Geometric Tilting theory and the Amitsur conjecture

Inaugural-Dissertation

Zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Heinrich-Heine-Universität Düsseldorf

vorgelegt von

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Düsseldorf, September 2014

Aus dem Mathematischen Institut
der Heinrich-Heine-Universität Düsseldorf

Gedruckt mit der Genehmigung der
der Mathematisch-Naturwissenschaftlichen Fakultät
der Heinrich-Heine-Universität Düsseldorf

Referent: Prof. Dr. Stefan Schröer
Koreferent: PD Dr. Markus Perling

Tag der mündlichen Prüfung: 02.12.2014

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Acknowledgements

The work on this dissertation has been a huge academic challenge and would not have been completed without the support, patience and love of the following people:
I am indebted to my advisor Stefan Schröer for the encouragement to work on this challenging project and who spent countless hours discussing this thesis with me. He taught me algebraic geometry and how to think about every little detail especially when writing an academic paper.
I also thank Markus Perling for all his comments and his patience and Nicolas Perrin and Alexander Samokhin for useful discussions. Next I thank all the members of the Düsseldorf algebraic research group for their time, their energy, kindness and for the interesting, helpful and amusing discussions: Philipp Gross, Christian Liedtke, Holger Partsch, André Schell, Benedikt Schilson, Felix Schüller and Leif Zimmermann.
The seminars at the Mathematical Institute of the University of Duisburg-Essen and the AG-Düsseldorf-Wuppertal greatly stimulated my mathematical research. The last years the Mathematical Institute at the Heinrich-Heine-University in Düsseldorf was the central part of my workaday life and I am grateful for the excellent working condition. I thank all colleagues and staff members for creating a warm and lovely atmosphere. In particular Ulrike Alba and Petra Simons supported me with their friendly help and delicious cakes.
Last, but not least, I thank my family: My parents Dušan and Borka Novaković, my sister Sandra, my parents in law Leonidas and Efthimia Kordelas, my brother in law Nikolaos Kordelas and my lovely $\gamma\iota\alpha\gamma\iota\alpha$ for their unconditional support, love, patience and sympathy. But special thanks goes to my wife Anastasia without whom my soul would not be able to breath and my mind would not be able to work and to whom i finally dedicate this thesis.

Summary

The present work has three parts, divided into four chapters. All three parts are dedicated to the investigation of a certain problem.
In the first two chapters we classify a certain class of locally free sheaves on proper k-schemes X. Concretely, we study and classify absolutely split locally free sheaves, i.e., locally free sheaves on X that split as a direct sum of invertible ones after base change to the algebraic closure $\bar{k}$. The first chapter is here dedicated to the discussion of the case X is a Brauer–Severi variety over k. We classify indecomposable and therefore all absolutely split locally free sheaves on X and investigate the relation between these sheaves for Brauer-equivalent and birational Brauer–Severi varieties. The results of this chapter generalize classification results obtained by Biswas and Nagaraj [29], [30], [31] and by the author [125], [126]. The ideas presented in the first chapter naturally generalize to the general case where X is an arbitrary proper k-scheme and lead us to the classification of absolutely split locally free sheaves on X. As an application of these general results, we investigate generalized Brauer–Severi varieties and classify all absolutely split locally free sheaves on them. We end up this chapter studying the behavior between these sheaves for birational generalized Brauer–Severi varieties.
The third chapter of the present work presents an overview of geometric tilting theory and is dedicated to the problem of classifying schemes, or more generally algebraic stacks, admitting a tilting object. Such a tilting object $\mathcal{T}$ gives rise to an equivalence $D^b(X) \xrightarrow{\sim} D^b(A)$, where $A = \mathrm{End}(\mathcal{T})$ is the endomorphism algebra of $\mathcal{T}$. This is a very useful equivalence since in many geometric situations the endomorphism algebra A is a Quiver-algebra with relations and one therefore has a representation-theoretical approach to study the derived geometry of X. There are of course more deeper ideas motivating the study of derived categories admitting tilting objects. We refer to Kontsevich's homological mirror symmetry conjecture [106], to mention only one of them. Tilting bundles have been found for several schemes including for instance the projective space [23], certain flag varieties and quadrics [96], [97], Grassmannians [95], rational surfaces [86], [87], Fano surfaces, various toric varieties and fibrations [56], [57]. We enlarge this list by proving the existence of tilting objects for some further schemes and algebraic stacks. In the stacky case, we consider quotient stacks obtained by actions of algebraic groups on schemes and investigate when such stacks admit tilting objects. Furthermore, we classify all curves having tilting objects and show that these are exactly the Brauer–Severi varieties of dimension one. In dimension two this is an open problem but it is a conjectured that a smooth projective surface admits a tilting object if and only if the surface is rational (see [86], [87]). As an application of the geometric tilting theory and the results obtained in the third chapter, we provide some further evidence for a conjecture formulated by Orlov [130]. More precise, the conjecture states that for a smooth proper and integral k-scheme X one has $\dim(X) = \dim D^b(X)$. The conjecture is known to be true for curves [130], affine schemes of finite type over k, quadrics and certain flags [137] and del Pezzo surfaces, certain Fano 3-folds, Hirzebruch surfaces plus certain toric surfaces and toric Deligne–Mumford stacks [19].
The last chapter is dedicated to the Amitsur conjecture for Brauer–Severi varieties. This conjecture was formulated by Amitsur [5] and states the following:

Two Brauer–Severi varieties X and Y are birational if and only if they generate the same cyclic subgroup in $\mathrm{Br}(k)$. The "only if" part was proved by Amitsur himself [5] whereas the "if" part turns out to be the harder part and is only known in some special cases (see [5], [107], [108], [136] and [149]). We give a new approach to this conjecture by considering the semiorthogonal decompositions of the derived categories of X and Y and present some ideas to tackle the conjecture via its derived geometry.

Zusammenfassung

Die vorliegende Arbeit besteht aus drei Teilen, aufgeteilt in vier Kapitel. Jeder dieser drei Teile ist jeweils der Untersuchung eines bestimmten Problems gewidmet.
In den ersten beiden Kapiteln klassifizieren wir eine bestimmte Klasse lokal freier Garben auf einem beliebigen eigentlichen k-Schema X. Konkret untersuchen und klassifizieren wir absolut spaltende lokal freie Garben, das heisst, lokal freie Garben auf X, welche nach Basiswechsel zum algebraischen Abschluss die direkte Summe von invertierbaren Garben sind. Hierbei behandelt das erste Kapitel den Fall, in dem X eine Brauer–Severi Varietät ist. Wir klassifizieren alle unzerlegbaren und damit auch alle absolut spaltenden lokal freien Garben auf X und untersuchen das Verhältnis zwischen solchen Garben für Brauer-äquivalente und birationale Brauer–Severi Varietäten. Die Resultate dieses Kapitels verallgemeinern Klassifikationsresultate Biswas' und Nagaraj's [29], [30], [31] und des Autors [125], [126]. Die im ersten Kapitel dargestellten Ideen lassen sich ohne weiteres für den Fall in dem X ein beliebiges eigentliches k-Schema ist verallgemeinern und liefern folglich eine Klassifikation absolut spaltender lokal freier Garben auf X. Als Anwendung der allgemeinen Resultate klassifizieren wir absolut spaltende lokal freie Garben auf verallgemeinerten Brauer–Severi Varietäten. Wir beenden schliesslich das Kapitel indem das Verhalten zwischen solchen Garben für birationale verallgemeinerte Brauer–Severi Varietäten untersucht wird.
Das dritte Kapitel der vorliegenden Arbeit präsentiert eine Übersicht über geometrische Kipp-Theorie und widmet sich dem Problem der Klassifikation bzw. des Auffindens von Schemata, oder allgemeiner, algebraischer Stacks, welche Kippobjekte besitzen. Solch ein Kippobjekt $\mathcal{T} \in D^b(X)$ induziert eine triangulierte Äquivalenz $D^b(X) \xrightarrow{\sim} D^b(A)$, wobei $A = \mathrm{End}(\mathcal{T})$ die Endomorphismenalgebra von $\mathcal{T}$ ist. Dies ist eine sehr nützliche Äquivalenz, da in vielen geometrischen Situationen die Endomorphismenalgebra A eine Köcher-Algebra mit Relationen ist und man daher einen darstellungstheoretischen Ansatz hat um die derivierte Kategorie $D^b(X)$ zu studieren. Es gibt natürlich noch tiefer liegende Ideen, welche eine Untersuchung von derivierten Kategorien zusammen mit der Existenz von Kippobjekten motivieren. Wir verweisen hier auf Kontsevich's Homologische Spiegelsymmetrie Vermutung [106], um nur eine zu nennen. Kippbündel wurden zum Beispiel für den Projektiven Raum [23], einige Flaggen Varietäten und Quadriken [96], [97], Grassmannsche [95], rationale Flächen [86], [87], Fano Flächen, verschiedene torische Varietäten und lokal triviale Faserungen [56], [57] gefunden. Wir erweitern diese Liste, indem wir die Existenz von Kippobjekten für weitere Schemata und algebraische Stacks beweisen. Im Falle algebraischer Stacks betrachten wir sogenannte Quotientenstacks, welche durch Wirkungen algebraischer Gruppen auf Schemata entstehen und untersuchen wann diese Kippobjekte besitzen. Desweiteren klassifizieren wir alle Kurven über einem Körper k, welche Kippobjekte besitzen und zeigen, dass diese Kurven genau die eindimensionalen Brauer–Severi Varietäten sind. Für Flächen ist das nach wie vor ein offenes Problem. Allerdings wird vermutet, dass die Existenz eines Kippobjekts für eine glatte projektive Fläche äquivalent zur Rationalität derselbigen ist [86], [87]. Als Anwendung der Kipp-Theorie und der in diesem Kapitel erhaltenen Resultate liefern wir weitere Hinweise für die Richtigkeit einer Vermutung von Orlov [130]. Genauer besagt diese Vermu-

tung, dass für glatte Varietäten über einem Körper k, $\dim(X) = \dim D^b(X)$ gilt. Diese Vermutung gilt für Kurven [130], affine Schemata von endlichem Typ über k, Quadriken, spezielle Flaggen Varietäten [137] und del Pezzo Flächen, einige Fano 3-Faltigkeiten, Hirzebruch Flächen, sowie bestimmte torische Flächen und torische Deligne–Mumford Stacks [19].
Das letzte Kapitel ist der Amitsur Vermutung für Brauer–Severi Varietäten gewidmet. Diese Vermutung wurde von Amitsur [5] formuliert und besagt folgendes: Zwei Brauer–Severi Varietäten X und Y sind birational genau dann, wenn sie die gleiche zyklische Untergruppe in $\mathrm{Br}(k)$ erzeugen. Der "genau dann"-Part wurde von Amitsur [5] bewiesen, wohingegen sich der "wenn"-Part als schwieriger erwiesen hat und nur in einigen Spezialfällen bekannt ist [5], [107], [108], [136] und [149]. Wir präsentieren eine neue Herangehensweise an diese Vermutung, indem wir die semiorthogonalen Zerlegungen der derivierten Kategorien von X and Y betrachten und formulieren einige neue Ideen um diese Vermutung zu beweisen.

Introduction

Grothendieck [74] classified all locally free sheaves on $\mathbb{P}^1$ and Atiyah [14] on elliptic curves. So it is natural to ask for other curves where a classification of such sheaves is possible. In [125] the author considered the case of a smooth non-degenerate conic C over k, that is, a Brauer–Severi variety of dimension one. In this case C becomes isomorphic to $\mathbb{P}^1$ after base change to a Galois extension $k \subset L$ of degree 2. Then one can apply descent theory to classify all locally free sheaves on C, exploiting the classification of locally free sheaves on the projective line. It is very natural to study a generalization of this problem, namely, consider Brauer–Severi varieties over k of arbitrary dimension and try to classify *absolutely split* locally free sheaves on them. An *absolutely split* locally free sheaf is a sheaf $\mathcal{E}$, that splits as a direct sum of invertible sheaves after base change to the algebraic closure of k. We simply refer to them as *AS-bundles.* This problem was considered by Biswas and Nagaraj [29], [30], [31] and partial results in favor of a complete classification of AS-bundles on Brauer–Severi varieties are obtained. More precise, Biswas and Nagaraj classified AS-bundles on Brauer–Severi varieties of dimension one and on Brauer–Severi varieties over $\mathbb{R}$. Chapter 1 is dedicated exactly to this problem and a complete classification of AS-bundles on arbitrary Brauer–Severi varieties is proved. The idea is to exploit the fact that AS-bundles are direct sums of invertible sheaves after base change to the algebraic closure. A Brauer–Severi variety over k becomes isomorphic to the projective space $\mathbb{P}^n$ after base change to $\bar{k}$ and therefore one has to investigate if the invertible sheaves $\mathcal{O}_{\mathbb{P}^n}(j)$ descent. It turns out that in general the sheaves $\mathcal{O}_{\mathbb{P}^n}(j)$ do not descent, but $\mathcal{O}_{\mathbb{P}^n}(j)^{\oplus d_j}$ do, for suitable d_j. In fact, one can show that for any $\mathcal{O}_{\mathbb{P}^n}(j)$, $j \in \mathbb{Z}$ there is an up to isomorphism unique indecomposable locally free sheaf $\mathcal{W}_j$, such that $\mathcal{W}_j \otimes_k \bar{k} \simeq \mathcal{O}_{\mathbb{P}^n}(j)^{\oplus \mathrm{rk}(\mathcal{W}_j)}$ (see Proposition 1.35, 1.37 and discussion right after Remark 1.38). With this notation we obtain the following result:

Theorem. *(Theorem 1.45) Let X be a n-dimensional Brauer–Severi variety over a field k and of period p. Then all indecomposable AS-bundles are up to isomorphism of the form*

$$\mathcal{W}_j \otimes \mathcal{O}_X(ap),$$

with unique $a \in \mathbb{Z}$ and unique $0 \leq j \leq p-1$.

As an immediate consequence of the above theorem we obtain:

Corollary. *Let X be a Brauer–Severi variety over a field k of period p. Then all AS-bundles $\mathcal{E}$ are of the form*

$$\mathcal{E} \simeq \bigoplus_{j=0}^{p-1} \left(\bigoplus_{i=0}^{r_j} \mathcal{W}_j \otimes \mathcal{O}_X(a_{i_j} p) \right),$$

with unique $a_{i_j} \in \mathbb{Z}$ and $r_j > 0$, where $0 \leq j \leq p-1$.

To have a complete understanding of the AS-bundles on Brauer–Severi varieties, one has to determine the ranks of the locally free sheaves $\mathcal{W}_j$. This leads us to consider the sequence of natural numbers $(d_j)_{j \in \mathbb{Z}}$, with $d_j = \mathrm{rk}(\mathcal{W}_j)$. In view of Proposition 1.43 and 1.44, we do not have to consider the hole sequence

$(d_j)_{j\in\mathbb{Z}}$. In fact, for a Brauer–Severi variety of period p it is enough to consider the $p+1$-tupel $(d_0, d_1, ..., d_{p-1}, d_p)$. Furthermore, we note that $\mathcal{W}_0 = \mathcal{O}_X$ and $\mathcal{W}_p = \mathcal{O}_X(p)$, where p is the period of X. This implies that $\mathrm{rk}(\mathcal{W}_0) = 1 = \mathrm{rk}(\mathcal{W}_p)$. Keeping this mind one can define the *AS-type* of a Brauer–Severi variety over k of period p to be the $p+1$-tuple $(1, d_1, ..., d_{p-1}, 1)$, with $d_j = \mathrm{rk}(\mathcal{W}_j)$ for $j = 1, ..., p-1$ (see Definition 1.47). The next result determines the AS-type.

Proposition. *(Proposition 1.51) Let X be a n-dimensional Brauer–Severi variety over a field k corresponding to a central simple k-algebra A. Then for every $j \in \mathbb{Z}$ one has*

$$\mathrm{rk}(\mathcal{W}_j) = \mathrm{ind}(A^{\otimes |j|}).$$

Theorem. *(Theorem 1.52) Let X be a Brauer–Severi variety over k and A the corresponding central simple k-algebra of period p. Then the AS-type of X is $(1, d_1, d_2, ..., d_{p-1}, 1) = (\mathrm{ind}(A^{\otimes j}))_{0\leq j\leq p}$.*

With this classification of AS-bundles on arbitrary Brauer–Severi varieties one gets the results obtained by Biswas and Nagaraj [29], [30], [31] and by the author [125], [126] as corollaries (see Corollary 1.53 and 1.54). Moreover, in the case the ground field k is local or global, the AS-type can concretely be computed in terms of the period of the considered Brauer–Severi variety (see Proposition 1.62 and 1.64). At the end of the first chapter we study the relation between the AS-types of Brauer-equivalent and birational Brauer–Severi varieties. Below we state the main results.

Proposition. *(Proposition 1.64) Let X and Y be two Brauer–Severi varieties over k that are Brauer-equivalent. Then X and Y have the same AS-type.*

Proposition. *(Proposition 1.67) Let X and Y be two birational Brauer–Severi varieties. Then they have the same AS-type.*

Note, that the converse of the above results does not hold in view of the following fact:

Proposition. *(Proposition 1.66) Every non-split* 1*-dimensional Brauer–Severi variety over k has the same AS-type.*

The ideas of the proofs presented in the first chapter very naturally generalize to the case where X is an arbitrary proper k-scheme. To work this out is the content of the second chapter and it turns out that a complete classification of AS-bundles on arbitrary proper k-schemes is possible. Consequently, all the results of chapter 1 are applications of these general classification theorems. We first assume the field k to be perfect. Let us denote by $\mathrm{Pic}^G(X \otimes_k \bar{k})$ the $G = \mathrm{Gal}(\bar{k}|k)$-invariant invertible sheaves. One can show that for all $\mathcal{L} \in \mathrm{Pic}^G(X \otimes_k \bar{k})$ there is an up to isomorphism unique indecomposable $\mathcal{M}_{\mathcal{L}}$ such that $\mathcal{M}_{\mathcal{L}} \otimes_k \bar{k} \simeq \mathcal{L}^{\oplus r}$ (see Corollary 2.7). Note that the natural number $r > 0$ is unique, since it equals the rank of $\mathcal{M}_{\mathcal{L}}$. We obtain the following result.

Theorem. *(Theorem 2.8) Let k be a perfect field and X a proper k-scheme. Then all indecomposable AS-bundles are of the form $\mathcal{M}_{\mathcal{L}}$ for a unique $\mathcal{L} \in \mathrm{Pic}^G(X \otimes_k \bar{k})$.*

This theorem has a very interesting geometric interpretation, at least if $H^0(X,\mathcal{O}_X)=k$. Considering the Picard scheme $\mathrm{Pic}_{X/k}$ of X, one can prove the following fact.

Theorem. *(Theorem 2.15) Let k be a perfect field and X a proper k-scheme with $H^0(X,\mathcal{O}_X)=k$. Then the k-rational points of $\mathrm{Pic}_{X/k}$ are in one-to-one correspondence with isomorphism classes of indecomposable AS-bundles on X.*

One can hold on to the fact that if X admits a k-rational point, $\mathrm{Pic}_{X/k}$ is the same as $\mathrm{Pic}(X)$ and the k-rational points of $\mathrm{Pic}_{X/k}$ are in one-to-one correspondence with the invertible sheaves on X. If X does not admit a k-rational point, the k-rational points of $\mathrm{Pic}_{X/k}$ correspond to the indecomposable AS-bundles on X. Exactly this can be observed in the case X is a Brauer–Severi variety. Indeed, if X admits a k-rational point, X is isomorphic to $\mathbb{P}^n$ and hence the indecomposable AS-bundles are the invertible sheaves of X. In view of the investigation of Brauer–Severi varieties in the first chapter and discussions in the second, we formulate the following conjecture, stating that the above theorem should also hold without the assumption on k being perfect.

Conjecture. *Let X be a proper k-scheme with $H^0(X,\mathcal{O}_X)=k$. Then the k-rational points of $\mathrm{Pic}_{X/k}$ are in one-to-one correspondence with isomorphism classes of indecomposable AS-bundles on X.*

Nonetheless, for arbitrary (not necessarily perfect) fields k one can show the following result that gives the desired generalization of Theorem 1.45.

Theorem. *(Theorem 2.19) Let X be a proper k-scheme with $\mathrm{Pic}(X\otimes_k\bar{k})\simeq\mathbb{Z}$ and period r. Suppose there is an indecomposable absolutely isotypical sheaf $\mathcal{M}_{\mathcal{L}}$ of type $\mathcal{L}$, where $\mathcal{L}$ is the generator of $\mathrm{Pic}(X\otimes_k\bar{k})$. Denote by $\mathcal{J}$ the generator of $\mathrm{Pic}(X)$. Then all indecomposable AS-bundles are up to isomorphism of the form*

$$\mathcal{M}_{\mathcal{L}^{\otimes j}}\otimes\mathcal{J}^{\otimes a}$$

with unique $a\in\mathbb{Z}$ and unique $0\le j\le r-1$.

Corollary. *(Corollary 2.20) Let X be a proper k-scheme with $\mathrm{Pic}(X\otimes_k\bar{k})\simeq\mathbb{Z}$ and period r. Suppose there is an indecomposable absolutely isotypical sheaf $\mathcal{M}_{\mathcal{L}}$ of type $\mathcal{L}$, where $\mathcal{L}$ is the generator of $\mathrm{Pic}(X\otimes_k\bar{k})$. Then all AS-bundles $\mathcal{E}$ are of the form*

$$\mathcal{E}\simeq\bigoplus_{j=0}^{r-1}\left(\bigoplus_{i=0}^{s_j}\mathcal{M}_{\mathcal{L}^{\otimes j}}\otimes\mathcal{J}^{\otimes a_{i_j}}\right),$$

with unique $a_{i_j}\in\mathbb{Z}$ and unique $s_j>0$, with $0\le j\le r-1$.

Following the classification of AS-bundles on Brauer–Severi varieties, a complete understanding of the AS-bundles would be obtained if one furthermore determines the ranks of $\mathcal{M}_{\mathcal{L}^{\otimes j}}$. For this, denote by $D(\mathcal{M}_{\mathcal{L}})$ the finite-dimensional semisimple k-algebra $\mathrm{End}(\mathcal{M}_{\mathcal{L}})/\mathrm{rad}(\mathrm{End}(\mathcal{M}_{\mathcal{L}}))$. One can show that $D(\mathcal{M}_{\mathcal{L}})$ is in fact a central simple k-algebra, provided $H^0(X,\mathcal{O}_X)=k$ and therefore one has the notion of the index of $D(\mathcal{M}_{\mathcal{L}})$. Under the assumption that the Picard group is cyclic, we have the following description of the ranks of $\mathcal{M}_{\mathcal{L}^{\otimes j}}$.

Proposition. *(Proposition 2.22) Let X be a proper k-scheme with $H^0(X,\mathcal{O}_X) = k$ and $\mathrm{Pic}(X\otimes_k \bar{k}) \simeq \mathbb{Z}$. Suppose there is an indecomposable absolutely isotypical sheaf $\mathcal{M}_{\mathcal{L}}$ of type $\mathcal{L}$, where $\mathcal{L}$ is the generator of $\mathrm{Pic}(X\otimes_k \bar{k})$. Then one has for all $j\in\mathbb{Z}$*

$$\mathrm{rk}(\mathcal{M}_{\mathcal{L}^{\otimes j}}) = \mathrm{ind}(D(\mathcal{M}_{\mathcal{L}})^{\otimes j}).$$

This last proposition together with the above theorem now classify all AS-bundles proper k-schemes with $H^0(X,\mathcal{O}_X) = k$ and $\mathrm{Pic}(X\otimes_k \bar{k}) \simeq \mathbb{Z}$. It is an easy consequence that all results obtained in chapter 1 for Brauer–Severi varieties are special cases of the above general results (see discussion after Remark 2.23). Another application of the classification of AS-bundles on proper k-schemes is the classification of AS-bundles on generalized Brauer–Severi varieties. As in the case of Brauer–Severi varieties there are up to isomorphism unique indecomposable locally free sheaves $\mathcal{W}_{\mathcal{L}^{\otimes j}}$ on the generalized Brauer–Severi variety $\mathrm{BS}(d,A)$, such that $\mathcal{W}_{\mathcal{L}^{\otimes j}}\otimes_k \bar{k} \simeq (\mathcal{L}^{\otimes j})^{\oplus \mathrm{rk}(\mathcal{W}_{\mathcal{L}^{\otimes j}})}$. Here $\mathcal{L}$ is the generator of $\mathrm{Pic}(\mathrm{BS}(d,A)\otimes_k \bar{k})) = \mathrm{Pic}(\mathrm{Grass}(d,n)) \simeq \mathbb{Z}$. Let us denote by $\mathcal{M}$ the generator of $\mathrm{Pic}(\mathrm{BS}(d,A)) \simeq \mathbb{Z}$. Then we have the following main result.

