k+1)^2\right) \cdot \log\left(k^2\right)\right)^{\frac{\alpha_p - \delta'}{\alpha_1}} = K \cdot \left(\exp(-2k-1) \cdot \log\left(k^2\right)\right)^{\frac{\alpha_p - \delta'}{\alpha_1}}$$

$$\leq \exp(-C_{41}\, k).$$

Hence, there exists $m_0 \in \mathbb{N}$ large enough such that for all $m > m_0$

$$\mathbb{P}\left(\bigcap_{k=m+1}^{2m} D_k\right) \leq \mathbb{P}\left(\bigcap_{k=m+1}^{2m} G_k\right) + \sum_{k=m+1}^{2m} \mathbb{P}(H_k)$$

$$= \prod_{k=m+1}^{2m} (1 - p_k) + \sum_{k=m+1}^{2m} q_k \leq \prod_{k=m+1}^{2m} \exp(-p_k) + \sum_{k=m+1}^{2m} \exp(-C_{41}\, k)$$

$$\leq \exp\left(-\sum_{k=m+1}^{2m} p_k\right) + C_5 \exp(-C_{41}\, m)$$

$$\leq \exp\left(-\sum_{k=m+1}^{2m} k^{-\frac{2}{3}}\right) + C_5 \exp(-C_{41}\, m)$$

$$\leq \exp\left(-\sum_{k=m+1}^{2m} (2m)^{-\frac{2}{3}}\right) + C_5 \exp(-C_{41}\, m)$$

$$= \exp\left(-2^{-\frac{2}{3}} m^{\frac{1}{3}}\right) + C_5 \exp(-C_{41}\, m) \leq \exp\left(-m^{\frac{1}{4}}\right).$$

Define

$$a_k = 2C_4 \psi(t_k) = 2C_4 \left(\log k^2\right)^{-\frac{1}{\alpha_1}} \cdot \exp\left(-k^2/\alpha_1\right).$$

Then for $m_0$ sufficiently large one can show that $a_{2m} > a_m^4$, for all $m \geq m_0$. By (4.23) and the properties of $\phi$, there now exists a positive constant $K_{13} > 0$ such that for $k$ large enough

$$t_k \sim \phi\left(\frac{a_k}{2C_4}\right) \geq 2K_{13} \cdot \phi(a_k).$$

Therefore, using $\{M(t) > a\} = \{P(a) < t\}$, we have for $m \geq m_0$ sufficiently large,

$$\mathbb{P}\left(\bigcap_{k=m+1}^{2m} D_k\right) = \mathbb{P}\left(\bigcap_{k=m+1}^{2m} \{M(t_k) > 2C_4\psi(t_k)\}\right)$$

$$= \mathbb{P}\left(\bigcap_{k=m+1}^{2m} \{P(2C_4\psi(t_k)) < t_k\}\right) \geq \mathbb{P}\left(\bigcap_{k=m+1}^{2m} \{P(a_k) < K_{13}\phi(a_k)\}\right)$$
$$\geq \mathbb{P}\left(\sup_{a_{2m}\leq a\leq a_{m+1}} \frac{P(a)}{\phi(a)} < K_{13}\right).$$

Let $\gamma_0 > 0$ be small enough such that $\gamma_0 \leq \exp(-5m_0^2/\alpha_1)$. Let $0 < \gamma < \gamma_0$, $\delta > \gamma^{1/6}$ and $m$ be the largest integer less than $\sqrt{-\frac{\alpha_1}{5}\log\gamma}$. Then

$$\gamma \leq \exp(-5m^2/\alpha_1) \leq a_m^4 < a_{2m} < a_{m+1}$$
$$= 2C_4\left(\log\left((m+1)^2\right)\right)^{-1/\alpha_1} \cdot \exp\left(-\frac{(m+1)^2}{\alpha_1}\right)$$
$$\leq 2C_4\left(\log\left((m+1)^2\right)\right)^{-1/\alpha_1} \cdot \exp\left(\frac{\alpha_1\log\gamma}{5\alpha_1}\right)$$
$$= 2C_4\left(\log\left((m+1)^2\right)\right)^{-1/\alpha_1} \cdot \gamma^{1/5}$$
$$\leq 2C_4\left(\log\log\left(\gamma^{-\alpha_1/5}\right)\right)^{-1/\alpha_1} \cdot \gamma^{1/5} \leq \gamma^{1/6} < \delta,$$

