Abstract

The modular group $\Gamma := \mathrm{PSL}(2,\mathbb{Z})$ of $2\times 2-$matrices over $\mathbb{Z}$ with determinant 1 operates on the upper half plain $\mathbb{H} := \{z = x+iy : x, y \in \mathbb{R}, y > 0\}$. The set of orbits $\Gamma \setminus \mathbb{H}$ is a non-compact Riemannian manifold, which can be compactified by adding the parabolic fixed points $\mathbb{P}^1(\mathbb{Q})$. Concerning subgroups $\Delta < \Gamma$ of finite index, the projective line decomposes into $r < \infty$ classes of cusps, thus enabling us to choose a vector of cusps $S(\Delta) := [s_1, \dots, s_r] \in (\mathbb{P}^1(\mathbb{Q}))^r$ as a complet set of inequivalent cusps obtaining a compact set of orbits $\Delta \setminus \mathbb{H} \cup \{s_1, \dots, s_r\}$. For the stabilizer $\Delta(s_i)$ of a representative $s_i \in \mathbb{P}^1(\mathbb{Q})$ with cusp width $w_i := [\Gamma(s_i) : \Delta(s_i)]$ there exist matrices $g_i \in \Gamma$ with $g_i\infty = s_i$ and $g_i^{-1}\Delta(s_i)g_i \subset \Gamma(\infty)$ as well as $h_i \in \mathrm{PSL}(2,\mathbb{R})$ with $h_i\infty = s_i$ and $h_i^{-1}\Delta(s_i)h_i = \Gamma(\infty)$. To each cusp s_i we asign the corresponding Eisenstein series:

$$E_i(z,s) := \sum_{\delta \in \Delta(s_i)\setminus\Delta} \mathrm{Im}(h_i^{-1}\delta z)^s,$$

which converges absolutely, if $\mathrm{Re}\, s > 1$, and which is $\Delta-$automorph. Each Eisenstein series $E_i(z,s)$ has a Fourier expansion at the cusp s_j for $1 \le i,j \le r$ with the constant terms:

$$\varphi_{ij}(s) := \pi^{\frac{1}{2}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{c\in\mathbb{N}} \frac{1}{(w_i w_j)^s c^{2s}} \left| \left\{ \left[\left(\begin{smallmatrix} \star & \star \\ c & \star \end{smallmatrix}\right)\right] \in g_i^{-1}\Delta(s_i)g_i \setminus g_i^{-1}\Delta g_j / g_j^{-1}\Delta(s_j)g_j \right\} \right|,$$

which are collected in the scattering matrix $\Phi(\Delta,s) := (\varphi_{ij}(s))_{1\le i,j\le r}$. We developed programms in GAP, which calculate for a subgroup $\Delta <_f \Gamma$ a complete set of inequivalent cusps, their cusp withs and the numbers b_{ij} of double cosets.

For groups $\Delta < \Lambda < \Gamma$ we introduce the notion of a relative cusp width of a cusp s_i^j of Δ with respect to Λ and construct the scattering matrix of Λ from the scattering matrix of Δ.

This concept allows for a wide field of applications: Firstly, we derive the scattering matrix of a congruence subgroup Δ from the scattering matrix of its principle congruence subgroup $\Gamma(n)$. Thus we estimate for $\Gamma(p)$ the structure of the scattering matrix and present their entries in an explicit way, before transfering our results as far as possible to principle congruence subgroups $\Gamma(n)$.

Secondly, we are able to determine the entries of the scattering matrix at least for such non-congruence subgroups, which are subgroups of a congruence subgroup and hold one or more analog cusps classes. Then the entries of the scattering matrix of Δ belonging to these cusps arise from the scattering matrix of Λ.

Knowledge of the structure of the scattering matrix enables us to construct the determinant $\det \Phi(\Gamma(p), s)$ as a quotient of products of the Riemannian $\zeta-$function and Dirichlet L-series.