Theorem. *(Theorem 2.43) Let $X = \mathrm{BS}(d,A)$ be a generalized Brauer–Severi variety over k for the central simple k-algebra A of degree n and period r. Then the AS-bundles $\mathcal{E}$ of finite rank are of the form:*

$$\mathcal{E} \simeq \bigoplus_{j=0}^{r-1}\left(\bigoplus_{i=0}^{m_j}\mathcal{M}^{\otimes a_{i_j}}\otimes\mathcal{W}_{\mathcal{L}^{\otimes j}}\right)$$

with unique a_{i_j}, s, m_j and $0\leq j\leq r-1$ and $\mathrm{rk}(\mathcal{W}_{\mathcal{L}^{\otimes j}}) = \mathrm{ind}(A^{\otimes j\cdot d})$.

As for Brauer–Severi varieties, one can define the AS-type for arbitrary proper k-schemes and try to study the relation between AS-types of schemes related to each other by certain morphisms. At the end of the second chapter this will be done for the generalized Brauer–Severi varieties related to each other by a birational map. It turns out that two birational generalized Brauer–Severi varieties have the same AS-type, whereas the converse of this fact does not hold (see Proposition 2.53 and Remark 2.54). Since the AS-bundles are direct sums of invertible ones after base change to the algebraic closure, one recognizes that for X being a Brauer–Severi variety, certain AS-bundles give rise to tilting objects. So the third chapter is dedicated to geometric tilting theory and to the problem of classifying schemes admitting a tilting object.
Classically, tilting theory was introduced to understand the module categories coming from representation theory of k-algebras. Central results are obtained by Brenner and Butler [44] and led to the work of Happel [80] and Rickard [135]. At this point we want to mention the 'Handbook of Tilting Theory' [81], which gives an excellent overview over the theory. The geometric tilting theory started with the work of Beilinson [23], where he investigated the derived category of $\mathbb{P}^n$ and proved the existence of tilting objects. Below we recall the definition of a tilting object. For this, let X be a noetherian quasiprojective scheme over a field k and $D(\mathrm{Qcoh}(X))$ the derived category of quasicoherent sheaves on X. An object $\mathcal{T}\in D(\mathrm{Qcoh}(X))$ is called *tilting object for $D(\mathrm{Qcoh}(X))$* if the following hold:

(i) $\mathrm{Hom}_{D(\mathrm{Qcoh}(X))}(\mathcal{T},\mathcal{T}[i]) = 0$ for $i\neq 0$.

(ii) If $\mathcal{N} \in D(\mathrm{Qcoh}(X))$ satisfies $\mathbb{R}\mathrm{Hom}_{D(\mathrm{Qcoh}(X))}(\mathcal{T},\mathcal{N}) = 0$, then $N = 0$.

(iii) $\mathrm{Hom}_{D(\mathrm{Qcoh}(X))}(\mathcal{T}, -)$ commutes with direct sums.

With this notation one has the following key-result for geometric tilting theory [81], p.172, Theorem 7.6. (see also [52], Theorem 1.8)

Theorem. *Let X be projective over a finitely generated k-algebra R. Suppose X admits a tilting object $\mathcal{T}$ and set $A = \mathrm{End}(\mathcal{T})$. Then the following hold:*

(i) *There is an equivalence* $\mathbb{R}\mathrm{Hom}(\mathcal{T}, -) : D(\mathrm{Qcoh}(X)) \xrightarrow{\sim} D(\mathrm{Mod}(A))$.

(ii) *The equivalence of (i) restricts to an equivalence* $D^b(X) \xrightarrow{\sim} D^b(A)$.

(iii) *If X is smooth over R, then A has finite global dimension.*

As mentioned above, Beilinson [23] started the geometric tilting theory and for the first time established a connection between the derived category of coherent sheaves and the representation theory of the endomorphism algebra of the tilting object. Roughly speaking, if X is projective, the endomorphism algebra $A = \mathrm{End}(\mathcal{T})$ of a tilting object $\mathcal{T}$ is a finite-dimensional k-algebra and therefore Morita-equivalent to the path algebra of a finite quiver with relations. In view of the equivalence $\mathbb{R}\mathrm{Hom}(\mathcal{T}, -) : D^b(X) \xrightarrow{\sim} D^b(A)$, one therefore has a representation-theoretical approach to study the derived category of coherent sheaves. Here the endomorphism algebra A can be thought of as a noncommutative coordinization of X. This representation-theoretical approach to understand the derived category of coherent sheaves is not the only motivation to study derived categories with tilting objects. Another motivation comes from Kontsevich's Homological Mirror Symmetry conjecture [106], see also [89], 13.2. Very roughly, this conjecture states that for an algebraic variety X, there exists an object Y, carrying a symplectic structure, such that the bounded derived category of coherent sheaves $D^b(X)$ is equivalent to the derived Fukaya-category of Y. Moreover, a conjecture of Dubrovin [62] states that the quantum cohomology of a smooth projective variety X is generically semisimple if and only if there exists a full exceptional collection in $D^b(X)$ and the validity of this conjecture would also provide evidence for the Homological Mirror Symmetry conjecture. We do not want to recall all the details of these conjectures and refer to [106] and [62] for further information and deeper understanding. Moreover, motivated by the Mirror Symmetry, in the recent past full strongly exceptional collections have also been considered in physics in the context of string theory, concretely in studying so-called D-branes (see for instance [11], [27]). Particular interest in exceptional collections also comes from local Calabi–Yau varieties. Consider the total space $\mathrm{Tot}(\omega_X) \to X$ for the canonical bundle ω_X. This is a local Calabi-Yau variety and it follows from results of Bridgeland [49], that a full, strongly exceptional collection on X can be extended to a cyclic strongly exceptional collection if and only if the pullbacks give rise to a tilting object on $\mathrm{Tot}(\omega_X)$. It is also an interesting observation that the endomorphism algebra of this titling object for $\mathrm{Tot}(\omega_X)$ gives an example of non-commutative resolutions in the sense of [150]. Other examples where Theorem 3.8 applies include some crepant resolutions of quotient singularities (see [85], 7 ff.). Finally, we want to note that the work of Bridgeland [48], Kuznetsov [110], Orlov [129] and others indicates that birational schemes should have bounded derived categories of coherent sheaves related by semiorthogonal decompositions. This philosophy will be the driving idea in the last chapter of the present work, where we want to study birational Brauer–Severi varieties in terms of their semiorthogonal decompositions.

The guiding problem in geometric tilting theory is the following:

Problem. *Classify schemes X (or more generally algebraic stacks) that have tilting objects (sheaves or bundles) for $D^b(X)$.*

It is a widely open question whether on a given smooth projective and integral k-scheme X a tilting object exits. In general the above problem is far from being solved and the existence of tilting objects is known only in some cases. These cases include projective spaces [23], flag varieties of type A_n [97], Grassmannians and quadrics over $\mathbb{C}$ [95], [96], rational surfaces [87], Fano surfaces, various toric varieties and fibrations [56], [57] and weighted projective lines [120], to name a few of them (see Chapter 3 for more examples). In the present work we want to enlarge the above list in proving the existence of tilting objects for some further schemes and algebraic stacks. Note that the existence of a tilting object for some scheme X imposes some necessary conditions on X, namely the Grothendieck group $K_0(X)$ has to be a free abelian group of finite rank and $H^q(X, \Omega^p_X) = 0$, for $q \neq p$ (see [52] and [118]). Below we list the main results of the third chapter.

Theorem. *(Theorem 3.36) Let X be a n-dimensional Brauer–Severi variety over an arbitrary field k and let A be the corresponding central simple k-algebra. Let $\mathcal{W}_i$ be the locally free sheaves of Definition 1.39. Then the locally free sheaf $\mathcal{T} = \bigoplus_{i=0}^n \mathcal{W}_i$ is a tilting bundle for $D^b(X)$.*

In the above theorem we have chosen the tilting bundle $\mathcal{T}$ in such a way that all the direct summands are indecomposable and have minimal rank with respect to the property that $\mathcal{W}_i \otimes_k \bar{k} \simeq \mathcal{O}_{\mathbb{P}^n}(i)^{\oplus \mathrm{rk}(\mathcal{W}_i)}$. That is, there is no indecomposable locally free sheaf $\mathcal{V}$ of smaller rank such that $\mathcal{V} \otimes_k \bar{k} \simeq \mathcal{O}_{\mathbb{P}^n}(i)^{\oplus d_i}$ for a suitable d_i. As a consequence of the existence of the tilting bundle in Theorem 3.36 we obtain a semiorthogonal decomposition for the derived category of a Brauer–Severi variety and therefore get back a result due to Bernardara [28]. As one expects, the generalized Brauer–Severi varieties also admit tilting objects. In fact we have the following result:

Theorem. *(Theorem 3.44) Let $X = \mathrm{BS}(d, A)$ be a generalized Brauer–Severi variety over a field k associated to the degree n central simple k-algebra A. Then the derived category $D^b(X)$ admits a tilting object.*

An interesting observation can be made while studying the existence of tilting objects on Brauer–Severi varieties. Although Brauer–Severi varieties always have tilting objects, they in general do not admit a full strongly exceptional collection (see Theorem 3.47). Such phenomenon also occurs while studying the existence of tilting objects on smooth projective rational surfaces (see [86]). Hille and Perling [87] proved that every such surface admits a tilting object, but in general no full strongly exceptional collection (see also [131], [132]). As a Brauer–Severi variety is a k-form of a projective space and therefore a k-form of a homogeneous variety, Theorem 3.47 shows that k-forms of homogeneous varieties in general do not admit a full strongly exceptional collection consisting of coherent sheaves. This shows that a conjecture of Catanese [35] stating the existence of full strongly exceptional collections on any homogeneous variety is may false when one considers k-forms of them. For such k-forms it is therefore reasonable to presume that any such k-form of a homogeneous variety should admit a tilting object.

The next source to produce some more examples of schemes admitting tilting objects is the class of relative flag varieties. For the projective and Grassmann bundle, Orlov [129] proved the existence of semiorthogonal decompositions. Böhning [35] furthermore showed that for relative flags of type A_n one has also a semiorthogonal decomposition. The existence of these semiorthogonal decompositions suggest the existence of a tilting object, provided the base scheme admits one. The main motivation for studying relative flag varieties is not only to generalize classical results obtained by Beilinson [23] and Kapranov [95], [96], [97], but is also to show that some homogeneous varieties of classical type admit tilting objects. In the theorems stated below, it is always assumed that X is a smooth projective and integral k-scheme.

Theorem. *(Theorem 3.50) Let X be as above and $\mathcal{E}$ a locally free of rank r and suppose that $D^b(X)$ admits a tilting bundle. Then $D^b(\mathbb{P}(\mathcal{E}))$ admits a tilting object.*

Note that Costa and Miró–Roig [56] proved that the projective bundle admits a full strongly exceptional collection consisting of invertible sheaves, provided the base scheme admits such. In view of the fact that some schemes admit tilting objects but no full strongly exceptional collection, we generalize their result as assuming the base scheme admits a tilting object. The above theorem has then the consequence that for all of the above mentioned cases where tilting bundles are proved to exist, the projective bundle admits also a tilting object. Especially in the case where X does not have a full strongly exceptional collection of invertible sheaves but a tilting bundle, a case where the results of Costa and Miró–Roig [56] do not apply, Theorem 3.50 still provides us with a tilting object for $\mathbb{P}(\mathcal{E})$. A further application of the above theorem is that partial isotropic flag varieties also admit tilting objects (see Corollary 3.55). The next result shows that the Grassmann bundles also admit tilting objects.

Theorem. *(Theorem 3.64) Let k be an algebraically closed field of characteristic zero and suppose X has a tilting bundle $\mathcal{T}_X$. Then the Grassmann bundle $\mathrm{Grass}_X(l, \mathcal{E})$ admits a tilting bundle.*

A direct consequence is the following:

Theorem. *(Theorem 3.65) Let k be an algebraically closed field of characteristic zero, X a smooth projective k-scheme and $\mathcal{E}$ a locally free sheaf of rank $r+1$ on X. Suppose that X admits a tilting bundle $\mathcal{T}$. Then the relative flag $\mathrm{Flag}_X(l_1, ..., l_t, \mathcal{E})$ admits a tilting bundle too.*

Note that under some technical assumptions on X and $\mathcal{E}$, one can show that the quadric bundle also admits a tilting object, provided X admits one (see Theorem 3.60, 3.70 and discussion right after). So more or less, we proved all the classical results concerning the existence of tilting objects in the relative setting.

As mentioned above, it is an open problem if any homogeneous variety admits a full (strongly) exceptional collection or a tilting object. As a consequence of the above results we can show the following:

Theorem. *Let G be a semisimple algebraic group of classical type, B a Borel subgroup and G/B the flag variety of G. Then $D^b(G/B)$ admits a tilting bundle.*

A complete solution of the guiding problem in geometric tilting theory can be obtained in the case the k-scheme X is a smooth proper and integral curve. Indeed, one can show that the curve X admits a tilting object if and only if it is a one-dimensional Brauer–Severi variety (see Theorem 3.71). As a well-known consequence one obtains that for curves C over $\mathbb{C}$ the existence of tilting objects is equivalent to C being rational. In dimension two the same statement is conjectured to be true and results in favor of this conjecture are due to Hille and Perling [86], [87] (see Chapter 3, Section 5). As a further source for examples where tilting objects may exist we consider a certain class of algebraic stacks, namely quotient stacks induced by the action of algebraic groups on schemes. For some stacks the existence of full (strongly) exceptional collections was already proved (see [90], [100]). As a first general result, we have a tilting correspondence that is completely analogously to that given for quasiprojective schemes. This fact follows from collecting the next two theorems.

Theorem. *(Theorem 3.87) Let X be a noetherian quasiprojective k-scheme and G a finite group acting on X, such that the characteristic of k does not divide the order of G. Suppose we are given a tilting object $\mathcal{T}$ for $D_G(\mathrm{Qcoh}(X))$ and let $A = \mathrm{End}_G(\mathcal{T})$. Then the following hold:*

(i) *The functor $\mathbb{R}\mathrm{Hom}_G(\mathcal{T}, -) : D_G(\mathrm{Qcoh}(X)) \to D(\mathrm{Mod}(A))$ is an equivalence.*

(ii) *If X is smooth, then the equivalence of (i) restricts to an equivalence*

$$D^b_G(X) \xrightarrow{\sim} \mathrm{perf}(A).$$

(iii) *If the global dimension of A is finite then $\mathrm{perf}(A) \simeq D^b(A)$.*

Theorem. *(Theorem 3.88) Let X, G and $\mathcal{T}$ be as in Theorem 3.87. If X is smooth, projective and integral, then $A = \mathrm{End}_G(\mathcal{T})$ has finite global dimension and the equivalence (i) of Theorem 3.87 restricts to an equivalence $D^b_G(X) \xrightarrow{\sim} D^b(A)$.*

For quotient stacks induced by a finite group action we have the following result:

Theorem. *(Theorem 3.89) Let X be a smooth projective and integral k-scheme, G a finite group acting on X such that $\mathrm{char}(k)$ does not divide $\mathrm{ord}(G)$. Suppose that $\mathcal{T}$ is a tilting sheaf for $D^b(X)$ and suppose furthermore, that $\mathcal{T}$ is G-equivariant. Let W_i be the irreducible representations of G. Then $\mathcal{T}_G = \bigoplus_i \mathcal{T} \otimes W_i$ is a tilting object for $D^b([X/G])$ and one has an equivalence*

$$\mathbb{R}\mathrm{Hom}_G(\mathcal{T}_G, -) : D^b([X/G]) \xrightarrow{\sim} D^b(\mathrm{End}_G(\mathcal{T}_G)).$$

This theorem enables us to find some quotient stacks admitting a tilting object. Moreover, exploiting the derived McKay correspondence one also finds some crepant resolutions that have tilting objects (see Chapter 3, section 6). The next result generalizes results of Bridgeland and Stern [50], Theorem 3.6 and Brav [43], Theorem 4.2.1. It also provides some evidence for a conjecture of King [103]. In what follows, the k-scheme X is supposed to be smooth, projective and integral.

Theorem. *(Theorem 3.96) Let X and G be as above and $\mathcal{E}$ a G-equivariant locally free sheaf of finite rank. Suppose $\mathcal{T}$ is a tilting bundle for $D^b(X)$ and suppose furthermore, that $\mathcal{T}$ is G-equivariant. If $H^i(X, \mathcal{T}^\vee \otimes \mathcal{T} \otimes S^l(\mathcal{E})) = 0$ for all $i \neq 0$ and all $l > 0$, then one has an equivalence*

$$D^b([\mathrm{Tot}(\mathcal{E})/G]) \xrightarrow{\sim} D^b(\mathrm{End}_G(\pi^*\mathcal{T}_G)).$$

Note that for X being a Fano variety and $G = 1$ one obtains the result of Bridgeland and Stern and if $X = \mathrm{Spec}(\mathbb{C})$ the result of Brav. In both cases the assumption $H^i(X, \mathcal{T}^\vee \otimes \mathcal{T} \otimes S^l(\mathcal{E})) = 0$ for all $i \neq 0$ and all $l > 0$ can be shown to be fulfilled. It is also natural to consider projective bundles with actions of finite groups. In this case we prove the following:

Theorem. *(Theorem 3.97) Let X and G be as above and suppose that $\mathcal{E}$ is G-equivariant. Furthermore, assume that $D^b(X)$ admits a tilting bundle $\mathcal{T}$ with G-equivariant structure. Then the quotient stack $[\mathbb{P}(\mathcal{E})/G]$ admits a tilting object too.*

All the above results provides us with many examples of quotient stacks admitting a tilting object. Investigating more closely the endomorphism algebras of the tilting objects of quotient stacks $[X/G]$, we find that the representation theory of $[X/G]$ is in some sense the G-equivariant representation theory of X (see Proposition 3.98 and Theorem 3.103). Problems occur in the case the group G is not supposed to be finite. In this case one can show that in general tilting objects for $[X/G]$ cannot exist. Indeed, we have the following result.

Theorem. *(Theorem 3.109) Let k be an algebraically closed field and X a smooth projective and integral k-scheme. Suppose G is an algebraic group acting on X. Suppose furthermore, that $D^b(X)$ admits an exceptional object $\mathcal{E}$, that is G-equivariant and that G does not admit finitely many irreducible representations. Then $D^b_G(X)$ does not admit a tilting object.*

For instance the Deligne–Mumford stack $[(\mathrm{Spec}[x_0, x_1, x_2] \smallsetminus 0)/\mathbb{G}_m]$, obtained by the $\mathbb{G}_m$ action of weight $(1,1,n)$, admits a tilting bundle, namely $\mathcal{T} = \bigoplus_{i=0}^{n+2} \mathcal{O}(i)$. So there are quotient stacks obtained by non-finite group actions that awfully well admit tilting objects. The question what kind of quotient stacks (or more generally algebraic stacks) admit tilting objects is a rather delicate one (see Chapter 6, Section 6) and seems at least as fascinating and difficult as the problem for schemes. But there is also another phenomenon that can occur. Loosely speaking, Theorem 3.89 states that the quotient stack $[X/G]$, induced by an action of a finite linear reductive group, admits a tilting object if X admits one. The question arises, if it is possible that a quotient stack $[X/G]$ has a tilting object although the scheme X does not admit one. It turns out that this is possible. Note that elliptic curves C cannot admit a tilting object, since $H^1(X, \mathcal{O}_C) \neq 0$. Having this in mind we observe the following fact:

Proposition. *(Proposition 3.114) Let C be an elliptic curve over an algebraically closed field of characteristic zero with $j \neq 0$ and $j \neq 1728$ and let $G = \{id, -id\} = \mathrm{Aut}(C)$ act on C. Furthermore, consider the induced action of $G' = G \times G$ on $C \times C$. Then both $D^b_G(C)$ and $D^b_{G'}(C \times C)$ admit a tilting object.*

At the end of the third chapter we give some application of geometric tilting theory and the results stated above, as we provide some further evidence for a conjecture formulated by Orlov [130].

Conjecture. *If X is a smooth integral and separated scheme of finite type over k, then $\dim D^b(X) = \dim(X)$.*

Note that the conjecture can also be formulated for smooth tame Deligne–Mumford stacks of finite type over k with quasiprojective coarse moduli space [19]. The conjecture is known to be true for smooth projective curves [130], affine schemes of finite type over k, certain flags and quadrics [137], del Pezzo surfaces, certain Fano 3-folds, Hirzebruch surfaces, certain toric surfaces and certain toric Deligne–Mumford stacks over $\mathbb{C}$ [19]. Our main result in favor of this conjecture is the following:

Theorem. *(Theorem 3.136) The dimension conjecture holds in the following cases:*

- **(i)** *Brauer–Severi varieties over arbitrary fields k.*
- **(ii)** *generalized Brauer–Severi varieties over fields of characteristic zero.*
- **(iii)** *finite products of the schemes in (i) and (ii).*
- **(iv)** *toric Fano 3-and 4-folds obtained as projective bundles of rank two locally free sheaves or as the product of projective spaces.*
- **(v)** *quotient stacks of the form $[\mathbb{P}^n/G]$ where G is a finite linearly reductive group.*
- **(vi)** *quotient stacks of the form $[\mathrm{Grass}(d,n)/G]$, provided $2d \neq n$ and the action of the finite group G is induced by a homomorphism $G \to \mathrm{PGL}_n(k)$.*
- **(vii)** *G-Hilbert schemes $\mathrm{Hilb}_G(\mathbb{P}^n)$ for $n \leq 3$, where G is a finite subgroup of $\mathrm{PGL}_{n+1}(k)$ such that $\omega_{\mathbb{P}^n}$ is locally trivial as a G-equivariant sheaf.*

The last chapter of the present work is dedicated to the Amitsur conjecture for Brauer–Severi varieties. The purpose is mainly to present a new point of view to consider this conjecture and to collect some ideas how to tackle it. In [5] Amitsur investigated generic splitting fields for central simple k-algebras and proved, among others, that if X and Y are birational Brauer–Severi varieties over k, the corresponding central simple k-algebras generate the same cyclic subgroup in $\mathrm{Br}(k)$. Moreover, Amitsur proved that if X has a cyclic splitting field the converse of above result holds. This led him to formulate the following conjecture:

Conjecture. *Let X and Y be two Brauer–Severi varieties of same dimension and A and B the corresponding central simple k-algebras. If A and B generate the same cyclic subgroup of $\mathrm{Br}(k)$, then X is birational to Y.*

The conjecture is known to be true for certain special cases but is still open in general (see Chapter 4). Notice that by the reconstruction theorem of Bondal and Orlov [39] one has for Brauer–Severi varieties X and Y, $D^b(X) \simeq D^b(Y)$ if and only if $X \simeq Y$. This is due to the fact that the anticanonical sheaf of Brauer–Severi varieties is ample. Hence the isomorphism class of Brauer–Severi varieties is completely understood in terms of their derived categories and it would be of interest to understand the birationality in terms of their derived categories too. As mentioned above, the aim of the last chapter is to consider more closely the semiorthogonal decompostion of the derived category of a Brauer–Severi variety and to present an idea how to tackle the Amitsur conjecture from this point of view. For this, let X and Y be n-dimensional Brauer–Severi varieties and A and B respectively the corresponding central simple k-algebras. Bernardara [28] proved that one has semiorthogonal decompositions:

$$D^b(X) = \langle D^b(k), D^b(A), ..., D^b(A^{\otimes n})\rangle$$
$$D^b(Y) = \langle D^b(k), D^b(B), ..., D^b(B^{\otimes n})\rangle$$

With this notation we make the following interesting observation.