and hence

$$\mathbb{P}\left(\sup_{\gamma\leq a\leq\delta} \frac{P(a)}{\phi(a)} < K_{13}\right) \leq \mathbb{P}\left(\bigcap_{k=m+1}^{2m} D_k\right) \leq \exp\left(-m^{\frac{1}{4}}\right)$$
$$\leq \exp\left(-K\cdot(m+1)^{\frac{1}{4}}\right) \leq \exp\left(-K\cdot\left(-\frac{\alpha_1}{5}\log\gamma\right)^{\frac{1}{8}}\right)$$
$$\leq \exp\left(-K_{14}\cdot(-\log\gamma)^{\frac{1}{8}}\right),$$

which concludes the proof for $X$ of type $A$ and $0 < \alpha_1 < 2$.

Now assume that $X$ is of type $B$ and $0 < \alpha_1 < 1$. Then $\phi(a) = a^{\alpha_1}\left(\log\log\frac{1}{a}\right)^{1-\alpha_1}$. Define $\psi(s) = s^{1/\alpha_1}\left(\log\log\frac{1}{s}\right)^{-\frac{1-\alpha_1}{\alpha_1}}$. Then $\phi$ and $\psi$ are asymptotically inverse to each other as $a, s \to 0$ in the same sense as in (4.23). Again consider the sequence $t_k = \exp(-k^2)$, $k \geq 1$. For all $k$ there exists an $i_k \in \mathbb{N}$ such that $c^{-(i_k+1)} \leq t_k < c^{-i_k}$. Furthermore, define $C_6 := (3K_{11})^{(1-\alpha_1)/\alpha_1}$, where $K_{11}$ is as in Lemma 4.10 (b), and let

$$D_k := \left\{\frac{M(t_k)}{\psi(t_k)} > 2\cdot C_6\right\},\ G_k := \left\{\frac{M'(t_k)}{\psi(t_k)} > C_6\right\} \text{ and } H_k := \left\{\frac{M(t_{k+1})}{\psi(t_k)} > C_6\right\}.$$

With the same methods as before one can show that for all $m \in \mathbb{N}$

$$\bigcap_{k=m+1}^{2m} D_k \subseteq \left( \bigcap_{k=m+1}^{2m} G_k \right) \cup \left( \bigcup_{k=m+1}^{2m} H_k \right).$$

Set $\mathbb{P}(G_k) = 1 - p_k$ and $\mathbb{P}(H_k) = q_k$. Applying Lemma 4.10(ii), we have for sufficiently large $k$

$$\begin{aligned} p_k &\geq \mathbb{P}\left( \sup_{t_{k+1} \leq t \leq t_k} \|X(t) - X(t_{k+1})\| > C_6 \psi(t_k) \right) \\ &\geq \mathbb{P}\left( \sup_{0 \leq t \leq t_k} \|X(t)\| \leq C_6 \cdot t_k^{1/\alpha_1} \left( \log\log \frac{1}{t_k} \right)^{-\frac{1-\alpha_1}{\alpha_1}} \right) \\ &\geq \exp\left( -K_{11} \left( C_6 \left( \log\log(1/t_k) \right)^{-\frac{1-\alpha_1}{\alpha_1}} \right)^{-\frac{\alpha_1}{1-\alpha_1}} \right) \\ &= \exp\left( -\frac{K_{11}}{3K_{11}} \log\log \frac{1}{t_k} \right) = k^{-\frac{2}{3}}. \end{aligned}$$

