Quantum cohomology of homogeneous spaces: curve neighborhoods and quantum to classical principles

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25. April 2016

Inaugural-Dissertation

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Tag der mündlichen Prüfung: 5. Juli 2016

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Abstract. The aim of this work is to develop formulas which express three point genus zero Gromov-Witten invariants on a homogeneous space of Picard rank one as classical intersection numbers on a different compact homogeneous space. Such formulas were first proved for the Grassmannian, the isotropic and the symplectic Grassmanian ([6], [7]) and were later generalized to cominuscule homogeneous spaces [11]. After reviewing the concepts already known, we study in more detail the isotropic Grassmannian. We give a new formula in this case which differs from the one already known from [7] by the use of different objects which relate rational curves passing through three Schubert varieties in general position and points in a threefold intersection of cohomology classes on an auxiliary space.

Das Ziel dieser Arbeit ist es, Formeln zu entwickeln, welche drei Punkt Geschlecht null Gromov-Witten Invarianten auf einem homogenen Raum von Picard-Rang eins als klassische Durchschnittszahlen auf einem anderen kompakten homogenen Raum ausdrücken. Solche Formeln wurden zuerst für die Graßmann-Varietät, die isotrope and die symplektische Graßmann-Varietät bewiesen ([6], [7]), und später auf kominusküle homogene Räume verallgemeinert [11]. Nachdem wir die bereits bekannten Konzepte durchdenken, studieren wir in größerem Detail die isotrope Graßmann-Varietät. Wir geben in diesem Fall eine neue Formel an, welche sich von der bereits aus [7] bekannten unterscheidet, durch den Gebrauch anderer Objekte welche rationale Kurven, die durch drei Schubert-Varietäten in allgemeiner Lage verlaufen, in Beziehung setzen zu Punkten in einem Durchschnitt von drei Kohomologieklassen auf einem Hilfsraum.

Contents

Introduction

The aim of this work is to work out quantum to classical formulas for computing Gromov-Witten invariants similar to formulas obtained in [6, Corollary 1]. Sereval steps in this direction were already done in [7] and [11]. Let us briefly present these results for the Grassmannian since they serve as a model for all quantum to classical formulas. Let $X = \mathbb{G}(k,n)$ and $1 \leq d \leq \min(k, n-k)$ be a degree. Let λ, μ and ν be three partitions of shape $k \times (n-k)$ such that $|\lambda| + |\mu| + |\nu| = k(n-k) + nd$. Denote by $Y = \mathbb{F}(k-d, k+d, n)$ the flag variety parametrizing partial flags $V_{k-d} \subseteq V_{k+d} \subseteq \mathbb{C}^n$ where V_{k-d} is of dimension $k-d$ and V_{k+d} is of dimension $k+d$. For a Schubert variety $X_\lambda(F_\bullet)$ where $F_\bullet$ is a complete flag in $\mathbb{C}^n$ we denote with $X_\lambda^{(d)}(F_\bullet)$ the transformed Schubert variety in Y defined by

$$X_\lambda^{(d)}(F_\bullet) = \{(V_{k-d}, V_{k+d}) \in Y \mid \exists V_k \in X_\lambda(F_\bullet) \colon V_{k-d} \subseteq V_k \subseteq V_{k+d}\}$$

and with $[X_\lambda^{(d)}]$ its cohomology class. Then we have the following formula

$$\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d = \int_Y [X_\lambda^{(d)}] \cdot [X_\mu^{(d)}] \cdot [X_\nu^{(d)}] \tag{1}$$

where on the left side stands the Gromov-Witten invariant of the Schubert cycles $\sigma_\lambda, \sigma_\mu$ and σ_ν. This invariant counts the number of rational curves of degree d which pass through general translates of the Schubert varieties parametrized by λ, μ and ν. We refer to a formula like (1) as a quantum to classical principle for X.

Let $X = G/P$ be a homogeneous space where P is a maximal parabolic subgroup and G is a simple affine algebraic group. Maximality of P means geometrically that $\mathrm{Pic}(X) \cong \mathbb{Z}$. While the quantum to classical principle (1) works for all degrees d we will focus our discussion on the degree d_X. We briefly recall the definition of this degree. For more details we refer to Section 7. First we define a metric d on X. For two points $x, y \in X$ we define $d(x,y)$ to be the degree d of a curve passing through x and y such that the degree d is minimal in the set of all degrees of curves passing through x and y. We further define

$$d_X = \max_{x,y \in X} d(x,y)\,.$$

It is a consequence of [13] that d_X is the minimal power of the quantum parameter q appearing in the quantum product $[\{\mathrm{pt}\}] \star [\{\mathrm{pt}\}]$, i.e. we can write

$$[\{\mathrm{pt}\}] \star [\{\mathrm{pt}\}] = \sigma \cdot q^{d_X} + \text{higher degree terms}$$

for some effective homogeneous cohomology class σ.

After recalling some basic notions we tackle the task of the algorithmic computation of d_X (Section 8.1). We may compute the number

d_X directly by exhibiting the root system of G: Let α_P be the simple root associated to P. If we denote with θ_1 the highest root in the root system $R = R_1$ of G, then all roots orthogonal to θ_1 form a root system, which may be reducible. We pick the irreducible component R_2 of the root system $\{\alpha \in R \mid (\theta_1, \alpha) = 0\}$ which contains the simple root α_P and repeat the procedure: we choose the highest root θ_2 in the root system R_2, etc. After finitely many steps we end up with k roots $\theta_1, \dots, \theta_k$ such that R_{k+1} is empty. We call the sequence of roots $\theta_1, \dots, \theta_k$ the θ-sequence. The reader will find many informations concerning the θ-sequence in Section 8, among them the following proposition concerning the algorithmic computation of d_X.

Proposition 0.1 (Proposition 8.16)**.** *Let $\theta_1, \dots, \theta_k$ be the θ-sequence as defined above. Then the maximal possible distance d_X is given by*

$$d_X = \sum_{i=1}^{k} \langle \theta_i^\vee, \omega \rangle$$

where ω denotes the fundamental weight dual to the simple coroot $\alpha_P^\vee$ associated to P.

The θ-sequence gives naturally rise to a rational curve f_Δ of degree d_X, the so called diagonal curve. We briefly recall the definition of the diagonal curve. Then we state our main result concerning the density of the diagonal curve. For more details we refer to Section 9. Let G' be the subgroup of G defined as $G' = \mathrm{SL}_2(\theta_1) \times \cdots \times \mathrm{SL}_2(\theta_k)$. To abbreviate we set $X' = G'/G' \cap P$. We clearly have $X' \cong \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. We can define a rational curve of degree d_X via the composition

$$f_\Delta \colon \mathbb{P}^1 \xrightarrow{\Delta} \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \cong X' \hookrightarrow X\,,$$

where Δ denotes the diagonal embedding. We call this curve the diagonal curve. Let $\mathcal{M} = \mathcal{M}_{0,3}(X, d_X)$ be the Kontsevich-Manin moduli space parametrizing all 3-pointed stable maps to X of degree d_X and genus zero. Our main result concerning the density of the diagonal curve reads as follows:

Theorem 0.2 (Theorem 9.5)**.** *Suppose that $X \neq G_2/P_1$ and that $X \neq B_\ell/P_\ell$ where $\ell > 1$ is odd. The diagonal curve has a dense open orbit under the action of G in $\mathcal{M}$. In other words, the diagonal curve is a general curve.*

This result is of importance since it says that every other general curve is in the G-orbit of f_Δ. Therefore we can reduce the main constructions to the diagonal curve. The main construction of this work is the construction of the irreducible subvariety $\hat{X}$ of X. We briefly recall the definition of $\hat{X}$ and state its main properties. For more details we refer to Section 11. Let $\mathcal{B}$ be Kostant's cascade of strongly orthogonal roots. For a precise definition of this maximal set of strongly orthogonal roots we refer to Section 4. From the definition of $\mathcal{B}$ it is clear that $\{\theta_1, \ldots, \theta_k\} \subseteq \mathcal{B}$. Let S be the set of all roots γ such that γ is not orthogonal to precisely two elements of $\{\theta_1, \ldots, \theta_k\}$. Let S_Δ be the set of simple roots contained in S. If $k = 1$ we set $R' = \emptyset$. If $k > 1$ we define R' to be the union of all irreducible components unequal to R_k of the root system consisting of all roots in R_{k-1} which are orthogonal to θ_{k-1}. Let $\hat{R}$ be the root subsystem of R generated by $\mathcal{B}$, S_Δ and R'. Let $\hat{G}$ be the algebraic subgroup of G with root system $\hat{R}$. Let $\hat{X} = \hat{G}/\hat{G} \cap P$. Then it is clear that $\hat{X}$ is an irreducible subvariety of X. Our main result concerning the variety $\hat{X}$ is the following proposition.

Proposition 0.3 (Proposition 11.67)**.** *Through three points of $\hat{X}$ in general position passes a unique rational curve of degree d_X which is contained in $\hat{X}$.*

The number d_X and its properties concerning the geometry of rational curves motivates the definition of the subvarieties $X_{d_X}(x, y)$ and $Y_{d_X}(f)$ of X where f is a general curve of degree d_X and x and y are two points in X. In general for a degree d with $1 \leq d \leq d_X$ we define $X_d(x, y)$ to be the union of all rational curves in X of degree d which pass through x and y and we define $Y_d(f)$ for a general curve f of degree d to be the intersection

$$Y_d(f) = \bigcap_{\substack{x,y \in f(\mathbb{P}^1) \\ \text{general}}} X_d(x, y)\,.$$

We will mostly restrict ourselves to the situation where $d = d_X$. Then we can restrict our attention to $Y_{d_X}(f_\Delta)$ since the properties of any $Y_{d_X}(f)$ for a general curve f of degree d_X follow from the properties of $Y_{d_X}(f_\Delta)$ via translation whenever the assumptions of Theorem 0.2 are satisfied.

We assume from now on that $X \neq G_2/P_1$ and $X \neq B_\ell/P_\ell$ where $\ell > 1$ is odd. Then we know that the diagonal curve has a dense open orbit in $\mathcal{M}$. By Proposition 0.3 and since $f_\Delta(\mathbb{P}^1) \subseteq \hat{X}$ by definition of $\hat{X}$ it follows fairly easily that $\hat{X} \subseteq Y_{d_X}(f_\Delta)$. Now we contemplate the following assumptions.

Assumption 0.4 (Assumption 13.3)**.**

- *We assume that $\hat{X}$ is an irreducible component of $Y_{d_X}(f_\Delta)$.*
- *We assume that all irreducible components of $Y_{d_X}(f_\Delta)$ are pairwise nonisomorphic.*
- *We assume that each irreducible component X_0 of $Y_{d_X}(f_\Delta)$ satisfies the following property: through three points of X_0 in general position passes a unique rational curve of degree d_X which is contained in X_0.*

Under these assumptions we are able to develop a quantum to classical principle for X. We will see later that these assumptions are satisfied for isotropic Grassmannians and lead to a quantum to classical principle which differs from the one known from [7]. We now introduce the abstract framework which makes sense whenever the Assumption 0.4 is satisfied. For more details we refer to Section 13.

Let Q be the stabilizer of $\hat{X}$ in G. Let $Y = G/Q$ and $Z = G/P \cap Q$. We have obvious projection maps $p\colon Z \to X$ and $q\colon Z \to Y$. Let w be a Weyl group element. Then we denote by X_w the Schubert variety parametrized by w and by $\sigma(w)$ the corresponding Schubert cycle parametrized by w. We write $F_w = qp^{-1}(X_w)$. The variety F_w is B-stable and irreducible. We define a surjective morphism $q_w\colon p^{-1}(X_w) \to F_w$ via restriction of q. Let N_w be the nonempty open subset of F_w where the fibers of q_w are of minimal dimension. Then N_w is an open dense B-stable subset of F_w. We can define a non negative integer $\bar{q}_w$ by the following equation in cohomology:

$$q_{w*}[p^{-1}(X_w)] = \bar{q}_w[F_w]\,.$$

With this notation we are able to formulate the following theorem.

Theorem 0.5 (Theorem 13.19)**.** *Suppose that the Assumption 0.4 is satisfied. Let g, g' and g'' be three general elements of G. Let u, v and w be three Weyl group elements such that*

$$\mathrm{codim}(X_u) + \mathrm{codim}(X_v) + \mathrm{codim}(X_w) = \dim(\mathcal{M})\,.$$

Then we have the following equality:

$$\langle\sigma(u), \sigma(v), \sigma(w)\rangle_{d_X} = \bar{q}_u\bar{q}_v\bar{q}_w\,\mathrm{card}(gF_u \cap g'F_v \cap g''F_w)$$

where $\langle\sigma(u), \sigma(v), \sigma(w)\rangle_{d_X}$ denotes the Gromov-Witten invariant of the Schubert cycles $\sigma(u)$, $\sigma(v)$ and $\sigma(w)$.

Let w be a Weyl group element. Then it is easy to see that we always have the following inequality:

$$\mathrm{codim}(F_w) \geq \mathrm{codim}(X_w) - \dim(\hat{X})\,.$$

Furthermore we know that the have $\bar{q}_w = 0$ whenever the previous inequality is strict. Therefore we get the following corollary of the previous theorem.

Corollary 0.6 (Corollary 13.20)**.** *Suppose that the Assumption 0.4 is satisfied. Let u, v and w be three Weyl group elements. Suppose that the inequality*

$$\mathrm{codim}(F_s) \geq \mathrm{codim}(X_s) + \dim(\hat{X})$$

is strict for at least one $s \in \{u, v, w\}$. Then we have the vanishing

$$\langle\sigma(u), \sigma(v), \sigma(w)\rangle_{d_X} = 0\,.$$

In Section 15 – 18 we work out this program in case of the isotropic Grassmannian $X = \mathbb{G}_Q(l, 2p)$ where l is odd and $l \leq p-2$. For this X we find using Proposition 0.1 that $d_X = l+1$. In this case we are able to give an explicit description of the irreducible components of $Y_{l+1}(f)$ for a general rational curve f of degree $d_X = l+1$ (cf. Lemma 17.1). This description yields in particular the following fact.

Fact 0.7 (Corollary 17.3)**.** *Let $X = \mathbb{G}_Q(l, 2p)$ where l is odd and $l \leq p-2$. Then the Assumption 0.4 is satisfied.*

For the convenience of the reader we briefly want to recall the description of $\hat{X}$ in terms of isotropic subspaces. To this end let $e_1, \dots, e_{2p}$ be a standard basis of $\mathbb{C}^{2p}$ which is compatible with B. We now define the $2(l-1)$-dimensional nondegenerated subspace W_{f_Δ} which parametrizes $\hat{X}$. Let

$$W_{f_\Delta} = \langle e_1, \dots, e_{l-1}, e_{2p-l+2}, \dots, e_{2p} \rangle \ .$$

Then we have the follwing fact.

Fact 0.8 (Corollary 17.2 and Lemma 17.4)**.** *The irreducible component $\hat{X}$ of $Y_{l+1}(f_\Delta)$ can be explicitely described in terms of isotropic subspaces via the following equation:*

$$\begin{aligned}\hat{X} &= \{V \in X \mid \dim(W_{f_\Delta} \cap V) = l-1, \dim(W_{f_\Delta}^\perp \cap V) = 1\} \\ &\cong \mathbb{G}_Q(l-1, 2(l-1)) \times \mathbb{Q}_{2(p-l)} \ .\end{aligned}$$

In case of the isotropic Grassmannian X we have a natural compactification $\bar{Y}$ of Y at hand. Let $\bar{Y} = \mathbb{G}(2(l-1), 2p)$. Then we see that Y embeds into $\bar{Y}$ as an open dense subvariety. We will identify the morphism $q\colon Z \to Y$ with the composition $Z \to Y \hookrightarrow \bar{Y}$.

We briefly explain how we parametrize Schubert varieties in X. For more details we refer to Section 14. Let $\tilde{\mathcal{P}}(l, p)$ be the set of all $(p-l)$-strict partitions of shape $l \times (2p-l-1)$ with type attached to them. Then we know that $\tilde{\mathcal{P}}(l, p)$ parametrizes the Schubert varieties in X. We denote by w_λ the minimal length representative corresponding to a partition λ. We write $X_\lambda = X_{w_\lambda}$ and $\sigma_\lambda = \sigma(w_\lambda) = [X_\lambda]$ for all partitions λ. Moreover we define $\bar{q}_\lambda = \bar{q}_{w_\lambda}$. With this notation Theorem 0.5 reads as follows.

Theorem 0.9 (Theorem 18.4)**.** *Let λ, μ and ν be elements of $\tilde{\mathcal{P}}(l, p)$ such that*

$$\operatorname{codim}(X_\lambda) + \operatorname{codim}(X_\mu) + \operatorname{codim}(X_\nu) = \dim(X) + (2p-l-1)(l+1) \ .$$

Then we have the following equality:

$$\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_{l+1} = \bar{q}_\lambda \bar{q}_\mu \bar{q}_\nu \int_{\bar{Y}} q_* p^* \sigma_\lambda \cdot q_* p^* \sigma_\mu \cdot q_* p^* \sigma_\nu \ .$$

Remark 0.10. The quantum to classical principle in Theorem 0.9 differs from the quantum to classical principle for cominuscule homogeneous spaces [11, Corollary 23] (formula (1) is an instance of this) by the following points:

- The transformed Schubert cycles $q_*p^*\sigma_\lambda$ are not any more Schubert cycles on $\bar{Y}$ but rather may decompose into a linear combination of Schubert cycles on $\bar{Y}$.
- The integer coefficients $\bar{q}_\lambda$ are not necessarily one so that the Gromov-Witten invariant is rather a positive multiple of an integral than the integral itself.
- The integrand $\bar{Y}$ is a homogeneous space which is not homogeneous under the action of $G = \mathrm{SO}_{2p}$ but rather homogeneous under the action of a larger group namely SL_{2p}.

The quantum to classical principles for many other examples, the isotropic Grassmannian included, in particular for homogeneous spaces homogeneous under the action of the exceptional group E_6 were intesively studied in [25, p. 55 f., p. 61 ff.]. In fact many ideas contained in this work as well as in its introduction already appear in some form in [25, cf. 1. Introduction]. One of the objectives of this thesis was to develop these ideas on a solid mathematical basis. Since the note [25] was never published, it was only available and accessible thanks to Prof. Nicolas Perrin, the adviser of this thesis. Through many useful discussions he patiently introduced the author to the subject, motivated and initiated this kind of research. So the author wants to take the opportunity to express his gratitude.

1. Homogeneous spaces

In this section we introduce the basic objects we are using throughout this work: algebraic homogeneous space. We use this occassion to set up the notation we are using from now on. In particular we will be interested in homogeneous fiber bundles, since we use this construction later on. It is supposed to be the counterpart in algebraic geometry of the concept of associated bundles in topology. We close this section with the investigation of a special class of homogeneous spaces, the so called cominuscule homogeneous spaces.

As a general reference for the theory of homogeneous spaces we use [30, Chapter 1]. For the theory of algebraic groups we refer to [15].

By a variety we mean a reduced k-scheme of finite type where k is an algebraically closed field of characteristic zero. Many of our results will even hold in positive characteristic, but to simplify the statements we always assume that the characteristic is zero. In the main part of this work, we will even focus on the case where $k = \mathbb{C}$, so that we do not lose anything.

By an algebraic group we mean a variety which has the structure of a group scheme. We always denote the Lie algebra of an algebraic group $G, H, \ldots$ by lowercase Gothic letters $\mathfrak{g}, \mathfrak{h}, \ldots$ We always denote the identity element of an algebraic group G by e and the character group of G by $X(G)$.

Let G be a linear algebraic group and H a closed subgroup of G.

Definition 1.1. *Let X be a variety with a transitive G-action. In this situation X is called a homogeneous space.*

A pointed homogeneous space is a pair (X, o) where X is a homogeneous space and $o \in X$. The natural map $\pi\colon G \to X$, $g \mapsto go$ is called the orbit map.

Definition 1.2. *The space G/H equipped with the quotient topology and a structure sheaf $\mathcal{O}_{G/H}$ which is the direct image of the sheaf $\mathcal{O}_G^H$ of H-invariant regular functions on G is called the (geometric) quotient of G modulo H.*

Theorem 1.3.
(1) *G/H is a smooth, quasiprojective homogeneous space. The quotient morphism $\pi\colon G \to G/H$ is seperable and loacally trivial in étale topology.*
(2) *For any pointed homogeneous space (X, o) such that $H \subseteq G_o$, the orbit map $\pi\colon G \to X$ factors through $\bar{\pi}\colon G/H \to X$.*
(3) *$\bar{\pi}$ is an isomorphism if and only if $H = G_o$ and π is separable.*

Proof. [30, Theorem 1.1] □

Since we assume that the ground field k is of characteristic zero the orbit map π will always be separable. We therefore can always assume that a pointed homogeneous space is of the form G/H.

Definition 1.4. *Let X be a G-homogeneous space. The isotropy representation at $x \in X$ is the natural representation T_xX of G_x given by differentials of right translation.*

Proposition 1.5. *$T_{eH}G/H \cong \mathfrak{g}/\mathfrak{h}$ as H-modules. The isomorphism is given by the differential of the seperable quotient map π. The right-hand representation of H is the quotient of the adjoint representation of H in $\mathfrak{g}$. The left-hand representation of H is the isotropy representation at eH.*

Proof. [30, Proposition 1.1] □

1.1. **Homogeneous fiber bundles.** In this subsection we introduce homogeneous fiber bundles following [30, Section 2] and collect the most important facts which we will be needed later on. Homogeneous fiber bundles can be used to compute the Picard group of a homogeneous space (Theorem 1.10).

Let Z be an H-variety. Then H acts on $G \times Z$ by $h(g, z) = (gh^{-1}, hz)$.

Definition 1.6. *The quotient space $G \times_H Z = (G \times Z)/H$ equipped with the quotient topology and a structure sheaf which is the direct image of the sheaf of H-invariant regular functions is called the homogeneous fiber bundle over G/H associated to Z.*

The G-action on $G \times Z$ by left translation of the first factor commutes with the H-action on $G \times Z$ and factors to a G-action on $G \times_H Z$. We denotes by $g * z$ the image of (g, z) in $G \times_H Z$ and identify $e * z$ with z.

The homogeneous bundle $G \times_H Z$ is G-equivariantly fibered over G/H with fibers gZ. The fiber map $G \times_H Z \to G/H$ is given by $g * z \mapsto gH$.

Theorem 1.7. *If Z is covered by H-stable quasiprojective open subsets, then $G \times_H Z$ is a G-variety and the fiber map $G \times_H Z \to G/H$ is locally trivial in étale topology.*

Proof. [30, Theorem 2.1] □

Fact 1.8. *If Z is quasiprojective or normal and H is connected, then Z is covered by H-stable quasiprojective open subsets. If H is reductive and Z is affine then $G \times_H Z$ is affine.*

Proof. [30, Theorem 2.1] □

In our application in Section 12 the assumptions of Theorem 1.7 are always satisfied. Therefore all our bundles will be locally trivial in étale topology.

Whenever we deal with homogeneous fiber bundles we will assume from now on that the assumptions of Theorem 1.7 are satisfied.

The next lemma indicates when a homogeneous bundle is trivial.

Lemma 1.9. *The homogeneous bundle $G \times_H Z$ is trivial, that is $G \times_H Z \cong G/H \times Z$ if the H-action on Z extends to a G-action.*

Proof. [30, Lemma 2.1] □

If the fiber Z is an H-module, then the homogeneous bundle is locally trivial in the Zariski topology ([30, page 11]). Any G-variety mapped onto G/H is a homogneous bundle, in particular any G-vector bundle over G/H is G-isomorphic to $G \times_H M$ where M is a finite-dimensional rational H-module. We denote the sheaf of sections of $G \times_H M$ by $\mathcal{L}(M)$.

We now use homogeneous bundles to compute the Picard group of a homogeneous space. Any G-line bundle over G/H is G-isomorphic to $G \times_H k_\chi$ for some character $\chi \in X(H)$ where H acts on k_χ via the character χ ([30, page 12]). We denote the sheaf of sections of $G \times_H k_\chi$ by $\mathcal{L}(\chi) = \mathcal{L}(k_\chi)$. This yields an isomorphism of abelian groups

$$X(H) \to \mathrm{Pic}_G(G/H)\,,\ \chi \mapsto \mathcal{L}(\chi)$$

where Pic_G denotes the group of G-linearized invertible sheaves. If G is simply connected we have a surjective homomorphism

$$X(H) \to \mathrm{Pic}(G/H)$$

which is defined in the same way by forgetting the G-linearization. Its kernel consists of the characters that correspond to different G-linearizations of the trivial line bundle $G/H \times k$ over G/H. If G is connected, then these characters are exactly the restrictions $\mathrm{Res}^G_H X(G)$ to H of characters of G. We obtain the following

Theorem 1.10. $\mathrm{Pic}_G(G/H) \cong X(H)$. *If G is connected and simply connected, then*

$$\mathrm{Pic}(G/H) \cong X(H)/\mathrm{Res}^G_H X(G)$$

.

Proof. For more details see [30, Theorem 2.2]. □

Example 1.11. Let G be a connected reductive linear algebraic group, $B \subseteq G$ a Borel subgroup. Then $\mathrm{Pic}(G/B)$ is isomorphic to the weight lattice of the root system of G.

1.2. **Parabolic subgroups.** In this subsection we give (according to [15, 30.1]) a description of parabolic subgroups of a reductive linear algebraic group. Moreover we describe the Levi decomposition of such parabolic subgroup (according to [15, 30.2]). We will use these concepts later on in the text, since we will be mostly concerned with reductive groups.

We start with the most greatest generality: let G be a linear algebraic group.

Definition 1.12. *A Borel subgroup of G is one which is maximal among the closed connected solvable subgroups. A closed subgroup P is called parabolic if P contains a Borel subgroup of G.*

Let H be an abitrary closed subgroup of G as before.

Theorem 1.13. *G/H is projective if and only if H is parabolic.*

Proof. [30, Theorem 3.1] □

For the rest of this subsection we assume that G is reductive. Whenever we work with a reductive group we use the notation we set up in the following.

We fix a maximal torus T and a Borel subgroup B containing T. We will describe all parabolic subgroups which contain B, the so called standard parabolic subgroups (relative to B). Every other parabolic subgroup is conjugated to a standard parabolic subgroup (since all Borel subgroups are conjugated to each other).

Let R be the root system associated to T and G, Δ the set of simple roots in R / the base of R (corresponding to the choice of the Borel subgroup B), R^+ the set of postive roots and R^- the set of negative roots.

Let W be the Weyl group associated to T and G. Let I be a subset of Δ. We denote by W_I the subgroup of W generated by all simple reflections s_α where $\alpha \in I$. Let $P_I = BW_IB$. The group P_I is a standard parabolic subgroup of G. Moreover we have the following

Theorem 1.14. (1) *Each parabolic subgroup of G is conjugate to one and only one subgroup P_I where $I \subseteq \Delta$.*
(2) *The roots of P_I are those in R^+ along with those roots in R^- which are $\mathbb{Z}$-linear combinations of I.*

Proof. [15, 30.1, Theorem] □

Example 1.15. Let P be a maximal parabolic subgroup of G containing B. By Theorem 1.14 there exists a subset $I \subseteq \Delta$ such that $P = P_I$. The complement $\Delta \setminus I$ of I in Δ consists of one element, which we denote by α_P. In this situation, we say that P is associated to α_P. If the simple roots Δ are ordered in some way such that $\alpha_P = \alpha_i$ is the ith element in this ordering, we write $P = P_i$. If the root system R is irreducible we always use the ordering of Δ given by the Bourbaki tables [4, Chapter VI, Table I-IX].

Example 1.16. Suppose that G is connected. Let $P = P_I$ be a standard parablic subgroup associated to some $I \subseteq \Delta$. By Theorem 1.10, the Picard group of $X = G/P$ is given by

$$\mathrm{Pic}(X) \cong \sum_{\alpha \in \Delta \setminus I} \mathbb{Z}\omega_\alpha$$

where ω_α is the fundamental weight associated to α. In particular we see that the Picard rank of X is equal to one if and only if P is maximal. In this case we write $\omega = \omega_{\alpha_P}$ and we have that $\mathrm{Pic}(X) \cong \mathbb{Z}\omega$.

Example 1.17. Let P be a maximal parabolic subgroup of G containing B. Let V_ω be the highest weight representation of G with highest weight ω. Let v_ω be a highest weight vector. Since the isotropy group in G of the line kv_ω is precisely P, we get a homogeneous emebedding $X = G/P \subseteq \mathbb{P}(V_\omega)$. We call this embedding the minimal homogeneous embedding of X.

For the purpose of this thesis the case where $k = \mathbb{C}$ and X is a projective homogeneous space of Picard rank one homogeneous under the action of a reductive, simple, linear algebraic group will be the most important one.

Let P be a parabolic subgroup of G containing B. In this sitation we will always use the following

Notation 1.18. Let R_P be the root system associated to T and P, Δ_P be the simple roots in R_P (corresponding to B), R_P^+ the positive roots of R_P and R_P^- the negative roots of R_P.

Let W_P be the Weyl group associated to T and P. Then we have $P = P_{\Delta_P}$, $W_P = W_{\Delta_P}$ and $P = BW_PB$. We denote by W^P the set of minimal length representatives of W modulo W_P.

Finally in this subsection we want to recall the Levi decomposition of P. To this end let V be the unipotent radical of P.

Definition 1.19. *If there exists a connected, reductive group L such that the product morphism $L \ltimes V \to P$ is an isomorphism then L is called a Levi factor and $P = LV$ is called the Levi decomposition of P.*

Theorem 1.20. *Any parabloic subgroup P of G has a Levi decomposition $P = LV$ and any two Levi factors are conjugate by an element of V.*

Proof. [15, 30.2, Theorem] □

If P is a parabolic subgroup containing B, we will usually assume that $T \subseteq L$, so that the Levi factor is unique. In this case we have direct sum decompositions of the Lie algebras as follows:

$$\mathfrak{v} = \bigoplus_{\alpha \in R^+ \setminus R_P^+} \mathfrak{g}_\alpha \,, \; \mathfrak{l} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R_P} \mathfrak{g}_\alpha \,. \tag{2}$$

Since V is normal in P, $\mathfrak{v}$ is an ideal in $\mathfrak{p}$. It is also clear that $\mathfrak{l}$ is a subalgebra of $\mathfrak{g}$. Moreover we see from $P = LV$ that $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{v}$.

1.3. **Cominuscule homogeneous spaces.** The class of cominuscule homogeneous spaces is important for us, since a quantum to classical principle is known for those spaces ([11]). In fact, many more general situations we will study will simplify drastically in the cominuscule case. Therefore it makes sense to introduce this class of spaces in this subsection, so that we can use it as a motivating example throughout the text. Furthermore we introduce some basic concepts, such as the symmetric space associated to a cominuscule homogeneous space and the rank of a cominuscule homogeneous space.

Let G be a simple, reductive, linear algebraic group and P a maximal parabolic subgroup containing B.

Lemma 1.21. *The following conditions are equivalent.*

1. *The unipotent radical V of P is abelian.*
2. *The simple root α_P occurs in the highest root θ_1 of R with coefficient one, i.e. $\langle \theta_1, \omega^\vee \rangle = 1$.*
3. *For all roots α we have that $\langle \alpha, \omega^\vee \rangle \in \{-1, 0, 1\}$.*
4. *$\omega^\vee$ is a nonzero minimal dominant weight, i.e. there is no weight $\lambda < \omega^\vee$ also dominant.*

Proof. The equivalence of (1) and (2) is proved in [27, Lemma 2.2]. The equivalence of (2) and (3) is obvious. The equivalence of (3) and (4) is proved in [14, 13., Exc. 13]. □

Remark 1.22. Since we assume that the ground field is of characteristic zero the equivalence of (1) and (2) is always valid. If the characteristic is positive the only exception where this equivalence fails to be true occurs in type G_2 when the characteristic is two. For more details see [27].

Definition 1.23. *We say that $X = G/P$ is a cominuscule homogeneous space if one of the equivalent conditions of Lemma 1.21 holds.*

The classification of cominuscule homogeneous spaces is easy since condition (2) of Lemma 1.21 restricts the possibility of the choice of α_P in a way we can directly read off the expression of the highest root θ_1 of R as a linear combination of simple roots.

Proposition 1.24. *Let G be a simply connected, simple, reductive, linear algebraic group. Let $X = G/P$ be a cominuscule homogeneous space. Then X is up to isomorphism either*

- *a Grassmannian $\mathbb{G}(k, n)$ where $n \geq 2$,*
- *a symplectic Grassmannian $\mathbb{G}_\omega(p, 2p)$ where $p \geq 2$,*
- *an isotropic Grassmannian $\mathbb{G}_Q(p, 2p)$ where $p \geq 3$,*
- *a quadric $\mathbb{Q}_m$ of dimension $m \geq 3$,*
- *the Cayley plane $\mathbb{OP}^2 = E_6/P_6$ or*
- *the Freudenthal variety E_7/P_7.*

On the other hand every of the varieties in the list is a cominuscule homogeneous space.

Proof. If G is of type A_{n-1}, then α_P may be chosen abitrary. If $\alpha_P = \alpha_k$, the quotient $X = G/P$ is isomorphic to the Grassmannian $\mathbb{G}(k, n)$. If G is of type B_ℓ where $\ell \geq 2$, then α_P is necessarily α_1. Therefore the quotient $X = G/P$ is isomorphic to a quadric $\mathbb{Q}_m$ of dimension $m = 2\ell - 1$. If G is of type C_p where $p \geq 2$, then α_P is necessarily α_p. Therefore the quotient $X = G/P$ is isomorphic to the symplectic Grassmannian $\mathbb{G}_\omega(p, 2p)$. If G is of type D_p where $p \geq 3$ then either $\alpha_P = \alpha_1, \alpha_{p-1}$ or $= \alpha_p$. The cases where $\alpha_P = \alpha_{p-1}$ and $= \alpha_p$ clearly lead to isomorphic spaces X where the isomorphism is induced by the automorphism which permutes α_{p-1} and α_p and fixes the other α_i's. If $\alpha_P = \alpha_1$, then the quotient $X = G/P$ is isomorphic to a quadric $\mathbb{Q}_m$ of dimension $m = 2(p-1)$. If $\alpha_P = \alpha_p$, then the quotient $X = G/P$ is isomorphic to the isotropic Grassmannian $\mathbb{G}_Q(p, 2p)$. If G is of type E_6, then α_P is necessarily α_1 or α_6. Both cases lead to isomorphic spaces X where the isomorphism is induced by the Weyl involution. If $\alpha_P = \alpha_6$ then the quotient $X = G/P$ equals the Cayley plane $\mathbb{OP}^2 = E_6/P_6$. If G is of type E_7, then α_P is necessarily α_7. The quotient $X = G/P$ equals the Freudenthal variety E_7/P_7. If G is of type E_8, of type F_4 or of type G_2, there is no choice of α_P so that condition (2) is satisfied.

On the other hand it is clear from what we said up to now that all varieties in the list occur as a cominuscule homogeneous space. □

Corollary 1.25. *Let $X = G/P$ be a cominuscule homogeneous space. Then α_P is a long root.*

Proof. If R is simply laced, the assertion is trivial. Assume that R is not simply laced. By the classification of cominuscule homogeneous spaces, G is either of type B_ℓ where $\ell \geq 2$ or of type C_p where $p \geq 2$. If G is of type B_ℓ where $\ell \geq 2$, then $\alpha_P = \alpha_1$ and $(\alpha_1, \alpha_1) > (\alpha_\ell, \alpha_\ell)$. Therefore α_P is long. If G is of type C_p where $p \geq 2$, then $\alpha_P = \alpha_p$ and $(\alpha_p, \alpha_p) > (\alpha_1, \alpha_1)$. Therefore α_P is long in all cases. □

We explain now how to associate to every cominuscule homogeneous space a symmetric space.

Let G be a connected, reductive, linear algebraic group and H a closed subgroup.

Definition 1.26 ([30, Definition 26.1])**.** *A symmetric space is a homogeneous space G/H equipped with a non-identical involution $\sigma \in \mathrm{Aut}(G)$ such that $(G^\sigma)^\circ \subseteq H \subseteq G^\sigma$.*

Let P be a parabolic subgroup of G containing B. Let L be a Levi factor of P and V the unipotent radical of P. Let P^- the parabloic subgroup of G opposite to P. The parabolic subgroup P^- contains the Borel subgroup B^- opposite to B. Let V^- be the unipotent radical of P^-. We denote the Lie algebras of $P^-, V^-, \ldots$ always with lowercase Gothic letters $\mathfrak{p}^-, \mathfrak{v}^-, \ldots$

Lemma 1.27. *Let $\mathfrak{q} = \mathfrak{v} \oplus \mathfrak{v}^-$. Then we have a direct sum composition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{q}$. If G/P is a cominuscule homogeneous space, then the three Cartan relations*

$$[\mathfrak{l}, \mathfrak{l}] \subseteq \mathfrak{l}\,,\ [\mathfrak{l}, \mathfrak{q}] \subseteq \mathfrak{q}\,,\ [\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{l}$$

are satisfied.

Proof. The direct sum decomposition of $\mathfrak{g}$ follows directly from (2) and since we have the direct sum decomposition

$$\mathfrak{v}^- = \bigoplus_{\alpha \in R^+ \setminus R_P^+} \mathfrak{g}_{-\alpha}\,.$$

The first two Cartan relations actually hold for every reductive group G and any parabolic subgroup P containing B (not only for cominuscule homogeneous spaces G/P). Since $\mathfrak{l}$ is a subalgebra of $\mathfrak{g}$ it is clear that $[\mathfrak{l}, \mathfrak{l}] \subseteq \mathfrak{l}$. Since $L = P \cap P^-$ we have that $\mathfrak{l} = \mathfrak{p} \cap \mathfrak{p}^-$. Since $\mathfrak{v}$ is an ideal in $\mathfrak{p}$ and $\mathfrak{v}^-$ is an ideal in $\mathfrak{p}^-$, it follows in particular that $[\mathfrak{l}, \mathfrak{v}] \subseteq \mathfrak{v}$ and $[\mathfrak{l}, \mathfrak{v}^-] \subseteq \mathfrak{v}^-$. Therefore we conclude that $[\mathfrak{l}, \mathfrak{q}] \subseteq \mathfrak{q}$.

We now prove the third Cartan relation. Let $\alpha, \beta \in R^+ \setminus R_P^+$ be two roots. By assumption we have that $\langle \alpha, \omega^\vee \rangle = \langle \alpha, \omega^\vee \rangle = 1$. Therefore $\alpha + \beta$ cannot be a root since $\langle \alpha + \beta, \omega^\vee, \rangle = 2$ which violates the cominuscule assumption. Thus $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ and by the same token $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_{-\beta}] = 0$. To see that $[\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{l}$ it therefore suffices to prove that if $\alpha - \beta, \beta - \alpha$ are roots then $\alpha - \beta, \beta - \alpha \in R_P$ (thanks to the direct sum decomposition (2) of $\mathfrak{l}$). But this is clear since $\langle \alpha - \beta, \omega^\vee \rangle = \langle \beta - \alpha, \omega^\vee \rangle = 0$. □

Corollary 1.28. *Let G/P be a cominuscule homogeneous space. Then G/L is a symmetric space. If G is semisimple and simply connected, then $G^\sigma = L$ is connected.*

Proof. Let σ_* be the involution of the vector space $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{q}$ with $(+1)$-eigenspace $\mathfrak{l}$ and (-1)-eigenspace $\mathfrak{q}$. The three Cartan relations guarantee that σ_* respects the Lie bracket and is thus a Lie algebra automorphism of $\mathfrak{g}$. Let $\sigma \in \mathrm{Aut}(G)$ be the unique non-identical involution of G such that $d_e\sigma = \sigma_*$. Since $\mathfrak{g}^{\sigma_*} = \mathfrak{l}$ it is clear that $(G^\sigma)^\circ \subseteq L \subseteq G^\sigma$. If G is semisimple and simply connected then G^σ is connected ([30, Remark 26.1]), whence it follows that $G^\sigma = L$ is connected. □

Definition 1.29. *Let $X = G/P$ be a cominuscule homogeneous space. Then we call G/L the symmetric space associated to X and σ the involution associated to X*

We now define the rank of a cominuscule homogeneous space, using the geometry of its associated symmetric space.

Definition 1.30. *Let G/H be a symmetric space with involution σ. We say a torus T_1 is σ-split if σ acts on T_1 as the inversion.*

Definition 1.31. *Let $X = G/P$ be a cominuscule homogeneous space and σ the involution associated to X. We define the rank of X to be the number*

$$\mathrm{rk}(X) = \sup_{T_1} \dim(T_1P/P)$$

where the supremum runs over all σ-split maximal tori T_1.

2. Cohomology of homogeneous spaces

In this section we describe the Schubert varieties in a homogeneous space and their relation to integral cohomology: the cohomology classes of the Schubert varieties form an integral bases of the cohomology ring. We briefly introduce the basic concepts, the Bruhat order and the Bialynicki-Birula decomposition, related to the cellular decompositon of the homogeneous space given by the Schubert cells. These concepts will be needed later on. Again we use this occasion to set up the notation we are using from now on.

As general references for these topics we use [2] and [17].

Let $X = G/P$ a homogeneous space where G is a reductive linear algebraic group and P is a parabolic subgroup.

Definition 2.1. *Let w be a Weyl group element. The Schubert cell Ω_w associated to w is defined to be the subvariety BwP/P of X. The Schubert variety X_w associated to w is defined to be the closure of the Schubert cell Ω_w associated to w. The opposite Schubert variety Y_w associated to w is defined to be the closure of the opposite Schubert cell B^-wP/P associated to w. The Schubert cycle σ_w associated to w is defined to be the cohomology class $[Y_w]$ of Y_w. The Schubert cycle $\sigma(w)$ associated to w is defined to be the cohomology class $[X_w]$ of X_w.*

Let $\ell_P(w)$ denote the P-length of w for some Weyl group element w.

Lemma 2.2. *Let w be a Weyl group element. The Schubert cell Ω_w is locally closed, irreducible and isomorphic, as a variety, to an affine space. The Schubert variety X_w is irreducible. Both Ω_w and X_w have equal dimension given by the P-length of w:*

$$\dim \Omega_w = \dim X_w = \ell_P(w)\,.$$

Proof. The Schubert cell Ω_w is a B-orbit by definition. Therefore it is locally closed. The Schubert cell Ω_w is the image of B under the orbit map which sends $b \in B$ to bwP/P. Since B is connected and hence irreducible the image Ω_w of B is also irreducible. That Ω_w is isomorphic to an affine space follows from [2, 14.12, Theorem (b)]. Since Ω_w is irreducible, it is clear that X_w is also irreducible. That the dimension of Ω_w and X_w is given by $\ell_P(w)$ follows from [2, 14.12]. □

It is clear that the Schubert cell Ω_w and the Schubert variety X_w only depend on the class of w modulo W_P. Therefore the set of all Schubert cells / Schubert varieties is parametrized by W/W_P and W^P.

Corollary 2.3. *The set of all Schubert cells $\{\Omega_w\}_{w\in W/W_P}$ forms an affine stratification*[1] *of X.*

Proof. We already saw in Lemma 2.2 that Ω_w is isomorphic to an affine space. From the Bruhat decomposition of G ([15, 28.3, Theorem]) we see that X decomposes into a disjoint union of all Schubert cells:

$$X = \coprod_{w\in W/W_P} \Omega_w$$

By the structure of the B-orbits in X, it is clear that the closure of any strata Ω_w is the union of strata of dimension less or equal than $\ell_P(w)$. Therefore $\{\Omega_w\}_{w\in W/W_P}$ forms an affine stratification of X. □

Corollary 2.4. *Let G be complex. The Schubert cycles σ_w where w runs through W/W_P form an integral basis of the cohomology ring $H^*(X,\mathbb{Z})$. In particular we have that*

$$\mathrm{Pic}(X) \cong \sum_{\alpha\in\Delta\setminus\Delta_P} \mathbb{Z}\sigma_{s_\alpha} \cong H^2(X,\mathbb{Z})\,.$$

[1] By affine stratification we mean in this context that all strata are isomorphic to an affine space.

The first isomorphism is given by sending a line bundle $\mathcal{L}(\chi)$ where $\chi \in X(P)$ to its first Chern class $c_1(\mathcal{L}(\chi))$.

Proof. Since $\{\Omega_w\}_{w \in W/W_P}$ is an affine stratification of X, it is also a cellular decomposition of X. It is a general fact from topology, that every cellular decomposition gives an integral basis of the cohomology ring, if we pass to the cohomology classes.

The cohomology $H^2(X, \mathbb{Z})$ is generated as a free $\mathbb{Z}$-module by the Schubert cycles σ_w parametrized by Weyl group elements w of P-length one wich are precisely the simple reflections s_α parametrized by $\alpha \in \Delta \setminus \Delta_P$. This proves the second isomorphism. The first isomophism follows from Example 1.16 and the following fact. □

Fact 2.5. *Let $\alpha \in \Delta \setminus \Delta_P$. Then $c_1(\mathcal{L}(\omega_\alpha)) = \sigma_{s_\alpha}$.*

Proof. [13, Lemma 3.3] □

Next we define a partial order $\preceq$ on W (or W^P), the so called Bruhat order.

Definition 2.6. *Let v and w be two Weyl group elements. Then $v \preceq w$ if and only if $\Omega_v \subseteq X_w$.*

It is clear that $\preceq$ is a partial order. Let w be a Weyl group element. Then we can write the Schubert variety X_w as a disjoint union:

$$X_w = \Omega_w \amalg \coprod_{v \prec w} \Omega_v \,.$$

By the strucutre of the B-orbits in X, it is clear that $\ell_P(v) < \ell_P(w)$ for all $v \prec w$.

Let w_o be the unique longest element of the Weyl group W and let w_P be the unique longest element of the Weyl group W_P. Let w_X be the minimal length representative of w_o modulo W_P.

Lemma 2.7. *The element w_X is the unique maximal element with respect to the Bruhat order on W^P. We have that $w_o w_P = w_X$. The Schubert cell Ω_{w_X} associated to w_X is an open dense subset of X.*

Proof. It follows from [15, 28.5, Proposition] that Ω_{w_X} is dense in X. Therefore $X_{w_X} = X$ and we have the decomposition of X as a disjoint union:

$$X = \Omega_{w_X} \amalg \coprod_{v \prec w_X} \Omega_v \,.$$

Thus it is clear that w_X is the unique maximal element with respect to the Bruhat order on W^P. In order to see that $w_o w_P = w_X$ it suffices to show that $w_o w_P$ is a minimal length representative modulo W_P since we already know that $w_o W_P = w_X W_P$. Since $\ell(w_X) = \ell_P(w_X) = \dim X$ it suffices to show that $\ell(w_o w_P) = \dim X$. But this is clear since

$$\ell(w_o w_P) = \ell(w_o) - \ell(w_P) = \dim G - \dim P = \dim X \,.$$

□

We will need an explicit description of Poincaré duality on the level of Weyl group elements in W^P. We denote the Poincaré dual of a cohomology class σ always by σ^*.

Fact 2.8. *Let w be a Weyl group element in W^P. The Poincaré dual σ_w^* of the Schubert cycle σ_w associated to w is a Schubert cycle associated to a Weyl group element in W^P which we denote by w^*. The involution from W^P to W^P which sends w to w^* is explicitly described by $w \mapsto w_o w w_o w_X = w_o w w_P$.*

Proof. [11, page 51] □

Let $\mathbb{G}_m \subseteq T$ correspond to an interior point of a Weyl chamber.

Lemma 2.9. *We have that*

$$X^{\mathbb{G}_m} = X^T \cong W/W_P$$

where $\cong$ means bijection. The bijection between the T-fixed points of X and W/W_P (or W^P) is given by sending a Weyl group element w to the T-fixed point $x(w) := wP/P$. In particular W acts transitively on the finite set X^T.

Proof. [17, page 3, claim (iii), Lemma 1, Lemma 2] □

We will need a slightly different description of the Schubert cells in terms of the Bialynicki-Birula decomposition of X.

Lemma 2.10. *Let w be a Weyl group element and $p = x(w)$ the associated T-fixed point. The Schubert cell Ω_w associated to w is isomorphic to the set A_p of all points $x \in X$ such that*

$$\lim_{t \to 0} tx = p\,.$$

Proof. In fact both Ω_w and A_p are isomorphic to affine space $\mathbb{A} = \mathbb{A}^{\ell_P(w)}$ of dimension $\ell_P(w)$. Under these isomorphisms $0 \in \mathbb{A}$ is sent to p. For more details see [17, page 5]. □

3. Stable maps and quantum cohomology in homogeneous spaces

We briefly want to recall the basic notions from the theory of quantum cohomology in homogeneous spaces which we will use throughout this work. For a systematic treatement, more details and further references we refer to [12].

The Kontsevich-Manin moduli space can be introduced for arbitrary smooth projective varieties and for arbitrary genus. A particular well-behaved theory which is immediately related to enumerative geometry can be developed for smooth convex spaces and genus zero. For our purposes the case of homogeneous spaces is completely sufficient.

Therefore we restrict our short introduction to the case of homogeneous spaces and genus zero and refer to [12] and the seminal papers of Kontsevich and Manin [18, 19, 20] for the more general theory.

Let $X = G/P$ be a homogeneous space where G is a complex connected semisimple algebraic group and P is a parabolic subgroup of G. Let $d \in H_2(X, \mathbb{Z})$ be a 1-cycle and N a non negative integer.

Definition 3.1. *We say that $(C, p_1, \ldots, p_N, \mu)$ is an N-pointed map to X of genus zero and degree d (or just is an N-pointed map to X if it is clear from the context which d is meant) if C a projective, connected, (at worst) nodal curve of arithmetic genus zero, the markings $p_1, \ldots, p_N$ are distinct nonsingular points of C, and μ is a morphism from C to X such that $\mu_*([C]) = d$. Two N-pointed maps $(C, p_1, \ldots, p_N, \mu)$ and $(C', p'_1, \ldots, p'_N, \mu')$ are said to be isomorphic if there exists an isomorphism $\tau\colon C \to C'$ taking p_i to p'_i for all $1 \leq i \leq N$ such that $\mu' \circ \tau = \mu$.*

Definition 3.2. *We say that $(C, p_1, \ldots, p_N, \mu)$ is an N-pointed stable map to X if $(C, p_1, \ldots, p_N, \mu)$ is an N-pointed map to X which has finite automorphism group or equivalent if Kontsevich's stability condition holds for every irreducible component E of C:*

- *If $E \cong \mathbb{P}^1$ and E is mapped to a point by μ then E must contain at least three special points (either marked points or singular points of C).*

Let $\mathcal{M} = \mathcal{M}_{0,N}(X, d)$[2] be the Kontsevich-Manin moduli space parametrizing all N-pointed stable maps to X of degree d and genus zero. The space $\mathcal{M}$ solves a specific coarse moduli problem which we will not spell out in detail. For us it suffices to know that the $\mathbb{C}$-valued points of $\mathcal{M}$ are in bijection with the isomorphism classes of N-pointed stable maps to X and that $\mathcal{M}$ is in a certain sense universal with this property. We list the main properties of $\mathcal{M}$ in the following theorem.

Theorem 3.3. *The space $\mathcal{M}$ is a normal projective irreducible variety with at worst quotient singularities. The dimension of $\mathcal{M}$ is given by the following formula:*

$$\dim(X) + \int_d c_1(X) + N - 3\,.$$

Here we denote by $c_1(X)$ the first Chern class $c_1(T_X)$ of the tangent bundle T_X on X. This class is sometimes also called the index of X. The previous formula is often called the expected dimension of the moduli space, in other words $\mathcal{M}$ is of expected dimension.

Proof. Every homogeneous space of the form $X = G/P$ is known to be convex. For a proof of this fact and the definition of a convex space

[2]This space is usually denoted by $\overline{M}_{0,N}(X, d)$. Our notation is somewhat nonstandard. But since we have no need to speak about other spaces which are usually denoted by $M_{0,N}(X, d)$, $\overline{\mathcal{M}}_{0,N}(X, d)$ or $\mathcal{M}_{0,N}(X, d)$ there will be no confusion.

we refer to [12, page 6]. Therefore all the properties of $\mathcal{M}$ except the irreducibility follow from [12, Theorem 2]. The irreducibilty is established in [17, Corollary 1]. □

Let $[C, p_1, \ldots, p_N, \mu]$ denote the isomorphism class of an N-pointed stable map $(C, p_1, \ldots, p_N, \mu)$. The space $\mathcal{M}$ is equipped with N evaluation morphisms $\mathrm{ev}_1, \ldots, \mathrm{ev}_N$ from $\mathcal{M}$ to X, where ev_i takes the point $[C, p_1, \ldots, p_N, \mu] \in \mathcal{M}$ to the point $\mu(p_i)$ in X. Given N arbitrary classes $\gamma_1, \ldots, \gamma_N$ in $H^*(X, \mathbb{Z})$, we can define a number $\langle \gamma_1, \ldots, \gamma_N \rangle_d$[3], called a Gromov-Witten invariant, by the following expression:

$$\langle \gamma_1, \ldots, \gamma_N \rangle_d = \int_{\mathcal{M}} \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_N^*(\gamma_N)\,.$$

This definition makes perfect sense in view of Theorem 3.3. It follows directly from the definition that $\langle \gamma_1, \ldots, \gamma_N \rangle_d$ is invariant under permutations of the classes $\gamma_1, \ldots, \gamma_N$. Moreover it is clear that if the classes $\gamma_1, \ldots, \gamma_N$ are homogeneous the Gromov-Witten invariant $\langle \gamma_1, \ldots, \gamma_N \rangle_d$ will be nonzero only if

$$\sum_{i=1}^{N} \mathrm{codim}(\gamma_i) = \dim(\mathcal{M})\,.$$

We have the following remark which is easy to prove (cf. [12, (I), page 35]): The Gromow Witten invariant $\langle \gamma_1, \ldots, \gamma_N \rangle_0$ is nonzero only if $N = 3$. In this case we have

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle_0 = \gamma_1 \cup \gamma_2 \cup \gamma_3\,. \tag{3}$$

Let $\Gamma_1, \ldots, \Gamma_N$ be pure dimensional subvarieties of X such that Γ_i represents the class γ_i for all $1 \leq i \leq N$. Then the classes γ_i are in particular homogeneous. Assume that

$$\sum_{i=1}^{N} \mathrm{codim}(\Gamma_i) = \sum_{i=1}^{N} \mathrm{codim}(\gamma_i) = \dim(\mathcal{M})\,.$$

Then we have the following lemma which relates the Gromov-Witten invariants to enumerative geometry.

Lemma 3.4. *Let $g_1, \ldots, _N$ be general elements of G. Then the scheme theoretic intersection*

$$\mathrm{ev}_1^{-1}(g_1\Gamma_1) \cap \cdots \cap \mathrm{ev}_N^{-1}(g_N\Gamma_N) \tag{4}$$

[3]Here again our notation is somewhat non-standard. This invariant is usually denoted by $I_d(\gamma_1 \cdots \gamma_N)$ whereas $\langle \gamma_1, \ldots, \gamma_N \rangle_d$ has a different meaning. Since we have no need to speak about both invariants at the same time there will be no confusion. For our investigations the case $N = 3$ is most important. Later on we will only consider three points (genus zero) Gromow-Witten invariants. In this case both invariants coincide: $I_d(\gamma_1, \gamma_2, \gamma_3) = \langle \gamma_1, \gamma_2, \gamma_3 \rangle_d$. In particular this case suffices to define the quantum product $\star$ on $QH^*(X, \mathbb{Z})$.

is a finite number of reduced points and we have

$$\langle \gamma_1, \ldots, \gamma_N \rangle_d = \mathrm{ev}_1^{-1}(g_1\Gamma_1) \cap \cdots \cap \mathrm{ev}_N^{-1}(g_N\Gamma_N)\,.$$

In particular the Gromov-Witten invariant $\langle \gamma_1, \ldots, \gamma_N \rangle_d$ *counts the number of* N*-pointed maps* μ *from* $\mathbb{P}^1$ *to* X *of degree* d *satisfying* $\mu(p_i) \in g_i\Gamma_i$ *for all* $1 \leq i \leq N$*.*

Proof. The first statement follows directly from [12, Lemma 14]. Let U be the open dense subvariety of $\mathcal{M}$ parametrizing smooth, rational, irreducible curves. By Kleiman-Bertini we know that the intersection (4) is supported in U. Hence the last statement follows. □

The Gromov-Witten invariants can be used to define the (small) quantum cohomology ring $QH^*(X,\mathbb{Z})$. This definition apparently depends on the choice of a basis of $H^*(X,\mathbb{Z})$. In case of a homogeneous space $X = G/P$ we have a canonical choice of a basis of $H^*(X,\mathbb{Z})$ given by the Schubert cycles σ_w parametrized by all $w \in W/W_P$. This choice of a basis makes the formulas particularly nice.

We write $\mathbb{Z}[q] = \mathbb{Z}[q_\alpha \mid \alpha \in \Delta \setminus \Delta_P]$ for the polynomial ring over $\mathbb{Z}$ with independent variables q_α indexed by $\alpha \in \Delta \setminus \Delta_P$. The index set $\Delta \setminus \Delta_P$ corresponds to our canonical choice of a basis of $H^2(X,\mathbb{Z})$. The variables q_α where $\alpha \in \Delta \setminus \Delta_P$ are graded by the following degree: $\deg(q_\alpha) = 2\int_{\sigma(s_\alpha)} c_1(X)$. For an effective degree d we write q^d for the monomial

$$q^d = \prod_{\alpha\in\Delta\setminus\Delta_P} q_\alpha^{\int_d \sigma_{s_\alpha}}$$

in other words if we write $d = \sum_{\alpha\in\Delta\setminus\Delta_P} d_\alpha \sigma(s_\alpha)$ where $d_\alpha \geq 0$ for all α then we have $q^d = \prod_{\alpha\in\Delta\setminus\Delta_P} q_\alpha^{d_\alpha}$. We now define a $\mathbb{Z}[q]$-module as the tensor product $H^*(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$. The same Schubert cycles σ_w which form a $\mathbb{Z}$-basis of $H^*(X,\mathbb{Z})$ form a $\mathbb{Z}[q]$-basis of $QH^*(X,\mathbb{Z})$. We equip the module $QH^*(X,\mathbb{Z})$ with the quantum product $\star$ which is defined by the rule

$$\sigma_u \star \sigma_v = \sum_d q^d \sum_w \langle \sigma_u, \sigma_v, \sigma_w^* \rangle_d\, \sigma_w$$

and extended $\mathbb{Z}[q]$-linearly to the whole module. Here the first sum runs over all effective degrees d and the second over all $w \in W/W_P$. In fact the summands in the sum over w will be nonzero only if

$$\ell_P(w) = \ell_P(u) + \ell_P(v) - \int_d c_1(X)\,.$$

Theorem 3.5. *The quantum product* $\star$ *makes* $QH^*(X,\mathbb{Z})$ *into a graded commutative associative* $\mathbb{Z}[q]$*-algebra with unit.*

Proof. The commutativity of the product is directly obvious from the definition. The associativity is nontrivial. A proof can be found in [12, 10.]. The unit $1 = \sigma_1 = [X] \in H^0(X,\mathbb{Z})$ of the ordinary cohomolgy ring is also a unit in the quantum cohomology ring. Indeed, this can be seen directly from property [12, (II), page 35]. Using this property we have

$$\sigma_u \star 1 = \sum_w \sigma_w \int_X \sigma_u \cup \sigma_w^* = \sigma_u \,.$$

It is easy to see that we have just defined the degree of the variables q_α in such a way that the resulting product preserves the grading. Therefore $QH^*(X,\mathbb{Z})$ is a graded $\mathbb{Z}[q]$-algebra as claimed. □

Remark 3.6.

- Note that the natural inclusion $H^*(X,\mathbb{Z}) \subseteq QH^*(X,\mathbb{Z})$ of abelian groups does not preserve the product structure.
- With the right adjustement of the definition of the quantum product one can define the quantum cohomology ring for an arbitrary basis of $H^*(X,\mathbb{Z})$. It turns out that the resulting algebras are up to isomorphism independent of the choice of a basis (cf. [12, 10.]).
- When $d \neq 0$ then $\mathcal{M}$ is empty unless d is the class of a curve, in particular effective. Therefore we can take the sum over d in the definition of the quantum product $\star$ over all d since the terms where d is not effective vanish anyway.
- The quantum cohomology is a deformation of $H^*(X,\mathbb{Z})$ in the usual sense: $H^*(X,\mathbb{Z})$ is recovered by setting the variables $q_\alpha = 0$. This is a easy consequence of the formula (3). The classical situation arises from something deeper in the limit $q \to 0$.

4. Chain cascades of orthogonal roots

In this section we introduce basic notions about chain cascades of orthogonal roots. We are closely following the reference [21]. We sometimes use these notions to „localize“ terminology in the sequel. In particular one specific chain cascade will give rise to the so called θ-sequence which is the major tool of this treatement.

Let $X = G/P$ be a homogeneous space where G is a simple and simply connected affine algebraic group and P is a maximal parabolic subgroup.

Notation 4.1. Let φ be a positive root. We always use the following notation. We write $\Delta(\varphi) = \mathrm{supp}(\varphi)$ for the set of simple roots which occur in the expression of φ as a linear combination of simple roots with positive integral coefficients. We denote by $R(\varphi)$ the root subsystem of R which has as set of simple roots the set $\Delta(\varphi)$. In other words $R(\varphi)$ is the root subsystem of R generated by $\Delta(\varphi)$. Since $\Delta(\varphi)$ is connected the root system $R(\varphi)$ is always irreducible. We denote by $G(\varphi)$ the simple subgroup of G which has $R(\varphi)$ as root system. We write $P(\varphi) = G(\varphi) \cap P$ and $B(\varphi) = G(\varphi) \cap B$. Then it is clear that $P(\varphi)$ is a maximal parabolic subgroup of $G(\varphi)$ whenever $\alpha_P \in$

$\Delta(\varphi)$. Moreover $B(\varphi)$ is always a Borel subgroup of $G(\varphi)$. We write $X(\varphi) = G(\varphi)/P(\varphi)$. We have a natural inclusion $X(\varphi) \subseteq X$. Note that $X(\varphi) = \{x(1)\}$ whenever $\alpha_P \notin \Delta(\varphi)$.