Proposition. *(Proposition 4.12) Let X, Y, A and B be as above and let furthermore p be the period of A. Then A and B generate the same cyclic subgroup in* $\mathrm{Br}(k)$ *if and only if there exists a bijection of sets $\phi : \{0,1,...,p-1\} \to \{0,1,...,p-1\}$ such that one has k-linear triangulated equivalences between $D^b(A^{\otimes i})$ and $D^b(B^{\otimes \phi(i)})$.*

As a consequence of this proposition one obtains

Corollary. *(Corollary 4.13) Let X and Y be Brauer–Severi varieties and A and B the corresponding central simple k-algebras. Furthermore, let p be the period of A. In all the cases where the Amitsur conjecture holds the following are equivalent.*

- **(i)** *X and Y are birational.*
- **(ii)** *A and B generate the same cyclic subgroup in* $\mathrm{Br}(k)$.
- **(iii)** *There is a bijection of sets $\phi : \{0,1,...,p-1\} \to \{0,1,...,p-1\}$ such that one has k-linear triangulated equivalences $D^b(A^{\otimes i}) \tilde{\to} D^b(B^{\otimes \phi(i)})$.*

This corollary suggests that the birationality between Brauer–Severi varieties should be reflected in interchanging the admissible subcategories of the respective semiorthogonal decompositions. The idea now how to tackle the Amitsur conjecture is the following: Let X and Y be as above and let A and B the corresponding central simple k-algebras generating the same subgroup in $\mathrm{Br}(k)$. Suppose we are able to understand the "glueing behavior" (see Chapter 4, Section 4) between the admissible subcategories of the semiorthogonal decomposition of $D^b(X)$ and $D^b(Y)$ respectively. Then one can try to exploit the interchanging equivalences $D^b(A^{\otimes i}) \tilde{\to} D^b(B^{\otimes \phi(i)})$ to construct a functor $\Phi : D^b(X) \to D^b(Y)$ compatible with the interchanging and respecting the glueing behavior. If it is possible to construct the functor Φ in that way, that $\mathrm{Hom}(\Phi(\mathcal{F}), \Phi(\mathcal{G})[i]) = 0$ for $i < 0$ for all coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, [54], Theorem 1.1 then implies that Φ is a Fourier–Mukai transform. This Fourier–Mukai transform can then be used to investigate if there is a birational map between X and Y. Note that the existence of a Fourier–Mukai transform $F : D^b(X) \to D^b(Y)$ with certain properties is necessary for the existence of a birational map between X and Y (see Proposition 4.17). To provides us with more flexibility for the construction of a Fourier–Mukai transform exploiting the interchanging equivalences from above, we want to consider dg-enhancements for the derived categories of X and Y. Then one can try to construct dg-functors between the dg-enhancements of $D^b(X)$ and $D^b(X)$, that induce Fourier–Mukai transforms between $D^b(X)$ and $D^b(Y)$. At the very end, the obtained Fourier–Mukai functor should induce a birational map between X and Y. We end up the last chapter by collecting the results obtained in the following theorem.

Theorem. *(Theorem 4.25) Let X and Y be Brauer–Severi varieties of same dimension corresponding to A and B respectively. Then the following are equivalent*

- **(i)** *A and B generate the same cyclic subgroup in* $\mathrm{Br}(k)$
- **(ii)** *$X \times \mathbb{P}^n$ is birational to $Y \times \mathbb{P}^n$ for some n*
- **(iii)** *There exists a bijective map $\phi : \{0, 1, ..., p-1\} \to \{0, 1, ..., p-1\}$ such that we have equivalences $D^b(A^{\otimes i}) \xrightarrow{\sim} D^b(B^{\otimes \phi(i)})$.*
- **(iv)** *There exists a bijective map $\phi : \{0, 1, ..., p-1\} \to \{0, 1, ..., p-1\}$ such that we have quasiequivalences $\mathcal{D}_i \to \mathcal{D}_{\phi(i)}$*
- **(v)** $\mathrm{ind}(A \otimes_k L) = \mathrm{ind}(B \otimes_k L)$ *for any field extension $k \subset L$.*
- **(vi)** *The motive of X is a direct summand of the motive of* $\mathrm{BS}(d, B)$ *for $[A] = \pm d[B]$ in* $\mathrm{Br}(k)$

Conjecturally, all the above statements are equivalent to the birationality of X and Y and in fact this is true for all cases where the Amitsur conjecture is proved to hold.

Chapter 1

AS-bundles on Brauer–Severi varieties

Grothendieck [74] classified all locally free sheaves on $\mathbb{P}^1$ and Atiyah [14] found a classification for elliptic curves. So it is natural to ask for other curves where a classification of locally free sheaves is possible. In [125], [126] the author considered the case of a smooth nondegenerate conic C over k, that is, a Brauer–Severi variety of dimension one. In this situation C becomes isomorphic to $\mathbb{P}^1$ after base change to a Galois extension $k \subset L$ of degree 2. Then one can apply descent theory to classify all locally free sheaves on C. We want to discuss a generalization of this problem. For this, we consider Brauer–Severi varieties over k of arbitrary dimension and try to classify absolutely split locally free sheaves on them. An absolutely split locally free sheaf is a sheaf $\mathcal{E}$, that splits as a direct sum of invertible sheaves after base change to the algebraic closure of k.

In this section we recall some facts about Brauer–Severi varieties and central simple k-algebras. We review briefly the theory of Brauer–Severi varieties and the one-to-one correspondence with isomorphism classes of central simple k-algebras. Note that Brauer–Severi varieties and central simple algebras can also be defined in the relative setting and are referred to as Brauer–Severi schemes and Azumaya algebras respectively (see [75] and [76]). The present work only deals with the absolute case and as main references we use [10], [71] and [139].

Definition 1.1. A *Brauer–Severi variety* of dimension n is a scheme X of finite type over k, such that $X \otimes_k L \cong \mathbb{P}^n_L$ for a finite field extension $k \subset L$.

A field extension $k \subset L$ for that $X \otimes_k L \cong \mathbb{P}^n_L$ is called *splitting field* of X. Clearly, the algebraic closure $\bar{k}$ is a splitting field for any Brauer–Severi variety. One can show that every Brauer–Severi variety always splits over a finite separable field extension of k (see [71], Corollary 5.1.4). By embedding the finite separable splitting field into its Galois closure, a Brauer–Severi variety splits over a finite Galois extension of the base field k (see [71], Corollary 5.1.5). It follows from [77], IV, Chapter II, Theorem 2.7.1 (see also [92], Lemma 2.12) that X is projective, integral and smooth over k.

To explain the one-to-one correspondence between isomorphism classes of Brauer–Severi varieties and isomorphism classes of central simple k-algebras we have to describe the isomorphism classes of Brauer–Severi varieties cohomologically. For this, let L be a finite Galois splitting field of X with Galois group G. Denote by $\mathrm{BS}_n(L|k)$ the pointed set of isomorphism classes of Brauer–Severi varieties of dimension n split by L, with base point being $\mathbb{P}^n_L$. Given an isomorphism $f: X\otimes_k L \to \mathbb{P}^n_L$ we consider for each $\sigma \in G$ the diagram:

$$\begin{array}{ccc} X\otimes_k L & \xrightarrow{f} & \mathbb{P}^n_L \\ {\scriptstyle id\otimes\sigma}\uparrow & & \uparrow{\scriptstyle id\otimes\sigma} \\ X\otimes_k L & \xrightarrow{f} & \mathbb{P}^n_L \end{array}$$

For $id\otimes\sigma$ we shortly write σ. Then $a_\sigma = f\circ\sigma\circ f^{-1}\circ\sigma^{-1}$ is an automorphism of $\mathbb{P}^n_L$ over L. This yields a map $G\to \mathrm{Aut}(\mathbb{P}^n_L)$ that sends σ to a_σ. One can show that a_σ is a cocycle. To see this, we consider $f\circ\sigma = a_\sigma\circ(\sigma\circ f)$ and calculate

$$\begin{aligned} a_{\sigma\tau}\circ(\sigma\tau\circ f) &= f\circ\sigma\tau \\ &= (f\circ\sigma)\circ\tau \\ &= a_\sigma\circ(\sigma\circ f)\circ\tau \\ &= a_\sigma\circ\sigma\circ(a_\tau\circ(\tau\circ f)) \\ &= a_\sigma\circ\sigma a_\tau\circ(\sigma\tau\circ f) \end{aligned}$$

and find that $a_{\sigma\tau} = a_\sigma\circ\sigma a_\tau$, what implies that a_σ is a cocycle. An invertible matrix $T\in \mathrm{GL}_{n+1}(L)$ yields an automorphism of $\mathbb{P}^n_L$ by acting on the polynomial ring $k[x_0,...,x_n]$. If $\lambda\in k^*$, then $\lambda\cdot T$ determines the same automorphism. Hence the group $\mathrm{PGL}_{n+1}(L)$ gives automorphisms of $\mathbb{P}^n_L$. One can show that every automorphism of $\mathbb{P}^n_L$ is an element of $\mathrm{PGL}_{n+1}(L)$ so that we have $\mathrm{Aut}(\mathbb{P}^n_L) = \mathrm{PGL}_{n+1}(L)$ (see [82] Example 7.1.1). Thus we get a class $[a_\sigma]$ in $H^1(G,\mathrm{PGL}_{n+1}(L))$. This is a pointed set too, with the distinguished element coming from the trivial cocycle $\sigma\mapsto 1$. Finally we get a map $\mathrm{BS}_n(L|k)\to H^1(G,\mathrm{PGL}_{n+1}(L))$ by sending the class of X to $[a_\sigma]$. To see that this map is well defined, we start with another isomorphism $f': X\otimes_k L\to\mathbb{P}^n_L$ that gives another cocycle $a'_\sigma = f'\circ\sigma\circ f'^{-1}\circ\sigma^{-1}$. Then there is some $b\in\mathrm{PGL}_{n+1}(L)$ such that $f = b\circ f'$ and $f\circ\sigma = a_\sigma\circ(\sigma\circ f)$ implies

$$f'\circ\sigma = b^{-1}\circ f\circ\sigma = b^{-1}\circ a_\sigma\circ(\sigma\circ(b\circ f')) = b^{-1}a_\sigma\circ\sigma b\circ(\sigma\circ f').$$

Since $f'\circ\sigma = a'_\sigma\circ(\sigma\circ f')$ we have shown that $a'_\sigma = b^{-1}a_\sigma\circ\sigma b$ what implies that a'_σ and a_σ are cohomologous. Hence the map $\mathrm{BS}_n(L|k)\to H^1(G,\mathrm{PGL}_{n+1}(L))$, sending the class of X to $[a_\sigma]$, is well defined. The following result is contained in [71], p.117, 5.2 (see also [142] or [92], Theorem 4.5).

Theorem 1.2. *Let L be a finite Galois extension of k and G the Galois group. Then the above constructed map is a base point preserving canonical bijection*

$$\mathrm{BS}_n(L|k)\xrightarrow{\simeq} H^1(G,\mathrm{PGL}_{n+1}(L))$$

with base point being the class of $\mathbb{P}^n_L$.

We denote by $\mathrm{BS}_n(k) = \bigcup_{L|k}\mathrm{BS}_n(L|k)$ the set of all isomorphism classes of Brauer–Severi varieties of dimension n. Here the union is taken over all finite Galois splitting fields $k\subset L$. In what follows, we want to explain the relation between $\mathrm{BS}_n(k) = \bigcup_{L|k}\mathrm{BS}_n(L|k)$ and the continuous cohomology group

$H^1(\mathrm{Gal}(k^{sep}|k), \mathrm{PGL}_{n+1}(k^{sep}))$. The main reference for the non-abelian cohomology groups H^0 and H^1 is [143].
For a tower of finite Galois extensions $k \subset L \subset E$ in the possibly infinite extension $k \subset k^{sep}$, Galois theory provides us with a canonical surjective group homomorphism $\phi_{E,L} : \mathrm{Gal}(E|k) \to \mathrm{Gal}(L|k)$. Moreover, if $k \subset F$ is another finite Galois extension containing E we have $\phi_{F,L} = \phi_{E,L} \circ \phi_{F,E}$. This yields an inverse system of groups and we can pass to the inverse limit to get the group $\mathrm{Gal}(k^{sep}|k)$. This group is endowed with a natural topology. For all finite Galois extensions $k \subset L$ endow the Galois groups $\mathrm{Gal}(L|k)$ with the discrete topology, their product with the product topology and the inverse limit $\mathrm{Gal}(k^{sep}|k) \subset \prod_L \mathrm{Gal}(L|k)$ with the subspace topology. Thus $G = \mathrm{Gal}(k^{sep}|k)$ is a profinite group. Now for two finite Galois extensions $k \subset L_\alpha \subset L_\beta \subset k^{sep}$ let $U_\alpha = \mathrm{Gal}(k^{sep}|L_\alpha)$ and $U_\beta = \mathrm{Gal}(k^{sep}|L_\beta)$ be closed subgroups of $\mathrm{Gal}(k^{sep}|k)$. Then $\mathrm{PGL}_{n+1}(k^{sep})^{U_\alpha}$ is a G/U_α group and $\mathrm{PGL}_{n+1}(k^{sep})^{U_\beta}$ a G/U_β group. The surjection $\phi_{\alpha,\beta} : G/U_\beta \to G/U_\alpha$ induces a map

$$\mathrm{Inf}_\alpha^\beta : H^1(G/U_\alpha, \mathrm{PGL}_{n+1}(k^{sep})^{U_\alpha}) \longrightarrow H^1(G/U_\beta, \mathrm{PGL}_{n+1}(k^{sep})^{U_\beta}).$$

We note that $\mathrm{PGL}_{n+1}(k^{sep})^{U_\alpha} = \mathrm{PGL}_{n+1}(L_\alpha)$. Indeed, for the intermediate field $k \subset L_\alpha \subset k^{sep}$ we have an exact sequence

$$1 \longrightarrow (k^{sep})^* \longrightarrow \mathrm{GL}_{n+1}(k^{sep}) \longrightarrow \mathrm{PGL}_{n+1}(k^{sep}) \to 1.$$

This induces the exact sequence on non-abelian cohomology (see [143], Proposition 36)

$$1 \longrightarrow (L_\alpha)^* \longrightarrow \mathrm{GL}_{n+1}(L_\alpha) \longrightarrow \mathrm{PGL}_{n+1}(k^{sep})^{U_\alpha} \longrightarrow H^1(U_\alpha, (k^{sep})^*).$$

But $H^1(U_\alpha, (k^{sep})^*) = 1$ by Hilbert 90 and hence $\mathrm{PGL}_{n+1}(k^{sep})^{U_\alpha} = \mathrm{PGL}_{n+1}(L_\alpha)$. The same is true for $\mathrm{PGL}_{n+1}(k^{sep})^{U_\beta}$. Since $G/U_\alpha \simeq \mathrm{Gal}(L_\alpha|k)$ and $G/U_\beta \simeq \mathrm{Gal}(L_\beta|k)$ we get with the bijection of Theorem 1.2 a commutative diagram of morphisms of pointed sets

$$\begin{array}{ccc} \mathrm{BS}_n(L_\alpha) & \longrightarrow & H^1(\mathrm{Gal}(L_\alpha|k), \mathrm{PGL}_{n+1}(L_\alpha)) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \mathrm{Inf}_\alpha^\beta} \\ \mathrm{BS}_n(L_\beta) & \longrightarrow & H^1(\mathrm{Gal}(L_\beta|k), \mathrm{PGL}_{n+1}(L_\beta)) \end{array}$$

Then, by passing to the direct limit, taken over all finite Galois extensions $k \subset L$, we get a unique natural bijection

$$\mathrm{BS}_n(k) \xrightarrow{\simeq} \varinjlim H^1(\mathrm{Gal}(L|k), \mathrm{PGL}_{n+1}(L)).$$

By definition of the continuous cohomology $\varinjlim H^1(\mathrm{Gal}(L|k), \mathrm{PGL}_{n+1}(L)) = H^1(\mathrm{Gal}(k^{sep}|k), \mathrm{PGL}_{n+1}(k^{sep}))$ and hence we have the natural bijection:

$$\mathrm{BS}_n(k) \xrightarrow{\simeq} H^1(\mathrm{Gal}(k^{sep}|k), \mathrm{PGL}_{n+1}(k^{sep})). \tag{1.1}$$

A detailed proof for this bijection can be found in [142], Proposition 8 (see also [92]).

To continue, we want to define central simple k-algebras following [71].

Definition 1.3. An associative finite-dimensional k-algebra A is called *central simple* if it has no two sided ideal other than 0 and A and if its center equals k.

There is a characterization of these algebras. It is the following result which ca be found in [71], Theorem 2.2.1.

Theorem 1.4. *Let A be a finite-dimensional associative k-algebra. Then the following are equivalent:*

(i) *A is central simple.*

(ii) *There is a finite field extension $k \subset L$, such that $A \otimes_k L \simeq M_n(L)$.*

(iii) $A \otimes_k \bar{k} \simeq M_n(\bar{k})$.

Proof. The equivalence of (i) and (ii) is [71], Theorem 2.2.1. It remains to show the equivalence of (i) and (iii). Suppose that $A \otimes_k \bar{k}$ is isomorphic to $M_n(\bar{k})$. Then in the proof of [71], Theorem 2.2.1 it is shown that there is a finite field extension $k \subset K$ contained in $\bar{k}$, such that $A \otimes_k K \simeq M_n(K)$. According to [71], Lemma 2.2.2 (or by the equivalence of (i) and (iii)), this implies that A is central simple. The other direction is clear. □

Definition 1.5. Let A be a central simple k-algebra. A field extension $k \subset L$ such that $A \otimes_k L \simeq M_n(L)$ is called *splitting field* of the central simple k-algebra A.

We now define a special class of central simple k-algebras.

Definition 1.6. Let A be a central simple k-algebra. Then A is called *central division algebra* if every nonzero element $a \in A$ has a multiplicative inverse.

Since division algebras over a field k are in general only central simple over their center, they are usually called division algebras, without the additional term central. If the center is k itself these division are called central division algebras. We want to note that in the literature central division algebras are sometimes referred to as division algebras, when it is clear from the context that the center is k itself. A standard example of a central division algebra are the *Hamiltonian quaternions*, denoted by $\mathbb{H}$. By definition it is a 4-dimensional $\mathbb{R}$-algebra with basis $1, i, j$ and ij subject to the relations $i^2 = -1$, $j^2 = -1$ and $ij = ji$ (see [71], 1.1).
It turns out that studying central simple k-algebras can be reduced to studying central division algebras. It is the following well-known *Wedderburn Theorem* [71], Theorem 2.1.3.

Theorem 1.7. *Let A be a finite-dimensional simple k-algebra. Then there is an integer $n > 0$ and a division algebra D, such that $A \simeq M_n(D)$. The central division algebra D is unique up to isomorphism.*

It is clear that if A is a central simple k-algebra, the unique division algebra D, for that $M_n(D) \simeq A$, is also central and hence a central division algebra.
By Theorem 1.4 from above, for every central simple k-algebra A there is a finite field extension L such that $A \otimes_k L \simeq M_n(L)$. Thus the dimension of A as a k-vector space is a square. Hence we get an integer

$$\deg(A) = \sqrt{\dim_k A}$$

that is called the *degree* of A. Finally, there is always a finite Galois field extension of k that splits A (see [71], Corollary 2.2.6). Now take such a finite Galois extension $k \subset L$ and let us denote the set of k-isomorphism classes of central simple k-algebras of degree $n+1$ that are split by L simply by $\mathrm{CSA}_{n+1}(L|k)$. It is a pointed set with base point being $M_{n+1}(L)$. To continue, we recall a basic fact for k-automorphisms of central simple k-algebras. It is the *Skolem–Noether Theorem* [71], Theorem 2.7.2.

Theorem 1.8. *Every k-automorphism of a central simple k-algebra A is inner.*

This theorem applied to a matrix algebra $M_n(k)$ yields that the automorphisms $\mathrm{Aut}(M_n(k))$ are isomorphic to $\mathrm{PGL}_n(k)$ (see [71], Corollary 2.4.2). Now for the Galois splitting field L, let $f : A \otimes_k L \to M_{n+1}(L)$ be an isomorphism and $\sigma \in G = \mathrm{Gal}(L|k)$. Then the diagram

$$\begin{array}{ccc} A \otimes_k L & \xrightarrow{f} & M_{n+1}(L) \\ {\scriptstyle id\otimes\sigma}\uparrow & & {\scriptstyle id\otimes\sigma}\uparrow \\ A \otimes_k L & \xrightarrow{f} & M_{n+1}(L) \end{array}$$

yields an automorphism $a_\sigma = f \circ \sigma \circ f^{-1} \circ \sigma^{-1}$ of $M_{n+1}(L)$ over L. As in the case for Brauer–Severi varieties one can show that a_σ is a cocycle. By the Skolem–Noether Theorem we have $\mathrm{Aut}(M_{n+1}(L)) = \mathrm{PGL}_{n+1}(L)$, so we get a class $[a_\sigma]$ in $H^1(G, \mathrm{PGL}_{n+1}(L))$ and hence a map $G \to H^1(G, \mathrm{PGL}_{n+1}(L))$, mapping σ to $[a_\sigma]$. As for Brauer–Severi varieties, where one has Theorem 1.2, one can prove the following result (see [71], Theorem 2.4.3).

Theorem 1.9. *The above constructed morphism is a base point preserving canonical bijection*

$$\mathrm{CSA}_{n+1}(L|k) \xrightarrow{\simeq} H^1(G, \mathrm{PGL}_{n+1}(L))$$

with base point being the class of $M_{n+1}(L)$.

As we have done in the case for the set of all isomorphism classes of Brauer–Severi schemes $\mathrm{BS}_n(k)$, we want to denote by $\mathrm{CSA}_{n+1}(k) = \bigcup_{L|k} \mathrm{CSA}_{n+1}(L|k)$ the set of all isomorphism classes of central simple k-algebras of degree $n+1$ split by L. As in (1.1) we can pass to the direct limit to get a unique natural bijection:

$$\mathrm{CSA}_{n+1}(k) \xrightarrow{\simeq} H^1(\mathrm{Gal}(k^{sep}|k), \mathrm{PGL}_{n+1}(k^{sep})) \tag{1.2}$$

Summarizing (1.1) and (1.2) we get canonical base point preserving bijections

$$\mathrm{CSA}_{n+1}(k) \xrightarrow{\simeq} H^1(\mathrm{Gal}(k^{sep}|k), \mathrm{PGL}_{n+1}(k^{sep})) \xleftarrow{\simeq} \mathrm{BS}_n(k)$$

and thus a one-to-one correspondence between isomorphism classes of central simple k-algebras of degree $n+1$ and isomorphism classes of Brauer–Severi varieties of dimension n.

Theorem 1.10. *There is a canonical identification*

$$\mathrm{CSA}_{n+1}(k) = \mathrm{BS}_n(k).$$

As we have seen above, studying central simple k-algebras can be reduced to studying central division algebras. In view of the one-to-one correspondence between Brauer–Severi varieties and central simple k-algebras, it turns out that one has also a geometric interpretation of central division algebras.