On the other hand, choosing $\delta' \in (0, \alpha_p)$, $\delta_1 := 0$ and $\delta_j = \frac{1}{\alpha_j} - \frac{1}{\alpha_1} \in (0, \alpha_j^{-1})$ for $j = 2, \ldots, p$, similarly to type A for sufficiently large $k \in \mathbb{N}$ we get by Lemma 4.13 and Lemma 4.11

$$\begin{aligned} q_k &\leq \sum_{j=1}^{p} \mathbb{P}\left( \sup_{0 \leq t \leq t_{k+1}} \|X^{(j)}(t)\| > \frac{C_6 \cdot t_k^{\frac{1}{\alpha_1}} (\log\log \frac{1}{t_k})^{-\frac{1-\alpha_1}{\alpha_1}}}{p} \right) \\ &= \sum_{j=1}^{p} \mathbb{P}\left( \sup_{0 \leq t \leq t_{k+1}} \|X^{(j)}(t)\| > \frac{C_6 \cdot t_k^{\frac{1}{\alpha_1}} (\log\log \frac{1}{t_k})^{-\frac{1-\alpha_1}{\alpha_1}} t_{k+1}^{-(\frac{1}{\alpha_j} - \delta_j)} t_{k+1}^{\frac{1}{\alpha_j} - \delta_j}}{p} \right) \\ &\leq \sum_{j=1}^{p} \mathbb{P}\left( \sup_{0 \leq t \leq c^{-i_{k+1}}} \|X^{(j)}(t)\| > \frac{C_6 \cdot t_k^{\frac{1}{\alpha_1}} \left(\log\log \frac{1}{t_k}\right)^{-\frac{1-\alpha_1}{\alpha_1}} t_{k+1}^{-(\frac{1}{\alpha_j} - \delta_j)} \cdot c^{-(i_{k+1}+1)(\frac{1}{\alpha_j} - \delta_j)}}{p} \right) \\ &\leq \sum_{j=1}^{p} K^{(j)} \cdot \mathbb{P}\left( \|X^{(j)}(c^{-i_{k+1}})\| > \frac{C_6 \cdot t_k^{\frac{1}{\alpha_1}} \left(\log\log \frac{1}{t_k}\right)^{-\frac{1-\alpha_1}{\alpha_1}} t_{k+1}^{-\frac{1}{\alpha_1}} \cdot c^{-\frac{1}{\alpha_1}} \cdot c^{-i_{k+1}(\frac{1}{\alpha_j} - \delta_j)}}{2p} \right) \end{aligned}$$

$$\le \sum_{j=1}^{p} \widetilde{K}^{(j)} \cdot \left( t_k^{\frac{1}{\alpha_1}} \left(\log\log \tfrac{1}{t_k}\right)^{-\frac{1-\alpha_1}{\alpha_1}} t_{k+1}^{-\frac{1}{\alpha_1}} \right)^{-(\alpha_j-\delta')}$$

$$\le \sum_{j=1}^{p} \widetilde{K}^{(j)} \left( t_k^{-1} t_{k+1} \left(\log\log \tfrac{1}{t_k}\right)^{1-\alpha_1} \right)^{\frac{\alpha_j-\delta'}{\alpha_1}} \le K \cdot \left( t_k^{-1} t_{k+1} \left(\log\log \tfrac{1}{t_k}\right)^{1-\alpha_1} \right)^{\frac{\alpha_p-\delta'}{\alpha_1}}$$

$$= K \cdot \left( \exp\left(k^2 - (k+1)^2\right) \cdot \left(\log\left(k^2\right)\right)^{1-\alpha_1} \right)^{\frac{\alpha_p-\delta'}{\alpha_1}}$$

$$= K \cdot \left( \exp(-2k-1) \cdot \left(\log\left(k^2\right)\right)^{1-\alpha_1} \right)^{\frac{\alpha_p-\delta'}{\alpha_1}} \le \exp(-C_{61}\, k).$$

Analogously to the calculations in type A above, we can now prove that there exist constants $K_{13}, K_{14}, \gamma_0 > 0$ such that

$$\mathbb{P}\left( \sup_{\gamma \le a \le \delta} \frac{P(a)}{\phi(a)} < K_{13} \right) \le \exp\left( -K_{14} \cdot (-\log\gamma)^{\frac{1}{8}} \right), \tag{4.24}$$

provided $0 < \gamma \le \gamma_0$ and $\delta \ge \gamma^{1/6}$. This concludes the proof. □

**Corollary 4.15**
*Let $X$ be a $(c^E, c)$-operator semistable Lévy process on $\mathbb{R}^d$ with diagonal principal exponent $E_1$ and $\alpha_1 < d_1$. Then for $\phi$ as in (4.21) there exist constants $K_{13}$, $K_{14}, \gamma_0 > 0$ such that*

$$\mathbb{P}\left( \sup_{\gamma \le a \le \delta} \frac{T(a,1)}{\phi(a)} < K_{13} \right) \le \exp\left( -K_{14} \cdot (-\log\gamma)^{\frac{1}{8}} \right), \tag{4.25}$$