We denote the Weyl group of $G(\varphi)$ by $W_{G(\varphi)}$. The group $W_{G(\varphi)}$ is clearly a subgroup of W. We denote the longest element of the Weyl group $W_{G(\varphi)}$ by $w_o(\varphi)$. We denote the Lie algebra of $G(\varphi)$ by $\mathfrak{g}(\varphi)$. We denote the Lie algebra of $P(\varphi)$ by $\mathfrak{p}(\varphi)$.

The following remark is of no importance for the rest of this work. The impatient reader may skip it.

Remark 4.2. Let R be the category with objects R^+, where we draw an arrow $\beta \to \alpha$ from $\beta \in R^+$ to $\alpha \in R^+$ if and only if $\operatorname{supp}(\beta) \supseteq \operatorname{supp}(\alpha)$. We can endow R with a pretopology by defining a covering family of $\alpha \in R^+$ to be a collection of morphisms $\{\beta_i \to \alpha\}_i$ which satisfies $\operatorname{supp}(\alpha) = \bigcap_i \operatorname{supp}(\beta_i)$.

To see this we only have to remark that for two morphism $\beta_1 \to \alpha$ and $\beta_2 \to \alpha$ the fiber product $\beta_1 \times_\alpha \beta_2$ always exists. Indeed, β_1 and β_2 are clearly non-separated roots since their supports are not disjoint: $\operatorname{supp}(\beta_1) \cap \operatorname{supp}(\beta_2) \supseteq \operatorname{supp}(\alpha) \neq \emptyset$. In this case $\beta_1 \vee \beta_2$ is a positive root ([9, Lemma 4.4]) and it is easy to see that $\beta_1 \vee \beta_2 = \beta_1 \times_\alpha \beta_2$. Using the fact that $\operatorname{supp}(\beta_1 \vee \beta_2) = \operatorname{supp}(\beta_1) \cup \operatorname{supp}(\beta_2)$ all axioms are immediate.

We can define two presheaves on R with values in the category of algebraic groups and in the category of varieties by the assignements $\alpha \mapsto G(\alpha)$ and $\alpha \mapsto X(\alpha)$ where a morphism $\beta \to \alpha$ is sent to the obvious maps $G(\alpha) \to G(\beta)$ and $X(\alpha) \to X(\beta)$ respectively. We denote these presheaves still by G and X respectively. Let $\{\beta_i \to \alpha\}_i$ be a covering family of α, then we clearly have $G(\alpha) = \bigcap_i G(\beta_i)$ and $X(\alpha) = \bigcap_i X(\beta_i)$. Therefore it is clear that the presheaves G and X are actually sheaves on R. Note that G and X are compatible in the sense that for every morphism $f\colon \beta \to \alpha$, every $g \in G(\alpha)$ and every $x \in X(\alpha)$ we have $X(f)(gx) = (G(f)(g))(X(f)(x))$.

If R is an irreducible root system we denote by R° the root subsystem of R consisting of all roots which are orthogonal to the highest root of R.

Definition 4.3. *A chain cascade C is a set $\{\alpha_1, \dots, \alpha_k\}$ of positive roots α_i such that α_1 is the highest root of R and such that for all $2 \leq i \leq k$ the root α_i is the highest root of any irreducible component of $R(\alpha_{i-1})^\circ$. If φ is an arbitrary positive root we uniquely associate to it a chain cascade $C(\varphi) = \{\alpha_1, \dots, \alpha_k\}$ such that $\varphi \in R(\alpha_i)$ for all $1 \leq i \leq k$ and such that φ is orthogonal to α_i for all $1 \leq i \leq k-1$ but not orthogonal to α_k. We call the union $\mathcal{B} = \bigcup_{\varphi \in R^+} C(\varphi)$ the cascade of strongly orthogonal roots.*

Definition 4.4. *A positive root φ is called locally high if it is the highest root of $R(\varphi)$. Two roots β and β' are called strongly orthogonal if β and β' are orthogonal and neither $\beta + \beta'$ nor $\beta - \beta'$ are roots. Two subsets of roots S and S' are called totally disjoint if every element of S is strongly orthogonal to every element of S'.*

Fact 4.5.

- *Any chain cascade is totally ordered with respect to the usual partial ordering on the root system.*
- *Let φ be a positive root. The highest root of any irreducible component of $R(\varphi)^\circ$ is locally high.*
- *All elements of any chain cascade are locally high. All elements of $\mathcal{B}$ are locally high.*
- *Two different elements of any chain cascade are strongly orthogonal.*
- *Let β and β' be two elements of $\mathcal{B}$ such that there exists no chain cascade which both contains β and β'. Then there exists a positive root $\varphi \in \mathcal{B}$ such that β and β' belong to different irreducible components of $R(\varphi)^\circ$. In particular $R(\beta)$ and $R(\beta')$ are totally disjoint.*
- *Two different elements of $\mathcal{B}$ are always strongly orthogonal.*
- *The set $\mathcal{B}$ is a maximal set of strongly orthogonal roots.*

Proof. Let $C = \{\alpha_1, \ldots, \alpha_k\}$ be a chain cascade where the roots are labeled as in the definition. Then we have $\alpha_1 \geq \cdots \geq \alpha_k$.

The second point is directly clear from the definition. The first sentence of the third point follows from the second point and the definition of a chain cascade. The second sentence in the third point follows since $\mathcal{B}$ is a union of chain cascades.

Let β and β' be two different elements of a chain cascade. From the definition of a chain cascade it is directly clear that β and β' are orthogonal. Since chain cascades are totally ordered we may assume that $\beta' \leq \beta$. Then we have $\beta, \beta' \in R(\beta)$. Since β is locally high we see that $\beta + \beta'$ cannot be a root. Suppose for a contradiction that $\beta - \beta'$ is a root. Since β is locally high we know that β is a long root of $R(\beta)$. Therefore it follows that $(\beta, \beta) \geq (\beta - \beta', \beta - \beta') = (\beta, \beta) + (\beta', \beta') > (\beta, \beta)$ which is absurd. Therefore we see that $\beta - \beta'$ cannot be a root. This means that β and β' are strongly orthogonal.

Let β and β' be two elements of $\mathcal{B}$ such that there exists no chain cascade which both contains β and β'. Let ϕ and ϕ' be positive roots such that $\beta \in C(\phi)$ and such that $\beta' \in C(\phi')$. The intersection $C = C(\phi) \cap C(\phi')$ is nonempty since it contains the highest root of R. Let φ be the smallest element of $C \subseteq \mathcal{B}$. By assumption we know that $\beta, \beta' \notin C$. Therefore it follows that $\beta, \beta' < \varphi$. By definition of φ we see that β and β' belong to different irreducible components of $R(\varphi)^\circ$ (otherwise the highest root of the common irreducible component would be smaller than φ and contained in C). Let $\alpha \in R(\beta)$ and $\alpha' \in R(\beta')$. We want to show that α and α' are strongly orthogonal. It is clear that α and α' are orthogonal. Suppose that $\alpha \pm \alpha'$ would be a root. Then it must be contained in $R(\varphi)^\circ$ and hence in the irreducible components $R(\beta)$ and $R(\beta')$ which is absurd since they are distinct.

Let β and β' be two different elements of $\mathcal{B}$. If there exists a chain cascade which both contains β and β' then the result follows from the fourth point. Otherwise the previous point shows that $R(\beta)$ and $R(\beta')$ are totally disjoint, in particular that β and β' are strongly orthogonal.

The very last point is [21, Theorem 1.8]. □

Fact 4.6.

- *If $\beta \in \mathcal{B}$, then $C(\beta)$ is a chain cascade with smallest element β. Conversely, let C be a chain cascade with smallest element β. Then $C = C(\beta)$.*
- *Let φ and φ' be two positive roots such that $\varphi \geq \varphi'$. Then $C(\varphi) \subseteq C(\varphi')$.*
- *Let φ be a positive root. Then $C(\varphi) = \{\beta \in \mathcal{B} \mid \varphi \leq \beta\}$.*
- *Let C be a chain cascade. Let $\beta \in \mathcal{B}$ and $\beta' \in C$ such that $\beta \geq \beta'$. Then $C(\beta) \subseteq C$. In particular $\beta \in C$.*
- *Let C and C' be two chain cascades such that $C \cup C'$ is also a chain cascade. Then either $C \subseteq C'$ or $C' \subseteq C$.*
- *We have the following identities:*

$$\mathcal{B} = \bigcup_{\varphi \in R^+} C(\varphi) = \bigcup_{\beta \in \mathcal{B}} C(\beta) = \bigcup_{\alpha \in \Delta} C(\alpha)\,.$$

Proof. Let $\beta \in \mathcal{B}$. Let β' be the smallest element of $C(\beta)$. By definition we have $\beta \leq \beta'$. From the definition it is also clear that β and β' are not orthogonal. Since two different elements of $\mathcal{B}$ are always (strongly) orthogonal, it follows that $\beta = \beta'$. Therefore $C(\beta)$ is a chain cascade with smallest element β.

Conversely, let C be a chain cascade with smallest element β. Let $C = \{\alpha_1, \ldots, \alpha_k\}$. Then $\beta = \alpha_k$. From the definition it is clear that $\beta \leq \alpha_i$ for all $1 \leq i \leq k$ and that β is orthogonal to α_i for all $1 \leq i \leq k-1$ but not orthogol to α_k. This implies that $C = C(\beta)$.

Let φ and φ' be two positive roots such that $\varphi \geq \varphi'$. Let $C(\varphi) = \{\alpha_1, \ldots, \alpha_k\}$. We clearly have $\varphi \leq \alpha_i$ for all $1 \leq i \leq k$ and thus $\varphi' \leq \alpha_i$ for all $1 \leq i \leq k$. This implies that $C(\varphi) \subseteq C(\varphi')$.

Let φ be a positive root. Let $\beta \in \mathcal{B}$ such that $\varphi \leq \beta$. Then $\beta \in C(\beta) \subseteq C(\varphi)$. This proves the inclusion from right to left. The other inclusion is an immediate consequence of the definition.

Let C be a chain cascade. Let $\beta \in \mathcal{B}$ and $\beta' \in C$ such that $\beta \geq \beta'$. Let β'' be the smallest element of C. Then $C = C(\beta'')$ and $\beta'' \leq \beta' \leq \beta$. Therefore we have $C(\beta) \subseteq C(\beta') \subseteq C(\beta'') = C$. In particular $\beta \in C$.

Let C and C' be two chain cascades. Let β be the smallest element of C and β' be the smallest element of C'. Since $C \cup C'$ is a chain cascade, we have either $\beta \leq \beta'$ or $\beta' \leq \beta$. This implies that either $C(\beta') = C' \subseteq C = C(\beta)$ or $C(\beta) = C \subseteq C' = C(\beta')$.

The first identity is just the definition of $\mathcal{B}$. The inclusion from right to left in the second identity is obvious. Let $\varphi \in R^+$ and let $\beta \in \mathcal{B}$ be the smallest element of $C(\varphi)$. Then $C(\varphi) = C(\beta)$. This fact proves the inclusion from left to right in the second identity. Next we show that $\bigcup_{\varphi \in R^+} C(\varphi) = \bigcup_{\alpha \in \Delta} C(\alpha)$. The inclusion from right to left is obvious. Let $\varphi \in R^+$. Let α be any simple root in the support of φ. Then $\alpha \leq \varphi$ and thus $C(\varphi) \subseteq C(\alpha)$. This proves the inclusion from left to right. $\square$

Lemma 4.7. *Suppose that G is not of type G_2 and not of type B_ℓ where ℓ is odd and greater than 1. All elements of $\mathcal{B}$ have equal length, i.e. all elements of $\mathcal{B}$ are contained in the same W-orbit. In particular, all elements of $\mathcal{B}$ are long roots; for all $\beta \in \mathcal{B}$ we have $\langle \beta^\vee, \alpha \rangle \in \{-1, 0, 1\}$ for all $\alpha \in R \setminus \{\pm\beta\}$.*

Proof. In order to prove that all elements of $\mathcal{B}$ have equal length, it suffices to show that an arbitrary element $\beta \in \mathcal{B}$ has equal length as θ_1. In other words this means, to show that β is long. By choosing $\alpha \in \operatorname{supp}(\beta)$ we may assume that $\beta \in C(\alpha)$. Let $C(\alpha) = \{\beta_1, \ldots, \beta_k\}$. Then $\beta_1 = \theta_1$ where θ_1 is the highest root in R. Let $\beta = \beta_i$ for some $1 \leq i \leq k$. We can clearly assume that $i \geq 2$ and that $k \geq 2$. Moreover we can assume that $R(\beta)$ is simply laced and that R is not simply laced. (If R is simply laced there is nothing to prove. If $R(\beta)$ is not simply laced, then β is clearly a long root, as it is the highest root of $R(\beta)$.) By analysing the non simply laced Dynkin diagrams it is easy to see that these assumptions imply that $i = k \geq 2$ and that $R(\beta)$ is of type A_1. This means that $\beta = \beta_k = \alpha$. If R is of type B_ℓ where $\ell \geq 2$ the only possibility is $\alpha \in \{\alpha_1, \alpha_3, \alpha_5 \ldots\}$. If R is of type C_p where $p \geq 2$ the only possibility is $\alpha = \alpha_p$. If R is of type F_4 the only possibility is $\alpha = \alpha_2$. If R is of type G_2 the only possibility is $\alpha = \alpha_1$. Since by assumption G is not of type G_2 and not of type B_ℓ where $\ell > 1$ is odd, we see that $\alpha = \beta$ is long in all cases.

The last statement is now obvious, since for all long roots β we have $\langle \beta^\vee, \alpha \rangle \in \{-1, 0, 1\}$ for all $\alpha \in R \setminus \{\pm\beta\}$. $\square$

Lemma 4.8. *Let φ be a positive root. Let $w \in W_{G(\varphi)}$. Then $X(\varphi)_w = X_w$. (Global and local Schubert varieties can be identified.) In particular, $W_{G(\varphi)}^{P(\varphi)} \subseteq W^P$ and $\ell_P = \ell_{P(\varphi)}$ on $W_{G(\varphi)}$. The Bruhat order on $W_{G(\varphi)}$ is compatible with the Bruhat order on W.*

Proof. We have the obvious inclusion $B(\varphi)wP(\varphi)/P(\varphi) \subseteq BwP/P$ which implies that $X(\varphi)_w \subseteq X_w$. On the other hand, it is clear that

$$\dim(X_w) = \ell_P(w) \leq \ell_{P(\varphi)}(w) = \dim(X(\varphi)_w)\,.$$

Since $X(\varphi)_w$ and X_w are both closed and irreducible, this implies that we have equalities $X(\varphi)_w = X_w$ and $\ell_P(w) = \ell_{P(\varphi)}(w)$. This means that a minimal length representative of w modulo $W_{P(\varphi)}$ is also a minimal length representative of w modulo W_P, in other words that $W_{G(\varphi)}^{P(\varphi)} \subseteq W^P$. The Bruhat order on $W_{G(\varphi)}$ is compatible with the Bruhat order on W since $X_v \subseteq X_w$ if and only if $X(\varphi)_v \subseteq X(\varphi)_w$ for all $v, w \in W_{G(\varphi)}$. □

Corollary 4.9. *Let $\beta \in \mathcal{B}$. Then $X(\beta) = X_{w_o(\beta)} = X_{w_o|_{\mathfrak{g}(\beta)}}$.*

Proof. The previous lemma implies that $X(\beta) = X(\beta)_{w_o(\beta)} = X_{w_o(\beta)}$. By [21, Proposition 1.10] we know that $w_o|_{\mathfrak{g}(\beta)} = w_o(\beta)|_{\mathfrak{g}(\beta)}$. Both equations together imply the statement. □

Corollary 4.10. *Let $\beta \to \alpha$ be an arrow from $\beta \in R^+$ to $\alpha \in R^+$. Then $\ell_P(w_o(\alpha)) \leq \ell_P(w_o(\beta))$.*

Proof. Since $\beta \to \alpha$ we have an inclusion $X(\alpha) \subseteq X(\beta)$. The previous corollary then implies that

$$\ell_P(w_o(\alpha)) = \dim(X(\alpha)) \leq \dim(X(\beta)) = \ell_P(w_o(\beta)) .$$

□

5. T-invariant curves

In this section we introduce T-invariant curves and the distance function δ according to [13]. This class of curves is of particular importance, since any curve converges to a T-invariant curve. Since T-invariant curves and their degree can be described purely combinatorially, this reduces many curve theoretic problems to computations with Weyl group elements.

Let $X = G/P$ be a homogeneous space where G is a reductive linear algebraic group and P is a parabolic subgroup.

Lemma 5.1. *Let α be a root in $R^+ \setminus R_P^+$. Then there is a unique irreducible T-invariant curve C_α in X that contains the points $x(1)$ and $x(s_\alpha)$.*

Proof. Let $\mathrm{SL}_2(\alpha)$ be the 3-dimensional simple subgroup of G with Lie algebra $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. Then $C_\alpha = \mathrm{SL}_2(\alpha)P/P$ is the desired unique T-invariant curve that contains the T-fixed points $x(1)$ and $x(s_\alpha)$. For more details, see [13, page 5]. □

Definition 5.2. *Let α be a root. Then we define the degree $d(\alpha)$ of α to be the following element of $H_2(X, \mathbb{Z})$:*

$$d(\alpha) = \sum_{\beta \in \Delta \setminus \Delta_P} \langle \alpha^\vee, \omega_\beta \rangle \, \sigma_{s_\beta^*} .$$

Lemma 5.3. *If w is in W_P, then $d(w(\alpha)) = d(\alpha)$ for all roots α.*

Proof. [13, Lemma 3.1] □

Lemma 5.4. *Let α be a root in $R^+ \setminus R_P^+$. Then the degree $[C_\alpha]$ of C_α is $d(\alpha)$.*

Proof. [13, Lemma 3.4] □

Lemma 5.5. *Let u and v be unequal elements in W/W_P. The following are equivalent:*

(1) *There is a reflection $s \in W$ such that $v = su$.*
(2) *There are representatives $\tilde{u}$ of u and $\tilde{v}$ of v, and a reflection $t \in W$ such that $\tilde{v} = \tilde{u}t$.*

The reflection s of (1) is uniquely determined. The reflection t of (2) is determined up to conjugation by an element of W_P.

Proof. [13, Lemma 4.1] □

Let u and v be two unequal elements in W/W_P. We call u and v adjacent if they are related as in Lemma 5.5. Note that this is a symmetic relation: u and v are adjacent if and only if v and u are adjacent. Moreover u and v are adjacent if and only if wu and wv are adjacent for any $w \in W$. In particular u and v are adjacent if and only if u^* and v^* are adjacent.

Lemma 5.6. *Let u and v be adjacent elements. Then u and v are comparable with respect to the Bruhat order, i.e. either $u \prec v$ or $v \prec u$.*

Proof. [10, Theorem F, (4)] □

A sequence $u_0, \ldots, u_r$ in W/W_P is called a chain if u_i and u_{i-1} are adjacent for all $1 \leq i \leq r$. Let $\beta_i \in R^+ \setminus R_P^+$ be a sequence of roots such that $u_{i-1}s_{\beta_i} = u_i$ for all $1 \leq i \leq r$. Then we define the degree of the chain $u_0, \ldots, u_r$ to be the sum $\sum_{i=1}^r d(\beta_i)$. This is well defined in the sense that it does not depend on the choice of the roots β_i since the roots β_i are determined up to conjugation by an element of W_P by u_{i-1} and u_i and since d is W_P-invariant. It is clear from the definition that each T-invariant curve is associated to a unique chain. In other words, T-invariant curves and chains are in one to one correspondence where the degree of a chain and the degree of the associated T-invariant curve are equal.

Let u and v be arbitrary elements of W/W_P. We call a chain $u_0, \ldots, u_r$ a chain from u to v if $u_0, \ldots, u_r$ is a chain which satisfies $u \preceq u_0$ and $u_r \preceq v^*$. We define $\delta(u, v)$ to be the degree of a chain $u_0, \ldots, u_r$ from u to v such that the degree of $u_0, \ldots, u_r$ is minimal in the set of all degrees of chains from u to v. We extend our notation and write $\delta(u) = \delta(u, w_X) = \delta(u, w_o)$ for all $u \in W/W_P$.

Remark 5.7. We actually think of δ as a function with functoriality $W \times W \to H_2(X, \mathbb{Z})$. But the reader should be aware that this function is not well defined in a strict sense, since there might be several minima

$\delta(u,v)$ in the set of all degrees of chains from u to v. These minima must not be comparable, except the parabolic subgroup P is maximal, in which case the function δ is well defined with unique values in $\mathbb{Z}$ which only depend on the arguments u and v. In the general case where P is not necessarily maximal, we still make statements about δ in the following sense: if z_d is for example an element in W and d a degree, we write $\delta(z_d) \leq d$. The statement $\delta(z_d) \leq d$ is supposed to mean, that there exists a chain from z_d to w_X of minimal degree $\delta(z_d)$ less or equal than d. In each case where we use the function δ in this way, it will be clear from the context which minima $\delta(z_d)$ is meant.

We list some immediate properties of the function δ in the following

Lemma 5.8.

- *The value of $\delta(u,v)$ depends only of the class of u and v modulo W_P.*
- *The function δ is commutative: $\delta(u,v) = \delta(v,u)$ for all u and v.*
- *$\delta(u,v) = 0$ if and only if $u \preceq v^*$.*
- *$\delta(u,v) = 0$ if either $u = 1$ or $v = 1$.*
- *$\delta(u,u^*) = \delta(u^*,u) = 0$ for all u.*
- *Let α be a root in $R^+ \setminus R_P^+$. Then $\delta(s_\alpha) \leq d(\alpha)$.*
- *$\delta(u,v) \leq \delta(u',v')$ for all $u \preceq u'$ and all $v' \succeq v$. In particular, $\delta(u) \leq \delta(u')$ for all $u \preceq u'$.*
- *Assume that P is maximal. Then we have*
$$\max_{u,v\in W/W_P} \delta(u,v) = \max_{u\in W/W_P} \delta(u) = \delta(w_X)\,.$$
- *A chain $u_0,\dots,u_r$ from u to w_X satisfies $u \preceq u_0$ and $u_r = 1$.*
- *Let $u_0,\dots,u_r$ be a chain from u to v of degree $\delta(u,v)$. Let u' and v' be such that $u \preceq u' \preceq u_0$ and such that $u_r^* \succeq v' \succeq v$. Then $\delta(u,v) = \delta(u',v')$.*
- *If P is maximal, then $\delta(u) \leq \ell_P(u)$ for all u.*

Proof. The definition of $\delta(u,v)$ depends only on the T-fixed points defined by u and v. Therefore it is clear that $\delta(u,v)$ does only depend on the class of u and v modulo W_P.

Let $u_0,u_1,\dots,u_r$ be a chain from u to v. Then $u_r^*,\dots,u_1^*,u_0^*$ determines a chain from v to u. Therefore $\delta(u,v) = \delta(v,u)$.

Suppose first that $u \preceq v^*$. Then every chain $u_0,\dots,u_r$ such that $u_0 = u_r$ and such that $u \preceq u_0 \preceq v^*$ defines a chain from u to v. In particular we can choose $r = 0$ to obtain a chain from u to v of degree zero. This shows that $\delta(u,v) = 0$. Conversely, suppose that $\delta(u,v) = 0$. Then there exists a chain from u to v of degree zero. Such a chain must consist of precisely one element. This means there exists an element u_0 such that $u \preceq u_0 \preceq v^*$. In particular we have $u \preceq v^*$.

If either $u = 1$ or $v = 1$ we obviously have $u \preceq v^*$. Therefore it follows from the previous point that $\delta(u,v) = 0$.

It suffices to show that $\delta(u,v)=0$ where $v=u^*$. The other equality follows since δ is commutative. But we obviously have $u \preceq v^* = u$. Therefore the conclusion follows from the third point.

Let $\alpha \in R^+ \setminus R_P^+$. A chain from s_α to w_X is given by $u_0 = s_\alpha, u_1 = 1$. This chain clearly has degree $d(\alpha)$. Therefore we conclude that $\delta(s_\alpha) \leq d(\alpha)$.

Let $u \preceq u'$ and $v' \succeq v$. Then every chain from u' to v' determines a chain from u to v. Therefore it is clear that $\delta(u,v) \leq \delta(u',v')$.

This point follows directly from the previous point, since w_X is the unique maximal element in the Bruhat order[4].

A chain $u_0,\ldots,u_r$ from u to w_X clearly satisfies $u \preceq u_0$ and $u_r \preceq w_X^*$. But $w_X^* \in W_P$ and thus $u_r \preceq 1$ which implies $u_r = 1$.

Let the notation be as in the statement. We know that $\delta(u,v) \leq \delta(u',v') \leq \delta(u_0,u_r^*)$. On the other hand $u_0,\ldots,u_r$ is a chain from u_0 to u_r^* of degree $\delta(u,v)$. This implies that $\delta(u_0,u_r^*) \leq \delta(u,v)$. Therefore we get the equality $\delta(u,v) = \delta(u',v')$.

Suppose that P is maximal. Let $u \in W$ be an arbitrary element. Let $\tilde{u}$ be a minimal length representative of u. Let $\tilde{u} = s_{\alpha_r}\cdots s_{\alpha_1}$ be a reduced expression of $\tilde{u}$. This means that α_i is simple for all i and that $\ell_P(u) = \ell(\tilde{u}) = r$. Let $i_1,\ldots,i_j$ be a sequence of integers such that $i_1 < \cdots < i_j$ and such that $\alpha_i = \alpha_P$ if and only if $i \in \{i_1,\ldots,i_j\}$. Since $\tilde{u}$ is a minimal length representative, we clearly have $i_1 = 1$. Then $\tilde{u} = s_{\alpha_r}\cdots s_{\alpha_{i_1}}, s_{\alpha_r}\cdots s_{\alpha_{i_2}},\ldots,s_{\alpha_r}\cdots s_{\alpha_{i_j}}, 1$ is a chain from u to w_X of degree j. Therefore we have $\delta(u) \leq j \leq r = \ell_P(u)$ as claimed. □

Example 5.9. We give an example for a root $\alpha \in R^+ \setminus R_P^+$ such that $\delta(s_\alpha) < d(\alpha)$. Let R be of type G_2 and let $\alpha_P = \alpha_2$. Let $\alpha = 2\alpha_1 + \alpha_2$ be the highest short root. Let $\beta_1 = \alpha_2$ and $\beta_2 = 3\alpha_1 + \alpha_2$. Then we have that $d(\alpha) = 3$ and $d(\beta_1) = d(\beta_2) = 1$. Moreover it is easy to check that $s_\alpha W_P = s_{\beta_2}s_{\beta_1}W_P$. Therefore we have $\delta(s_\alpha) \leq 2$. Since $d(\alpha) = 3 > 1$ we must have $\delta(s_\alpha) > 1$ and thus $\delta(s_\alpha) = 2 < d(\alpha) = 3$.

Lemma 5.10. *Let f be a curve of degree d which passes through a finite number of Schubert cells $\Omega_{u_0},\ldots,\Omega_{u_r}$. Then f converges to a T-invariant curve f_0 of degree d which passes through the T-fixed points*

$$x(u_0),\ldots,x(u_r)\,.$$

In other words, there is a chain u with members $u_0,\ldots,u_r$ such that f_0 is associated to u.

Proof. Let $\mathbb{G}_m$ correspond to an interior point of a Weyl chamber. Let $f_0 = \lim_{t\to 0} tf$. Then f_0 is clearly a T-invariant curve. Since

[4]Note that we need to assume here that P is maximal, since only then two degrees are always comparable. In general we can only say that

$$\max_{u,v\in W/W_P} \delta(u,v) \supseteq \max_{u\in W/W_P} \delta(u) \supseteq \delta(w_X)\,.$$

f passes through $\Omega_{u_0}, \ldots, \Omega_{u_r}$, Lemma 2.10 implies that f_0 passes through $x(u_0), \ldots, x(u_r)$. In other words f_0 is associated to a chain with members $u_0, \ldots, u_r$. □

Fact 5.11. *Let $(a_1, \ldots, a_l) \in \mathbb{Z}_{>0}^l$ be a sequence of positive integers such that the following inequality is satisfied*

$$\sum_{i=1}^{l} a_i \left(a_i + \sum_{j=0}^{l-i-1} 2^j a_{i+j+1} \right) > \left(\sum_{i=1}^{l} a_i \right)^2 .$$

Then we have that $l \geq 4$ and that $\sum_{i=1}^{l} a_i \geq 5$.

Proof. If $l = 1$ then the inequality becomes $a_1^2 > a_1^2$ which can never be satisfied. If $l = 2$ then the inequality becomes $a_1(a_1 + a_2) + a_2^2 > a_1^2 + 2a_1a_2 + a_2^2$ which is equivalent to $0 > a_1a_2$. The later inequality can never be satisfied. If $l = 3$ then the inequality becomes

$$a_1(a_1+a_2+2a_3)+a_2(a_2+a_3)+a_3^2 > a_1^2+a_2^2+a_3^2+2a_1a_2+2a_1a_3+2a_2a_3$$

which is equivalent to $0 > a_1a_2 + a_2a_3$. The later inequality can never be satisfied. Therefore we conclude that $l \geq 4$. If $l \geq 5$ then it is clear that $\sum_{i=1}^{l} a_i \geq 5$. We are left to check that if $l = 4$ then also $\sum_{i=1}^{l} a_i \geq 5$. Assume we had $l = 4$ and $\sum_{i=1}^{l} a_i < 5$. It then follows that $a_1 = a_2 = a_3 = a_4 = 1$. By plugging in these values the initial inequality becomes $15 > 16$ – a contradiction. Therefore we conclude that in all cases $\sum_{i=1}^{l} a_i \geq 5$. □

Fact 5.12. *Suppose that P is maximal and that R is simply laced. Let $\beta_1, \ldots, \beta_l$ be a sequence of positive roots. Then we have the following inequality:*

$$\omega - s_{\beta_l} \cdots s_{\beta_1}(\omega) \leq \sum_{i=1}^{l} d(\beta_i) \left(\beta_i + \sum_{j=0}^{l-i-1} 2^j \beta_{i+j+1} \right) .$$

Proof. The proof is based on the following inequality $s_\gamma(\delta) \leq \gamma + \delta$ for all positive roots γ and δ. We prove the formula by induction on l. The case where $l = 1$ is plain. Suppose that $l > 1$ and that the formula is true for $l - 1$. Then we have

$$\omega - d(\beta_l)\beta_l - s_{\beta_l} \cdots s_{\beta_1}(\omega) = s_{\beta_l}(\omega - s_{\beta_{l-1}} \cdots s_{\beta_1}(\omega)) .$$

If we apply the induction hypotheses and repeatedly the simple inequality stated in the beginning of the proof, we obtain the following upper bound for the previous expression:

$$\leq \sum_{i=1}^{l-1} d(\beta_i) \left(\beta_i + \sum_{j=0}^{(l-1)-i-1} 2^j \beta_{i+j+1} + \left(1 + \sum_{j=0}^{(l-1)-i-1} 2^j \right) \beta_l \right) .$$

Using the fact that $1 + \sum_{j=0}^{(l-1)-i-1} 2^j = 2^{l-i-1}$ and rearanging the inequality the result follows. □

Lemma 5.13. *Suppose that P is maximal and that R is simply laced. Let $\gamma, \beta_1, \ldots, \beta_l \in R^+ \setminus R_P^+$ be roots such that $s_\gamma W_P = s_{\beta_l} \cdots s_{\beta_1} W_P$ and such that $\delta(s_\gamma) = \sum_{i=1}^l d(\beta_i) < d(\gamma)$. Then it follows that $d(\gamma) = 6$ and that $\delta(s_\gamma) = 5$ and that $l \in \{4, 5\}$.*

Proof. To abbreviate, let $a_i = d(\beta_i)$ for all $1 \leq i \leq l$. Using the previous fact we obtain the following inequality:

$$d(\gamma)\gamma = \omega - s_{\beta_l} \cdots s_{\beta_1}(\omega) \leq \sum_{i=1}^{l} a_i \left(\beta_i + \sum_{j=0}^{l-i-1} 2^j \beta_{i+j+1} \right) .$$

If we apply $\langle -, \omega^\vee \rangle$ to this inequality we obtain:

$$\left(\sum_{i=1}^{l} a_i \right)^2 < d(\gamma)^2 \leq \sum_{i=1}^{l} a_i \left(a_i + \sum_{j=0}^{l-i-1} 2^j a_{i+j+1} \right) .$$

The fact above then implies that $l \geq 4$ and that $\sum_{i=1}^l a_i \geq 5$. Since $d(\gamma) \leq 6$ in any root system, we conclude that $d(\gamma) = 6$ and that $\delta(s_\gamma) = \sum_{i=1}^l a_i = 5$. The later equality immediately implies that $l \leq 5$ and thus $l \in \{4, 5\}$ as claimed. □

Lemma 5.14. *Suppose that P is maximal and that R is simply laced. For all positive roots γ we have the following equality: $\delta(s_\gamma) = d(\gamma)$.*

Proof. If $\gamma \in R_P^+$ then the statement is obvious. Assume that $\gamma \in R^+ \setminus R_P^+$. Assume for a contradiction that $\delta(s_\gamma) < d(\gamma)$. Then there exist roots $\beta_1, \ldots, \beta_l \in R^+ \setminus R_P^+$ such that $s_\gamma W_P = s_{\beta_l} \cdots s_{\beta_1} W_P$ and such that $\delta(s_\gamma) = \sum_{i=1}^l d(\beta_i)$. From the previous lemma it follows that $d(\gamma) = 6$ and that $\delta(s_\gamma) = 5$ and that $l \in \{4, 5\}$. Let $a_i = d(\beta_i)$ for all $1 \leq i \leq l$. Then we have $\sum_{i=1}^l a_i = 5$. This means that either $l = 5$ and $a_1 = \cdots = a_5 = 1$ or that $l = 4$ and $a_p = 2$ for some $p \in \{1, \ldots, 4\}$ and $a_i = 1$ for all $i \in \{1, \ldots, 4\} \setminus \{p\}$. From the proof of the previous lemma we see that the sequence $a_1, \ldots, a_l$ must satisfy the following inequality:

$$d(\gamma)^2 = 36 \leq \sum_{i=1}^{l} a_i \left(a_i + \sum_{j=0}^{l-i-1} 2^j a_{i+j+1} \right) .$$

But each of the five possibilities we have for the sequence $a_1, \ldots, a_l$ violates the previous inequality. Therefore we end up with a contradiction. This shows that we have $\delta(s_\gamma) = d(\gamma)$ for all $\gamma \in R^+ \setminus R_P^+$ and proves the lemma. □

6. Curve neighborhoods

Let $X = G/P$ be a homogeneous space defined by a connected, simply connected, semisimple linear algebraic group G and an arbitrary parabolic subgroup P. In this section we investigate curve neighborhoods according to the references [8] and [9]. We start by recalling basic notions and basic theorems of this theory.

Lemma 6.1. *There exists a unique binary operation $\cdot$ such that $(W, \cdot)$ is a monoid which satisfies the following properties:*

- $us_\alpha \preceq u \cdot s_\alpha$ *for all* $u \in W$ *and all* $\alpha \in \Delta$.
- $u = u \cdot 1 \preceq u \cdot s_\alpha$ *for all* $u \in W$ *and all* $\alpha \in \Delta$.
- *The monoid* $(W, \cdot)$ *is minimal with respect to this properties, i.e. if* $(W, *)$ *is a second monoid which satisfies the two previous properties and in addition* $u * s_\alpha \preceq u \cdot s_\alpha$ *for all* $u \in W$ *and all* $\alpha \in \Delta$ *then we have* $(W, *) = (W, \cdot)$.

The binary operation $\cdot$ is called the Demazure product.

Proof. Let $\cdot$ be the Demazure product as defined in [9, 3.]. By [9, Proposition 3.1] the Demazure product satisfies the first two properties stated above. Let $(W, *)$ be a monoid which satisfies the first two properties listed above. We first show that $u \cdot s_\alpha \preceq u * s_\alpha$ for all $u \in W$ and all $\alpha \in \Delta$. Indeed, if $us_\alpha \succeq u$ then $u \cdot s_\alpha = us_\alpha \preceq u * s_\alpha$ by the first property. If $us_\alpha \preceq u$ then $u \cdot s_\alpha = u = u * 1 \preceq u * s_\alpha$ by the second property.

If either $(W, *)$ is a monoid which in addition satisfies the third property or $(W, *)$ is a monoid which satisfies $u * s_\alpha \preceq u \cdot s_\alpha$ for all $u \in W$ and all $\alpha \in \Delta$ we obtain $u * s_\alpha = u \cdot s_\alpha$. In order to show that the Demazure product satisfies the third property and in order to establish the uniquenes part of the lemma, it suffices to show that a monoid $(W, *)$ which satisfies $u * s_\alpha = u \cdot s_\alpha$ for all $u \in W$ and all $\alpha \in \Delta$ is equal to $(W, \cdot)$.

Let $(W, *)$ be such a monoid. To prove $u * v = u \cdot v$ we proceed by induction on the length of v. The case where $\ell(v) = 0$ is trivial and the case where $\ell(v) = 1$ follows from the assumption. Assume that $\ell(v) > 1$ and that the equation is known for all elements with length less than $\ell(v)$. Let $v = s_{\alpha_1} \cdots s_{\alpha_r}$ be a reduced expression of v where $r = \ell(v) > 1$. By the induction hypothesis and the definition of the Demazure product it follows that $u \cdot v = u \cdot vs_{\alpha_r} \cdot s_{\alpha_r} = (u * vs_{\alpha_r}) \cdot s_{\alpha_r} = u * vs_{\alpha_r} * s_{\alpha_r} = u * (vs_{\alpha_r} \cdot s_{\alpha_r}) = u * v$ as required. □

For basic properties of the Demazure product we refer to [8, 4.] and [9, 3.]. In particular we will tacitly use that the Demazure product induces a product $W \times W/W_P \to W/W_P$ given by $u \cdot (wW_P) = (u \cdot w)W_P$.

Definition 6.2. *Given any subvariety $\Omega \subseteq X$ and any effective degree d, we define the degree d curve neighborhood of Ω to be $\Gamma_d(\Omega) = \mathrm{ev}_1(\mathrm{ev}_2^{-1}(\Omega))$. If Ω is a Schubert variety, then $\Gamma_d(\Omega)$ is a Schubert variety as well (*[9, Theorem 5.1]*). For each effective degree d, we denote* *the unique element of W^P parametrizing the Schubert variety $\Gamma_d(X_1)$ by z_d.*

From this definition it is immediately obvious that for two degrees $0 \leq d \leq d'$ we have $z_d \preceq z_{d'}$ since we have $\Gamma_d(X_1) \subseteq \Gamma_{d'}(X_1)$.

The following theorem is of major importance. We quote it once and then often use it without explicitely referring to it.

Theorem 6.3. *For any $w \in W$ and any effective degree d we have $\Gamma_d(X_w) = X_{w \cdot z_d}$.*

Proof. [8, Theorem 1] or [9, Theorem 5.1] □

Corollary 6.4. *Let d and d' be two effective degrees. Then we have $z_d \cdot z_{d'} \preceq z_{d+d'}$.*

Proof. [8, Corollary 5.1] or [9, Corollary 4.12(b)] □

Definition 6.5. *Given a degree d a root which is a maximal element of the set $\{\alpha \in R^+ \setminus R_P^+ \mid d(\alpha) \leq d\}$ is called a maximal root of d. Let $d \geq 0$ be a degree then we define a greedy decomposition of d to be a sequence of roots $(\alpha_1, \ldots, \alpha_r)$ such that α_1 is a maximal root of d and $(\alpha_2, \ldots, \alpha_r)$ is a greedy decomposition of $d - d(\alpha_1)$. The empty sequence is the only greedy decomposition of 0.*

Fact 6.6. *Any greedy decomposition is unique up to reordering. Let $(\alpha_1, \ldots, \alpha_r)$ be a greedy decomposition of an effective degree d. Then we have the following equality:*

$$z_d w_P = s_{\alpha_1} \cdot \ldots \cdot s_{\alpha_r} \cdot w_P \,.$$

Any element α_i of a greedy decomposition is a maximal root of $d(\alpha_i)$.

Proof. [9, 4.2] □

Lemma 6.7. *For all $u \in W$ we have $x(u) \in \Gamma_{\delta(u)}(X_1)$ or equivalent $u \preceq z_{\delta(u)}$. In particular we can find a chain $u_0, \ldots, u_r$ from u to w_X of degree $\delta(u)$ which satisfies $u_0 = u$ and $u_r = 1$.*

Proof. By [13, Theorem 9.1] $\delta(u)$ gives the minimal power of q in $\sigma_u \star [\{\mathrm{pt}\}]$. This means in particular that there exists a rational curve $\varphi\colon \mathbb{P}^1 \to X$ of degree $\delta(u)$ which meets Y_u and $x(1)$. This is equivalent to saying that $\mathrm{ev}_1^{-1}(x(1)) \cap \mathrm{ev}_2^{-1}(Y_u)$ is nonempty in $\mathcal{M}_{0,2}(X, \delta(u))$ which is again equivalent to $\Gamma_{\delta(u)}(X_1) \cap Y_u \neq \emptyset$. Therefore we conclude that $x(u) \in \Gamma_{\delta(u)}(X_1)$ or equivalent $u \preceq z_{\delta(u)}$. The very last statement is now obvious. □

Corollary 6.8. *For all $\alpha \in R^+ \setminus R_P^+$ we have $s_\alpha \preceq z_{d(\alpha)}$. For all $d \geq \delta(w_X)$ we have $z_d = w_X$.*

Proof. Indeed, let $\alpha \in R^+ \setminus R_P^+$ then we have $\delta(s_\alpha) \leq d(\alpha)$. Therefore the previous lemma yields $s_\alpha \preceq z_{\delta(s_\alpha)} \preceq z_{d(\alpha)}$. Let $d \geq \delta(w_X)$. Then we have by the previous lemma that $w_X \preceq z_{\delta(w_X)} \preceq z_d$ and thus $z_d = w_X$ as claimed. □

Corollary 6.9. *For all $u \in W$ we have $\delta(u) = \delta(u^{-1})$.*

Proof. We can find a chain $u_0, \dots, u_r$ from u to w_X of degree $\delta(u)$ which satisfies $u_0 = u$ and $u_r = 1$. Then $u_0^{-1}, \dots, u_r^{-1}$ is a chain from u^{-1} to w_X of the same degree $\delta(u)$ which satisfies $u_0 = u^{-1}$ and $u_r = 1$. Thus $\delta(u^{-1}) \leq \delta(u)$. By replacing u with u^{-1} we also conclude that $\delta(u) = \delta((u^{-1})^{-1}) \leq \delta(u^{-1})$. □

Corollary 6.10. *The function δ is W_P-invariant: for all $u \in W$ and all $w \in W_P$ we have $\delta(wu) = \delta(u)$.*

Proof. By the previous corollary, we have $\delta(wu) = \delta(u^{-1}w^{-1})$. Since $\delta(u^{-1})$ depends only on the class of u^{-1} modulo W_P we have $\delta(u^{-1}w^{-1}) = \delta(u^{-1})$. Using the previous corollary again, we find $\delta(u^{-1}) = \delta(u)$ which gives in total $\delta(wu) = \delta(u)$. □

Corollary 6.11. *Let $u_0, \dots, u_r$ be a chain from u to w_X of degree $\delta(u)$. For all $1 \leq i \leq r$ let α_i be the unique root in $R^+ \setminus R_P^+$ such that $s_{\alpha_i} u_{i-1} = u_r$. Then*

$$\delta(u_i) = \sum_{j=i+1}^{r} d(\alpha_j)\,.$$

In particular we have

$$\delta(u) = \delta(u_0) > \delta(u_1) > \cdots > \delta(u_{r-1}) > \delta(u_r) = 0$$

and $u \not\preceq u_1$ and

$$u_0 \succ u_1 \succ \cdots \succ u_{r-1} \succ u_r\,.$$

Proof. The chain $u_0, \dots, u_r$ clearly satisfies $u_0 \succeq u$ and $u_r = 1$. Therefore we have $\delta(u_0) \geq \delta(u)$ and $\delta(u_r) = 0$. On the other hand $u_0, \dots, u_r$ is a chain from u_0 to w_X of degree $\delta(u)$, thus $\delta(u_0) \leq \delta(u)$ and we get $\delta(u) = \delta(u_0)$. Since $u_i, \dots, u_r$ is a chain from u_i to w_X of degree $\sum_{j=i+1}^r d(\alpha_j)$ we get $\sum_{j=i+1}^r d(\alpha_j) \geq \delta(u_i)$. On the other hand there exists a chain $u'_i, \dots, u'_s$ from u_i to w_X of degree $\delta(u_i)$ which satisfies $u_i = u'_i$ and $u'_s = 1$. If $\sum_{j=i+1}^r d(\alpha_j) > \delta(u_i)$ then $u_0, \dots, u_i = u'_i, \dots, u'_s$ would be a chain from u to w_X of degree

$$\sum_{j=1}^{i} d(\alpha_j) + \delta(u_i) < \sum_{j=1}^{r} d(\alpha_j) = \delta(u)$$

which contradicts the definition of $\delta(u)$. Therefore we conclude that $\delta(u_i) = \sum_{j=i+1}^r d(\alpha_j)$ for all $0 \leq i \leq r$. In particular it follows that $\delta(u_i) - \delta(u_{i+1}) = d(\alpha_i) > 0$ and thus $\delta(u_i) > \delta(u_{i+1})$ for all $0 \leq i \leq r-1$. If $u_i \preceq u_{i+1}$ for some $0 \leq i \leq r-1$ then $\delta(u_i) \leq \delta(u_{i+1})$ which contradicts the previous results. Since two element u_i and u_{i+1} which are adjacent are always comparable in the Bruhat order, we get $u_i \succ u_{i+1}$ for all $0 \leq i \leq r-1$. Similar, if $u \preceq u_1$ then $\delta(u) \leq \delta(u_1)$ which again contradicts the previous results. Therefore it follows that $u \not\preceq u_1$. □

Lemma 6.12. *For all degrees d we have $\delta(z_d) \leq d$. For all $u \in W$ we have that $\delta(z_{\delta(u)}) = \delta(u)$.*

Proof. Let d be a degree. Then clearly $z_d \in \Gamma_d(X_1) = X_{z_d}$. Thus we can find a chain $u_0, \ldots, u_r$ from z_d to w_X of degree d which even satisfies $u_0 = z_d$ and $u_r = 1$. It follows that $\delta(z_d) \leq d$.

Since $u \preceq z_{\delta(u)}$ it is clear that $\delta(u) \leq \delta(z_{\delta(u)})$. The other inequality follows from the previous claim applied to $d = \delta(u)$. □

Lemma 6.13. *Let $u \in W$. Then $\delta(u)$ is the smallest degree d such that $u \preceq z_d$.*

Proof. Let d be the smallest degree such that $u \preceq z_d$. Since $u \preceq z_{\delta(u)}$ it follows that $d \leq \delta(u)$. On the other hand, we can find a chain $u_0, \ldots, u_r$ of degree d which satisfies $u_0 = z_d$ and $u_r = 1$ since $z_d \in \Gamma_d(X_1)$. Since $u \preceq z_d$ this chain defines a chain from u to w_X of degree d. Thus $\delta(u) \leq d$. In total $\delta(u) = d$. □

Lemma 6.14. *Let $u, v \in W$. Then $\delta(uv) \leq \delta(u \cdot v) \leq \delta(u) + \delta(v)$.*

Proof. Note that $uv \preceq u \cdot v$. Therefore $\delta(uv) \leq \delta(u \cdot v)$. From $u \preceq z_{\delta(u)}$ and $v \preceq z_{\delta(v)}$ we conclude that

$$uv \preceq u \cdot v \preceq z_{\delta(u)} \cdot z_{\delta(v)} \preceq z_{\delta(u)+\delta(v)} \,.$$

Since $\delta(u \cdot v)$ is the smallest degree d such that $u \cdot v \preceq z_d$ we conclude that $\delta(u \cdot v) \leq \delta(u) + \delta(v)$. □

Corollary 6.15. *Let $u_0, \ldots, u_r$ be a chain from u to w_X of degree $\delta(u)$. Let β_i be a root in $R^+ \setminus R^+_P$ such that $u_{i-1} s_{\beta_i} = u_i$ for all $1 \leq i \leq r$. Then $\delta(s_{\beta_i}) = d(\beta_i)$ for all i.*

Proof. We clearly have $\delta(u) = \delta(u_0)$ and $u_r \in W_P$. By definition we have $u_0 W_P = s_{\beta_1} \cdots s_{\beta_r} W_P$. Since $\delta(s_{\beta_i}) \leq d(\beta_i)$ is always satisfied, we get from the triangle inequality of δ that

$$\delta(u) = \delta(u_0) = \delta(s_{\beta_1} \cdots s_{\beta_r}) \leq \sum_{i=1}^{r} \delta(s_{\beta_i}) \leq \sum_{i=1}^{r} d(\beta_i) = \delta(u)$$

where the last equation follows from the definition of the chain $u_0, \ldots, u_r$. Therefore we get equality everywhere which means that $\delta(s_{\beta_i}) = d(\beta_i)$ for all i. □

6.1. P-cosmall roots.

Definition 6.16. *A root $\alpha \in R^+ \setminus R^+_P$ is P-cosmall if*

$$\ell_P(s_\alpha) = \int_{d(\alpha)} c_1(T_X) - 1$$

.

Definition 6.17. *A Weyl group element u is called P-cosmall if*

$$\ell(z_{\delta(u)}) = \int_{\delta(u)} c_1(T_X) - 1 \,.$$

Theorem 6.18. *Let $\alpha \in R^+ \setminus R_P^+$ be a root. Then the following are equivalent.*

- *The root α is P-cosmall.*
- *The root α is a maximal root of $d(\alpha)$.*
- *We have $(R^+ \setminus R_P^+) \cap s_\alpha(R_P^+) = \emptyset$ and $\langle \alpha^\vee, \gamma \rangle = 1$ for all $\gamma \in I(s_\alpha) \setminus (R_P^+ \cup \{\alpha\})$ where $I(s_\alpha)$ denotes the inversion set of s_α, i.e. $I(s_\alpha)$ consists of all positive roots β such that $s_\alpha(\beta) < 0$.*

Proof. [9, Theorem 6.1] □

Example 6.19. We always denote by θ_1 the highest root of R. The highest root θ_1 is P-cosmall, since it is a maximal root of $d(\theta_1)$ ([9, Theorem 6.1]). It is the unique maximal root in the set $\{\alpha \in R^+ \setminus R_P^+ \mid d(\alpha) \leq d(\theta_1)\}$.

Remark 6.20. Suppose that P is maximal, and that $\alpha \in R^+ \setminus R_P^+$ is a P-cosmall root. Then we have necessarily that $\mathrm{supp}(\alpha) = \mathrm{supp}(\theta_1) = \Delta$.

Indeed, since $\alpha \in R^+ \setminus R_P^+$ we have $d(\alpha) > 0$ and also $\langle \alpha, \omega^\vee \rangle > 0$. Therefore we must have $\alpha_P \in \mathrm{supp}(\alpha)$. Suppose that $\mathrm{supp}(\alpha) \neq \Delta$, then we can find a simple root $\beta \in \Delta_P$ such that $\beta \notin \mathrm{supp}(\alpha)$. Then $s_\beta(\alpha)$ is a root which is strictly larger than α and satisfies $d(s_\beta(\alpha)) = d(\alpha)$ since $s_\beta \in W_P$. This contradicts the fact that α is a maximal root of $d(\alpha)$. Therefore we conclude that $\mathrm{supp}(\alpha) = \mathrm{supp}(\theta_1) = \Delta$ as claimed.

Example 6.21. Let R be of type A_{n-1} and let P be maximal. Then the only P-cosmall root is the highest root θ_1. This follows directly from the previous remark.

Example 6.22. Every P-cosmall root is also B-cosmall. A complete list of all B-cosmall roots in type A_{n-1}, B_ℓ, C_p, D_p, F_4 and G_2 is given in [9, Example 4.1, 4.2 and 4.3].

Lemma 6.23. *Let α and β be B-cosmall roots. (This is in particular the case if they are P-cosmall.) Suppose that $\alpha \leq \beta$. Then $d(\alpha) \leq d(\beta)$.*

Proof. If α and β are B-cosmall roots, then $\alpha \leq \beta$ is equivalent to $\alpha^\vee \leq \beta^\vee$ ([9, Lemma 4.7]). In particular it follows that $d(\alpha) \leq d(\beta)$ if $\alpha \leq \beta$. □

Lemma 6.24. *Let α and β be roots such that β is P-cosmall and such that $d(\alpha) \leq d(\beta)$. Then $s_\alpha \preceq s_\beta$.*

Proof. Indeed, we have $s_\alpha \preceq z_{\delta(s_\alpha)} \preceq z_{d(\alpha)} \preceq z_{d(\beta)}$. Since β is P-cosmall, we have $s_\beta W_P = z_{d(\beta)} W_P$ and thus $s_\alpha \preceq s_\beta$. □

Lemma 6.25. *Let $\alpha \in R^+ \setminus R_P^+$ be a P-cosmall root. Then*

$$\langle [\{\mathrm{pt}\}], \sigma_{s_\alpha} \rangle_{d(\alpha)} = 1\,.$$

Conversely, let $\alpha \in R^+ \setminus R_P^+$ be a root wich satisfies $\langle [\{\mathrm{pt}\}], \sigma_{s_\alpha} \rangle_{d(\alpha)} \neq 0$. Then α is P-cosmall.

Proof. Let $\alpha \in R^+ \setminus R_P^+$ be a P-cosmall root. Let $u = w_X$ and $w = s_\alpha^*$. Then

$$\ell_P(ws_\alpha) = \dim(X) = \ell_P(w) + \ell_P(s_\alpha) = \ell_P(w) + c_1(X)d(\alpha) - 1$$

and $uW_P = ws_\alpha W_P$. Therefore it follows from [9, Corollary 7.3] that $\langle [Y_u], [X_w] \rangle_{d(\alpha)} = \langle [\{\mathrm{pt}\}], \sigma_{s_\alpha} \rangle_{d(\alpha)} = 1$.

Conversely, let $\alpha \in R^+ \setminus R_P^+$ be a root which satisfies $\langle [\{\mathrm{pt}\}], \sigma_{s_\alpha} \rangle_{d(\alpha)} \neq 0$, then there exists a P-cosmall root $\alpha' \in R^+ \setminus R_P^+$ such that $uW_P = ws_{\alpha'} W_P$ ([9, Corollary 7.3]). It immediately follows that $\alpha = \alpha'$. Therefore α is P-cosmall. □

Lemma 6.26. *Suppose that P is maximal. Let d be a degree and let α be a P-cosmall root. If $s_\alpha \preceq z_d$ then $d(\alpha) \leq d$.*

Proof. Suppose that $s_\alpha \preceq z_d$. Then $X_{s_\alpha} \subseteq X_{z_d}$. By comparing the dimensions of these varieties, we that $\ell_P(s_\alpha) = c_1(X)d(\alpha) - 1 \leq \ell(z_d)$. By [8, Proposition 5.3] we see that $\ell(z_d) \leq c_1(X)d - 1$. We conclude that $d(\alpha) \leq d$.

□

Corollary 6.27. *Suppose that P is maximal. Let α be a P-cosmall root. Then $\delta(s_\alpha) = d(\alpha)$.*

Proof. We clearly have $s_\alpha \preceq z_{\delta(s_\alpha)}$. The previous lemma then implies $d(\alpha) \leq \delta(s_\alpha)$. The other inequality $\delta(s_\alpha) \leq d(\alpha)$ is always satisfied. Therefore we get equality $\delta(s_\alpha) = d(\alpha)$ as claimed. □

Corollary 6.28. *Suppose that P is maximal. Let $\alpha \in R^+ \setminus R_P^+$ be a P-cosmall root. Then s_α is P-cosmall.*

Proof. By the above we have that $\delta(s_\alpha) = d(\alpha)$. Since α is P-cosmall we have that $z_{d(\alpha)} W_P = s_\alpha W_P$. Therefore we get, again since α is P-cosmall, that

$$\ell(z_{\delta(s_\alpha)}) = \ell(z_{d(\alpha)}) = \ell_P(s_\alpha) = c_1(X)d(\alpha) - 1 = c_1(X)\delta(s_\alpha) - 1$$

as claimed. □

Lemma 6.29. *Let $\alpha \in R^+ \setminus R_P^+$ be a root such that $z_{d(\alpha)} W_P = s_\alpha W_P$. This is in particular the case if α is P-cosmall. A chain $u_0, \ldots, u_r$ from s_α to w_X of minimal degree $\delta(s_\alpha)$ satisfies $u_0 = s_\alpha$ and $u_r = 1$.*

Proof. Let $u_0, \ldots, u_r$ be a chain from s_α to w_X of degree $\delta(s_\alpha)$. Then we have $u_0 \succeq s_\alpha$ and $u_r = 1$. We show that actually $u_0 = s_\alpha$ in this situation. Indeed, by definition $x(u_0) \in \Gamma_{\delta(s_\alpha)}(X_1) \subseteq \Gamma_{d(\alpha)}(X_1)$. By assumption we know that $\Gamma_{d(\alpha)}(X_1) = X_{s_\alpha}$. We conclude that $u_0 \preceq s_\alpha$. Therefore we can assume that $u_0 = s_\alpha$. □

Lemma 6.30. *Let u be a P-cosmall Weyl group element. Then there exists a unique root $\alpha \in R^+ \setminus R_P^+$ such that $z_{\delta(u)} W_P = s_\alpha W_P$. This root α is P-cosmall and satisfies $\delta(u) = d(\alpha)$ and $u \preceq s_\alpha$.*

Proof. By [8, Proposition 5.3] there exists a unique $\alpha \in R^+ \setminus R_P^+$ such that $z_{\delta(u)} W_P = s_\alpha W_P$. This root α is P-cosmall and satisfies $\delta(u) = d(\alpha)$. We only have to show that $u \preceq s_\alpha$. To this end, let $u_0, \ldots, u_r$ be a chain from u to w_X. This chain satisfies $u_0 \succeq u$ and $u_r = 1$. By definition, we see that $x(u_0) \in \Gamma_{\delta(u)}(X_1) = X_{s_\alpha}$. Therefore we conclude that $u_0 \preceq s_\alpha$ and thus $u \preceq u_0 \preceq s_\alpha$. □

6.2. **The set $\mathcal{U}$.** We now study the subset $\mathcal{U}$ of W/W_P defined as follows:

$$\mathcal{U} = \{u \in W/W_P \mid \delta(u) + \delta(u^*) = \delta(w_X)\}$$

Lemma 6.31. *We have the following description of $\mathcal{U}$ in terms of the geometry of chains:*

$$\mathcal{U} = \bigcup_{\mathsf{u}} \{u_0, \ldots, u_r\}$$

where the union runs over all chains u: $u_0, \ldots, u_r$ from w_X to w_X of degree $\delta(w_X)$.

Proof. Let $u_0, \ldots, u_r$ be a chain from w_X to w_X of degree $\delta(w_X)$. Then we necessarily have $u_0 = w_o$ and $u_r = 1$. Let j be an index between 0 and r. We have to show that $u = u_j \in \mathcal{U}$. Let $\beta_i \in R^+ \setminus R_P^+$ be a root such that $u_{i-1} s_{\beta_i} = u_i$ for all $1 \leq i \leq r$. Then we have $\delta(u) = \sum_{i=j+1}^{r} d(\beta_i)$. Moreover $u^* = w_o u = s_{\beta_1} \cdots s_{\beta_j}$ and thus $\delta(u^*) \leq \sum_{i=1}^{j} d(\beta_i)$. Suppose that $\delta(u^*) < \sum_{i=1}^{j} d(\beta_i)$. There exists a chain $u'_0, \ldots, u'_s$ from u^* to w_X which satisfies $u'_0 = u^*$ and $u'_s = 1$ and has degree $\delta(u^*)$. Let $\beta'_i \in R^+ \setminus R_P^+$ be a root such that $u'_{i-1} s_{\beta'_i} = u'_i$ for all $1 \leq i \leq s$. Then we have $\delta(u^*) = \sum_{i=1}^{s} d(\beta'_i)$ and $u^* s_{\beta'_1} \cdots s_{\beta'_s} = 1$ which gives $u = w_o s_{\beta'_s} \cdots s_{\beta'_1}$. Therefore $w_o, w_o s_{\beta'_s}, \ldots, w_o s_{\beta'_s} \cdots s_{\beta'_1} = u = u_j, \ldots, u_r$ is a chain from w_X to w_X of degree

$$\sum_{i=1}^{s} d(\beta'_i) + \sum_{i=j+1}^{r} d(\beta_i) < \sum_{i=1}^{r} d(\beta_i) = \delta(w_X)$$

which contradicts the definition of $\delta(w_X)$. Therefore we conclude that $\delta(u^*) = \sum_{i=1}^{j} d(\beta_i)$ and thus

$$\delta(u) + \delta(u^*) = \sum_{i=1}^{r} d(\beta_i) = \delta(w_X)\,.$$

This proves that $u \in \mathcal{U}$ and thus the inclusion from right to left.

In order to prove the inclusion from left to right, let $u \in \mathcal{U}$. Let $u_0, \ldots, u_r$ be a chain from u to w_X which satisfies $u_0 = u$ and $u_r = 1$ and has degree $\delta(u)$. Let $u'_0, \ldots, u'_s$ be a chain from u^* to w_X which satisfies $u'_0 = u^*$ and $u'_s = 1$ and has degree $\delta(u^*)$. Let $\beta'_i \in R^+ \setminus R_P^+$ be a root such that $u'_{i-1}s_{\beta'_i} = u'_i$ for all $1 \leq i \leq s$. Then we have $u^* s_{\beta'_1} \cdots s_{\beta'_s} = 1$ and thus $w_o s_{\beta'_s} \cdots s_{\beta'_1} = u$. Therefore $w_o, w_o s_{\beta'_s}, \ldots, w_o s_{\beta'_s} \cdots s_{\beta'_1} = u = u_0, \ldots, u_r$ is a chain from w_X to w_X of degree

$$\sum_{i=1}^{s} d(\beta'_i) + \delta(u) = \delta(u^*) + \delta(u) = \delta(w_X)\,.$$

This shows that u is contained in the right set and proves the inclusion from left to right. □

Corollary 6.32. *For all $u \in W/W_P$ we have $\delta(u) + \delta(u^*) \geq \delta(w_X)$. The set $\mathcal{U}$ can also described as the set of all elements $u \in W/W_P$ which satisfy $\delta(u) + \delta(u^*) \leq \delta(w_X)$.*

Proof. The first claim is an immediate consequence of the previous proof. The second claim follows from the first. □

For the sake of completeness, we list further immediate properties of the set $\mathcal{U}$.