Definition 1.11. Let X be a Brauer–Severi variety of dimension n over k. Then a *linear subvariety* of X is a closed subscheme $Y \subset X$, such that for a splitting field $k \subset L$ of X one has that $Y \otimes_k L \subset X \otimes_k L \cong \mathbb{P}^n_L$ is a linear subspace of the projective space $\mathbb{P}^n_L$. We call a Brauer–Severi variety *minimal* if it has no proper linear subvariety.

We now end up this section in citing a result that shows, that in the geometric interpretation of the central simple k-algebras, the minimal linear subvarieties exactly play the role of the central division algebras (see [71], Corollary 5.3.5).

Proposition 1.12. *A central simple k-algebra is a central division algebra if and only if the corresponding Brauer–Severi variety is minimal.*

1.1 Invariants of Brauer–Severi varieties

In this section we recall the standard invariants of Brauer–Severi varieties and central simple k-algebras respectively, namely the *period* and the *index* and discuss some of their properties. The main references are [10], [71], [139] and [142].
We first recall the definition of the the Brauer group $\mathrm{Br}(k)$ of a field k. Elements of the group are equivalence classes of central simple k-algebras. Two central simple k-algebras $A \simeq M_n(D_1)$ and $B \simeq M_m(D_2)$ are called *Brauer-equivalent* if $D_1 \simeq D_2$, where D_i are central division algebras according to Theorem 1.7. The group operation is given by the tensor product and one can show that $A \otimes A^{op} \simeq M_n(k)$. Hence the tensor product gives a group structure on $\mathrm{Br}(k)$ with neutral element being the equivalence class of k. The obtained group is called *Brauer group* of k. Since for two central simple k-algebras one has $A \otimes_k B \simeq B \otimes_k A$ the Brauer group is an abelian group. From the above definition of Brauer-equivalence we see that D and $M_n(D)$ represent the same element in $\mathrm{Br}(k)$. Hence, understanding this group means understanding central division algebras over k. We now give some examples where the Brauer group is trivial. These and others, together with further discussion concerning the Brauer group, can be found in [142], §7.

Example 1.13. Let k be a finite field or an algebraic extension of $\mathbb{Q}$ containing all roots of unity, then $\mathrm{Br}(k)$ is trivial (see [142], §7 Example a) and d)).

Furthermore, since every Brauer–Severi variety, and hence every central simple algebra, splits over an algebraically closed field, the Brauer group of an algebraically closed field is trivial too.

Example 1.14. Since every central simple $\mathbb{R}$-algebra is isomorphic to $\mathbb{R}$ or the Hamilton quaternions the Brauer group $\mathrm{Br}(\mathbb{R})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (see [142], §7 Example e)).

Remark 1.15. Understanding the Brauer group of a field k is rather delicate issue. The Brauer group has applications in Class field theory and plays a prominent role in understanding global fields. We refer to [142] and references therein for further details on this topic.

To continue, we recall some facts about Galois Coverings (see [121], §5) and apply them to Brauer–Severi varieties to define the *period*. We shall say a morphism $f : Z \to Y$, that is locally of finite type, is *unramified* at $z \in Z$ if $\mathcal{O}_{Z,z}/m_y\mathcal{O}_{Z,z}$ is a finite separable field extension of $k(y)$, where $y = f(z)$. The morphism is *unramified* if it is unramified at every point $z \in Z$. A morphism of schemes is defined to be *étale* if it is flat and unramified. Now let (Cov/Y) denote the category of finite and étale Y-schemes. For the category (Cov/Y) equipped with the Grothendieck topology, defined by coverings, we write simply Y_{et}. Furthermore, for any Y-scheme Z, we write $\mathrm{Aut}_Y(Z)$ for the group of automorphisms of Z. This group acts on $F(Z)$ on the right, where F is the functor $F = \mathrm{Hom}(\bar{y}, -) : (\mathrm{Cov}/Y) \to \mathbf{Sets}$ for a geometric point $\bar{y}$ in Y. If Y is connected this action is faithful. If furthermore the action is transitive, then Z is said to be *Galois* over Y. If G is a finite group, then G_Z denotes the scheme $\coprod_{\sigma \in G} Z_\sigma$, where $Z_\sigma \cong Z$. Let G act on Z, then we say that $Z \to Y$ is Galois with Galois group G, if the map is faithfully flat and if $\psi : G_Z \to Z \times Z$, defined by $\psi_{|Z_\sigma} = (z \mapsto (z, z\sigma))$, is an isomorphism with respect to the action of G. For such a Galois covering with Galois group G and a sheaf $\mathcal{F}$ in the étale topology $Y_{ét}$ there is the Hochschild–Serre spectral sequence (see [121], Chapter III, Theorem 2.20):

$$H^p(G, H^q(Z_{ét}, \mathcal{F})) \Longrightarrow H^{p+q}(Y_{ét}, \mathcal{F}).$$

Applying this to the Galois covering $\mathbb{P}^n_L \to X$, with X a Brauer–Severi variety over k, $G = \mathrm{Gal}(L|k)$ the Galois group for the finite Galois splitting field L and sheaf $\mathbb{G}_m \simeq k^*$ we get the four term exact sequence (see also [10], p.197):

$$0 \longrightarrow H^1(G, H^0(\mathbb{P}^n_L, \mathbb{G}_m)) \longrightarrow H^1(X, \mathbb{G}_m) \longrightarrow H^0(G, H^1(\mathbb{P}^n_L, \mathbb{G}_m)) \longrightarrow$$

$$\longrightarrow H^2(G, H^0(\mathbb{P}^n_L, \mathbb{G}_m)).$$

Keeping in mind that

$$H^1(G, H^0(\mathbb{P}^n_L, \mathbb{G}_m)) = H^1(G, k^*) = 0,$$

according to Hilbert 90, $H^1(X, \mathbb{G}_m) = \mathrm{Pic}(X)$,

$$H^0(G, H^1(\mathbb{P}^n_L, \mathbb{G}_m)) = \mathrm{Pic}(\mathbb{P}^n_L)^G = \mathbb{Z}$$

and

$$H^2(G, H^0(\mathbb{P}^n_L, \mathbb{G}_m)) = H^2(G, k^*) = \mathrm{Br}(k),$$

we finally get the following exact sequence:

$$0 \longrightarrow \mathrm{Pic}(X) \xrightarrow{deg} \mathbb{Z} \longrightarrow \mathrm{Br}(k).$$

A Theorem due to Lichtenbaum (see [71], Theorem 5.4.10.) now states that the boundary map $\delta : \mathbb{Z} \to \mathrm{Br}(k)$ is given by sending 1 to the class of X in $\mathrm{Br}(k)$. Here the class of X is that of the corresponding central simple k-algebra. It follows that $\mathrm{Pic}(X)$ is identified with some subgroup $r\mathbb{Z}$ of $\mathbb{Z}$. So r is the order of A in $\mathrm{Br}(k)$, where A is the central simple k-algebra corresponding to X. This order is called the *period* of X and is also the smallest positive integer r such that $\mathcal{O}_X(r)$ exists in $\mathrm{Pic}(X)$. We denote the period of X by $\mathrm{per}(X)$ or $\mathrm{per}(A)$. The *index* of X is defined to be the smallest integer $d > 0$ such that there is a linear subvariety Y with $d - 1 = \dim(Y)$. It is denoted by $\mathrm{ind}(X)$ and clearly $\mathrm{ind}(X) \leq n + 1$, where equality holds if and only if there is no proper linear subvariety $Y \subset X$. On the other hand, there is also a definition of the index of a central simple k-algebra. For this, let A be a central simple k-algebra. Then by Theorem 1.7 above there is an isomorphism $A \cong M_n(D)$, with a unique central division algebra D. The *index* of A is now defined to be $\deg(D)$ and is denoted by $\mathrm{ind}(A)$.

With the above notation we can state a result that is originally due to Châtelet and can be found in [71], Corollary 5.3.6.

Proposition 1.16. *Let X be a Brauer–Severi variety over k and A the corresponding central simple k-algebra. Then all minimal linear subvarieties Y of X have the same dimension satisfying the equality*

$$\dim(Y) = \mathrm{ind}(A) - 1.$$

Remark 1.17. From Proposition 1.16 we now conclude that the index of X equals the index of the corresponding central simple k-algebra A. Furthermore, the index of a central simple k-algebra A is the smallest among the degrees of finite separable field extensions that split A (see [71], Corollary 4.5.9). Another important fact is that the period divides the index and both, period and index, have the same prime factors (see [71], Proposition 4.5.13).

Lemma 1.18. *Let X be a Brauer–Severi variety over k and A the corresponding central simple k-algebra. Then the following holds:*

$$\mathrm{per}(A)|\mathrm{ind}(A)|\deg(A).$$

Proof. Because of the fact that the period divides the index by Remark 1.17, we only have to prove that $\mathrm{ind}(A)$ divides $\deg(A)$. According to Theorem 1.7 we have $A \simeq M_m(D)$ for some m. This implies $\dim_k A = m^2 \cdot \dim_k D$ and hence $\deg(A) = m \cdot \mathrm{ind}(A)$. This completes the proof. □

For every triple $(p, i, d) \in \mathbb{N}^3$, satisfying the relations $p|i$ and $i|d$, one can ask for the existence of a Brauer–Severi variety X corresponding to a central simple k-algebra A with $\mathrm{per}(A) = p$, $\mathrm{ind}(A) = i$ and $\deg(A) = d$. In view of this question we want to give the following definition:

Definition 1.19. A Brauer–Severi variety X over k corresponding to a central simple k-algebra A, with $\mathrm{per}(A) = p$, $\mathrm{ind}(A) = i$ and $\deg(A) = d$ is called a *Brauer–Severi variety of type* (p, i, d).

Example 1.20. If the index of a Brauer–Severi variety X is one, we conclude with Remark 1.17 that k itself is a splitting field. Hence $X \simeq \mathbb{P}^n_k$ and thus X is of type $(1, 1, n + 1)$. This implies that $X \simeq \mathbb{P}^n_k$ if and only if X is of type $(1, 1, n + 1)$.

Recall that for a field k with $\mathrm{char}(k) \neq 2$, for any two elements $a, b \in k^*$ one can define a *quaternion algebra* $(a,b)_k$ as the 4-dimensional k-algebra with basis $1, i, j, ij$, where the multiplication is determined by $i^2 = a, j^2 = b$ and $ij = -ji$. If this quaternion algebra does not split it is a division algebra (see [71], Proposition 1.1.7). For $a = b = -1$ and $k = \mathbb{R}$ we get the usual Hamilton quaternions. In characteristic 2 one defines the *quaternion algebra* by the presentation

$$(a,b)_k = \langle i, j | i^2 + i = a, j^2 = b, ij = ji + i \rangle.$$

This algebra has properties analogous to those in $\mathrm{char}(k) \neq 2$ (see [71], Chapter 1, Exercise 4). Furthermore, in both cases the quaternion algebra $(a,b)_k$ has a degree two separable splitting field and therefore $\mathrm{ind}((a,b)_k) = 2$.
For the next observation we first need a fact about central division algebras of dimension four. It can be found in [71], Proposition 1.2.1.

Proposition 1.21. *Let k be a field of $\mathrm{char}(k) \neq 2$. Then every non-split central division algebra over k of dimension four is isomorphic to a quaternion algebra.*

As a consequence of this fact we immediately get:

Proposition 1.22. *Let k be a field of $\mathrm{char}(k) \neq 2$. Then a Brauer–Severi variety X over k is of type $(2,2,2m)$ if and only if the corresponding central simple k-algebra is isomorphic to $M_m(D)$ for a unique quaternion algebra $D = (a,b)_k$.*

Proof. Let X be the Brauer–Severi variety corresponding to $A = M_m(D)$, where $D = (a,b)_k$. Then $\mathrm{ind}(X) = 2$ and by Lemma 1.18 we have that $\mathrm{per}(X) = 2$, since D is assumed to be non-split. Since $\deg(A) = m \cdot \mathrm{ind}(X) = 2m$, we immediately see that X is of type $(2,2,2m)$. Now assume that we are given a Brauer–Severi variety of type $(2,2,2m)$. Then the corresponding central simple k-algebra A is non-split and has index two. But $\mathrm{ind}(A) = 2$ implies that $\dim_k D = 4$ for the unique division algebra D with $A \simeq M_m(D)$. But every non-split central division algebra of dimension four is a quaternion algebra according to Proposition 1.21. From this follows $A \simeq M_m((a,b)_k)$. □

Remark 1.23. For a Brauer–Severi variety X of type $(2,2,2m)$ there exists a separable field extension $k \subset L$ of degree two that splits X (see Remark 1.17). Since degree two field extensions are normal, $k \subset L$ is a Galois extension. Hence for every Brauer–Severi variety of index two there is a degree two Galois extension that splits X. This holds without the restriction of k being of $\mathrm{char}(k) \neq 2$.

The next Theorem states some facts about the index of $A^{\otimes r}$ and can be found in [139], Theorem 5.5. We will need this fact later on. Denote by (m,n) the greatest common divisor of the natural numbers m and n. Then one has the following fact:

Theorem 1.24. *Let A be a central simple k-algebra of index i. Then for $r > 0$ one has:*

- **(i)** *The index of $A^{\otimes r}$ divides $(\binom{i}{r}, i)$.*
- **(ii)** *Suppose i and r are coprime. Then $A^{\otimes r}$ has index i.*
- **(iii)** *Let e be (r,i). Then $A^{\otimes r}$ has index dividing i/e.*

The following results answer the question if for a prescribed triple (p, i, d), with $p|i$ and $i|d$, there exists a Brauer–Severi variety of type (p, i, d). In fact it is a consequence of the following Theorem (see [91], Theorem 2.8.12).

Theorem 1.25. *Let p and i be natural numbers with the property that p divides i and they have the same prime factors. Then there exists a field k and a central division algebra over k, such that D has period p and index i.*

Now as an easy consequence we obtain:

Theorem 1.26. *Let p, i and d be a natural numbers with the property that p divides i, i divides d and p and i have the same prime factors. Then there exists a field k and a Brauer–Severi variety X over k that is of type (p, i, d).*

Proof. With Theorem 1.25 one gets the existence of a division algebra D over a field k with period p and index i. The degree of D is i too. Since $d = mi$ for some natural number m, we find a central simple k-algebra $A = M_m(D)$ that also has period p and index i. By Theorem 1.10 we finally get the existence of a Brauer–Severi variety X corresponding to A that is exactly of type (p, i, d). □

We want to mention that the existence of a Brauer–Severi variety X of prescribed period, index and degree hardly depends on the field k over which X is defined. Since the Brauer group $\mathrm{Br}(k)$ is trivial for a finite field, the only Brauer–Severi varieties over such fields are of type $(1, 1, n + 1)$. The same phenomenon appears if the field k is an algebraic extension of $\mathbb{Q}$ containing all roots of unity. Again, the Brauer group is trivial in this case and the only Brauer–Severi varieties we find are of type $(1, 1, n + 1)$. In the case $k = \mathbb{R}$ we have that $\mathrm{Br}(k) \simeq \mathbb{Z}/2\mathbb{Z}$ and the only two isomorphism classes of central simple $\mathbb{R}$-algebras are $M_n(\mathbb{R})$ and the Hamilton quaternions. Hence all Brauer–Severi varieties over $\mathbb{R}$ are of type $(1, 1, n + 1)$ or $(2, 2, 2m)$. This is the reason why Theorem 1.25 above, together with the existence of a central division algebra D, also yields the existence of some field k over which D is defined. More precisely, one has to make some assumptions on the given ground field and obtains k as a field extension of it. Then one can construct a central division algebra D over k with the desired properties. So the question from above only makes sense if one asks if there exists a field k and a Brauer–Severi variety X over that k of type (p, i, d) for a prescribed triple $(p, i, d) \in \mathbb{N}^3$, with $p|i$ and $i|d$. As we have seen above, this question can be answered affirmatively. For a more detailed explanation of how the field k and the division algebra is constructed, we refer to [91].

1.2 Absolutely isotypical sheaves

In this section we investigate a certain class of locally free sheaves, called absolutely isotypical and prove some properties of the endomorphism algebra of this sheaves.

We first need some facts of simple and semisimple rings that can be found in [7] or [42]. We recall that for a ring R with unity a R-module M is called *simple* if M has no non-trivial submodules. A R-module M is called *semisimple* if M is isomorphic to the direct sum of simple modules (see [7], Chapter 3, § 9).

Note that a ring R is called simple (semisimple) if it is a simple (semisimple) left module over itself. One has the following fundamental characterization of semisimple modules (see [7], Theorem 9.6)

Theorem 1.27. *Let R be a ring with unity and M a R-module. Then M is semisimple if and only if every submodule of M is a direct summand.*

We note that a simple ring R is of course semisimple. For semisimple rings one has another very important characterization. We recall, the *Jacobson radical* of a ring R is by definition the intersection of all maximal left ideals in R and is denoted by $\mathrm{rad}(R)$. With this notation one has the following fact (see [7], Proposition 15.16):

Proposition 1.28. *Let R be a left artinian ring. Then R is semisimple if and only if* $\mathrm{rad}(R) = 0$. *In particular, $R/\mathrm{rad}(R)$ is semisimple.*

Since central simple k-algebras are isomorphic to $M_n(D)$, for some unique central division algebra D, they are simple in the above sense (see [7], § 13, Example 13.1). Hence the above theorem and proposition applies. For central simple k-algebras A, one therefore has $\mathrm{rad}(A) = 0$. This fact will became important later on.

On Brauer–Severi varieties one has a special class of locally free sheaves. We want to call them *absolutely isotypical* referring to the isotypical decomposition in representation theory. In the work of Arason, Elman and Jacob [9], they are called *pure* and are closely related to central simple algebras.

Definition 1.29. A locally free sheaf $\mathcal{E}$ of finite rank on a proper k-scheme X is called *absolutely isotypical* if on $X \otimes_k \bar{k}$ there is an indecomposable locally free sheaf $\mathcal{W}$, such that $\mathcal{E} \otimes_k \bar{k} \simeq \bigoplus_{i=0}^n \mathcal{W}$. The sheaf $\mathcal{W}$ is called the *type* of the absolutely isotypical sheaf. If the indecomposable locally free sheaf $\mathcal{W}$ is invertible, we say that $\mathcal{E}$ is *absolutely rank-one-isotypical.*

We give a first example of such an absolutely isotypical sheaf that later on became important for further considerations.

Example 1.30. Let X be a n-dimensional Brauer–Severi variety over k. We consider the Euler sequence on $X \otimes_k \bar{k} \simeq \mathbb{P}^n$ (see [82], Theorem 8.13):

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0. \tag{1.3}$$

This short exact sequence does not split since $\mathcal{O}_{\mathbb{P}^n}(-1)$ has no global sections. Applying the the functor $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}, -)$ to this short exact sequence yields $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^n}, \Omega^1_{\mathbb{P}^n}) \simeq \bar{k}$ and hence the middle term of the Euler sequence is unique up to isomorphism. Furthermore, since the sheaves $\mathcal{O}_X$ and $\Omega^1_{X/k}$ exist on X and $\mathrm{Ext}^1(\mathcal{O}_X, \Omega^1_{X/k}) = k$, there is also a non-split short exact sequence on X

$$0 \longrightarrow \Omega^1_{X/k} \longrightarrow \mathcal{V} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

where the locally free sheaf $\mathcal{V}$ is unique up to isomorphism. After base change to $\bar{k}$ one gets back the sequence (1.3) for the projective space $\mathbb{P}^n$ and therefore $\mathcal{V} \otimes_k \bar{k} \simeq \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)}$. Thus the locally free sheaf $\mathcal{V}$ is absolutely isotypical of type $\mathcal{O}_{\mathbb{P}^n}(-1)$. Furthermore, we have $(\mathcal{V} \otimes \bar{k}) \otimes \mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}$ and hence $\mathbb{P}(\mathcal{V} \otimes \bar{k}) \simeq \mathbb{P}(\mathcal{V} \otimes \bar{k} \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}) \simeq \mathbb{P}^n \times (X \otimes \bar{k})$.

The locally free sheaf $\mathcal{V}$ from Example 1.30 has an interesting property. To illustrate this, we consider the endomorphism algebra $\mathrm{End}(\mathcal{V})$, that is a finite-dimensional associative k-algebra. After base change to the algebraic closure $\bar{k}$ we find $\mathrm{End}(\mathcal{V}) \otimes_k \bar{k} \simeq \mathrm{End}(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)}) \simeq \mathrm{End}(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}) \simeq M_{n+1}(\bar{k})$. By Theorem 1.4 we conclude that $\mathrm{End}(\mathcal{V})$ is a central simple k-algebra. More generally, we make the following observation:

Proposition 1.31. *Let X be a geometrically integral and proper k-scheme. Suppose $\mathcal{E}$ is absolutely rank-one-isotypical. Then* $\mathrm{End}(\mathcal{E})$ *is a central simple k-algebra.*

Proof. By definition there exists an invertible sheaf $\mathcal{L}$ on $X \otimes_k \bar{k}$ such that $\mathcal{E} \otimes_k \bar{k} \simeq \bigoplus_{i=0}^n \mathcal{L}$. Writing $X_{\bar{k}}$ for the scheme $X \otimes_k \bar{k}$, we conclude $\mathcal{E} \otimes_k \bar{k} \otimes \mathcal{L}^\vee \simeq \mathcal{O}_{X_{\bar{k}}}^{\oplus(n+1)}$. Since X is supposed to be proper, the endomorphism algebra $\mathrm{End}(\mathcal{E})$ is a finite-dimensional k-algebra. Furthermore, since X is geometrically integer, we have $\mathrm{End}(\mathcal{O}_{X_{\bar{k}}}) \simeq H^0(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}}) \simeq \bar{k}$, from what we finally conclude

$$\begin{aligned}
\mathrm{End}(\mathcal{E}) \otimes_k \bar{k} &\simeq \mathrm{End}(\bigoplus_{i=0}^n \mathcal{L}) \\
&\simeq \mathrm{Hom}(\mathcal{L} \otimes (\bigoplus_{i=0}^n \mathcal{O}_{X_{\bar{k}}}), \mathcal{L} \otimes (\bigoplus_{i=0}^n \mathcal{O}_{X_{\bar{k}}})) \\
&\simeq \mathrm{Hom}((\bigoplus_{i=0}^n \mathcal{O}_{X_{\bar{k}}}), \mathcal{L}^\vee \otimes \mathcal{L} \otimes (\bigoplus_{i=0}^n \mathcal{O}_{X_{\bar{k}}})) \\
&\simeq \mathrm{End}(\mathcal{O}_{X_{\bar{k}}}^{\oplus(n+1)}) \\
&\simeq M_{n+1}(\bar{k}).
\end{aligned}$$

Now Theorem 1.4 implies that $\mathrm{End}(\mathcal{E})$ has to be a central simple k-algebra. □

We note that the rank of such an absolutely rank-one-isotypical sheaf on a scheme X as in Proposition 1.31 is equal to the degree of the central simple k-algebra $\mathrm{End}(\mathcal{E})$. Indeed, the proof above shows that

$$\mathrm{rk}(\mathcal{E}) = n + 1 = \deg(\mathrm{End}(\mathcal{E})). \tag{1.4}$$

Now we want to investigate what happens if the absolutely rank-one-isotypical sheaf in question is indecomposable.