*for all $0 < \gamma \le \gamma_0$ and $\delta \ge \gamma^{1/6}$.*

*Proof.* Obviously, $T(a,1) \le t < 1$ implies that $P(a) \le t$. This gives us

$$\left\{ \sup_{\gamma \le a \le \delta} \frac{T(a,1)}{\phi(a)} < K_{13} \right\} \subseteq \left\{ \sup_{\gamma \le a \le \delta} \frac{P(a)}{\phi(a)} < K_{13} \right\},$$

provided $\delta$ and therefore $\gamma$ small enough to ensure that $\phi$ is increasing on $(0, \delta)$ and $K_{13}\phi(\delta) < 1$. Lemma 4.14 then concludes the proof. □

Let $K_3 > 0$ be a fixed constant. A family $\Lambda(a)$ of cubes of side $a$ in $\mathbb{R}^d$ is called $K_3$-nested if no balls of radius $a$ in $\mathbb{R}^d$ can intersect more than $K_3$ cubes of $\Lambda(a)$. Here, we will choose $\Lambda(a)$ to be the family of all cubes in $\mathbb{R}^d$ of the form $[k_1 a, (k_1+1)a] \times \ldots \times [k_d a, (k_d+1)a]$ with $K_3 = 3^d$. The following covering lemma is due to Pruitt and Taylor [48, Lemma 6.1]:

**Lemma 4.16**
*Let $X = \{X(t)\}_{t\geq 0}$ be a Lévy process in $\mathbb{R}^d$ and let $\Lambda(a)$ be a fixed $K_3$-nested family of cubes in $\mathbb{R}^d$ of side $a$ with $0 < a \leq 1$. For any $u \geq 0$ let $M_u(a,s)$ be the number of cubes in $\Lambda(a)$ hit by $X(t)$ at some time $t \in [u, u+s]$. Then*

$$\mathbb{E}\left[M_u(a,s)\right] \leq 2\,K_3 s \cdot \left(\mathbb{E}\left[T\left(\tfrac{a}{3}, s\right)\right]\right)^{-1}.$$

For $u = 0$ we simply write $M(a,s) := M_0(a,s)$. The following result is a direct consequence of Lemma 4.2 and Lemma 4.16. Although part (ii) is not needed here, it might be useful to show that $\phi - m(X([0,1])) < \infty$ for $\phi$ as in (4.11) in case $X$ is of type $A$, $d \geq 2$ and $\alpha_1 > d_1 = 1$.

**Lemma 4.17**
*Let $X$ be a $(c^E, c)$-operator semistable Lévy process on $\mathbb{R}^d$ with diagonal principal exponent $E_1$.*

*(i) If $\alpha_1 < d_1$, there exists a constant $K_{15} > 0$ such that for all $a \leq 1$*

$$\mathbb{E}[M(a,1)] \leq K_{15} a^{-\alpha_1}.$$

*(ii) If $d \geq 2$ and $\alpha_1 > d_1$, then $d_1 = 1$ and we further assume that $E_2$ is diagonal. Then there exists a constant $K_{16} > 0$ such that for all $a > 0$ small enough*

$$\mathbb{E}[M(a,1)] \leq K_{16} a^{-\rho},$$

*where $\rho = 1 + \alpha_2(1 - 1/\alpha_1)$.*

Let $\Lambda_k$ be the set of cubes of side $2^{1-k}$ and centered at $(j_1/2^k, \ldots, j_d/2^k)$, where $j_l$, $1 \leq l \leq d$, are integers, closed on the left and open on the right. The following result is taken from Lemma 3.9 in [30] and based on Lemma 9 in [53].