Lemma 6.33.

- *$\mathcal{U}$ is closed under under taking Poincaré duals: $u \in \mathcal{U}$ if and only if $u^* \in \mathcal{U}$. $\mathcal{U}$ contains 1 and w_X.*
- *Let d be a degree. Then we have*

  $$\left(X_{z_d} \cap Y_{z^*_{\delta(w_X)-d}}\right)^T = \{x(u) \mid u \in \mathcal{U} \text{ such that } \delta(u) = d\}\,.$$

  and $X_{z_{d'}} \cap Y_{z^*_{\delta(w_X)-d}} = \emptyset$ for all degrees $d' < d$.
- *For all $u \in \mathcal{U}$ we have $z^*_{\delta(u^*)} \preceq z_{\delta(u)}$.*
- *For all $u \in \mathcal{U}$ the power $q^{\delta(u)}$ is the smallest power of q in the quantum product $[\{\mathrm{pt}\}] \star \sigma_{z^*_{\delta(u^*)}}$. Therefore we have $\delta(z^*_{\delta(u^*)}) = \delta(u)$.*
- *If $u \in \mathcal{U}$, then $z_{\delta(u)} \in \mathcal{U}$.*
- *For all $u \in \mathcal{U}$ we have $w_o W_P = z_{\delta(u)} \cdot z_{\delta(u^*)} W_P = z_{\delta(u^*)} \cdot z_{\delta(u)} W_P$.*

Proof. The very first point follows immediatly from the definition of $\mathcal{U}$.

Let d be a degree. Since $Y_{z^*_{\delta(w_X)-d}} = w_o \Gamma_{\delta(w_X)-d}(X_1)$ and $X_{z_d} = \Gamma_d(X_1)$, a T-fixed point $x(u)$ in the intersection of both varieties is part of a curve of degree $\delta(w_X)$ passing through $x(1)$ and $x(w_X)$, in other words u is part of a chain from w_X to w_X of degree $\delta(w_X)$. Therefore we conclude that $u \in \mathcal{U}$. On the other hand, we have $u \preceq z_d$ and $u^* \preceq z_{\delta(w_X)-d}$ and thus $\delta(u) \leq \delta(z_d) \leq d$ and $\delta(u^*) \leq \delta(z_{\delta(w_X)-d}) \leq \delta(w_X) - d$. Since $\delta(u) + \delta(u^*) = \delta(w_X)$ we get equality in both cases, in particular $\delta(u) = d$. This means that $x(u)$ is contained in the right side. This proves the inclusion from left to right.

Let $u \in \mathcal{U}$ such that $\delta(u) = d$. Then $u \preceq z_{\delta(u)} = z_d$ and $u^* \preceq z_{\delta(u^*)} = z_{\delta(w_X)-\delta(u)} = z_{\delta(w_X)-d}$. Thus $x(u)$ is in the intersection on the left side. This proves the inclusion from right to left.

Let d' be a degree strictly smaller than d. If the intersection $X_{z_{d'}} \cap Y_{z^*_{\delta(w_X)-d}}$ is nonempty, then we can find a point x which is part of a curve of degree $d' + \delta(w_X) - d$ passing through $x(1)$ and $x(w_X)$. This curves gives rise to a chain from w_X to w_X of degree $d' + \delta(w_X) - d < \delta(w_X)$, which contradicts the definition of $\delta(w_X)$. Therefore we conclude that the intersection is empty.

If we apply the previous results to $d = \delta(u)$ for some $u \in \mathcal{U}$, then we see that the intersection $X_{z_{\delta(u)}} \cap Y_{z^*_{\delta(w_X)-\delta(u)}}$ is nonempty, since it contains the point $x(u)$. This immediately implies that $z^*_{\delta(u^*)} = z^*_{\delta(w_X)-\delta(u)} \preceq z_{\delta(u)}$.

Let $u \in \mathcal{U}$. By what we saw up to now, $\delta(u)$ is the smallest degree d, such that $X_{z_d} \cap Y_{z^*_{\delta(u^*)}}$ is nonempty. By [13, Theorem 9.1] this precisely means that $q^{\delta(u)}$ is the smallest power of q in the quantum product $[\{\mathrm{pt}\}] \star \sigma_{z^*_{\delta(u^*)}}$. On the other hand, again by [13, Theorem 9.1] this power of q is given by $q^{\delta(z^*_{\delta(u^*)})}$. Therefore we conclude that $\delta(z^*_{\delta(u^*)}) = \delta(u)$.

Let $u \in \mathcal{U}$. Then $\delta(z_{\delta(u)}) = \delta(u)$ (this holds even for arbitrary elements in W) and $\delta(z^*_{\delta(u)}) = \delta(u^*)$ by what we saw in the previous item applied to $u^* \in \mathcal{U}$. It follows that $\delta(z_{\delta(u)}) + \delta(z^*_{\delta(u)}) = \delta(u) + \delta(u^*) = \delta(w_X)$ since $u \in \mathcal{U}$ and thus $z_{\delta(u)} \in \mathcal{U}$.

For all $u \in \mathcal{U}$ we can choose a chain $u_0, \ldots, u_r$ from w_X to w_X which satisfies $u_0 = w_0, u_r = 1$ and $u_j = z_{\delta(u)}$ for some index j between 0 and r. (This is the content of the previous item.) Let α_i be the unique root in $R^+ \setminus R^+_P$ such that $s_{\alpha_i} u_{i-1} = u_i$ for all $1 \le i \le r$. Then we have $\sum_{i=1}^j d(\alpha_i) = \delta(u_j^*)$ since $u_j \in \mathcal{U}$. Let $u = s_{\alpha_1} \cdots s_{\alpha_j}$ so that $w_0 = u u_j$. It then follows that

$$
\begin{aligned}
w_0 = u u_j \preceq s_{\alpha_1} \cdot \ldots \cdot s_{\alpha_j} \cdot u_j \preceq z_{\delta(s_{\alpha_1})} \cdot \ldots \cdot z_{\delta(s_{\alpha_j})} \cdot u_j \preceq \\
z_{d(\alpha_1)} \cdot \ldots \cdot z_{d(\alpha_j)} \cdot u_j \preceq z_{\delta(u_j^*)} \cdot u_j \preceq w_0
\end{aligned}
$$

and thus $w_o W_P = z_{\delta(u_j^*)} \cdot u_j W_P$. But $\delta(u_j^*) = \delta(z^*_{\delta(u)}) = \delta(u^*)$, since $u^* \in \mathcal{U}$. The equation then reads as $w_o W_P = z_{\delta(u^*)} \cdot z_{\delta(u)} W_P$. By replacing u with u^* we also get $w_o W_P = z_{\delta(u)} \cdot z_{\delta(u^*)} W_P$. This proves the last claim. □

6.3. *P*-indecomposable roots.

Definition 6.34. *Let α be a root in $R^+ \setminus R^+_P$. We say that α is P-indecomposable if $\delta(s_\alpha) = d(\alpha)$ and if for every sequence of roots $\beta_1, \ldots, \beta_r \in R^+ \setminus R^+_P$ such that $s_{\beta_1} \cdots s_{\beta_r} W_P = s_\alpha W_P$ and such that $d(\alpha) = \sum_{i=1}^r d(\beta_i)$ it follows that $r = 1$ (and thus $\beta_1 = \alpha$).*

We say that a Weyl group element $u \in W$ is P-indecomposable, if there exists a P-indecomposable root $\alpha \in R^+ \setminus R_P^+$ such that $uW_P = s_\alpha W_P$. (If such an α exists it is of course unique.)

Example 6.35. All simple roots $\alpha \in \Delta \setminus \Delta_P$ are P-indecomposable. In particular, if P is maximal, then α_P is P-indecomposable.

Example 6.36. Let P be maximal. Then all roots $\alpha \in R^+ \setminus R_P^+$ such that $d(\alpha) = 1$ are P-indecomposable.

Example 6.37. Let X be a simply laced cominuscule homogeneous space. Then all roots in $R^+ \setminus R_P^+$ are P-indecomposable.

Lemma 6.38. *Let $\alpha \in R^+ \setminus R_P^+$ be a P-indecomposable root and let $w \in W_P$. Then $w(\alpha)$ is also P-indecomposable.*

Proof. Let α be P-indecomposable, let $w \in W_P$ and let $\beta_1, \dots, \beta_r \in R^+ \setminus R_P^+$ be a sequence of roots such that $s_{\beta_1} \cdots s_{\beta_r} W_P = s_{w(\alpha)} W_P$ and such that $d(w(\alpha)) = \sum_{i=1}^r d(\beta_i)$. It is easy to see that this implies $s_{w^{-1}(\beta_1)} \cdots s_{w^{-1}(\beta_r)} W_P = s_\alpha W_P$. On the other hand $d(\alpha) = d(w(\alpha)) = \sum_{i=1}^r d(\beta_i) = \sum_{i=1}^r d(w^{-1}(\beta_i))$ since d is W_P-invariant. Since α is P-indecomposable, this implies that $r = 1$. Therefore $w(\alpha)$ is also P-indecomposable. □

Lemma 6.39. *Let $\alpha \in R^+ \setminus R_P^+$ be a P-cosmall root which satisfies $\delta(s_\alpha) = d(\alpha)$. (If P is maximal every P-cosmall root α satisfies $\delta(s_\alpha) = d(\alpha)$.) Then α is P-indecomposable.*

Proof. Let $\beta_1, \dots, \beta_r \in R^+ \setminus R_P^+$ be a sequence of roots such that $s_{\beta_1} \cdots s_{\beta_r} W_P = s_\alpha W_P$ and such that $d(\alpha) = \sum_{i=1}^r d(\beta_i)$. By the triangle inequality for ℓ_P we have $\ell_P(s_\alpha) \leq \sum_{i=1}^r \ell_P(s_{\beta_i})$. Since α is P-cosmall, this inequality gives

$$\int_{d(\alpha)} c_1(X) - 1 \leq \sum_{i=1}^r \left(\int_{d(\beta_i)} c_1(X) - 1 \right) = \int_{d(\alpha)} c_1(X) - r\,.$$

But this inequality can only be satisfied if $r = 1$. Therefore α is P-indecomposable. □

Example 6.40 ([9, Example 6.6]). We give an example for a root $\alpha \in R^+ \setminus R_P^+$ which is neither P-cosmall nor P-indecomposable but satisfies $\delta(s_\alpha) = d(\alpha)$. Let R be of type B_2 and let $\alpha_P = \alpha_1$. Let $\alpha = \alpha_1 + \alpha_2$, let $\theta_1 = \alpha_1 + 2\alpha_2$ and let $\theta_2 = \alpha_1$. Then we have $s_\alpha W_P = s_{\theta_1} s_{\theta_2} W_P$. We will see later that $\delta(s_{\theta_1} s_{\theta_2}) = d(\theta_1) + d(\theta_2) = 1 + 1 = 2$. Therefore we have $\delta(s_\alpha) = d(\alpha) = d(\theta_1) + d(\theta_2) = 2$. This shows that α cannot be P-indecomposable. The previous lemma then shows that α cannot be P-cosmall.

Lemma 6.41. *An element $u \in W/W_P$ is P-indecomposable if and only if for every sequence of elements $u_1, \dots, u_r \in W$ such that $uW_P = u_1 \cdots u_r W_P$ and such that $\delta(u) = \sum_{i=1}^r \delta(u_i)$ it follows that $r = 1$.*

Proof. Suppose first that u is P-indecomposable. Let $u_1, \dots, u_r \in W$ be a sequence of elements such that $uW_P = u_1 \cdots u_r W_P$ and such that $\delta(u) = \sum_{i=1}^{r} \delta(u_i)$. For each i we can find a sequence of roots $\beta_{i1}, \dots, \beta_{ij_i} \in R^+ \setminus R_P^+$ such that $u_i W_P = s_{\beta_{i1}} \cdots s_{\beta_{ij_i}} W_P$ and such that $\delta(u_i) = \sum_{j=1}^{j_i} d(\beta_{ij})$. If we possibly replace β_{ij} with an element in its W_P-orbit this gives $uW_p = s_{\beta_{11}} \cdots s_{\beta_{1j_1}} \cdots s_{\beta_{r1}} \cdots s_{\beta_{rj_r}} W_P$. Let α be a P-indecomposable root such that $uW_P = s_\alpha W_P$. It follows that

$$d(\alpha) = \delta(s_\alpha) = \delta(u) = \sum_{i=1}^{r} \delta(u_i) = \sum_{i,j} d(\beta_{ij})$$

and thus $r = 1$ since α is P-indecomposable. This proves one implication.

To prove the other implication, let $\beta_1, \dots, \beta_r \in R^+ \setminus R_P^+$ be a sequence of roots such that $uW_P = s_{\beta_1} \cdots s_{\beta_r} W_P$ and such that $\delta(u) = \sum_{i=1}^{r} d(\beta_i)$. Every β_i clearly satisfies $\delta(s_{\beta_i}) = d(\beta_i)$. If we apply the condition to $u_i = s_{\beta_i}$ we see that $r = 1$ and thus $uW_P = s_\alpha W_P$ where $\alpha = \beta_1$ satisfies $\delta(s_\alpha) = d(\alpha) = \delta(u)$. Since the sequence $\beta_1, \dots, \beta_r$ was chosen arbitrary, we see that α and thus u is P-indecomposable. $\square$

Example 6.42. Let P be maximal. Then all elements $u \in W/W_P$ such that $\delta(u) = 1$ are P-indecomposable.

Lemma 6.43. *Let d be a degree. Let $(\alpha_1, \dots, \alpha_r)$ be a greedy decomposition of d. Then*

$$z_d W_P = s_{\alpha_1} \cdot \ldots \cdot s_{\alpha_r} W_P = z_{d(\alpha_1)} \cdot \ldots \cdot z_{d(\alpha_r)} W_P \,.$$

In terms of curve neighborhoods this equation becomes

$$\Gamma_d(X_1) = \Gamma_{d(\alpha_r)}(\cdots(\Gamma_{d(\alpha_1)}(X_1))\cdots) \,.$$

Proof. From the main theorem on curve neighborhoods it is clear that

$$X_{z_{d(\alpha_1)} \cdot \ldots \cdot z_{d(\alpha_r)}} = \Gamma_{d(\alpha_r)}(\cdots(\Gamma_{d(\alpha_1)}(X_1))\cdots) \,.$$

Since all members of a greedy decomposition are P-cosmall, we have $s_{\alpha_i} W_P = z_{d(\alpha_i)} W_P$ for all i. Therefore it follows that $\Gamma_{d(\alpha_1)}(X_1) = X_{z_{d(\alpha_1)}} = X_{s_{\alpha_1}}$ and thus $\Gamma_{d(\alpha_2)}(\Gamma_{d(\alpha_1)}(X_1)) = X_{s_{\alpha_1} \cdot z_{d(\alpha_2)}} = X_{s_{\alpha_1} \cdot s_{\alpha_2}}$. By repreating this process we find that

$$\Gamma_{d(\alpha_r)}(\cdots(\Gamma_{d(\alpha_1)}(X_1))\cdots) = X_{s_{\alpha_1} \cdot \ldots \cdot s_{\alpha_r}} \,.$$

Both displayed euqations together yield $s_{\alpha_1} \cdot \ldots \cdot s_{\alpha_r} W_P = z_{d(\alpha_1)} \cdot \ldots \cdot z_{d(\alpha_r)} W_P$. From the definition of a greedy decomposition it follows directly that $z_d W_P = s_{\alpha_1} \cdot \ldots \cdot s_{\alpha_r} W_P$. This proves the first statement. The last statement is clear now since we have $\Gamma_d(X_1) = X_{z_d}$. $\square$

Lemma 6.44. *Let $\alpha \in R^+ \setminus R_P^+$ be a P-indecomposable root. Every greedy decomposition of $d(\alpha)$ consists of precisely one element. Moreover there exists a P-cosmall root β such that $z_{d(\alpha)} W_P = s_\beta W_P$ and such that $d(\alpha) = d(\beta)$. In particular, $z_{d(\alpha)}$ is P-indecomposable.*

Let $\alpha \in R^+ \backslash R_P^+$ be a P-indecomposable root which satisfies $z_{d(\alpha)} W_P = s_\alpha W_P$. Then α is also P-cosmall.

Proof. Let α be a P-indecomposable root. Let $(\alpha_1, \dots, \alpha_r)$ be a greedy decomposition of $d(\alpha)$. We clearly have $s_\alpha \preceq z_{\delta(s_\alpha)} = z_{d(\alpha)}$ and thus

$$x(s_\alpha) \in \Gamma_{d(\alpha)}(X_1) = \Gamma_{d(\alpha_r)}(\cdots(\Gamma_{d(\alpha_1)}(X_1))\cdots).$$

Therefore we can find elements $u_1, \dots, u_r \in W$ such that $s_\alpha W_P = u_r \dots u_1 W_P$ and such that $\delta(u_i) = d(\alpha_i)$. Now we have

$$\delta(s_\alpha) = d(\alpha) = \sum_{i=1}^{r} d(\alpha_i) = \sum_{i=1}^{r} \delta(u_i)$$

which implies that $r = 1$ since α is P-indecomposable. Therefore it follows that $z_{d(\alpha)} W_P = s_{\alpha_1} W_P$. If we put $\beta = \alpha_1$ the first statement follows since α_1 is P-cosmall as it is part of a greedy decomposition. Moreover we have $d(\alpha) = d(\beta)$. To see that $z_{d(\alpha)}$ is P-indecomposable it suffices to show that $\delta(s_\beta) = d(\beta)$ since β is P-cosmall. But this is clear since $\delta(s_\beta) = \delta(z_{d(\alpha)}) = \delta(z_{\delta(s_\alpha)}) = \delta(s_\alpha) = d(\alpha) = d(\beta)$ where we used that α is P-indecomposable and thus $\delta(s_\alpha) = d(\alpha)$.

If α is P-indecomposable and satisfies in addition $z_{d(\alpha)} W_P = s_\alpha W_P$, then it clearly follows that $s_\alpha W_P = s_\beta W_P$ and thus $\alpha = \beta$ which means that α is P-cosmall. □

Corollary 6.45. *Let $\alpha \in R^+ \setminus R_P^+$ be a root which satisfies $\delta(s_\alpha) = d(\alpha)$. Then α is P-cosmall if and only if α is P-indecomposable and $z_{d(\alpha)} W_P = s_\alpha W_P$.*

Proof. This is just a reformulation of the results we proved up to now. □

Lemma 6.46. *Let $\alpha \in R^+ \setminus R_P^+$ be a P-cosmall root which satisfies $\delta(s_\alpha) = d(\alpha)$. (If P is maximal this is always the case.) Then there exists a unique maximal root of $d(\alpha)$ and this unique maximal root is given by α.*

Proof. Let α_1 be a maximal root of $d(\alpha)$. By definition we know that $d(\alpha_1) \leq d(\alpha)$. We first show that $d(\alpha_1) = d(\alpha)$. Suppose for a contradiction that $d(\alpha_1) < d(\alpha)$. Then we can find a greedy decomposition $(\alpha_1, \dots, \alpha_r)$ of $d(\alpha)$ such that $r > 1$. On the other hand, we know that α is a P-cosmall root which satisfies $\delta(s_\alpha) = d(\alpha)$. Therefore α is P-indecomposable and every greedy decomposition of $d(\alpha)$ must consist of precisely one element. This contradiction proves that $d(\alpha_1) = d(\alpha)$. Therefore α_1 is a P-cosmall root. It follows that $s_{\alpha_1} W_P = z_{d(\alpha)} W_P = s_\alpha W_P$ and thus $\alpha = \alpha_1$. This proves that α is the unique maximal root of $d(\alpha)$. □

6.4. **The set $\mathcal{U}_1$.** Let $\mathcal{U}_1$ be the subset of $\mathcal{U}$ consisting of all $u \in \mathcal{U}$ such that $\delta(u)$ is maximal in the set $\{\delta(u) \mid u \in \mathcal{U}$ such that $\delta(u) < \delta(w_X)\}$ or equivalent such that $\delta(u^*)$ is minimal in the set $\{\delta(u) \mid u \in \mathcal{U}$ such that $\delta(u) > 0\}$.

We have an obvious inclusion:

$$\mathcal{U}_1 \subseteq \{u_1 \mid u_0, \ldots, u_r \text{ chain from } w_X \text{ to } w_X \text{ of degree } \delta(w_X)\}$$

Every element $u \in \mathcal{U}_1$ is part of a chain $u_0, \ldots, u_r$ from w_X to w_X of degree $\delta(w_X)$. Let j be the index such that $u = u_j$. We clearly have that $j \geq 1$ since $\delta(u_0) = \delta(w_X)$. On the other hand we know that $\delta(w_X) > \delta(u_1) > \delta(u_i)$ for all $i \geq 2$. Since $u_i \in \mathcal{U}$ for all i, we follow from the maximality of $\delta(u)$ that $u = u_1$ and $j = 1$. This proves the stated inclusion.

From this inclusion, we see in particular that for all $u \in \mathcal{U}_1$ there exists a unique root $\alpha \in R^+ \setminus R_P^+$ such that $u^* W_P = s_\alpha W_P$.

Lemma 6.47. *We have the following inclusion of sets:*

$$\mathcal{U}_1 \subseteq \{u \in \mathcal{U} \mid u^* \text{ is } P\text{-indecomposable}\}$$

Proof. Let $u \in \mathcal{U}_1$. Let $\alpha \in R^+ \setminus R_P^+$ be the unique root such that $u^* W_P = s_\alpha W_P$. By Corollary 6.15 we know that $\delta(s_\alpha) = d(\alpha)$. Let $\beta_1, \ldots, \beta_r \in R^+ \setminus R_P^+$ be a sequence of roots such that

$$s_\alpha W_P = s_{\beta_1} \cdots s_{\beta_r} W_P$$

and such that $\delta(\alpha) = \sum_{i=1}^r d(\beta_i)$. In order to see that α and u^* are P-indecomposable it suffices to show that $r = 1$. Let $u_0, \ldots, u_r$ be a chain from w_X to w_X of degree $\delta(w_X)$ which satisfies $u_1 = u$. The chain $u_0, \ldots, u_r$ necessarily satisfies $u_0 = w_o$ and $u_r = 1$. Then $w_o, w_o s_{\beta_1}, \ldots, w_o s_{\beta_1} \cdots s_{\beta_r}, u_2, \ldots, u_r$ is a chain from w_X to w_X since $w_o s_{\beta_1} \cdots s_{\beta_r} W_P = u W_P$. The degree of this chain is clearly

$$\delta(u) + \sum_{i=1}^r d(\beta_i) = \delta(u) + d(\alpha) = \delta(u) + \delta(u^*) = \delta(w_X)\,.$$

Therefore we have that $w_o s_{\beta_1} \in \mathcal{U}$. Suppose that $r > 1$. Then we have by Corollary 6.11 that $\delta(w_X) > \delta(w_o s_{\beta_1}) > \delta(w_o s_{\beta_1} \cdots s_{\beta_r}) = \delta(u)$. But this contradicts the fact that $u \in \mathcal{U}_1$. Therefore we conclude that $r = 1$ and that α and u^* are P-indecomposable. ☐

Lemma 6.48. *Let $u \in \mathcal{U}_1$. Then there exists a P-cosmall root $\alpha \in R^+ \setminus R_P^+$ such that $z_{\delta(u^*)} W_P = s_\alpha W_P$. In particular $\delta(u^*) = d(\alpha)$ and $\delta(w_X) = \delta(u) + d(\alpha)$. We can write $w_o W_P = z_{\delta(u)} \cdot s_\alpha W_P$.*

Proof. Let $d = \delta(u^*)$ for short. Let $\alpha \in R^+ \setminus R_P^+$ be a maximal root of d. By [9, Corollary 4.12(c)] we have $s_\alpha \cdot z_{d-d(\alpha)} W_P = z_d W_P$ and thus $w_o W_P = z_{\delta(u)} \cdot z_d W_P = z_{\delta(u)} \cdot s_\alpha \cdot z_{d-d(\alpha)} W_P$. Since $z_{\delta(u)} \cdot s_\alpha \preceq z_{\delta(u)+d(\alpha)}$

by [9, Corollary 4.12(b)] it follows that $w_oW_P = z_{\delta(u)+d(\alpha)} \cdot z_{d-d(\alpha)}W_P$. In the language of curve neighborhoods this reads as

$$\Gamma_{d-d(\alpha)}(\Gamma_{\delta(u)+d(\alpha)}(X_1)) = X \,.$$

This means that we can find a v such that $v \preceq z_{\delta(u)+d(\alpha)}$ and such that there exists a chain from $v_0, \dots v_r$ of degree $d-d(\alpha)$ which satisfies $v_0 = w_X$ and $v_r = v$. Therefore we have $\delta(v) \leq \delta(z_{\delta(u)+d(\alpha)}) \leq \delta(u) + d(\alpha)$ and $\delta(v^*) \leq d - d(\alpha)$. This shows that $\delta(v) + \delta(v^*) \leq \delta(u) + \delta(u^*) = \delta(w_X)$. Since this inequaltiy can never be strict, we get equalities $\delta(v) = \delta(u) + d(\alpha)$ and $\delta(v^*) = d - d(\alpha)$ and $\delta(v) + \delta(v^*) = \delta(w_X)$. Therefore $v \in \mathcal{U}$ and $\delta(v) > \delta(u)$ (since $d(\alpha) > 0$). By the maximality of u we conclude that $\delta(v) = \delta(w_X)$. This implies that $\delta(w_X) = \delta(u) + d(\alpha)$ and $d = d(\alpha)$.

By our choice, $\alpha \in R^+ \setminus R_P^+$ was a maximal root in d. Since $d = d(\alpha)$, this implies that α is P-cosmall. The equation $s_\alpha \cdot z_{d-d(\alpha)}W_P = z_dW_P$ becomes $s_\alpha W_P = z_dW_P$. The equation $w_oW_P = z_{\delta(u)} \cdot z_dW_P$ becomes $w_oW_P = z_{\delta(u)} \cdot s_\alpha W_P$. □

Corollary 6.49. *Let $u \in \mathcal{U}_1$. Let $\alpha \in R^+ \setminus R_P^+$ be a maximal root of $\delta(u^*)$. Then $s_\alpha^* \in \mathcal{U}_1$. In particular $s_\alpha \in \mathcal{U}$. Moreover we have $\delta(s_\alpha^*) = \delta(u)$, $\delta(s_\alpha) = \delta(u^*) = d(\alpha)$ and $s_\alpha^* \preceq z_{\delta(u)}$.*

Proof. We know in general that $z_{\delta(u^*)}^* \preceq z_{\delta(u)}$ for all $u \in \mathcal{U}$. In particular for u as in the statement, this gives $s_\alpha^* \preceq z_{\delta(u)}$ and thus $\delta(s_\alpha^*) \leq \delta(z_{\delta(u)}) = \delta(u)$. On the other hand we know that $\delta(s_\alpha) = d(\alpha) = \delta(u^*)$ by the previous result. This gives $\delta(s_\alpha)+\delta(s_\alpha^*) \leq \delta(u)+\delta(u^*) = \delta(w_X)$. Since this inequality can never be strict, we get the equality $\delta(s_\alpha^*) = \delta(u)$. Moreover we see that $s_\alpha \in \mathcal{U}$ and $s_\alpha^* \in \mathcal{U}_1$. □

Let B be the following set[5] of roots:

$$B = \{\alpha \in R^+ \setminus R_P^+ \mid \alpha \text{ maximal root of } \delta(u^*) \text{ for some } u \in \mathcal{U}_1\}\,.$$

We denote the highest root of R by θ_1.

Lemma 6.50.

- *All elements of B are P-cosmall and P-indecomposable.*
- *Different roots in B are incomparable: if $\alpha, \beta \in B$ such that $\alpha \leq \beta$ then $\alpha = \beta$.*
- *Different degrees of roots in B are incomparable: if $\alpha, \beta \in B$ such that $d(\alpha) \leq d(\beta)$ then $d(\alpha) = d(\beta)$.*
- *Let $\alpha, \beta \in B$ such that $s_\alpha \preceq s_\beta$ then $s_\alpha W_P = s_\beta W_P$.*
- *For all $\alpha \in B$ we have $d(\alpha) \leq d(\theta_1)$ and $s_\alpha \preceq s_{\theta_1}$.*
- *For all $\alpha \in B$ there is a unique maximal root of $d(\alpha)$ and this unique maximal root is given by α.*

[5] There should not be any confusion with the set B and the Borel subgroup of G which is also denoted by B.

- *If $\theta_1 \in B$ then $B = \{\theta_1\}$. If there is a root $\alpha \in B$ such that $d(\alpha) = d(\theta_1)$ then $B = \{\theta_1\}$.*
- *Suppose that w_o is P-indecomposable. Let α be the unique P-indecomposable root such that $w_o W_P = s_\alpha W_P$. Then $B = \{\alpha\}$.*
- *For all $u \in \mathcal{U}_1$ we have $u^* \preceq s_\alpha \preceq s_{\theta_1}$ for a unique element $\alpha \in B$. This element α satisfies $\delta(u^*) = d(\alpha)$.*
- *Let $\alpha \in B$. Then there exists a chain $u_0, \dots, u_r$ from w_X to w_X of degree $\delta(w_X)$ which satisfies $u_0 = w_o$ and $u_1 = w_o s_\alpha = s_\alpha^*$.*
- *If $u \in \mathcal{U}_1$ then $z_{\delta(u)} \in \mathcal{U}_1$. In particular for all $\alpha \in B$ we have that $s_\alpha^* \in \mathcal{U}_1$ and that $z_{\delta(s_\alpha^*)} \in \mathcal{U}_1$.*

Proof. We already saw that all elements of B are P-cosmall. We also know that $s_\alpha^* \in \mathcal{U}_1$ for all $\alpha \in B$. But this implies that s_α and α are P-indecomposable.

Let $\alpha, \beta \in B$ such that $\alpha \leq \beta$. Since α and β are both P-cosmall, it follows that $d(\alpha) \leq d(\beta)$. By the minimality of $d(\beta)$ we conclude that $d(\alpha) = d(\beta)$. Since α is a maximal root of $d(\alpha)$ we conclude that $\alpha = \beta$.

Let $\alpha, \beta \in B$ such that $d(\alpha) \leq d(\beta)$. By the minimality of $d(\alpha)$ it follows that $d(\alpha) = d(\beta)$.

Let $\alpha, \beta \in B$ such that $s_\alpha \preceq s_\beta$. Then we have $z_{d(\alpha)} \preceq z_{d(\beta)}$ and thus $d(\alpha) \leq d(\beta)$. (Note that $\delta(s_\alpha) = d(\alpha)$ and $\delta(s_\beta) = d(\beta)$.) By the previous point, it follows that $d(\alpha) = d(\beta)$ and thus $s_\alpha W_P = s_\beta W_P$ since $s_\alpha W_P = z_{d(\alpha)} W_P$ and $s_\beta W_P = z_{d(\beta)} W_P$.

Let $\alpha \in B$. Since α and θ_1 are P-cosmall, it follows that $d(\alpha) \leq d(\theta_1)$. Consequently, we have $z_{d(\alpha)} \preceq z_{d(\theta_1)}$. Again, since α and θ_1 are P-cosmall, this impies $s_\alpha \preceq s_{\theta_1}$.

Let $\alpha \in B$. Let $\beta \in R^+ \setminus R_P^+$ a maximal root of $d(\alpha)$. Since $s_\alpha^* \in \mathcal{U}_1$ we conclude that $\beta \in B$. By definition $d(\beta) \leq d(\alpha)$. Since different degrees of roots in B are incomparable, we conclude that $d(\alpha) = d(\beta)$. Since α and β are P-cosmall, it follows that $s_\alpha W_P = s_\beta W_P$. Since $\alpha \in R^+ \setminus R_P^+$ is uniquely determined by the coset $s_\alpha W_P$ ([9, Lemma 2.2]), it follows that $\alpha = \beta$. Therefore α is the unique maximal root of $d(\alpha)$. A different way to see that α is the unique maximal root of $d(\alpha)$ is to note that α is P-cosmall and satisfies $\delta(s_\alpha) = d(\alpha)$ (Lemma 6.46).

Suppose that $\theta_1 \in B$. Every root $\alpha \in B$ can be compared with the highest root θ_1: $\alpha \leq \theta_1$. Since different roots in B are incomparable, it follows that $\alpha = \theta_1$ and thus $B = \{\theta_1\}$. Suppose that there is a root $\alpha \in B$ such that $d(\alpha) = d(\theta_1)$. For all $\beta \in B$, we have that $d(\beta) \leq d(\alpha) = d(\theta_1)$. Since different degrees of roots in B are incomparable, we conclude that $d(\beta) = d(\theta_1)$ for all $\beta \in B$. On the other hand we know that β is the unique maximal root of $d(\beta) = d(\theta_1)$. But this unique maximal root is given by θ_1. Therefore we have $\beta = \theta_1$ for $\beta \in B$ which means that $B = \{\theta_1\}$.

Suppose that w_o is P-indecomposable. Let α be as in the statement. By definition it is clear that $\mathcal{U}_1 = \{1\}$. Therefore it is clear that B

consists of precisely one element α' which is the unique maximal root of $\delta(w_o)$. We also know that $\delta(w_o) = d(\alpha')$ and that α' is P-cosmall. Therefore it follows that $s_{\alpha'}W_P = z_{d(\alpha')}W_P = z_{\delta(w_o)}W_P = w_oW_P = s_\alpha W_P$ and thus $\alpha = \alpha'$. This means that $B = \{\alpha\}$ as claimed.

We already saw that $s_\alpha \preceq s_{\theta_1}$ for all $\alpha \in B$. Let $u \in \mathcal{U}_1$ and let α be a maximal root of $\delta(u^*)$. We have that $u^* \preceq z_{\delta(u^*)}$ and that $z_{\delta(u^*)}W_P = s_\alpha W_P$. This implies that $u^* \preceq s_\alpha$ as claimed. It also clear by what we saw up to now that $\delta(u^*) = d(\alpha)$. Next we prove the uniqueness of α. Let $\alpha, \beta \in B$ such that $u^* \preceq s_\alpha$ and $u^* \preceq s_\beta$. Then it follows that from the minimality of $d(\alpha)$ and $d(\beta)$ that $\delta(u^*) = d(\alpha)$ and that $\delta(u^*) = d(\beta)$. This implies $d(\alpha) = d(\beta)$. But since different degrees of roots in B are incomparable we must have $\alpha = \beta$.

Let $\alpha \in B$. Since $s_\alpha^* \in \mathcal{U}_1$ there exists a chain $u_0, \ldots, u_r$ from w_X to w_X of degree $\delta(w_X)$ which satisfies $u_1 = s_\alpha^*$. It is clear that we also can chose $u_0 = w_o$.

Let $u \in \mathcal{U}_1$. We already saw that this implies that $z_{\delta(u)} \in \mathcal{U}$. But since $\delta(z_{\delta(u)}) = \delta(u)$ we also have $z_{\delta(u)} \in \mathcal{U}_1$. □

Remark 6.51. We already see from the previous lemma that if P is maximal then B consists of precisely one element. Indeed, if P is maximal any two degrees are comparable, therefore $d = d(\alpha)$ is independent of the choice of $\alpha \in B$, since different degrees of roots in B are incomparable. On the other hand there is a unique maximal root α of d, which must be the unique element of B: $B = \{\alpha\}$. (Note that B is nonempty by construction.)

Lemma 6.52. *Let $(\alpha_1, \ldots, \alpha_r)$ be a greedy decomposition of $\delta(w_X)$. Then there exist elements $\alpha, \beta \in B$ such that $d(\alpha) \leq d(\alpha_1)$ and $d(\beta) \leq d(\alpha_r)$.*

Proof. Let $(\alpha_1, \ldots, \alpha_r)$ be a greedy decomposition of $\delta(w_X)$. Then we can write

$$X = \Gamma_{d(\alpha_r)}(\cdots(\Gamma_{d(\alpha_2)}(X_{s_{\alpha_1}}))\cdots)\,.$$

Therefore we can find $u, v \in \mathcal{U}$ such that $\delta(u^*) = d(\alpha_1)$, $\delta(v^*) = d(\alpha_r)$ and $u^* \preceq s_{\alpha_1}$ ($\mathcal{U}$ is closed under taking Poincaré duals). By definition we can find $\alpha, \beta \in B$ such that $d(\alpha) \leq d(\alpha_1)$ and $d(\beta) \leq d(\alpha_r)$. □

Lemma 6.53. *Let $u \in \mathcal{U}$ and $v \in W$ such that $\delta(v) \leq \delta(u)$ and $u \preceq v$. Then $v \in \mathcal{U}$ and $\delta(u) = \delta(v)$.*

Proof. Since $u \preceq v$, we have that $u^* \succeq v^*$ and thus $\delta(u^*) \geq \delta(v^*)$. Since $u \in \mathcal{U}$ we obtain

$$\delta(v) + \delta(v^*) \leq \delta(u) + \delta(u^*) = \delta(w_X)\,.$$

This means that $v \in \mathcal{U}$. Moreover we get the equality $\delta(u) = \delta(v)$. □

Lemma 6.54. *Let P be maximal. Then we have that $s_{\theta_1} \in \mathcal{U}$. This means that $\delta(w_X) = d(\theta_1) + \delta(s_{\theta_1}^*)$.*

Proof. Let $(\alpha_1, \dots, \alpha_r)$ be a greedy decomposition of $\delta(w_X)$. Since P is maximal and θ_1 is P-cosmall, we know that $\delta(s_{\theta_1}) = d(\theta_1)$. Therefore we have that $d(\theta_1) \leq \delta(w_X)$ and thus $\alpha_1 = \theta_1$. We can write

$$z_{\delta(w_X)} W_P = s_{\alpha_1} \cdot \ldots \cdot s_{\alpha_r} W_P = s_{\theta_1} \cdot z_{\delta(w_X) - d(\theta_1)} W_P \, .$$

In terms of curve neighborhoods this becomes $X = \Gamma_{\delta(w_X)-d(\theta_1)}(X_{s_{\theta_1}})$. Therefore we can find a chain $u_0, \dots, u_r$ from w_X to w_X of degree $\delta(w_X)$ such that there exists an index j between 0 and r such that $\delta(u_j) = d(\theta_1)$ and $u_j \preceq s_{\theta_1}$. Since $\delta(s_{\theta_1}) = d(\theta_1)$ (P maximal, θ_1 P-cosmall) and since $u_j \in \mathcal{U}$, the previous lemma implies that $s_{\theta_1} \in \mathcal{U}$. The final equation follows from the definition of $\mathcal{U}$ and since $\delta(s_{\theta_1}) = d(\theta_1)$. □

We now introduce a further set of roots. Let $\alpha \in B$. We saw that $z_{\delta(s^*_\alpha)} \in \mathcal{U}_1$ and thus that $z^*_{\delta(s^*_\alpha)}$ is P-indecomposable. Therefore there exists a unique P-indecomposable root $\beta \in R^+ \setminus R^+_P$ such that $z^*_{\delta(s^*_\alpha)} W_P = s_\beta W_P$. Since β is uniquely determined by α we can write $\beta = \varphi(\alpha)$. Let B^* be the following set of roots

$$B^* = \{\beta \in R^+ \setminus R^+_P \mid s_\beta W_P = z^*_{\delta(s^*_\alpha)} W_P \text{ for some } \alpha \in B\} \, .$$

We have natural surjective map $\varphi \colon B \to B^*$ which sends an element $\alpha \in B$ to $\varphi(\alpha) \in B^*$ where $\varphi(\alpha)$ is as defined above.

Lemma 6.55.

- *φ induces a bijection between B and B^*.*
- *All element of B^* are P-indecomposable.*
- *For all $\alpha \in B$ we have that $d(\varphi(\alpha)) = d(\alpha)$, $\varphi(\alpha) \leq \alpha$ and that $s_{\varphi(\alpha)} \preceq s_\alpha$.*
- *For all $u \in \mathcal{U}_1$ we have $s_{\varphi(\alpha)} \preceq u^* \preceq s_\alpha$ for a unique element $\alpha \in B$. This element α satisfies $\delta(u^*) = d(\alpha)$. Conversely, every element $u \in W$ such that $s_{\varphi(\alpha)} \preceq u^* \preceq s_\alpha$ for some $\alpha \in B$ satisfies $u \in \mathcal{U}_1$.*

Proof. For all $\alpha \in B$ it is clear that $s^*_{\varphi(\alpha)} \in \mathcal{U}_1$ and that $s_{\varphi(\alpha)} \preceq s_\alpha$. Therefore we get that α is uniquely determined by $\varphi(\alpha)$ or in other words that φ is injective and induces therefore a bijection between B and B^*.

This is true just because every $\beta \in B^*$ satisfies $s^*_\beta \in \mathcal{U}_1$.

We already saw that $s_{\varphi(\alpha)} \preceq s_\alpha$ for all $\alpha \in B$. This immediately implies that $\delta(s_{\varphi(\alpha)}) = d(\alpha)$ since $s^*_{\varphi(\alpha)} \in \mathcal{U}_1$. Since $s_{\varphi(\alpha)}$ is a reflection which occurs in a chain from w_X to w_X of degree $\delta(w_X)$ we also know that $\delta(s_{\varphi(\alpha)}) = d(\varphi(\alpha))$ and thus $d(\varphi(\alpha)) = d(\alpha)$. Since α is the unique maximal root of $d(\alpha)$ it follows that $\varphi(\alpha) \leq \alpha$.

Let $u \in \mathcal{U}_1$. We already know that there exists a unique $\alpha \in B$ such that $u^* \preceq s_\alpha$ and such that $\delta(u^*) = d(\alpha)$. We clearly have $\delta(u) = \delta(s^*_\alpha)$ since $u, s_\alpha \in \mathcal{U}_1$. But this implies that $u \preceq z_{\delta(s^*_\alpha)}$ which gives by taking Poincaré duals that $s_{\varphi(\alpha)} \preceq u^*$ as claimed. Conversely, let $u \in W$ be an element which satisfies $s_{\varphi(\alpha)} \preceq u^* \preceq s_\alpha$ for some $\alpha \in B$. Then we have $\delta(u^*) \leq d(\alpha)$ and $\delta(u) \leq \delta(s^*_{\varphi(\alpha)}) = \delta(z_{\delta(s^*_\alpha)}) = \delta(s^*_\alpha)$ which implies that $\delta(u) + \delta(u^*) \leq \delta(w_X)$. Therefore we get equalities $\delta(u) = \delta(s^*_\alpha)$ and $\delta(u^*) = d(\alpha)$. But this means that $u \in \mathcal{U}_1$. □

Corollary 6.56. *We have the following bijection of sets*

$$\mathcal{U}_1 \cong \mathcal{U}_1^* := \coprod_{\alpha \in B} \{\beta \in R^+ \setminus R_P^+ \mid s_{\varphi(\alpha)} \preceq s_\beta \preceq s_\alpha\}$$

which sends an element $\beta \in \mathcal{U}_1^*$ to the element $s^*_\beta \in \mathcal{U}_1$. This bijection induces via restriction a further bijection of sets as follows:

$$\{u \in \mathcal{U}_1 \mid \langle [\{\text{pt}\}], \sigma_{u^*} \rangle_{\delta(u^*)} \neq 0\} \cong B\,.$$

Proof. The first bijection follows directly from the previous lemma. To see the second bijection, note that all elements $\beta \in \mathcal{U}_1^* \setminus B$ are not P-cosmall, since they are not maximal in $d(\beta)$. Therefore those β's have vanishing Gromov-Witten invariant: $\left\langle [\{\text{pt}\}], \sigma_{s_\beta} \right\rangle_{d(\beta)} = 0$. On the other hand, all elements α of B are P-cosmall and therefore have nonvanishing Gromov-Witten invariant: $\langle [\{\text{pt}\}], \sigma_{s_\alpha} \rangle_{d(\alpha)} = 1$. This proves the second bijection. □

Corollary 6.57. *All elements of $\mathcal{U}_1^*$ are P-indecomposable. Let $\beta \in \mathcal{U}_1^*$ then there exists a unique $\alpha \in B$ such that $\delta(s_\beta) = d(\beta) = d(\alpha)$. This root α satisfies $\beta \leq \alpha$.*

Proof. Let $\beta \in \mathcal{U}_1^*$. It is clear that $s^*_\beta \in \mathcal{U}_1$. Therefore β and s_β are P-indecomposable. Let α be the unique root in B such that $s_{\varphi(\alpha)} \preceq s_\beta \preceq s_\alpha$. Then we have $\delta(s_\beta) = d(\alpha)$. Since all reflections s_β where $\beta \in \mathcal{U}_1^*$ occur in a chain from w_X to w_X of degree $\delta(w_X)$ we know that $\delta(s_\beta) = d(\beta)$ and thus $d(\beta) = d(\alpha)$. Since α is the unique maximal root of $d(\alpha)$ it follows that $\beta \leq \alpha$. The uniqueness of $\alpha \in B$ such that $d(\beta) = d(\alpha)$ is clear since different degrees of roots in B are incomparable. □

6.5. **Local curve neighborhoods.** Let $\beta \in \mathcal{B}$ and let d be a degree in $H_2(X(\beta), \mathbb{Z})$. Let w be a Weyl group element in $W_{G(\beta)}$. We define the local degree d curve neighborhood $\Gamma^\beta_d(X_w)$ of X_w with respect to $X(\beta)$ in the same way we defined the (global) degree d curve neighborhood $\Gamma_d(X_w)$ of X_w with respect to X. This makes sense since global and local Schubert varieties can be identified for $\beta \in \mathcal{B}$. Moreover we clearly identify $H_2(X(\beta), \mathbb{Z})$ with a sublattice of $H_2(X, \mathbb{Z})$. We define z^β_d to be the unique element in $W^{P(\beta)}_{G(\beta)}$ which satisfies $X_{z^\beta_d} = \Gamma^\beta_d(X_1)$. With this notation we clearly have $z^{\theta_1}_d = z_d$ for all degrees d, $z^\beta_d \preceq z_d$ for all β and all degrees d in $H_2(X(\beta), \mathbb{Z})$ and $z^\beta_0 = 1$ for all β.

Let β still denote a root in $\mathcal{B}$. Then we define a function δ_β with functoriality $W_{G(\beta)} \times W_{G(\beta)} \to H_2(X(\beta), \mathbb{Z})$ with respect to $X(\beta)$ in the same way as we defined the function δ with functoriality $W \times W \to H_2(X,\mathbb{Z})$ with respect to X. Again we extend our notation by writing $\delta_\beta(u) = \delta_\beta(u, w_o(\beta))$. With this notation we clearly have $z_d^\beta W_P = w_o(\beta)W_P$ for all degrees d in $H_2(X(\beta),\mathbb{Z})$ such that $d \geq \delta_\beta(w_o(\beta))$.

Lemma 6.58. *Let $\beta \in \mathcal{B}$. Let d be a degree in $H_2(X(\beta),\mathbb{Z})$ and let $w \in W_{G(\beta)}$. Then we have $\Gamma_d^\beta(X_w) = \Gamma_d(X_w) \cap X(\beta)$. In particular we have $\Gamma_d^\beta(X_1) = \Gamma_d(X_1) \cap X(\beta)$.*

Proof. With the notation as in the statement, we have an obvious inclusion $\Gamma_d^\beta(X_w) \subseteq \Gamma_d(X_w) \cap X(\beta)$. We prove the other inclusion by induction on $d \in H_2(X(\beta),\mathbb{Z})$. If $d = 0$ then $\Gamma_d^\beta(X_w) = \Gamma_d(X_w) = X_w \subseteq X(\beta)$ and there is nothing to prove. Suppose that $d > 0$ and that the inclusion from right to left is true for all $d' < d$. Since $\Gamma_d(X_w) \cap X(\beta)$ and $\Gamma_d^\beta(X_w)$ are B-stable it suffices to prove that every T-fixed point $x(u) \in \Gamma_d(X_w) \cap X(\beta)$ is contained in $\Gamma_d^\beta(X_w)$. Since $x(u) \in X(\beta)$ we know that $u \preceq w_o(\beta)$ or equivalent $u \in W_{G(\beta)}$. Since $x(u) \in \Gamma_d(X_w)$ there exists a chain $u_0, \ldots, u_r$ from u to w^* of degree d which satisfies $u_0 = u$. Let $\beta_i \in R^+ \setminus R_P^+$ be roots such that $u_{i-1}s_{\beta_i} = u_i$ for all $1 \leq i \leq r$. Then $u_1, \ldots, u_r$ is a chain from u_1 to w^* of degree $d' := d - d(\beta_1) < d$. Therefore $x(u_1) \in \Gamma_{d'}(X_w)$. Since u_0 and u_1 are adjacent, we have either $u_0 \prec u_1$ or $u_1 \prec u_0$. If $u = u_0 \prec u_1$ if follows that $x(u) \in \Gamma_{d'}(X_w)$ since $\Gamma_{d'}(X_w)$ is a Schubert variety. The induction hypothesis implies that $x(u) \in \Gamma_{d'}(X_w) \cap X(\beta) = \Gamma_{d'}^\beta(X_w) \subseteq \Gamma_d^\beta(X_w)$ as required. Therefore we can assume that $u_1 \prec u_0 = u$. In this case it follows that $u_1 \prec u \preceq w_o(\beta)$ and thus $u_1 \in W_{G(\beta)}$ and $x(u_1) \in X(\beta)$. The induction hypothesis implies that $x(u_1) \in \Gamma_{d'}(X_w) \cap X(\beta) = \Gamma_{d'}^\beta(X_w)$. Therefore there exists a chain $u_1', \ldots, u_s'$ from u_1 to w^* of degree d' which satisfies $u_1' = u_1$ such that the associated T-invariant curve is completely contained in $X(\beta)$. Since $u_0 \in W_{G(\beta)}$ and $u_1 \in W_{G(\beta)}$ it clearly follows that $\beta_1 \in R(\beta)$. Therefore $u_0, u_1 = u_1', \ldots, u_s'$ is a chain from u to w^* of degree $d = d(\beta_1) + d'$ which satisfies $u_0 = u$ such that the associated T-invariant curve is completely contained in $X(\beta)$. But this means that $x(u) \in \Gamma_d^\beta(X_w)$ as required. This completes the proof of the inclusion from right to left. $\square$

Lemma 6.59. *Let $u \in W$ be a Weyl group element. Let $\beta \in \mathcal{B}$. Let $u_0, \ldots, u_r$ be a chain of degree $\delta(u)$ from u to w_X such that $u_0 \in W_{G(\beta)}$. Let $\alpha_i \in R^+ \setminus R_P^+$ be the unique root such that $s_{\alpha_i}u_{i-1} = u_i$ for all i. Then $u_i \in W_{G(\beta)}$ for all i and $\alpha_i \in R(\beta)$ for all i.*

Proof. Let $u_0, \ldots, u_r$ be as in the statement. By Corollary 6.11 we know that $u_0 \succ u_1 \succ \cdots \succ u_r$. Since $u_0 \in W_{G(\beta)}$ and $\beta \in \mathcal{B}$ this implies that $u_i \in W_{G(\beta)}$ for all i. Since $s_{\alpha_i}u_{i-1} = u_i$ this implies that $\alpha_i \in R(\beta)$ for all i. $\square$

Lemma 6.60. *Let $\beta \in \mathcal{B}$. Then δ_β and δ coincide on $W_{G(\beta)}$, i.e. $\delta_\beta(u) = \delta(u)$ for all $u \in W_{G(\beta)}$.*

Proof. By definition we have $\delta \leq \delta_\beta$ since the minimum in δ runs through a larger set of chains which includes the set of chains concerned in δ_β. To prove the other equality, let $u \in W_{G(\beta)}$. Let $u_0, \ldots, u_r$ be a chain from u to w_X of degree $\delta(u)$ which satisfies $u_0 = u$ and $u_r = 1$. Since $u_0 \in W_{G(\beta)}$ and $\beta \in \mathcal{B}$ the previous lemma implies that the T-invariant curve associated to the chain $u_0, \ldots, u_r$ is completely contained in $X(\beta)$, so that it is part of the set of chains concerned in $\delta_\beta(u)$. This implies that $\delta_\beta(u) \leq \delta(u)$. In total, we get equality $\delta_\beta(u) = \delta(u)$ for all $u \in W_{G(\beta)}$.

A different way to see the statement is to use (local) curve neighborhoods. Let $u \in W_{G(\beta)}$ and $\beta \in \mathcal{B}$. On the one hand we have $u \preceq z^\beta_{\delta_\beta(u)} \preceq z_{\delta_\beta(u)}$ which implies that $\delta(u) \leq \delta_\beta(u)$. This means in particular that $\delta(u) \in H_2(X(\beta), \mathbb{Z})$ since $\delta_\beta(u) \in H_2(X(\beta), \mathbb{Z})$ by definition. On the other hand $u \preceq z_{\delta(u)}$ and $u \preceq w_o(\beta)$ implies that $u \preceq z^\beta_{\delta(u)}$ since $\Gamma^\beta_{\delta(u)}(X_1) = \Gamma_{\delta(u)}(X_1) \cap X(\beta)$. But $u \preceq z^\beta_{\delta(u)}$ implies that $\delta_\beta(u) \leq \delta(u)$. In total, we again find that $\delta_\beta(u) = \delta(u)$ for all $u \in W_{G(\beta)}$. □

Definition 6.61. *A root $\alpha \in R^+ \setminus R^+_P$ is called locally P-cosmall if there exists a root $\beta \in \mathcal{B}$ such that* $\mathrm{supp}(\alpha) \subseteq \mathrm{supp}(\beta)$ *and such that α is $P(\beta)$-cosmall in $X(\beta)$.*

Definition 6.62. *A root $\alpha \in R^+ \setminus R^+_P$ is called locally P-indecomposable if there exists a root $\beta \in \mathcal{B}$ such that* $\mathrm{supp}(\alpha) \subseteq \mathrm{supp}(\beta)$ *and such that α is $P(\beta)$-indecomposable in $X(\beta)$.*

Example 6.63. Every P-cosmall root / P-indecomposable root is in particular locally P-cosmall / locally P-indecomposable. We just apply the defintion to $\beta = \theta_1 \in \mathcal{B}$.

Every root $\beta \in \mathcal{B}$ is locally P-cosmall. Indeed, let $\beta \in \mathcal{B}$ then β is locally high, which means that β is the highest root of $R(\beta)$. This implies that β is $P(\beta)$-cosmall in $X(\beta)$. Therefore it follows from the definition that β is locally P-cosmall.

Lemma 6.64. *A root $\alpha \in R^+ \setminus R^+_P$ is P-indecomposable if and only if it is locally P-indecomposable.*

Proof. A root which is P-indecomposable is clearly locally P-indecomposable. Suppose that α is locally P-indecomposable. Let $\beta \in \mathcal{B}$ such that $\mathrm{supp}(\alpha) \subseteq \mathrm{supp}(\beta)$ and such that α is $P(\beta)$-indecomposable in $X(\beta)$. Then we have $\delta_\beta(s_\alpha) = d(\alpha)$. But since δ and δ_β coincide on $W_{G(\beta)}$ we also have $\delta(s_\alpha) = d(\alpha)$. Let $\beta_1, \ldots, \beta_r \in R^+ \setminus R^+_P$ be a sequence of roots such that $s_\alpha W_P = s_{\beta_1} \cdots s_{\beta_r} W_P$ and such that $d(\alpha) = \sum_{i=1}^r d(\beta_i)$. We have to show that $r = 1$. The sequence $u_0 = s_{\beta_1} \cdots s_{\beta_r}, u_1 = s_{\beta_2} \cdots s_{\beta_r}, \ldots, u_{r-1} = s_{\beta_r}, u_r = 1$ defines a chain of degree $\delta(s_\alpha) = d(\alpha)$ from s_α to w_X such that $u_0 \in W_{G(\beta)}$. This implies that $u_i \in W_{G(\beta)}$ for all i and that $\beta_i \in R(\beta)$ for all i. The equation $s_\alpha W_P = u_0 W_P$ clearly also holds modulo $W_{P(\beta)}$. Since α is $P(\beta)$-indecomposable in $X(\beta)$ this implies $r = 1$ as required. □

Lemma 6.65. *Let α be a locally P-cosmall root. If P is maximal, then $\delta(s_\alpha) = d(\alpha)$. In particular, if P is maximal we have $\delta(s_\beta) = d(\beta)$ for all $\beta \in \mathcal{B}$.*

Proof. Suppose that α is locally P-cosmall. Let $\beta \in \mathcal{B}$ such that $\mathrm{supp}(\alpha) \subseteq \mathrm{supp}(\beta)$ and such that α is $P(\beta)$-cosmall in $X(\beta)$. If P-maximal, we can assume that $P(\beta)$ is also maximal. Otherwise we had $G(\beta) = P(\beta) \subseteq P$ and thus $\delta(s_\alpha) = d(\alpha) = 0$ for all $\alpha \in R(\beta) \subseteq R_P$. If $P(\beta)$ is maximal, it follows that $\delta_\beta(s_\alpha) = d(\alpha)$ since α is $P(\beta)$-cosmall in $X(\beta)$. Since δ_β and δ coincide on $W_{G(\beta)}$ for all $\beta \in \mathcal{B}$ it follows that $\delta(s_\alpha) = \delta_\beta(s_\alpha) = d(\alpha)$ as claimed.

The last statement is clear since every element of $\mathcal{B}$ is locally P-cosmall. □

Example 6.66. Suppose that P is maximal. Every root $\beta \in \mathcal{B}$ is P-indecomposable. Indeed, every root $\beta \in \mathcal{B}$ is locally P-cosmall and satisfies $\delta(s_\beta) = d(\beta)$ since P is maximal, therefore every $\beta \in \mathcal{B}$ is locally P-indecomposable, therefore P-indecomposable.

Lemma 6.67. *Let $\alpha \in R^+ \setminus R_P^+$ be a P-indecomposable root. Let $\beta \in \mathcal{B}$ be a root such that* $\mathrm{supp}(\alpha) \subseteq \mathrm{supp}(\beta)$ *or equivalent such that $\alpha \leq \beta$. Then $d(\alpha) \leq d(\beta)$. In particular, if P is maximal and if $\beta, \beta' \in \mathcal{B}$ are two roots such that $\beta \leq \beta'$. Then $d(\beta) \leq d(\beta')$.*

Proof. Let α and β be as in the statement. Since α is P-indecomposable and since $\alpha \in R(\beta)$ we know that α is also $P(\beta)$-indecomposable in $X(\beta)$. By Lemma 6.44 there exists a root γ which is $P(\beta)$-cosmall in $X(\beta)$ such that $z^\beta_{d(\alpha)} W_P = s_\gamma W_P$ and such that $d(\alpha) = d(\gamma)$. Since $\gamma \in R(\beta)$ and β is locally high, it follows that $\gamma \leq \beta$. Since $\beta \in \mathcal{B}$ we know that β is $P(\beta)$-cosmall in $X(\beta)$. Therefore we conclude that $d(\gamma) \leq d(\beta)$ and thus $d(\alpha) \leq d(\beta)$.

The last statement is clear since if P is maximal every element of $\mathcal{B}$ is P-indecomposable. □

Theorem 6.68 ([9, Conjecture 6.5])**.** *Assume that R is simply laced and let $\alpha \in R^+ \setminus R_P^+$. Then α is P-cosmall if and only if $z_{d(\alpha)} W_P = s_\alpha W_P$.*

Proof. If α is P-cosmall then $z_{d(\alpha)} W_P = s_\alpha W_P$ is always satisfied (even if R is not necessarily simply laced). Suppose that $z_{d(\alpha)} W_P = s_\alpha W_P$. Let β be a maximal element of the set $\{\gamma \in R^+ \setminus R_P^+ \mid d(\gamma) \leq d(\alpha) \text{ and } \gamma \geq \alpha\}$. Then β is clearly also a maximal element of the set $\{\gamma \in R^+ \setminus R_P^+ \mid d(\gamma) \leq d(\alpha)\}$ since any element $\gamma \in R^+ \setminus R_P^+$ which satisfies $\gamma \geq \beta$ and $d(\gamma) \leq d(\alpha)$ also satisfies $\gamma \geq \alpha$. Since $\beta \geq \alpha$ and since R is simply laced it follows that $\beta^\vee \geq \alpha^\vee$ and thus $d(\beta) \geq d(\alpha)$. On the other hand we know that $d(\beta) \leq d(\alpha)$ by the choice of β. Therefore we have $d(\alpha) = d(\beta)$ and that β is P-cosmall. This implies that $z_{d(\alpha)}W_P = s_\beta W_P = s_\alpha W_P$ and thus $\alpha = \beta$. This means that α is P-cosmall. □

Corollary 6.69. *Suppose that R is simply laced and that P is maximal. Let $\alpha \in R^+ \setminus R_P^+$. If $z_{d(\alpha)}W_P = s_\alpha W_P$, then α is P-indecomposable.*

Proof. Suppose that $z_{d(\alpha)}W_P = s_\alpha W_P$. Since R is simply laced we know by the previous theorem that α is P-cosmall. Since P is maximal this implies $\delta(s_\alpha) = d(\alpha)$. Lemma 6.39 implies that α is P-indecomposable. □

Example 6.70 ([9, Example 6.6]). We give an example for a root $\alpha \in R^+ \setminus R_P^+$ such that $z_{d(\alpha)}W_P = s_\alpha W_P$ and such that α is not P-cosmall. By the previous theorem this is only possible if R is not simply laced. Let R be of type B_2 and let $\alpha_P = \alpha_1$. Let $\alpha = \alpha_1 + \alpha_2$, let $\theta_1 = \alpha_1 + 2\alpha_2$ and let $\theta_2 = \alpha_1$. We already saw in Example 6.40 that α is not P-cosmall. We will see later (and it is easy to prove directly) that $s_{\theta_1}s_{\theta_2}W_P = w_oW_P$. So it follows as in Example 6.40 that $\delta(w_X) = \delta(s_{\theta_1}s_{\theta_2}) = 2$. On the other hand we know that $d(\alpha) = 2$ and thus $z_{d(\alpha)}W_P = w_oW_P$. Together with the fact that $s_\alpha W_P = s_{\theta_1}s_{\theta_2}W_P$ this yields $z_{d(\alpha)}W_P = s_\alpha W_P$.

7. The distance function d

In this section we introduce the distance function d according to [11, Definition 3.2] and relate it to the function δ.

As before, $X = G/P$ denotes a homogeneous space where G is a connected, simply connected linear algebraic group and P is a parabolic subgroup.