Proposition 1.32. *Let X be a geometrically integral and proper k-scheme and $\mathcal{E}$ an indecomposable absolutely rank-one-isotypical sheaf. Then* $\mathrm{End}(\mathcal{E})$ *is a central division algebra over k.*

Proof. According to [9], p.1324, on a proper k-scheme X a locally free sheaf $\mathcal{E}$ of finite rank is indecomposable if and only if $\mathrm{End}(\mathcal{E})/\mathrm{rad}(\mathrm{End}(\mathcal{E}))$ is a division algebra over k. Since $\mathrm{End}(\mathcal{E})$ is a central simple k-algebra according to Proposition 1.31, we conclude with Proposition 1.28 and the comments afterwards that $\mathrm{rad}(\mathrm{End}(\mathcal{E})) = 0$. Hence $\mathrm{End}(\mathcal{E})$ is a central division algebra. □

Now suppose we are given a geometrically integral and proper k-scheme X and an absolutely rank-one-isotypical sheaf $\mathcal{E}$. Suppose furthermore, we are given an indecomposable direct summand $\mathcal{F}$ of $\mathcal{E}$. We want to understand the relation between the central simple k-algebras $\mathrm{End}(\mathcal{E})$ and $\mathrm{End}(\mathcal{F})$. For this, we first cite a very useful result that we need later on need regularly for our further investigations. Recall that in the category of coherent sheaves on X an object $\mathcal{F}$ is called *indecomposable* if $\mathcal{F} \simeq \mathcal{G} \oplus \mathcal{H}$ implies $\mathcal{G} \simeq 0$ or $\mathcal{H} \simeq 0$. The next result is due to Atiyah [13], Theorem 2 and states the following:

Theorem 1.33. *Let* $\mathrm{coh}(X)$ *be the category of coherent sheaves for some proper* k*-scheme* X*. Then every non-zero object* $\mathcal{F}$ *has a decomposition as a direct sum of indecomposable objects and if*

$$\mathcal{F} = \bigoplus_{i=1}^{n} \mathcal{F}_i \quad \text{and} \quad \mathcal{F} = \bigoplus_{i=1}^{m} \mathcal{G}_i$$

are two such decompositions, then $n = m$ *and the indecomposable direct summands are unique up to isomorphism and permutation. Shortly we say that the* Krull–Schmidt Theorem *holds for the category of coherent sheaves on* X*.*

We note that the above theorem is derived from a more general statement that states, that in an exact category where a certain technical assumption, called the bi-chain condition, is satisfied, the Krull–Schmidt Theorem holds. This technical assumption is fulfilled whenever the exact category $\mathcal{C}$ is k-linear and $\mathrm{Hom}(A, B)$ is a finite dimensional k-vector space for all $A, B \in \mathcal{C}$ (see [13], Corollary to Lemma 3). The above theorem especially applies in the case when X is a proper k-scheme and we consider the class of locally free sheaves of finite rank. Since a locally free sheaf $\mathcal{E}$ is coherent, we can decompose it according to the above theorem. Furthermore, since every direct summand of $\mathcal{E}$ is again locally free (see [13], Lemma 9), one obtains that in the class of locally free sheaves of finite rank on X the Krull–Schmidt Theorem holds. Furthermore, we want to note that in the case of coherent sheaves the theorem become false if one removes the properness condition. Take for example a Dedekind ring R that is not a principal ideal domain and consider the affine curve $\mathrm{Spec}(R)$. For two fractional ideals $\mathfrak{A}$ and $\mathfrak{B}$, one has $\mathfrak{A} \oplus \mathfrak{B} = R \oplus \mathfrak{A}\mathfrak{B}$, where both sides are considered as R-modules. Thus, if $\mathfrak{A}$ represents an element of order 2 in the class group of R, we have $\mathfrak{A} \oplus \mathfrak{A} = R \oplus \mathfrak{A}\mathfrak{A} \simeq R \oplus R$. Here both, R and $\mathfrak{A}$ are indecomposable R-modules and since $\mathfrak{A}$ is not principal, one has $\mathfrak{A} \neq R$. Hence there are two different ways of decomposing the free module $R^{\oplus 2}$ and the Krull–Schmidt Theorem fails to hold for the free sheaf $\widetilde{R}^{\oplus 2}$ on $\mathrm{Spec}(R)$.

To prove the next result we first need a lemma that is proved in detail in [31], Lemma 3.4. Later (see Proposition 2.5 below) we will show that this result holds without the restriction of k being infinite. Furthermore it is enough to assume that X is proper over k.

Lemma 1.34. *Let* X *be a projective and geometrically integral scheme over an infinite field* k *and let* $\mathcal{E}$ *and* $\mathcal{E}'$ *be two locally free sheaves of finite rank over* X*. If* $\mathcal{E} \otimes_k L \simeq \mathcal{E}' \otimes_k L$ *for a field extension* $k \subset L$*, then they are already isomorphic over* X*.*

In the next chapter we will give two different proofs for the above lemma. The first uses Galois cohomology and the second is a direct one. In fact the assumption on k being infinite is not needed. Furthermore, we want to note that for our prospective investigation the assumption on the field being infinite is not a restriction, since we will be interested in non-trivial Brauer–Severi varieties. Note that over finite fields the only Brauer–Severi variety is the projective space (see Example 1.13) and the indecomposable absolutely rank-one-isotypical sheaves are just the invertible sheaves. Now as a consequence of the above lemma (or Proposition 2.5) we observe the following:

Proposition 1.35. *Let X be a projective and geometrically integral k-scheme over an infinite field k and $\mathcal{E}$ and $\mathcal{E}'$ two indecomposable absolutely rank-one-isotypical sheaves of the same type, then $\mathcal{E} \simeq \mathcal{E}'$.*

Proof. Since $\mathcal{E}$ and $\mathcal{E}'$ are two absolutely rank-one-isotypical of the same type, there is a locally free sheaf $\mathcal{L}$, such that we have $\mathcal{E} \otimes_k \bar{k} \simeq \bigoplus_{i=1}^r \mathcal{L}$ and $\mathcal{E}' \otimes_k \bar{k} \simeq \bigoplus_{i=1}^s \mathcal{L}$. Hence $(\mathcal{E}^{\oplus s}) \otimes_k \bar{k} \simeq (\mathcal{E}'^{\oplus r}) \otimes_k \bar{k}$. With Lemma 1.34 (or Proposition 2.5) we obtain $\mathcal{E}^{\oplus s} \simeq \mathcal{E}'^{\oplus r}$. Since the Krull–Schmidt Theorem holds for locally free sheaves on X, we finally get that $\mathcal{E} \simeq \mathcal{E}'$. □

With the above proposition, we are now able to understand the relation between the central simple k-algebras $\mathrm{End}(\mathcal{E})$ and $\mathrm{End}(\mathcal{F})$.

Proposition 1.36. *Let k be an infinite field and X a geometrically integral and projective k-scheme. Let $\mathcal{E}$ be an indecomposable absolutely rank-one-isotypical sheaf and $\mathcal{F}$ an indecomposable direct summand of $\mathcal{E}$. Then one has $\mathrm{End}(\mathcal{E}) \simeq M_n(\mathrm{End}(\mathcal{F}))$ and hence both endomorphism algebras are Brauer-equivalent.*

Proof. Since X is geometrically integral and proper, Proposition 1.31 yields that $\mathrm{End}(\mathcal{E})$ is a central simple k-algebra. Furthermore, since X is geometrically integral and proper, Theorem 1.33 applies and we can decompose $\mathcal{E}$ as a direct sum of indecomposable locally free sheaves. Now let $\mathcal{E} = \bigoplus_{i=1}^n \mathcal{E}_i$ be the Krull–Schmidt decomposition of $\mathcal{E}$. Since $\mathcal{E}$ is absolutely rank-one-isotypical, there is an invertible sheaf $\mathcal{L}$ such that $\mathcal{E} \otimes_k \bar{k} \simeq \bigoplus_{j=1}^r \mathcal{L}$. Together with $\mathcal{E} \otimes_k \bar{k} \simeq \bigoplus_{i=1}^n (\mathcal{E}_i \otimes_k \bar{k})$ we have an isomorphism

$$\bigoplus_{j=1}^r \mathcal{L} \simeq \mathcal{E} \otimes_k \bar{k} \simeq \bigoplus_{i=1}^n (\mathcal{E}_i \otimes_k \bar{k})$$

and hence, by applying Theorem 1.33 for locally free sheaves on $X \otimes_k \bar{k}$, we obtain that $\mathcal{E}_i$ is also absolutely rank-one-isotypical of type $\mathcal{L}$. Since all the locally free sheaves $\mathcal{E}_i$ are indecomposable, Proposition 1.35 yields that they are all isomorphic. Hence $\mathcal{E} \simeq \bigoplus_{i=1}^n \mathcal{E}_1$ and thus $\mathcal{F} \simeq \mathcal{E}_1$ by Krull–Schmidt. By Proposition 1.32, the endomorphism algebra $\mathrm{End}(\mathcal{F})$ is a central division algebra and hence $\mathrm{End}(\mathcal{E}) \simeq \mathrm{End}(\bigoplus_{i=1}^n \mathcal{F}) \simeq M_n(\mathrm{End}(\mathcal{F}))$ what furthermore implies that they are Brauer-equivalent. □

In the proof of the above proposition we observed the following fact.

Proposition 1.37. *Let k be an infinite field and X a projective k-scheme and $\mathcal{E}$ be a absolutely rank-one-isotypical sheaf. Then all the indecomposable direct summands in the Krull–Schmidt decomposition of $\mathcal{E}$ are isomorphic and thus we have $\mathcal{E} \simeq \mathcal{G}^{\oplus n}$ for a up to isomorphism unique locally free sheaf $\mathcal{G}$.*

Proof. We denote by r the rank of $\mathcal{E}$. Since the locally free sheaf $\mathcal{E}$ is absolutely rank-one-isotypical, there is an invertible sheaf $\mathcal{L}$ on $X \otimes_k \bar{k}$ such that $\mathcal{E} \otimes_k \bar{k} \simeq \mathcal{L}^{\oplus r}$. Now considering the Krull–Schmidt decomposition of $\mathcal{E}$ we have $\mathcal{E} \simeq$

$\bigoplus_{i=1}^n \mathcal{G}_i$ for indecomposable locally free sheaves $\mathcal{G}_i$. After base change to the algebraic closure $\bar{k}$ we get

$$\mathcal{E} \otimes_k \bar{k} \simeq \bigoplus_{i=1}^{n} \mathcal{G}_i \otimes_k \bar{k} \simeq \mathcal{L}^{\oplus r}.$$

Applying Krull–Schmidt Theorem on $X \otimes_k \bar{k}$ for $\bigoplus_{i=1}^n \mathcal{G}_i \otimes_k \bar{k}$ yields that all the $\mathcal{G}_i$ are absolutely rank-one-isotypical of the same type $\mathcal{L}$. With Proposition 1.35 we conclude that all summands $\mathcal{G}_i$ must be isomorphic and hence $\mathcal{E} \simeq \mathcal{G}^{\oplus n}$, where $\mathcal{G} \simeq \mathcal{G}_i$ is the up to isomorphism unique locally free sheaf. □

Remark 1.38. At this point we want to mention that taking Proposition 2.5 of the next section into account, Proposition 1.35, 1.36 and 1.37 hold for arbitrary proper k-schemes X with $H^0(X, \mathcal{O}_X) = k$ (or more generally for geometrically integral proper k-schemes) and one in fact does not have to make the assumption on k being infinite. The proves are exactly the same with the difference of applying Proposition 2.5 instead of Lemma 1.34.

We now want to study the absolutely rank-one-isotypical sheaves on non-trivial Brauer–Severi varieties. Note that the base field k in this case has to be infinite, otherwise the Brauer–Severi variety is isomorphic to the projective space and all indecomposable absolutely rank-one-isotypical sheaves are all of the elements in the Picard group. Clearly, since a Brauer–Severi variety X over k is projective, the Krull–Schmidt Theorem holds for locally free sheaves on X. In the exact sequence for a n-dimensional Brauer–Severi variety X of Example 1.30

$$0 \longrightarrow \Omega^1_{X/k} \longrightarrow \mathcal{V} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

we have seen that $\pi^*\mathcal{V} \simeq \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)}$, where π is the projection $\pi : X \otimes_k \bar{k} \to X$. Hence $\pi^*\mathcal{V}^\vee$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)}$ and thus absolutely rank-one-isotypical. We want to denote the sheaf $\mathcal{V}^\vee$ by $\mathcal{E}$. Since in the category of locally free sheaves on a Brauer–Severi variety X the Krull–Schmidt Theorem holds, we can decompose $\mathcal{E}$ as a direct sum of indecomposable locally free sheaves. Thus we have

$$\mathcal{E} \simeq \bigoplus_{i=1}^{m} \mathcal{E}_i.$$

By Proposition 1.37 all the $\mathcal{E}_i$ are isomorphic and thus we have $\mathcal{E} \simeq \mathcal{E}_1^{\oplus m}$, where $\mathcal{E}_1$ is unique up to isomorphism. If we take the locally free sheaf $\mathcal{E}$ and consider the tensor power for some $j > 1$ we get the locally free sheaf $\mathcal{E}^{\otimes j}$ on X. Again, since the Krull–Schmidt Theorem holds, we can decompose this sheaf into a direct sum of indecomposables. Since $\pi^*\mathcal{E}^{\otimes j}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(j)^{\oplus d^j}$, where $d = \mathrm{rk}(\mathcal{E})$, we conclude that $\mathcal{E}^{\otimes j}$ is absolutely rank-one-isotypical of type $\mathcal{O}_{\mathbb{P}^n}(j)$. Again with Proposition 1.37 we obtain that all the direct summands are isomorphic. Hence we have $\mathcal{E}^{\otimes j} \simeq \mathcal{E}_j^{\oplus m_j}$, for some indecomposable $\mathcal{E}_j$ that is unique up to isomorphism. In the case the Brauer–Severi variety is the projective space, the locally free sheaf $\mathcal{V}$ from above is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)}$ and the above sheaf $\mathcal{E}_1$ becomes $\mathcal{O}_{\mathbb{P}^n}(1)$. Hence the sheaves $\mathcal{E}_j$ are simply $\mathcal{O}_{\mathbb{P}^n}(j)$ for all $j \in \mathbb{Z}$. Keeping this in mind we give the following definition:

Definition 1.39. Let X be a Brauer–Severi variety over k. For all $j \in \mathbb{Z}$ we define $\mathcal{W}_j$ to be the indecomposable locally free sheaf $\mathcal{E}_j$ from above if $j > 0$, $\mathcal{E}^{\vee}_{|j|}$ if $j < 0$ and $\mathcal{O}_X$ if $j = 0$.

We summarize the above discussion in the following proposition.

Proposition 1.40. *Let X be a n-dimensional Brauer–Severi variety over k and $\pi : X \otimes_k \bar{k} \to X$ the projection. Then the locally free sheaves $\mathcal{W}_j$ of Definition 1.39 are indecomposable absolutely rank-one-isotypical and one has*

$$\pi^*\mathcal{W}_j \simeq \mathcal{O}_{\mathbb{P}^n}(j)^{\oplus \mathrm{rk}(\mathcal{W}_j)}.$$

We end up this section in stating that the locally free sheaves $\mathcal{W}_j$ are the up to isomorphism only possible indecomposable absolutely rank-one-isotypical sheaves.

Proposition 1.41. *Let X be a n-dimensional Brauer–Severi variety over k. Let $\mathcal{O}_{\mathbb{P}^n}(m)$ be an invertible sheaf on $X \otimes_k \bar{k}$. Then, up to isomorphism, the only indecomposable absolutely rank-one-isotypical sheaf of type $\mathcal{O}_{\mathbb{P}^n}(m)$ is $\mathcal{W}_m$.*

Proof. Since $\mathcal{W}_m \otimes_k \bar{k} \simeq \mathcal{O}_{\mathbb{P}^n}(m)^{\oplus \mathrm{rk}(\mathcal{W}_m)}$, we conclude by Proposition 1.35 that this $\mathcal{W}_m$ is unique up to isomorphism and hence the only indecomposable absolutely rank-one-isotypical sheaf of type $\mathcal{O}_{\mathbb{P}^n}(m)$. □

1.3 Classification of AS-bundles

In this section we introduce absolutely split locally free sheaves on a scheme X. Moreover, we want to classify them for Brauer–Severi varieties. In the next chapter we will see how the ideas presented in this section naturally generalize to arbitrary proper k-schemes. Furthermore, we define the AS-type of a Brauer–Severi variety and determine it in the next section.

Definition 1.42. Let X be a k-scheme. A locally free sheaf $\mathcal{E}$ of finite rank on X is called *absolutely split* if it splits after base change as a direct sum of invertible sheaves on $X \otimes_k \bar{k}$. For an absolutely split locally free sheaf we shortly write *AS-bundle*.

For a given k-scheme X we can formulate the following problem:

Problem. *Classifiy all indecomposable AS-bundles on the scheme X.*

In order to give a solution to the problem in the case X is a Brauer–Severi variety, we start with an observation. We remind that the Picard group of a Brauer–Severi variety X is cyclic and generated by $\mathcal{O}_X(p)$, where p is the period of X. Furthermore, one has $\mathcal{O}_X(ap) \otimes_k L \simeq \mathcal{O}_{\mathbb{P}^n}(ap)$ for all splitting fields L of X.

Proposition 1.43. *Let X be a n-dimensional Brauer–Severi variety over k of period p. Then $\mathcal{W}_{ap} \simeq \mathcal{O}_X(ap)$ for all $a \in \mathbb{Z}$.*

Proof. By Proposition 1.40 we have $\mathcal{W}_{ap} \otimes_k \bar{k} \simeq \mathcal{O}_{\mathbb{P}^n}(ap)^{\oplus \mathrm{rk}(\mathcal{W}_{ap})}$. Since the Picard group $\mathrm{Pic}(X)$ is cyclic and generated by $\mathcal{O}_X(p)$, we conclude that

$$(\mathcal{O}_X(ap)^{\oplus \mathrm{rk}(\mathcal{W}_{ap})}) \otimes_k \bar{k} \simeq \mathcal{O}_{\mathbb{P}^n}(ap)^{\oplus \mathrm{rk}(\mathcal{W}_{ap})}.$$

In the case the field k is infinite, by Lemma 1.34 (or Proposition 2.5) we obtain $\mathcal{W}_{ap} \simeq \mathcal{O}_X(ap)^{\oplus \mathrm{rk}(\mathcal{W}_{ap})}$. Since $\mathcal{W}_{ap}$ is by definition indecomposable, the Krull–Schmidt Theorem yields $\mathcal{W}_{ap} \simeq \mathcal{O}_X(ap)$. In the case the field k is finite, we have that X is isomorphic to $\mathbb{P}^n$ and hence $\mathcal{W}_j \simeq \mathcal{O}_{\mathbb{P}^n}(j)$ for all $j \in \mathbb{Z}$ by definition. This completes the proof. □

The last proposition shows that in the set of the locally free sheaves $\mathcal{W}_j$ one has a periodicity relative to the period of the Brauer–Severi variety. Furthermore, there is also a symmetry concerning the ranks of $\mathcal{W}_j$ as will be explained in the next proposition.

Proposition 1.44. *Let X be a n-dimensional Brauer–Severi variety over k of period p. Then for all $a, j \in \mathbb{Z}$ the locally free sheaves $\mathcal{W}_j$ have the following properties*

- **(i)** $\mathrm{rk}(\mathcal{W}_j) = \mathrm{rk}(\mathcal{W}_{-j})$.
- **(ii)** $\mathrm{rk}(\mathcal{W}_{j+ap}) = \mathrm{rk}(\mathcal{W}_j)$.

Proof. In the case the field k is finite, X is isomorphic to $\mathbb{P}^n$ and $\mathcal{W}_j \simeq \mathcal{O}_{\mathbb{P}^n}(j)$ by definition. Hence the ranks of $\mathcal{W}_j$ have the desired property. For infinite fields k, the first property is clear since $\mathcal{W}_{-j}$ is by definition $\mathcal{W}_j^{\vee}$. The second property can be seen as follows: The locally free sheaf $\mathcal{W}_{j+ap}$ is according to Proposition 1.40 absolutely rank-one-isotypical of type $\mathcal{O}_{\mathbb{P}^n}(j+ap)$. Furthermore, the locally free sheaf $\mathcal{W}_j \otimes \mathcal{O}_X(ap)$ is also absolutely rank-one-isotypical of type $\mathcal{O}_{\mathbb{P}^n}(j+ap)$ and indecomposable. Applying Proposition 1.35 yields that $\mathcal{W}_{j+ap} \simeq \mathcal{W}_j \otimes \mathcal{O}_X(ap)$. This implies that $\mathrm{rk}(\mathcal{W}_{j+ap}) = \mathrm{rk}(\mathcal{W}_j)$. □

This motivates the following consideration: Taking a Brauer–Severi variety X of period p, we can twist the locally free sheaves $\mathcal{W}_j$ with $\mathcal{O}_X(ap)$, where a is an arbitrary integer. The obtained locally free sheaves $\mathcal{W}_j \otimes \mathcal{O}_X(ap)$ are again indecomposable absolutely rank-one-isotypical and hence AS-bundles. Furthermore they are isomorphic to $\mathcal{W}_{j+ap}$. Therefore, to obtain a classification of indecomposable AS-bundles it is enough to consider only a certain subset of all $\mathcal{W}_j$. To show this is the content of the next result that also classifies all indecomposable AS-bundles on Brauer–Severi varieties.

Theorem 1.45. *Let X be a n-dimensional Brauer–Severi variety over a field k and of period p. Then all indecomposable AS-bundles are up to isomorphism of the form*

$$\mathcal{W}_j \otimes \mathcal{O}_X(ap),$$

with unique $a \in \mathbb{Z}$ and unique $0 \le j \le p-1$.

Proof. We start with the case $X = \mathbb{P}^n$. In this case the period p of X is one and $\mathcal{W}_0$ is by definition $\mathcal{O}_X$. Obviously, all indecomposable AS-bundles are of the form $\mathcal{O}_X \otimes \mathcal{O}_X(a \cdot 1)$. Now we consider the non-split case. For this, let $\mathcal{E}$ be an arbitrary, not necessarily indecomposable AS-bundle and $\pi : X \otimes_k \bar{k} \to X$ the projection. Now consider the locally free sheaves $\mathcal{W}_j$, for $j = 0, ..., p-1$. We set $d = \mathrm{lcm}(\mathrm{rk}(\mathcal{W}_0), \mathrm{rk}(\mathcal{W}_1), ..., \mathrm{rk}(\mathcal{W}_{p-1}))$ the least common multiple and consider the bundle $\pi^*(\mathcal{E}^{\oplus d})$. Since $\mathcal{E}$ is an AS-bundle, the direct sum $\mathcal{E}^{\oplus d}$ is an AS-bundle too. Hence we can decompose $\pi^*(\mathcal{E}^{\oplus d})$ into a direct sum of invertible sheaves and find that, after reordering mod p, the sheaf $\pi^*(\mathcal{E}^{\oplus d})$ is isomorphic to

$$\left(\bigoplus_{i=0}^{r_0} \mathcal{O}_{\mathbb{P}^n}(a_{i_0}p)^{\oplus d}\right) \oplus \left(\bigoplus_{i=0}^{r_1} \mathcal{O}_{\mathbb{P}^n}(a_{i_1}p+1)^{\oplus d}\right) \oplus \dots$$
$$\oplus \left(\bigoplus_{i=0}^{r_{p-1}} \mathcal{O}_{\mathbb{P}^n}(a_{i_{p-1}}p+(p-1))^{\oplus d}\right).$$

By definition of d, there are h_j such that $h_j \cdot \mathrm{rk}(\mathcal{W}_j) = d$, for $0 \leq j \leq p-1$. Furthermore, by Proposition 1.40 the locally free sheaves $\mathcal{W}_j$ have the property that $\pi^*\mathcal{W}_j \simeq \mathcal{O}_{\mathbb{P}^n}(j)^{\oplus d_j}$, where $d_j = \mathrm{rk}(\mathcal{W}_j)$. Since X is non-split the field k is infinite and applying Lemma 1.34 yields that $\mathcal{E}^{\oplus d}$ on X is of the form

$$\left(\bigoplus_{i=0}^{r_0} \mathcal{O}_X(a_{i_0}p)^{\oplus d}\right) \oplus \left(\bigoplus_{i=0}^{r_1} \mathcal{O}_X(a_{i_1}p) \otimes \mathcal{W}_1^{\oplus h_1}\right) \oplus \dots$$
$$\oplus \left(\bigoplus_{i=0}^{r_{p-1}} \mathcal{O}_X(a_{i_{p-1}}p) \otimes \mathcal{W}_{p-1}^{\oplus h_{p-1}}\right),$$

since for the locally free sheaves $\mathcal{O}_X(a_{i_j}p) \otimes \mathcal{W}_j$ we have

$$\pi^*(\mathcal{O}_X(a_{i_j}p) \otimes \mathcal{W}_j) \simeq \mathcal{O}_{\mathbb{P}^n}(a_{i_j}p+j)^{\oplus d_j}$$

and hence

$$\pi^*(\mathcal{O}_X(a_{i_j}p) \otimes \mathcal{W}_j^{\oplus h_j}) \simeq \mathcal{O}_{\mathbb{P}^n}(a_{i_j}p+j)^{\oplus d_j \cdot h_j}.$$

And because the Krull–Schmidt Theorem holds for locally free sheaves on X, we conclude that $\mathcal{E}$ is isomorphic to the direct sum of these $\mathcal{W}_j \otimes \mathcal{O}_X(ap)$ for unique j and a. Furthermore, since all these bundles are indecomposable by Proposition 1.40, we finally get that all the indecomposable AS-bundles are of the form $\mathcal{W}_j \otimes \mathcal{O}_X(ap)$ with unique $a \in \mathbb{Z}$ and $0 \leq 1 \leq p-1$. This completes the proof. □

As an immediate consequence of the above classification we obtain:

Corollary 1.46. *Let X be a Brauer–Severi variety over a field k of period p. Then all AS-bundles $\mathcal{E}$ are of the form*

$$\mathcal{E} \simeq \bigoplus_{j=0}^{p-1}\left(\bigoplus_{i=0}^{r_j} \mathcal{W}_j \otimes \mathcal{O}_X(a_{i_j}p)\right),$$

with unique $a_{i_j} \in \mathbb{Z}$ and $r_j > 0$, where $0 \leq j \leq p-1$.