**Lemma 4.18**
*If $E = \bigcup_{i=1}^m I_i$, where each $I_i$ is a cube of $\Lambda_k$ for some integer $k$, then we can find a subset $\{j_r\}$ such that $E \subseteq \bigcup I_{j_r}$ and no point of $E$ is contained in more than $2^d$ of the cubes $I_{j_r}$.*

**Theorem 4.19**
*Let $X$ be a $(c^E, c)$-operator semistable Lévy process on $\mathbb{R}^d$ with diagonal principal exponent*

*$E_1$ and $\alpha_1 < d_1$. Then for*

$$\phi(a) = \begin{cases} a^{\alpha_1} \log\log \frac{1}{a}, & \text{if } X \text{ is of type } A \text{ and } 0 < \alpha_1 < 2 \\ \\ a^{\alpha_1} \left(\log\log \frac{1}{a}\right)^{1-\alpha_1}, & \text{if } X \text{ is of type } B \text{ and } 0 < \alpha_1 < 1 \end{cases}$$

*we have almost surely $\phi - m(X([0,1])) < \infty$.*

*Proof.* Let $r$ be a positive integer and $\delta := 2^{-r}$. Furthermore, let $n$ be an integer with $2^{-n} \leq \min(\gamma_0, 2^{-6r})$, where $\gamma_0$ is as in Lemma 4.14 and $\overline{\Lambda}_n$ the collection of cubes of side $2^{-n}$ with centers the same as in $\Lambda_n$. Define $\tau^I = \inf\{t \geq 0 : X(t) \in I\}$ for any cube $I$ and $\overline{\Lambda'}_n = \{I \in \overline{\Lambda}_n : \tau^I \leq 1\}$, the cubes hit by $X$ over the time interval $[0,1]$. Then $M(2^{-n}, 1) = |\overline{\Lambda'}_n|$. Let $\gamma_n := 2^{-n}$. We say that a cube $I$ in $\overline{\Lambda'}_n$ is bad if for all $a \in [\gamma_n, \delta]$

$$\int_{\tau^I}^{\tau^I+1} 1_{B(X(\tau^I),a)}(X(t))dt \leq K_{13}\phi(a),$$

and good otherwise. For any cube $I \in \overline{\Lambda}_n$ we have

$$\begin{aligned} &\mathbb{P}\left(I \text{ is bad } | 0 \leq \tau^I \leq 1\right) \\ &= \mathbb{P}\left( \sup_{\gamma_n \leq a \leq \delta} \left\{ \frac{\int_{\tau^I}^{\tau^I+1} 1_{B(X(\tau^I),a)}(X(t))dt}{\phi(a)} \right\} \leq K_{13} \middle| 0 \leq \tau^I \leq 1 \right) \\ &= \mathbb{P}\left( \sup_{\gamma_n \leq a \leq \delta} \left\{ \frac{\int_0^1 1_{B(0,a)}(X(t+\tau^I) - X(\tau^I))dt}{\phi(a)} \right\} \leq K_{13} \middle| 0 \leq \tau^I \leq 1 \right). \end{aligned}$$

Note that $\{X(t+\tau^I) - X(\tau^I)\}_{t\geq 0}$ is identical in law with $\{X(t)\}_{t\geq 0}$ on $\{\tau^I < \infty\}$ by the strong Markov property (see e.g. Corollary 40.11 in [50]). Hence, we get by applying Corollary 4.15

$$\mathbb{P}\left(I \text{ is bad } | 0 \leq \tau^I \leq 1\right) \leq \exp\left(-K_{14} \cdot (-\log \gamma_n)^{\frac{1}{8}}\right) = \exp\left(-C_7 \cdot n^{\frac{1}{8}}\right),$$

where $C_7 > 0$ is a constant independent from $n$. Now let $N_n$ denote the number of bad cubes in $\overline{\Lambda'}_n$. Then by Lemma 4.17

$$\mathbb{E}[N_n] \leq \exp\left(-C_7 \cdot n^{\frac{1}{8}}\right) \mathbb{E}\left[M(2^{-n}, 1)\right] \leq K_{15} 2^{n\alpha_1} \exp\left(-C_7 \cdot n^{\frac{1}{8}}\right).$$

Hence, by the Markov inequality for $n$ sufficiently large there exists a constant $C_8 > 0$

$$\begin{aligned}\mathbb{P}\left(N_n \geq 2^{n\alpha_1}\exp\left(-n^{\frac{1}{10}}\right)\right) &\leq \frac{\mathbb{E}\left[|N_n|\right]}{2^{n\alpha_1}\exp\left(-n^{1/10}\right)} \\ &\leq K_{15}\exp\left(-C_7\cdot n^{\frac{1}{8}} + n^{\frac{1}{10}}\right) \leq K_{15}\exp\left(-C_8\cdot n^{\frac{1}{10}}\right).\end{aligned}$$