Let $x, y \in X$, we define $d(x, y)$ to be the degree d of a curve passing through x and y such that the degree d is minimal in the set of all degrees of curves passing through x and y. We further define

$$d_X = \max_{x,y\in X} d(x, y)\,.$$

Note that the nature of the function d is similar to that of δ explained in Remark 5.7.

We list immediate properties of the function d in the following

Lemma 7.1.

- *The function d is a metric on X.*
- *The function d is G-invariant: for all $x, y \in X$ and all $g \in G$ we have $d(x, y) = d(gx, gy)$.*
- *Let $u, v \in W$. Then $d(x(u), x(v)) = \delta(v^{-1}u) = \delta(u^{-1}v)$.*
- *Let $x \in \Omega_u$ and $y \in \Omega_v$. Then $d(x, y) \geq d(x(u), x(v))$*
- *Let $x \in \Omega_u$. Then $d(x, x(1)) = d(x(u), x(1)) = \delta(u)$.*

- *Assume that P is maximal. Then we have the equality $d_X = \delta(w_X) = d(x(w_X), x(1))$.*

Proof. It is clear from the definition of d that d is symmetric and that $d(x,x) = 0$ for all $x \in X$. Let x, y, z be arbitrary in X. Let f be a curve of degree $d(x,y)$ passing through x and y and let g be a curve of degree $d(y,z)$ passing through y and z. Then $f \cup g$ is a curve of degree $d(x,y) + d(y,z)$ passing through x and z. Therefore we have $d(x,z) \leq d(x,y) + d(y,z)$. This proves that d is a metric on X.

Let $x, y \in X$ and $g \in G$. Let f be a curve of degree $d(x,y)$ passing through x and y. Then gf is a curve of degree $d(x,y)$ passing through gx and gy. Therefore we have $d(gx, gy) \leq d(x,y)$. By replacing g with g^{-1} / x with gx / y with gy the other inequality $d(x,y) \leq d(gx, gy)$ follows. In total, we get equality $d(x,y) = d(gx, gy)$.

Let $u, v \in W$. Since $d(x(u), x(v)) = d(x(v^{-1}u), x(1))$ by the G-ivariance of d, we can assume from the beginning that $v = 1$. We already saw that $\delta(u) = \delta(u^{-1})$. We are left to show that $d(x(u), x(1)) = \delta(u)$. We saw earlier that we can find a chain $u_0, \ldots, u_r$ from u to w_X of degree $\delta(u)$ which satisfies $u_0 = u$ and $u_r = 1$. The T-invariant curve associated to the chain $u_0, \ldots, u_r$ is a curve of degree $\delta(u)$ which passes through $x(u)$ and $x(1)$. Therefore we find that $d(x(u), x(1)) \leq \delta(u)$. On the other hand, let f be a curve of degree $d(x(u), x(1))$ which passes through $x(u)$ and $x(1)$. Then f converges to a T-invariant curve which still passes through $x(u)$ and $x(1)$. This T-invariant curve is associated to a chain from u to w_X of degree $d(x(u), x(1))$. It follows that $d(x(u), x(1)) \geq \delta(u)$. In total, we get equality $d(x(u), x(1)) = \delta(u)$ as claimed.

Let $x \in \Omega_u$ and $y \in \Omega_v$. Let f be a curve of degree $d(x,y)$ passing through x and y. Then f coverges to a T-invariant curve of degree $d(x,y)$ passing through $x(u)$ and $x(v)$. Therefore we have $d(x,y) \geq d(x(u), x(v))$.

Let $x \in \Omega_u$. By the previous point, we already know that $d(x, x(1)) \geq d(x(u), x(1))$. On the other hand, we know that $d(x(u), x(1)) = \delta(u)$. Since $u \preceq z_{\delta(u)}$ we have $\Omega_u \subseteq \Gamma_{\delta(u)}(X_1)$. Since $x \in \Omega_u$ we can find a curve of degree $\delta(u)$ passing through x and $x(1)$. This means that $d(x, x(1)) \leq \delta(u) = d(x(u), x(1))$. In total, we get equality $d(x, x(1)) = d(x(u), x(1))$.

By the G-invariance of d, the previous point and Lemma 5.8, we have

$$d_X = \max_{x,y \in X} d(x,y) = \max_{x \in X} d(x, x(1)) = \max_{u \in W/W_P} d(x(u), x(1))$$
$$= \max_{u \in W/W_P} \delta(u) = \delta(w_X) = d(x(w_X), x(1)) \,.$$

This proves the desired equality. □

Conjecture 7.2. *The degree d_X gives a maximal power q^{d_X} of the quantum parameter q which can occur in an arbitrary quantum product of two Schubert cycles.*

Remark 7.3. We will see later that this conjecture is satisfied for a specific class of homogeneous spaces where P is maximal (cf. Lemma 13.21, Remark 13.22). More generally it is known that this conjecture is satisfied for all cominuscule homogeneous spaces (cf. [11, Proposition 28]).

8. The θ-sequence

Let $X = G/P$ denote a homogeneous space where G is a connected, simply connected, linear algebraic group and P is a maximal parabolic subgroup. Since P is maximal we have a distinguished chain cascade $C(\alpha_P)$ associated to α_P. We always write $C(\alpha_P) = \{\theta_1, \ldots, \theta_k\}$ where $\theta_1 \geq \cdots \geq \theta_k$ are ordered according to their indices. We call the sequence of roots $\theta_1, \ldots, \theta_k$ the θ-sequence. The θ-sequence consists precisely of the elements of $\mathcal{B}$ which are contained in $R^+ \setminus R_P^+$. We always denote with k the length of the θ-sequence, that is the cardinality of $C(\alpha_P)$. The θ-sequence was first introduced and intensively studied in [25].

To simplify notation, we use the following notation adapted to the θ-sequence.

Notation 8.1. For all $1 \leq i \leq k$ we write $R_i = R(\theta_i)$, $\Delta_i = \Delta(\theta_i)$, $G_i = G(\theta_i)$, $P_i = P(\theta_i)$, $B_i = B(\theta_i)$, $X^i = X(\theta_i)$, $\mathfrak{g}_i = \mathfrak{g}(\theta_i)$, $\mathfrak{p}_i = \mathfrak{p}(\theta_i)$ and $d_i = d(\theta_i)$.

Note that for all $1 \leq i \leq k$ we have $H_2(X, \mathbb{Z}) = H_2(X^i, \mathbb{Z}) = \mathbb{Z}$. Therefore every degree in $H_2(X, \mathbb{Z})$ is also a degree in $H_2(X^i, \mathbb{Z})$. For each $w \in W_{G_i}$ and each degree d we write $\Gamma_d^i(X_w) = \Gamma_d^{\theta_i}(X_w)$ and $z_d^i = z_d^{\theta_i}$. We also write $\delta_i = \delta_{\theta_i}$ although this notation is only of technical nature since $\delta = \delta_i$ on W_{G_i}.

We know that every element of $\mathcal{B}$ is locally P-cosmall. More concretely, for all $1 \leq i \leq k$ the root θ_i is P_i-cosmall in X^i. Therefore we have $s_{\theta_i} W_P = z_{d_i}^i W_P$ and thus $\Gamma_{d_i}^i(X_1) = X_{s_{\theta_i}}$. The dimension of $\Gamma_{d_i}^i(X_1)$ is given by $\ell_P(s_{\theta_i}) = c_1(X^i)d_i - 1$. Moreover, we have $\delta(s_{\theta_i}) = d_i$.

Remark 8.2. Note that we have $\delta(s_{\theta_i}) = d_i$ for $i \geq 2$ although the roots θ_i for $i \geq 2$ are not P-cosmall, since $\mathrm{supp}(\theta_i) \neq \Delta$ for $i \geq 2$. If P is maximal P-cosmalleness of a root α is sufficient to guarantee that $\delta(s_\alpha) = d(\alpha)$ but it is not necessary.

Also, note that all elements of $\mathcal{B}$ are locally P-cosmall and P-indecomposable. In particular, all θ_i for $i \geq 2$ are locally P-cosmall and P-indecomposable although the roots θ_i for $i \geq 2$ are not P-cosmall, since $\mathrm{supp}(\theta_i) \neq \Delta$ for all $i \geq 2$. If P is maximal P-cosmalleness of a root is sufficient for P-indecomposablility but not necessary. Indeed, for all $i \geq 2$ we must have $z_{d_i}^i \prec z_{d_i}$ or equivalent $s_{\theta_i} \prec z_{d_i}$.

Lemma 8.3. *Let $1 \leq i \leq k$. At most one irreducible component of R_i° is different from A_1. Moreover R_i° has at most three irreducible components which happens if and only if R_i is of type D_4.*

Proof. We can clearly assume that $i = 1$. If R is of type A_n where $n \geq 1$, C_p where $p \geq 2$, E_6, E_7, E_8, F_4 or G_2 then there is at most one irreducible component of R° and there is nothing to prove. If R is of type B_ℓ where $\ell \geq 2$ then there are at most two irreducible components of R° with simple roots α_1 and $\alpha_3, \dots, \alpha_\ell$. If R is of type D_p where $p > 4$ then there are two irreducible components of R° with simple roots α_1 and $\alpha_3, \dots, \alpha_p$. If R is of type D_4 then there are three irreducible components of R° with simple roots α_1 and α_3 and α_4. In each case the assertion is true. □

Lemma 8.4. *We have three integer sequences associated to X:*

$$d_1 \geq d_2 \geq \cdots \geq d_k \geq 1$$
$$\ell_P(s_{\theta_1}) > \ell_P(s_{\theta_2}) > \cdots > \ell_P(s_{\theta_k}) > 0$$
$$c_1(X) > c_1(X^2) > \cdots > c_1(X^k) > 1$$

Proof. It is clear that $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_k$ and that $\theta_i \in \mathcal{B}$ for all i. Therefore Lemma 6.67 implies $d_1 \geq d_2 \geq \cdots \geq d_k$. Since $\theta_k \notin R_P$ it is clear that $d_k \geq 1$. This proves the first integer sequence. Let $2 \leq i \leq k$. Since $\operatorname{supp}(\theta_i) \subsetneq \operatorname{supp}(\theta_{i-1})$ we know that θ_i is not P_{i-1}-cosmall in X^{i-1} (Remark 6.20) which means that $\ell_P(s_{\theta_i}) < c_1(X^{i-1})d_i - 1 \leq c_1(X^{i-1})d_{i-1} - 1 = \ell_P(s_{\theta_i})$. Since $s_{\theta_k} \notin W_P$ it is clear that $\ell_P(s_{\theta_k}) > 0$. This proves the second integer sequence. The third one follows from the inequality $\ell_P(s_{\theta_i}) = c_1(X^i)d_i - 1 < c_1(X^{i-1})d_i - 1$. It is clear that $c_1(X^k) > 0$ since otherwise $\ell_P(s_{\theta_k}) < 0$. To prove that $c_1(X^k) > 1$, note that θ_k is locally P-cosmall as it is an element of $\mathcal{B}$ and thus $\delta(s_{\theta_k}) = d_k$. On the other hand we know that $\delta(s_{\theta_k}) \leq \ell_P(s_{\theta_k}) = c_1(X^k)d_k - 1$. Putting these inequalities together, we get $1 \leq (c_1(X^k) - 1)d_k$ which implies $c_1(X^k) > 1$. □

Corollary 8.5. *If $X = G/P$ is a cominuscule homogeneous space, then $d_1 = \cdots = d_k = 1$.*

Proof. Let $X = G/P$ be a cominuscule homogeneous space. Then α_P and θ_1 are both long roots. Therefore we have $(\theta_1, \theta_1) = (\alpha_P, \alpha_P) = (\omega, \omega)$ and thus $d_1 = \langle \theta_1^\vee, \omega \rangle = \langle \theta_1, \omega^\vee \rangle = 1$. Since $d_1 \geq \cdots \geq d_k \geq 1$ it follows that $d_1 = \cdots = d_k = 1$ as claimed. □

Lemma 8.6. *Suppose that $X \neq G_2/P_1$ and that $X \neq B_\ell/P_\ell$ where $\ell > 1$ is odd. All elements of the θ-sequence are long. In particular, for all $1 \leq i \leq k$ we have $\langle \theta_i^\vee, \alpha \rangle \in \{-1, 0, 1\}$ for all $\alpha \in R \setminus \{\pm\theta_i\}$.*

Proof. This follows directly from Lemma 4.7 and its proof. □

Lemma 8.7. *For all* $1 \le i \le k$*, we have the following equation:*

$$w_o W_P\big|_{\mathfrak{g}_i/\mathfrak{p}_i} = s_{\theta_i} \cdots s_{\theta_k} W_P\big|_{\mathfrak{g}_i/\mathfrak{p}_i} .$$

Proof. By [21, Proposition 1.10] we know that $w_o = \prod_{\beta \in \mathcal{B}} s_\beta$. All elements $\beta \in \mathcal{B} \setminus C(\alpha_P)$ are contained in R_P^+. Therefore we obtain $w_o W_P = s_{\theta_1} \cdots s_{\theta_k} W_P$. Since s_{θ_j} acts trivially on $\mathfrak{g}_i$ for all $j < i$ the result follows if we restrict this equation to $\mathfrak{g}_i/\mathfrak{p}_i$. □

Corollary 8.8. *For all* $1 \le i \le k$*, we have that* $X^i = X_{w_o(\theta_i)} = X_{s_{\theta_i} \cdots s_{\theta_k}}$.

Proof. This is an immediate consequence of the previous lemma and Corollary 4.9. □

Lemma 8.9 ([25, Lemma 9.4])**.** *Let* $1 \le i \le k$*. Then the* P*-length is additive in the sense that*

$$\ell_P(s_{\theta_i} \cdots s_{\theta_k}) = \sum_{j=i}^{k} \ell_P(s_{\theta_j}) .$$

For each $w \in W$ we let $I(w) = \{\alpha \in R^+ \mid w(\alpha) < 0\}$ denote the inversion set of w. We then have $\ell(w) = \mathrm{card}(I(w))$ or more generally $\ell_P(w) = \mathrm{card}(I(w) \setminus R_P^+)$.

Proof. Note that $\ell_P = \ell_{P_i}$ on W_{G_i}. By replacing R_i with R it therefore suffices to show that

$$\ell_P(s_{\theta_1} \cdots s_{\theta_k}) = \sum_{i=1}^{k} \ell_P(s_{\theta_i}) .$$

Let $u = s_{\theta_1} \cdots s_{\theta_k}$ for short. Each of the inversion sets $I(s_{\theta_i})$ is contained in R_i since $s_{\theta_i} \in W_{G_i}$ and since $\ell_B = \ell_{B_i}$ on W_{G_i}. For all $1 \le i \le k$ let $I_i = I(s_{\theta_i}) \setminus R_{P_i}^+$. To prove the claimed equality it suffices to show that $I(u) \setminus R_P^+$ can be written as a disjoint union of the sets I_i as follows: $I(u) \setminus R_P^+ = \coprod_{i=1}^k I_i$. We first prove that the sets I_i are pairwise disjoint for different i.

Let $\gamma \in I_i$ for some i. To see that the sets I_i are pairwise disjoint, it is clearly sufficient to prove that $\gamma \notin I_j$ for all $j > i$. By [9, Theorem 6.1(c)] we know that $\langle \theta_i^\vee, \gamma \rangle \in \{1, 2\}$ and that the value 2 occurs if and only if $\gamma = \theta_i$. In particular γ is not orthogonal to θ_i and thus not contained in R_j for any $j > i$, a fortiori not contained in I_j for any $j > i$ $(I_j \subseteq R_j)$.

Next we prove that $I_i \subseteq I(u) \setminus R_P^+$ for all i. Let $\gamma \in I_i$. To see that $\gamma \in I(u) \setminus R_P^+$ it is sufficent to prove that $u(\gamma) < 0$. Since $\theta_1, \ldots, \theta_k$ are pairwise orthogonal and γ is orthogonal to θ_j for all $j < i$ as γ is an element of R_i we get that

$$u(\gamma) = s_{\theta_i} \cdots s_{\theta_k}(\gamma) = \gamma - \sum_{j=i}^{k} \left\langle \theta_j^\vee, \gamma \right\rangle \theta_j = s_{\theta_i}(\gamma) - \sum_{j=i+1}^{k} \left\langle \theta_j^\vee, \gamma \right\rangle \theta_j \,.$$

From this equation it follows immediately that

$$\langle \theta_i^\vee, u(\gamma) \rangle = \langle \theta_i^\vee, s_{\theta_i}(\gamma) \rangle = - \langle \theta_i^\vee, \gamma \rangle$$

which implies that

$$s_{\theta_i} u(\gamma) = u(\gamma) + \langle \theta_i^\vee, \gamma \rangle \, \theta_i \,.$$

By [9, loc. cit.] we know that $\langle \theta_i^\vee, \gamma \rangle \in \{1, 2\}$ and that the value 2 occurs if and only if $\gamma = \theta_i$. Since $\gamma \in R_i$ we also know that $s_{\theta_i} u(\gamma) = s_{\theta_{i+1}} \cdots s_{\theta_k}(\gamma) \in R_i$. Suppose for a contradiction that $u(\gamma) > 0$. By what we said before, we then have $s_{\theta_i} u(\gamma) > \theta_i$ and $s_{\theta_i} u(\gamma) \in R_i$. Since this contradicts the fact that θ_i is the highest root of R_i we must have $u(\gamma) < 0$. This shows the desired inclusion $I_i \subseteq I(u) \setminus R_P^+$ for all i.

Finally we prove the inclusion $I(u) \setminus R_P^+ \subseteq \bigcup_{i=1}^{k} I_i$. Let $\gamma \in I(u) \setminus R_P^+$. The root γ cannot be orthogonal to θ_i for all i since otherwise we had $u(\gamma) = \gamma > 0$. Let i be the smallest index such such that γ is not orthogonal to θ_i. Then we have that $\gamma \in R^+ \setminus R_P^+$ is orthogonal to $\theta_1, \ldots, \theta_{i-1}$ but not orthogonal to θ_i. This means that $\gamma \in R_i^+ \setminus (R_{i+1}^+ \cup R_{P_i}^+)$ which in particular implies that $s_{\theta_i}(\gamma) \in R_i \setminus R_{i+1}$. Therefore we know that $\gamma \leq \theta_i$ and that $s_{\theta_i}(\gamma) = \gamma - \langle \theta_i^\vee, \gamma \rangle \, \theta_i \leq \theta_i$.These two inequalities imply that the nonzero number $\langle \theta_i^\vee, \gamma \rangle$ is positive and thus that $s_{\theta_i}(\gamma) < 0$. This means that $\gamma \in I_i$ as claimed. This completes the proof. □

Remark 8.10. Let F be a subset of $C(\alpha_P)$. Let $u = \prod_{\beta \in F} s_\beta$. Let r be the cardinality of the set F and let $1 \leq i_1 < \cdots < i_r \leq k$ be the sequence of integers such that $F = \{\theta_{i_1}, \ldots, \theta_{i_r}\}$. The proof of the previous lemma actually shows that $I(u) \setminus R_P^+ = \coprod_{j=1}^{r} I_{i_j}$ and in particular that

$$\ell_P \left(\prod_{\beta \in F} s_\beta \right) = \sum_{\beta \in F} \ell_P(s_\beta) \,.$$

To simplify notation and to avoid the use of double indices we have written out the arguments only for $F = C(\alpha_P)$ but the idea of the proof works analogously for any $F \subseteq C(\alpha_P)$. The P-length additivity for arbitrary subsets of $C(\alpha_P)$ directly generalizes to arbitrary subsets of $\mathcal{B}$ (cf. proof of Corollary 8.17).

Remark 8.11. The proof of the previous lemma also shows that we have $I_i = R_i^+ \setminus (R_{i+1}^+ \cup R_{P_i}^+)$ for all $1 \leq i \leq k$. Here we set by convention $R_{k+1} = \emptyset$. We will recover this equation in a more geometric context later on (cf. Section 8.3).

For all $1 \leq i \leq k$, we know that $z^i_{d_i}$ is the minimal length representative of s_{θ_i}. It is convenient to write $\tilde{s}_{\theta_i} = z^i_{d_i}$.

Lemma 8.12. *For all $1 \leq i \leq k$ the element $\tilde{s}_{\theta_k} \cdots \tilde{s}_{\theta_i}$ is a minimal length representative of $s_{\theta_i} \cdots s_{\theta_k}$. Moreover we have*

$$\tilde{s}_{\theta_k} \cdots \tilde{s}_{\theta_i} = \tilde{s}_{\theta_k} \cdot \ldots \cdot \tilde{s}_{\theta_i}\,.$$

Proof. Let $v_i \in W_P$ such that $\tilde{s}_{\theta_i} = s_{\theta_i} v_i$. Since $\tilde{s}_{\theta_i}, s_{\theta_i} \in W_{G_i}$ it is clear that $v_i \in W_{G_i} \cap W_P = W_{P_i}$. Since θ_i is orthogonal to all simple roots in Δ_j for all $j > i$, it follows that $s_{\theta_i} v_j = v_j s_{\theta_i}$ for all $j > i$. Using this we obtain

$$s_{\theta_i} \cdots s_{\theta_k} v_k \cdots v_i = s_{\theta_k} \cdots s_{\theta_i} v_k \cdots v_i = s_{\theta_k} v_k \cdots s_{\theta_i} v_i = \tilde{s}_{\theta_k} \cdots \tilde{s}_{\theta_i}$$

which means that $s_{\theta_i} \cdots s_{\theta_k}$ and $\tilde{s}_{\theta_k} \cdots \tilde{s}_{\theta_i}$ represent the same class modulo W_P. Using the P-length additivity this gives

$$\sum_{j=i}^{k} \ell(\tilde{s}_{\theta_j}) = \sum_{j=i}^{k} \ell_P(s_{\theta_j}) = \ell_P(s_{\theta_i} \cdots s_{\theta_k}) = \ell_P(\tilde{s}_{\theta_k} \cdots \tilde{s}_{\theta_i}) \leq \ell(\tilde{s}_{\theta_k} \cdots \tilde{s}_{\theta_i})$$

Since the other inequality is always satisfied, we get equality in the previous inequality, which means that $\tilde{s}_{\theta_k} \cdots \tilde{s}_{\theta_i}$ is a minimal length representative of $s_{\theta_i} \cdots s_{\theta_k}$ and that this element is length additive. By [9, Proposition 3.2] we get that $\tilde{s}_{\theta_k} \cdots \tilde{s}_{\theta_i} = \tilde{s}_{\theta_k} \cdot \ldots \cdot \tilde{s}_{\theta_i}$. ☐

Corollary 8.13. *We can express X^i for all $1 \leq i \leq k$ as iterated curve neighborhood of a point as follows:*

$$X^i = \Gamma^i_{d_i}(\Gamma^{i-1}_{d_{i-1}}(\cdots(\Gamma^k_{d_k}(X_1))\cdots))\,.$$

Proof. By the previous lemma and Corollary 8.8, we know that $X^i = X_{s_{\theta_i} \cdots s_{\theta_k}} = X_{\tilde{s}_{\theta_k} \cdot \ldots \cdot \tilde{s}_{\theta_i}}$. Since $\tilde{s}_{\theta_i} = z^i_{d_i}$ by definition, the statement is now just a reformulation in terms of curve neighborhoods. ☐

Lemma 8.14. *For all $1 \leq i \leq k$ the cohomology class $\sigma_{s_{\theta_i} \cdots s_{\theta_k}}$ is Poincaré dual to the cohomology class $\sigma_{s_{\theta_1} \cdots s_{\theta_{i-1}}}$ where we set for $i = 1$ the empty product equal to 1. In particular $[X^2]$ is Poincaré dual to $[\Gamma_{d_1}(X_1)]$.*

Proof. We already know that $w_o(\theta_i) W_P = s_{\theta_i} \cdots s_{\theta_k} W_P$. Since the minimal length representative of $s_{\theta_i} \cdots s_{\theta_k}$ is given by $\tilde{s}_{\theta_k} \cdots \tilde{s}_{\theta_i}$ it follows that $w_{X^i} = \tilde{s}_{\theta_k} \cdots \tilde{s}_{\theta_i}$. It is clear that w_{X^i} is Poincaré dual in X^i to the element 1. Therefore we have $w_o(\theta_i) w_{X^i} w_o(\theta_i) w_{X^i} = 1$. Since $w_o|_{\mathfrak{g}_i} = w_o(\theta_i)|_{\mathfrak{g}_i}$ it follows also that $w_o w_{X^i} w_o w_{X^i} = 1$. If we multiply this equation with $\tilde{s}_{\theta_{i-1}} \cdots \tilde{s}_{\theta_1}$ from the right, we obtain $w_o w_{X^i} w_o w_X = \tilde{s}_{\theta_{i-1}} \cdots \tilde{s}_{\theta_1}$ which means that w_{X^i} is Poincaré dual to $\tilde{s}_{\theta_{i-1}} \cdots \tilde{s}_{\theta_1}$ or equivalent that $s_{\theta_i} \cdots s_{\theta_k}$ is Poincaré dual to $s_{\theta_1} \cdots s_{\theta_{i-1}}$.

The statement in the last sentence is now obvious, since $X^2 = X_{s_{\theta_2} \cdots s_{\theta_k}}$ and $\Gamma_{d_1}(X_1) = X_{s_{\theta_1}}$. ☐

Example 8.15. Let $X = \mathbb{G}(3,6)$ and denote with $\alpha_1, \ldots, \alpha_5$ the simple roots of G. The θ-sequence is given by

$$\theta_1 = \alpha_1 + \cdots + \alpha_5\,,\ \theta_2 = \alpha_2 + \alpha_3 + \alpha_4\,,\ \theta_3 = \alpha_3\,.$$

We have the following Weyl group elements (with the notation explained bellow)

$$\begin{aligned} s_{\theta_1} &= (16) = (123654) = (2,2,0)\\ s_{\theta_2} &= (25) = (2354) = (3,2,1)\\ s_{\theta_3} &= (34) = (34) = (3,3,2)\\ s_{\theta_1}s_{\theta_2} &= (16)(25) = (1364)(25) = (1,0,0)\\ s_{\theta_2}s_{\theta_3} &= (25)(34) = (24)(35) = (3,1,1)\\ s_{\theta_1}s_{\theta_2}s_{\theta_3} &= (16)(25)(34) = (14)(25)(36) = (0,0,0) \end{aligned}$$

where the first equality is strict, the second equality in each line gives the minimal length representative modulo $W_P = S_3 \times S_3$ and the third equality gives the corresponding partition. Since $\mathcal{B} = C(\alpha_P)$ we know that $w_o = s_{\theta_1}s_{\theta_2}s_{\theta_3} = (16)(25)(34)$. Therefore we see that $w_P = w_o w_X = (13)(46)$. The homogeneous spaces X^2 and X^3 and the curve neighborhoods in X and X^2 are given by

$$X^2 = X_{s_{\theta_2}s_{\theta_3}} = X_{(3,1,1)} = \mathbb{G}(2,4)\,,\ X^3 = X_{s_{\theta_3}} = X_{(3,3,2)} = \mathbb{G}(1,2)\,,$$
$$\Gamma_1(X_1) = X_{s_{\theta_1}} = X_{(2,2,0)}\,,\ \Gamma_1^2(X_1) = X_{s_{\theta_2}} = X_{(3,2,1)}\,.$$

We can verify in this example the general results we obtained so far: $[\Gamma_1(X_1)]$ and $[X^2]$ are Poincaré dual to each other, $s_{\theta_1}s_{\theta_2}$ and s_{θ_3} are Poincaré dual to each other, we have that $\Gamma_1^2(X_1) = \Gamma_1(X_1) \cap X^2$ and that $\Gamma_1^3(X_1) = X^3 = \Gamma_1(X_1) \cap X^3$.

8.1. **Algorithmic computation of d_X.**

Proposition 8.16. *We have the following identities:*

$$d_X = \sum_{i=1}^{k} d_i \text{ and } \delta(w_X) = \sum_{i=1}^{k} \delta(s_{\theta_i})\,.$$

In particular, the chain $s_{\theta_1}\cdots s_{\theta_k}, s_{\theta_2}\cdots s_{\theta_k}, \ldots, s_{\theta_k}, 1$ from w_X to w_X is of degree $\delta(w_X)$.

Proof. The first formula is clearly equivalent to the second one by the results we obtain up to now. We prove these formulas by induction on k. If $k = 1$, then $s_{\theta_1}W_P = w_o W_P$ and we get $d_X = \delta(w_X) = \delta(s_{\theta_1}) = d_1$. Suppose that $k > 1$ and that the formula is proven for all integers smaller than k. By Lemma 6.54, we know that $\delta(w_X) = d_1 + \delta(s_{\theta_1}^*)$. On the other hand we have that $s_{\theta_1}^* W_P = s_{\theta_2}\cdots s_{\theta_k} W_P = w_o(\theta_2)W_P$. The induction hypotheses implies that $\delta_2(w_o(\theta_2)) = \sum_{i=2}^k d_i$. Moreover we know that $\delta_2 = \delta$ on W_{G_2}. Putting these facts together, we get

$$d_X = \delta(w_X) = d_1 + \delta(s^*_{\theta_1}) = d_1 + \delta(w_o(\theta_2)) = \sum_{i=1}^{k} d_i \,.$$

The very last statement is now obvious. □

Corollary 8.17. *Let F be a subset of $\mathcal{B}$. Then δ is additive in the sense that*

$$\delta\left(\prod_{\beta\in F} s_\beta\right) = \sum_{\beta\in F} \delta(s_\beta) = \sum_{\beta\in F} d(\beta)\,.$$

Proof. We only need to prove the first equality in the statement. The second equality is then obvious since we know that $\delta(s_\beta) = d(\beta)$ for all $\beta \in \mathcal{B}$. We first reduce to the case where F is a subset of $C(\alpha_P)$. Note that all elements of $\mathcal{B}\setminus C(\alpha_P)$ must be contained in R^+_P since they do not contain α_P in their support. Since s_β and $s_{\beta'}$ commute for all $\beta, \beta' \in \mathcal{B}$ this implies that

$$\left(\prod_{\beta\in F} s_\beta\right) W_P = \left(\prod_{\beta\in F\cap C(\alpha_P)} s_\beta\right) W_P\,.$$

Since δ depends only on the class modulo W_P the previous equation implies in particular that

$$\delta\left(\prod_{\beta\in F} s_\beta\right) = \delta\left(\prod_{\beta\in F\cap C(\alpha_P)} s_\beta\right)\,.$$

Moreover we have $\delta(s_\beta) = 0$ for all $\beta \in \mathcal{B}\setminus C(\alpha_P)$ since $s_\beta \in W_P$. Therefore it is clear that we have

$$\sum_{\beta\in F} \delta(s_\beta) = \sum_{\beta\in F\cap C(\alpha_P)} \delta(s_\beta)\,.$$

By replacing F with $F\cap C(\alpha_P)$ we may assume that $F \subseteq C(\alpha_P)$. Next we reduce to the case where $F = C(\alpha_P)$. The case $F = C(\alpha_P)$ is clear from the previous proposition since $\delta(w_X) = \delta(s_{\theta_1}\cdots s_{\theta_k})$. The triangle inequality for δ gives

$$\delta(s_{\theta_1}\cdots s_{\theta_k}) \le \delta\left(\prod_{\beta\in F} s_\beta\right) + \delta\left(\prod_{\beta\in C(\alpha_P)\setminus F} s_\beta\right) \le \sum_{i=1}^{k} \delta(s_{\theta_i})\,.$$

Since the left side is known to be equal to the right side, we get equality everywhere. In particular the claim follows. □

Corollary 8.18. *Let $\beta, \beta' \in \mathcal{B}$ such that $\beta \le \beta'$. Then $\delta(w_o(\beta)) \le \delta(w_o(\beta'))$.*

Proof. If $\beta \in R_P^+$ then $w_o(\beta) \in W_P$ and $\delta(w_o(\beta)) = 0$. There is nothing to prove in this case. Suppose that $\beta \in R^+ \backslash R_P$, then also $\beta' \in R^+ \backslash R_P$. Therefore we have $\beta, \beta' \in C(\alpha_P)$. Let $\beta = \theta_i$ and $\beta' = \theta_j$. Then $i \geq j$ since $\beta \leq \beta'$. It follows that

$$\delta(w_o(\beta)) = \sum_{l=i}^{k} \delta(s_{\theta_l}) \leq \sum_{l=j}^{k} \delta(s_{\theta_l}) = \delta(w_o(\beta')) \, .$$

□

Corollary 8.19. *Let f be a curve of degree d_X which passes through $\Omega_{s_{\theta_i} \cdots s_{\theta_k}}$ for all $1 \leq i \leq k$. Then f converges to the T-invariant curve associated to the chain $s_{\theta_1} \cdots s_{\theta_k}, s_{\theta_2} \cdots s_{\theta_k}, \ldots, s_{\theta_k}, 1$.*

Proof. We know that f converges to a T-invariant curve of degree d_X which passes through the T-fixed points

$$x(s_{\theta_1} \cdots s_{\theta_k}), x(s_{\theta_2} \cdots s_{\theta_k}), \ldots, x(s_{\theta_k}), x(1) \, .$$

But since s_{θ_i} are P-indecomposable for all $1 \leq i \leq k$, there is only one T-invariant curve passing through those T-fixed points, namely the T-invariant curve associated to the chain $s_{\theta_1} \cdots s_{\theta_k}, s_{\theta_2} \cdots s_{\theta_k}, \ldots, s_{\theta_k}, 1$. □

Proposition 8.20. *If $X = G/P$ is a cominuscule homogeneous space the following numbers are equal:*

- $d_X = k$
- *The dimension of a Cartan subspace of $\mathfrak{p}$, i.e. the dimension of a maximal abelian subspace of $\mathfrak{p}$ consisting of semisimple elements. (This is well defined since all the Cartan subspaces of $\mathfrak{p}$ are L-conjugates.)*
- *The number of occurences of s_{α_P} in a reduced expression of w_X. (This well defined since X is cominuscule.)*
- *The number of orbits of the isotropy representation.*
- *The rank* $\mathrm{rk}(X)$ *of X.*

Proof. That $d_X = k$ is clear since $d_1 = \cdots = d_k = 1$ for a cominuscule homogeneous space X. Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{p}$. That $k = \dim(\mathfrak{a})$ was proved in [28, Proposition 2.1(3)]. That $d_X = \delta(w_X)$ equals the number of occurences of s_{α_P} in a reduced expression of w_X was proved in [11, Lemma 21]. That $\dim(\mathfrak{a})$ equals the number of orbits of the isotropy representation was proved in [16, 6.2]. That $\dim(\mathfrak{a})$ equals the rank $\mathrm{rk}(X)$ of X was proved in [30, Proposition 26.7]. □

Since P is maximal, we know that B consists of a unique root α and that B^* consists also of a unique root $\beta = \varphi(\alpha)$. We denote the degree by $d(B) = d(\alpha) = d(\beta)$. In the same way we associated to X the sets B and B^* we can associate to X^i sets B^i and B^{i*}. We denote the degree of the unique element of B^i (or B^{i*}) by $d(B^i)$. Since θ_k is

P-indecomposable, it is easy to see that $B^k = B^{k*} = \{\theta_k\}$ and thus $d(B^k) = d_k$.

Lemma 8.21. *We have the following ascending integer sequence:*

$$d(B) \leq d(B^2) \leq \cdots \leq d(B^k) = d_k \,.$$

Proof. It is clearly enough to prove that $d(B) \leq d(B^2)$. If we want to prove that $d(B^i) \leq d(B^{i+1})$ we just replace X with X^i. Let $\alpha' \in B^2$. Let $u'_0, \ldots, u'_r$ be a chain in X^2 from w_{X^2} to w_{X^2} of degree $d_{X^2} = \sum_{i=2}^k d_i$ such that $u'_1 = w_o(\theta_2)s_{\alpha'}$. Then $u_0 = s_{\theta_1}u'_0, \ldots, u_r = s_{\theta_1}u'_r, u_{r+1} = u'_r$ is a chain in X from w_X to w_X of degree $d_X = \sum_{i=1}^k d_i$. Then we have $u_1 \in \mathcal{U}$ and $\delta(u_1^*) = d(\alpha') = d(B^2)$. From the definition it then follows that $d(B) \leq d(B^2)$. □

Corollary 8.22. *If X is cominuscule we have $B^i = \{\theta_i\}$ for all $1 \leq i \leq k$. In particular, $d(B) = \cdots = d(B^k) = d_k = \cdots = d_1 = 1$.*

Proof. Since X is cominuscule, we know that $d_k = 1$. Since $d(B) > 0$ by definition, it follows that $d(B) = \cdots = d(B^k) = d_k = \cdots = d_1 = 1$. By definition the unique element of B^i is the unique maximal root of $d(B^i) = 1$ contained in $R_i^+ \setminus R_{P_i}^+$. Therefore it is clear that $B^i = \{\theta_i\}$ as claimed. □

Lemma 8.23. *Let β be the unique element of B^*. Then we have $s_\beta \preceq s_{\theta_i}$ for all $1 \leq i \leq k$. This means in particular that $\beta \in R_k^+ \setminus R_{P_k}^+$, $\beta \leq \theta_k$ and that β is orthogonal to $\theta_1, \ldots, \theta_{k-1}$ but not orthogonal to θ_k.*

Proof. Let us first prove that $s_\beta \preceq s_{\theta_k}$. Let α be the unique element of B. Since $d(B) \leq d_k$ we have that $\delta(s_\alpha^*) \geq \sum_{i=1}^{k-1} d_i$. This shows that $s_{\theta_1} \cdots s_{\theta_{k-1}} \preceq z_{\delta(s_\alpha^*)}$. By dualizing this inequality, we get that $s_\beta \preceq s_{\theta_k}$. This means that $\beta \in R_k^+ \setminus R_{P_k}^+$, in particular that $\beta \in R_i^+ \setminus R_{P_i}^+$ for all $1 \leq i \leq k$. Once we know this it is clear that $s_\beta \preceq z_{d(B)}^i$. Since $d(B) \leq d_k \leq d_i$ this implies that $s_\beta \preceq z_{d_i}^i$ which means that $s_\beta \preceq s_{\theta_i}$. The last sentence of the statement is now obvious. □

Lemma 8.24. *Let $z = z_{d(B)}^k$. We clearly have $z \preceq s_{\theta_k}$. Moreover we have $z^* \in \mathcal{U}_1$. Let β' be the unique root in $R^+ \setminus R_P^+$ such that $zW_P = s_{\beta'}W_P$ (which exists since z is P-indecomposable). The root β' is locally P-cosmall and contained in $R_k^+ \setminus R_{P_k}^+$. Moreover we have $\beta \leq \beta' \leq \alpha$ and $\beta \leq \beta' \leq \theta_k$.*

Proof. Let $z = z_{d(B)}^k$. From the definition it is clear that $z \preceq s_{\theta_k}$. We prove that $z^* \in \mathcal{U}_1$. Since $\delta = \delta_k$ on W_{G_k} we clearly have $\delta(z) = \delta_k(z)$. On the other hand we know that $\delta_k(z) \leq d(B)$ from the definition of z. Therefore we get $\delta(z) \leq d(B)$. Since $\delta_k(s_\beta) = \delta(s_\beta) = d(\beta) = d(B)$, we get $s_\beta \preceq z$. On the other hand it is clear that $z \preceq z_{d(B)}$ and thus

$z \preceq s_\alpha$. This means that z is in the interval $s_\beta \preceq z \preceq s_\alpha$. By what we saw before this implies that $z^* \in \mathcal{U}_1$.

Let $\beta' \in R^+ \setminus R_P^+$ be as in the statement. From the definition it is clear that $\beta' \in \mathcal{U}_1^*$ and thus β' is P-indecomposable. It is also clear that $\beta' \in R_k^+ \setminus R_{P_k}^+$. Since β' is P-indecomposable and since $s_{\beta'}W_P = zW_P$, it follows that β' is P_k-cosmall in X^k and thus locally P-cosmall. That $\beta' \leq \theta_k$ is clear since $\beta' \in R_k$. For all roots $\gamma \in \mathcal{U}_1^*$ we know that $\gamma \leq \alpha$. In particular, we know that $\beta' \leq \alpha$. We are left to show that $\beta \leq \beta'$. By definition, β' is the unique maximal root of $d(\beta') = d(\beta) = d(B)$ in $R_k^+ \setminus R_{P_k}^+$. Therefore it follows that $\beta \leq \beta'$. □

Remark 8.25. It is unclear if it can happen that $d(B) < d_k$. We are not able to provide an example with justification of this behaviour. If $d(B) = d_k$ (which happens for example if $k = 1$, if X is cominuscule or more generally if $d_k = 1$) we can say that $\theta_k \in \mathcal{U}_1^*$, since $zW_P = s_{\theta_k}W_P$ in this case.

Example 8.26. Suppose that $d_X = k = 2$. Then it is clear that $d_1 = d_2 = 1$ and thus $d(B) = 1$. Therefore B consists of the unique maximal root of 1 which is the highest root θ_1. Since θ_1 is P-cosmall we have that $s_{\theta_1}W_P = z_1W_P$. Moreover $s_{\theta_1}^*W_P = s_{\theta_2}W_P$. Using this we see from the definition of B^* that $B^* = \{\theta_2\}$ in other words that $\varphi(\theta_1) = \theta_2$. Corollary 6.56 then gives us that

$$\mathcal{U}_1^* = \{\beta \in R^+ \setminus R_P^+ \mid s_{\theta_2} \preceq s_\beta \preceq s_{\theta_1}\}$$

or equivalently that

$$\mathcal{U}_1 = \{u \in W/W_P \mid s_{\theta_2} \preceq u^* \preceq s_{\theta_1}\}.$$

In particular we see that the intervall in the Bruhat order which describes $\mathcal{U}_1$ and $\mathcal{U}_1^*$ is nontrivial in the sense that it consists of more than one element.

8.2. **The structure of Ω.** In this subsection we will always suppose that $k \geq 2$. For all $1 \leq i \leq k$ let $I_i = \Delta_i \setminus \Delta_{i+1}$ where we set $\Delta_{k+1} = \emptyset$. Let P_i' be the parabolic subgroup of G_i associated to Δ_{i+1}: $P_i' = (P_i)_{\Delta_{i+1}} = (P_i)_{\Delta_i \setminus I_i}$. Let $\Omega_i = P_i' w_o(\theta_i)P/P$ be the P_i'-orbit in X parametrized by $w_o(\theta_i)$.

Note that Ω_i is open and dense in X^i. Indeed, Ω_i is dense in X^i since it contains the open and dense subvariety $\Omega_{w_o(\theta_i)}$ of X^i. Moreover Ω_i is open since it is locally closed and the closure of Ω_i is X^i.

To simplify notation we write $I = I_1$, $P' = P_1'$ and $\Omega = \Omega_1 = P'w_oP/P$. Let P'^- be the parabolic subgroup of G opposite to P'. Let L' be the Levi factor of P' (or P'^-). Let V' be the unipotent radical of P' and let V'^- be the unipotent radical of P'^-.

Since $w_o|_{\mathfrak{g}_i} = w_o(\theta_i)|_{\mathfrak{g}_i}$ for all $1 \leq i \leq k$ we know that the Weyl involution ι leaves stable Δ_i. In particular ι leaves stable I_i.

For a root $\beta \in R$ we may write β as an integral linear combination of simple roots. We denote the coefficient of a simple root α in this expression by $n_\alpha(\beta)$. With this notation we have

$$\beta = \sum_{\alpha \in \Delta} n_\alpha(\beta)\alpha \,.$$

Depending wether β is a positive or a negative root, either all coefficients $n_\alpha(\beta)$ are non negative or non positive. If $S \subseteq \Delta$ is a subset of simple roots, we set

$$n_S(\beta) = \sum_{\alpha \in S} n_\alpha(\beta) \,.$$

Fact 8.27. *We have two trivial identities:*

$$\{\beta \in R \mid n_I(\beta) > 0\} = R^+ \setminus R_2^+$$
$$\{\beta \in R \mid n_I(\beta) < 0\} = R^- \setminus R_2^-$$

Proof. The second identity is a trivial consequence of the first one and vice versa. We prove the first identity. Note first that

$$\{\beta \in R \mid n_I(\beta) > 0\} = \{\beta \in R^+ \mid n_I(\beta) > 0\}$$

since every root β which has at least one positive coefficient $n_\alpha(\beta)$ must be positive. Therefore the desired equality follows from the equality

$$\{\beta \in R^+ \mid n_I(\beta) = 0\} = R_2^+$$

by taking complements. The inclusion from right to left is obvious. A root $\beta \in R^+$ which satisfies $n_I(\beta) = 0$ must be a linear combination of simple roots in Δ_2 (because $\Delta = I \amalg \Delta_2$). This proves that a root $\beta \in R^+$ which satisfies $n_I(\beta) = 0$ is contained in R_2^+. This shows the inclusion from left to right. □

Corollary 8.28. *We have the following identities:*

$$V' = \prod_{\beta \in R^+ \setminus R_2^+} U_\beta \,,\; V'^- = \prod_{\beta \in R^- \setminus R_2^-} U_\beta \,,$$

where U_β denotes as usual the unique T-stable subgroup of G having Lie algebra $\mathfrak{g}_\beta$.

Proof. The first identity clearly follows from the second one and vice versa since $w_o(R^+ \setminus R_2^+) = R^- \setminus R_2^-$. We prove the first identity. From [3, Proposition 4.7] it follows that

$$V' = \prod_{\beta \in R:\, n_I(\beta) > 0} U_\beta \,.$$

The result is now a trivial consequence of the previous fact. □

Corollary 8.29. *The groups V' and V'^- are stable under conjugation with elements of W_{G_2}: for all $w \in W_{G_2}$ we have $wV'w^{-1} = V'$, $wV'^-w^{-1} = V'^-$ or equivalent $wV' = V'w$, $wV'^- = V'^-w$.*

Proof. Let $w \in W_{G_2}$. By the previous corollary it clearly suffices to prove that $w(R^+ \setminus R_2^+) = R^+ \setminus R_2^+$ and $w(R^- \setminus R_2^-) = R^- \setminus R_2^-$. The first equation is a trivial consequence of the second one and vice versa. We prove the first equation. Since $w \in W_{G_2}$ we know that w leaves stable R, R_2 and therefore also $R \setminus R_2$. Since $\ell_B = \ell_{B_2}$ on W_{G_2} it is clear that no positive root in $R^+ \setminus R_2^+$ can be mapped to a negative root. Therefore the desired equality follows. □

Lemma 8.30. *Let α be a positive root such that $s_\alpha s_{\theta_1} \cdots s_{\theta_k} \in W_{G_2}$ (or equivalently such that $s_\alpha s_{\theta_1} \in W_{G_2}$). Then $\alpha = \theta_1$.*

Proof. By assumption $s_\alpha s_{\theta_1}$ is an element of W_{G_2}, therefore it follows from the definition that $s_\alpha s_{\theta_1}(\theta_1) = \theta_1$. On the other hand we have by direct computation that $s_\alpha s_{\theta_1}(\theta_1) = -\theta_1 + \langle \alpha^\vee, \theta_1 \rangle \alpha$. So we conclude that $2\theta_1 = \langle \alpha^\vee, \theta_1 \rangle \alpha$. Since θ_1 is the highest root and α is positive, this is only posible if $\langle \alpha^\vee, \theta_1 \rangle \geq 2$. If $\langle \alpha^\vee, \theta_1 \rangle = 2$ it follows that $\alpha = \theta_1$ as claimed. Assume that $\langle \alpha^\vee, \theta_1 \rangle > 2$. Then we have necessarily that $\langle \alpha^\vee, \theta_1 \rangle = 3$ and R is of type G_2. By assumption $k \geq 2$ so that necessarily $\alpha_P = \alpha_1$. Therefore R_2 is of type A_1 and $W_{G_2} = \{1, s_{\alpha_1}\}$. If $s_\alpha s_{\theta_1} = 1$ it follows that $\alpha = \theta_1$ as claimed. Assume that $s_\alpha s_{\theta_1} = s_{\alpha_1}$ then we get by evaluating this equation at α_1 that $\alpha_1 - \langle \alpha^\vee, \alpha_1 \rangle \alpha = -\alpha_1$ and thus $2\alpha_1 = \langle \alpha^\vee, \alpha_1 \rangle \alpha$. This is only possible if $\langle \alpha^\vee, \alpha_1 \rangle = 2$ and $\alpha = \alpha_1$. The equation $s_\alpha s_{\theta_1} = s_{\alpha_1}$ then becomes $s_{\theta_1} = 1$ which is absurd. □

Lemma 8.31. *For all elements $w \in W_{G_2}$ we have that $s_{\theta_1} \preceq w s_{\theta_1} \cdots s_{\theta_k}$.*

Proof. Let $w \in W_{G_2}$. Since s_{θ_1} and $s_{\theta_2} \cdots s_{\theta_k}$ are Poincaré dual to each other, the statement $s_{\theta_1} \preceq w s_{\theta_1} \cdots s_{\theta_k}$ is equivalent to the statement $s_{\theta_2} \cdots s_{\theta_k} \succeq w_o w s_{\theta_1} \cdots s_{\theta_k}$. Since $s_{\theta_1} \cdots s_{\theta_k}$ is Poincaré dual to 1 the expression $w_o w s_{\theta_1} \cdots s_{\theta_k}$ is congruent to $w_o w w_o$ modulo W_P. Since $w_o|_{\mathfrak{g}_2} = w_o(\theta_2)|_{\mathfrak{g}_2}$ we have that $w_o w w_o \in W_{G_2}$. The statement $s_{\theta_2} \cdots s_{\theta_k} \succeq w_o w w_o$ therefore becomes obvious since $s_{\theta_2} \cdots s_{\theta_k}$ is the maximal element with respect to the Bruhat order on W_{G_2} (in other words since $X_{s_{\theta_2} \cdots s_{\theta_k}} = X^2$). □

Corollary 8.32. *For all elements $w \in W_{G_2}$ we have that $s_{\theta_1} \preceq w s_{\theta_1}$.*

Proof. This is an immediate consequence of the lemma since $w s_{\theta_1} = (w s_{\theta_2} \cdots s_{\theta_k}) s_{\theta_1} \cdots s_{\theta_k}$ and $w s_{\theta_2} \cdots s_{\theta_k} \in W_{G_2}$. □

Lemma 8.33. *The open dense P'-orbit Ω parametrized by w_o decomposes into a finite disjoint union of Schubert cells as follows:*

$$\Omega = \coprod_{s_{\theta_1} \preceq v} \Omega_v \,.$$

Proof. By [23, Proposition 2] Ω can be written as a disjoint union of Schubert cells parametrized by an intervall in the Bruhat order:

$$\Omega = \coprod_{w_m \preceq v \preceq w_M} \Omega_v \,.$$

where $w_m, w_M \in W/W_P$. Since $\Omega_{w_X} \subseteq \Omega$ and w_X is the unique maximal element in the Bruhat order, we only have to determine w_m. To this end, let $\Omega^* = w_o\Omega$. Then we can write

$$\Omega^* = \coprod_{v \preceq w_m^*} B^- vP/P\,.$$

On the other hand the B^--orbits of $\Omega^* = P'^-P/P$ are parametrized by $W_{G_2}W_P/W_P \cong W_{G_2}/W_{P_2}$ since $W_{P'^-} = W_{P'} = W_{L'} = W_{G_2}$ and $W_{G_2} \cap W_P = W_{P_2}$. The unique maximal element in W_{G_2} with respect to the Bruhat order is given by $s_{\theta_2} \cdots s_{\theta_k}$ since $X_{s_{\theta_2} \cdots s_{\theta_k}} = X^2$. Therefore we conclude that $w_m^* = s_{\theta_2} \cdots s_{\theta_k}$ and thus $w_m = s_{\theta_1}$ as claimed. □

Corollary 8.34. *We have that*

$$\Gamma_{d_1}(X_1) \cap \Omega = \Omega_{s_{\theta_1}}\,.$$

Proof. Since $\Gamma_{d_1}(X_1) = X_{s_{\theta_1}}$ the intersection $\Gamma_{d_1}(X_1) \cap \Omega = X_{s_{\theta_1}} \cap \Omega$ is B-stable, thus a union of Schubert cells parametrized by Weyl group elements v which both satisfy $v \preceq s_{\theta_1}$ (since $\Omega_v \subseteq X_{s_{\theta_1}}$) and $v \succeq s_{\theta_1}$ (since $\Omega_v \subseteq \Omega$). There is only one such Schubert cell, namely $\Omega_{s_{\theta_1}}$. The claim follows. □

Corollary 8.35. *The T-fixed points of Ω and the T-fixed points of X^2 are in natural bijection:*

$$(X^2)^T = W_{G_2}/W_{P_2} \cong \{v \in W/W_P \mid s_{\theta_1} \preceq v\} = \Omega^T\,.$$

A bijection is given by sending $vW_{P_2} \in (X^2)^T$ to the element $vs_{\theta_1}W_P = s_{\theta_1}vW_P$. In particular, for every element $w \succeq s_{\theta_1}$ there exists a $v \in W_{G_2}$ such that $wW_P = vs_{\theta_1}W_P = s_{\theta_1}vW_P$.

Proof. We already saw that we can identify the T-fixed points of Ω and the T-fixed points of X^2 with the sets described in the statement. Moreover it is clear that we can write

$$W_{G_2}/W_{P_2} = \{w \in W/W_P \mid w \preceq s_{\theta_2} \cdots s_{\theta_k}\}\,.$$

Therefore it is obvious that Poincaré duality induces a bijection between the T-fixed points of Ω and the T-fixed points of X^2. In particular both sets have the same cardinality. Corollary 8.32 guarantees that the map which sends vW_{P_2} to $vs_{\theta_1}W_P$ is a well defined map from $(X^2)^T$ to Ω^T. Moreover this maps is clearly injective. Since source and target have the same cardinality, it must be also bijective. The statement in the last sentence is just a reformulation of the surjectivity of this map. □

According to this results we can write Ω as a disjoint union of Schubert cells as follows:

$$\Omega = \coprod_{v \in W_{G_2}} \Omega_{vs_{\theta_1}}$$

or more generally

$$\Omega_i = \coprod_{s_{\theta_i} \preceq v \preceq s_{\theta_i} \cdots s_{\theta_k}} \Omega_v = \coprod_{v \in W_{G_{i+1}}} \Omega_{v s_{\theta_i}}$$

for all $1 \leq i \leq k-1$.

Lemma 8.36. *Let w be a Weyl group element such that $w \succeq s_{\theta_1}$ and let $\alpha \in R^+ \setminus R_P^+$ be a root such that $s_\alpha w \in W_{G_2}$. Then $\alpha = \theta_1$.*

Proof. Let w and α be as in the statement. If $s_\alpha w \in W_{G_2}$ then we also have $s_\alpha w' \in W_{G_2}$ for every other element w' in the class wW_P. Therefore we can assume by the previous corollary that $w = s_{\theta_1} v$ for some $v \in W_{G_2}$ such that $s_\alpha w \in W_{G_2}$. But this immediately implies that $s_\alpha s_{\theta_1} \in W_{G_2}$. Lemma 8.30 then shows that $\alpha = \theta_1$ as claimed. □

8.3. **The morphism g_1.** In this subsection we still assume that $k \geq 2$.

Lemma 8.37. *Ω is an open and dense subvariety of X with complement of codimension at least two which is contained in $X \setminus X^2$. Moreover there exists a surjective morphism*

$$g_1 \colon \Omega \to X^2 .$$

Proof. We already saw that Ω is an open and dense subvariety of X. We have a trivial decomposition of Δ into a disjoint union: $\Delta = I \amalg \Delta_2$. Since ι leaves stable I and since $\alpha_P \in \Delta_2$, it follows that $\iota(I) \cap \{\alpha_P\} = I \cap \{\alpha_P\} = \emptyset$. [24, Proposition 6] then implies that the complement of Ω in X is of codimension at least two. In order to see that $\Omega \subseteq X \setminus X^2$, we prove the stronger statement that $\Omega \subseteq X \setminus \overline{P' u_2 P/P}$ where $u_2 = s_{\theta_2} \cdots s_{\theta_k}$ for short. Suppose this inclusion does not hold, then it follows that $\Omega \subseteq \overline{P' u_2 P/P}$ because of the structure of the P'-orbits in X. This implies that $\Omega = P' u_2 P/P$ since Ω is the maximal P'-orbit in X. By the structure of Ω this means that $s_{\theta_1} \preceq u_2$ which implies that $s_{\theta_1} \in W_{G_2}$ and that $\theta_1 \in R_2^+$ – a contradiction.

We now introduce the surjective morphism $g_1 \colon \Omega \to X^2$. We consider the composition of surjective morphisms

$$\Omega \xrightarrow{\sim} P'^- P/P \cong P'^-/P'^- \cap P \to L'/L' \cap P$$

where the first morphism is given by multiplication with $w_o^{-1} = w_o$ from the left and the third morphism is the projection onto the Levi factor L'. Because of the decomposition of $\Delta = I \amalg \Delta_2$ it is clear that L' decomposes as

$$L' \cong (\mathbb{G}_m)^I \times G_2 .$$

This isomorphism sends $L' \cap P$ to a product of $(\mathbb{G}_m)^I$ and the parabolic subgroup of G_2 associated to α_P (which makes sense, since $\alpha_P \in \Delta_2$) which is P_2. Thus $L'/L' \cap P \cong G_2/P_2 = X^2$. In total this defines a surjective morphism $g_1 \colon \Omega \to X^2$. □

It is clear that we can introduce analogously for each $1 \leq i \leq k-1$ surjective morphisms

$$g_i \colon \Omega_i \to X^{i+1}\,.$$

These morphism satisfy analogous properties as we will state for g_1. To simplify notation we will usually consider only the case $i = 1$ in what follows.

Corollary 8.38. *For all* $1 \leq i \leq k$ *we have the following formulas*

$$\ell_P(s_{\theta_i}) = \operatorname{card}(R_i^+ \setminus (R_{i+1}^+ \cup R_{P_i}^+))$$

where we set $R_{k+1} = \emptyset$*.*

Proof. The formula for $i = k$ follows since $X^k = X_{s_{\theta_k}}$. By replacing X with X^i we can assume without loss of generality that $i = 1$. From the definition of g_1 it is clear that the fiber of g_1 is isomorphic to $V'^-/V'^- \cap P$. From the structure of V'^- we see that

$$\dim(V'^-/V'^- \cap P) = \operatorname{card}(R^- \setminus (R_2^- \cup R_P^-)) = \operatorname{card}(R^+ \setminus (R_2^+ \cup R_P^+))\,.$$

Since Ω is dense in X, we know that $\dim(X) = \dim(\Omega)$. Therefore the surjective morphism g_1 yields the equation

$$\dim(X) = \dim(V'^-/V'^- \cap P) + \dim(X^2)\,.$$

From the P-length additvity, we already know that

$$\dim(X^i) = \ell_P(s_{\theta_i} \cdots s_{\theta_k}) = \sum_{j=i}^{k} \ell_P(s_{\theta_j})$$

for all $1 \leq i \leq k$. Therefore we have

$$\dim(X) - \dim(X^2) = \ell_P(s_{\theta_1})\,.$$

Putting all these equation together the desired equation with $i = 1$ follows. □

Corollary 8.39. *For all* $1 \leq i \leq k$ *we have an equality:*

$$I(s_{\theta_i}) \setminus R_{P_i}^+ = R_i^+ \setminus (R_{i+1}^+ \cup R_{P_i}^+)$$

where we set $R_{k+1} = \emptyset$*.*

Proof. We already saw that $I(s_{\theta_i}) \subseteq R_i^+$. Since all elements of R_{i+1} are orthogonal to θ_i they can not be part of the inversion set. It follows that $I(s_{\theta_i}) \subseteq R_i^+ \setminus R_{i+1}^+$ and thus $I(s_{\theta_i}) \setminus R_{P_i}^+ \subseteq R_i^+ \setminus (R_{i+1}^+ \cup R_{P_i}^+)$. By the previous corollary both sets of this inclusion have cardinality equal to $\ell_P(s_{\theta_i})$. Therefore the desired equality follows. □

From the previous equality we see once more that we have a disjoint union:

$$R^+ \setminus R_P^+ = \coprod_{i=1}^{k} I(s_{\theta_i}) \setminus R_{P_i}^+\,.$$

We already saw this in the proof of Lemma 8.9. It is of course also possible to prove the previous corollary directly without using the morphism g_1. As a direct consequence of the previous corollary and [9, Theorem 6.1(c)] we can say that $\langle \theta_i^\vee, \gamma\rangle \in \{1,2\}$ for all $\gamma \in R_i^+ \setminus (R_{i+1}^+ \cup R_{P_i}^+)$ and that the value 2 occurs if and only if $\gamma = \theta_i$.

Lemma 8.40. *The fiber of g_1 is isomorphic to the degree d_1 curve neighborhood of a point in X intersected with Ω:*

$$V'^-/V'^- \cap P \cong V' w_o P/P = g_1^{-1}(x(1)) \cong \Omega_{s_{\theta_1}} = \Gamma_{d_1}(X_1) \cap \Omega\,.$$

Proof. It is clear from the definition of g_1 that the fiber of g_1 is isomorphic to $V'^-/V'^- \cap P$. We have an isomorphism

$$V'^-/V'^- \cap P \cong V'^- P/P \cong V' w_o P/P$$

where the second isomorphism is given by multiplication with w_o from the left. From the definition of g_1 it is clear that $g_1^{-1}(x(1)) = V' w_o P/P$. We already saw that $\Omega_{s_{\theta_1}} = \Gamma_{d_1}(X_1) \cap \Omega$. We are left to show that $V' w_o P/P \cong \Omega_{s_{\theta_1}}$. Since V' is stable under conjugation with elements of W_{G_2} we have

$$V' w_o P/P = V' s_{\theta_1} \cdots s_{\theta_k} P/P = s_{\theta_2} \cdots s_{\theta_k} V' s_{\theta_1} P/P\,.$$

This defines an isomorphism

$$V' w_o P/P \cong V' s_{\theta_1} P/P$$

which is given by multiplication with $s_{\theta_2} \cdots s_{\theta_k}$ from the left. Since $V' \subseteq B$ this gives an injective morphism $V' w_o P/P \hookrightarrow \Omega_{s_{\theta_1}}$. From the previous corollary we know that both varieties have equal dimension $\ell_P(s_{\theta_1})$. The unipotent radical V' is closed and connected hence irreducible. Therefore $V' w_o P/P$ is also irreducible. Since $\Omega_{s_{\theta_1}}$ is an irreducible Schubert cell, it follows that the injective morphism $V' w_o P/P \hookrightarrow \Omega_{s_{\theta_1}}$ is actually an isomorphism as claimed. □

Remark 8.41. The morphism g_1 was first introduced in a more general setup in [24, Proposition 5]. It was proved there that g_1 is a tower of affine bundles.