To have a complete understanding of the AS-bundles on Brauer–Severi varieties one has to determine the ranks of the locally free sheaves $\mathcal{W}_j$. This leads us to consider the sequence of natural numbers $(d_j)_{j\in\mathbb{Z}}$, with $d_j = \mathrm{rk}(\mathcal{W}_j)$. Proposition 1.43 and 1.44 show that we do not have to consider the hole sequence $(d_j)_{j\in\mathbb{Z}}$. Furthermore, we note that $\mathcal{W}_0 = \mathcal{O}_X$ and $\mathcal{W}_p = \mathcal{O}_X(p)$, where p is the period of X. This implies that $\mathrm{rk}(\mathcal{W}_0) = 1 = \mathrm{rk}(\mathcal{W}_p)$. Keeping this mind we give the following definition:

Definition 1.47. Let X be a Brauer–Severi variety over k of period p. We call the $p+1$-tuple $(1, d_1, \dots, d_{p-1}, 1)$, with $d_j = \mathrm{rk}(\mathcal{W}_j)$ for $j = 1, \dots, p-1$, the *AS-type* of X.

Thus, to have a complete understanding of the indecomposable AS-bundles on Brauer–Severi varieties one has to determine the AS-type. Note that if X is isomorphic to $\mathbb{P}^n$, the AS-type is $(1,1)$. Before we examine more closely the sheaves $\mathcal{W}_j$ in the next section, we first want to give a criterion for a locally free sheaf to be an AS-bundle. It is an application of the Horrocks criterion that can be found in [128], Theorem 2.3.2. We recall the *Horrocks criterion.*

Theorem 1.48. *Let k be an algebraically closed field and $\mathcal{E}$ a locally free sheaf over $\mathbb{P}^n_k$ of finite rank. Then $\mathcal{E}$ is the direct sum of invertible sheaves if and only if $H^i(\mathbb{P}^n, \mathcal{E}(l)) = 0$, for every $l \in \mathbb{Z}$ and $0 < i < n$.*

Lemma 1.49. *Let X be a Brauer–Severi variety over k and $\mathcal{W}_j$ the locally free sheaves of Definition 1.39. Then for all integers j and l the locally free sheaves $\mathcal{W}_j \otimes \mathcal{W}_l$ are AS-bundles.*

Proof. Let $\pi : X \otimes_k \bar{k} \to X$ the projection. Since $\pi^*(\mathcal{W}_j \otimes \mathcal{W}_l) \simeq \pi^*\mathcal{W}_j \otimes \pi^*\mathcal{W}_l$, we have

$$\pi^*(\mathcal{W}_j \otimes \mathcal{W}_l) \simeq (\mathcal{O}_{\mathbb{P}^n}(j)^{\oplus d_j}) \otimes (\mathcal{O}_{\mathbb{P}^n}(l)^{\oplus d_l})$$

where $d_j = \mathrm{rk}(\mathcal{W}_j)$ and $d_l = \mathrm{rk}(\mathcal{W}_l)$. Thus we find

$$\pi^*(\mathcal{W}_j \otimes \mathcal{W}_l) \simeq \mathcal{O}_{\mathbb{P}^n}(j+l)^{\oplus (d_j \cdot d_l)}$$

what implies that $\mathcal{W}_j \otimes \mathcal{W}_l$ is an absolutely rank-one-isotypical sheaf of type $\mathcal{O}_{\mathbb{P}^n}(j+l)$ and hence an AS-bundle. □

Theorem 1.50. *(AS-criterion) Let X be a n-dimensional Brauer–Severi variety over k and period p. A locally free sheaf $\mathcal{E}$ of finite rank is an AS-bundle if and only if for $0 < i < n$ one has*

$$H^i(X, \mathcal{E} \otimes \mathcal{O}_X(ap) \otimes \mathcal{W}_j) = 0$$

for every $a \in \mathbb{Z}$ and every $0 \leq j \leq p-1$.

Proof. Suppose $\mathcal{E}$ is an AS-bundle. Without loss of generality we can assume that $\mathcal{E}$ is indecomposable. Then by Theorem 1.45 the sheaf $\mathcal{E}$ is of the form $\mathcal{O}_X(bp) \otimes \mathcal{W}_l$ with unique $b \in \mathbb{Z}$ and $0 \leq l \leq p-1$. Therefore, applying Lemma 1.49 and Theorem 1.45 we get that

$$\mathcal{E} \otimes \mathcal{O}_X(ap) \otimes \mathcal{W}_j \simeq \mathcal{O}_X((a+b)p) \otimes \mathcal{W}_l \otimes \mathcal{W}_j$$

is again an AS-bundle and hence splits as a direct sum of invertible sheaves after base change to the algebraic closure. We denote by d_j the rank of the locally free sheaves $\mathcal{W}_j$. From this we get with Lemma 1.49 and Horrocks criterion from above, for $0 < i < n$:

$$\begin{aligned} H^i(X, \mathcal{E} \otimes \mathcal{O}_X(ap) \otimes \mathcal{W}_j) \otimes_k \bar{k} &\simeq H^i(\mathbb{P}^n, \mathcal{E} \otimes_k \bar{k} \otimes \mathcal{O}_{\mathbb{P}^n}(ap) \otimes \mathcal{O}_{\mathbb{P}^n}(j)^{\oplus d_j}) \\ &\simeq H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}((a+b)p) \otimes \mathcal{O}_{\mathbb{P}^n}(j+l)^{\oplus d_j \cdot d_l}) \\ &= 0 \end{aligned}$$

for every $a \in \mathbb{Z}$ and every $0 \leq j \leq p-1$. Conversely, assume that the cohomology $H^i(X, \mathcal{E} \otimes \mathcal{O}_X(ap) \otimes \mathcal{W}_j)$ vanishes for every $a \in \mathbb{Z}$ and $0 \leq j \leq p-1$. Considering the locally free sheaf $\mathcal{E} \otimes \mathcal{O}_X(ap) \otimes \mathcal{W}_j$, we find after base change to the algebraic closure

$$(\mathcal{E} \otimes \mathcal{O}_X(ap) \otimes \mathcal{W}_j) \otimes_k \bar{k} \simeq \mathcal{E} \otimes_k \bar{k} \otimes \mathcal{O}_{\mathbb{P}^n}(a)^{\otimes p} \otimes \mathcal{O}_{\mathbb{P}^n}(j)^{\oplus d_j}.$$

Applying the Horrocks criterion we conclude that $\mathcal{E} \otimes_k \bar{k}$ splits as a direct sum of invertible sheaves and hence $\mathcal{E}$ is an AS-bundle. □

1.4 AS-type of Brauer–Severi varieties

In this section we want to determine the AS-type of Brauer–Severi varieties. Furthermore, we investigate the case of Brauer–Severi varieties corresponding to certain cyclic division algebras.

Considering the locally free sheaf $\mathcal{V}^\vee$ of Example 1.30 for the n-dimensional Brauer–Severi vartiety X, corresponding to the central simple k-algebra A, we notice that $\mathrm{End}(\mathcal{V}^\vee) \simeq A$ (see [133], p.144 or [154], §3, 3.6). Note that in the case $X \simeq \mathbb{P}^n$ one has $\mathcal{V} \simeq \mathcal{O}_X(-1)^{\oplus(n+1)}$ and hence $\mathrm{End}(\mathcal{V}^\vee) \simeq M_{n+1}(k)$. With this fact we can state the next proposition, that determines the AS-type of Brauer–Severi varieties.

Proposition 1.51. *Let X be a n-dimensional Brauer–Severi variety over a field k corresponding to a central simple k-algebra A. Then for every $j \in \mathbb{Z}$ one has*

$$\mathrm{rk}(\mathcal{W}_j) = \mathrm{ind}(A^{\otimes |j|}).$$

Proof. Note that in the case $X \simeq \mathbb{P}^n$ we have $\mathcal{W}_j \simeq \mathcal{O}_{\mathbb{P}^n}(j)$. Since the corresponding central simple k-algebra is $M_{n+1}(k)$, one trivially has $\mathrm{rk}(\mathcal{W}_j) = 1 = \mathrm{ind}(M_{n+1}(k)) = \mathrm{ind}(M_{n+1}(k)^{\otimes|j|})$ for all $j \in \mathbb{Z}$. Now we consider the case X is a non-trivial Brauer–Severi variety. As mentioned above, for the locally free sheaf $\mathcal{V}$ on X of Example 1.30 one has $\mathrm{End}(\mathcal{V}^\vee) \simeq A$ (see [133], p.144). By Proposition 1.40, $\mathcal{W}_1$ is indecomposable absolutely rank-one-isotypical of type $\mathcal{O}_{\mathbb{P}^n}(1)$. Hence by Proposition 1.32 we have that $\mathrm{End}(\mathcal{W}_1) = D$ is a central division algebra. By definition, $\mathcal{W}_1$ is an indecomposable direct summand of $\mathcal{E} = \mathcal{V}^\vee$ and since $\mathcal{E}$ is also absolutely rank-one-isotypical of type $\mathcal{O}_{\mathbb{P}^n}(1)$, we conclude with Proposition 1.36 that $A \simeq \mathrm{End}(\mathcal{V}^\vee) \simeq M_n(D)$. Now for $j > 1$ we have $\mathrm{End}(\mathcal{E}^{\otimes j}) \simeq A^{\otimes j}$. Since the locally free sheaves $\mathcal{W}_j$ are indecomposable absolutely rank-one-isotypical of type $\mathcal{O}_{\mathbb{P}^n}(j)$, Proposition 1.32 implies that $\mathrm{End}(\mathcal{W}_j) = D_j$ is a central division algebra. By definition, the sheaf $\mathcal{W}_j$ is an indecomposable direct summand of the locally free sheaf $\mathcal{E}^{\otimes j}$. Since the sheaf $\mathcal{E}^{\otimes j}$ is also absolutely rank-one-isotypical of type $\mathcal{O}_{\mathbb{P}^n}(j)$, we conclude with Proposition 1.36 that $\mathrm{End}(\mathcal{E}^{\otimes j}) \simeq A^{\otimes j} \simeq M_{m_j}(D_j)$. But since the rank of $\mathcal{W}_j$ is exactly the degree of the division algebra D_j by (1.4), what by definition is the index of $A^{\otimes j}$, we have shown that $\mathrm{rk}(\mathcal{W}_j) = \mathrm{ind}(A^{\otimes j})$. For $j < 0$ we defined $\mathcal{W}_j$ to be $\mathcal{W}_{-j}^\vee$ and hence $\mathrm{rk}(\mathcal{W}_j) = \mathrm{rk}(\mathcal{W}_{-j}^\vee) = \mathrm{rk}(\mathcal{W}_{|j|}^\vee) = \mathrm{rk}(\mathcal{W}_{|j|}) = \mathrm{ind}(A^{\otimes|j|})$. Finally, we have to consider the case $j = 0$, which of course is trivial since $\mathcal{W}_0 = \mathcal{O}_X$ and hence $\mathrm{rk}(\mathcal{O}_X) = 1 = \mathrm{ind}(k)$. This completes the proof. □

Combining the last proposition with Theorem 1.45 we are finally able to determine the AS-type of Brauer–Severi varieties.

Theorem 1.52. *Let X be a Brauer–Severi variety over k and A the corresponding central simple k-algebra of period p. Then the AS-type of X is $(1, d_1, d_2, ..., d_{p-1}, 1) = (\mathrm{ind}(A^{\otimes j}))_{0 \le j \le p}$.*

Theorem 1.52 together with Theorem 1.45 now gives a complete classification all AS-bundles on Brauer–Severi varieties and thus we get the results obtained by Biswas and Nagaraj [29], [30], [31] and by the author [125], [126] as corollaries.

Corollary 1.53. *([126], Theorem 5.1 and [30], Theorem 1.1) Let X be a n-dimensional Brauer–Severi variety over k of index two and $\pi : X \otimes_k \bar{k} \to X$ the projection. Then the AS-bundles are of the form*

$$\mathcal{E} \simeq \left(\bigoplus_{i=1}^{r} \mathcal{O}_X(2a_i)\right) \oplus \left(\bigoplus_{j=1}^{s} \mathcal{O}_X(2b_j) \otimes \mathcal{W}_1\right)$$

with unique r, s, a_i and b_j and $\pi^*\mathcal{W}_1 \simeq \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus 2}$.

Proof. Since the index of X is two and, by Remark 1.17, the period divides the index, we conclude that the period is also two. Note that the period cannot be one, since this would imply that X is the projective space what contradicts the fact that the index of X is two. Hence the AS-type of X is $(1,2,1)$ according to Theorem 1.52. Now the assertion follows from Corollary 1.46. □

As a special case of Corollary 1.53, or more generally of Theorem 1.52 one obtains:

Corollary 1.54. *(Biswas and Nagaraj [29], [31] and [125]) Let X be a non-split 1-dimensional Brauer–Severi variety over k and $\mathcal{E}$ a locally free sheaf of finite rank. Then $\mathcal{E}$ is of the form*

$$\mathcal{E} \simeq \left(\bigoplus_{i=1}^{r} \omega_X^{\otimes a_i}\right) \oplus \left(\bigoplus_{j=1}^{s} \omega_X^{\otimes b_j} \otimes \mathcal{W}_1\right)$$

with uniquely determined r, s, a_i and b_j and $\mathcal{W}_1 \otimes_k L \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, where $k \subset L$ is a degree two Galois extension that splits X.

Proof. Note that for a 1-dimensional non-trivial Brauer–Severi variety the degree of the corresponding central simple k-algebra is two according to Theorem 1.10. Since X is assumed to be non-split, the period of X is two and $\mathcal{O}_X(2) \simeq \omega_X$ exists. Then the assertion follows from Theorem 1.52, Corollary 1.46 and Remark 1.23. □

In the case that the Brauer–Severi variety corresponds to a central simple k-algebra with period equals the index, the AS-type can be determined very explicitly. To illustrate this we first state a well-known fact from group theory.

Proposition 1.55. *Let G be a group and $a \in G$ an element of of finite order r, then $\mathrm{ord}(a^l) = r/(r,l)$.*

Proof. We first show that for an integer m one has $a^m = e$ if and only if r divides m. Of course if r divides m this is clear. Conversely, suppose $a^m = e$. Dividing m by r yields $m = qr + s$, where $q, s \in \mathbb{Z}$ and $0 \leq s < r$. But this implies that $e = a^m = a^{qr+s} = a^s$. But this contradicts the fact that r is the order of a. Hence $s = 0$ and r divides m. Now consider a^l and $r/(r,l)$. Clearly, since (r,l) is a factor of l, $l/(r,l)$ is an integer and hence $(a^l)^{r/(r,l)} = (a^r)^{l/(r,l)} = e$. In order to prove that the order of a^l is $r/(r,l)$, we only have to prove that $r/(r,l)$ divides all integers m such that $(a^l)^m = e$. Compare the integers $r/(r,l)$ and $l/(r,l)$

and divide out all common factors so that we suppose that they are coprime. Then, since $a^l m = e$, we have that that r divides lm and hence $rs = lm$. Now we consider the integers $(l/(r,l))m$ and $(r/(r,l))s$. Since $r/(r,l)$ and $l/(r,l)$ are assumed to be coprime, $r/(r,l)$ has to divide m. This completes the proof. □

Corollary 1.56. *Let A be a central simple k-algebra of period p. Then one has $p/(p,r) = \mathrm{per}(A^{\otimes r})$.*

Proof. By definition, the period of A is the order of A in $\mathrm{Br}(k)$. From Proposition 1.55 we get that the order of $A^{\otimes r}$ in $\mathrm{Br}(k)$ is exactly $p/(p,r)$. □

Proposition 1.57. *Let A be a central simple k-algebra with period p and index i. Then for all $r > 0$ one has that $p/(p,r)$ divides $\mathrm{ind}(A^{\otimes r})$ and $\mathrm{ind}(A^{\otimes r})$ divides $i/(i,r)$. In particular one has $p/(p,r) \leq \mathrm{ind}(A^{\otimes r}) \leq i/(i,r)$.*

Proof. By Corollary 1.56 we have $p/(p,r) = \mathrm{per}(A^{\otimes r})$ and since the period always divides the index, according to Lemma 1.18, we have that $p/(p,r)$ divides $\mathrm{ind}(A^{\otimes r})$. The second inequality $\mathrm{ind}(A^{\otimes r}) \leq i/(i,r)$ and the fact that $\mathrm{ind}(A^{\otimes r})$ divides $i/(i,r)$ is (iii) of Theorem 1.24. □

The last proposition now enables us to determine the *AS*-type of Brauer–Severi varieties in the case where the period equals the index.

Proposition 1.58. *Let X be a Brauer–Severi variety over k corresponding to a central simple k-algebra A such that the period p equals the index i. Then the AS-type of X is $(1, p, p/(p,2), ..., p/(p,p-2), p, 1)$.*

Proof. Since the period p equals the index i, we conclude with Proposition 1.57, that $p/(p,r) = \mathrm{ind}(A^{\otimes r})$, for $2 \leq r \leq p-2$. Hence by Theorem 1.52 the *AS*-type is $(1, p, p/(p,2), ..., p/(p,p-2), p, 1)$. □

Remark 1.59. The problem for which fields k the period equals the index is called *period-index problem*. For further discussion of this problem we refer to [15] and to the work of de Jong [61].

After determining the *AS*-type of a Brauer–Severi variety X with period p and index i a natural question came up. In view of Proposition 1.57 one can ask if for a prescribed $p+1$-tuple $(1, d_1, ..., d_{p-1}, 1)$ with the property that $p/(p,r)$ divides d_r and d_r divides $i/(i,r)$ there exists a Brauer–Severi variety over a field k such that the *AS*-type is exactly the prescribed $p+1$-tuple. It turns out that this is possible in the case where the prescribed $p+1$-tuple is of the form $(1, p, p/(p,2), ..., p/(p,p-2), p, 1)$ for an arbitrary natural number $p \geq 1$ (see Proposition 1.62 below). It is not yet clear to the author if this is possible for arbitrary $p+1$-tuple $(1, d_1, ..., d_{p-1}, 1)$ with the above described property.

We now investigate a special class of Brauer–Severi varieties and determine the *AS*-type. We start with the following definition that is contained in [71].

Definition 1.60. Let A be a central simple k-algebra of degree m containing a subalgebra K which is a cyclic Galois field extension $k \subset K$ of degree m. Then the central simple k-algebra A is called a *cyclic algebra*. If the central simple k-algebra A is a central division algebra it is called *cyclic division algebra*.

A cyclic algebra A can explicitly be described as follows: For the cyclic Galois field extension $k \subset K$ we fix $b \in k^*$ and $y \in A$ and consider the k-algebra generated by K and $y \in A$ subject to the relations $y^m = b$ and $\lambda y = y\sigma(\lambda)$, where σ is the generator of $G = \mathrm{Gal}(K|k)$. This finite-dimensional k-algebra is denoted by $B_{b,y}$. One can prove that for every cyclic algebra A, there exist $b \in k^*$ and $y \in A$ such that $A \simeq B_{b,y}$ (see [71], Proposition 2.5.3). A crucial fact is that the cyclic algebra A is split by the cyclic Galois extension K (see [71], Corollary 2.2.10). For some kind of fields every central division algebra is cyclic. We recall, a field is called *global* if it is a finite field extension of $\mathbb{Q}$ or $\mathbb{F}_p(t)$ and *local* if it is a finite extension of $\mathbb{R}$, $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ (see [142], p.194). The next result is contained in [139], Theorem 10.7.

Theorem 1.61. *Let k be a global or local field, then every central division algebra over k is cyclic and the period equals the index.*

With this theorem we make the following observation:

Proposition 1.62. *Let X be a minimal Brauer–Severi variety over a global or local field k of period p. Then the AS-type is $(1, p, p/(p,2), ..., p/(p, p-2), p, 1)$.*

Proof. Let A be the central simple k-algebra corresponding to X. Since X is minimal, the central simple k-algebra A is a central division algebra according to Proposition 1.12. Applying Theorem 1.61 yields that the period equals the index. Finally, Proposition 1.58 yields the assertion. □

Now we can give a partial answer to the question from above, asking for the existence of a Brauer–Severi variety of prescribed *AS*-type.

Proposition 1.63. *Let $r \geq 1$ be a natural number and consider the $r+1$-tupel $(1, r, r/(r,2), ..., r/(r, r-2), r, 1)$. Then there exists a field k and a Brauer–Severi variety over k of period r with AS-type being exactly $(1, r, r/(r,2), ..., r/(r, r-2), r, 1)$.*

Proof. According to Theorem 1.25, for natural numbers r and i with the property that r divides i and both have the same prime factors, there exists a field k and a Brauer–Severi variety over k with period r and index i. Especially for the case $r = i$ there is a field k and a Brauer–Severi variety X over k such that X has period and index equal to r. Now Proposition 1.58 yields the assertion. □

At the end of this section we discuss the relation of the *AS*-types between two Brauer-equivalent and birational Brauer–Severi varieties. We start with the following simple observation.

Proposition 1.64. *Let X and Y be two Brauer–Severi varieties over k that are Brauer-equivalent. Then X and Y have the same AS-type.*

Proof. Let A be the central simple k-algebra corresponding to X and B the central simple k-algebra corresponding to Y. Since they are Brauer-equivalent, there is a unique central division algebra D such that $A \simeq M_n(D)$ and $B \simeq M_m(D)$ for some n and m. Hence $\mathrm{ind}(A^{\otimes j}) = \mathrm{ind}(D^{\otimes j}) = \mathrm{ind}(B^{\otimes j})$ for all $j \in \mathbb{Z}$. Since Brauer-equivalent Brauer–Severi varieties have the same period, Theorem 1.52 yields the assertion. □

We note that Proposition 1.62 together with Proposition 1.64 implies that arbitrary (not necessarily minimal) Brauer–Severi varieties of period p over a local or global field have AS-type $(1, p, p/(p,2), ..., p/(p,p-2), p, 1)$. It is natural to ask if the converse of Proposition 1.63 holds. In what follows we give an answer to this question and investigate what happens in the case of two birational Brauer–Severi varieties. First we cite a fact that is contained in [71], Theorem 1.4.2.