This implies that

$$\sum_{n=1}^{\infty}\mathbb{P}\left(N_n \geq 2^{n\alpha_1}\exp\left(-n^{1/10}\right)\right) \leq K + K_{15}\sum_{n=1}^{\infty}\exp\left(-C_8\cdot n^{1/10}\right) < \infty.$$

By the Borel-Cantelli lemma, there now exists an $\Omega_0$ with $\mathbb{P}(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$ we can find an integer $n_1 = n_1(\omega)$ such that for $n \geq n_1$

$$N_n(\omega) < 2^{n\alpha_1}\exp\left(-n^{1/10}\right).$$

Furthermore, by regular variation techniques, there exists a constant $C_9 > 0$ such that

$$\phi\left(\sqrt{d}\cdot 2^{-n}\right) = \left(\sqrt{d}\cdot 2^{-n}\right)^{\alpha_1}\log\log\left(\sqrt{d}\cdot 2^{n}\right) = C_9\; 2^{-n\alpha_1}\log(n).$$

Thus, for $\omega \in \Omega_0$ and $n \geq n_1(\omega)$

$$\sum_{I\text{ bad}}\phi(|I|) = N_n(\omega)\cdot\phi\left(\sqrt{d}\cdot 2^{-n}\right) \leq C_9\exp\left(-n^{1/10}\right)\log(n). \tag{4.26}$$

Now consider the good cubes $I$ in $\overline{\Lambda'}_n$. Our aim is to show that the good cubes can be covered economically. For a good cube $I$ there exists $a \in [\gamma_n, 2^{-r}]$ such that

$$\phi(a) < \frac{1}{K_{13}}\int_{\tau^I}^{\tau^I+1} 1_{B(X(\tau^I),a)}(X(t))dt.$$

We can find an integer $k$ with $2^{-k} > 5a \geq 2^{-k-1}$ and a cube $I'$ in $\Lambda_k$ such that $I'$ contains $I$ and $B(X(\tau^I),a)$. Then, one can easily show that $k > r-4$ and, since $\tau^I \leq 1$ by definition of $\overline{\Lambda'}_n$, we get

$$\begin{aligned}\phi\left(|I|\right) &= \phi\left(\sqrt{d}\cdot 2^{-k+1}\right) = \phi\left(\sqrt{d}\cdot 4\cdot 2^{-k-1}\right) \leq \phi\left(\sqrt{d}\cdot 4\cdot 5a\right) \\ &\leq K\phi(a) < K\int_{\tau^I}^{\tau^I+1} 1_{B(X(\tau^I),a)}(X(t))dt \leq K\int_0^2 1_{I'}(X(t))dt.\end{aligned}$$

Applying Lemma 4.18 to the collection $\{I' : I \text{ is good}\}$, we can show that there is a subset $\Lambda'$, which still covers $\bigcup_{I \text{ good}} I$, but no point is covered more than $2^d$ times. Hence, $\sum_{I' \in \Lambda'} 1_{I'} \leq 2^d$ and there exists a constant $C_{10} > 0$ such that

$$\sum_{I' \in \Lambda'} \phi(|I'|) \leq \sum_{I' \in \Lambda'} K \int_0^2 1_{I'}(X(t))dt = K \int_0^2 \sum_{I' \in \Lambda'} 1_{I'}(X(t))dt \leq C_{10} \cdot 2^{d+1}. \tag{4.27}$$

Using all the bad cubes together with the covering of good cubes defined above, we obtain a covering of $X([0,1])$ by cubes with diameters less than $\sqrt{d} \cdot 2^{-r+5}$. This means

$$X([0,1]) \subseteq \left( \bigcup_{I \text{ bad}} I \right) \cup \left( \bigcup_{I' \in \Lambda'} I' \right).$$

For sufficiently large $n$ applying (4.26) and (4.27) we finally arrive at

$$\sum_{I:\text{ bad}} \phi\left(|I|\right) + \sum_{I' \in \Lambda'} \phi(|I'|) \leq C_9 \exp\left(-n^{1/10}\right) \log(n) + C_{10} \cdot 2^{d+1} \leq C_{10} \cdot 2^{d+1} + 1.$$

Thus, $\phi - m(X([0,1])) \leq C_{10} \cdot 2^{d+1} + 1 < \infty$ almost surely, which concludes the proof. □

### 4.4.3 Proof of the main result

The proof of Theorem 4.4 now follows directly from Theorem 4.8 and Theorem 4.19.