8.4. **The sets O_i.** In this subsection we still suppose that $k \geq 2$. Moreover we assume that $X \neq G_2/P_1$ and that $X \neq B_\ell/P_\ell$ where $\ell > 1$ is odd so that we can freely use Lemma 8.6.

For all $2 \leq i \leq k$ we define the following sets of roots:

$$O_i = \{\beta \in R^+ \setminus (R_2^+ \cup R_P^+) \mid s_{\theta_i}(\beta) \in R_P\}\,.$$

We set $O_1 = \{\theta_1\}$.

Lemma 8.42. *Let $2 \leq i \leq k$. Then we have the following equality of sets:*

$$O_i = \{\beta \in R^+ \setminus (R_2^+ \cup R_P^+) \mid \beta = \theta_i + \alpha \text{ where } \alpha \in R_P\}\,.$$

For all $\beta \in O_i$ we have $\langle \theta_i^\vee, \beta \rangle = 1$. The relation between α and β is given by $s_{\theta_i}(\beta) = \alpha \in R_P^+$. The degree of $\beta \in O_i$ is given by $d(\beta) = d_i(\theta_i, \theta_i)/(\beta, \beta) \geq d_i$.

Proof. It is clear that $\theta_i \notin R^+ \setminus (R_2^+ \cup R_P^+)$. By Lemma 8.6 we have $\langle \theta_i^\vee, \beta \rangle \in \{-1, 0, 1\}$ for all $\beta \in R^+ \setminus (R_2^+ \cup R_P^+)$ in particular for all $\beta \in O_i$. If β satisfies in addition $s_{\theta_i}(\beta) \in R_P$, then it is clear that we must have $\langle \theta_i^\vee, \beta \rangle = 1$. It follows that $s_{\theta_i}(\beta) = \beta - \theta_i$ and thus $\beta = \theta_i + s_{\theta_i}(\beta)$ where $s_{\theta_i}(\beta) \in R_P$. This proves the inclusion from left to right. To prove the other inclusion, let $\beta \in R^+ \setminus (R_2^+ \cup R_P^+)$ such that $\beta = \theta_i + \alpha$ for some $\alpha \in R_P$. Then $\langle \theta_i^\vee, \beta \rangle = 2 + \langle \theta_i^\vee, \alpha \rangle$. Since $\alpha \neq \pm\theta_i$ as $\alpha \in R_P$ we see that $\langle \theta_i^\vee, \beta \rangle, \langle \theta_i^\vee, \alpha \rangle \in \{-1, 0, 1\}$. The only possibility that the equation is satisfied occurs if $\langle \theta_i^\vee, \beta \rangle = 1$ and $\langle \theta_i^\vee, \alpha \rangle = -1$. We conclude that $s_{\theta_i}(\beta) = \beta - \theta_i = \alpha \in R_P$. This proves the inclusion from right to left. It also proves that $\langle \theta_i^\vee, \beta \rangle = 1$ for all $\beta \in O_i$. The relation between α and β was already established. It is also clear that α is always positive since $\beta - \theta_i$ must be positive.

Let $\beta \in O_i$. We prove the equation for the degree of β. Since $s_{\theta_i}(\beta) \in R_P$ we have $d(s_{\theta_i}(\beta)) = 0$. On the other hand we have

$$\langle s_{\theta_i}(\beta)^\vee, \omega \rangle = \langle \beta^\vee, \omega - d_i\theta_i \rangle = d(\beta) - d_i \langle \beta^\vee, \theta_i \rangle$$

by the W-invariance of $\langle -, - \rangle$. Since $\langle \theta_i^\vee, \beta \rangle = 1$ we also have $\langle \beta^\vee, \theta_i \rangle = (\theta_i, \theta_i)/(\beta, \beta)$. Putting these equations together the desired equality $d(\beta) = d_i(\theta_i, \theta_i)/(\beta, \beta)$ follows. Since θ_i is long, it is clear that $(\theta_i, \theta_i) \geq (\beta, \beta)$ and thus $d(\beta) \geq d_i$. □

Lemma 8.43. *Let $2 \leq i \leq k$. The set $R^+ \setminus (R_2^+ \cup R_P^+ \cup O_i)$ is stable under the action of s_{θ_i}:*

$$s_{\theta_i}(R^+ \setminus (R_2^+ \cup R_P^+ \cup O_i)) = R^+ \setminus (R_2^+ \cup R_P^+ \cup O_i)\,.$$

Proof. We know that the inversion set $I(s_{\theta_i})$ is contained in R_i. Therefore we see that $s_{\theta_i}(R^+ \setminus R_i^+) = R^+ \setminus R_i^+$, in particular $s_{\theta_i}(R^+ \setminus (R_2^+ \cup R_P^+ \cup O_i)) \subseteq R^+$. Let $\beta \in R^+ \setminus (R_2^+ \cup R_P^+ \cup O_i)$. Since $\beta \notin O_i$ we know that $s_{\theta_i}(\beta) \notin R_P$ and thus $s_{\theta_i}(\beta) \in R^+ \setminus R_P^+$. Since $\beta \in R^+ \setminus (R_2^+ \cup R_P^+)$ we know that β is not orthogonal to θ_1. The equation $(s_{\theta_i}(\beta), \theta_1) = (\beta, \theta_1)$ shows that $s_{\theta_i}(\beta)$ is also not orthogonal to θ_1. Therefore we conclude that $s_{\theta_i}(\beta) \in R^+ \setminus (R_2^+ \cup R_P^+)$. If $s_{\theta_i}(\beta) \in O_i$ then $\beta \in R_P$ which is inpossible by the choice of β. Therefore we conclude that $s_{\theta_i}(\beta) \in R^+ \setminus (R_2^+ \cup R_P^+ \cup O_i)$ as claimed. □

Corollary 8.44. *Let $2 \leq i \leq k$. We have the following equation:*

$$\sum_{\beta \in R^+ \setminus (R_2^+ \cup R_P^+ \cup O_i)} \langle \theta_i^\vee, \beta \rangle = 0\,.$$

Proof. By the previous lemma we have

$$\sum_{\beta\in R^+\setminus(R_2^+\cup R_P^+\cup O_i)} \langle\theta_i^\vee,\beta\rangle = \sum_{\beta\in R^+\setminus(R_2^+\cup R_P^+\cup O_i)} \langle\theta_i^\vee, s_{\theta_i}(\beta)\rangle = -\sum_{\beta\in R^+\setminus(R_2^+\cup R_P^+\cup O_i)} \langle\theta_i^\vee,\beta\rangle \ .$$

This immediately implies the desired equality. □

Corollary 8.45. *Let $2\le i\le k$. We have the following identity:*

$$\mathrm{card}(O_i) = (c_1(X) - c_1(X^2))d_i \,.$$

Proof. By [9, page 5, equation (3)] we know that

$$(c_1(X) - c_1(X^2))d_i = \sum_{\beta\in R^+\setminus(R_2^+\cup R_P^+)} \langle\theta_i^\vee,\beta\rangle \ .$$

Using the previous corollary, the fact that $\langle\theta_i^\vee,\beta\rangle = 1$ for all $\beta\in O_i$ and the fact that $O_i\subseteq R^+\setminus(R_2^+\cup R_P^+)$ we get

$$\sum_{\beta\in R^+\setminus(R_2^+\cup R_P^+)} \langle\theta_i^\vee,\beta\rangle = \sum_{\beta\in O_i}\langle\theta_i^\vee,\beta\rangle = \mathrm{card}(O_i)\,.$$

Both equation together yield the desired result. □

Lemma 8.46. *Let $\beta\in\bigcup_{i=2}^k O_i$. Then $\langle\theta_i^\vee,\beta\rangle = 1$ for precisely one index $2\le i\le k$. If $\beta\in O_i$ then $\left\langle\theta_j^\vee,\beta\right\rangle\in\{-1,0\}$ for all $2\le i\ne j\le k$. In particular the sets O_i are pairwise disjoint for all $1\le i\le k$.*

Proof. Suppose that $\beta\in O_i$ for some $2\le i\le k$. Let j be an index between 2 and k such that $\left\langle\theta_j^\vee,\beta\right\rangle = 1$. We first prove that $i=j$ in this situation. Suppose for a contradiction that $i\ne j$. Then we know that $s_{\theta_i}s_{\theta_j}(\beta) = \beta-\theta_i-\theta_j$ since $\langle\theta_i^\vee,\beta\rangle = \left\langle\theta_j^\vee,\beta\right\rangle = 1$. Moreover $\beta-\theta_i-\theta_j$ is a positive root since $n_I(\beta-\theta_i-\theta_j) = n_I(\beta) > 0$. It follows that $d(s_{\theta_i}s_{\theta_j}(\beta))\ge 0$. On the other hand, we have

$$\left\langle s_{\theta_i}s_{\theta_j}(\beta)^\vee,\omega\right\rangle = \langle\beta^\vee,\omega-d_i\theta_i-d_j\theta_j\rangle = d(\beta) - d_i\frac{(\theta_i,\theta_i)}{(\beta,\beta)} - d_j\frac{(\theta_j,\theta_j)}{(\beta,\beta)}$$

where last equality follows since $\langle\theta_i^\vee,\beta\rangle = \left\langle\theta_j^\vee,\beta\right\rangle = 1$. We know from Lemma 8.42 that $d(\beta) = d_i(\theta_i,\theta_i)/(\beta,\beta)$. Therefore it follows that $d(s_{\theta_i}s_{\theta_j}(\beta)) = -d_j(\theta_j,\theta_j)/(\beta,\beta) < 0$ – a contradiction.

By what we proved up to now the statement in the first sentence of the lemma is reduced to the statement that the union $\bigcup_{i=2}^k O_i$ is disjoint. Suppose that $\beta\in O_i\cap O_j$ where $2\le i,j\le k$. Then we have $\beta\in O_i$ such that $\left\langle\theta_j^\vee,\beta\right\rangle = 1$. By what we proved in the first paragraph we must have $i=j$. Therefore the sets O_i are pairwise disjoint for all $2\le i\le k$. This proves the statement in the first sentence of the lemma.

Since O_1 is clearly disjiont from O_i for all $2\le i\le k$ we also see that the sets O_i are pairwise disjoint for all $1\le i\le k$. Since we know that always $\left\langle \theta_j^\vee, \beta \right\rangle \in \{-1, 0, 1\}$ for all $\beta \in \bigcup_{i=2}^k O_i$ and all $2 \leq j \leq k$ the second statement follows immediately from the first. □

Corollary 8.47. *We have the following identity:*

$$\operatorname{card}\left(\bigcup_{i=1}^k O_i\right) = (c_1(X)d_X - \dim(X)) - (c_1(X^2)d_{X^2} - \dim(X^2)) .$$

Proof. Recall that we have

$$\dim(X) - \dim(X^2) = \ell_P(s_{\theta_1}) = c_1(X)d_1 - 1$$

and $d_X = d_1 + d_{X^2}$. Therefore the right side of the identity is equal to

$$1 + (c_1(X) - c_1(X^2))d_{X^2} = 1 + \sum_{i=2}^k (c_1(X) - c_1(X^2))d_i .$$

If we plug in the identity from Corollary 8.45 and use the disjointness from the previous lemma this expression becomes

$$1 + \sum_{i=2}^k \operatorname{card}(O_i) = \sum_{i=1}^k \operatorname{card}(O_i) = \operatorname{card}\left(\bigcup_{i=1}^k O_i\right) .$$

□

Corollary 8.48. *Let $2 \leq i, j \leq k$ be two indices. Then*

$$\sum_{\beta \in O_i} \left\langle \theta_j^\vee, \beta \right\rangle = \sum_{\beta \in O_j} \left\langle \theta_i^\vee, \beta \right\rangle .$$

Proof. We may assume that $i \neq j$. By be the previous lemma it suffices to prove that the sets $A_{ij} = \{\beta \in O_i \mid \left\langle \theta_j^\vee, \beta \right\rangle = -1\}$ and $A_{ji} = \{\beta \in O_j \mid \langle \theta_i^\vee, \beta \rangle = -1\}$ are in bijection. We define a map φ_{ij} from A_{ij} to A_{ji} by sending β to $s_{\theta_j} s_{\theta_i}(\beta)$. If we write $\beta = \theta_i + \alpha$ where $\alpha = s_{\theta_i}(\beta) \in R_P^+$ this map is given by $\theta_i + \alpha \mapsto \theta_j + \alpha$. Therefore it is obvious that $\varphi_{ij}(\beta) \in O_j$. It is clear that $\langle \theta_i^\vee, \varphi_{ij}(\beta) \rangle = \left\langle \theta_i^\vee, s_{\theta_j} s_{\theta_i}(\beta) \right\rangle = -\langle \theta_i^\vee, \beta \rangle = -1$. Therefore the map φ_{ij} is well defined. Similarly we define a map φ_{ji} from A_{ji} to A_{ij} by sending β to $s_{\theta_i} s_{\theta_j}(\beta)$. Then φ_{ij} and φ_{ji} are inverse to each other. This proves that A_{ij} and A_{ji} are in bijection as required. □

Conjecture 8.49. *The sum $\sum_{\beta \in \bigcup_{j=1}^k O_j} \langle \theta_i^\vee, \beta \rangle$ is independent of i for all $2 \leq i \leq k$.*

9. The diagonal curve

Let $X = G/P$ be a homogeneous space where G is a connected, simply connected, linear algebraic group and P is a maximal parabolic subgroup.

Let G' be the subgroup of G defined as $G' = \mathrm{SL}_2(\theta_1) \times \cdots \times \mathrm{SL}_2(\theta_k)$. We call G' the diagonal group. To abbreviate we set $X' = G'x(1) = G'/G' \cap P$. We clearly have $X' = C_{\theta_1} \times \cdots \times C_{\theta_k} \cong \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$.

Let $u = s_{\theta_1} \cdots s_{\theta_k}$ for short. We can define a rational curve of degree d_X passing through $x(1)$ and $x(u)$ via the composition

$$f_\Delta \colon \mathbb{P}^1 \xrightarrow{\Delta} \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \cong X' \hookrightarrow X \,,$$

where Δ denotes the diagonal embedding. We call this curve the diagonal curve and denote its image by C_Δ. The diagonal curve was first introduced in [25, 3.2].

Let α be a root. We denote by x_α the root vector corresponding to α. We write $x_\theta = \sum_{i=1}^k x_{\theta_i}$ for short.

Fact 9.1. *The diagonal curve has the following description in terms of the exponential map:*

$$f_\Delta \colon \mathbb{P}^1 \to X \,,\; t \mapsto \exp(tx_\theta)x(1) \,.$$

In particular, the diagonal curve has tangent direction at $x(1)$ given by x_θ.

Proof. Let $\alpha \in R^+ \setminus R_P^+$. The isomorphism $\mathbb{P}^1 \cong C_\alpha$ is given explicitely by the exponential map $t \mapsto \exp(tx_\alpha)x(1)$. Using this description we see from the definition of the diagonal curve that the diagonal curve is given by

$$t \mapsto \exp(tx_{\theta_1}) \cdots \exp(tx_{\theta_k})x(1) \,.$$

Let i and j be two indices between 1 and k. Suppose that $i \le j$. Then $\theta_i + \theta_j$ is never a root, since otherwise it is contained in R_i and larger than θ_i. Therefore we conclude that the elements $x_{\theta_1}, \ldots, x_{\theta_k}$ are pairwise commutative. This implies that

$$\exp(tx_{\theta_1}) \cdots \exp(tx_{\theta_k}) = \exp(tx_\theta) \,.$$

for all $t \in \mathbb{P}^1$. Therefore the diagonal curve is given by $t \mapsto \exp(tx_\theta)x(1)$ as claimed. The last sentence in the statement is obvious from the definition of the exponential map. □

Let O^i be the set of all roots $\beta \in R_i^+ \setminus R_{P_i}^+$ such that $\beta = \theta_j + \alpha$ for some $i \le j \le k$ and some $\alpha \in R_{P_i}^+ \cup \{0\}$. With this notation it is clear that we have $O^k \subseteq \cdots \subseteq O^1$ and $O^k = \{\theta_k\}$.

Fact 9.2. *For all $1 \le i \le k-1$ we have the following identity: $O^{i+1} = O^i \cap (R_{i+1}^+ \setminus R_{P_{i+1}}^+) = O^i \cap R_{i+1}$.*

Proof. The inclusions $O^{i+1} \subseteq O^i \cap (R_{i+1}^+ \setminus R_{P_{i+1}}^+) \subseteq O^i \cap R_{i+1}$ are obvious. Let $\beta \in O^i \cap R_{i+1}$. It is clear that $\beta \in (R_i^+ \setminus R_{P_i}^+) \cap R_{i+1} = R_{i+1}^+ \setminus R_{P_{i+1}}^+$ since $R_i^+ \cap R_{i+1} = R_{i+1}^+$ and $R_{P_i}^+ \cap R_{i+1}^+ = R_{P_{i+1}}^+$. Let $\beta = \theta_j + \alpha$ for some $i \le j \le k$ and some $\alpha \in R_{P_i}^+ \cup \{0\}$. Since $\beta \in R_{i+1}$ we have $n_{I_i}(\beta) = n_{I_i}(\theta_j) + n_{I_i}(\alpha) = 0$ and thus $n_{I_i}(\theta_j) = n_{I_i}(\alpha) = 0$ which implies $j \ge i+1$ and $\alpha \in R_{P_{i+1}}^+ \cup \{0\}$. This means that $\beta \in O^{i+1}$. □

Lemma 9.3. *Suppose that $X \neq G_2/P_1$ and that $X \neq B_\ell/P_\ell$ where $\ell > 1$ is odd. Then we have*

$$O^1 = O^2 \amalg \coprod_{i=1}^{k} O_i\,.$$

Proof. By the previous fact we have $O^1 = O^1 \cap (R^+ \setminus R_P^+) = O^2 \amalg (O^1 \cap (R^+ \setminus (R_2^+ \cup R_P^+)))$ since $R^+ \setminus R_P^+ = (R_2^+ \setminus R_{P_2}^+) \amalg (R^+ \setminus (R_2^+ \cup R_P^+))$. In Section 8.4 we saw that for all $1 \leq i \leq k$ we have the uniform description

$$O_i = \{\beta \in R^+ \setminus (R_2^+ \cup R_P^+) \mid \beta = \theta_i + \alpha \text{ where } \alpha \in R_P^+ \cup \{0\}\}\,.$$

From this description it is obvious that $O^1 \cap (R^+ \setminus (R_2^+ \cup R_P^+)) = \coprod_{i=1}^k O_i$. □

Let $1 \leq i \leq k$. Let $\mathcal{M}^i = \mathcal{M}_{0,3}(X^i, d_{X^i})$. We denote by $\mathcal{M}^i(2)$ the fiber of the evaluation map $\mathrm{ev}_1 \times \mathrm{ev}_2 \colon \mathcal{M}^i \to X^i \times X^i$ over

$$(x(1), x(w_{X^i})) = (x(1), x(w_o(\theta_i)))\,.$$

The dimension of $\mathcal{M}^i(2)$ is given by

$$\dim(\mathcal{M}^i(2)) = c_1(X^i)d_{X^i} - \dim(X^i)\,.$$

To abbreviate we write $\mathcal{M} = \mathcal{M}^1$ and $\mathcal{M}(2) = \mathcal{M}^1(2)$.

Lemma 9.4. *Suppose that $X \neq G_2/P_1$ and that $X \neq B_\ell/P_\ell$ where $\ell > 1$ is odd. For all $1 \leq i \leq k$ we have the following identity:*

$$\mathrm{card}(O^i) = \dim(\mathcal{M}^i(2))\,.$$

Proof. By replacing X with X^i we may assume that $i = 1$. We prove the statement by induction on k. If $k = 1$ then $\dim(X) = \ell_P(s_{\theta_1}) = c_1(X)d_1 - 1 = c_1(X)d_X - 1$ and thus $\dim(\mathcal{M}(2)) = 1$. On the other hand we have $O^1 = \{\theta_1\}$ and thus $\mathrm{card}(O^1) = 1$. Assume that $k > 1$ and that the statement is known for all integers strictly smaller than k. By induction hypothesis we then know that $\dim(\mathcal{M}^2(2)) = \mathrm{card}(O^2)$. By Corollary 8.47 we know that

$$\mathrm{card}\left(\bigcup_{i=1}^{k} O_i\right) = \dim(\mathcal{M}(2)) - \dim(\mathcal{M}^2(2))\,.$$

From the previous lemma we know that

$$\mathrm{card}(O^1) = \mathrm{card}\left(\bigcup_{i=1}^{k} O_i\right) + \mathrm{card}(O^2)\,.$$

Altogether this implies that $\mathrm{card}(O^1) = \dim(\mathcal{M}(2))$ as desired. □

Theorem 9.5 ([25, Proposition 3.1])**.** *Suppose that $X \neq G_2/P_1$ and that $X \neq B_\ell/P_\ell$ where $\ell > 1$ is odd. The diagonal curve has a dense open orbit under the action of G in $\mathcal{M}$. In other words, the diagonal curve is a general curve.*

Proof. First note that the claim is equivalent to saying that the diagonal curve has a dense orbit under the action of L in $\mathcal{M}(2)$ where L is the Levi factor of P. Let T_{f_Δ} be the tangent space at f_Δ of the orbit Lf_Δ of the diagonal curve under the action of L. In order to prove that Lf_Δ is dense in $\mathcal{M}(2)$ it suffices to prove that $\dim(T_{f_\Delta}) = \dim(\mathcal{M}(2))$. We know that the tangent direction of f_Δ at $x(1)$ is given by x_θ. By letting the maximal torus $T \subseteq L$ act on f_Δ we see that the linearly independent root vectors $x_{\theta_1}, \ldots, x_{\theta_k}$ are contained in the tangent space T_{f_Δ}. By letting L act on $x_{\theta_1}, \ldots, x_{\theta_k}$ we see that the tangent space T_{f_Δ} is spanned by the vectors x_β where $\beta = \theta_i + \alpha$ for some $1 \leq i \leq k$ and some $\alpha \in R_P^+ \cup \{0\}$. Let $\beta = \theta_i + \alpha$ for some $1 \leq i \leq k$ and some $\alpha \in R_P^+$. If $\beta \notin R$ then $x_\beta = 0$. If $\beta \in R$ then clearly also $\beta \in R^+ \setminus R_P^+$. Therefore we conclude that the tangent space T_{f_Δ} is spanned by the linearly independent root vectors x_β where $\beta \in O^1$. In other words this means that $\dim(T_{f_\Delta}) = \mathrm{card}(O^1)$. By the previous lemma we know that $\mathrm{card}(O^1) = \dim(\mathcal{M}(2))$. Together this implies that $\dim(T_{f_\Delta}) = \dim(\mathcal{M}(2))$ as required. □

Remark 9.6. The idea of the previous result goes back to [25, Proposition 3.1]. It is stated there in a more general context. However no proof is given.

Remark 9.7. The author has checked that the previous proposition is optimal in the sense that the diagonal curve has *not* a dense orbit under action of G in $\mathcal{M}$ if $X = G_2/P_1$ or $X = B_\ell/P_\ell$ where $\ell > 1$ is odd.

Indeed, let $X = G_2/P_1$ or $X = B_\ell/P_\ell$ where $\ell > 1$. We will later introduce a notion of dualizing variety and will see that X is an instance of a dualizing variety (Lemma 11.59). All we need to know for now is the following formula for the dimension of X:

$$\dim(X) = \frac{c_1(X)d_X}{2}$$

which is proved in Corollary 11.62. This formula is obviously equivalent to $\dim(\mathcal{M}) = 3\dim(X)$ which is in turn equivalent to $\dim(\mathcal{M}(2)) = \dim(X)$. In order to disprove the previous theorem it clearly suffices to show that $\dim(T_{f_\Delta}) < \dim(\mathcal{M}(2))$ which is in view of the preceeding discussion and the proof of Theorem 9.5 equivalent to show that $O^1 \neq R^+ \setminus R_P^+$. Therefore we only have to find a root $\beta \in R^+ \setminus R_P^+$ which is not contained in O^1. Let β be any root which has α_P-coefficient equal to two and all other coefficients less or equal than one. Since X is a dualizing variety we know that $\theta_k = \alpha_P$ (cf. Corollary 11.3, this fact is also easy to check directly for our specific X). Since $\theta_k + \alpha$ has α_P-coefficient equal to one for all $\alpha \in R_P^+ \cup \{0\}$ we see that there is no choice of $\alpha \in R_P^+ \cup \{0\}$ such that $\beta = \theta_k + \alpha$. On the other hand for $i < k$ the root θ_i has more than one coefficient greater than one, in particular there is no choice of $\alpha \in R_P^+ \cup \{0\}$ such that $\beta = \theta_i + \alpha$. This shows that $\beta \notin O^1$ as desired.

Conjecture 9.8. *The diagonal curve has always a dense open orbit under the action of* $\mathrm{Aut}(X)$ *in* $\mathcal{M}$*.*

Conjecture 9.9. *Let* $\ell > 1$ *be odd and let* $p = \ell + 1 \geq 3$*. Under the isomorphism* $G_2/P_1 \cong \mathbb{Q}_5 \cong B_3/P_1$ *and the isomorphism*

$$B_\ell/P_\ell \cong \mathbb{G}_Q(\ell, 2\ell + 1) \cong \mathbb{G}_Q(p, 2p) \cong D_p/P_p$$

the diagonal curve of the homogeneous space in the source is mapped to the diagonal curve of the homogeneous space in the target. Both diagonal curves have the same degree.

Remark 9.10. In view of Theorem 9.5 the proof of Conjecture 9.8 is reduced to the proof of Conjecture 9.9. Indeed, Conjecture 9.8 is a trivial consequence of Theorem 9.5 for all $X \neq G_2/P_1$ and for all $X \neq B_\ell/P_\ell$ where $\ell > 1$ is odd since G is a subgroup of $\mathrm{Aut}(X)$. In case that $X = G_2/P_1$ or $X = B_\ell/P_\ell$ where $\ell > 1$ is odd the isomorphism from Conjecture 9.9 shows us that the diagonal curve has a dense open orbit under the action of $\mathrm{Aut}(X)$ in $\mathcal{M}$.

Remark 9.11. We will see later at least that the degree of the diagonal curve of the homogeneous space in the source of the isomorphism from Conjecture 9.9 is equal to the degree of the diagonal curve of the homogeneous space in the target of the isomorphism (cf. proof of Corollary 11.61).

Corollary 9.12. *For any homogeneous* X*, the group* $\mathrm{Aut}(X)$ *has a dense open orbit on* $\mathcal{M}$*.*

Proof. If $X \neq G_2/P_1$ and $X \neq B_\ell/P_\ell$ where $\ell > 1$ is odd this is a direct consequence of Theorem 9.5. In the other cases it follows (again by Theorem 9.5) that the inverse image of the diagonal curve under the isomorphism from Conjecture 9.9 has a dense open orbit under the action of $\mathrm{Aut}(X)$. □

10. The set S

Let $X = G/P$ be a homogeneous space where G is a connected, simply connected, linear algebraic group and P is a maximal parabolic subgroup.

Let S be the set of all roots γ such that γ is not orthogonal to precisely two elements of $C(\alpha_P)$. Let $S^+ = S \cap R^+$ the set of positive roots in S. Let $S_\Delta = S \cap \Delta$ the set of simple roots in S. With this notation we have $S_\Delta \subseteq \Delta_P$ since α_P is orthogonal to $\theta_1, \ldots, \theta_{k-1}$ and therefore not contained in S.

Let $\gamma \in S$. We denote by $i(\gamma)$ the smallest index i such that γ is not orthogonal to θ_i. We denote by $j(\gamma)$ the largest index j such that γ is not orthogonal to θ_j. With this notation we have $i(\gamma) < j(\gamma)$ for all $\gamma \in S$.

Let W_θ be the subgroup of W generated by $s_{\theta_1}, \dots, s_{\theta_k}$. From the definition it is clear that W_θ acts on S. With this notation we have $i(w(\gamma)) = i(\gamma)$ and $j(w(\gamma)) = j(\gamma)$ for all $\gamma \in S$ and all $w \in W_\theta$.

Let $u = s_{\theta_1} \cdots s_{\theta_k}$ for short. For all $1 \le i \le k$ we write $h_i = \mathrm{ht}(\theta_i)$ for short. We then have a decreasing integer sequence $h_1 > \cdots > h_k$.

Lemma 10.1. *Assume that R is simply laced. A root γ is contained in S if and only if $u(\gamma)$ is orthogonal to γ.*

Proof. Assume that R is simply laced. Let γ be a root. Then $u(\gamma)$ is orthogonal to γ if and only if $\langle \gamma^\vee, u(\gamma)\rangle = 0$ if and only if

$$2 - \sum_{i=1}^{k} \langle \theta_i^\vee, \gamma\rangle \langle \gamma^\vee, \theta_i\rangle = 0$$

if and only if $\langle \theta_i^\vee, \gamma\rangle \langle \gamma^\vee, \theta_i\rangle = \left\langle \theta_j^\vee, \gamma\right\rangle \langle \gamma^\vee, \theta_j\rangle = 1$ for precisely two different indices i and j between 1 and k (since R is simply laced) if and only if γ is not orthogonal two precisely two different indices i and j between 1 and k if and only if $\gamma \in S$. □

Lemma 10.2. *Let $\gamma \in S$ and let $i = i(\gamma)$. Then $\gamma \in R_i \setminus R_{i+1}$.*

Proof. Let γ and i be as in the statement. Let l be the largest index such that $\gamma \in R_l$. By definition we have $\gamma \in R_l \setminus R_{l+1}$. We have to show that $i = l$. Suppose for a contradiction that γ is orthogonal to θ_l. Then we clearly have $\gamma \in R_P$ since otherwise we have $\gamma \in R_{l+1}$. Moreover it follows that $i > l$. It is also clear that $s_{\theta_i}(\gamma) \in R_l$. In order to show that $s_{\theta_i}(\gamma) \in R_l \setminus R_{l+1}$ it suffices to show that $n_{I_l}(s_{\theta_i}(\gamma)) \neq 0$. The later statement is clear since $\gamma \in R_l \setminus R_{l+1}$ and $i > l$ and thus $n_{I_l}(s_{\theta_i}(\gamma)) = n_{I_l}(\gamma) \neq 0$. Therefore we conclude that $s_{\theta_i}(\gamma) \in R_l \setminus R_{l+1}$. On the other hand we have that $s_{\theta_i}(\gamma)$ is orthogonal to $\theta_1, \dots, \theta_l$ and that $\langle s_{\theta_i}(\gamma), \omega^\vee\rangle = -\langle \theta_i^\vee, \gamma\rangle \langle \theta_i, \omega^\vee\rangle \neq 0$ and thus $s_{\theta_i}(\gamma) \in R \setminus R_P$. Both facts together mean that $s_{\theta_i}(\gamma) \in R_{l+1}$ – a contradiction. Therefore we conclude that γ is not orthogonal to θ_l. Since $\gamma \in R_l$ we know that γ is orthogonal to $\theta_1, \dots, \theta_{l-1}$. In other words this means that $i = l$ as required. □

Corollary 10.3. *Let $\gamma \in S_\Delta$ and let $i = i(\gamma)$. Then $\gamma \in I_i$.*

Proof. Let γ and i be as in the statement. By the previous lemma we have $\gamma \in (R_i \setminus R_{i+1}) \cap \Delta = I_i$. □

Corollary 10.4. *Let $\gamma \in S^+$ and let $i = i(\gamma)$. Then we have $\langle \theta_i^\vee, \gamma\rangle =$ 1*

Proof. By the previous lemma we know that $\gamma \in R_i$ and thus $s_{\theta_i}(\gamma) \in R_i$. Since θ_i is the highest root of R_i it is clear that $\langle \theta_i^\vee, \gamma\rangle \in \{-1, 1\}$. Since γ is positive we must have $\langle \theta_i^\vee, \gamma\rangle = 1$ as claimed. □

Lemma 10.5. *Let $\gamma \in S_\Delta$, let $i = i(\gamma)$ and $j = j(\gamma)$. Then $j = i+1$, $\langle \theta_i^\vee, \gamma\rangle = 1$, $\left\langle \theta_{i+1}^\vee, \gamma\right\rangle < 0$ and $2 \le h_i - h_{i+1}$.*

Proof. Let $\gamma \in S_\Delta$. Since $\gamma \in R_i \setminus R_{i+1}$ we know that $\gamma \in I_i$. The root γ and the root θ_{i+1} are non-separated, since otherwise γ and θ_l for all $i+1 \leq l \leq k$ are separated and thus γ is orthogonal to θ_l for all $i+1 \leq l \leq k$. Since γ and θ_{i+1} are non-separated and since γ and θ_{i+1} have disjoint support, γ and θ_{i+1} must be not orthogonal to each other. It follows that $j = i+1$. We already saw that $\langle \theta_i^\vee, \gamma \rangle = 1$. Suppose that $\left\langle \theta_{i+1}^\vee, \gamma \right\rangle > 0$. Then $\delta := s_{\theta_{i+1}}(\gamma)$ would be a root which satisfies $n_{I_i}(\delta) = 1$ and $n_{\Delta_{i+1}}(\delta) = -\left\langle \theta_{i+1}^\vee, \gamma \right\rangle h_{i+1} < 0$ – a contradiction. Therefore we conclude that $\left\langle \theta_{i+1}^\vee, \gamma \right\rangle < 0$, in particular δ is a positive root in R_i. Therefore $s_{\theta_i} s_{\theta_{i+1}}(\gamma) = \delta - \theta_i$ is a negative root. We find that $1 \leq h_i - \mathrm{ht}(\delta) = h_i - 1 + \left\langle \theta_{i+1}^\vee, \gamma \right\rangle h_{i+1} \leq h_i - 1 - h_{i+1}$ and thus $2 \leq h_i - h_{i+1}$. □

Let $\mathcal{P}$ be the set consisting of all root $\gamma \in S^+$ such that $\left\langle \theta_{j(\gamma)}^\vee, \gamma \right\rangle < 0$. By the previous lemma we know that $S_\Delta \subseteq \mathcal{P}$.

Lemma 10.6. *The set $\mathcal{P}$ is a set of representatives of the orbits of the action of W_θ on S. Each W_θ-orbit representated by some $\gamma \in \mathcal{P}$ consists of the elements*

$$\gamma > 0, \gamma - \theta_i < 0, \gamma - \left\langle \theta_j^\vee, \gamma \right\rangle \theta_j > 0, \gamma - \theta_i - \left\langle \theta_j^\vee, \gamma \right\rangle \theta_j < 0$$

where $i = i(\gamma)$ and $j = j(\gamma)$.

Proof. Let $\gamma \in S$. The orbit $W_\theta(\gamma)$ contains a positive root. Indeed, if $\gamma < 0$ then $s_{\theta_i}(\gamma) = \gamma + \theta_i$ where $i = i(\gamma)$ is positive. Therefore we can choose a root $\delta \in W_\theta(\gamma)$ such that the height of δ is minimal in the set $\{\mathrm{ht}(\alpha) \mid \alpha \in W_\theta(\gamma), \mathrm{ht}(\alpha) > 0\}$. Let $j = j(\gamma) = j(\delta)$. Suppose that $\left\langle \theta_j^\vee, \delta \right\rangle > 0$. Then $s_{\theta_j}(\delta) = \delta - \left\langle \theta_j^\vee, \delta \right\rangle \theta_j$ is a positive roots (since $\delta \in R_i \setminus R_{i+1}$ and thus $n_{I_i}(s_{\theta_j}(\delta)) = n_{I_i}(\delta) > 0$) with strictly smaller height than δ – contrary to the choice of δ. Therefore we conclude that $\left\langle \theta_j^\vee, \delta \right\rangle < 0$ and that $s_{\theta_j}(\delta)$ is a positive root in R_i. This proves that every W_θ-orbit in S contains at least one element in $\mathcal{P}$. Moreover it proves that the W_θ-orbit of some element $\gamma \in \mathcal{P}$ is given by the four elements described in the statement together with their sign. We are left to show that every W_θ-orbit of some $\gamma \in S$ contains a unique element in $\mathcal{P}$. To this end, suppose that $W_\theta(\gamma) = W_\theta(\delta)$ where $\gamma, \delta \in \mathcal{P}$. Suppose in addition that $\gamma \neq \delta$. Since δ is positive we must have $\delta = \gamma - \left\langle \theta_j^\vee, \gamma \right\rangle \theta_j$ where $j = j(\gamma) = j(\delta)$. It follows that $\left\langle \theta_j^\vee, \delta \right\rangle = -\left\langle \theta_j^\vee, \gamma \right\rangle > 0$ – contrary to the fact that $\delta \in \mathcal{P}$. Therefore we must have $\gamma = \delta$. This proves the desired uniqueness. □

Corollary 10.7. *Suppose that $\mathcal{P} \subseteq R_P^+$. The set $S \cap (R^+ \setminus R_P^+)$ is a set of representatives of the orbits of the action of W_θ on S. If in addition $X \neq B_\ell / P_\ell$ where $\ell > 1$ is odd, then we have $\left\langle \sum_{i=1}^k \theta_i^\vee, \gamma \right\rangle = 2$ for all $\gamma \in S \cap (R^+ \setminus R_P^+)$.*

Proof. Suppose that $\mathcal{P} \subseteq R_P^+$. By the previous lemma a W_θ-orbit representated by some $\gamma \in \mathcal{P}$ contains a unique element in $R^+ \setminus R_P^+$ namely $\gamma - \langle \theta_j^\vee, \gamma \rangle \theta_j$ where $j = j(\gamma)$. This proves that $S \cap (R^+ \setminus R_P^+)$ is a set of representatives of the orbits of the action of W_θ on S.

The assumption $\mathcal{P} \subseteq R_P^+$ implies that $X \neq G_2/P_1$, since if $X = G_2/P_1$ we have $\alpha_1+\alpha_2 \in \mathcal{P}$ but $\alpha_1+\alpha_2 \notin R_P^+$. If in addition $X \neq B_\ell/P_\ell$ where $\ell > 1$ is odd, we can use Lemma 8.6. Therefore we have for all $\gamma \in \mathcal{P}$ that $\left\langle \theta_{j(\gamma)}^\vee, \gamma \right\rangle = -1$ and consequently $\left\langle \sum_{i=1}^k \theta_i^\vee, \gamma \right\rangle = 0$.

Let $\gamma \in S \cap (R^+ \setminus R_P^+)$. By what we proved up to now we can find a unique element $\delta \in \mathcal{P}$ such that $\gamma = \delta + \theta_j$ where $j = j(\gamma) = j(\delta)$. It follows that $\left\langle \sum_{i=1}^k \theta_i^\vee, \gamma \right\rangle = 0 + 2 = 2$ as claimed. □

Corollary 10.8. *Suppose that X is a cominuscule homogeneous space. Then we have $\mathcal{P} \subseteq R_P^+$. Moreover the set $S \cap (R^+ \setminus R_P^+)$ and the set $S \cap (R^- \setminus R_P^-)$ are sets of representatives of the orbits of the action of W_θ on S. Moreover we have that $\langle \eta^\vee, \gamma \rangle = 2$ for all $\gamma \in S \cap (R^+ \setminus R_P^+)$ and that $\langle \eta^\vee, \gamma \rangle = -2$ for all $\gamma \in S \cap (R^- \setminus R_P^-)$ where $\eta^\vee = \sum_{i=1}^k \theta_i^\vee$.*

Proof. Since X is cominuscule we know that $X \neq G_2/P_1$ and that $X \neq B_\ell/P_\ell$ where $\ell > 1$ is odd. Therefore we can use Lemma 8.6.

Suppose there exists $\gamma \in \mathcal{P}$ such that $\gamma \notin R_P^+$. Then the root $s_{\theta_{j(\gamma)}}(\gamma) = \gamma + \theta_{j(\gamma)}$ has α_P-coefficient larger than one which contradicts the fact that X is a cominuscule homogeneous space. Therefore we conclude that $\mathcal{P} \subseteq R_P^+$. By the previous corollary we then know that $S \cap (R^+ \setminus R_P^+)$ is a set of representatives of the orbits of W_θ in S and that $\langle \eta^\vee, \gamma \rangle = 2$ for all $\gamma \in S \cap (R^+ \setminus R_P^+)$.

Moreover a W_θ-orbits representated by some $\gamma \in \mathcal{P}$ contains a unique element in $R^- \setminus R_P^-$ namely $\gamma - \theta_{i(\gamma)}$. (To exclude that $\gamma - \theta_{i(\gamma)} + \theta_{j(\gamma)}$ is also in $S \cap (R^- \setminus R_P^-)$ we use the cominuscule assumption.) Therefore $S \cap (R^- \setminus R_P^-)$ is a set of representatives of the orbits of W_θ in S.

Finally let $\gamma \in S \cap (R^- \setminus R_P^-)$. By what we proved up to know we can find a unique element $\delta \in \mathcal{P}$ such that $\gamma = \delta - \theta_i$ where $i = i(\gamma) = i(\delta)$. As in the previous corollary it follows that $\langle \eta^\vee, \gamma \rangle = 0 - 2 = -2$ since $\langle \eta^\vee, \delta \rangle = 1 - 1 = 0$. □

11. The group $\hat{G}$

Let $X = G/P$ be a homogeneous space where G is a connected, simply connected, linear algebraic group and P is a maximal parabolic subgroup.

If $k = 1$ we set $R' = \emptyset$. If $k > 1$ let R' be the union of all irreducible components $\neq R_k$ of the root system consisting of all roots in R_{k-1} orthogonal to θ_{k-1}. Let $\hat{R}$ be the root subsystem of R generated by $\mathcal{B}$, S_Δ and R'. Let $\hat{G}$ be the algebraic subgroup of G with root system $\hat{R}$.

Let $\hat{X} = \hat{G}/\hat{G} \cap P$. The variety $\hat{X}$ is a subvariety of X wich satisfies $C_\Delta \subseteq X' \subseteq \hat{X}$.

Definition 11.1. *We call X a dualizing variety if $\hat{R} = R$ and thus $\hat{X} = X$.*

Lemma 11.2. *The root system $\hat{R}_i$*[6] *is generated by $\mathcal{B} \cap R_i$, $S_\Delta \cap R_i$ and $R' \cap R_i$. In particular we have that*

$$\{\pm\theta_k\} = \hat{R}_k \subseteq \hat{R}_{k-1} \subseteq \cdots \subseteq \hat{R}\,.$$

Proof. It is clear that $\mathcal{B} \cap R_i$ is the cascade of orthogonal roots associated to R_i. Since all elements of $S_\Delta \cap R_i$ are orthogonal to $\theta_1, \ldots, \theta_{i-1}$ it is clear that $S_\Delta \cap R_i$ is the set S_Δ associated to R_i. Note that $R'_k = \emptyset$ and also $R' \cap R_k = \emptyset$ and thus $R' \cap R_k = R'_k$. If $i < k$ then it is clear that $R' \subseteq R_i$ and thus $R' \cap R_i = R' = R'_i$. Therefore we get for all $1 \leq i \leq k$ that $R' \cap R_i = R'_i$. All these facts together imply that $\hat{R}_i$ is generated by $\mathcal{B} \cap R_i$, $S_\Delta \cap R_i$ and $R' \cap R_i$. The sequence of inclusions is now obvious. Note that $\mathcal{B} \cap R_k = \{\theta_k\}$, $S_\Delta \cap R_k = \emptyset$ and $R' \cap R_k = \emptyset$. Therefore we have $\hat{R}_k = \{\pm\theta_k\}$. □

Corollary 11.3. *If $\hat{R} = R$ then $\hat{R}_i = R_i$ for all $1 \leq i \leq k$ and $\theta_k = \alpha_P$ (or equivalently $\alpha_P \in C(\alpha_P)$).*

Proof. Since $R_i = R \cap R_i = \hat{R} \cap R_i$ it is clear that R_i is generated by $\mathcal{B} \cap R_i$, $S_\Delta \cap R_i$ and $R' \cap R_i$. By the previous lemma $\hat{R}_i$ has the same set of generators. Therefore it follows that $\hat{R}_i = R_i$ as claimed.

In particular, this shows that $\hat{R}_k = R_k = \{\pm\theta_k\}$. On the other hand we know that $\alpha_P \in R_k$. Therefore we conclude that $\theta_k = \alpha_P$ as claimed. □

Lemma 11.4. *If $\hat{R} = R$ then for all $i \in \{1, \ldots, k-1\}$ there exists a $\gamma \in S_\Delta$ such that $i = i(\gamma)$.*

Proof. Let $i \in \{1, \ldots, k-1\}$ and suppose for a contradiction that $i \neq i(\gamma)$ for all $\gamma \in S_\Delta$. Then the irreducible component of R containing R' cannot contain $\theta_1, \ldots, \theta_i$ (Lemma 10.5) which contradicts the fact that R is irreducible. □

Let $\mathfrak{R}^1, \ldots, \mathfrak{R}^r$ be the irreducible components of $\hat{R}$. For each $1 \leq i \leq r$ let $\mathfrak{G}^i$ be the simple linear algebraic subgroup of G with root system $\mathfrak{R}^i$. We have a natural choice of a Borel subgroup of $\mathfrak{G}^i$ and a natural choice of a maximal parabolic subgroup $\mathfrak{P}^i = \mathfrak{G}^i \cap P$ of $\mathfrak{G}^i$. In particular it makes sense to speak about the positive roots in $\mathfrak{R}^i$, the simple roots in $\mathfrak{R}^i$, the θ-sequence of $\mathfrak{R}^i$, etc. Let $\mathfrak{X}_i = \mathfrak{G}^i/\mathfrak{P}^i$. Then the number $d_{\mathfrak{X}_i}$ is well defined. It makes sense to write $d_{\hat{X}} = \sum_{i=1}^r d_{\mathfrak{X}_i}$.

[6] With $\hat{R}_i$ we mean the root system associated to R_i in the same way we associated $\hat{R}$ to R. More explicitely we have $\hat{R}_i = \widehat{R_i}$. We do not mean $\hat{R}_i = (\hat{R})_i$ which makes no sense in general since $\hat{R}$ is not necessarily irreducible.

Lemma 11.5. *Let $1 \leq i \leq r$. Let θ_* be the smallest element of $C(\alpha_P) \cap \mathfrak{R}^i$ and let θ^* be the largest element of $C(\alpha_P) \cap \mathfrak{R}^i$. Then we have*

$$C(\alpha_P) \cap \mathfrak{R}^i = \{\theta \in C(\alpha_P) \mid \theta_* \leq \theta \leq \theta^*\}.$$

Proof. Let $\theta^* = \theta_{j_1}$ and let $\theta_* = \theta_{j_2}$ for some $1 \leq j_1, j_2 \leq k$. We clearly have $j_1 \leq j_2$. The inclusion from left to right is obvious from the definition. We prove the inclusion from right to left by induction on $j_2 - j_1$. If $j_1 = j_2$ there is nothing to prove. Suppose that $j_1 < j_2$. Using the induction hypothesis, it suffices to prove that $\theta_{j_1+1} \in C(\alpha_P) \cap \mathfrak{R}^i$. Suppose for a contradiction that $\theta_{j_1+1} \notin C(\alpha_P) \cap \mathfrak{R}^i$. Since $\hat{R} = \mathfrak{R}^1 \times \cdots \times \mathfrak{R}^r$ and since $C(\alpha_P) \subseteq \hat{R}$ we have $C(\alpha_P) = \coprod_{i=1}^r C(\alpha_P) \cap \mathfrak{R}^i$. From this disjoint union we see that $S_\Delta \cap \mathfrak{R}^i$ consists of the simple roots which are not orthogonal to precisely two elements of $C(\alpha_P) \cap \mathfrak{R}^i$. Using the assumption and Lemma 10.5 we conclude that θ_{j_1} is orthogonal to all elements of $S_\Delta \cap \mathfrak{R}^i$. In addition θ_{j_1} is orthogonal to all elements of $\mathcal{B} \cap \mathfrak{R}^i \setminus \{\theta_{j_1}\}$ and to all elements of $R' \cap \mathfrak{R}^i$. Since $\mathfrak{R}^i$ is generated by $\mathcal{B} \cap \mathfrak{R}^i$, $S \cap \mathfrak{R}^i$ and $R' \cap \mathfrak{R}^i$ we conclude that θ_{j_1} is orthogonal to all elements of $\mathfrak{R}^i \setminus \{\pm\theta_{j_1}\}$ which contradicts the fact that $\mathfrak{R}^i$ is irreducible. Therefore we conclude that $\theta_{j_1+1} \in C(\alpha_P) \cap \mathfrak{R}^i$ which completes the proof. □

Lemma 11.6. *Assume that $R' \neq \emptyset$. There exists one and only one index $1 \leq i \leq r$ such that $R' \subseteq \mathfrak{R}^i$.*

Proof. Since $R' \subseteq \hat{R}$ we clearly have $R' = \coprod_{i=1}^r R' \cap \mathfrak{R}^i$. If R' is irreducible it follows that $R' = R' \cap \mathfrak{R}^i$ and thus $R' \subseteq \mathfrak{R}^i$ for one and only one index $1 \leq i \leq r$. Suppose that R' is not irreducible. By Lemma 8.3 we then know that R_{k-1}° has three irreducible components and that R_{k-1} is of type D_4. It immediately follows that $R_{k-1} \subseteq \mathfrak{R}^i$ for one and only one index $1 \leq i \leq r$, in particular $R' \subseteq \mathfrak{R}^i$. In each case the assertion is true. □

Lemma 11.7. *Assume that $R' \neq \emptyset$. Let $1 \leq i \leq r$ such that $R' \subseteq \mathfrak{R}^i$. Then $\theta_{k-1}, \theta_k \in C(\alpha_P) \cap \mathfrak{R}^i$.*

Proof. Let i be as in the statement. Then there exists an element $\gamma \in S_\Delta \cap \mathfrak{R}^i$ such that γ is not orthogonal to the highest root of an irreducible component of R' since otherwise $\mathfrak{R}^i$ is not irreducible. By Lemma 10.5 it follows that $i(\gamma) = k - 1$ and that $j(\gamma) = k$ and thus $\theta_{k-1}, \theta_k \in C(\alpha) \cap \mathfrak{R}^i$. □

Lemma 11.8. *Let $1 \leq i \leq r$. Let θ^* be the largest element of $C(\alpha_P) \cap \mathfrak{R}^i$. Then we have $\mathfrak{R}^i \subseteq R(\theta^*)$.*

Proof. Let $\theta^* = \theta_j$ for some $1 \leq j \leq k$. It is clear from the definition that $\mathcal{B} \cap \mathfrak{R}^i \subseteq R_j$. The set $S_\Delta \cap \mathfrak{R}^i$ consists of all simple roots which are not orthogonal to precisely two elements of $C(\alpha_P) \cap \mathfrak{R}^i$. Therefore we have $i(\gamma) \geq j$ for all $\gamma \in S_\Delta \cap \mathfrak{R}^i$ and thus $S_\Delta \cap \mathfrak{R}^i \subseteq R_j$ (Lemma

10.2). If $j = k$ we have $C(\alpha_P) \cap \mathfrak{R}^i = \{\theta_k\}$ and $S_\Delta \cap \mathfrak{R}^i = \emptyset$. Since all elements of R' are orthogonal to θ_k and since $\mathfrak{R}^i$ is irreducible, we conclude that $R' \cap \mathfrak{R}^i = \emptyset$ and thus $\mathfrak{R}^i = \{\pm\theta_k\}$ in particular $\mathfrak{R}^i \subseteq R_k$. If $j < k$ then we have $R' \subseteq R_j$ in particular $R' \cap \mathfrak{R}^i \subseteq R_j$. Therefore we know in all cases that $R' \cap \mathfrak{R}^i \subseteq R_j$. Since $\mathfrak{R}^i$ is generated by $\mathcal{B} \cap \mathfrak{R}^i$, $S_\Delta \cap \mathfrak{R}^i$ and $R' \cap \mathfrak{R}^i$ we conclude that $\mathfrak{R}^i \subseteq R_j$. □

Lemma 11.9. *Let $1 \leq i \leq r$. The set $C(\alpha_P) \cap \mathfrak{R}^i$ is the θ-sequence of $\mathfrak{R}^i$.*

Proof. Let θ_{j_1} be the largest element of $C(\alpha_P) \cap \mathfrak{R}^i$ and let θ_{j_2} be the smallest element of $C(\alpha_P) \cap \mathfrak{R}^i$. Let $\hat{\theta}_1, \dots, \hat{\theta}_{k_i}$ be the θ-sequence of $\mathfrak{R}^i$. Since $\mathfrak{R}^i \subseteq R_{j_1}$ by the previous lemma we see that $\hat{\theta}_1 \leq \theta_{j_1}$. On the other hand we have $\theta_{j_1} \in \mathfrak{R}^i$ and thus $\theta_{j_1} \leq \hat{\theta}_1$ which means that $\hat{\theta}_1 = \theta_{j_1}$. From this it follows immediately that $\mathfrak{R}^i_2 = \mathfrak{R}^i \cap R_{j_1+1}$ which implies that $\hat{\theta}_2 = \theta_{j_1+1}$ since $\theta_{j_1+1} \in \mathfrak{R}^i$ and therefore also $\theta_{j_1+1} \in \mathfrak{R}^i_2$. By repeating this process we see that $\mathfrak{R}^i_j = \mathfrak{R}^i \cap R_{j_1+j}$ for all $0 \leq j \leq j_2 - j_1$ and that $k_i \geq j_2 - j_1$ and that $\hat{\theta}_j = \theta_{j_1+j}$ for all $0 \leq j \leq j_2 - j_1$. From the definition of $\mathfrak{R}^i$ it is rather clear that $\mathfrak{R}^i_{j_2-j_1} = \mathfrak{R}^i \cap R_{j_2} = \{\pm\theta_{j_2}\}$. From this it follows that $k_i = j_2 - j_1$ which completes the proof. □

Corollary 11.10. *We have the equality $d_X = d_{\hat{X}}$.*

Proof. By the previous lemma we know that

$$d_{\mathfrak{X}_i} = \sum_{\theta \in C(\alpha_P) \cap \mathfrak{R}^i} d(\theta)$$

for all $1 \leq i \leq r$. On the other hand we have $C(\alpha_P) = \coprod_{i=1}^{r} C(\alpha_P) \cap \mathfrak{R}^i$. This implies that

$$d_{\hat{X}} = \sum_{i=1}^{r} d_{\mathfrak{X}_i} = \sum_{\theta \in C(\alpha_P)} d(\theta) = d_X\,.$$

□

Corollary 11.11. *Let X be a cominuscule homogeneous space. Then $\mathfrak{X}_i$ is also a cominuscule homogeneous space for all $1 \leq i \leq r$ such that $C(\alpha_P) \cap \mathfrak{R}^i \neq \emptyset$.*

Proof. Let i be an index such that $C(\alpha_P) \cap \mathfrak{R}^i \neq \emptyset$. Let θ^* be the largest element of $C(\alpha_P) \cap \mathfrak{R}^i$. Then we know that θ^* is the highest root of $\mathfrak{R}^i$. Since X is cominuscule it is clear that $\langle \theta^*, \omega^\vee \rangle = 1$. Therefore $\mathfrak{X}_i$ is also cominuscule. □

Lemma 11.12. *Let i be an index such that $C(\alpha_P) \cap \mathfrak{R}^i = \emptyset$. Then $\mathfrak{R}^i = \{\pm\theta\}$ for some root $\theta \in \mathcal{B} \cap R_P^+$.*

Proof. Let i be an index such that $C(\alpha_P) \cap \mathfrak{R}^i = \emptyset$. Then $S_\Delta \cap \mathfrak{R}^i = \emptyset$ and $R' \cap \mathfrak{R}^i = \emptyset$. It follows that $\mathfrak{R}^i$ is generated by $\mathcal{B} \cap \mathfrak{R}^i$. Since $\mathfrak{R}^i$ is irreducible the set $\mathcal{B} \cap \mathfrak{R}^i$ consists of precisely one element θ. Since $\theta \notin C(\alpha_P)$ it follows that $\theta \in R_P^+$. Therefore we have $\mathfrak{R}^i = \{\pm\theta\}$ where $\theta \in \mathcal{B} \cap R_P^+$. □

Lemma 11.13. *For all $1 \leq i \leq r$ such that $C(\alpha_P) \cap \mathfrak{R}^i \neq \emptyset$ we have that $\hat{\mathfrak{R}}^i = \mathfrak{R}^i$. This means that $\mathfrak{X}_i$ is a dualizing variety.*

Proof. Let i be an index such that $C(\alpha_P) \cap \mathfrak{R}^i \neq \emptyset$. Assume first that $R' \cap \mathfrak{R}^i = \emptyset$. Then $\mathfrak{R}^i$ is generated by $\mathcal{B} \cap \mathfrak{R}^i$ and $S_\Delta \cap \mathfrak{R}^i$. Let θ^* be the highest root of $\mathfrak{R}^i$. Then we know that $\theta^* \in C(\alpha_P) \cap \mathfrak{R}^i$ and that $\mathfrak{R}^i \subseteq R(\theta^*)$. Therefore it is clear that $\mathcal{B} \cap \mathfrak{R}^i$ is the cascade of strongly orthogonal roots associated to $\mathfrak{R}^i$. Moreover $S_\Delta \cap \mathfrak{R}^i$ consists of all simple roots which are not orthogonal to precisely two elements of $C(\alpha_P) \cap \mathfrak{R}^i$ which is the θ-sequence associated to $\mathfrak{R}^i$. Since $\theta_{k-1}, \theta_k \notin C(\alpha_P) \cap \mathfrak{R}^i$ we know by Lemma 8.3 that $\mathfrak{R}^{i\prime}$ is either empty or of type A_1 (or of type $\mathsf{A}_1 \times \mathsf{A}_1$) and thus $\mathfrak{R}^{i\prime} \subseteq \mathcal{B} \cap \mathfrak{R}^i$. In total we see that $\hat{\mathfrak{R}}^i$ is generated by $\mathcal{B} \cap \mathfrak{R}^i$ and by $S_\Delta \cap \mathfrak{R}^i$. Therefore we conclude that $\hat{\mathfrak{R}}^i = \mathfrak{R}^i$.

Assume now that $R' \cap \mathfrak{R}^i \neq \emptyset$. Then $R' \subseteq \mathfrak{R}^i$ and $\theta_{k-1}, \theta_k \in C(\alpha_P) \cap \mathfrak{R}^i$. It follows that $\mathfrak{R}^{i\prime} = R'$. By what we proved up to now we see that $\hat{\mathfrak{R}}^i$ is generated by $\mathcal{B} \cap \mathfrak{R}^i$, $S_\Delta \cap \mathfrak{R}^i$ and $R' \cap \mathfrak{R}^i$. Since $\mathfrak{R}^i$ has the same set of generators it follows that $\hat{\mathfrak{R}}^i = \mathfrak{R}^i$. □

Corollary 11.14. *The variety $\hat{X}$ is a product of dualizing varieties.*

Proof. By definition we know that $\hat{X} = \prod_{i=1}^r \mathfrak{X}_i$. If i is an index such that $C(\alpha_P) \cap \mathfrak{R}^i = \emptyset$ then $\mathfrak{R}^i \subseteq R_P$ and thus $\mathfrak{X}_i = \{\mathrm{pt}\}$. It follows that $\hat{X} = \prod_{i\colon C(\alpha_P) \cap \mathfrak{R}^i \neq \emptyset} \mathfrak{X}_i$. By the previous lemma each factor $\mathfrak{X}_i$ where i is an index such that $C(\alpha_P) \cap \mathfrak{R}^i \neq \emptyset$ is a dualizing variety. Therefore $\hat{X}$ is a product of dualizing varieties. □

11.1. **Computation of $\hat{X}$.** Let d be a degree and let $x, y \in X$. We denote by $X_d(x,y)$ the union all rational curves of degree d which pass through x and y. Let f be a general rational curve of degree d. We denote by $Y_d(f)$ the intersection

$$Y_d(f) = \bigcap_{\substack{x,y \in f(\mathbb{P}^1) \\ \text{general}}} X_d(x,y)\,.$$

Lemma 11.15. *Let f be a general rational curve of degree d. Let Y be an irreducible subvariety of X such that through three points of Y in general position passes a unique rational curve of degree d which is contained in Y. Suppose that $f(\mathbb{P}^1) \subseteq Y$. Then $Y \subseteq Y_d(f)$.*

Proof. By assumption there exists an open dense subset U of Y such that for two general points $x, y \in f(\mathbb{P}^1)$ there exists a unique rational curve of degree d which passes through x, y and U. It follows that $U \subseteq X_d(x, y)$ for all general points $x, y \in f(\mathbb{P}^1)$ and thus $U \subseteq Y_d(f)$. Since $Y_d(f)$ is closed it follows that $Y = \bar{U} \subseteq Y_d(f)$ as claimed. $\square$

Lemma 11.16. *Let $X = G/P$ be a cominuscule homogeneous space. Then we have an isomorphism $\hat{X} \cong Y_{d_X}(f_\Delta)$. Up to isomorphism we get the following table:*

X	d_X	$\hat{X} \cong Y_{d_X}(f_\Delta)$
$\mathbb{G}(l, n)$, $n \geq 2$	$\min(l, n-l)$	$\mathbb{G}(d_X, 2d_X)$
$\mathbb{G}_\omega(p, 2p)$, $p \geq 2$	p	$\mathbb{G}_\omega(d_X, 2d_X)$
$\mathbb{G}_Q(p, 2p)$, $p \geq 3$	$[p/2]$	$\mathbb{G}_Q(2d_X, 4d_X)$
$\mathbb{Q}_m$, $m \geq 3$	2	$\mathbb{Q}_m$
E_6/P_1	2	$\mathbb{Q}_8$
E_7/P_7	3	E_7/P_7

Proof. By Theorem 9.5 we know that the diagonal curve f_Δ is general for a cominuscule homogeneous space since a cominuscule homogeneous space X satisfies $X \neq G_2/P_1$ and $X \neq B_\ell/P_\ell$ where $\ell > 1$ is odd. It follows from [11, Proposition 19] that $Y_{d_X}(f_\Delta) = X_{d_X}(x, y)$ where x and y are two general points in C_Δ. The isomorphism types of $X_{d_X}(x, y)$ as well as the values of d_X are known from [11, Proposition 18]. (The value of d_X can also be computed directly using the algorithm discussed in this work.) They are given as described in the table. Therefore it suffices to show that $\hat{X}$ has the same isomorphism types as described in the table. We prove this by studying each case separately.