Theorem 1.65. *Let X and Y be two non-split 1-dimensional Brauer–Severi varieties over k. Then X is birational to Y if and only if they are isomorphic.*

The next fact is a simple consequence of Theorem 1.52.

Proposition 1.66. *Every non-split 1-dimensional Brauer–Severi variety over k has the same AS-type.*

Proof. A non-split 1-dimensional Brauer–Severi variety over k is of type $(2,2,2)$. Since the period and the index of X is two, we conclude with Theorem 1.52 that the AS-type of X is $(1,2,1)$. This implies that all non split 1-dimensional Brauer–Severi varieties have the same AS-type. □

The last proposition together with Theorem 1.65 yields that it is possible that two Brauer–Severi varieties over k have the same AS-type even if they are not Brauer equivalent. Furthermore, we see that two 1-dimensional Brauer–Severi varieties have the same AS-type even if they are not birational. But still it is not clear if two birational Brauer–Severi varieties may have the same AS-type. The next result will clarify this.

Proposition 1.67. *Let X and Y be two birational Brauer–Severi varieties. Then they have the same AS-type.*

Proof. Let A be the central simple k-algebra corresponding to X and B that corresponding to Y. Since X and Y are supposed to be birational, [71], Corollary 5.4.2 implies that A and B generate the same cyclic subgroup in $\mathrm{Br}(k)$. Proposition 1.57 yields that $\mathrm{ind}(A^{\otimes r})$ divides $i/(i,r)$. Since $i/(i,r)$ divides i, we conclude that $\mathrm{ind}(A^{\otimes r})$ divides $\mathrm{ind}(A)$. The same holds for B and we have that $\mathrm{ind}(B^{\otimes s})$ divides $\mathrm{ind}(B)$. In what follows we prove that $\mathrm{ind}(A^{\otimes r}) = \mathrm{ind}(B^{\otimes r})$ for all r. Since A and B generate the same cyclic subgroup in $\mathrm{Br}(k)$, we have that A is Brauer-equivalent to $B^{\otimes l}$ and B to $A^{\otimes m}$ for some l and m. Hence $\mathrm{ind}(A^{\otimes m}) = \mathrm{ind}(B)$ divides $\mathrm{ind}(A)$ and $\mathrm{ind}(B^{\otimes l}) = \mathrm{ind}(A)$ divides $\mathrm{ind}(B)$. Thus $\mathrm{ind}(B)$ divides $\mathrm{ind}(A)$ and vice verse and therefore they have to be equal. The same argument applied to $\mathrm{ind}(A^{\otimes r})$ and $\mathrm{ind}(B^{\otimes r})$ now yields that $\mathrm{ind}(A^{\otimes rm}) = \mathrm{ind}(B^{\otimes r})$ divides $\mathrm{ind}(A^{\otimes r})$ and that $\mathrm{ind}(B^{\otimes rl}) = \mathrm{ind}(A^{\otimes r})$ divides $\mathrm{ind}(B^{\otimes r})$. This shows that $\mathrm{ind}(A^{\otimes r}) = \mathrm{ind}(B^{\otimes r})$ for all r. As mentioned above, A and B generate the same cyclic subgroup and thus they have the same period. Now Theorem 1.52 yields that X and Y have the same AS-type. □

Remark 1.68. Notice that if $k = \mathbb{R}$, two birational Brauer–Severi varieties X and Y have to be isomorphic. This is due to the fact that the corresponding central simple $\mathbb{R}$-algebras have to generate the same subgroup in $\mathrm{Br}(\mathbb{R})$ (see [71], Corollary 5.4.2). But since the Brauer group $\mathrm{Br}(\mathbb{R})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, generated by the Hamilton quaternions (see Example 1.14), we conclude that the corresponding central simple algebras of X and Y have to be isomorphic. In view of Theorem 1.10, this implies that $X \simeq Y$. The fact that the classes of two birational Brauer–Severi varieties over k generate the same subgroup in $\mathrm{Br}(k)$ is highly non-trivial and in fact a theorem due to Amitsur [5]. The question whether the converse is true is an open question and will be discussed in the last chapter of the present work.

In the proof of Proposition 1.67 we actually showed the following:

Proposition 1.69. *Let A and B be two central simple k algebras generating the same subgroup in $\mathrm{Br}(k)$ and X and Y the corresponding Brauer–Severi varieties. Then X and Y have the same AS-type.*

We continue the above discussion with the following proposition.

Proposition 1.70. *Let X and Y be two non-split Brauer–Severi varieties over k of type (p, p, d_1) and (p, p, d_2) respectively. Then X and Y have the same AS-type.*

Proof. By assumption, X and Y are of type (p, p, d_1) and (p, p, d_2) respectively. Let A be the central simple k-algebra corresponding to X and B the central simple k-algebra corresponding to Y. Now with Proposition 1.57 we conclude that for both A and B we have $p/(p, r) = \mathrm{ind}(A^{\otimes r}) = \mathrm{ind}(B^{\otimes r})$. Applying Theorem 1.52 yields that X and Y have the same AS-type. □

Corollary 1.71. *Let X and Y be two non-split n-dimensional Brauer–Severi varieties over k. Suppose that $n + 1$ is a prime number. Then X and Y have the same AS-type.*

Proof. By Theorem 1.10 the degree of X and Y is $n+1$. Since $n+1$ is supposed to be a prime number, we conclude that X and Y are both of type $(n+1, n+1, n+1)$. Applying Proposition 1.70 yields the assertion. □

Corollary 1.72. *Let X and Y be two n-dimensional minimal Brauer–Severi varieties over a global or local field k. Then they have the same AS-type.*

Proof. Since X and Y are minimal and have same dimension, we conclude by Theorem 1.61 that X and Y are both of type $(n + 1, n + 1, n + 1)$. Applying Proposition 1.70 yields the assertion. □

To show that there really exist Brauer–Severi varieties where the previews results apply is the purpose of the next example.

Example 1.73. Let k be a field of $\mathrm{char}(k) \neq 2$. Now consider the quaternion algebras $(a, b)_k$ that were mentioned directly after Example 1.20. Now taking matrix algebras $M_n((a, b)_k)$ we get in view of Theorem 1.10 a Brauer–Severi variety X corresponding to $M_n((a, b)_k)$. Note that these Brauer–Severi varieties are of type $(2, 2, 2n)$ if $(a, b)_k$ is non-split. Considering another non-split quaternion algebra (a', b'), not isomorphic to $(a, b)_k$, the Brauer–Severi variety Y corresponding to $M_n((a', b')_k)$ is neither Brauer-equivalent nor birational to X but both have the same AS-Type according to Proposition 1.70.

The above discussion and especially Theorem 1.65 together with Proposition 1.66 shows that the question if the converse of Proposition 1.64 holds has a negative answer. Furthermore, the converse of Proposition 1.67 also does not hold and it really is possible to construct Brauer–Severi varieties of same AS-type that are not Brauer-equivalent or birational (see Example 1.73).

1.5 AS-bundles and the Grothendieck group

In this section we briefly recall some basic facts about the Grothendieck group and show that for Brauer–Severi varieties it is generated by the AS-bundles. For a general introduction to higher K-theory we refer to the work of Quillen [133], where, as an application of the general theory, the K-theory of Brauer–Severi varieties is calculated. The main reference for the Grothendieck group is [68].

For a noetherian scheme X we write $\mathrm{VB}(X)$ for the category of locally free sheaves of finite rank on X. By definition, the *Grothendieck group* of $\mathrm{VB}(X)$ is the free abelian group on the set of isomorphism classes of locally free sheaves modulo the relations $[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}'']$, whenever

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

is an exact sequence of locally free sheaves. It is denoted by $K^0(X)$. The tensor product makes $K^0(X)$ into a ring. Furthermore, let $\mathrm{Coh}(X)$ be the category of coherent sheaves on X. We denote by $K_0(X)$ the Grothendieck group of $\mathrm{Coh}(X)$. An arbitrary morphism of noetherian schemes $f : X \to Y$ induces a group homomorphism $f^* : K^0(Y) \to K^0(X)$, via $\mathcal{E} \mapsto f^*\mathcal{E}$. Let $f : X \to Y$ be a projective morphism of noetherian schemes and $\mathcal{F}$ a coherent sheaf. The $\mathcal{O}_X$-modules $\mathbb{R}^i f_*\mathcal{F}$ are coherent, because f is proper and Y is noetherian. If Y is smooth we define the push-forward by setting $f_*[\mathcal{F}] = \sum_i (-1)^i [\mathbb{R}^i f_*\mathcal{F}]$. For a scheme X, there is a canonical homomorphism

$$\delta : K^0(X) \longrightarrow K_0(X)$$

induced by the exact embedding of the categories. If $\mathcal{F}$ is a coherent sheaf equipped with a finite resolution

$$0 \longrightarrow \mathcal{E}_n \longrightarrow ... \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

where each $\mathcal{E}_i$ is locally free, the element $r(\mathcal{F}) = \sum_{i=0}^n (-1)^i [\mathcal{E}_i] \in K^0(X)$ does not depend on the resolution. We have $\delta \circ r(\mathcal{F}) = \mathcal{F}$. Now consider a smooth integral scheme X. Any coherent sheaf $\mathcal{F}$ admits a finite resolution by locally free sheaves and therefore, by the above consideration, we conclude that δ is an isomorphism. Hence we can identify the groups $K^0(X)$ and $K_0(X)$. For higher K-theory see the work of Quillen [133].

To continue, we consider a n-dimensional Brauer–Severi variety X corresponding to the central simple k-algebra A and denote by $\mathbb{P}^n$ the projective space $X \otimes_k \bar{k}$. Let $h \in K^0(\mathbb{P}^n)$ be the class of $\mathcal{O}_{\mathbb{P}^n}(-1)$. Then the ring $K^0(\mathbb{P}^n)$ is generated by h subject to the relation $(h-1)^{n+1} = 0$, where $n+1$ is the degree of A (see [68], Example 8.3.4). With the above notation one has the following result (see [133], Section 8, Theorem 4.1 see also [98], Theorem 3.1).

Theorem 1.74. *The restriction map $K^0(X) \to K^0(\mathbb{P}^n)$ is injective and its image is additively generated by* $\mathrm{ind}A^{\otimes l} \cdot h^l$ *with* $l \geq 0$.

The next result illustrates the importance of the AS-bundles and shows that the Grothendieck group is generated by the duals of some of them.

Theorem 1.75. *Let X be a n-dimensional Brauer–Severi variety over k, corresponding to the central simple k-algebra A. Then the duals of the indecomposable AS-bundles $\mathcal{W}_j$, with $j \geq 0$, generate the Grothendieck group $K_0(X)$.*

Proof. Note that for a Brauer–Severi variety X one has $K^0(X) = K_0(X)$. We consider the projection map $\pi : \mathbb{P}^n \simeq X \otimes_k \bar{k} \to X$. Via pullback this induces the restriction map $\mathrm{res} : K^0(X) \to K^0(\mathbb{P}^n)$. Now we consider the sheaves $\mathcal{W}_j$ as in Definition 1.39 and their duals $\mathcal{W}_j^\vee$. Pulling them back yields $\pi^*\mathcal{W}_j^\vee \simeq \mathcal{O}_{\mathbb{P}^n}(-j)^{\oplus d_j}$, where $d_j = \mathrm{ind}(A^{\otimes|j|})$ is the rank of $\mathcal{W}_j$ by Proposition 1.51. But $[\mathcal{O}_{\mathbb{P}^n}(-j)^{\oplus d_j}] = d_j \cdot [\mathcal{O}_{\mathbb{P}^n}(-1)]^j$ in $K^0(\mathbb{P}^n)$ and applying Theorem 1.74 we find that the duals of the AS-bundles $\mathcal{W}_j$ generate the Grothendieck group. □

In Chapter 3 we will see that the AS-bundles that generate the Grothendieck group of a Brauer–Severi variety very naturally fit into the theory of tilting objects. One just has to take the direct sum of them to obtain a tilting bundle for the Brauer–Severi variety. This a posteriori encourages the previous study of the AS-bundles on Brauer–Severi varieties. As a last comment in this section we want to note that we will see in the next chapter that the ideas applied to classify the AS-bundles on Brauer–Severi varieties very natural generalize to the general case where X is an arbitrary proper k-scheme. To work this out and to give another examples where a complete classification of AS-bundles is possible is the purpose of the next chapter.

Chapter 2

AS-bundles on proper schemes

2.1 Classification of AS-bundles

As mentioned at the end of the first chapter, the ideas applied to classify the AS-bundles on Brauer–Severi varieties work also in the general case where X is supposed to be an arbitrary proper k-scheme. With the techniques of Galois descent, the generalization only works for k assumed to be perfect. But it is still possible to get some results without assuming k to be perfect. This chapter is dedicated to this problem and the main goal is to classify AS-bundles on arbitrary proper k-schemes. Furthermore, as an application of the general classification of AS-bundles we will study twisted forms of general hypersurfaces or complete intersections in projective space and the generalized Brauer–Severi varieties. At the end we classify all AS-bundles on generalized Brauer–Severi varieties and obtain very naturally a generalization of the results obtained for Brauer–Severi varieties in the last chapter.

As we have seen in the first chapter, some results of Galois descent were applied. To study the general case of proper k-schemes we very roughly recall the basic facts of Galois descent. The main reference is [143].
As pointed out in [143], the main idea of Galois descent is the following: Let k be a field and $k \subset L$ a Galois extension. We fix an object X defined over k. Then we want to consider the set of objects Y, defined over k, such that X and Y become isomorphic after base change to L. Such objects Y are called *k-forms* of X. Following [143] let us denote by $E(k,X)$ the set of isomorphism classes of k-forms of X. The main idea now is to establish a bijection between $E(k,X)$ and $H^1(G,\mathrm{Aut}_L(X_L))$, where G is the Galois group $\mathrm{Gal}(L|k)$ and X_L the object obtained from X after base change to L. We now very roughly recall the very basics.
For this, let G be a finite group and M a left G-module, that is a group such that the action of G is compatible with the group structure on M. A *cocycle* is a map $f: G \to M$ such that $f(\sigma\tau) = f(\sigma) \star (\sigma \cdot f(\tau))$, where $\star$ is the operation in M. The set of cocycles is denoted by $Z^1(G,M)$. We say two cocycles f and g are *cohomologous* and write $f \sim g$, if there exists an element $m \in M$

such that $g(\sigma) = m^{-1} \star f(\sigma) \star (\sigma \cdot m)$. Now the first *group cohomology* is the set $Z^1(G, M)/\sim$ and is denoted by $H^1(G, M)$. An *A-torsor* is a left G-set N admitting a compatible freely and transitive right action of the left G-group A. Two A-torsors are isomorphic if they admit a G and A-equivariant bijection. Let us denote by $A\text{-Tors}_{\mathrm{G}}$ the set of isomorphism classes of A-torsors. One has the following result (see [143], Proposition 33)

Proposition 2.1. *There is a one-to-one correspondence between* $A\text{-Tors}_{\mathrm{G}}$ *and* $H^1(G, A)$.

Now for a left G-set M we consider the group of automorphisms $\mathrm{Aut}(M)$. This group can be given a structure of a G-module via $\sigma \cdot h = \sigma \cdot h \cdot \sigma^{-1}$, where h is an element of $\mathrm{Aut}(M)$. In view of the above proposition, we want to look at $\mathrm{Aut}(M)$-torsors. One can now show that the set $\mathrm{Aut}(M)$-torsors is exactly the set of G-sets M' with isomorphisms $\phi : M \to M'$, such that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\phi} & M' \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ M & \xrightarrow{\phi} & M' \end{array}$$

for all $\sigma \in G$. The above ideas can be applied to the case where we consider a Galois group G acting on some scheme X. We will be mainly interested in the category of coherent sheaves on X obtaining Galois descent for coherent sheaves on X. Now let k be a field and X a smooth projective k-scheme. For a Galois extension $k \subset L$ we have the scheme $X_L = X \otimes_k L$ obtained by base change. For the Galois group $G = \mathrm{Gal}(L|k)$ we have a right G-action on X_L as G acts on the extension L. For a coherent sheaf $\mathcal{F}$ on X_L, one has the following well-known fact:

Proposition 2.2. *The set of isomorphism classes of coherent sheaves on X becoming isomorphic to a coherent sheaf $\mathcal{F}$ on X_L is in one-to-one correspondence with the set* $H^1(G, \mathrm{Aut}_L(\mathcal{F}))$.

We now state a fact from Galois cohomology that can be found in [142], p.152. We sketch the proof.

Proposition 2.3. *Let A be a finite-dimensional k-algebra and $k \subset L$ a Galois field extension. Then* $H^1(G, (A \otimes_k L)^*) = 1$.

Proof. We first note that it is enough to consider finite Galois extensions, since $H^1(G, (A \otimes_k L)^*) = \varinjlim H^1(\mathrm{Gal}(M|k), (A \otimes_k M)^*)$, where the limit is taken over all finite Galois extensions $k \subset M$. Now we consider the k-forms of the k-algebra A. The k-forms of A are left A-modules B of finite dimension over k such that $A \otimes_k M \simeq B \otimes_k M$. The k-forms are in one-to-one correspondence with $H^1(\mathrm{Gal}(M|k), (A \otimes_k M)^*)$ and hence to show that $H^1(\mathrm{Gal}(M|k), (A \otimes_k M)^*) = 1$, we only have to verify that $A \simeq B$. Now we have $B \otimes_k M \simeq A \otimes_k M$ as $A \otimes_k M$-modules and we can consider this isomorphism as an isomorphism of A-modules. For this we note that as an A-module $A \otimes_k M$ is isomorphic to $A^{\oplus[M:k]}$. This is due to the fact that $A \otimes_k 1 \simeq A$. The same is true for the $A \otimes_k M$-module $B \otimes_k M$ and hence as an A-module it is isomorphic to $B^{\oplus[M:k]}$. Then the isomorphism $A \otimes_k M \simeq B \otimes_k M$ yields an isomorphism of A-modules between $A^{\oplus[M:k]}$ and

$B^{\oplus[M:k]}$. Applying the Krull–Schmidt Theorem for A-modules (see for instance [7], Theorem 12.9) yields that $A \simeq B$ and hence $H^1(\mathrm{Gal}(M|k), (A \otimes_k M)^*) = 1$. This completes the proof. □

With this fact we can prove the next proposition which in some sense is a generalization of Lemma 1.34 and shows that the assumption on k being infinite is not needed.

Proposition 2.4. *Let X be a proper k-scheme and $\mathcal{F}$ and $\mathcal{G}$ two coherent sheaves. If $\mathcal{F} \otimes_k L \simeq \mathcal{G} \otimes_k L$ for some separable extension $k \subset L$, then $\mathcal{F}$ is isomorphic to $\mathcal{G}$.*

Proof. It is enough to check this for finite separable extensions, otherwise we take the absolute Galois group for the argumentation. Taking the Galois closure, we can assume $k \subset L$ is a finite Galois extension. Now for the Galois group $G = \mathrm{Gal}(L|k)$ we consider the first cohomology $H^1(G, \mathrm{Aut}_L(\mathcal{F} \otimes_k L))$. According to Proposition 2.3, $H^1(G, \mathrm{Aut}_L(\mathcal{F} \otimes_k L))$ is trivial. This, together with Proposition 2.2, immediately implies that $\mathcal{F} \simeq \mathcal{G}$. □

One can go still a bit further and show that the above proposition holds also in the case where the extension L is an arbitrary finite extension or the algebraic closure of k. This is proved by Wiegand [156], Lemma 2.3 under the assumption that X is projective, but, as the author mentioned at the end of the proof, everything holds provided the endomorphism ring of a coherent $\mathcal{O}_X$-module is finite-dimensional in order to apply Krull–Schmidt. This is the case when X is a proper k-scheme as pointed out in the first chapter. For convenience to the reader, we sketch the proof.

Proposition 2.5. *Let X be a proper k-scheme and $\mathcal{F}$ and $\mathcal{G}$ two coherent sheaves. If $\mathcal{F} \otimes_k \bar{k} \simeq \mathcal{G} \otimes_k \bar{k}$, then $\mathcal{F}$ is isomorphic to $\mathcal{G}$.*

Proof. We follow exactly the proof in [156]. Since the extension $k \subset \bar{k}$ is a direct limit of finite extensions, it suffices to prove the statement for finite field extensions. Now suppose $k \subset L$ is a finite field extension and $\pi : X \otimes_k L \to X$ the projection. Choose a basis $\{\alpha_1, ..., \alpha_d\}$ for L over k. By assumption we have $\pi^*\mathcal{F} \simeq \pi^*\mathcal{G}$. For the coherent sheaf $\mathcal{A} = \pi_*\pi^*\mathcal{F}$ we have over any affine open set $U \subset X$, $\mathcal{A}(U) = \mathcal{F}(U) \otimes_k L$ and there is a unique $\mathcal{O}_X(U)$-module isomorphism from $\mathcal{A}(U)$ to $\mathcal{F}(U)^{\oplus d}$, assigning $m \otimes \alpha_i$ to $(0, 0, ..., m, ..., 0)$, where m is located at the i^{th} entry. This yields $\pi_*\pi^*\mathcal{F} \simeq \mathcal{F}^{\oplus d}$ and obviously the same holds for $\pi_*\pi^*\mathcal{G}$. Hence $\pi_*\pi^*\mathcal{F} \simeq \mathcal{F}^{\oplus d} \simeq \pi_*\pi^*\mathcal{G} \simeq \mathcal{G}^{\oplus d}$ and we conclude from Krull–Schmidt Theorem that $\mathcal{F} \simeq \mathcal{G}$. □

We note that the above proposition implies that the canonical map $\mathrm{Pic}(X) \to \mathrm{Pic}(X \otimes_k L)$, where $\mathcal{L}$ is mapped to $\mathcal{L} \otimes_k L$, is injective in the case L is a finite extension of k or even the algebraic closure. We now cite the for our further investigations crucial fact that is proved in [9], Proposition 3.4. We sketch the proof for convenience to the reader.

Proposition 2.6. *Let k be a perfect field and X a proper k-scheme. Suppose that $\mathcal{E}$ is an indecomposable locally free sheaf on $X \otimes_k \bar{k}$ such that the isomorphism class is $\mathrm{Gal}(\bar{k}|k)$-invariant. Then there is an indecomposable absolutely isotypical sheaf on X that is of type $\mathcal{E}$ and unique up to isomorphism.*

Proof. Let $k \subset M$ be a finite Galois extension inside of $\bar{k}$ such that $\mathcal{E} \simeq \mathcal{N} \otimes_M \bar{k}$ for some locally free sheaf $\mathcal{N}$ on $X \otimes_k M$. Then let $\pi_* \mathcal{N}$ be the sheaf on X obtained by the projection $\pi : X \otimes_k M \to X$. As the $\mathrm{Gal}(M|k)$-conjugates of $\mathcal{N} \otimes_M \bar{k}$ are all isomorphic to $\mathcal{E}$, we have $\pi^* \pi_* \mathcal{N} \simeq \mathcal{E}^{\oplus [M:k]}$. Thus $\pi_* \mathcal{N}$ is absolutely isotypical of type $\mathcal{E}$ and applying Krull–Schmidt Theorem we can consider a direct summand $\mathcal{M}$ of $\pi_* \mathcal{N}$. This locally free sheaf $\mathcal{M}$ is also absolutely isotypical of type $\mathcal{E}$. To prove the uniqueness, we thus have to assume that there is another indecomposable absolutely isotypical sheaf $\mathcal{M}'$ of type $\mathcal{E}$. Let $r = \mathrm{rk}(\mathcal{M})$ and $s = \mathrm{rk}(\mathcal{M}')$. Then $(\mathcal{M}^{\oplus s}) \otimes_k \bar{k} \simeq (\mathcal{M}'^{\oplus r}) \otimes_k \bar{k}$, what with Proposition 2.5 implies that $\mathcal{M}^{\oplus s} \simeq \mathcal{M}'^{\oplus r}$ and hence, applying Krull–Schmidt Theorem, $\mathcal{M} \simeq \mathcal{M}'$. □

Under the assumption on k being perfect, let us denote by $\mathrm{Pic}^G(X \otimes_k \bar{k})$ the $G = \mathrm{Gal}(\bar{k}|k)$-invariant invertible sheaves. The above proposition has the following easy consequence:

Corollary 2.7. *Let k be a perfect field and X a proper k-scheme. For all $\mathcal{L} \in \mathrm{Pic}^G(X \otimes_k \bar{k})$ there is an up to isomorphism unique indecomposable $\mathcal{M}_{\mathcal{L}}$ such that $\mathcal{M}_{\mathcal{L}} \otimes_k \bar{k} \simeq \mathcal{L}^{\oplus r}$.*

We now have all together to classify all indecomposable AS-bundles on proper k-schemes, at least when k is assumed to be perfect.