# Chapter 5

# Conclusion and outlook

Operator semistable Lévy processes are stochastic processes with a selfsimilarity property on a discrete scale. They generalize the better known class of (operator) stable Lévy processes, which have a continuous scaling property. Operator semistable Lévy processes have an application in various scientific fields, such as physics and financial mathematics, where they prove to be useful when developing mathematical models describing long-term dependencies. For this reason, it is important to analyze the fractal properties of this class of stochastic processes in order to get a better understanding of the path behavior.

Starting from the work of Kern and Wedrich [36], who calculated the Hausdorff dimension $\dim_H X(B)$ for the range $X(B) = \{X(t) : t \in B\}$ of an operator semistable Lévy process $X$ and an arbitrary Borel set $B \subseteq \mathbb{R}_+$, this thesis offers further results on the dimension properties of operator semistable Lévy processes.

Based on the examination of a specific non-strictly semistable Lévy process in the first manuscript (Chapter 2), the problem of generating a formula for the graph $\mathrm{Gr}_X(B) = \{(t, X(t)) : t \in B\}$ of an arbitrary operator semistable Lévy process $X$ in $\mathbb{R}^d$ and any Borel set $B \subseteq \mathbb{R}_+$ has been completely solved in Chapter 3 of this thesis. Finally, based on the works of Taylor [53] and Hou and Ying [30], the third manuscript in Chapter 4 refines the results on the path behavior of certain operator semistable Lévy processes by dealing with the subject of exact Hausdorff measure functions. In particular, for the range of an operator semistable Lévy process with a partially diagonal exponent and $\alpha_1 < d_1$ exact Hausdorff measure functions are calculated over the time interval $[0, 1]$ .

Although the results in this thesis are an important step to fully understand the fractal properties of selfsimilar stochastic processes (both on a continuous and a discrete scale), there are still open research questions that have yet to be addressed.

For instance, if $X$ is of type $B$, an additional assumption is needed to prove the result in the third manuscript (compare Assumption 4.3). Hence, in a next step, one can further investigate if either the assumption is valid for all operator semistable Lévy processes of type $B$ with principal diagonal exponent and $\alpha_1 < d_1$ or if it is possible to find an alternative way of proof that doesn't depend on the assumption. Also, for $\alpha_1 \geq d_1$ as well as operator (semi)stable Lévy processes with non-diagonal principal exponent $E$ the problem of finding a representation of exact Hausdorff measure functions is still open. For the calculation of exact Hausdorff measure functions for operator semistable Lévy processes with principal diagonal exponent, sharp upper and lower bounds for the expected sojourn times $\mathbb{E}[T(a,s)]$ were needed (compare Lemma 4.2 in the third manuscript). In case that the principal exponent is non-diagonal, it is not clear if a corresponding result even exists. Hence, if this turns out not to be the case, one has to either investigate if there is an alternative way to calculate exact Hausdorff measure functions or, by generating a counterexample, prove that exact Hausdorff measure functions do not necessarily exist for arbitrary operator semistable Lévy processes.

Furthermore, the proofs of the above mentioned results depend, in particular, on the Lévy properties of the process $X$. Consequently, another line of further research would be to try and generalize the above results on stochastic processes which, for instance, fulfill the semi-selfsimilarity condition (1.1) and have stationary increments but lack the independence of the increments. First results in this direction can be found in [40] and [60] for selfsimilar Markov processes (cf. also the survey article [59]).

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# Erklärung

Die hier vorgelegte Dissertation habe ich eigenständig und ohne unerlaubte Hilfe angefertigt. Die Dissertation wurde in der vorgelegten oder in ähnlicher Form noch bei keiner anderen Institution eingereicht. Ich habe bisher keine erfolglosen Promotionsversuche unternommen.

Lina Wedrich Köln, den 26.04.2016