Let $X = \mathbb{G}(l, n)$ where $n \geq 2$. Then we have $\mathcal{B} \cap (R^+ \setminus R_P^+) = C(\alpha_P)$ and $S_\Delta = \{\alpha_1, \ldots, \alpha_{k-1}, \alpha_{n-k+1}, \ldots, \alpha_{n-1}\}$ and $R' = \emptyset$. It is easy to see that the root system generated by $\alpha_1, \ldots, \alpha_{k-1}, \alpha_{n-k+1}, \ldots, \alpha_{n-1}, \theta_k$ contains the roots $\theta_1, \ldots, \theta_{k-1}$. The irreducible component of $\hat{R}$ which is not contained in R_P is therefore generated by

$$\alpha_1, \ldots, \alpha_{k-1}, \alpha_{n-k+1}, \ldots, \alpha_{n-1}, \theta_k$$

and is thus of type A_{2k-1} with the kth node marked. It follows that $\hat{X} = \mathbb{G}(k, 2k) = \mathbb{G}(d_X, 2d_X)$ as claimed.

Let $X = \mathbb{G}_\omega(p, 2p)$ where $p \geq 2$. Then we have $\mathcal{B} = C(\alpha_P)$ and $S_\Delta = \{\alpha_1, \ldots, \alpha_{p-1}\}$ and $R' = \emptyset$. Therefore $\hat{R}$ contains all simple roots Δ and thus $\hat{R} = R$. It follows that $\hat{X} = X = \mathbb{G}_\omega(p, 2p) = \mathbb{G}_\omega(d_X, 2d_X)$ as claimed.

Let $X = \mathbb{G}_Q(p, 2p)$ where $p \geq 3$. We distinguish the cases where p is odd and where p is even. Suppose that p is odd. We can assume that $p \geq 5$ since the case $p = 3$ is already covered by type A_{n-1}. Then we have that $\mathcal{B} = C(\alpha_P) \cup \{\alpha_1, \alpha_3, \ldots, \alpha_{p-2}\}$ and $S_\Delta = \{\alpha_2, \alpha_4, \ldots, \alpha_{p-3}\}$ and $R' = \{\alpha_{p-4}\} \subseteq \mathcal{B}$. It is easy to see that the roots system generated by $\alpha_1, \alpha_2, \ldots, \alpha_{p-2}, \theta_1$ contains the roots $\theta_2, \ldots, \theta_k$. Therefore $\hat{R}$ is generated by $\alpha_1, \alpha_2, \ldots, \alpha_{p-2}, \theta_1$. It follows that $\hat{R}$ is of type D_{p-1} with the $(p-1)$th node marked. Therefore we have $\hat{X} = \mathbb{G}_Q(p-1, 2(p-1)) = \mathbb{G}_Q(2d_X, 4d_X)$ as claimed.

Suppose next that $X = \mathbb{G}_Q(p, 2p)$ where $p \geq 3$ and p is even. Then we have $\mathcal{B} = C(\alpha_P) \cup \{\alpha_1, \alpha_3, \ldots, \alpha_{p-1}\}$ and $S_\Delta = \{\alpha_2, \alpha_4, \ldots, \alpha_{p-2}\}$ and $R' = \{\alpha_{p-3}, \alpha_{p-1}\} \subseteq \mathcal{B}$. Therefore $\hat{R}$ contains all simple roots Δ and thus $\hat{R} = R$. It follows that $\hat{X} = X = \mathbb{G}_Q(p, 2p) = \mathbb{G}_Q(2d_X, 4d_X)$ as claimed.

Let $X = \mathbb{Q}_m$ where $m \geq 3$. We distinguish the cases where m is even and where m is odd. Suppose that m is even. Let $p = m/2 + 1$. We can assume that $p > 3$ since the case $p = 3$ is already covered by type A_{n-1}. We can even assume that $p > 4$ since the case $p = 4$ is already covered by type D_p with the pth node marked. Then we have $\mathcal{B} \cap (R^+ \setminus R_P^+) = C(\alpha_P)$ and $S_\Delta = \{\alpha_2\}$ and R' is of type D_{p-2} with simple roots $\alpha_3, \ldots, \alpha_p$. Therefore $\hat{R}$ contains all simple roots Δ and thus $\hat{R} = R$. It follows that $\hat{X} = X = \mathbb{Q}_m$.

Suppose next that $X = \mathbb{Q}_m$ where $m \geq 3$ and m is odd. Let $\ell = (m+1)/2$. Then we have $\mathcal{B} \cap (R^+ \setminus R_P^+) = C(\alpha_P)$ and $S_\Delta = \{\alpha_2\}$ and R' is generated by $\alpha_3, \ldots, \alpha_\ell$. ($R'$ is of type $\mathsf{B}_{\ell-2}$ if $\ell > 2$ and of type A_1 if $\ell = 2$.) Therefore $\hat{R}$ contains all simple roots Δ and thus $\hat{R} = R$. It follows that $\hat{X} = X = \mathbb{Q}_m$.

Let $X = E_6/P_1$. Then we have $\mathcal{B} = C(\alpha_P) \cup \{\alpha_3 + \alpha_4 + \alpha_5, \alpha_4\}$ and $S_\Delta = \{\alpha_2\}$ and $R' = \emptyset$. It is easy to see that the root system generated by $\alpha_2, \alpha_4, \theta_1, \theta_2$ contains the root $\alpha_3 + \alpha_4 + \alpha_5$. Therefore $\hat{R}$ is generated by $\alpha_2, \alpha_4, \theta_1, \theta_2$. It follows that $\hat{R}$ is of type D_4 with the first node marked. Therefore we have $\hat{X} = \mathbb{Q}_8$ as claimed.

Let $X = E_7/P_7$. Then we have $S_\Delta = \{\alpha_1, \alpha_6\}$. The root system R_2 is of type D_6 with the first node marked and its set of simple roots is given by $\{\alpha_2, \ldots, \alpha_7\}$. We already figured out that $\hat{R}_2 = R_2$. Since $\hat{R}_2 \subseteq \hat{R}$ and $S_\Delta \subseteq \hat{R}$ it follows that all simple roots Δ are contained in $\hat{R}$ and thus $\hat{R} = R$. It follows that $\hat{X} = X = E_7/P_7$ as claimed. □

Corollary 11.17. *Let $X = G/P$ be a cominuscule homogeneous space. Then $\hat{X}$ is also a cominuscule homogeneous space. Through three points of $\hat{X}$ in general position passes a unique rational curve of degree d_X which is contained in $\hat{X}$.*

Proof. Let X be a cominuscule homogeneous space. From the table in the previous lemma it is clear that $\hat{X}$ is also a cominuscule homogeneous space. From [11, Fact 20] and the table above we know that through three points of $\hat{X}$ in general position passes a unique rational curve of degree d_X which is contained in $\hat{X}$. □

Corollary 11.18. *Let $X = G/P$ be a cominuscule homogeneous space. Then we have an equality $\hat{X} = Y_{d_X}(f_\Delta)$.*

Proof. By the previous corollary the assumptions of Lemma 11.15 are satisfied for $Y = \hat{X}$, $d = d_X$ and $f = f_\Delta$. It follows that $\hat{X} \subseteq Y_{d_X}(f_\Delta)$. On the other hand we already know from the previous lemma that $\hat{X} \cong Y_{d_X}(f_\Delta)$. Therefore it follows that $\hat{X} = Y_{d_X}(f_\Delta)$ as claimed. □

Lemma 11.19. *Let $X = \mathbb{G}_Q(l, 2p)$ where $l \leq p-2$ and l odd. Then $\hat{R}$ is of type $\mathsf{D}_{l-1} \times \mathsf{D}_{p-l+1}$ and we have an isomorphism*

$$\hat{X} \cong \mathbb{G}_Q(l-1, 2(l-1)) \times \mathbb{Q}_{2(p-l)} .$$

Proof. Let $X = \mathbb{G}_Q(l, 2p)$ where $l \leq p-2$ and l is odd. We distinguish the cases where $l < p-2$ and where $l = p-2$. Assume first that $l < p-2$. The root system R' is generated by $\alpha_{l+2}, \dots, \alpha_p$ and thus of type D_{p-l-1}. The root system R' is irreducible if and only if $l+2 < p-1$. If R' is irreducible, let $\mathcal{B}'$ be the cascade of orthogonal roots associated to R'. If R' is not irreducible then $l+2 = p-1$ and we set $\mathcal{B}' = \{\alpha_{p-1}, \alpha_p\}$. With this notation we have a disjoint union $\mathcal{B} = C(\alpha_P) \amalg \{\alpha_1, \alpha_3, \dots, \alpha_{l-2}\} \amalg \mathcal{B}'$. Moreover we have $S_\Delta = \{\alpha_2, \alpha_4, \dots, \alpha_{l-3}, \alpha_{l+1}\}$. Let D_{l-1} be the root system generated by $\alpha_1, \alpha_2, \dots, \alpha_{l-2}, \theta_1$. It is easy to see that the root system D_{l-1} contains the roots $\theta_2, \dots, \theta_{k-1}$. By definition it is clear that the root system D_{l-1} is of type D_{l-1} with the $(l-1)$th node marked. Let D_{p-l+1} be the root system generated by θ_k, α_{l+1} and R'. Then we know that D_{p-l+1} is generated by $\alpha_l, \dots, \alpha_p$ since $\theta_k = \alpha_l$ and R' is generated by $\alpha_{l+2}, \dots, \alpha_p$. Therefore the root system D_{p-l+1} is of type D_{p-l+1} with the first node marked. From the definition it is clear that $\hat{R} = D_{l-1} \times D_{p-l+1}$ and thus $\hat{X} = \mathbb{G}_Q(l-1, 2(l-1)) \times \mathbb{Q}_{2(p-l)}$ as claimed.

Next we treat the case where $l = p-2$. Then we have $\mathcal{B} = C(\alpha_P) \cup \{\alpha_1, \alpha_3, \dots, \alpha_{l-2}\}$ and $S_\Delta = \{\alpha_2, \alpha_4, \dots, \alpha_{l-3}, \alpha_{p-1}, \alpha_p\}$ and $R' = \emptyset$. Let D_{l-1} be the root system generated by $\alpha_1, \alpha_2, \dots, \alpha_{l-2}, \theta_1$. It is easy to see that the root system D_{l-1} contains the roots $\theta_2, \dots, \theta_{k-2}$. By definition it is clear that the root system D_{l-1} is of type D_{l-1} with $(l-1)$th node marked. Let $D_{p-l+1} = D_3$ be the root system generated by $\alpha_{p-1}, \alpha_p, \theta_k$. Then it is clear that the root system D_3 is generated by $\alpha_{p-2}, \alpha_{p-1}, \alpha_p$ since $\theta_k = \alpha_{p-2}$ and contains the root θ_{k-1}. Therefore D_3 is of type D_3 with the first node marked. From the definition it is clear that $\hat{R} = D_{l-1} \times D_3$ and thus $\hat{X} = \mathbb{G}_Q(l-1, 2(l-1)) \times \mathbb{Q}_4 = \mathbb{G}_Q(l-1, 2(l-1)) \times \mathbb{Q}_{2(p-l)}$ as claimed. □

Corollary 11.20. *Let $X = \mathbb{G}_Q(l, 2p)$ with $l \leq p-2$ and l odd. Then we have $d_X = l+1$ and $k = (l+3)/2$.*

Proof. By the previous lemma we know that $\hat{X}$ is the product of two dualizing varieties $\mathfrak{X}_1$ and $\mathfrak{X}_2$ where $\mathfrak{X}_1 \cong \mathbb{G}_Q(l-1, 2(l-1))$ and $\mathfrak{X}_2 \cong \mathbb{Q}_{2(p-l)}$. Since $d_1 = \cdots = d_{k-2} = 2$ we have that $d_{\mathfrak{X}_1} = 2d_{\mathbb{G}_Q(l-1,2(l-1))} = 2(l-1)/2 = l-1$. Since $d_{k-1} = d_k = 1$ we have that $d_{\mathfrak{X}_2} = d_{\mathbb{Q}_{2(p-l)}} = 2$. Both equation together lead to the result that $d_X = d_{\hat{X}} = (l-1)+2 = l+1$.

Using $d_X = l + 1$ we can easily reformulate k in terms of l. Indeed, the equation $d_X = d_{\mathfrak{X}_1} + d_{\mathfrak{X}_2}$ can be reformulated as $l+1 = 2(k-2)+2$ and thus $k = (l+3)/2$. $\square$

11.2. **The group $\hat{W}$.** Let $u = s_{\theta_1} \cdots s_{\theta_k}$. For a root γ we write $t_\gamma = s_{u(\gamma)} s_\gamma$. Let W_δ be the subgroup of W generated by the set $\{t_\gamma \mid \gamma \perp u(\gamma)\}$. Let $\hat{W}$ be the subgroup of W generated by W_θ, W_δ and the set $\{s_\beta \mid \beta \perp \theta_i \text{ for all } 1 \leq i \leq k\}$.

Let Θ be the free $\mathbb{Z}$-module with generators $\theta_1, \ldots, \theta_k$. Let $\tilde{W}$ be the subgroup of W consisting of all elements $w \in W$ such that $w(\theta_i) \in \Theta$ for all $1 \leq i \leq k$.

Fact 11.21. *We have an inclusion of subgroups $\hat{W} \subseteq \tilde{W}$.*

Proof. To prove the inclusion $\hat{W} \subseteq \tilde{W}$ it is clearly sufficient to prove that a set of generators of $\hat{W}$ is contained in $\tilde{W}$. Since $W_\theta \subseteq \tilde{W}$ and $\{s_\beta \mid \beta \perp \theta_i \text{ for all } 1 \leq i \leq k\} \subseteq \tilde{W}$ is obvious it suffices to prove that $t_\gamma \in \tilde{W}$ for all roots γ which are orthogonal to $u(\gamma)$. Let γ be a root which is orthogonal to $u(\gamma)$. By the W-invariance of $\langle -, - \rangle$ we have $\langle u(\gamma)^\vee, \theta_i \rangle = - \langle \gamma^\vee, \theta_i \rangle$ and thus

$$t_\gamma(\theta_i) = \theta_i - \langle \gamma^\vee, \theta_i \rangle \gamma + \langle \gamma^\vee, \theta_i \rangle u(\gamma) .$$

Since $u(\gamma) = \gamma - \sum_{j=1}^{k} \left\langle \theta_j^\vee, \gamma \right\rangle \theta_j$ it follows that

$$t_\gamma(\theta_i) = \theta_i - \langle \gamma^\vee, \theta_i \rangle \sum_{j=1}^{k} \left\langle \theta_j^\vee, \gamma \right\rangle \theta_j \in \Theta .$$

Therefore we have that $t_\gamma \in \tilde{W}$ for all roots γ which are orthogonal to $u(\gamma)$. This completes the proof. $\square$

Fact 11.22. *The element u is contained in the center of $\hat{W}$.*

Proof. It clearly suffices to show that u commutes with a set of generator of $\hat{W}$. Since u clearly commutes with all elements of W_θ and all elements of $\{s_\beta \mid \beta \perp \theta_i \text{ for all } 1 \leq i \leq k\}$ it suffices to show that u commutes with t_γ for all roots γ which are orthogonal to $u(\gamma)$. Let γ be a root which is orthogonal to $u(\gamma)$. Then we have $ut_\gamma = (us_{u(\gamma)}u^{-1})(us_\gamma u^{-1})u = s_\gamma s_{u(\gamma)} u = t_\gamma u$ since $u^2 = 1$ and since $t_\gamma = s_\gamma s_{u(\gamma)}$ for a root γ which is orthogonal to $u(\gamma)$. This completes the proof. $\square$

Corollary 11.23. *Let γ be a root which is orthogonal to $u(\gamma)$. Let δ be an arbitrary root. Then we have $t_\gamma = t_\gamma^{-1}$ and $t_\gamma(\gamma) = -\gamma$ and $t_\gamma t_\delta t_\gamma^{-1} = t_{t_\gamma(\delta)}$. More generally let w be an element which commutes with u (which is in particular the case if $w \in \hat{W}$). Then we have $wt_\delta w^{-1} = t_{w(\delta)}$.*

Proof. Let γ and δ be as in the statement. The first identity is obvious since $t_\gamma = s_\gamma s_{u(\gamma)}$. The second identity is obvious since $s_{u(\gamma)}(\gamma) = \gamma$. Since $u \in Z(\hat{W})$ we know that u commutes with t_γ. Therefore the third identity follows from the very last statement. To prove the last statement, let w be an element which commutes with u. Then we have $wt_\delta w^{-1} = s_{wu(\delta)}s_{w(\delta)} = s_{uw(\delta)}s_{w(\delta)} = t_{w(\delta)}$ as claimed. □

Corollary 11.24. *Let γ be a root which is orthogonal to $u(\gamma)$. Then we have a strict inclusion of subgroups of $\hat{W}$ as follows:*

$$\mathrm{Stab}_{\hat{W}}(\gamma) \subsetneq C_{\hat{W}}(t_\gamma)\,.$$

Proof. Let $w \in \mathrm{Stab}_{\hat{W}}(\gamma)$. Since w commutes with u we have $wt_\gamma w^{-1} = t_{w(\gamma)} = t_\gamma$ and thus $w \in C_{\hat{W}}(t_\gamma)$. The element t_γ is clearly contained in $C_{\hat{W}}(t_\gamma)$ but not contained in $\mathrm{Stab}_{\hat{W}}(\gamma)$ since $t_\gamma(\gamma) = -\gamma$. Therefore the inclusion in the statement is strict. □

Corollary 11.25. *Let γ and δ be roots which are orthogonal to $u(\gamma)$ and $u(\delta)$. Then $t_\gamma(\delta)$ is a root which is orthogonal to $ut_\gamma(\delta)$.*

Proof. Indeed, let γ and δ be as in the statement. Since $u \in Z(\hat{W})$ it follows that $(ut_\gamma(\delta), t_\gamma(\delta)) = (t_\gamma u(\delta), t_\gamma(\delta)) = (u(\delta), \delta) = 0$. Therefore $t_\gamma(\delta)$ is orthogonal to $ut_\gamma(\delta)$ as claimed. □

Corollary 11.26. *Suppose that R is simply laced. The group $\hat{W}$ acts on the set S.*

Proof. Since W_θ obviously acts on the set S and since $s_\beta(\gamma) \in S$ for all $\gamma \in S$ and all roots β which are orthogonal to $\theta_1, \ldots, \theta_k$ (this holds even if R is arbitrary and not necessarily simply laced), it suffices to prove that $t_\gamma(\delta) \in S$ for all $\gamma, \delta \in S$. But this follows directly from the previous corollary. □

Let α be a root. We write $\alpha' = \alpha + u(\alpha)$. Let γ be a root which is orthogonal to $u(\gamma)$. Then we have $t_\gamma(\gamma') = -\gamma'$ since $t_\gamma(\gamma) = -\gamma$ and since $t_\gamma u(\gamma) = ut_\gamma(\gamma) = -u(\gamma)$. Moreover note that $u(\eta) = -\eta$ for all $\eta \in \Theta$, i.e. u acts as $-\mathrm{id}$ on Θ.

Lemma 11.27. *Let γ and δ be roots which are orthogonal to $u(\gamma)$ and $u(\delta)$ respectively. Suppose that t_γ and t_δ commute. Suppose further that δ is not orthogonal to γ'. Then we have $\delta = n\gamma + \eta$ for some $n \in \{\pm 1\}$ and some $\eta \in \Theta$.*

Proof. Let γ and δ be roots which satisfy the assumptions in the statement. To abbreviate let $\hat{\delta} = t_\gamma(\delta)$. For a root α we then have

$$\begin{aligned} \hat{\delta} &= \delta - \langle \gamma^\vee, \delta' \rangle\, \gamma + \Theta^{7} \\ t_\delta(\alpha) &= \alpha - \langle \delta^\vee, \alpha' \rangle\, \delta + \Theta \\ t_{\hat{\delta}}(\alpha) &= \alpha - \left\langle \hat{\delta}^\vee, \alpha' \right\rangle \hat{\delta} + \Theta\,. \end{aligned}$$

The equation $t_\gamma t_\delta = t_\delta t_\gamma$ is equivalent to $t_{\hat\delta} = t_\delta$. Evaluating this equation at α we find that

$$\langle \delta^\vee, \alpha' \rangle \delta = \left\langle \hat\delta^\vee, \alpha' \right\rangle \hat\delta + \Theta = \left\langle \hat\delta^\vee, \alpha' \right\rangle \delta - \left\langle \hat\delta^\vee, \alpha' \right\rangle \langle \gamma^\vee, \delta' \rangle \gamma + \Theta \, .$$

If we plug in $\alpha = \gamma$ and use that

$$\left\langle \hat\delta^\vee, \gamma' \right\rangle = \langle \delta^\vee, t_\gamma(\gamma') \rangle = - \langle \delta^\vee, \gamma' \rangle$$

we find that

$$2 \langle \delta^\vee, \gamma' \rangle \delta = \langle \delta^\vee, \gamma' \rangle \langle \gamma^\vee, \delta' \rangle \gamma + \Theta \, .$$

Since δ is not orthogonal to γ' we therefore may write $\delta = n\gamma + \eta$ for some $n \in \mathbb{Z}$ and some $\eta \in \Theta$. If we plug in this identity for δ in the equation $(u(\delta), \delta) = 0$ we find that

$$0 = n^2(u(\gamma), \gamma) + n(u(\gamma), \eta) + n(u(\eta), \gamma) + (u(\eta), \eta) = -2n(\eta, \gamma) - (\eta, \eta) \, .$$

If $\eta \neq 0$ this equation implies that $-n \langle \eta^\vee, \gamma \rangle = 1$ and thus $n \in \{\pm 1\}$. If $\eta = 0$ we have $\delta = n\gamma$ which again implies that $n \in \{\pm 1\}$ since the root system R is reduced. In all cases we find that $n \in \{\pm 1\}$. Therefore δ has the desired expression in terms of γ. □

Corollary 11.28. *Let γ and δ be two roots which are orthogonal to $u(\gamma)$ and $u(\delta)$. Suppose that $t_\gamma = t_\delta$. Then we have $\delta = n\gamma + \eta$ for some $n \in \{\pm 1\}$ and some $\eta \in \Theta$.*

Proof. Let γ and δ be roots which satisfy the assumptions in the statement. We clearly know that t_γ and t_δ commute. By the previous lemma it suffices to show that δ is not orthogonal to γ'. Suppose for a contradiction that δ is orthogonal to γ'. Then we have that

$$-\gamma = t_\gamma(\gamma) = t_\delta(\gamma) = \gamma - \langle \delta^\vee, \gamma' \rangle \delta + \Theta = \gamma + \Theta$$

which implies that $\gamma \in \Theta$ and thus $(u(\gamma), \gamma) = -(\gamma, \gamma) \neq 0$ – a contradiction. □

Lemma 11.29. *Let X be a simply laced dualizing variety, i.e. $\hat R = R$ and R is simply laced. Let $w \in W$ be an element which commmutes with all elements of $\hat W$. Then the element w is an involution.*

Proof. Let $w \in W$ be an element which commutes with all elements of $\hat W$. Since w commutes with s_{θ_i} we have $s_{\theta_i} w(\theta_i) = -w(\theta_i)$ and thus $w(\theta_i) = n_i \theta_i$ for some $n_i \in \{\pm 1\}$. Hence $w^2(\theta_i) = \theta_i$ for all $1 \leq i \leq k$. Similarly, since w commutes with all s_β where β is orthogonal to all θ_i, it follows that $w(\beta) = n_\beta \beta$ for some $n_\beta \in \{\pm 1\}$ and hence $w^2(\beta) = \beta$. Let $\gamma \in S$. Since w commutes with t_γ and with u it follows that $w t_\gamma w^{-1} = t_{w(\gamma)} = t_\gamma$. From the previous corollary it follows that

[7] We write here and in what follows Θ as a placeholder for an element of Θ.

$w(\gamma) = n\gamma + \eta$ for some $n \in \{\pm 1\}$ and $\eta \in \Theta$. Write $\eta = \sum_{i=1}^{k} b_i\theta_i$ for some $b_i \in \mathbb{Z}$. Then we find that

$$\begin{aligned} ws_{\theta_i}(\gamma) &= n\gamma + \eta - \langle \theta_i^\vee, \gamma \rangle \, n_i\theta_i \\ s_{\theta_i}w(\gamma) &= n\gamma + s_{\theta_i}(\eta) - \langle \theta_i^\vee, \gamma \rangle \, n\theta_i \end{aligned}$$

which gives that

$$\eta - s_{\theta_i}(\eta) = 2b_i\theta_i = \langle \theta_i^\vee, \gamma \rangle \, (n_i - n)\theta_i \, .$$

From the last equation we see that if $n = n_i$ then $b_i = 0$ and $b_i(n+n_i) = 0$. If $n \neq n_i$ then $n + n_i = 0$ and $b_i(n + n_i) = 0$ since $n, n_i \in \{\pm 1\}$. Therefore we get that $b_i(n + n_i) = 0$ for all i and thus

$$0 = \sum_{i=1}^{k} b_i\theta_i(n + n_i) = n\eta + w(\eta) \, .$$

Using the last equation, we find that

$$w^2(\gamma) = \gamma + n\eta + w(\eta) = \gamma$$

for all $\gamma \in S$.

In total we have that $w^2(\alpha) = \alpha$ for all roots α which are either equal to θ_i for some i or orthogonal to θ_i for all i or contained in S. Since the root system generated by all such α contains the root system $\hat{R}$ and since $\hat{R} = R$ by assumption, we conclude that $w^2(\alpha) = \alpha$ for all roots α. Therefore w is an involution. □

Corollary 11.30. *Let X be a simply laced dualizing vatiety. Then the center of $\hat{W}$ and the center of $\tilde{W}$ consist of involutions.*

Proof. An element $w \in Z(\hat{W}) \subseteq W$ clearly commutes with all elements of $\hat{W}$. An element $w \in Z(\tilde{W}) \subseteq W$ clearly commutes with all elements of $\tilde{W}$ and therefore also with all elements of $\hat{W}$ since $\hat{W} \subseteq \tilde{W}$. In each case the previous lemma implies that w is an involution. □

Conjecture 11.31. *Let X be a dualizing variety (not necessarily simply laced). Then the center of $\tilde{W}$ consists of involutions.*

Example 11.32. Let $X = \mathbb{G}(2, 4)$, so that $\theta_1 = \alpha_1 + \alpha_2 + \alpha_3$ and $\theta_2 = \alpha_2$ is the chain cascade associated to $\alpha_P = \alpha_2$. Then $W = S_4$, $u = (14)(23)$ and

$$\tilde{W} = \hat{W} = \{\mathrm{id}, (14), (23), (14)(23), (12)(34), (1243), (1342), (13)(24)\}$$

and the center of $\tilde{W}$ is given by $Z = Z(\tilde{W}) = Z(\hat{W}) = \{\mathrm{id}, (14)(23)\}$. We see that in this example the inclusion $\hat{W} \subseteq \tilde{W}$ is an equality, in particular $C_{\tilde{W}}(u) = \tilde{W}$. The set of positive roots γ orthogonal to $u(\gamma)$ is

$$S^+ = \{\alpha_1, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3\}$$

so that the group generated by all $s_{u(\gamma)}s_\gamma$ for all such γ is

$$W_\delta = \langle s_{-\alpha_3}s_{\alpha_1}, s_{-\alpha_2-\alpha_3}s_{\alpha_1+\alpha_2} \rangle = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\} \, .$$

Moreover we have

$$W_\theta = \langle s_{\theta_1}, s_{\theta_2} \rangle = \{\mathrm{id}, (14), (23), (14)(23)\}\,.$$

Then we see that $W_\theta \cap W_\delta = Z$, $W_\theta/Z \cong \mathbb{Z}/2\mathbb{Z}$ and $W_\delta/Z \cong \mathbb{Z}/2\mathbb{Z}$. Moreover

$$\tilde{W}/Z = \hat{W}/Z = W_\theta/Z \rtimes W_\delta/Z \cong \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$$

where the semi direct product is of course a direct product in this example. Furthermore we see that $\hat{W}$ acts transitively on S.

11.3. **The dimension formula for dualizing varieties.** Let β be a root. Then we write $J(\beta)$ for the set of indices $i \in \{1, \ldots, k\}$ such that β is not orthogonal to θ_i. For all $\gamma \in S$ we know that $J(\gamma) = \{i(\gamma), j(\gamma)\}$ with $i(\gamma) < j(\gamma)$.

Fact 11.33. *Suppose that R_k is of type A_1. Then we have the following equality:*

$$\{\beta \in R \mid J(\beta) = \emptyset\} = (\pm\mathcal{B} \cup R') \setminus \pm C(\alpha_P)\,.$$

Proof. The inclusion from right to left is obvious. We prove the inclusion from left to right. We do an induction on k. Suppose first that $k = 1$. Then R is of type A_1 by assumption and both sets under consideration are empty. Therefore the inclusion from left to right is obvious. Suppose that $k > 1$. Let $\beta \in R$ such that $J(\beta) = \emptyset$. It is clear that $\beta \notin \pm C(\alpha_P)$ since then we had $\mathrm{card}(J(\beta)) = 1$. Therefore it suffices to show that $\beta \in \pm\mathcal{B} \cup R'$. We distinguish the cases where R_2 is of type A_1 and where R_2 is not of type A_1. Suppose first that R_2 is of type A_1. Then it is clear that $k = 2$ and thus $R^\circ = R' \amalg R_2$. Since β is orthogonal to θ_1 we know that $\beta \in R^\circ$. Since $R_2 = \{\pm\theta_2\}$ we know that $\beta \notin R_2$ and thus $\beta \in R'$, in particular $\beta \in \pm\mathcal{B} \cup R'$. Assume next that R_2 is not of type A_1. By Lemma 8.3 we know that all irreducible components of R° different from R_2 are of type A_1 and thus contained in $\pm\mathcal{B}$. Therefore we see that $R^\circ \subseteq \pm\mathcal{B} \cup R_2$. Since β is orthogonal to θ_1 we know that $\beta \in R^\circ$. If in addition $\beta \in R_2$ the induction hypothesis yields that $\beta \in \pm\mathcal{B} \cup R'$ and we are done. If $\beta \notin R_2$ it follows from the inclusion $R^\circ \subseteq \pm\mathcal{B} \cup R_2$ that $\beta \in \pm\mathcal{B}$, in particular $\beta \in \pm\mathcal{B} \cup R'$. This completes the induction step. □

Example 11.34. Note that we cannot expect the equality in the fact to be true if the assumption that R_k is of type A_1 is dropped. To see this, we may take R of type D_p with p sufficiently large and $\alpha_P = \alpha_2$. Then it is clear that $k = 1$ and thus $R' = \emptyset$. The root α_4 is orthogonal to θ_1 but not contained in $\pm\mathcal{B}$. The inclusion from left to right fails, since $R_k = R$ is not of type A_1.

Corollary 11.35. *In complete generality, we have the following equality:*

$$\{\beta \in \hat{R} \mid J(\beta) = \emptyset\} = (\pm\mathcal{B} \cup R') \setminus \pm C(\alpha_P)\,.$$

Proof. The inclusion from right to left is obvious. We prove the inclusion from left to right. Let $\beta \in \hat{R}$ such that $J(\beta) = \emptyset$. It is clear that $\beta \notin \pm C(\alpha_P)$ since otherwise we had $\mathrm{card}(J(\beta)) = 1$. Therefore it suffices to show that $\beta \in \pm\mathcal{B} \cup R'$. Let $\beta \in \mathfrak{R}^i$ for some $1 \leq i \leq r$. If $C(\alpha_P) \cap \mathfrak{R}^i = \emptyset$ then $\mathfrak{R}^i \subseteq \pm\mathcal{B} \cap R_P$ and the assertion is obvious. Suppose that $C(\alpha_P) \cap \mathfrak{R}^i \neq \emptyset$. Then we know that $\hat{\mathfrak{R}}^i = \mathfrak{R}^i$ and thus the previous fact applies to the root system $\mathfrak{R}^i$ (Corollary 11.3). Therefore we get that $\beta \in \pm(\mathcal{B} \cap \mathfrak{R}^i) \cup \mathfrak{R}^{i\prime}$. If $R' \cap \mathfrak{R}^i = \emptyset$ then $\mathfrak{R}^{i\prime} \subseteq \mathcal{B} \cap \mathfrak{R}^i$ and if $R' \cap \mathfrak{R}^i \neq \emptyset$ then $\mathfrak{R}^{i\prime} = R'$ (Lemma 11.13). In both cases we get $\pm(\mathcal{B} \cap \mathfrak{R}^i) \cup \mathfrak{R}^{i\prime} \subseteq \pm\mathcal{B} \cup R'$ which completes the proof. □

Corollary 11.36. *Let X be a dualizing variety. Then we have the following inequality:*

$$\{\beta \in R \mid J(\beta) = \emptyset\} = (\pm\mathcal{B} \cup R') \setminus \pm C(\alpha_P)\,.$$

Proof. This follows either directly from the previous corollary since $\hat{R} = R$ for a dualizing variety, or it follows from the original fact since R_k is of type A_1 for a dualizing variety by Corollary 11.3. □

Lemma 11.37. *A simply laced dualizing variety is a cominuscule homogeneous space.*

Proof. We argue by contradiction. Suppose that there is a simply laced dualizing variety X which is not cominuscule. First we note that $k > 1$ since otherwise $k = 1$ and $X \cong \mathbb{P}^1$ were a cominuscule homogeneous space. By Corollary 11.3 we know that $d_k = 1$ and thus we can choose an index i minimal with the property that $d_{i+1} = 1$. Since $\hat{R}_i = R_i$ we may replace R with R_i (or X with X^i) and therefore may assume that $d_1 > d_2 = \cdots = d_k = 1$.

We now do a case by case analysis to show that such a variety cannot exists. It is clear that R cannot be of type A_{n-1} since all all quotients of GL_n by a maximal parabolic subgroup are cominuscule homogeneous spaces. Assume that R is of type D_p where $p \geq 4$. Since X is not cominuscule and since $k > 1$ we know that $\alpha_P \in \{\alpha_3, \ldots, \alpha_{p-2}\}$ and that $p > 4$. Since $d_2 = 1$ we conclude that $\alpha_P = \alpha_3$. Let $l = 3$. Then $X = \mathbb{G}_Q(l, 2p)$ where $l \leq p - 2$ and l odd. By Lemma 11.19 we know that this variety is not a dualizing variety. Next assume that R is of type E_6. Since X is not cominuscule we know that $\alpha_P \notin \{\alpha_1, \alpha_6\}$. Since $k > 1$ we know that $\alpha_P \neq \alpha_2$. Therefore we conclude that $\alpha_P \in \{\alpha_3, \alpha_4, \alpha_5\}$. Then it follows that $\alpha_2 \notin S_\Delta$ and thus that θ_1 generates an irreducible component of $\hat{R}$. This contradicts the fact that $\hat{R} = R$ is irreducible. Next assume that R is of type E_7. Since $k > 1$ and X is not cominuscule, we conclude that $\alpha_P \in \{\alpha_2, \ldots, \alpha_6\}$. Since $d_2 = \cdots = d_k = 1$ we conclude further that $\alpha_P \in \{\alpha_2, \alpha_3\}$. Then it follow that $\alpha_1 \notin S_\Delta$ and thus that θ_1 generates an irreducible component of $\hat{R}$. This contradicts the fact that $\hat{R} = R$ is irreducible. Next assume that R is of type E_8. Since $k > 1$ and $d_2 = \cdots = d_k = 1$

we conclude that $\alpha_P = \alpha_7$. Then it follows that $\alpha_8 \notin S_\Delta$ and thus that θ_1 generates an irreducible component of $\hat{R}$. This contradicts the fact that $\hat{R} = R$ is irreducible.

In each case follows a contradiction. Therefore a simply laced dualizing variety which is not cominuscule cannot exists. In other words: every simply laced dualizing variety is a cominuscule homogeneous space. □

Lemma 11.38. *Suppose that R is simply laced. Let $\gamma \in S$. Let $i = i(\gamma)$ and $j = j(\gamma)$. Then we have that $s_{\theta_i} s_\gamma \preceq s_{\theta_i}$ and thus $\delta(s_{\theta_i} s_\gamma) \leq d_i$. Moreover we have that $s_{\theta_j} s_\gamma \succeq s_{\theta_j}$ and thus $\delta(s_{\theta_j} s_\gamma) \geq d_j$.*

Proof. We statement in the third sentence holds even if R is not necessarily simply laced. Indeed, we know that $s_{\theta_j} s_\gamma$ and s_{θ_j} must be comparable in the Bruhat order. Suppose for a contradiction that $s_{\theta_j} s_\gamma \preceq s_{\theta_j}$. Then it follows that $s_\gamma \in W_{G_j}$ and thus $\gamma \in R_j$. This would mean that γ is orthogonal to θ_i – a contradiction. Therefore we conclude that $s_{\theta_j} s_\gamma \succeq s_{\theta_j}$ and thus $\delta(s_{\theta_j} s_\gamma) \geq d_j$.

Next suppose that R is simply laced. We first show that $\delta(s_{\theta_i} s_\gamma) \leq d_i$. For this purpose we may well assume that γ is positive. We then know that $\langle \theta_i^\vee, \gamma \rangle = \langle \gamma^\vee, \theta_i \rangle = 1$. Furthermore we know that $\gamma \in R_i$ and thus $\gamma < \theta_i$ or equivalent $\theta_i - \gamma > 0$. Since $s_{\theta_i} s_\gamma = s_\gamma s_{s_\gamma(\theta_i)} = s_\gamma s_{\theta_i - \gamma}$ and since R is simply laced it then follows that

$$\delta(s_{\theta_i} s_\gamma) \leq d(\gamma) + d(\theta_i - \gamma) = d_i \,.$$

Since $\gamma, \theta_i \in R_i$ this inequality implies that $s_{\theta_i} s_\gamma \preceq z^i_{d_i}$. The statement $s_{\theta_i} s_\gamma \preceq s_{\theta_i}$ is now obvious since $s_{\theta_i} W_P = z^i_{d_i} W_P$. □

Lemma 11.39. *Let X be a cominuscule homogeneous space. Let $\gamma \in S$ and $i = i(\gamma)$ and $j = j(\gamma)$. Then we have that $s_{\theta_i} s_\gamma \preceq s_{\theta_i}$ and $\delta(s_{\theta_i} s_\gamma) = d_i$. Moreover we have that $s_{\theta_j} s_\gamma \succeq s_{\theta_j}$ and $\delta(s_{\theta_j} s_\gamma) = d_j$.*

Proof. It clearly suffices to show that $\delta(s_{\theta_i} s_\gamma) = d_i$ and that $\delta(s_{\theta_j} s_\gamma) = d_j$. The relations in the Bruhat order follow immediately as in the proof of the previous lemma. Let δ be the unique element of $\mathcal{P}$ such that $\gamma = w(\delta)$ for some $w \in W_\theta$. We now distinguish the four possibilities we have for w.

If w is the identity, then $\gamma = \delta \in R_P$ and the equalities are obvious. If $w = s_{\theta_i}$ then $s_{\theta_i} s_\gamma = s_\delta s_{\theta_i}$ and thus $\delta(s_{\theta_i} s_\gamma) = \delta(s_{\theta_i}) = d_i$ since $\delta \in R_P$. Moreover $s_{\theta_j} s_\gamma = s_{s_{\theta_i} s_{\theta_j}(\delta)} s_{\theta_j}$ and thus $\delta(s_{\theta_j} s_\gamma) = \delta(s_{\theta_j}) = d_j$ since $s_{\theta_i} s_{\theta_j}(\delta) \in R_P$. If $w = s_{\theta_j}$ then the equalities follow in exactly the same way as in the case that $w = s_{\theta_i}$ by replacing the indices i and j. If $w = s_{\theta_i} s_{\theta_j}$ then $\gamma \in R_P$ and the equalities are obvious. Therefore in all cases the equalities follow. □

We already know that $W_\theta \subseteq \mathcal{U}$. The following corollary produces other elements of $\mathcal{U}$ in the case that X is a cominuscule homogeneous space.

Corollary 11.40. *Let X be a cominuscule homogeneous space. Let $\gamma \in S$ and let F be a subset of $\{1,\dots,k\}$ such that $F \cap J(\gamma)$ consists of precisely one element. Then we have that $\left(\prod_{l\in F} s_{\theta_l}\right) s_\gamma \in \mathcal{U}$ and that $\delta\left(\left(\prod_{l\in F} s_{\theta_l}\right) s_\gamma\right) = \sum_{l\in F} d_l$. In particular if $i \in J(\gamma)$ we have that $s_{\theta_i} s_\gamma \in \mathcal{U}$ and that $\delta(s_{\theta_i} s_\gamma) = d_i$.*

Proof. Let F be a subset of $\{1,\dots,k\}$ such that $F \cap J(\gamma) = \{i\}$ for some i. Let j be the index such that $J(\gamma) = \{i,j\}$. Let $u = \left(\prod_{l\in F} s_{\theta_l}\right) s_\gamma$. We then have

$$\delta(u) \leq \delta\left(\prod_{l\in F\setminus\{i\}} s_{\theta_l}\right) + \delta(s_{\theta_i} s_\gamma) = \sum_{l\in F} d_l$$

by the previous lemma. Write F^c for the complement of F in $\{1,\dots,k\}$. Then we have

$$\delta(u^*) = \delta\left(\left(\prod_{l\in F^c} s_{\theta_l}\right) s_\gamma\right) \leq \delta\left(\prod_{l\in F^c\setminus\{j\}} s_{\theta_l}\right) + \delta(s_{\theta_j} s_\gamma) = \sum_{l\in F^c} d_l$$

by the previous lemma. Both inequalities together yield that $\delta(u) + \delta(u^*) \leq d_X$. Therefore it follows that $u \in \mathcal{U}$ and that we have equalities everywhere. This means in particular that $\delta(u) = \sum_{l\in F} d_l$ as claimed. The very last statement follows by takting $F = \{i\} \subseteq J(\gamma)$. □

Fact 11.41. *Let X be a cominuscule homogeneous space. Suppose that R is simply laced. Let $\gamma \in S$. Then we have the following result:*

$$\delta(t_\gamma) = \begin{cases} 2 & \text{if } \gamma \in R \setminus R_P \\ 0 & \text{if } \gamma \in R_P \end{cases}.$$

Moreover we see that $t_\gamma \in \mathcal{U}$ for all $\gamma \in S$.

Proof. Write $u = s_{\theta_1} \cdots s_{\theta_k}$ for short. Suppose first that $\gamma \in R_P$. Then it is clear that also $u(\gamma) \in R_P$ and thus $t_\gamma \in W_P$ and thus $\delta(t_\gamma) = 0$. Suppose next that $\gamma \in R \setminus R_P$. We may assume that γ is positive without changing the situation. Let $i = i(\gamma)$ and let $j = j(\gamma)$. By the previous corollary we have that $\delta(t_\gamma) \leq \delta(s_{u(\gamma)} s_{\theta_i}) + \delta(s_{\theta_i} s_\gamma) = 2d_i = 2$. Since $\gamma \in R^+ \setminus R_P^+$ we know that $u(\gamma) \in R^- \setminus R_P^-$. Since γ and $u(\gamma)$ are orthogonal we therefore compute that $t_\gamma(\omega) = \omega + u(\gamma) - \gamma = \omega - \theta_i - \theta_j$. If $\delta(t_\gamma) = 0$ then $t_\gamma \in W_P$ and thus $t_\gamma(\omega) = \omega$. With the prvious computation we conclude that $\theta_i = -\theta_j$ which is absurd. Therefore we conclude that $\delta(t_\gamma) > 0$. If $\delta(t_\gamma) = 1$ then t_γ is P-indecomposable. In particular there exists a P-indecomposable root $\beta \in R^+ \setminus R_P^+$ such that $s_\beta W_P = t_\gamma W_P$. With the previous computation we conclude that $\beta = \theta_i + \theta_j$ which is absurd since X is cominuscule. Therefore we conclude that $\delta(t_\gamma) > 1$ and thus $\delta(t_\gamma) = 2$ as claimed.

Finally we want to prove that $t_\gamma \in \mathcal{U}$ for all $\gamma \in S$. If $\gamma \in R_P$ then there is nothing to prove. Assume that $\gamma \in R^+ \setminus R_P^+$. In view of the previous computation, to see the result it clearly suffices to prove that $\delta(ut_\gamma) \leq k-2$. Now $ut_\gamma = s_\gamma u s_\gamma = s_{\theta_1} \cdots s_{s_\gamma(\theta_i)} \cdots s_{s_\gamma(\theta_j)} \cdots s_{\theta_k}$. Since both $s_\gamma(\theta_i), s_\gamma(\theta_j) \in R_P$ we conclude that $\delta(ut_\gamma) \leq k-2$ as claimed. This completes the proof. □

Lemma 11.42. *Suppose that X is a cominuscule homogeneous space and that R is simply laced. Let $\gamma, \delta \in S$ such that $J(\gamma) \cap J(\delta) = \emptyset$. Then γ is orthogonal to δ.*

Proof. Let $\gamma, \delta \in S$ such that $J(\gamma) \cap J(\delta) = \emptyset$. Suppose for a contradiction that γ is not orthogonal to δ. We may assume that $\gamma, \delta \in R^+ \setminus R_P^+$ otherwise we replace if necessary γ and δ by other roots in their W_θ-orbit without changing the sets $J(\gamma)$ and $J(\delta)$ and without changing the property that γ is not orthogonal to δ. In this situation we must have $\langle \gamma^\vee, \delta \rangle > 0$ since otherwise the root $s_\gamma(\delta)$ had α_P-coefficient greater than one which contradicts the fact that X is a cominuscule homogneous space. Since $\gamma \neq \pm\delta$ and since R is simply laced we conclude that $\langle \gamma^\vee, \delta \rangle = \langle \delta^\vee, \gamma \rangle = 1$. This means that $s_\gamma(\delta) = \delta - \gamma$ and $s_\delta(\gamma) = \gamma - \delta$ are both contained in R_P. We may assume that $i(\gamma) < i(\delta) < j(\delta)$ by replacing γ with δ if necessary. It then follows that the root $\delta - \gamma$ is smaller than $\delta < \theta_{i(\delta)}$ and thus contained in $R_{i(\delta)}$ which means that the root $\delta - \gamma$ is orthogonal to $\theta_1, \ldots, \theta_{i(\delta)-1}$, in particular orthogonal to $\theta_{i(\gamma)}$. On the other hand we have $(\delta - \gamma, \theta_{i(\gamma)}) = -(\gamma, \theta_{i(\gamma)}) = -1$ – a contradiction. Therefore we conclude that the initial assumption is false and thus that γ is orthogonal to δ. □

Remark 11.43. The converse of the previous lemma is false. For instance, let R be of type A_5 and $\alpha_P = \alpha_3$. Let $\gamma = \alpha_1$ and $\delta = \alpha_5$. Then $\gamma, \delta \in S_\Delta \subseteq \mathcal{P} \subseteq S$ such that γ is orthogonal to δ and such that $J(\gamma) = J(\delta) = \{1, 2\}$.

Lemma 11.44. *Suppose that X is a cominuscule homogeneous space and that R is simply laced. Let $\gamma \in S$ and let $\delta \in R$ such that $J(\delta) = \{i\}$. If $i \notin J(\gamma)$ then γ is orthogonal to δ.*

Proof. The proof follows among the same lines as the proof of the previous lemma. □

Lemma 11.45. *Let R be simply laced. Let γ and δ be two different elements in S^+ such that γ is not orthogonal to δ and such that $J(\gamma) \cap J(\delta) \neq \emptyset$. Let $i \in J(\gamma) \cap J(\delta)$. Then either we have that $(s_\gamma(\delta), \theta_i) = (s_\delta(\gamma), \theta_i) = 0$ or (exclusively) that $s_\gamma(\delta), s_\delta(\gamma) \in \{\pm\theta_i\}$.*

Proof. Since R is simply laced we have that $s_\gamma(\delta) = -\langle \gamma^\vee, \delta \rangle s_\delta(\gamma)$. Thus $(s_\gamma(\delta), \theta_i) = 0$ is equivalent to $(s_\delta(\gamma), \theta_i) = 0$ and $s_\gamma(\delta) \in \{\pm\theta_i\}$ is equivalent to $s_\delta(\gamma) \in \{\pm\theta_i\}$. It is obvious that the logical or is exclusive. Therefore we only have to prove one of each statements. Since R is simply laced we know that $\langle \gamma^\vee, \delta \rangle, \langle \theta_i^\vee, \gamma \rangle, \langle \theta_i^\vee, \delta \rangle \in \{\pm 1\}$ and thus that $\langle \theta_i^\vee, s_\gamma(\delta) \rangle \in \{-2, 0, 2\}$. If the later bracket is zero the first case occurs, if the later bracket is ± 2 then $s_\gamma(\delta) \in \{\pm\theta_i\}$ and the second case occurs. $\square$

Corollary 11.46. *Let R be simply laced. Let $\gamma, \delta \in S$ be two elements such that γ is not orthogonal to δ. Suppose that $J(\gamma) \cap J(\delta) = \{i\}$. Then $(s_\gamma(\delta), \theta_i) = (s_\delta(\gamma), \theta_i) = 0$.*

Proof. We may assume that γ and δ are positive without changing the situation. Then the previous lemma applies to γ and δ. Suppose that $s_\gamma(\delta) = \pm\theta_i$. Let $j \in J(\delta) \setminus \{i\}$. Then $(\pm\theta_i, \theta_j) = 0$ but $(s_\gamma(\delta), \theta_j) = (\delta, \theta_j) \neq 0$ – a contradiction. Therefore the second case does not occure. The conclusion follows. $\square$

Lemma 11.47. *Let X be a cominuscule homogeneous space and let R be simply laced. Let $\gamma, \delta \in S$. Then* $\mathrm{card}(J(s_\gamma(\delta))) \leq 2$.

Proof. If γ and δ are orthogonal, then $s_\gamma(\delta) = \delta \in S$ and the result is trivial. Therefore we may assume that γ and δ are not orthogonal. If $\gamma = \pm\delta$, then $s_\gamma(\delta) = -\delta \in S$ and the result is trivial. Therefore we may assume that $\gamma \neq \pm\delta$. Moreover we may assume that γ and δ are both positive without changing the situation. Since γ is not orthogonal to δ we know by Lemma 11.42 that $J(\gamma) \cap J(\delta) \neq \emptyset$. There are two possibility: either $J(\gamma) \cap J(\delta) = \{i\}$ or $J(\gamma) = J(\delta)$. Assume first that the first possibility occurs. Then the previous corollary implies that $J(s_\gamma(\delta)) = \{j, l\}$ where $J(\gamma) = \{i, j\}$ and $J(\delta) = \{i, l\}$, in particular the condition in the statement is satisfied. Suppose next that $J(\gamma) = J(\delta)$. Then it is directly clear that $J(s_\gamma(\delta)) \subseteq J(\gamma) = J(\delta)$ and the condition in the statement is again satisfied. $\square$

Lemma 11.48. *Let X be a cominuscule homogeneous space and let R be simply laced. Let $\gamma \in S$ and let $\delta \in R$ such that $J(\delta) = \{i\}$. Then* $\mathrm{card}(J(s_\gamma(\delta))) \leq 2$.

Proof. If γ and δ are orthogonal, then $s_\gamma(\delta) = \delta$ and the result is trivial. Assume that γ and δ are not orthogonal. By Lemma 11.44 we then know that $i \in J(\gamma)$. But then it is obvious that $J(s_\gamma(\delta)) \subseteq J(\gamma)$ and the result follows. $\square$

Lemma 11.49. *Let X be a cominuscule homogeneous space and let R be simply laced. Let $\gamma \in \mathcal{B} \cup S \cup R'$ and let $\delta \in R$ such that* $\mathrm{card}(J(\delta)) \leq 2$. *Suppose that either $\delta \in \hat{R}$ or that R_k is of type A_1. Then* $\mathrm{card}(J(s_\gamma(\delta))) \leq 2$.

Proof. If $\mathrm{card}(J(\delta)) = 2$ then $\delta \in S$. If $\gamma \in S$ also, then Lemma 11.47 yields the result. The reflections along roots in $\mathcal{B} \cup R'$ clearly act on S. Therefore $s_\gamma(\delta) \in S$ if $\gamma \in \mathcal{B} \cup R'$ and the result is clear. Assume next that $\mathrm{card}(J(\delta)) = 1$. If $\gamma \in S$, then Lemma 11.48 yields the result. The reflections along roots in $\mathcal{B} \cup R'$ clearly act on the set $\{\beta \in R \mid \mathrm{card}(J(\beta)) = 1\}$. Thus $J(s_\gamma(\delta))$ consists of one element if

$\gamma \in \mathcal{B} \cup R'$ and the result is clear. Finally assume that $J(\delta) = \emptyset$. Then $\delta \in (\pm\mathcal{B} \cup R') \setminus \pm C(\alpha_P)$. If $\gamma \in S$ the result follows since reflections along roots in $\mathcal{B} \cup R'$ act on S. If $\gamma \in \mathcal{B} \cup R'$ the result follows since $\pm\mathcal{B} \cup R'$ is a roots system whose elements β all satisfy $\mathrm{card}(J(\beta)) \leq 1$. In all cases the result follows. □

Lemma 11.50. *Let X be a cominuscule homogeneous space and let R be simply laced. Then we have the following inclusion:*

$$\hat{R} \subseteq \{\alpha \in R \mid \mathrm{card}(J(\alpha)) \leq 2\}\,.$$

Proof. Let $\alpha \in \hat{R}$. Then there exists roots $\gamma_1, \ldots, \gamma_n, \delta \in \mathcal{B} \cup S_\Delta \cup R'$ such that $\alpha = s_{\gamma_1} \cdots s_{\gamma_n}(\delta)$. We prove by induction on n that $\mathrm{card}(J(\alpha)) \leq 2$. If $n = 0$ there is nothing to prove. If $n > 0$ then the induction hypotheses implies that $\mathrm{card}(J(\alpha')) \leq 2$ where $\alpha' = s_{\gamma_2} \cdots s_{\gamma_n}(\delta)$. The previous lemma then implies that also $\mathrm{card}(J(\alpha)) \leq 2$ where $\alpha = s_{\gamma_1}(\alpha')$. This completes the proof. □

Corollary 11.51. *Let X be a simply laced dualizing variety. Then we have* $\mathrm{card}(J(\alpha)) \leq 2$ *for all $\alpha \in R$.*

Proof. A simply laced dualizing variety is a cominuscule homogeneous space. Therefore the previous lemma applies and yields the result since $R = \hat{R}$. □

Lemma 11.52. *Let X be a cominuscule homogeneous space and let R be simply laced. Then we have the following equality:*

$$\{\alpha \in \hat{R} \mid \mathrm{card}(J(\alpha)) = 1\} = \pm C(\alpha_P)\,.$$

Proof. The inclusion from right to left is obvious. We prove the inclusion from left to right. Let $\alpha \in \hat{R}$ such that $\mathrm{card}(J(\alpha)) = 1$. By definition of $\hat{R}$ there exists $\gamma_1, \ldots, \gamma_n, \delta \in \mathcal{B} \cup S_\Delta \cup R'$ such that $\alpha = s_{\gamma_1} \cdots s_{\gamma_n}(\delta)$. Since reflections along roots in $\mathcal{B} \cup R'$ act on S we may assume that $\gamma_1, \ldots, \gamma_l \in \mathcal{B} \cup R'$ and that $\gamma_{l+1}, \ldots, \gamma_n \in S \cap \hat{R}$ for some $0 \leq l \leq n$. Note that $\alpha \in \pm C(\alpha_p)$ if and only if $s_{\gamma_l} \cdots s_{\gamma_1}(\alpha) \in \pm C(\alpha_P)$. Since reflections along roots in $\mathcal{B} \cup R'$ clearly act on the set $\{\beta \in R \mid \mathrm{card}(J(\beta)) = 1\}$, we have $J(\alpha) = J(s_{\gamma_l} \cdots s_{\gamma_1}(\alpha))$. Therefore we may assume that $l = 0$ in other words that $\gamma_1, \ldots, \gamma_n \in S \cap \hat{R}$.

We now proceed by induction on n. If $n = 0$ the assertion is obvious. Let $n > 0$ and let $\alpha' = s_{\gamma_2} \cdots s_{\gamma_n}(\delta)$. If $J(\alpha')$ consists of one element, then the induction hypothesis yieds that $\alpha' \in \pm C(\alpha_P)$ and consequently that $\alpha = s_{\gamma_1}(\alpha') \in S$ since α' acts on S. This contradicts the fact that $J(\alpha)$ consists of one element. Similar, if $J(\alpha')$ is empty, then $\alpha = s_{\gamma_1}(\alpha') \in S$ since $(\mathcal{B} \cup R') \setminus C(\alpha_P)$ acts on S. This contradicts the fact that $J(\alpha)$ consists of one element. From the definition of α' it is clear that $\alpha' \in \hat{R}$. Lemma 11.50 yields that $\mathrm{card}(J(\alpha')) = 2$ and thus $\alpha' \in S$. Since $\alpha \notin S$ we may assume that γ_1 and α' are not orthogonal. Moreover we may assume that γ_1 and α' are both positive without changing the situation. The fact that γ_1 and α' are not orthogonal implies that $J(\gamma_1) \cap J(\alpha') \neq \emptyset$. Since $J(\gamma_1) \triangle J(\alpha') \subseteq J(\alpha)$ we see that $J(\alpha') = J(\gamma_1)$. Let $J(\alpha) = \{i\}$. Then we obviously have $i \in J(\alpha') \cap J(\gamma_1)$. We are now in the situation to apply Lemma 11.45. Since $(\alpha, \theta_i) = (s_{\gamma_1}(\alpha'), \theta_i) \neq 0$ we must have $\alpha \in \{\pm\theta_i\}$ as desired. □

Corollary 11.53. *Let X be a simply laced dualizing variety. Then we have the following equality:*

$$\{\alpha \in R \mid \operatorname{card}(J(\alpha)) = 1\} = \pm C(\alpha_P)\,.$$

Proof. A simply laced dualizing variety is a cominuscule homogeneous space. Therefore the previous lemma applies and yields the result since $R = \hat{R}$. □

Lemma 11.54 (Dimension formula for dualizing varieties)**.** *Let X be a simply laced dualizing variety. Then we have the following formula for the dimension of X:*

$$\dim(X) = k + \operatorname{card}(\mathcal{P})\,.$$

Proof. By Corollary 11.51 we have

$$R^+ \setminus R_P^+ = \coprod_{i=0}^{2} \{\alpha \in R^+ \setminus R_P^+ \mid \operatorname{card}(J(\alpha)) = i\}\,.$$

Since $\{\beta \in R \mid \operatorname{card}(J(\beta)) = 0\} = (\pm\mathcal{B} \cup R') \setminus \pm C(\alpha_P) \subseteq R_P$ the first of the three summands is empty. By the previous corollary the second summand is equal to $C(\alpha_P)$. The third summand is by definition equal to $(R^+ \setminus R_P^+) \cap S$. The later set is a set of representatives of the W_θ-orbits in S. The set $\mathcal{P}$ is also a set of representatives of the W_θ-orbits in S. Therefore the set $(R^+ \setminus R_P^+) \cap S$ is in bijection with $\mathcal{P}$ which means that $\operatorname{card}((R^+ \setminus R_P^+) \cap S) = \operatorname{card}(\mathcal{P})$. Putting these facts together we conclude that

$$\dim(X) = \operatorname{card}(R^+ \setminus R_P^+) = k + \operatorname{card}(\mathcal{P})\,.$$

□

Lemma 11.55. *Let X be a cominuscule homogeneous space and let R be simply laced. Then we have the following upper bound for the dimension of $\hat{X}$:*

$$\dim(\hat{X}) \leq k + \operatorname{card}(\mathcal{P})\,.$$

Proof. In view of the previous results, the proof of this upper bound for $\dim(\hat{X})$ follows among the same lines as the proof of the previous lemma. The only difference is that we only have an inclusion $\{\alpha \in \hat{R}^+ \setminus R_P^+ \mid \operatorname{card}(J(\alpha)) = 2\} \subseteq (R^+ \setminus R_P^+) \cap S$ where we do not know if always equality holds. □

Lemma 11.56. *Suppose that R is simply laced. Let $\gamma \in \mathcal{P}$. Let $i = i(\gamma)$ and $j = j(\gamma)$. Then we have that γ is not a rational multiple of $\theta_i - \theta_j$.*

Proof. Suppose for a contradiction that $\gamma = (\theta_i - \theta_j)q$ for some $q \in \mathbb{Q}^\times$. Then we know that $s_{\theta_j}(\gamma) = \gamma + \theta_j = q\theta_i + (1-q)\theta_j$ is a root. Since R is simply laced it follows that $\left\langle \gamma^\vee, s_{\theta_j}(\gamma) \right\rangle = 2q^2 - 2q(1-q) = 2q(2q-1)$. On the other hand we compute that $\left\langle \gamma^\vee, s_{\theta_j}(\gamma) \right\rangle = \langle \gamma^\vee, \gamma + \theta_j \rangle = 2 - 1 = 1$. Both equation together yield that $1 = 2q(2q-1)$. Since this equation has no rational solutions we end up with a contradiction. $\square$

Corollary 11.57. *Suppose that R is simply laced. Then we know that the cardinality of $\mathcal{P}$ is divisible by two.*

Proof. We first prove that the element $-s_{\theta_1} \cdots s_{\theta_k}$ acts on $\mathcal{P}$. Indeed, let $\gamma \in \mathcal{P}$ and let $i = i(\gamma)$ and $j = j(\gamma)$. Then we know that $\delta = -s_{\theta_1} \cdots s_{\theta_k}(\gamma) = \theta_i - (\gamma + \theta_j)$ is a positive roots in S which satisfies $i = i(\delta)$ and $j = j(\delta)$. Furthermore we see by direct computation that $\left\langle \theta_j^\vee, \delta \right\rangle = -1 < 0$. All together this means by definition that $\delta \in \mathcal{P}$.

From this discussion we see that $\mathcal{P}$ is paritioned by the orbits of the group $\{1, -s_{\theta_1} \cdots s_{\theta_k}\} \cong \mathbb{Z}/2\mathbb{Z}$. We are left to show that each of the orbits consists of precisely two elements, in other words we are left to show that $-s_{\theta_1} \cdots s_{\theta_k}(\gamma) \neq \gamma$ for all $\gamma \in \mathcal{P}$. Suppose for a contradiction that $-s_{\theta_1} \cdots s_{\theta_k}(\gamma) = \gamma$ for some $\gamma \in \mathcal{P}$. Then it immediately follows that $\gamma = (\theta_{i(\gamma)} - \theta_{j(\gamma)})/2$ which contradicts the previous lemma. $\square$

Lemma 11.58. *Suppose that R is simply laced. Let $\gamma \in S$. Then the sum of two roots in $W_\theta(\gamma)$ is never a root: let $\gamma', \delta' \in W_\theta(\gamma)$ then $\gamma' + \delta'$ is not a root.*

Proof. To prove the lemma we clearly can assume that $\gamma \in \mathcal{P}$ since $\mathcal{P}$ is a set of representatives of the W_θ-orbits in S. Then we have that

$$W_\theta(\gamma) = \{\gamma, \gamma - \theta_i, \gamma + \theta_j, \gamma - \theta_i + \theta_j\}$$

where $i = i(\gamma)$ and $j = j(\gamma)$. In order to prove the statement it clearly suffices to show that none of the following elements:

$$2\gamma - \theta_i, 2\gamma + \theta_j, 2\gamma - \theta_i + \theta_j, 2\gamma - 2\theta_i + \theta_j, 2\gamma - \theta_i + 2\theta_j$$

is a root. Suppose that $2\gamma - \theta_i$ is a root. Then we have that

$$\langle \gamma^\vee, 2\gamma - \theta_i \rangle = 4 - 1 = 3$$

which is impossible since R is simply laced. Suppose that $2\gamma + \theta_j$ is a root. Then we have that $\langle \gamma^\vee, 2\gamma + \theta_j \rangle = 4 - 1 = 3$ which is impossible since R is simply laced. Suppose that $2\gamma - \theta_i + \theta_j$ is a root. Then we have that $\langle \gamma^\vee, 2\gamma - \theta_i + \theta_j \rangle = 4 - 1 - 1 = 2$. Since R is simply laced, this implies that $2\gamma - \theta_i + \theta_j = \pm\gamma$ which means that either $\gamma = \theta_i - \theta_j$ or $\gamma = (\theta_i - \theta_j)/3$. Both cases imply that there exists a rational number q such that $\gamma = (\theta_i - \theta_j)q$ which contradicts the previous lemma. Suppose that $2\gamma - 2\theta_i + \theta_j$ is a root. Then we have that $\langle \gamma^\vee - \theta_i^\vee, 2\gamma - 2\theta_i + \theta_j \rangle = 4-2-1-2+4 = 3$ which is impossible since R is simply laced. Suppose that $2\gamma - \theta_i + 2\theta_j$ is a root. Then we have that $\left\langle \gamma^\vee + \theta_j^\vee, 2\gamma - \theta_i + 2\theta_j \right\rangle = 4-1-2-2+4 = 3$ which is impossible since R is simply laced. Therefore none of the proceeding elements is a root as claimed. ☐

11.4. **Classification of dualizing varieties.**

Lemma 11.59. *Let X be a dualizing variety. Up to isomorphism X is given by one of the varieties in the following list:*

$X = \hat{X}$
$\mathbb{G}(p, 2p)$
$\mathbb{G}_\omega(p, 2p)$, $p \geq 2$
$\mathbb{G}_Q(p, 2p)$, $p \geq 3$, *p even*
$\mathbb{Q}_m$, $m \geq 3$
E_7/P_7

Proof. From Lemma 11.16 it is clear that the varieties in the list are all dualizing varieties. Moreover Lemma 11.16 says that the cominuscule dualizing varieties are up to isomorphism precisely the varieties occuring in the previous list. We have to show that there are no other dualizing varieties.

Let X be a dualizing variety. By what we said up to know we may assume that X is not cominuscule. Since a simply laced dualizing variety is a cominuscule homogeneous space, it follows that R is not simply laced. We now do a case by case analysis.

Suppose first that G is of type B_ℓ where $\ell \geq 2$ and let $\alpha_P = \alpha_n$ for some $1 \leq n \leq \ell$. Since X is not cominuscule we know that $n > 1$. Suppose that $n < \ell$. Since X is a dualizing variety we know that $\theta_k = \alpha_n$ (Corollary 11.3). This immediately implies that $k > 2$. Since $\alpha_{n-1} \notin S_\Delta$ it is easy to see that θ_1 and R' generate two different irreducible components of $\hat{R}$. Since $\hat{R} = R$ this is a contradiction. We conclude that $\alpha_P = \alpha_\ell$. Again since we must have $\theta_k = \alpha_\ell$ we conclude that ℓ is odd. This means that $X = B_\ell/P_\ell \cong \mathbb{G}_Q(\ell, 2\ell+1) \cong \mathbb{G}_Q(p, 2p)$ where $p = \ell + 1$. Since $\ell \geq 2$ is odd we know that $p \geq 3$ is even. Therefore it follows that X already appears in the list of dualizing varieties in the statement.

Next suppose that G is of type C_p where $p \geq 2$ and let $\alpha_P = \alpha_n$ for some $1 \leq n \leq p$. Since X is not cominuscule we know that $n < p$. Then it is easy to see that $\alpha_p \notin \hat{R}$ which contradicts the fact that $\hat{R} = R$. Therefore we see that there is no dualizing variety of type C_p which is not cominuscule.

Next suppose that G is of type F_4. Since $\theta_k = \alpha_P$ we conclude that $k > 1$ and that $\alpha_P = \alpha_n$ for some $2 \leq n \leq 4$. Therefore we know that $R_2 = \hat{R}_2$ is of type C_3. Since there is only one dualizing variety of type

C_3 by the previous analysis we conclude that $\alpha_P = \alpha_2$. Since $\alpha_1 \notin S_\Delta$ is is easy to see that $\alpha_1 \notin \hat{R}$. Since $\hat{R} = R$ this is a contradiction. Therefore we see that there is no dualizing variety of type F_4.

Next suppose that G is of type G_2. Since $\theta_k = \alpha_P$ we immediately conclude that $\alpha_P = \alpha_1$. This means that $X = G_2/P_1 \cong \mathbb{Q}_5$ already appears in the list of dualizing varieties in the statement. □

Corollary 11.60. *Every dualizing variety is isomorphic to a cominuscule dualizing variety.*

Proof. Let X be a dualizing variety which is not cominuscule. The proof of the previous lemma then shows that X is either equal to B_ℓ/P_ℓ where $\ell > 1$ is odd or is equal to G_2/P_1. Both varieties where seen to be isomorphic to a cominuscule dualizing variety in the proof of the previous lemma. □

Corollary 11.61. *Let X be a dualizing variety. Through three points of X in general position passes a unique rational curve of degree d_X.*

Proof. For a cominuscule dualizing variety this is a direct consequence of Corollary 11.17. For a dualizing variety which is not cominuscule the result will follows from the previous corollary once we have checked that the invariant d_X is preserved under the isomorphism under consideration. Since we know from Lemma 11.16 that $d_{\mathbb{G}_Q(p,2p)} = [p/2]$ where $p \geq 3$ and that $d_{\mathbb{Q}_m} = 2$ where $m \geq 3$, we only have to check that $d_{B_\ell/P_\ell} = (\ell+1)/2$ for all $\ell > 1$ odd and that $d_{G_2/P_1} = 2$. If $X = B_\ell/P_\ell$ where $\ell > 1$ is odd it is immediate that $k = (\ell+1)/2$ and that $d_1 = \cdots = d_k = 1$ and thus $d_X = (\ell+1)/2$. If $X = G_2/P_1$ it is immediate that $k = 2$ and that $d_1 = d_2 = 1$ and thus $d_X = 2$. In both cases we get the desired value of d_X. □

Corollary 11.62. *Let X be a dualizing variety. Then we have the following formula for the dimension of X:*

$$\dim(X) = \frac{c_1(X)d_X}{2}\,.$$

Proof. Indeed, from the previous corollary it is immediate that

$$\dim(\mathcal{M}_{0,3}(X, d_X)) - 3\dim(X) = 0\,.$$

But we know that $\dim(\mathcal{M}_{0,3}(X, d_X)) = c_1(X)d_X + \dim(X)$. Therefore the desired formula follows. □

Corollary 11.63. *Let X be a dualizing variety (not necessarily simply laced). Then we have $d_1 = \cdots = d_k = 1$.*

Proof. We already proved that a simply laced dualizing variety is cominuscule. Therefore we may assume that X is not simply laced. In the case that X is a dualizing variety which is not cominuscule, we have checked the assertion in the proof of Corollary 11.61. We are left to check the case of a non simply laced dualizing variety which is cominuscule. By the classification of dualizing varieties this means that either $X = \mathbb{G}_\omega(p, 2p)$ where $p \geq 2$ or $X = \mathbb{Q}_m$ where $m \geq 3$ is odd. In both cases it is clear that $d_X = k$ and thus $d_1 = \cdots = d_k = 1$. □

The following conjecture is concerned with the converse of Corollary 11.61.

Conjecture 11.64. *Through three points of X in general position passes a unique rational curve of degree d_X if and only if X is a dualizing variety.*

Remark 11.65. The previous conjecture is known for all cominuscule homogeneous spaces X. Indeed, let X be a cominuscule homogeneous space such that through three points of X in general position passes a unique rational curve of degree d_X. Then it is clear from what we proved up to now that $\hat{X} = Y_{d_X}(f_\Delta) = X_{d_X}(x(1), x(w_o)) = X$ where the last equality follows from the assumption on X. Thus it follows that X is a dualizing variety.

We use the same notation as introduced in the beginning of Section 11: we denote with $\mathfrak{R}^1, \ldots, \mathfrak{R}^r$ the irreducible components of $\hat{R}$, etc. Consistently with the previous notation, we denote with $W_{\hat{G}}$ the Weyl group of $\hat{G}$.

Definition 11.66. *We call an irreducible component $\mathfrak{R}^i$ of $\hat{R}$ nontrivial if $C(\alpha_P) \cap \mathfrak{R}^i \neq \emptyset$ or equivalent if $\mathfrak{R}^i \not\subseteq R_P$. Otherwise we call an irreducible component trivial.*

We denote the number of nontrivial irreducible components with s. It is clear that $r \geq s$.

The following proposition is a generalization of Corollary 11.17.

Proposition 11.67. *Through three points of $\hat{X}$ in general position passes a unique rational curve of degree d_X which is contained in $\hat{X}$. Consequently we have that $\hat{X} \subseteq Y_{d_X}(f_\Delta)$ and that $W_{\hat{G}} \subseteq \mathcal{U}$.*

Proof. For all $1 \leq i \leq r$ such that $\mathfrak{R}^i$ is nontrivial we know that $\mathfrak{X}_i$ is a dualizing variety and thus that through three points of $\mathfrak{X}_i$ in general position passes a unique rational curve of degree $d_{\mathfrak{X}_i}$ which is contained in $\mathfrak{X}_i$. Therefore we see that through three points of $\hat{X} = \prod_{i:\, C(\alpha_P) \cap \mathfrak{R}^i \neq \emptyset} \mathfrak{X}_i$ in general position passes a unique rational curve of degree $d_X = d_{\hat{X}}$ which is contained in $\hat{X}$. This proves the first statement. That $\hat{X} \subseteq Y_{d_X}(f_\Delta)$ follows now from Lemma 11.15 and the fact that $C_\Delta \subseteq \hat{X}$. From the inclusion $\hat{X} \subseteq Y_{d_X}(f_\Delta)$ follows in particular that $\hat{X} \subseteq X_{d_X}(x(1), x(w_o))$ which implies that $W_{\hat{G}} \subseteq \mathcal{U}$. □

Corollary 11.68. *Let X be a dualizing variety. Then we have that $\mathcal{U} = W/W_P$.*

Proof. Indeed, by the previous proposition we know that $W_{\hat{G}} \subseteq \mathcal{U}$. But since $\hat{G} = G$ the result follows. $\square$

Lemma 11.69. *We have the following inequality between the number s of nontrivial irreducible components of $\hat{R}$ and and the number of different values of d_i for $1 \leq i \leq k$:*

$$s \geq \operatorname{card}\{d_1, \ldots, d_k\}\,.$$

Proof. Let $\mathfrak{R}^i$ be a nontrivial irreducible component of $\hat{R}$. By the previous results we know that $C(\alpha_P) \cap \mathfrak{R}^i$ is the θ-sequence of $\mathfrak{R}^i$ and that the value of $d(\theta)$ is independent of the choice of $\theta \in C(\alpha_P) \cap \mathfrak{R}^i$[8]. Therefore we have a map from $\{1 \leq i \leq r \mid \mathfrak{R}^i \text{ nontrivial}\}$ to $\{d_1, \ldots, d_k\}$ which sends an index i to the value $d(\theta)$ for some $\theta \in C(\alpha_P) \cap \mathfrak{R}^i$. Since $C(\alpha_P) = \coprod_{i\colon C(\alpha_P)\cap\mathfrak{R}^i \neq \emptyset} C(\alpha_P) \cap \mathfrak{R}^i$ this map is surjective. It follows that $s \geq \operatorname{card}\{d_1, \ldots, d_k\}$ as claimed. $\square$

Lemma 11.70. *The number s of nontrivial irreducible components of $\hat{R}$ and the number of different values of d_i for $1 \leq i \leq k$ coincide:*

$$s = \operatorname{card}\{d_1, \ldots, d_k\}\,.$$

Proof. The author has checked the previous statement in all possible types by direct computation of $\hat{X}$ and its representation as a product of dualizing varieties. Since we are lacking a neat type independent proof, we omit the details. In all cases where $s = 1$, where $\hat{X}$ is itself a dualizing variety, which include the cases where X is a cominuscule homogeneous space, the statement is obvious anyway. For the successive discussion the statement will be of no importance, although it completes the picture we have from the relation between X and $\hat{X}$. To prove the inequality „$\leq$“ it clearly suffices to show that for all $i \in \{1, \ldots, k-1\}$ such that $d_i = d_{i+1}$ there exists a root $\gamma \in S_\Delta$ such that $i(\gamma) = i$. $\square$

Fact 11.71. *Let X be a simply laced dualizing variety. Then we have the following equality:*

$$(\ell_P(s_{\theta_1}) - 1)k = 2\operatorname{card}(\mathcal{P})\,.$$

Proof. Since through three points of X in general position passes a unique rational curve of degree d_X, we know that $\dim(\mathcal{M}_{0,3}(X, d_X)) - 3\dim(X) = c_1(X)d_X - 2\dim(X) = 0$ and thus $c_1(X)d_X = 2\dim(X)$. If we plug in the dimension formula for dualizing varieties, we obtain $c_1(X)d_X = 2k + 2\operatorname{card}(\mathcal{P})$ and thus $(c_1(X) - 2)k = 2\operatorname{card}(\mathcal{P})$ since $d_X = k$. Since $\ell_P(s_{\theta_1}) = c_1(X)d_1 - 1 = c_1(X) - 1$ the result follows. $\square$

[8] Note that this value $d(\theta)$ need not necessarily be equal to one, although $\mathfrak{X}_i$ is isomorphic to a dualizing variety. The value $d(\theta)$ depends on the embedding of $\mathfrak{X}_i$ and $\hat{X}$ into X but is independent of the choice of $\theta \in C(\alpha_P) \cap \mathfrak{R}^i$.

Corollary 11.72. *Let X be a simply laced dualizing variety. Then we know that the number $(\ell_P(s_{\theta_1}) - 1)k$ is divisible by four.*

Proof. We already know that the cardinality of $\mathcal{P}$ is divisible by two. Thus the formula from the previous fact implies the claim. □

Fact 11.73. *Let $\mathfrak{R}^1, \ldots, \mathfrak{R}^s$ be the nontrivial irreducible components of $\hat{R}$. Then the diagonal curve can be written as $f_\Delta = (f_1, \ldots, f_s)$ where each f_i is the diagonal curve associated to $\mathfrak{X}_i$. In particular each f_i is of degree $d_{\mathfrak{X}_i}$ and has image in $\mathfrak{X}_i$.*

Proof. Since the diagonal curve has image in $\hat{X}$ and since $\hat{X}$ is the product of $\mathfrak{X}_1, \ldots, \mathfrak{X}_s$ it is clear that f_Δ can be written as $f_\Delta = (f_1, \ldots, f_s)$ where each f_i has image in $\mathfrak{X}_i$. Since $C(\alpha_P) \cap \mathfrak{R}^i$ is the θ-sequence of $\mathfrak{R}^i$ it follows from the definition of f_Δ that each f_i is the diagonal curve associated to $\mathfrak{X}_i$. The very last statement follows from general properties of diagonal curves. □

12. Bundles over X'

Let $X = G/P$ be a homogeneous space where G is a simple, simply connected linear algebraic group and P is a maximal parabolic subgroup. In this section we assume that R is simply laced.

Let $\gamma \in S$. We denote by $\mathcal{O}_\gamma$ the orbit of the action of G' on U_γ. Let U be the unipotent radical of B. Let U^- be the unipotent radical of B^-. With this notation we have $\mathcal{O}_\gamma \subseteq U \times U^-$ for all $\gamma \in S$.

Let $\gamma \in S$. By Lemma 11.58 we know that $\mathcal{O}_\gamma$ is a subgroup of G. Therefore we can define $\mathcal{E}_\gamma = G' \times_{G' \cap P} \mathcal{O}_\gamma / \mathcal{O}_\gamma \cap P$. We write $X'_\gamma = \overline{G'\mathcal{O}_\gamma x(1)}$. With this notation X'_γ is the closure of the image of the total space of $\mathcal{E}_\gamma$ in X.

Fact 12.1. *Let $\gamma \in S$. The group G' and the orbit $\mathcal{O}_\gamma$ form a semidirect product $G'\mathcal{O}_\gamma = G' \rtimes \mathcal{O}_\gamma$. In particular we have that*

$$G' = \bigcap_{\gamma \in S} G'\mathcal{O}_\gamma$$

whenever $\mathcal{P}$ is not empty. Moreover we have the weaker inclusion

$$X' \subseteq \bigcap_{\gamma \in S} X'_\gamma$$

which holds for all $\mathcal{P}$.

Proof. Let $\gamma \in S$. To see that G' and $\mathcal{O}_\gamma$ form a semidirect product it suffices to prove that $G' \cap \mathcal{O}_\gamma = 1$ or equivalently that

$$\left(\prod_{\theta \in \pm C(\alpha_P)} U_\theta \right) \cap \mathcal{O}_\gamma = 1$$

since $\mathcal{O}_\gamma \subseteq U \times U^-$. Since $\theta_i \notin S$ for all $1 \leq i \leq k$ the before mentioned intersection is obviously trivial.

Let $\gamma, \delta \in S$. Then we clearly have $\mathcal{O}_\gamma = \mathcal{O}_\delta$ if and only if γ and δ are in the same orbit of the action of W_θ on S. Therefore we have

$$\bigcap_{\gamma \in S} \mathcal{O}_\gamma = \bigcap_{\gamma \in \mathcal{P}} \mathcal{O}_\gamma \,.$$

Since G' and $\mathcal{O}_\gamma$ form a semidirect product it follows that

$$\bigcap_{\gamma \in S} G'\mathcal{O}_\gamma = G' \bigcap_{\gamma \in \mathcal{P}} \mathcal{O}_\gamma \,.$$

Since the set $\mathcal{P}$ is a set of representatives of the orbits of the action of W_θ on S we have a direct product $\prod_{\gamma \in \mathcal{P}} \mathcal{O}_\gamma$ and thus $\bigcap_{\gamma \in \mathcal{P}} \mathcal{O}_\gamma = 1$ whenever $\mathcal{P}$ consists of more than one element. Since the cardinality of $\mathcal{P}$ is divisible by two, we know that $\mathcal{P}$ will consist of more than one element whenever $\mathcal{P}$ is not empty. The previous equality therefore yields $\bigcap_{\gamma \in S} G'\mathcal{O}_\gamma = G'$ whenever $\mathcal{P}$ is not empty. The inclusion in the statement follows by applying the natural projection $G \to X$ to the equality we proved just before whenever $\mathcal{P}$ is not empty. If $\mathcal{P}$ is empty we interpret the intersection $\bigcap_{\gamma \in S} X'_\gamma$ as the whole space X. Therefore the desired inclusion becomes trivial whenever $\mathcal{P}$ is empty. □

Since we have a direct product $\prod_{\gamma \in \mathcal{P}} \mathcal{O}_\gamma$ we also have a direct sum $\bigoplus_{\gamma \in \mathcal{P}} \mathcal{E}_\gamma$. We denote this direct sum by $\mathcal{E} = \bigoplus_{\gamma \in \mathcal{P}} \mathcal{E}_\gamma$ and this direct product by $\mathcal{O} = \prod_{\gamma \in \mathcal{P}} \mathcal{O}_\gamma$. We denote the closure of the image of the total space of $\mathcal{E}$ in X by $\tilde{X}$. With this notation we have $\tilde{X} = \overline{G'\mathcal{O}x(1)}$. We have a commutative diagram

$$\begin{array}{ccc} & \mathcal{E} & \\ & \uparrow & \searrow \\ \prod_{i=1}^{k} U_{-\theta_i} \times & \prod_{\gamma \in S \cap (R^- \setminus R_P^-)} U_\gamma & \longrightarrow \tilde{X} \subseteq X \end{array}$$

where the horizontal arrow is an injection. From this diagram we infer that

$$k + \operatorname{card}(S \cap (R^- \setminus R_P^-)) \leq \dim(\tilde{X}) \,.$$

Fact 12.2. *Let X be a cominuscule homogeneous space and let R be simply laced. Then we have the following inequalities:*

$$\dim(\hat{X}) \leq k + \operatorname{card}(\mathcal{P}) \leq \dim(\tilde{X}) \,.$$

Proof. The first inequality was already proved before for every cominuscuel homogeneous space X such that R is simply laced. Since X is cominuscule we know that the set $S \cap (R^- \setminus R_P^-)$ as well as the set $\mathcal{P}$ are sets of representatives of the orbits of the action of W_θ on S. In particular $\operatorname{card}(S \cap (R^- \setminus R_P^-)) = \operatorname{card}(\mathcal{P})$. Therefore the second inequality follows from the inequality we deduced directly before the fact. □

Fact 12.3. *Let X be a cominuscule homogeneous space and let R be simply laced. Let $\gamma \in S$. Then $\mathcal{E}_\gamma$ is a line bundle on X' such that $f_\Delta^*(\mathcal{E}_\gamma) \cong \mathcal{O}_{\mathbb{P}^1}(2)$.*

Proof. We may assume that $\gamma \in \mathcal{P}$. Then we know that the orbit of γ under the action of W_θ contains precisely one element of $S \cap (R^+ \setminus R_P^+)$, namely $\gamma + \theta_{j(\gamma)}$, and precisely one element of $S \cap (R^- \setminus R_P^-)$, namely $\gamma - \theta_{i(\gamma)}$. These roots satisfy $\langle \eta^\vee, \gamma + \theta_{j(\gamma)} \rangle = 2$ and $\langle \eta^\vee, \gamma - \theta_{i(\gamma)} \rangle = -2$ where $\eta^\vee = \sum_{i=1}^k \theta_i^\vee$. Therefore it is clear from the definition that $\mathcal{E}_\gamma$ is a line bundle. Moreover we see that the highest weight of the action of the onedimensional maximal torus in $P' = G' \cap P$ on $\mathcal{O}_\gamma / \mathcal{O}_\gamma \cap P$ is given by 2. In formulas this means that $\mathcal{E}_\gamma = \mathcal{L}(2)$. From this description it obviously follows that $f_\Delta^*(\mathcal{E}_\gamma) \cong \mathcal{O}_{\mathbb{P}^1}(2)$. □

Lemma 12.4. *Let X be a cominuscule homogeneous space and let R be simply laced. Then we have $\tilde{X} = \hat{X}$. Consequently the dimension formula holds:*

$$\dim(\hat{X}) = k + \mathrm{card}(\mathcal{P})\,.$$

Moreover we have $S \subseteq \hat{R}$.

Proof. We first prove that through three points of $\tilde{X}$ in general position passes a rational curve of degree d_X. Since $\mathcal{E}$ maps onto a dense subset of $\tilde{X}$ it clearly suffices to prove that through three points of $\mathcal{E}$ in general position passes a rational curve of degree d_X. Since $\mathcal{E}$ is a direct sum it clearly suffices to show that for all $\gamma \in \mathcal{P}$ through three points of $\mathcal{E}_\gamma$ in general position passes a rational curve of degree d_X. Since $X' = G'C_\Delta$ it is clear that through three points of X' in general position passes a rational curve of degree d_X. Since $f_\Delta^*(\mathcal{E}_\gamma) \cong \mathcal{O}_{\mathbb{P}^1}(2)$ this curve will lift uniquely to a rational curve in $\mathcal{E}_\gamma$ of the same degree passing through three points of $\mathcal{E}_\gamma$ in general position. This proves the claim.

We know that $\tilde{X}$ is irreducible as the closure of the image of the irreducible variety $\mathcal{E}$. Therefore it now follows from the first paragraph and Lemma 11.15 that $\tilde{X} \subseteq Y_{d_X}(f_\Delta) = \hat{X}$. By Fact 12.2 we know that $\dim(\hat{X}) \leq \dim(\tilde{X})$. Since $\hat{X}$ is irreducible it therefore follows that $\tilde{X} = \hat{X}$. Using Fact 12.2 once more we see that the dimension formula is satisfied. From the equality $\tilde{X} = \hat{X}$ it immediately follows that $S \subseteq \hat{R}$. □

Corollary 12.5. *Let X be a simply laced dualizing variety. Then we have $\tilde{X} = \hat{X} = X$.*

Proof. Obvious. □

13. General quantum to classical principle

Let $X = G/P$ be a homogeneous space where G is a simple, simply connected linear algebraic group and P is a maximal parabolic subgroup. In this section we suppose that $X \neq G_2/P_1$ and that $X \neq B_\ell/P_\ell$ where $\ell > 1$ is odd. Then we know that the diagonal curve has a dense open orbit under the action of G in $\mathcal{M} = \mathcal{M}_{0,3}(X, d_X)$. In other words a curve which is in the orbit of the diagonal curve is a general curve.

Let $Q = \mathrm{Stab}_G(\hat{X})$. Then we know that Q is a closed subgroup of G. Moreover we have the obvious inclusions $\hat{G} \subseteq Q \subseteq \hat{G}P$. We write $Y = G/Q$ for the quotient.

Fact 13.1. *Suppose in addition that X is a cominuscule homogeneous space. Then Q is a parabolic subgroup of G.*

Proof. In view of Lemma 11.16 this follows from [11, Proposition 18]. □

Remark 13.2. Even if X is a cominuscule homogeneous space, the parabolic subgroup Q will not necessarily be a standard parabolic subgroup. Already in type A_n we find examples where $B \not\subseteq Q$.

Assumption 13.3. *In the rest of this section we will make the following assumptions:*

- *We assume that $\hat{X}$ is an irreducible component of $Y_{d_X}(f_\Delta)$.*
- *We assume that all irreducible components of $Y_{d_X}(f_\Delta)$ are pairwise nonisomorphic.*
- *We assume that each irreducible component X_0 of $Y_{d_X}(f_\Delta)$ satisfies the following property: through three points of X_0 in general position passes a unique rational curve of degree d_X which is contained in X_0.*

Remark 13.4. We already know that this assumption is satisfied for all cominuscule homogeneous spaces, since then $\hat{X} = Y_{d_X}(f_\Delta)$. We will see later that this assumption is also satisfied if $X = \mathbb{G}_Q(l, 2p)$ where $l \leq p - 2$ and l odd.

Conjecture 13.5. *$\hat{X}$ is an irreducible component of $Y_{d_X}(f_\Delta)$ if and only if $S \subseteq \hat{R}$.*

Conjecture 13.6. *Suppose that R is simply laced. Then the following are equivalent:*

- *$\hat{X}$ is an irreducible component of $Y_{d_X}(f_\Delta)$.*
- *We have the inclusion $S \subseteq \hat{R}$.*
- *We have the equality $\tilde{X} = \hat{X}$.*

Remark 13.7. By what we proved up to now the first of the two previous conjectures is known for all cominuscule homogeneous spaces and the second one is known for all cominuscule homogeneous spaces such that R is simply laced.

Fact 13.8. *Let $X_0, \dots, X_n$ be the irreducible components of $Y_{d_X}(f_\Delta)$. Then we have*

$$\mathrm{Stab}_G(Y_{d_X}(f_\Delta)) = \bigcap_{i=0}^{n} \mathrm{Stab}_G(X_i)\,.$$

Proof. The inclusion from right to left is obvious. Let $g \in G$ be an element wich stabilizes $Y_{d_X}(f_\Delta)$. Then X_i and gX_i are two isomorphic irreducible components of $Y_{d_X}(f_\Delta)$. By the assumption it follows that $gX_i = X_i$ for all i. Therefore g is contained in the intersection on the right side. This proves the inclusion from left to right. □

Corollary 13.9. *We have an inclusion:* $\mathrm{Stab}_G(f_\Delta) \subseteq Q$.

Proof. For every $g \in G$ we obviously have $gY_{d_X}(f_\Delta) = Y_{d_X}(gf_\Delta)$. Therefore every element which stabilizes the diagonal curve will also stabilize $Y_{d_X}(f_\Delta)$. The previous fact then implies that every element which stabilizes the diagonal curve also stabilizes each irreducible component of $Y_{d_X}(f_\Delta)$ in particular $\hat{X}$ by assumption. This proves the desired inclusion. □

Fact 13.10. *Let $g \in G$ be an element such that $g\hat{X} \subseteq Y_{d_X}(f_\Delta)$. Then we have that $g\hat{X}$ is an irreducible component of $Y_{d_X}(f_\Delta)$.*

Proof. Indeed, let X_0 be the irreducible component of $Y_{d_X}(f_\Delta)$ which contains $g\hat{X}$. Then we have $C_\Delta \subseteq \hat{X} \subseteq g^{-1}X_0$. By our assumption X_0 and $g^{-1}X_0$ satisfy the three point property. Therefore Lemma 11.15 implies that $g^{-1}X_0 \subseteq Y_{d_X}(f_\Delta)$. Again by the assumption we know that $\hat{X}$ is an irreducible component. Since $g^{-1}X_0$ is irreducible it follows that $\hat{X} = g^{-1}X_0$ and thus $g\hat{X} = X_0$ which means that $g\hat{X}$ is an irreducible component of $Y_{d_X}(f_\Delta)$ as claimed. □

Corollary 13.11. *Let $g \in G$ such that $C_\Delta \subseteq g\hat{X} \cap \hat{X}$. Then it follows that $g \in Q$.*

Proof. Through three points of $g\hat{X}$ and $\hat{X}$ in general position passes a unique rational curve of degree d_X. In addition we have $C_\Delta \subseteq g\hat{X} \cap \hat{X}$. Therefore Lemma 11.15 implies that $g\hat{X}, \hat{X} \subseteq Y_{d_X}(f_\Delta)$. By the previous fact we know that $g\hat{X}$ and $\hat{X}$ are two isomorphic irreducible components of $Y_{d_X}(f_\Delta)$ and thus must be equal by assumption. This means $g\hat{X} = \hat{X}$ or equivalent $g \in Q$. □

Let f be a general curve in $\mathcal{M}$. Then there exists a $g \in G$ such that $f = gf_\Delta$. Then we can define $\hat{X}_f = g\hat{X}$. This is well defined since we know that $\mathrm{Stab}_G(f_\Delta) \subseteq Q$. With this notation we obviously have $\hat{X}_{f_\Delta} = \hat{X}$. Moreover we know that the image of any general curve f is contained in $\hat{X}_f$. The homogeneous space Y parametrizes the set $\{\hat{X}_f \mid f \text{ general}\}$.

Fact 13.12. *We have a birational map:*

$$\{(y, x_1, x_2, x_3) \in Y \times X^3 \mid x_1, x_2, x_3 \in y\hat{X}\} \to \mathcal{M}\,.$$

Proof. Let us denote by $\mathcal{N}$ the potential source of the birational map. Let (y, x_1, x_2, x_3) be a general point in $\mathcal{N}$. Then we know that x_1, x_2 and x_3 are three points of $y\hat{X}$ in general position. By Proposition 11.67 there exists a unique rational curve f which is contained in $y\hat{X}$ and passes through the three points x_1, x_2 and x_3. The assignement $(y, x_1, x_2, x_3) \mapsto f$ with marked points x_1, x_2 and x_3 then defines a rational map from $\mathcal{N}$ to $\mathcal{M}$. The inverse of this rational map can be defined as follows: let f be a general curve in $\mathcal{M}$. Then we know that f is in the orbit of the diagonal curve. Therefore it makes sense to send f to the point in Y associated to $\hat{X}_f$ where we keep the marked points of f as marked points of the image.

We can easily check that the defined maps are inverse to each other. Let (y, x_1, x_2, x_3) be a general point in $\mathcal{N}$ with image f. Since the original point is general, we see that f is also general. Therefore there exists a $g \in G$ such that $f = gf_\Delta$. We have to show that $gQ/Q = y$. Let $h \in G$ such that $hQ/Q = y$. We know that $f(\mathbb{P}^1) \subseteq y\hat{X} = h\hat{X}$ and thus $gC_\Delta \subseteq h\hat{X}$ and thus $C_\Delta \subseteq g^{-1}h\hat{X} \cap \hat{X}$. The previous corollary then implies $g^{-1}h \in Q$ which means $gQ/Q = hQ/Q = y$ as desired.

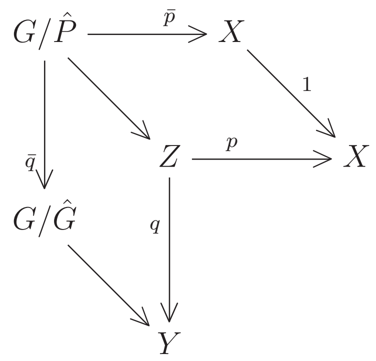
Finally, let f be a general curve in $\mathcal{M}$ with image $(y, x_1, x_2, x_3) \in \mathcal{N}$. Then we know by definition that $y\hat{X} = \hat{X}_f$. Since f is general, we see that the image is also general. Therefore there exists a unique rational curve in $y\hat{X}$ which passes through the points x_1, x_2 and x_3. This curve is given by f – hence the assignement $(y, x_1, x_2, x_3) \mapsto f$. □

Corollary 13.13. *We have the following equality for the dimension of $\mathcal{M}$:*

$$\dim(\mathcal{M}) = \dim(Y) + 3\dim(\hat{X})\,.$$

Proof. Indeed, the previous fact implies that $\dim(\mathcal{M}) = \dim(\mathcal{N})$. But we obviously have $\dim(\mathcal{N}) = \dim(Y) + 3\dim(\hat{X})$. □

We now introduce the incidence variety Z. Let $Z = \{(x, y) \in X \times Y \mid x \in y\hat{X}\}$. Since $\hat{G}$ acts transitively on $\hat{X}$ we know that Q also acts transitively on $\hat{X}$. It follows that $\hat{X} \cong Q/Q \cap P$ and that $Z \cong G/P \cap Q$. It is convenient to write $\hat{P} = \hat{G} \cap P$. With this notation we have the following commutative diagram:



In this diagram the morphisms $p, q, \bar{p}$ and $\bar{q}$ are the obvious projections. The morphisms which are not labeled are also the obvious projections.

The fibers of q are all (independently from the point in Y) isomorphic to $\hat{X}$. Therefore we get the equality of dimensions:

$$\dim(Z) = \dim(Y) + \dim(\hat{X}) \,.$$

The fibers of p are all (independently from the point in X) isomorphic to $P/P \cap Q$. Therefore we get the equality of dimensions:

$$\dim(Z) = \dim(X) + \dim(P/P \cap Q) \,.$$

Both equalities together yield the following equality of dimensions:

$$\dim(Y) - \dim(P/P \cap Q) = \dim(X) - \dim(\hat{X}) \,. \tag{5}$$

Let w be a Weyl group element. Then we write $F_w = qp^{-1}(X_w)$. For an abritrary element $g \in G$ we obviously have $gF_w = qp^{-1}(gX_w)$ since both morphisms p and q are G-equivariant. We define a surjective morphism $q_w \colon p^{-1}(X_w) \to F_w$ via restriction of q.

Fact 13.14. *Let w be a Weyl group element. The morphism q_w is proper.*

Proof. To see that q_w is proper it clearly suffices to show that $p^{-1}(X_w) \to Y$ is proper (since the inclusion $F_w \subseteq Y$ is separated). Since $p^{-1}(X_w) \to Y$ is the composition of the closed immersion $p^{-1}(X_w) \to Z$ and $q \colon Z \to Y$ it suffices to show q is proper in order to see that $p^{-1}(X_w) \to Y$ is proper. Since q is the composition of the closed immersion $Z \to X \times Y$ and the natural projection $X \times Y \to Y$ it suffices to show that $X \times Y \to Y$ is proper in order to see that q is proper. Since X is projective we know that the structure morphism $X \to \mathrm{Spec}(\mathbb{C})$ is proper. Since proper is stable under base change and since $X \times Y \to Y$ is the base change of $X \to \mathrm{Spec}(\mathbb{C})$ along $Y \to \mathrm{Spec}(\mathbb{C})$ we conclude that $X \times Y \to Y$ is proper as desired. □

Fact 13.15. *Let w be a Weyl group element. The variety $qp^{-1}(\Omega_w)$ is a dense subset of F_w.*

Proof. It clearly suffices to show that $\overline{p^{-1}(\Omega_w)}$ is a dense subset of $p^{-1}(X_w)$ since we have $q(\overline{p^{-1}(\Omega_w)}) \subseteq \overline{qp^{-1}(\Omega_w)}$. Let $\pi \colon G \to X$ be the natural projection. Since $p^{-1}(\Omega_w)$ resp. $p^{-1}(X_w)$ is the image of $\pi^{-1}(\Omega_w)$ resp. $\pi^{-1}(X_w)$ under the natural projection $G \to Z$ it suffices to prove that $\pi^{-1}(\Omega_w)$ is a dense subset of $\pi^{-1}(X_w)$. Since $\pi^{-1}(\Omega_w) = BwP$ is stable under the right action of P, the closure A of $\pi^{-1}(\Omega_w)$ in G is also stable under the right action of P. In other words this means that $A = \pi^{-1}(B)$ where $B = \pi(A)$. Since A is closed and X carries the quotient topology we conclude that B is also closed. Since π is surjective we see that $B = \pi(A) \supseteq \pi\pi^{-1}(\Omega_w) = \Omega_w$. Since B is closed we conclude that $B \supseteq X_w$ and thus $A \supseteq \pi^{-1}(X_w)$. Since the other inclusion $A \subseteq \pi^{-1}(X_w)$ is obvious it follows that $A = \pi^{-1}(X_w)$ which means that $\pi^{-1}(\Omega_w)$ is a dense subset of $\pi^{-1}(X_w)$ as desired. □

Fact 13.16. *Let w be a Weyl group element. The variety F_w is B-stable and irreducible.*

Proof. It is clear that F_w is B-stable since X_w is B-stable and the morphisms p and q are G-equivariant. Since $qp^{-1}(\Omega_w)$ is a dense subset of F_w it suffices to show that $qp^{-1}(\Omega_w)$ is irreducible. Since p and q are G-equivariant we have $qp^{-1}(\Omega_w) = Bqp^{-1}(x(w)) = Bwqp^{-1}(x(1))$. Therefore it suffices to show that $qp^{-1}(x(1))$ is irreducible in order to see that $qp^{-1}(\Omega_w)$ is irreducible. By definition we have of p and q we have $qp^{-1}(x(1)) = PQ/Q$. From this expression it is clear that $qp^{-1}(x(1))$ is irreducible. The claim follows. □

Let N_w be the nonempty open subset of F_w where the fibers of q_w are of minimal dimension. Since F_w is irreducible, the nonempty open subset N_w of F_w is dense. Since q_w is B-equivariant the open dense subset N_w of F_w is B-stable.

Fact 13.17. *Let w be a Weyl group element. We have the following fundamental inequality:*

$$\operatorname{codim}(F_w) \geq \operatorname{codim}(X_w) - \dim(\hat{X}) \,.$$

Moreover the following statements are equivalent:

- *The fundamental inequality is an equality.*
- *The equality $\dim(F_w) = \dim(p^{-1}(X_w))$ holds.*
- *The fiber of q_w over some point in N_w is finite.*
- *The fiber of q_w over all points in N_w is finite.*

Proof. We morphism q_w is obviously surjective. Therefore we get that $\dim(F_w) \leq \dim(p^{-1}(X_w))$. On the other hand we already know that

$$\dim(p^{-1}(X_w)) = \dim(X_w) + \dim(P/P \cap Q) \,.$$

If we put this together we find that

$$\operatorname{codim}(F_w) \geq \dim(Y) - \dim(P/P \cap Q) - \ell_P(w) = \operatorname{codim}(X_w) - \dim(\hat{X})$$

where we used the equality (5). We also see that equality holds if and only if $\dim(F_w) = \dim(p^{-1}(X_w))$. This proves the fundamental inequality and the equivalence of the first two statements.

The equivalence of the third and the fourth statement is clear from the definition of N_w, since the dimension of the fibers of q_w over points in N_w is independent from the point in N_w.

Assume that we have equality $\dim(F_w) = \dim(p^{-1}(X_w))$. Then there exists a nonempty open subset U of F_w such that all fibers of q_w over points in U are zerodimensional / finite. Since N_w is a dense open subset of F_w it is clear that $N_w \cap U \neq \emptyset$, hence the fiber of q_w over some point in N_w is finite. This proves that the second statement implies the third statement.

Similarly, assume that we have $\dim(F_w) < \dim(p^{-1}(X_w))$. Then there exists a nonempty open subset U of F_w such that all fibers of q_w over points in U have positive dimension. Since N_w is a dense open subset of F_w it is clear that $N_w \cap U \neq \emptyset$, hence the fiber of q_w over some point in N_w is infinite. This proves that the fourth statement implies the second statement. □

Let w be a Weyl group element. We are now ready to define the non negative integer $\bar{q}_w$. Suppose that one of the four equivalent statements of Fact 13.17 is satisfied. Then we define $\bar{q}_w = \mathrm{card}(q_w^{-1}(y))$ for some $y \in N_w$. This is a well defined non negative integer since the fibers of q_w over points in N_w are all finite and of the same cardinality. If one of the four equivalent statements of Fact 13.17 is violated then we define $\bar{q}_w = 0$. More concisely we can say that $\bar{q}_w$ is the unique non negative integer defined by the following equation in cohomology:

$$q_{w*}[p^{-1}(X_w)] = \bar{q}_w[F_w]\,.$$

Fact 13.18. *Let g, g' and g'' be three general elements of G. Let u, v and w be three Weyl group elements. Then we have an isomorphism between*

$$\{(y, x_1, x_2, x_3) \in gF_u \cap g'F_v \cap g''F_w \times X^3 \mid \\ x_1 \in y\hat{X} \cap gX_u, x_2 \in y\hat{X} \cap g'X_v, x_3 \in y\hat{X} \cap g''X_w\}$$

and

$$\{f \in \mathcal{M} \mid \mathrm{ev}_1(f) \in gX_u, \mathrm{ev}_2(f) \in g'X_v, \mathrm{ev}_3(f) \in g''X_w\}$$

which is given by restricting the birational map from Fact 13.12.

Proof. Let (y, x_1, x_2, x_3) be a point in the first variety. Since g, g' and g'' are general elements of G, we know that x_1, x_2 and x_3 are three points of $y\hat{X}$ in general position. Thus there exists a unique rational curve f which is contained in $y\hat{X}$ such that $\mathrm{ev}_1(f) = x_1 \in gX_u$, $\mathrm{ev}_2(f) = x_2 \in g'X_v$ and such that $\mathrm{ev}_3(f) = x_3 \in g''X_w$. This means that we have a well defined morphism between the varieties under consideration which is given by restricting the birational map from Fact 13.12.

Let f be a curve in the second scheme. Since g, g' and g'' are general elements of G, we know that f is a general curve. Thus there exists a $h \in G$ such that $f = hf_\Delta$. Let x_1, x_2 and x_3 be the marked points of f. Let $y = hQ/Q$. Since $f(\mathbb{P}^1) \subseteq y\hat{X}$ we see that the points x_1, x_2 and x_3 satisfy the defining condition of the first variety in the statement. Therefore (y, x_1, x_2, x_3) is a well defined point in this variety. Therefore we have a well defined inverse morphism which is given by restricting the inverse of the birational map from Fact 13.12.

That the two defined morphism are inverse to each other was already checked in the proof of Fact 13.12. □

Theorem 13.19. *Let g, g' and g'' be three general elements of G. Let u, v and w be three Weyl group elements such that*

$$\operatorname{codim}(X_u) + \operatorname{codim}(X_v) + \operatorname{codim}(X_w) = \dim(\mathcal{M}) \,.$$

Then we have the following equality:

$$\langle \sigma(u), \sigma(v), \sigma(w) \rangle_{d_X} = \bar{q}_u \bar{q}_v \bar{q}_w \operatorname{card}(gF_u \cap g'F_v \cap g''F_w) \,.$$

Proof. Suppose first that one of the four equivalent statements of Fact 13.17 is satisfied for each of the Weyl group elements u, v and w. Then we find that

$$\begin{aligned} \operatorname{codim}(gF_u \cap g'F_v \cap g''F_w) &= \sum_{s \in \{u,v,w\}} \operatorname{codim}(F_s) \\ &= \sum_{s \in \{u,v,w\}} \operatorname{codim}(X_s) + 3\dim(\hat{X}) \\ &= \dim(\mathcal{M}) + 3\dim(\hat{X}) = \dim(Y) \end{aligned}$$

where the last line follows from Corollary 13.13. This means that the cardinality of $gF_u \cap g'F_v \cap g''F_w$ is finite. Moreover for sufficiently general elements g, g' and g'' we can assume that

$$gF_u \cap g'F_v \cap g''F_w = gN_u \cap g'N_v \cap g''N_w \,,$$

i.e that the the intersection thakes place in the open dense B-stable subsets. Therefore we see from the previous fact that the number of curves passing through three general translates of X_u, X_v and X_w is finite and that this number is given by the expression in the statement of the theorem.

Next assume that one of the four equivalent statements of Fact 13.17 is violated for at least one $s \in \{u, v, w\}$. Then we know by definition that $\bar{q}_s = 0$. Therefore the right side of the claimed formula is always zero. To prove the desired equality it suffices to show that there are either no or there are infinitely many curves passing through three general translates of X_u, X_v and X_w. Suppose that there exists at least one such curve corresponding to a point $y \in gN_u \cap g'N_v \cap g''N_w$.

We may assume that $s = u$. Then we know by assumption that the fiber $y\hat{X} \cap gX_u$ is infinite. The previous fact then implies that there are infinitely many curves passing through three general translates of X_u, X_v and X_w. □

Corollary 13.20. *Let u, v and w be three Weyl group elements. Suppose that the inequality*

$$\operatorname{codim}(F_s) \geq \operatorname{codim}(X_s) + \dim(\hat{X})$$

is strict for at least one $s \in \{u, v, w\}$. Then we have the vanishing

$$\langle \sigma(u), \sigma(v), \sigma(w) \rangle_{d_X} = 0\,.$$

Proof. If $\sum_{s \in \{u,v,w\}} \operatorname{codim}(X_s) \neq \dim(\mathcal{M})$ then the Gromov-Witten invariant is zero anyway. Otherwise we know from the previous theorem that the Gromov-Witten invariant is zero since we have $\bar{q}_s = 0$ for at least one $s \in \{u, v, w\}$. □

Lemma 13.21. *Let X be a dualizing variety. Then we have*

$$\langle \sigma_u, \sigma_v, \sigma_w \rangle_{d_X} = \begin{cases} 1 & \textit{if } u = v = w = w_X \\ 0 & \textit{otherwise} \end{cases}$$

For all $d > d_X$ we have $\langle \sigma_u, \sigma_v, \sigma_w \rangle_d = 0$ for all Weyl group elements u, v and w.

Proof. The first claim follows directly from Corollary 11.61, since if one of the Schubert cycles σ_u, σ_v or σ_w is positive dimensional then there exist infinitely many rational curves passing through general translates of Y_u, Y_v and Y_w, and if all Schubert cycles are zero dimensional then there exists precisely one rational curve. Let $d > d_X$. Since we know that $c_1(X)d_X - 2\dim(X) = 0$ it follows that $c_1(X)d - 2\dim(X) > 0$. This means that there are infinitely many rational curves passing through three general points of X. This gives the desired vanishing $\langle \sigma_u, \sigma_v, \sigma_w \rangle_d = 0$ for all Weyl group elements u, v and w. □

Remark 13.22. From the previous lemma we see that Conjecture 7.2 is satisfied for all dualizing varieties. More generally it was proved in [11, Proposition 28] that Conjecture 7.2 is also true for all cominuscule homogeneous spaces.

Corollary 13.23. *Let X be a dualizing variety. Then we have*

$$[\{\mathrm{pt}\}] \star [\{\mathrm{pt}\}] = [X] \cdot q^{d_X}\,.$$

Proof. Since $d_X = \delta(w_X)$ we kow that q^{d_X} is the minimal power of the quantum parameter q occuring in the quantum product $[\{\mathrm{pt}\}] \star [\{\mathrm{pt}\}]$. But the previous lemma also shows that q^{d_X} is a maximal power of the quantum parameter q which can occur in any quantum product $\sigma_u \star \sigma_v$. Therefore we obtain the desired formula in view of the previous lemma. □

14. Schubert varieties in isotropic Grassmannians

Let $X = G/P = \mathbb{G}_Q(l, 2p)$ where $l \leq p-2$ and $p \geq 3$. Let $q = p-l$. In this section we describe the Schubert varieties in X using the parametrization by q-strict partitions as in [7, 4.] and [29, 6.].

Let V be the real vector space spanned by the root system R. Let $\epsilon_1, \ldots, \epsilon_p$ be the standard basis of V which represents the root system as in Bourbaki (cf. [4, Chapter VI, Table I-IX]). The Weyl group W is the semidirect product $S_p \rtimes (\mathbb{Z}/2\mathbb{Z})^{p-1}$ where S_p acts by permutation of the ϵ_i's and $(\mathbb{Z}/2\mathbb{Z})^{p-1}$ acts by $\epsilon_i \mapsto (\pm 1)_i \epsilon_i$ where $\prod_{i=1}^{p} (\pm 1)_i = 1$. We think of elements of W either as permutations with an even number of bars or as permutation matrices with negative signs attached to an even number of entries. The simple reflections s_{α_i} which generate W can be described as barred permutations as follows: $s_{\alpha_i} = (i(i+1))$ for all $1 \leq i \leq p-1$ (cycle notation) and $s_{\alpha_p} = (1, \ldots, p-2, \overline{p}, \overline{p-1})$ (array notation). The order of the Weyl group W is obviously $2^{p-1}p!$. The Weyl group W_P of P is generated by all simple reflections s_{α_i} where $i \neq l$. Consequently we can write $W_P = S_l \times (S_{p-l} \rtimes (\mathbb{Z}/2\mathbb{Z})^{p-l-1})$ where the first factor is generated by $s_{\alpha_1}, \ldots, s_{\alpha_{l-1}}$ and the second factor is generated by $s_{\alpha_{l+1}}, \ldots, s_{\alpha_p}$. The order of the group W_P is thus $2^{p-l-1} l!(p-l)!$. The order of the set W^P of minimal length representatives is then the order of W divided by the order of W_P which is $2^l \binom{p}{l}$.

Fact 14.1. *The set W^P of minimal length representatives is given by the set of all barred permutations*

$$w = (u_1, \ldots, u_t, \overline{u_{t+1}}, \ldots, \overline{u_l}, u_{l+1}, \ldots, u_{p-1}, \hat{u}_p)$$

where $0 \leq t \leq l$, $u_1 < \cdots < u_t$, $u_{t+1} > \cdots > u_l$, $u_{l+1} < \cdots < u_p$ and where $\hat{u}_p = u_p$ if $l-t$ is even and $\hat{u}_p = \overline{u_p}$ if $l-t$ is odd.

Proof. Let w be a barred permutation as in the statement. It is easy to see that $w(\alpha) > 0$ for all $\alpha \in \Delta_P$. Therefore it follows that w is a minimal length representative (cf. [22, 9.1]). To see that the set of all barred permutations as in the statement is the complete set of minimal length representatives it suffices to show that the cardinality of all barred permutations as in the statement equals $2^l \binom{p}{l}$. But the cardinality of all barred permutations as in the statement is by definition

$$\sum_{t=0}^{l} \binom{l}{t} \binom{p}{l} = 2^l \binom{p}{l}$$

as desired. □

Remark 14.2. Let $w \in W^P$ be a minimal length representative with representation as barred permutation as in Fact 14.1. Since we know that $l \leq p-2$ by assumption and since $u_{l+1} < \cdots < u_p$ we can always say that $u_p > 1$.

Definition 14.3 ([7, Definition 1.1])**.** *We say that a partition λ is q-strict if no part greater than q is repeated, i.e. $\lambda_j > q$ implies $\lambda_{j+1} < \lambda_j$. We say that a partition λ is strict if it is* 0*-strict.*

We now introduce various sets of partitions. We denote by $\mathcal{R}(q,l)$ the set of all partitions of shape $q \times l$. We denote by $\mathcal{D}(l,p-1)$ the set of all strict partitions of shape $l \times (p-1)$. We denote by $\mathcal{P}(l,p)$ the set of all q-strict partitions of shape $l \times (2p-l-1) = l \times (p+q-1)$. We denote the length of a partition λ by $\ell(\lambda)$. We denote the weight of a partition λ by $|\lambda|$. We denote the conjugate (or transpose) of a partition α by α'. We denote by $\mathcal{Q}(l,p)$ the set of all partition pairs $(\alpha,\lambda) \in \mathcal{R}(q,l) \times \mathcal{D}(l,p-1)$ such that $\alpha_q \geq \ell(\lambda)$.

We now associate a number in $\{0,1,2\}$ called the type and denoted by type$(-)$ to any element of $\mathcal{P}(l,p)$ and $\mathcal{Q}(l,p)$. If a partition $\lambda \in \mathcal{P}(l,p)$ has no part equal to q we set type$(\lambda) = 0$, otherwise we have type$(\lambda) = 1$ or type$(\lambda) = 2$. If a partition pair $(\alpha,\lambda) \in \mathcal{Q}(l,p)$ satisfies $\alpha_q = \ell(\lambda)$ we set type$(\alpha,\lambda) = 0$, otherwise we have type$(\alpha,\lambda) = 1$ or type$(\alpha,\lambda) = 2$. We denote by $\tilde{\mathcal{P}}(l,p)$ the set of all partitions in $\mathcal{P}(l,p)$ with their type attached to them. We denote by $\tilde{\mathcal{Q}}(l,p)$ the set of all partition pairs in $\mathcal{Q}(l,p)$ with their type attached to them.

Fact 14.4. *The map φ wich sends a partition pair $(\alpha,\lambda) \in \mathcal{Q}(l,p)$ to the partition $\alpha' + \lambda$ defines a well defined injective map from $\mathcal{Q}(l,p)$ to $\mathcal{P}(l,p)$. The map φ extends to a well defined injective map from $\tilde{\mathcal{Q}}(l,p)$ to $\tilde{\mathcal{P}}(l,p)$ (which we still denote by φ) in such a way that* type$(\alpha,\lambda) =$ type$(\varphi(\alpha,\lambda))$ *for all partition pairs $(\alpha,\lambda) \in \tilde{\mathcal{Q}}(l,p)$.*

Proof. It is obvious that the image $\varphi(\alpha,\lambda)$ of a partition pair $(\alpha,\lambda) \in \mathcal{Q}(l,p)$ is of shape $l \times (2p-l-1)$. To see that the map φ is well defined it therefore suffices to show that the image is a q-strict partition. Suppose that two succesive parts are repeated: $\alpha'_j + \lambda_j = \alpha'_{j+1} + \lambda_{j+1}$ for some j. Then we have to show that $\alpha'_j + \lambda_j \leq q$. Indeed, it follows that $\lambda_j - \lambda_{j+1} = \alpha'_{j+1} - \alpha'_j \leq 0$ and thus $\lambda_j = \lambda_{j+1} = 0$ since λ is a strict partition. This means that $\alpha'_j + \lambda_j = \alpha'_j \leq q$ as claimed.

Next we prove that the map φ is injective. Suppose that $\varphi(\alpha,\lambda) = \varphi(\beta,\mu)$ for two partition pairs $(\alpha,\lambda),(\beta,\mu) \in \mathcal{Q}(l,p)$. Suppose that $\ell(\mu) \leq \ell(\lambda)$. From the condition $\alpha_q \geq \ell(\lambda)$ it follows that $\alpha'_1 = \cdots = \alpha'_{\ell(\lambda)} = q$ and similar that $\beta'_1 = \cdots = \beta'_{\ell(\mu)} = q$. From $\varphi(\alpha,\lambda)_j = \varphi(\beta,\mu)_j$ for all $1 \leq j \leq \ell(\mu)$ it then follows that $\lambda_j = \mu_j$ for all $1 \leq j \leq \ell(\mu)$ which means in particular that $\mu \subseteq \lambda$. From $\alpha' + \mu \subseteq \alpha' + \lambda = \beta' + \mu$ it then follows that $\alpha' \subseteq \beta'$ in particular $\alpha'_{\ell(\lambda)} = \beta'_{\ell(\lambda)} = q$. The equality $\alpha'_{\ell(\lambda)} + \lambda_{\ell(\lambda)} = \beta'_{\ell(\lambda)} + \mu_{\ell(\lambda)}$ then gives $\lambda_{\ell(\lambda)} = \mu_{\ell(\lambda)} > 0$ and thus $\ell(\mu) \geq \ell(\lambda)$ which means $\ell(\mu) = \ell(\lambda)$ by assumption. This immediately implies $\lambda = \mu$ and thus $\alpha' = \beta'$ and thus $(\alpha,\lambda) = (\beta,\mu)$ as desired.

To see that the map φ extends to a well defined injective map from $\tilde{\mathcal{Q}}(l,p)$ to $\tilde{\mathcal{P}}(l,p)$ in such a way that $\mathrm{type}(\alpha,\lambda) = \mathrm{type}(\varphi(\alpha,\lambda))$ for all partition pairs $(\alpha,\lambda) \in \tilde{\mathcal{Q}}(l,p)$ we only have to show that the image of a type zero partition pair $(\alpha,\lambda) \in \tilde{\mathcal{Q}}(p,l)$ is mapped under φ to a type zero partition in $\tilde{\mathcal{P}}(l,p)$, in other words we have to show that if $\varphi(\alpha,\lambda)_j = q$ for a partition pair $(\alpha,\lambda) \in \mathcal{Q}(l,p)$ and some j then $\alpha_q > \ell(\lambda)$. Indeed, suppose that $\alpha'_j + \lambda_j = q$ for a partition pair $(\alpha,\lambda) \in \mathcal{Q}(l,p)$ and some j. Suppose that $\lambda_j > 0$ then $\alpha'_j < q$ and thus $\ell(\lambda) \geq j > \alpha_q$ – a contradiction. Therefore we conclude that $\lambda_j = 0$ and thus $\alpha'_j = q$ and thus $\alpha_q \geq j > \ell(\lambda)$ as required. □

We define a second length function $\mathring{\ell}$ on the set of minimal length representatives W^P. Let $w \in W^P$ be a minimal length representative with representation as barred permutation as in Fact 14.1. Then we define $\mathring{\ell}(w)$ by the following assignement:

$$\mathring{\ell}(w) = \begin{cases} 0 & \text{if } t = l \\ l - t & \text{if } t < l \text{ and } u_l > 1 \\ l - t - 1 & \text{if } t < l \text{ and } u_l = 1 \end{cases}$$

Lemma 14.5. *Let $w \in W^P$ be a minimal length representative with representation as barred permutation as in Fact 14.1. Let $l < i \leq p$ be an index. Then we have the following inequality:*

$$u_i - 1 + \mathrm{card}\{t < j \leq l \mid u_j > u_i\} \geq l - t\,.$$

Proof. Indeed, the inequality in question is equivalent to the inequality

$$u_i - 1 \geq \mathrm{card}\{t < j \leq l \mid u_j < u_i\}$$

which is obvious. □

Corollary 14.6. *Let $w \in W^P$ be a minimal length representative with representation as barred permutation as in Fact 14.1. Let $l < i \leq p$ be an index. Suppose that we have the following equality:*

$$u_i - 1 + \mathrm{card}\{t < j \leq l \mid u_j > u_i\} = \mathring{\ell}(w)\,.$$

Then we have that $\mathring{\ell}(w) = l - t$ which means that $u_l > 1$ if $t < l$.

Proof. This is obvious from the previous lemma and the definition of $\mathring{\ell}$. □

Lemma 14.7. *Let $w \in W^P$ be a minimal length representative with representation as barred permutation as in Fact 14.1. Let $d_i = \mathrm{card}\{t < j \leq l \mid u_j > u_{p-i+1}\}$ for all $1 \leq i \leq q$. Let $\alpha_i = u_{p-i+1} + i - q - 1 + d_i$ for all $1 \leq i \leq q$. Then we have*

$$l \geq \alpha_1 \geq \cdots \geq \alpha_q \geq l - t \geq \mathring{\ell}(w) \geq 0$$

which means that $\alpha = (\alpha_1, \ldots, \alpha_q)$ is a well defined partition in $\mathcal{R}(q,l)$.

Proof. In general we have $d_i \leq p - u_{p-i+1}$ for all $1 \leq i \leq q$, in particular we have $d_1 \leq p - u_p$ from which it follows that $\alpha_1 \leq p - q = l$. By definition and by the previous lemma we have that $\alpha_q \geq l - t$. Let $1 \leq i < q$. We are left to show that $\alpha_i - \alpha_{i+1} \geq 0$. By definition this inequality is equivalent to $u_{p-i+1} - u_{p-i} - 1 \geq d_{i+1} - d_i$. Again by definition we have

$$d_{i+1} - d_i = \operatorname{card}\{t < j \leq l \mid u_{p-i} < u_j < u_{p-i+1}\}$$

so that the previous inequality becomes obvious. □

We now associate to each element $w \in W^P$ an element $\psi(w) = (\alpha, \lambda) \in \tilde{\mathcal{Q}}(l, p)$. Suppose that w is represented as barred permutation as in Fact 14.1. We then define α to be the element of $\mathcal{R}(l, p)$ which depends on w as described in the previous lemma. To define λ we suppose first that we have equality $\alpha_q = \mathring{\ell}(w)$. Then we know by the previous corollary that $u_l > 1$ if $t < l$. Therefore we can define a strict partition λ of length $\mathring{\ell}(w) = l - t$ by the assignement $\lambda_i = u_{t+i} - 1$ for all $1 \leq i \leq \mathring{\ell}(w)$. It is clear that λ is a well defined element of $\mathcal{D}(l, p-1)$ and that the pair (α, λ) is a well defined element of $\tilde{\mathcal{Q}}(l, p)$ of type zero. Suppose next that $\alpha_q \neq \mathring{\ell}(w)$, i.e. $\alpha_q > \mathring{\ell}(w)$ by Lemma 14.5. Then we can define a strict partition λ of length $\mathring{\ell}(w)$ by the assignement $\lambda_i = u_{t+i} - 1$ for all $1 \leq i \leq \mathring{\ell}(w)$ which makes sense since $u_{t+i} > 1$ for all $1 \leq i \leq \mathring{\ell}(w)$. It is clear that λ is a well defined element of $\mathcal{D}(l, p-1)$ and that the pair (α, λ) is a well defined element of $\mathcal{Q}(l, p)$. We set $\operatorname{type}(\alpha, \lambda) = 1$ if $\hat{u}_p = u_p$ and $\operatorname{type}(\alpha, \lambda) = 2$ if $\hat{u}_p = \overline{u_p}$ to produce a well defined element $(\alpha, \lambda) \in \tilde{\mathcal{Q}}(l, p)$. With these definitions we always have $\ell(\lambda) = \mathring{\ell}(w)$.

Lemma 14.8. *Let w and w' be two elements of W^P which have the following representation as barred permutations:*

$$w = (u_1, \ldots, u_t, \overline{u_{t+1}}, \ldots, \overline{u_l}, u_{l+1}, \ldots, \hat{u}_p)\,,$$
$$w' = (v_1, \ldots, v_t, \overline{u_{t+1}}, \ldots, \overline{u_l}, v_{l+1}, \ldots, \hat{v}_p)\,.$$

Let $1 \leq i \leq q$ be some index. Let $d_i = \operatorname{card}\{t < j \leq l \mid u_j > u_{p-i+1}\}$ and let $e_i = \operatorname{card}\{t < j \leq l \mid u_j > v_{p-i+1}\}$. Suppose that $u_{p-i+1} + d_i = v_{p-i+1} + e_i$. Then we have that $u_{p-i+1} = v_{p-i+1}$.

Proof. Suppose for a contradiction that we have $u_{p-i+1} < v_{p-i+1}$. Then we have

$$n := v_{p-i+1} - u_{p-i+1} = d_i - e_i = \operatorname{card}\{t < j \leq l \mid u_{p-i+1} < u_j < v_{p-i+1}\}$$

which means that there exist n indices $t < j_n < \cdots < j_1 \leq l$ such that $u_{p-i+1} < u_{j_1} < \cdots < u_{j_n} < v_{p-i+1}$. But the later sequence of inequalities immediately implies that $n = v_{p-i+1} - u_{p-i+1} > n$ – a contradiction. Therefore we conclude that $u_{p-i+1} = v_{p-i+1}$ as desired. □

Lemma 14.9. *The map ψ from W^P to $\tilde{\mathcal{Q}}(l,p)$ as defined above is injective.*

Proof. Suppose that $\psi(w) = \psi(w') = (\alpha, \lambda)$ for two elements $w, w' \in W^P$. First we observe that we know by construction that $\ell(\lambda) = \mathring{\ell}(w) = \mathring{\ell}(w')$. It is easy to see that we can reconstruct from the type of (α, λ) and the length of λ the value of t of a barred permutation in the preimage of (α, λ) which therefore must coincide for both barred permutations w and w'. Indeed, if (α, λ) is of type zero then we have $t = l - \ell(\lambda)$ (Corollary 14.6). Suppose next that (α, λ) is of type one. Then we know that $t - l$ must be even. In addition it is clear that $t - l - \ell(\lambda)$ is zero if $\ell(\lambda)$ is even and one if $\ell(\lambda)$ is odd. Finally suppose that (α, λ) is of type two. Then we know that $t - l$ must be odd. In addition it is clear that $t - l - \ell(\lambda)$ is zero if $\ell(\lambda)$ is odd and one if $\ell(\lambda)$ is odd. Once we know how to reconstruct the value of t from (α, λ) we also know how to reconstruct the values of $u_{t+1}, \ldots, u_l$. Indeed, we must have $u_{t+i} = \lambda_i + 1$ for all $1 \leq i \leq \ell(\lambda)$ and $u_l = 1$ if $\ell(\lambda) < t - l$. In total this means that w and w' have a representations as barred permutations as in the previous lemma. The conclusion of the previous lemma then immediately implies that $u_{p-i+1} = v_{p-i+1}$ for all $1 \leq i \leq q$ since w and w' have the same image under ψ. This in turn immediately implies that $w = w'$ as desired. □

Corollary 14.10. *The maps ψ and φ induce bijcetions:*

$$W^P \cong_\psi \tilde{\mathcal{Q}}(l,p) \cong_\varphi \tilde{\mathcal{P}}(l,p)\,.$$

Proof. Since ψ and φ are injective we know that the composition $\varphi \circ \psi$ is also injective. Since by [7, 4.3] the set $\tilde{\mathcal{P}}(l,p)$ parametrizes the B-orbits as well as the set W^P, we know that they must have the same cardinality. Therefore the map $\varphi \circ \psi$ is bijective and thus also the maps ψ and φ are bijective as claimed. □

Lemma 14.11. *Let $w \in W^P$ be a minimal length representative with representation as barred permutation as in Fact 14.1. Let the numbers d_i for all $1 \leq i \leq q$ be defined as in Lemma 14.7. Then we have the following formula for the length of w:*

$$\ell(w) = \sum_{i=1}^{t} u_i - \sum_{i=1}^{q} d_i + (p+q)(l-t) - \frac{l(l+1)}{2}\,.$$

Proof. Let $a_i = \operatorname{card}\{j > i \mid u_j < u_i\}$ and let $b_i = \operatorname{card}\{j > i \mid u_j > u_i\}$. By [26, 2.(1)] we know that

$$\ell(w) = \sum_{i=1}^{p} a_i + 2\sum_{i=t+1}^{l} b_i\,.$$

To prove the desired formula we only have to simplify this expression. If $i > l$ then $a_i = 0$. If $t < i \leq l$ then $a_i = l - i + \operatorname{card}\{l < j \leq p \mid u_j < u_i\}$

and thus
$$\sum_{i=t+1}^{l} a_i = \frac{(l-t-1)(l-t)}{2} + \sum_{i=1}^{q} d_i\,.$$
If $i \le t$ then $a_i = u_i - i$ and thus
$$\sum_{i=1}^{t} a_i = \sum_{i=1}^{t} u_i - \frac{t(t+1)}{2}\,.$$
In total this gives that
$$\sum_{i=1}^{p} a_i = \sum_{i=1}^{t} u_i + \sum_{i=1}^{q} d_i + \frac{l(l-1)}{2} - lt\,.$$
On the other hand we have
$$\sum_{i=t+1}^{l} b_i = \operatorname{card}\{t < i \le l, l < j \le p \mid u_j > u_i\} = (p-l)(l-t) - \sum_{i=1}^{q} d_i\,.$$
Putting all formulas together we obtain the desired result. □

Let $w \in W^P$ be a minimal length representative with representation as barred permutation as in Fact 14.1. Let $u'_i = p+1-u_i$ for all $1 \le i \le p$. Then we can define a new minimal length representative w' by setting:
$$w' = (u'_t, \dots, u'_1, \overline{u'_l, \dots, u'_{t+1}}, u'_p, \dots, \hat{u}'_{l+1})\,.$$
If we set $\phi(w) = w'$ this construction defines us a map from W^P to W^P. From the definition it is clear that ϕ is an involution, in particular bijective. We denote the image of an element $w \in W^P$ under $\varphi \circ \psi \circ \phi$ by λ_w. We denote the preimage of an element $\lambda \in \tilde{\mathcal{P}}(l,p)$ under $\varphi \circ \psi \circ \phi$ by w_λ. If $w \in W$ is an arbitrary Weyl group element we define $\lambda_w = \lambda_{\tilde{w}}$ where $\tilde{w}$ is the minimal length representative of w. If $\lambda \in \tilde{\mathcal{P}}(l,p)$ then it makes sense to denote the dual partition of λ by λ^*. The element $^* \in \tilde{\mathcal{P}}(l,p)$ is defined by the formula $\lambda^* = \lambda_{w^*_\lambda}$.

Lemma 14.12. *For all $w \in W$ we have $\ell_P(w) = |\lambda_w|$ or equivalent for all $w \in W^P$ we have $\ell(w) = |\lambda_w|$.*

Proof. For a partition pair (α, λ) we define the weight of (α, λ) in the obvious way as $|(\alpha,\lambda)| = |\alpha| + |\lambda|$. From the definition of φ it is obvious that φ is weight preserving, that is $|\varphi(\alpha,\lambda)| = |(\alpha,\lambda)|$ for all partition pairs (α,λ). Let $w \in W^P$ be a minimal length representative with representation as barred permutation as in Fact 14.1. Let $w' = \phi(w)$ and let $(\alpha,\lambda) = \psi(w')$. Since φ is weight preserving we only have to show that $\ell(w) = |(\alpha,\lambda)|$. Let the numbers d_i for all $1 \le i \le q$ be defined as in Lemma 14.7 with respect to the entries of w and let the numbers d'_i for all $1 \le i \le q$ be defined as in Lemma 14.7 but now with respect to the entries of w'. From the definition of these numbers it is rather clear that we have

$$\sum_{i=1}^{q} d'_i = q(l-t) - \sum_{i=1}^{q} d_i \,.$$

Let u'_i for all $1 \le i \le p$ be defined as before in terms of u_i. By reordering the summands the definition of α gives

$$|\alpha| = \sum_{i=1}^{q}(u'_{p-i+1} + i - q - 1 + d'_i)\,.$$

If we plug in the identities we know for the primed variables this sum becomes

$$\sum_{i=1}^{q}(l + i - u_{p-i+1} + d'_i) = lq + \frac{q(q+1)}{2} - \sum_{i=l+1}^{p} u_i + q(l-t) - \sum_{i=1}^{q} d_i\,.$$

By reordering the summands the definition of λ gives on the other hand

$$|\lambda| = \sum_{i=t+1}^{l} u'_i - 1 = p(l-t) - \sum_{i=t+1}^{l} u_i\,.$$

If we use the trivial identity

$$\sum_{i=1}^{t} u_i = \frac{p(p+1)}{2} - \sum_{i=t+1}^{p} u_i$$

then we immediately see that we have

$$|(\alpha,\lambda)| = \sum_{i=1}^{t} u_i - \sum_{i=1}^{q} d_i + (p+q)(l-t) + lq + \frac{q(q+1)}{2} - \frac{p(p+1)}{2}\,.$$

If we plug in the symbolic identity

$$lq + \frac{q(q+1)}{2} - \frac{p(p+1)}{2} + \frac{l(l+1)}{2} = 0$$

in the previous equation then the expression precisely becomes $\ell(w)$ according to the previous lemma. □

Fact 14.13. *The dimension of X is given by the following formula:*

$$\dim(X) = (p+q)l - \frac{l(l+1)}{2} = 2lq + \frac{l(l-1)}{2}\,.$$

Proof. There is a unique element of $\tilde{\mathcal{P}}(l,p)$ of maximal weight, namely the partition

$$\rho = (2p - l - 1, 2p - l - 2, \ldots, 2(p-l))\,.$$

Since every part of ρ is greater than q, we necessarily have $\mathrm{type}(\rho) = 0$. On the other hand there is also a unique element of W^P of maximal length, namely the element w_X. Therefore we conclude that $w_X = w_\rho$ or equivalent $\rho = \lambda_{w_X}$. This implies that

$$\dim(X) = \ell(w_X) = |\rho| = (p+q)l - \frac{l(l+1)}{2}\,.$$

The second equality in the statement is just a symbolic identity. □

Example 14.14. We have the following description of w_o, w_X and w_P in terms of barred permutations:

$$\begin{aligned} w_o &= (\overline{1}, \overline{2}, \dots, \overline{p-1}, \hat{p})\,, \\ w_X &= (\overline{l}, \overline{l-1}, \dots, \overline{1}, l+1, \dots, p-1, \hat{p})\,, \\ w_P &= (l, l-1, \dots, 1, \overline{l+1}, \dots, \overline{p-1}, \hat{p})\,, \end{aligned}$$

where in each case the hat indicates that the number of bars is completed to an even number of bars. Indeed, the description of w_o is well known. The barred permutation which desribes w_X is obtained by taking the unique element of W^P of maximal length equal to $\dim(X)$. The barred permutation which describes w_P arises from the identity $w_P = w_o w_X$ and the previous descriptions.

Example 14.15. In this example we assume that l is odd. Let $\rho_i = \lambda_{s_{\theta_i}\cdots s_{\theta_k}}$ for all $1 \le i \le k$. In this example we want to compute the partitions ρ_i. We start with the case $i = k$. Then we have $s_{\theta_k} = s_{\alpha_l}$ and thus $\tilde{s}_{\theta_k} = s_{\theta_k} = s_{\alpha_l}$. This means that we have $\ell_P(s_{\theta_k}) = 1$. Therefore ρ_k must be the unique parition of weight 1. We conclude that $\rho_k = (1, 0^{l-1})$. The partition ρ_k is necessarily of type zero.

Assume now that $1 \le i < k$. Then we have

$$s_{\theta_i} = (1, \dots, 2i-2, \overline{2i}, \overline{2i-1}, 2i+1, \dots, p)\,.$$

By multiplying those elements we obtain that

$$\begin{aligned} s_{\theta_i}\cdots s_{\theta_k} = (1, \dots, 2i-2, \overline{2i}, \overline{2i-1}, \overline{2(i+1)}, \overline{2(i+1)-1}, \\ \dots, \overline{l-1}, \overline{l-2}, \overline{l}, \overline{l+1}, l+2, \dots, p)\,. \end{aligned}$$

If we take minimal length representatives of the above elements this results in the formula:

$$w_{X^i} = \tilde{s}_{\theta_i}\cdots\tilde{s}_{\theta_k} = (1, \dots, 2i-2, \overline{l}, \dots, \overline{2i-1}, l+1, \dots, \overline{p})\,.$$

From this formula it is easy to compute the corresponding paritions as

$$\rho_i = (p+q-2i+1, p+q-2i, \dots, p+q-l, 0^{2i-2})\,.$$

The parition ρ_i is necessarily of type zero. According to the computation in the previous example we find that $\rho = \rho_1$. Note that we have inclusions

$$\rho_k \subseteq \rho_{k-1} \subseteq \cdots \subseteq \rho_1$$

corresponding to the general fact that

$$s_{\theta_k} \preceq s_{\theta_{k-1}} s_{\theta_k} \preceq \cdots \preceq s_{\theta_1}\cdots s_{\theta_k}\,.$$

Let W_G be the set of all permutations $\sigma \in S_{2p}$ such that $\sigma(i) + \sigma(2p+1-i) = 2p+1$ for all $1 \leq i \leq 2p$ (or for all $1 \leq i \leq p$) and such that $\mathrm{card}\{j \leq p \mid \sigma(j) > p\}$ is even. We think of elements of W_G as permutation matrices.

Fact 14.16. *The set W_G is a subgroup of S_{2p} of order $\mathrm{card}(W) = 2^{p-1}p!$. For all $\sigma \in W_G$ we have $\mathrm{sgn}(\sigma) = 1$.*

Proof. It is clear that the identity is part of W_G. Let $\sigma, \tau \in W_G$. It is clear that we have $\tau\sigma(i) + \tau\sigma(2p+1-i) = 2p+1$ for all $1 \leq i \leq 2p$. In order to see that W_G is a subgroup of S_{2p} we therefore only have to check that $\mathrm{card}\{j \leq p \mid \tau\sigma(j) > p\}$ is even. To this end we compute:

$$\begin{aligned}\mathrm{card}\{j \leq p \mid \tau\sigma(j) > p\} &= \mathrm{card}\{j \mid \tau(j) > p\} - \mathrm{card}\{j > p \mid \tau\sigma(j) > p\} \\ &\equiv \mathrm{card}\{j > p \mid \tau(j) > p\} - \mathrm{card}\{j > p \mid \tau\sigma(j) > p\}\end{aligned}$$

where the last congruence follows since $\tau \in W_G$. Now we have

$$\mathrm{card}\{j > p \mid \tau(j) > p\} =$$
$$\mathrm{card}\{j > p \mid \tau\sigma(j) > p, \sigma(j) > p\} + \mathrm{card}\{j \leq p \mid \tau\sigma(j) > p, \sigma(j) > p\}$$

and

$$\mathrm{card}\{j > p \mid \tau\sigma(j) > p\} =$$
$$\mathrm{card}\{j > p \mid \tau\sigma(j) > p, \sigma(j) > p\} + \mathrm{card}\{j > p \mid \tau\sigma(j) > p, \sigma(j) \leq p\} =$$
$$\mathrm{card}\{j > p \mid \tau\sigma(j) > p, \sigma(j) > p\} + \mathrm{card}\{j \leq p \mid \tau\sigma(j) \leq p, \sigma(j) > p\}\,.$$

If we plug in the two latter identities in the first congruence we get

$$\mathrm{card}\{j \leq p \mid \tau\sigma(j) > p\} \equiv \mathrm{card}\{j \leq p \mid \sigma(j) > p\} \equiv 0$$

where the last congruence follows since $\sigma \in W_G$. In total this proves that $\tau\sigma \in W_G$ and that W_G is a subgroup of S_{2p}.

We next check that the $\mathrm{sgn}(\sigma) = 1$ for all $\sigma \in W_G$. Let $\sigma \in W_G$. Then there exists a unique product $\pi \in W_G$ of an even number of the transitions $(1(2p)), (2(2p-1)), \ldots, (p(p+1))$ such that $\sigma\pi$ has a permutation matrix of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A and B are permutation matrices of permutations $\tau, \tau' \in S_p$. Since $\sigma\pi$ is an element of W_G we know that $\tau(i) + \tau(p+1-i) = p+1$ for all $1 \leq i \leq p$. But this equation immediately implies that $\mathrm{sgn}(\tau) = \mathrm{sgn}(\tau')$. On the other hand it is obvious that $\mathrm{sgn}(\pi) = 1$. Therefore we get $\mathrm{sgn}(\sigma) = \mathrm{sgn}(\sigma\pi) = \mathrm{sgn}(\tau)\,\mathrm{sgn}(\tau') = 1$ as claimed.

Finally we compute the order of W_G. To this end, let $\pi_i = (i(2p+1-i))((i+1)(2p-i))$ for all $1 \leq i \leq p-1$. The subgroup of W_G generated by all products of an even number of the transitions $(i(2p+1-i))$ for all $1 \leq i \leq p$ is obviously the same as the subgroup generated by $\pi_1, \ldots, \pi_{p-1}$ which is isomophic to $(\mathbb{Z}/2\mathbb{Z})^{p-1}$. Since we already know that every element of W_G can be uniquely written as a product of an element of S_p and an element of $\langle \pi_1, \ldots, \pi_{p-1} \rangle$ we conclude that W_G is as a set in bijection to $S_p \times (\mathbb{Z}/2\mathbb{Z})^{p-1}$ and thus of order $\mathrm{card}(W) = 2^{p-1}p!$. □

Corollary 14.17. *If we embed S_p in the obvious way into W_G then we have a decomposition of W_G as a semidirect product:*

$$W_G = S_p \rtimes \langle \pi_1, \ldots, \pi_{p-1} \rangle$$

where $\pi_i = (i(2p+1-i))((i+1)(2p-i))$ for all $1 \leq i \leq p-1$.

Proof. By what we saw up to now it suffices to show that S_p acts on $\langle \pi_1, \ldots, \pi_{p-1} \rangle$ via conjugation. But this is clear since we have that $\sigma(i(2p+1-i))\sigma^{-1} = (\sigma(i)(2p+1-\sigma(i)))$ for all $\sigma \in S_p$ and all $1 \leq i \leq p$, that is every of the transitions $(i(2p+1-i))$ where $1 \leq i \leq p$ is mapped again to such a transition. □

It is clear that an element of W_G is already determined by the first p entries of the permutation. Threrefore we sometimes write an element of W_G as a tuple of the first p entries and not as a tuple of all $2p$ entries.

Let $w \in W$. Then we write $w = (\hat{w}_1, \ldots, \hat{w}_p)$ where $\hat{w}_i \in \{w_i, \overline{w_i}\}$ for all $1 \leq i \leq p$. We now define a map $f \colon W \to W_G$ by the following assignement:

$$f(w)(i) = \begin{cases} w_i & \text{if } \hat{w}_i = w_i \\ 2p+1-w_i & \text{if } \hat{w}_i = \overline{w_i} \end{cases}$$

for all $1 \leq i \leq p$.

Fact 14.18. *The map f is an isomorphism of groups.*

Proof. It is clear that f maps the identity to the identity. Let $w, v \in W$. We distinguish four cases to check that $f(vw) = f(v)f(w)$. Let $1 \leq i \leq p$ be arbitrary. Assume first that $\hat{w}_i = w_i$ and $\hat{v}_{w_i} = v_{w_i}$. Then we have $f(vw)(i) = v_{w_i}$ and $f(v)f(w)(i) = f(v)(w_i) = v_{w_i}$ as claimed. Assume next that $\hat{w}_i = w_i$ and $\hat{v}_{w_i} = \overline{v_{w_i}}$. Then we have $f(vw)(i) = 2p+1-v_{w_i}$ and $f(v)f(w)(i) = f(v)(w_i) = 2p+1-v_{w_i}$ as claimed. Assume next that $\hat{w}_i = \overline{w_i}$ and $\hat{v}_{w_i} = v_{w_i}$. Then we have $f(vw)(i) = 2p+1-v_{w_i}$ and $f(v)f(w)(i) = f(v)(2p+1-w_i) = 2p+1-f(v)(w_i) = 2p+1-v_{w_i}$ as claimed. Assume finally that $\hat{w}_i = \overline{w_i}$ and $\hat{v}_{w_i} = \overline{v_{w_i}}$. Then we have $f(vw)(i) = v_{w_i}$ and $f(v)f(w)(i) = f(v)(2p+1-w_i) = 2p+1-f(v)(w_i) = v_{w_i}$ as claimed. This shows that f is a group homomorphism. The injectivity of f is obvious from the definition. The surjectivity of f follows since we already know that $\mathrm{card}(W) = \mathrm{card}(W_G)$. □

Example 14.19. Let $\pi_1, \ldots, \pi_{p-1}$ be the elements of W_G as defined in Corollary 14.17. Then we have:

$$f(1, \ldots, i-1, \overline{i}, \overline{i+1}, i+2, \ldots, p) = \pi_i$$

for all $1 \leq i \leq p-1$.

Moreover it is clear from the definition of f that we have $f(w) = w$ for all $w \in S_p$ where S_p is as usually embedded into W and W_G. In particular we have $f(s_{\alpha_i}) = s_{\alpha_i}$ for all $1 \leq i \leq p-1$. The image of s_{α_p} under f is seen to be $f(s_{\alpha_p}) = (1, \ldots, p-2, p+1, p+2)$.

Let $\mathbb{C}^{2p}$ be a $2p$-dimensional complex vector space equipped with a nondegenerated symmetric bilinear form Q. We can choose a basis $v_1, \ldots, v_{2p}$ of $\mathbb{C}^{2p}$ such that $Q(v_i, v_j) = 0$ if $i + j \neq 2p + 1$ and $Q(v_i, v_{2p+1-i}) = 1$ for all $1 \leq i \leq p$ ([7, Lemma 4.1]). With respect to this basis Q is represented by the matrix E, the antidiagonal $(1, \ldots, 1)$ of size $2p \times 2p$. By changing the basis of $\mathbb{C}^{2p}$ we may assume from the beginning that Q is represented by E with respect to the canonical basis $e_1, \ldots, e_{2p}$. With these choices the group G is given by matrices $A \in \mathrm{SL}_{2p}(\mathbb{C})$ which satisfy ${}^tAEA = E$. If we identify elements of W_G with permutation matrices then we see that the defining condition of W_G and the fact that all elements of W_G have signature one imply that $W_G \subseteq G$. On the other hand a permutation matrix clearly normalizes a diagonal matrix, so that we get $W_G \subseteq N_G(T)$.

Fact 14.20. *The morphism f is a section of the natural projection $N_G(T) \to N_G(T)/T = W$.*

Proof. Since f is a group homomorphism it clearly suffices to show that the simple reflections in W have the right images in $N_G(T)$. But we already computed these images in Example 14.19. To check that these images project to the correspoding simple reflections we only have to check their action (via conjugation) on the spaces $\mathfrak{g}_{\alpha_i}$ for all $1 \leq i \leq p$. By choosing a Chevalley basis of $\mathfrak{g}$ as in [1, 3.5] we immediately see that we have the right action on these spaces. □

Definition 14.21. *According to* [7, page 39] *we define an index set P to be a subset of $\{1, \ldots, 2p\}$ of cardinality l such that $i + j \neq 2p + 1$ for all $i, j \in P$.*

To every element $w \in W$ we can associate an index set P_w by setting $P_w = \{\sigma(1), \ldots, \sigma(l)\}$ where $\sigma = f(w)$. From the definition of $\sigma \in W_G$ it is obvious that P_w is an index set. Moreover it is clear that P_w only depends on the class of w modulo W_P since W_P acts on P_w by permuting the entries. If in addition $w \in W^P$ then the description of minimal length representatives shows that $\sigma(1) < \cdots < \sigma(l)$. Since an element of W^P is completely determined by its first l entries this shows that $w \mapsto P_w$ defines a bijections between W^P and the set of all index sets. Since P is the stabilizer of $\langle e_1, \ldots, e_l \rangle$ in G we know that $x(w) = \langle e_i \mid i \in P_w \rangle$.

Definition 14.22. *An isotropic flag in $(\mathbb{C}^{2p}, Q)$ is a complete flag $F_\bullet$ such that $F_{p+i} = F_{p-i}^{\perp}$ for all $0 \leq i \leq p$.*

For all $1 \leq i \leq 2p$ let $F_i = \langle e_1, \dots, e_i \rangle$ and let $F_0 = 0$. Then $F_\bullet$ is an isotropic flag in the above sense. The stabilizer of $F_\bullet$ in G is the Borel group B.

Lemma 14.23. *Let $w \in W$ and let $\sigma = f(w)$. Then we can describe the isotropic subspaces parametrized by Ω_w in terms of the following formula:*

$$\Omega_w = \{V \in X \mid \dim(V \cap F_i) = \operatorname{card}\{j \leq l \mid \sigma(j) \leq i\} \text{ for all } 1 \leq i \leq 2p\}.$$

If w is in addition in W^P then Ω_w parametrizes all $V \in X$ which satisfy the incidences $V \subseteq F_{\sigma(l)}$ and $\dim(V \cap F_i) = j$ for all $\sigma(j) \leq i < \sigma(j+1)$ and all $1 \leq j < l$.

Proof. Denote by Ω'_w the right side of the first formula in the lemma. Since P_w only depends on the class of w modulo W_P we see that Ω'_w also only depends on the classe of w modulo W_P. In order to show that $\Omega_w = \Omega'_w$ we therefore may assume that $w \in W^P$. From the definition of Ω'_w it is clear that Ω'_w is B-stable and that $x(w) \in \Omega'_w$. It therefore suffices to show that the only T-fixed point contained in Ω'_w is equal to $x(w)$. Let $v \in W^P$ such that $x(v) \in \Omega'_w$. Then we have to show that $w = v$. Let $\pi = f(v)$. Since $x(v) \in \Omega'_w$ we see that

$$\operatorname{card}\{j \leq l \mid \sigma(j) \leq i\} = \operatorname{card}\{j \leq l \mid \pi(j) \leq i\}$$

for all $1 \leq i \leq 2p$. But this immediately implies that $P_w = P_v$ and thus $w = v$ since the set of all index sets in bijection with W^P. This proves the first formula. The second formula is an immediate consequence of the first and the fact that $\sigma(1) < \cdots < \sigma(l)$ if $w \in W^P$. □

Proposition 14.24. *Let $w \in W$ and let $P_w = \{p_1 < \cdots < p_l\}$ be the corresponding index set. If $p+1 \notin P_w$ then*

$$X_w = \{V \in X \mid \dim(V \cap F_{p_j}) \geq j \text{ for all } 1 \leq j \leq l\}$$

while if $p+1 \in P_w$ then

$$X_w = \{V \in X \mid V \cap F_{p-1} = V \cap F_p\,,\ \dim(V \cap F_{p_j}) \geq j \text{ for all } 1 \leq j \leq l\}.$$

Proof. [7, Proposition 4.5] □

Remark 14.25. The Bruhat order and Poincaré duality can be explicitely described in terms of index sets. The reader finds the formulas in [7, page 43].

Let $\lambda \in \tilde{\mathcal{P}}(l,p)$. Then it makes sense to write $X_\lambda = X_{w_\lambda}$. In this situation it is clear that the Schubert variety is computed with respect to the flag $F_\bullet$ corresponding to the Borel B. Latter on we will also consider translates of Schubert varieties which are computed with respect to a different flag. Therefore it is often necesserary to mention the flag in the notation. If $G_\bullet$ is an arbitrary isotropic flag we write $X_\lambda(G_\bullet)$ for the Schubert variety relative to the flag $G_\bullet$. With this notation we clearly have $X_\lambda = X_\lambda(F_\bullet)$. We denote the Schubert cycle associated to w_λ by $\sigma_\lambda = [X_\lambda] = [X_\lambda(G_\bullet)]$ where $G_\bullet$ is an arbitrary isotropic flag.

Example 14.26. In this example we assume again that l is odd. Let $1 \leq i < k$. Then it is clear from Example 14.15 that we have

$$P_{s_{\theta_i}\cdots s_{\theta_k}} = \{1, \ldots, 2i-2, 2p+1-l, 2p+2-l, \ldots, 2p+2-2i\}\,.$$

It is clear that $p+1 \notin P_{s_{\theta_i}\cdots s_{\theta_k}}$. Therefore the first formula from the previous proposition applies. We find that all incidences for isotropic subspaces parametrized by X_{ρ_i} are redundant except the $(2i-2)$th and the lth one. This gives us the formula

$$X_{\rho_i} = \{V \in X \mid F_{2(i-1)} \subseteq V \subseteq F_{2(p-(i-1))}\}\,.$$

By definition it is clear that $F_{2(p-(i-1))} = F_{2(i-1)}^{\perp}$. Therefore Q induces a nondegenerate symmetric bilinear form on $F_{2(p-(i-1))}/F_{2(i-1)}$ which we still denote by Q. Consequently the map $V \mapsto V/F_{2(i-1)}$ defines an isomorphism $X_{\rho_i} \cong \mathbb{G}_Q(l-2(i-1), 2(p-2(i-1)))$. This isomorphism corresponds to the canonical identification $X_{\rho_i} = X^i$. In particular we have $X^{k-1} = \mathbb{G}_Q(1, 2(p-l+1)) = \mathbb{Q}_{2q}$. Concerning the codimension we find that

$$\operatorname{codim}(X^i) = |\rho| - |\rho_i| = 2(i-1)(p+q) - (i-1)(2(i-1)+1)\,.$$

Finally we treat the case where $i = k$. Then we have $P_{\theta_k} = \{1, \ldots, l-1, l+1\}$. It is clear that $p+1 \notin P_{\theta_k}$. Therefore the first formula from the previous proposition applies. We find that all incidences for isotropic subspaces parametrized by X_{ρ_k} are redundant except the $(l-1)$th and the lth one. This gives us the formula

$$X_{\rho_k} = \{V \in X \mid F_{l-1} \subseteq V \subseteq F_{l+1}\}\,.$$

Since any l-dimensional subspace V which satisfies $F_{l-1} \subseteq V \subseteq F_{l+1}$ is automatically totally isotropic, we see that the map $V \mapsto V/F_{l-1}$ defines an isomorphism $X_{\rho_k} \cong \mathbb{G}(1,2) = \mathbb{P}^1$ corresponding to the canonical identification $X_{\rho_k} = X^k$.

Example 14.27. Assume again that l is odd. We know that $\hat{X}$ is the product of two dualizing varieties $\mathfrak{X}_1$ and $\mathfrak{X}_2$ where $\mathfrak{X}_1 \cong \mathbb{G}_Q(l-1, 2(l-1))$ and $\mathfrak{X}_2 \cong \mathbb{Q}_{2(p-l)}$. Thanks to the previous example we can express which isotropic subspaces are parametrized by $\mathfrak{X}_2$. Indeed, we know that the root system $\mathfrak{R}^2$ is generated by the simple roots $\alpha_l, \ldots, \alpha_p$ (cf. proof of Lemma 11.19). On the other hand, from the definition it is clear that the root system R_{k-1} is also generated by the simple roots $\alpha_l, \ldots, \alpha_p$. Therefore we conclude that $\mathfrak{X}_2 = X^{k-1}$. The previous example then shows that we have

$$\mathfrak{X}_2 = \{V \in X \mid F_{l-1} \subseteq V \subseteq F_{2p-l+1}\}\,.$$

Example 14.28. Assume again that l is odd. We already know that $d_1 = \cdots = d_{k-2} = 2$, $d_{k-1} = d_k = 1$, $d_X = l+1$ and $k = (l+3)/2$. In this example we want to compute other invariants of X. We claim that $c_1(X^i) = p+q-2i+1$ for all $1 \leq i < k$. This means in particular that we always have $c_1(X) = p+q-1$. To prove this, suppose first that $1 \leq i \leq k-2$. Then we know that $d_i = 2$. Using the formula for the codimension of X^i from the example above we get that

$$\ell_P(s_{\theta_i}) = \operatorname{codim}(X^{i+1}) - \operatorname{codim}(X^i) = 2(p+q) - 4i + 1 \,.$$

The formula $\ell_P(s_{\theta_i}) = c_1(X^i)d_i - 1$ then immediately yields that $c_1(X^i) = p+q-2i+1$. Next we consider the case $i = k-1$. Then we have by the example above that

$$\ell_P(s_{\theta_{k-1}}) = \dim(X^{k-1}) - \dim(X^k) = 2q - 1 \,.$$

Since $d_{k-1} = 1$ the formula $\ell_P(s_{\theta_{k-1}}) = c_1(X^{k-1})d_{k-1} - 1$ yields that $c_1(X^{k-1}) = 2q$. But $p+q-2(k-1)+1 = p+q-l = 2q$. Therefore the formula is also satisfied for $i = k-1$. (Note that the index of $X^k = \mathbb{P}^1$ is trivially $c_1(X^k) = 2$ and that $\ell_P(s_{\theta_k}) = \dim(X^k) = 1$.)

15. Rational curves in isotropic Grassmannians

Let $X = G/P = \mathbb{G}_Q(l, 2p)$ where $l \leq p-2$ and $p \geq 3$. Let $q = p - l$. We stick to all the notation which was introduced in the previous section.

A rational curve of degree d to X is a morphism $f \colon \mathbb{P}^1 \to X$ such that

$$\int_X f_*[\mathbb{P}^1] \cdot \sigma_{s_{\alpha_P}} = d \,.$$

For a given degree d and three q-strict paritions λ, μ and ν in $\tilde{\mathcal{P}}(l,p)$ such that

$$|\lambda^*| + |\mu^*| + |\nu^*| = \dim(X) + c_1(X)d$$

the (three-point genus zero) Gromov-Witten invariant $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d$ is the number of rational curves $f \colon \mathbb{P}^1 \to X$ of degree d such that $f(0) \in X_\lambda(F_\bullet)$, $f(1) \in X_\mu(G_\bullet)$ and $f(\infty) \in X_\nu(H_\bullet)$ for three isotropic flags $F_\bullet, G_\bullet$ and $H_\bullet$ in general position. This is equivalent to the general definition from Section 3, since a general member of $\mathcal{M}_{0,3}(X,d)$ will be a rational curve in the above sense (cf. [13, 7.]).

We will use the following proposition from [7, Proposition 1.1] during the text.

Proposition 15.1. *Let U, V and W be three points of $\mathbb{G}_Q(2d, 4d)$, which are in pairwise general position. Then there exists a unique rational curve $f \colon \mathbb{P}^1 \to \mathbb{G}_Q(2d, 4d)$ of degree d such that $f(0) = U, f(1) = V$ and $f(\infty) = W$.*

Proof. The variety $\mathbb{G}_Q(2d,4d)$ is a dualizing variety which satisfies $d_{\mathbb{G}_Q(2d,4d)} = d$. Therefore the assertion follows from Corollary 11.61. For a direct proof which uses the geometry of isotropic subspaces we refer to [7, Proposition 1.1] □

If f is a rational curve to X then we define, according to [5], the kernel of f, denoted by $\ker(f)$, to be the largest linear subspace contained in all the linear subspaces given by points in $f(\mathbb{P}^1)$ and we define the span of f, denoted by $\mathrm{Span}(f)$, to be the smallest linear subspace containing all the linear subspaces given by points in $f(\mathbb{P}^1)$, i.e.

$$\ker(f) = \bigcap_{x\in\mathbb{P}^1} f(x),\ \mathrm{Span}(f) = \sum_{x\in\mathbb{P}^1} f(x)\,.$$

For the kernel and the span of a rational curve we have the following dimension bounds. A proof can be found in [5, Lemma 1].

Proposition 15.2. *Let f be a rational curve of degree d in X. Then the dimension of the span of f is at most $l+d$ and the dimension of the kernel of f is at least $l-d$.*

Proof. [5, Lemma 1] □

Corollary 15.3. *Let f be a general curve of degree d in X. Then the dimension of the span of f is equal to* $\min(l+d,2p)$ *and the dimension of the kernel of f is equal to* $\max(l-d,0)$.

Proof. Since the curve f is general this is a direct consequence of the previous proposition. □

16. The subspaces W_f

Let $X = \mathbb{G}_Q(l,2p)$ where $l \le p-2$, $p \ge 3$ and l is odd. Let $q = p-l$. During the text we worked out the following invariants of X:

$$d_X = l+1\,,\ k = (l+3)/2\,,\ c_1(X) = p+q-1\,.$$

We also worked out the isomorphism

$$\hat{X} \cong \mathbb{G}_Q(l-1,2(l-1)) \times \mathbb{Q}_{2(p-l)}\,.$$

We denote by $\mathfrak{R}^1$ and $\mathfrak{R}^2$ the nontrivial irreducible components of $\hat{R}$ where $\mathfrak{R}^1$ is of type D_{l-1} and $\mathfrak{R}^2$ is of type D_{p-l+1}. Then we have $\mathfrak{X}_1 \cong \mathbb{G}_Q(l-1,2(l-1))$ and $\mathfrak{X}_2 \cong \mathbb{Q}_{2(p-l)}$ and $\hat{X} = \mathfrak{X}_1 \times \mathfrak{X}_2$. Moreover we have $d_{\mathfrak{X}_1} = l-1$ and $d_{\mathfrak{X}_2} = 2$ and $d_X = d_{\hat{X}} = d_{\mathfrak{X}_1} + d_{\mathfrak{X}_2}$. Furthermore note that we know from Theorem 9.5 that the diagonal curve is a general curve in X.

The ideas of this section and the following sections appear first in [25, 9.7.6 and 9.7.7]. For a $2(l-1)$-dimensional subspace W, we define a subvariety X_W of X as

$$X_W = \{V \in X \mid \dim(W \cap V) = l-1, \dim(W^\perp \cap V) = 1\}\,.$$

Lemma 16.1. *Let W be a nondegenerated subspace of dimension $2(l-1)$. If V is a l-dimensional totally isotropic subspace which satisfies* $\dim(W \cap V) = l-1$*, then we have a direct sum decomposition*

$$V = (W \cap V) \oplus (W^{\perp} \cap V) \text{ where } \dim(W^{\perp} \cap V) = 1$$

so that X_W parametrizes l-dimensional isotropic subspaces which satisfy $\dim(W \cap V) = l-1$*. We get an isomorphism*

$$X_W \cong \mathbb{G}_Q(l-1, 2(l-1)) \times \mathbb{Q}_{2(p-l)}$$

given by sending a totally isotropic subspace $V \in X_W$ to the pair $(W \cap V, W^{\perp} \cap V)$. The inverse of this isomorphism is given by sending the pair (V', U) to the direct sum $V' \oplus U$. In particular X_W is an irreducible subvariety of X of dimension

$$\dim(X_W) = \frac{(l-1)(l-2)}{2} + 2(p-l)\,.$$

Proof. We only need to proof that an isotropic subspace V of dimension l which satisfies $\dim(W \cap V) = l-1$ always also satisfies $\dim(W^{\perp} \cap V) = 1$. Indeed, $W^{\perp}$ and $W \cap V$ are in direct sum, since W is nondegenerated and are both contained in $(W \cap V)^{\perp}$ since V is totally isotropic. Since $W^{\perp} \oplus (W \cap V)$ and $(W \cap V)^{\perp}$ are of equal dimension $2p - l + 1$ we get equality. Intersecting the equality $(W \cap V)^{\perp} = W^{\perp} \oplus (W \cap V)$ with V we get the decomposition $(W \cap V) \oplus (W^{\perp} \cap V)$ of V. Since $\dim(W \cap V) = l-1$ this implies that $\dim(W^{\perp} \cap V) = 1$. □

Corollary 16.2. *Let W be a nondegenerated subspace of dimension $2(l-1)$. Then X_W parametrizes l-dimensional isotropic subspaces V which satisfy* $\dim(W \cap V) \geq l-1$.

Proof. Since W is a nondegenerated subspace of dimension $2(l-1)$, a maximal isotropic subspace contained in W is of dimension $l-1$. If V is a l-dimensional isotropic subspace we therefore must have $\dim(W \cap V) < l$ since otherwise $V \subseteq W$ and we obtain a maximal isotropic subspace contained in W of dimension l. Therefore the claim follows from the previous which says that X_W parametrizes l-dimensional isotropic subspaces V which satisfy $\dim(W \cap V) = l-1$. □

Corollary 16.3. *Let W be a nondegenerated subspace of dimension $2(l-1)$. Then X_W satisfies the three point property with respect to the degree d_X.*

Proof. From the previous lemma we see that we have an isomorphism $X_W \cong \hat{X}$. Moreover we see that this isomorphism preserves the degrees $d_{\mathfrak{X}_1}$ and $d_{\mathfrak{X}_2}$ of the factors of $\hat{X}$. Since $\hat{X}$ satisfies the three point property with respect to the degree d_X it therefore follows that X_W also satisfies the three point property with respect to the same degree. □

Lemma 16.4. *Let* $f\colon \mathbb{P}^1 \to X$ *be a general rational curve in* X *of degree* $d_X = l+1$. *Then there exists a subspace* W_f *associated to* f *of dimension* $2(l-1)$ *such that* $\dim(W_f \cap f(x)) = l-1$ *and* $\dim(W_f^{\perp} \cap f(x)) = 1$ *for all* $x \in \mathbb{P}^1$.

Proof. Let K be the tautological vector bundle on X. Since $f^*(K)$ has degree $c_1(\wedge^l f^*(K)) = f^*(\mathcal{O}_X(-1)) = -(l+1)$ and rank l, the vector bundle $f^*(K)$ over $\mathbb{P}^1$ slpits into a direct sum $\bigoplus_{i=1}^{l} \mathcal{O}_{\mathbb{P}^1}(-a_i)$ of line bundles $\mathcal{O}_{\mathbb{P}^1}(-a_i)$ with $\sum_{i=1}^{l} a_i = l+1$ and $a_1 \geq \ldots \geq a_l \geq 0$. Since the differences between a_i and a_{i+1} are minimal for general f, we have $f^*(K) = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(l-1)}$.

Let $W_f = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(l-1)})^{\vee}$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(l-1)} & \longrightarrow & \mathcal{O}_{\mathbb{P}^1} \otimes \mathbb{C}^{2p} \\ \uparrow & & \uparrow \\ \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(l-1)} & \longrightarrow & \mathcal{O}_{\mathbb{P}^1} \otimes W_f \end{array}$$

where the left vertical arrow is the unique canonical morhism

$$\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(l-1)} \to \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(l-1)}$$

given by inclusion of the direct factor, the right vertical arrow and the top horizontal arrow are the obvious morphisms and the horizontal arrow below is the dual of the morphism

$$\mathcal{O}_{\mathbb{P}^1} \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(l-1)}) \to \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(l-1)}$$

resulting from the fact that $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(l-1)}$ is globally generated. From the commutativity of the diagram, we see that $\dim(W_f \cap f(x)) \geq l-1$ for all $x \in \mathbb{P}^1$. From the definition of W_f it is clear that $\dim(W_f) = 2(l-1)$. Since f is general it is also clear that W_f is nondegenerated. Therefore Corollary 16.2 applies and it follows that $f(x) \in X_{W_f}$ for all $x \in \mathbb{P}^1$. The definition of X_{W_f} then implies that $\dim(W_f \cap f(x)) = l-1$ and $\dim(W_f^{\perp} \cap f(x)) = 1$ for all $x \in \mathbb{P}^1$ as claimed. ☐

Corollary 16.5. *Let* f *be a general curve in* X *of degree* $d_X = l+1$. *Then* f *can be written as* (f', f'') *with* $f'\colon \mathbb{P}^1 \to \mathbb{G}_Q(l-1, 2(l-1))$ *and* $f''\colon \mathbb{P}^1 \to \mathbb{Q}_{2(p-l)}$ *of bidegree* $(l-1, 2)$.

Proof. Since f takes values in X_{W_f} it can be written as (f', f'') with $f'\colon \mathbb{P}^1 \to \mathbb{G}_Q(l-1, 2(l-1))$ and $f''\colon \mathbb{P}^1 \to \mathbb{Q}_{2(p-l)}$ of bidegree (d', d''). We know that $d' \leq d_{\mathfrak{X}_1} = l-1$ and that $d'' \leq d_{\mathfrak{X}_2} = 2$ since if either $d' > l-1$ or $d'' > 2$ then there would exist infinitely many curves of degree $d_X = l+1$ through three general points in $f(\mathbb{P}^1)$ which are contained in X_{W_f}. But this contradicts Corollary 16.3. Since the sum $d' + d''$ equals the degree $l+1$ of f we get equality in both cases: $d' = l-1$ and $d'' = 2$. ☐

Remark 16.6. Note that this corollary also follows directly from Fact 11.73 once we know that $X_{W_{f_\Delta}} = \hat{X}$.

Corollary 16.7. *Let f be a general curve in X of degree $d_X = l+1$. Then W_f can be written as the intersection of the span of pairs of general points in $f(\mathbb{P}^1)$, i.e.*

$$W_f = \bigcap_{\substack{x,y\in\mathbb{P}^1 \\ \text{general}}} (f(x)+f(y))\,.$$

Proof. Let $f = (f', f'')$ be a decomposition of f as in the previous corollary. Let x and y be a pair of general points in $\mathbb{P}^1$. Then we know that $f'(x)$ are $f'(y)$ are in direct sum and hence $f'(x) + f'(y)$ is of dimension $2(l-1)$. By definition it is clear that $f'(x)+f'(y) \subseteq W_f$. Therefore we get the equality $f'(x)+f'(y) = W_f$. Using this equality we get the following line of equations for pairs of general points $x, y \in \mathbb{P}^1$:

$$\begin{aligned}\bigcap_{x,y}(f(x)+f(y)) &= \bigcap_{x,y}((f'(x)+f'(y)) \oplus (f''(x)+f''(y))) \\ &= \bigcap_{x,y}(W_f \oplus (f''(x)+f''(y)))\end{aligned}$$

Using that $\sum_{x,y}(f''(x)+f''(y)) \subseteq W_f^\perp$ we see that W_f and $\sum_{x,y}(f''(x)+f''(y))$ are in direct sum since W_f is nondegenerated for a general curve f. Using this fact we get the equality:

$$\bigcap_{x,y}(W_f \oplus (f''(x)+f''(y))) = W_f \oplus \bigcap_{x,y}(f''(x)+f''(y))\,.$$

Finally it is clear that four general points x, x', y, y' in $\mathbb{P}^1$ satisfy the equation

$$(f''(x)+f''(y)) \cap (f''(x')+f'(y')) = 0\,.$$

In particular it follows from this that we get for pairs of general points $x, y \in \mathbb{P}^1$ that

$$\bigcap_{x,y}(f''(x)+f''(y)) = 0\,.$$

Putting all equations together the corollary follows. □

Lemma 16.8. *Let W be a subspace of dimension $2(l-1)$. If V_1, V_2 and V_3 are three elements of X_W in pairwise general position, then we have the inclusion*

$$W \subseteq (V_1+V_2) \cap (V_1+V_3) \cap (V_2+V_3)\,. \tag{6}$$

and equality holds if either the span of V_1, V_2 and V_3 is contained in a $(2l+1)$-dimensional subspace which is for example the case if V_1, V_2 and V_3 lie on a rational curve in X of degree $l+1$ or if W is nondegenerated which is for example the case if there exists a general rational curve f in X of degree $l+1$ such that $W = W_f$.

Proof. Since V_1, V_2 and V_3 are in general position, the intersections $V_1 \cap W, V_2 \cap W$ and $V_3 \cap W$ are in general position and of dimension $l-1$, hence they are pairwise in direct sum and we get that

$$(V_i + V_j) \cap W = (V_i \cap W) \oplus (V_j \cap W) = W$$

for all $1 \leq i < j \leq 3$. Denote with W' the intersection on the right side of the inclusion (6). Intersecting W' with W, we see that this intersection is equal to W because of the previous equalities. This shows the inclusion $W \subseteq W'$.

If V_1, V_2 and V_3 lie on a rational curve of degree $l+1$, then their span is contained in the span of the rational curve which is of dimension less or equal than $2l+1$ by Proposition 15.2. And if the span of V_1, V_2 and V_3 is contained in a subspace L of dimension less or equal than $2l+1$, then it follows that $\dim(W') \leq 2(l-1)$ since V_1, V_2 and V_3 are in pairwise general position and hence pairwise in direct sum as subspaces of L. In view of the inclusion $W \subseteq W'$ it then follows that $W = W'$ as claimed.

Finally, suppose that W is in addition nondegenerated. By Lemma 16.4 this happens if there exists a general rational curve f such that $W = W_f$. By Lemma 16.1 we have a direct sum decomposition of V_i into $U_i = W^{\perp} \cap V_i$ and $V_i' = W \cap V_i$ for $i = 1, 2, 3$. Using the fact that W is nondegenerated, we see that $U_1 + U_2 + U_3 \subseteq W^{\perp}$ and $V_1' + V_2' + V_3 \subseteq W$ are in direct sum. This gives us the equality

$$W' = (U_1 + U_2) \cap (U_1 + U_3) \cap (U_2 + U_3) + \\ (V_1' + V_2') \cap (V_1' + V_3') \cap (V_2' + V_3')$$

where the first summand is equal to 0 since U_1, U_2 and U_3 are of dimension one and in pairwise general position. The inclusion $W' \subseteq W$ then follows. In total we get equality $W = W'$ as claimed. □

Corollary 16.9. *Let f be a general curve in X of degree $d_X = l+1$. Let x, y and z be three points in $\mathbb{P}^1$ which are in pairwise general position. Then we get the equality:*

$$W_f = (f(x) + f(y)) \cap (f(x) + f(z)) \cap (f(y) + f(z)) .$$

Proof. This follows directly from the previous lemma. □

17. Irreducible components of $Y_{d_X}(f)$

Let f be a general curve in X of degree $d_X = l+1$. We define the following subvariety of X:

$$X_{\mathrm{Span}(f)} = \{V \in X \mid V \subseteq \mathrm{Span}(f)\} .$$

By Corollary 15.3 we know that $\ker(f) = 0$ and that $\mathrm{Span}(f)$ is of dimension $2l+1$. Therefore $X_{\mathrm{Span}(f)}$ parametrizes all isotropic subspaces which lie between the kernel and the span of f. Moreover since f is a general curve we know that Span(f) is nondegenerated. Therefore we get the following isomorphism:

$$X_{\mathrm{Span}(f)} \cong \mathbb{G}_Q(l, 2l+1) \cong \mathbb{G}_Q(l+1, 2(l+1))\,.$$

This shows that $X_{\mathrm{Span}(f)}$ is isomorphic to a dualizing variety. In particular this means that $X_{\mathrm{Span}(f)}$ satisfies the three point property with respect to the degree d_X. From Lemma 11.15 it follows that $X_{\mathrm{Span}(f)} \subseteq Y_{d_X}(f)$.

Lemma 17.1. *Let f be a general rational curve in X of degree $d_X = l+1$. Then $Y_{l+1}(f)$ decomposes into two irreducible components given by $X_{\mathrm{Span}(f)}$ and X_{W_f}.*

Proof. We already saw that both X_{W_f} and $X_{\mathrm{Span}(f)}$ are irreducible and satisfy the three point property with respect to the degree d_X (cf. Corollary 16.3). Therefore it follows from Lemma 11.15 that $X_{W_f} \subseteq Y_{d_X}(f)$, $X_{\mathrm{Span}(f)} \subseteq Y_{d_X}(f)$ and thus $X_{\mathrm{Span}(f)} \cup X_{W_f} \subseteq Y_{d_X}(f)$. To see that $Y_{l+1}(f)$ decomposes into two irreducible components given by $X_{\mathrm{Span}(f)}$ and X_{W_f} it therefore suffices to show that $Y_{l+1} \subseteq X_{\mathrm{Span}(f)} \cup X_{W_f}$.

Let V be an element of $Y_{l+1}(f)$ which is not contained in Span(f). Let x and y be two general points of $\mathbb{P}^1$. Then there exists a rational curve g of degree $l+1$ passing through $f(x)$, $f(y)$ and V. Since $f(x)+f(y)$ is of dimension $2l$ and V is not contained in Span(f), it follows from Proposition 15.2 that the span of g is of dimension $2l+1$ and that the intersection $(f(x)+f(y)) \cap V$ is of dimension $l-1$. If the intesection $(f(x)+f(y)) \cap V$ varies with x and y then $V \subseteq \mathrm{Span}(f)$. Thus $(f(x)+f(y)) \cap V$ is independent of x and y. By Corollary 16.7, it follows that $W_f \cap V$ is of dimension $l-1$. This implies that V is an element of X_{W_f} by Lemma 16.1. □

Corollary 17.2. *We have the following equality: $\hat{X} = X_{W_{f_\Delta}}$.*

Proof. We already know that $\hat{X}$ is irreducible and contained in $Y_{d_X}(f_\Delta)$. By the previous lemma it follows that $\hat{X}$ is either contained in $X_{\mathrm{Span}(f_\Delta)}$ or in $X_{W_{f_\Delta}}$. Since the inclusion $\hat{X} \subseteq X_{\mathrm{Span}(f_\Delta)}$ is not possible it follows that $\hat{X} \subseteq X_{W_{f_\Delta}}$. Since we already know that $\hat{X} \cong X_{W_{f_\Delta}}$ it immediately follows that $\hat{X} = X_{W_{f_\Delta}}$ as claimed. □

Corollary 17.3. *The Assumption 13.3 is satisfied. In particular we get a quantum to classical principle for X as described in Section 13.*

Proof. This is clear now from the previous lemma and the previous corollary. □

Lemma 17.4. *The nondegenerated $2(l-1)$-dimensional subspace W_{f_Δ} which is associated to the diagonal curve can be explicitely described in* *terms of the standard basis* $e_1, \dots, e_{2p}$ *by the equation:*

$$W_{f_\Delta} = \langle e_1, \dots, e_{l-1}, e_{2p-l+2}, \dots, e_{2p} \rangle \ .$$

Proof. Since the diagonal curve is a general curve, we have a well defined nondegenerated subspace $W = W_{f_\Delta}$ of dimension $2(l-1)$. Since $X_W = \hat{X}$ we see from the explicit description of $\mathfrak{X}_2$ in terms of isotropic subspaces (Example 14.27) that

$$W^\perp = \langle e_l, \dots, e_{2p-l+1} \rangle \ .$$

Since $W = W^{\perp\perp}$ the result follows.

We provide a second proof of this lemma. By Example 14.26 we know that

$$x(w_o) = \langle e_{2p+1-l}, \dots, e_{2p} \rangle \ , \ x(s_{\theta_k}) = \langle e_1, \dots, e_{l-1}, e_{l+1} \rangle \ .$$

Furthermore it is trivial that $x(1) = \langle e_1, \dots, e_l \rangle$. By Corollary 16.7 it obviously follows that $W \subseteq x(1) \oplus x(w_o)$. Since $x(s_{\theta_k}) \in \hat{X} = X_W$ it follows that $\dim(W \cap x(s_{\theta_k})) = l-1$. Since $e_{l+1} \notin W$ (this must be the case since even $e_{l+1} \notin x(1) \oplus x(w_o)$) it follows from this equation that

$$\langle e_1, \dots, e_{l-1} \rangle \subseteq W \subseteq \langle e_1, \dots, e_l, e_{2p+1-l}, \dots, e_{2p} \rangle \ .$$

But there is one and only one nondegenerated subspace W of dimension $2(l-1)$ which satisfies these two inclusions, namely the W described in the statement of the lemma. □

18. Compactification of Y

Fact 18.1. *We have the equality* $Q = \mathrm{Stab}_G(W_{f_\Delta})$.

Proof. Let $W = W_{f_\Delta}$ for short. By definition we have $g \in Q$ if and only if $g\hat{X} = \hat{X}$. Since $\hat{X} = X_W$ this is equivalent to $X_{g^{-1}W} = X_W$ which is equivalent to $g^{-1}W = W$. By definition we have $g^{-1}W = W$ if and only if $g^{-1} \in \mathrm{Stab}_G(W)$ if and only if $g \in \mathrm{Stab}_G(W)$. □

Let $\bar{Y} = \mathbb{G}(2(l-1), 2p)$. By the previous fact we have a welldefined open immersion $Y \hookrightarrow \bar{Y}$ which sends a point $y \in Y$ to the $2(l-1)$-dimensional nondegenerate subspace yW_{f_Δ}. The image of this morphism is the open dense subvariety of $\bar{Y}$ consisting of all $2(l-1)$-dimensional nondegenerate subspaces. Therefore we can think of Y as the variety parametrizing the set $\{X_W \mid W \in \bar{Y} \text{ nondegenerated}\}$ and we can think of the morphism $Y \hookrightarrow \bar{Y}$ as the assignement $X_W \mapsto W$. Since $\bar{Y}$ is a projective homogeneous space we can think of $\bar{Y}$ as a natural compactification of Y. We will often identify Y with the open dense subvariety of $\bar{Y}$. We will often write $Y \subseteq \bar{Y}$.

Recall the morphisms p and q from Section 13. We will identify the morphism $q\colon Z \to Y$ with the composition $Z \to Y \hookrightarrow \bar{Y}$. Let w be a Weyl group element. Recall the non negative numbers $\bar{q}_w$ from Section 13. Let $\lambda \in \tilde{\mathcal{P}}(l,p)$. Then we can define a non negative number associated to λ by the equality $\bar{q}_\lambda = \bar{q}_{w_\lambda}$. With this notation we have for example $\bar{q}_\emptyset = \bar{q}_1 = 1$.

Fact 18.2. *Let f be a general curve of degree $d_X = l + 1$. Then we have the equality $\hat{X}_f = X_{W_f}$.*

Proof. Let $g \in G$ be an element such that $f = gf_\Delta$. Then we have by definition that $\hat{X}_f = g\hat{X}$. We already know that $\hat{X} = X_{W_{f_\Delta}}$. Therefore it follows that $g\hat{X} = X_{gW_{f_\Delta}}$. Therefore it suffices to show that $gW_{f_\Delta} = W_f$. But this clear from Corollary 16.7 since $f = gf_\Delta$. □

Lemma 18.3. *Let λ, μ and ν be elements of $\tilde{\mathcal{P}}(l, p)$. Let $F_\bullet, G_\bullet$ and $H_\bullet$ be three isotropic flags in general position. Then the map $f \mapsto W_f$ gives a bijection between the set of general rational curves f of degree $d_X = l + 1$ satisfying $f(0) \in X_\lambda(F_\bullet)$, $f(1) \in X_\mu(G_\bullet)$ and $f(\infty) \in X_\nu(H_\bullet)$ and the set of points W in the intersection $qp^{-1}(X_\lambda(F_\bullet)) \cap qp^{-1}(X_\mu(G_\bullet)) \cap qp^{-1}(X_\nu(H_\bullet))$ together with three points in the fibers $X_W \cap X_\lambda(F_\bullet)$, $X_W \cap X_\mu(G_\bullet)$ and $X_W \cap X_\nu(H_\bullet)$.*

Proof. By Fact 13.18 we already know that the two sets in question are in bijection. We only have to show that the bijection is given by the assignement $f \mapsto W_f$. But this is clear from the previous fact and the definition of the assignement in the proof of Fact 13.18 □

Theorem 18.4. *Let λ, μ and ν be elements of $\tilde{\mathcal{P}}(l, p)$ such that*

$$|\lambda^*| + |\mu^*| + |\nu^*| = \dim(X) + c_1(X)d_X \,.$$

Then we have the following equality:

$$\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_{l+1} = \bar{q}_\lambda \bar{q}_\mu \bar{q}_\nu \int_{\bar{Y}} q_* p^* \sigma_\lambda \cdot q_* p^* \sigma_\mu \cdot q_* p^* \sigma_\nu \,.$$

Proof. Let $u = w_\lambda$, $v = w_\mu$ and $w = w_\nu$ be the Weyl group elements corresponding to the partitions λ, μ and ν. Let $\bar{F}_u$, $\bar{F}_v$ and $\bar{F}_w$ be the closures of F_u, F_v and F_w in $\bar{Y}$ where we consider F_u, F_v and F_w as subvarieties of $\bar{Y}$ via the natural inclusion $Y \subseteq \bar{Y}$. Let g, g' and g'' be general elements of G. Then we have

$$gF_u \cap g'F_v \cap g''F_w = g\bar{F}_u \cap g'\bar{F}_v \cap g''\bar{F}_w \,.$$

The cardinality of the right side equals the integral over $\bar{Y}$ which is written in the statement. Therefore the claim follows from Theorem 13.19. □

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Ich versichere an Eides Statt, dass die Dissertation von mir selbständig und ohne unzulässige fremde Hilfe unter Beachtung der „Grundsätze zur Sicherung guter wissenschaftlicher Praxis an der Heinrich-Heine-Universität Düsseldorf" erstellt worden ist.

Düsseldorf, 25. April 2016