Theorem 2.8. *Let k be a perfect field and X a proper k-scheme. Then all indecomposable AS-bundles are of the form $\mathcal{M}_{\mathcal{L}}$ for a unique $\mathcal{L} \in \mathrm{Pic}^G(X \otimes_k \bar{k})$.*

Proof. Let $\mathcal{M}$ be an indecomposable AS-bundle. Then by definition we have

$$\mathcal{M} \otimes_k \bar{k} \simeq \bigoplus_{i=1}^{m} \mathcal{L}_i^{\oplus r_i},$$

where $\mathcal{L}_i$ are invertible sheaves on $X \otimes_k \bar{k}$. Note that since $\mathcal{M} \otimes_k \bar{k}$ is G-invariant, we have $\mathcal{M} \otimes_k \bar{k} \simeq \sigma^*(\mathcal{M} \otimes_k \bar{k})$ what implies $\bigoplus_{i=1}^m \sigma^*(\mathcal{L}_i^{\oplus r_i}) \simeq \bigoplus_{i=1}^m \mathcal{L}_i^{\oplus r_i}$, for all $\sigma \in G$. Krull–Schmidt Theorem now implies that all $\mathcal{L}_i$ are also G-invariant. Now by Proposition 2.6, for all these $\mathcal{L}_i$ there is a up to isomorphism unique indecomposable isotypical sheaf $\mathcal{M}_{\mathcal{L}_i}$ of type $\mathcal{L}_i$. Thus we have $\mathcal{M}_{\mathcal{L}_i} \otimes_k \bar{k} \simeq \mathcal{L}_i^{\oplus s_i}$. Now we consider the least common multiple $d = \mathrm{lcm}(s_1, ..., s_m)$ of all s_i. Then by the definition of the least common multiple, there are integers n_i such that $n_i s_i = d$. Considering the AS-bundle $\mathcal{M}^{\oplus d}$ we find

$$(\mathcal{M}^{\oplus d}) \otimes_k \bar{k} \simeq (\bigoplus_{i=1}^{m} \mathcal{L}_i^{\oplus r_i})^{\oplus d} \simeq \bigoplus_{i=1}^{m} \mathcal{L}_i^{\oplus s_i \cdot (n_i r_i)}.$$

Since the locally free sheaves $\mathcal{L}_i^{\oplus s_i}$ descent to $\mathcal{M}_{\mathcal{L}_i}$, we find that

$$(\bigoplus_{i=1}^{m} \mathcal{M}_{\mathcal{L}_i}^{\oplus n_i r_i}) \otimes_k \bar{k} \simeq (\mathcal{M}^{\oplus d}) \otimes_k \bar{k}.$$

Applying Proposition 2.5 yields that

$$\mathcal{M}^{\oplus d} \simeq \bigoplus_{i=1}^{m} \mathcal{M}_{\mathcal{L}_i}^{\oplus n_i r_i}.$$

Finally, applying the Krull–Schmidt Theorem to $\mathcal{M}^{\oplus d}$ yields that $\mathcal{M}$ has to be of the form $\mathcal{M}_{\mathcal{L}}$ for some unique $\mathcal{L} \in \mathrm{Pic}^G(X \otimes_k \bar{k})$. □

Corollary 2.9. *Let k be a perfect field, X a proper k-scheme and $\mathcal{E}$ an AS-bundle. The $\mathcal{E}$ is of the form*

$$\mathcal{E} \simeq \bigoplus_{i=1}^{n} \mathcal{M}_{\mathcal{L}_i}^{\oplus r_i}$$

with unique natural numbers n and r_i and unique $\mathcal{L}_i \in \mathrm{Pic}^G(X \otimes_k \bar{k})$.

We note that Proposition 2.6 and Theorem 2.8 remain valid if we give a variation of the definition of absolutely isotypical sheaf and *AS*-bundle respectively.

Definition 2.10. Let X be a proper k-scheme. A locally free sheaf $\mathcal{E}$ of finite rank is called *separably isotypical* if on $X \otimes_k k^{sep}$ there is a indecomposable locally free sheaf $\mathcal{W}$ such that $\mathcal{E} \otimes_k k^{sep} \simeq \mathcal{W}^{\oplus n}$. The sheaf $\mathcal{W}$ is called the *type* of the separably isotypical sheaf.

Definition 2.11. Let X be a proper k-scheme. A locally free sheaf $\mathcal{E}$ of finite rank is called *separably split* if it splits as a direct sum of invertible sheaves on $X \otimes_k k^{sep}$.

One can check that Proposition 2.6 and Theorem 2.8 still hold in the setting of Definition 2.10 and 2.11.

Proposition 2.12. *Let X a proper k-scheme and $\mathcal{W}$ an indecomposable locally free sheaf on $X \otimes_k k^{sep}$ such that the isomorphism class is $\mathrm{Gal}(k^{sep}|k)$-invariant. Then there is a indecomposable separably isotypical sheaf on X that is of type $\mathcal{W}$ and unique up to isomorphism.*

Proof. The proof goes exactly as in Proposition 2.6. For the uniqueness use again Proposition 2.5. Note that the proof of Proposition 2.5 shows that the proposition holds for arbitrary finite field extensions and especially in the case the field extension is finite Galois. Since the separable closure is the direct limit of the finite Galois extensions, we therefore conclude that Proposition 2.5 holds also for the separable closure. □

With this proposition we now obtain that for every invertible sheaf $\mathcal{L} \in \mathrm{Pic}^G(X \otimes_k k^{sep})$, where G is the absolute Galois group $\mathrm{Gal}(k^{sep}|k)$, there is an up to isomorphism unique separably isotypical sheaf $\mathcal{M}_{\mathcal{L}}$ of type $\mathcal{L}$. With this notation we obtain the following result, analogous to Theorem 2.8.

Theorem 2.13. *Let X be a proper k-scheme. Then all indecomposable separably split locally free sheaves are of the form $\mathcal{M}_{\mathcal{L}}$ for a unique $\mathcal{L} \in \mathrm{Pic}^G(X \otimes_k k^{sep})$.*

Proof. The proof is exactly the same as for Theorem 2.8, with the difference that one needs Proposition 2.12 instead of 2.6. □

Remark 2.14. Obviously, Corollary 2.9 also holds for separably split locally free sheaves.

To state the main theorem in this chapter we want to investigate the four-term exact sequence obtained from the Hochschild–Serre spectral sequence for Galois coverings as explained in the first chapter. In general, for a proper k-scheme X with $H^0(X, \mathcal{O}_X) = k$ one has the four-term exact sequence for the

Galois covering $X' = X \otimes_k k^{sep} \to X$:

$$0 \longrightarrow H^1(G, H^0(X'_{et}, \mathbb{G}_m)) \longrightarrow H^1(X_{et}, \mathbb{G}_m) \longrightarrow H^0(G, H^1(X'_{et}, \mathbb{G}_m)) \longrightarrow$$

$$\longrightarrow H^2(G, H^0(X'_{et}, \mathbb{G}_m)).$$

This yields the following exact sequence

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}^G(X') \longrightarrow H^2(G, H^0(X'_{et}, \mathbb{G}_m)).$$

Under the assumption $H^0(X, \mathcal{O}_X) = k$, one has $H^2(G, H^0(X'_{et}, \mathbb{G}_m)) = \mathrm{Br}(k)$ and hence

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}^G(X') \longrightarrow \mathrm{Br}(k). \tag{2.1}$$

In order to get a nice interpretation of the group $\mathrm{Pic}^G(X')$, we recall the basics of the Picard scheme. The main references are [77], [78] and [104].
For a scheme X, the Picard group $\mathrm{Pic}(X)$ is the same as $H^1(X, \mathcal{O}_X^*)$ (see [82], p.224). This group is also called the absolute Picard group. To get some relative version of this group, we fix a S-scheme X with structural morphism $f : X \to S$. Now for a S-scheme T we have the following base change diagram:

$$\begin{array}{ccc} X_T = X \times_S T & \longrightarrow & X \\ \downarrow{\scriptstyle f_T} & & \downarrow{\scriptstyle f} \\ T & \longrightarrow & S \end{array}$$

The idea now is the following: For the S-scheme T we form the presheaf $T \mapsto H^1(X_T, \mathcal{O}_{X_T}^*)$. The associated sheaf is $\mathbb{R}^1 f_{T_*} \mathcal{O}_{X_T}^*$ (see [82], Proposition 8.1, p.250). Therefore, in the Zariski topology one defines the relative Picard functor as $\mathrm{Pic}_{(X/S)(Zar)}(T) = H^0(T, \mathbb{R}^1 f_{T_*} \mathcal{O}_{X_T}^*)$. We now can try to imitate this for the fppf or étale-topology. For this, consider the Leray spectral sequence

$$E_2^{pq} = H^p(T, \mathbb{R}^q f_{T_*} \mathcal{F}_{X_T}) \Longrightarrow H^{p+q}(X_T, \mathcal{F}_{X_T}),$$

that yields the exact sequence of low degree

$$0 \longrightarrow H^1(T, f_{T_*}\mathcal{F}) \longrightarrow H^1(X_T, \mathcal{F}) \longrightarrow H^0(T, \mathbb{R}^1 f_{T_*}\mathcal{F}) \longrightarrow H^2(T, f_{T_*}\mathcal{F}).$$

The cohomology groups occurring in the exact sequence are meant with respect to the fppf-topology. Now for the sheaf $\mathbb{G}_m$ in the fppf-topology, that is also a sheaf in the étale-topology, the above exact sequence becomes:

$$0 \longrightarrow H^1(T, f_{T_*}\mathbb{G}_m) \longrightarrow H^1(X_T, \mathbb{G}_m) \longrightarrow H^0(T, \mathbb{R}^1 f_{T_*}\mathbb{G}_m) \longrightarrow$$

$$\longrightarrow H^2(T, f_{T_*}\mathbb{G}_m).$$

Under the assumption that $f_*\mathcal{O}_X \simeq \mathcal{O}_S$, one has $f_{T_*}\mathbb{G}_m = \mathbb{G}_m$ and hence $H^1(T, f_{T_*}\mathbb{G}_m) = H^1(T, \mathbb{G}_m)$, what by [78], p.190-216 implies that $H^1(T, f_{T_*}\mathbb{G}_m) = \mathrm{Pic}(T)$. Thus the above exact sequence becomes

$$0 \longrightarrow \mathrm{Pic}(T) \longrightarrow \mathrm{Pic}(X_T) \longrightarrow H^0(T, \mathbb{R}^1 f_{T_*}\mathbb{G}_m) \longrightarrow H^2(T, f_{T_*}\mathbb{G}_m).$$

Following the above idea, one defines the *relative Picard functor* $\mathrm{Pic}_{X/S}(T)$ simply as $H^0(T, \mathbb{R}^1 f_{T_*}\mathbb{G}_m)$. Although this functor is defined in the fppf-topology, it can also be defined in the étale topology, since the above exact sequence also exists in this topology. It is a very delicate problem under what kind of assumptions the Picard functor is representable in dependence of the given topology. We refer to [78] and [104] for all the details. The main Theorem of Grothendieck about the Picard functor is that, under the assumption that $f : X \to S$ is projective and flat and its geometric fibers are integral, the Picard functor $\mathrm{Pic}_{(X/S)(et)}$ is representable (see [104], Theorem 4.8). The techniques to prove that result are developed in EGA I [77], (0,4.5.5), p.106 and also apply to prove that for example the Grassmannian functor is representable. The object that represents the Picard functor is a separated scheme locally of finite type over S and is denoted by $\mathrm{Pic}_{X/S}$. Since we will be primary interested in the case $S = \mathrm{Spec}(k)$ and $H^0(X, \mathcal{O}_X) = k$, we note that in this case for a proper k-scheme X, the Picard functor $\mathrm{Pic}_{(X/k)(\acute{e}t)}$ is also representable (see [104], Theorem 4.18.2, Corollary 4.18.3 or [78] p.236 ff.). Specializing further, the above exact sequence becomes for $S = \mathrm{Spec}(k) = T$ and $H^0(X, \mathcal{O}_X) = k$, what is the same as $f_*\mathcal{O}_X \simeq \mathcal{O}_S$,

$$0 \longrightarrow \mathrm{Pic}(\mathrm{Spec}(k)) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}_{(X/k)(fppf)}(k) \longrightarrow \mathrm{Br}(k).$$

As mentioned above, if X is supposed to be proper over $S = \mathrm{Spec}(k) = T$ and $H^0(X, \mathcal{O}_X) = k$, then the above sequence also holds in the étale-topology and we consider the Picard functor $H^0(T, \mathbb{R}^1 f_{T_*}\mathbb{G}_m)$ as given in this topology and write $\mathrm{Pic}_{(X/k)(et)}(T)$. From this we finally get the following exact sequence

$$0 \longrightarrow \mathrm{Pic}(\mathrm{Spec}(k)) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}_{(X/k)(et)}(k) \longrightarrow \mathrm{Br}(k). \tag{2.2}$$

This means that we can represent elements of $\mathrm{Pic}_{(X/k)(et)}(k)$ by elements of the Brauer group $\mathrm{Br}(k)$. If X has a k-rational point, one can show that $\mathrm{Pic}(X) = \mathrm{Pic}_{(X/k)(et)}(k)$ and the elements are represented by the class of k in $\mathrm{Br}(k)$. Thus, the k-rational points of $\mathrm{Pic}_{X/k}$ are in one-to-one correspondence with invertible sheaves on X. The question arises what happens if in general X does not admit a k-rational point. Comparing the two exact sequences (2.1) and (2.2) yields that the G-invariant invertible sheaves on $X \otimes_k k^{sep}$ are in one-to-one correspondence with k-rational points in the Picard scheme $\mathrm{Pic}_{X/k}$, since the Picard functor is representable and one has $\mathrm{Hom}(\mathrm{Spec}(k), \mathrm{Pic}_{X/k}) = \mathrm{Pic}_{(X/k)(et)}(k)$. The above discussion now yields the following interpretation of Theorem 2.8:

Theorem 2.15. *Let k be a perfect field and X a proper k-scheme such that $H^0(X, \mathcal{O}_X) = k$. Then the k-rational points of $\mathrm{Pic}_{X/k}$ are in one-to-one correspondence with isomorphism classes of indecomposable AS-bundles on X.*

Proof. We want to construct a map

$$\mathrm{Pic}_{X/k}(k) \longrightarrow \mathrm{AS}_X,$$

where AS_X denotes the set of isomorphism classes of indecomposable AS-bundles on X, and show that this map is bijective. Note that for perfect fields $k^{sep} = \bar{k}$. From the above discussion we get that for every k-rational point $y \in \mathrm{Pic}_{X/k}$ we have up to isomorphism a unique G-invariant invertible sheaf $\mathcal{L}_y \in \mathrm{Pic}^G(X \otimes_k k^{sep})$. According to Proposition 2.6 and Theorem 2.8, for this

$\mathcal{L}_y$ there is up to isomorphism a unique indecomposable AS-bundle $\mathcal{M}_{\mathcal{L}_y}$. We define the above map by assigning to a k-rational point y the isomorphism class $[\mathcal{M}_{\mathcal{L}_y}]$. This map is well defined and according to Theorem 2.8 it is bijective. This completes the proof. □

Remark 2.16. Summarizing, one can hold on to the fact that in the case X has a k-rational point, the rational points of $\mathrm{Pic}_{X/k}$ are in one-to-one correspondence with invertible sheaves. If X does not admit a k-rational point, the rational points of $\mathrm{Pic}_{X/k}$ correspond to indecomposable AS-bundles, provided k is perfect. Note furthermore, that in the case the field k is not assumed to be perfect, the k-rational points of $\mathrm{Pic}_{X/k}$ are anyhow in one-to-one correspondence with indecomposable separably split locally free sheaves. The proof for this is exactly the same as the proof of Theorem 2.15, with the difference of applying Proposition 2.12 and Theorem 2.13 instead of Proposition 2.6 and Theorem 2.8.

Since separably split locally free sheaves are of course AS-bundles, one immediately gets that, without assuming k being perfect, the above map $\mathrm{Pic}_{X/k} \to AS_X$ is injective. Proposition 2.6 should also hold for k not supposed to be perfect and instead of Galois descent, faithfully flat descent may is the right tool to investigate this situation. In view of this, the investigation of Brauer–Severi varieties in the first chapter and the above discussion, we formulate the following conjecture:

Conjecture. *Let X be a proper k-scheme with $H^0(X, \mathcal{O}_X) = k$. Then the k-rational points of $\mathrm{Pic}_{X/k}$ are in one-to-one correspondence with isomorphism classes of indecomposable AS-bundles on X.*

We now continue as in Chapter 1 and try to determined the ranks of the locally free sheaves $\mathcal{M}_{\mathcal{L}}$. For the rest of this section we assume k to be an arbitrary, not necessarily perfect field. Suppose we are given an invertible sheaf $\mathcal{L} \in \mathrm{Pic}(X \otimes_k \bar{k})$ such that there is an indecomposable absolutely rank-one-isotypical sheaf $\mathcal{M}$ of type $\mathcal{L}$. By Proposition 2.5, the sheaf $\mathcal{M}$ is unique up to isomorphism and hence we can write $\mathcal{M}_{\mathcal{L}}$ for it. Clearly, if for $\mathcal{L}$ there exists an indecomposable absolutely rank-one-isotypical sheaf $\mathcal{M}_{\mathcal{L}}$ of type $\mathcal{L}$, then for the invertible sheaves $\mathcal{L}^{\otimes j}$ there are also indecomposable absolutely isotypical sheaves of type $\mathcal{L}^{\otimes j}$. This is due to the following fact: Let $r = \mathrm{rk}(\mathcal{M}_{\mathcal{L}})$ and consider $(\mathcal{L}^{\oplus r})^{\otimes j} \simeq (\mathcal{L}^{\otimes j})^{\oplus r^j}$. From this one gets $\mathcal{M}_{\mathcal{L}}^{\otimes j} \otimes_k \bar{k} \simeq (\mathcal{L}^{\oplus r})^{\otimes j} \simeq (\mathcal{L}^{\otimes j})^{\oplus r^j}$. Considering the Krull–Schmidt decomposition of $\mathcal{M}_{\mathcal{L}}^{\otimes j}$ and taking into account that all indecomposable direct summands are isomorphic according to Proposition 1.37 and Remark 1.38, we get an up to isomorphism unique indecomposable locally free sheaf $\mathcal{M}_{\mathcal{L}^{\otimes j}}$ such that $\mathcal{M}_{\mathcal{L}^{\otimes j}} \otimes_k \bar{k} \simeq (\mathcal{L}^{\otimes j})^{\oplus r_j}$, where r_j is the rank of $\mathcal{M}_{\mathcal{L}^{\otimes j}}$. Remind, that we denoted by $D(\mathcal{M}_{\mathcal{L}})$ the finite-dimensional semisimple k-algebra $\mathrm{End}(\mathcal{M}_{\mathcal{L}})/\mathrm{rad}(\mathrm{End}(\mathcal{M}_{\mathcal{L}}))$. With this notation and the above observation we can state the next result.

Proposition 2.17. *Let X be a proper k-scheme with $H^0(X, \mathcal{O}_X) = k$ and $\mathcal{M}_{\mathcal{L}}$ a indecomposable absolutely isotypical sheaf of type $\mathcal{L}$, where $\mathcal{L}$ is an invertible sheaf on $X \otimes_k \bar{k}$. Then one has*

$$\mathrm{rk}(\mathcal{M}_{\mathcal{L}}) = \mathrm{ind}(D(\mathcal{M}_{\mathcal{L}})).$$

Proof. By assumption we have $\mathcal{M}_{\mathcal{L}}\otimes_k\bar{k}\simeq\mathcal{L}^{\oplus r}$, where r is the rank of $\mathcal{M}_{\mathcal{L}}$. Note that $D(\mathcal{M}_{\mathcal{L}}\otimes_k\bar{k})\simeq \mathrm{End}(\mathcal{M}_{\mathcal{L}}\otimes_k\bar{k})/\mathrm{rad}(\mathrm{End}(\mathcal{M}_{\mathcal{L}}\otimes_k\bar{k}))$. We now consider $\mathrm{End}(\mathcal{M}_{\mathcal{L}}\otimes_k\bar{k})\simeq\mathrm{End}(\mathcal{L}^{\oplus r})$ and see that $\mathrm{End}(\mathcal{L}^{\oplus r})\simeq M_r(\mathrm{End}(\mathcal{L}))$. Since $H^0(X,\mathcal{O}_X)\otimes_k\bar{k}\simeq H^0(X\otimes_k\bar{k},\mathcal{O}_{X\otimes_k\bar{k}})\simeq k\otimes_k\bar{k}\simeq\bar{k}$ by assumption, we conclude that $\mathrm{End}(\mathcal{L}^{\oplus r})\simeq M_r(\bar{k})$, what by Theorem 1.4 implies that $\mathrm{End}(\mathcal{M}_{\mathcal{L}})$ is a central simple k-algebra and hence $\mathrm{rad}(\mathrm{End}(\mathcal{M}_{\mathcal{L}}))=0$ by the discussion after Proposition 1.28. But this yields that $D(\mathcal{M}_{\mathcal{L}})$ is isomorphic to $\mathrm{End}(\mathcal{M}_{\mathcal{L}})$ and hence is central simple. Since $\mathcal{M}_{\mathcal{L}}\otimes_k\bar{k}\simeq\mathcal{L}^{\oplus r}$, we conclude that the degree of $D(\mathcal{M}_{\mathcal{L}})$ is equal to the rank of $\mathcal{M}_{\mathcal{L}}$. Finally, since $\mathcal{M}_{\mathcal{L}}$ is supposed to be indecomposable, $D(\mathcal{M}_{\mathcal{L}})$ is a division algebra according to Proposition 1.32. And since for division algebras the degree equals the index, we have $\mathrm{rk}(\mathcal{M}_{\mathcal{L}})=\mathrm{ind}(D(\mathcal{M}_{\mathcal{L}}))$. This completes the proof. □

Corollary 2.18. *Let X be a proper k-scheme with $H^0(X,\mathcal{O}_X)=k$ and $\mathcal{M}_{\mathcal{L}}$ a indecomposable absolutely isotypical sheaf of type $\mathcal{L}$, where $\mathcal{L}$ is an invertible sheaf on $X\otimes_k\bar{k}$. Then the rank of $\mathcal{M}_{\mathcal{L}}$ equals the minimal degree of a finite separable field extension $k\subset L$ that splits $D(\mathcal{M}_{\mathcal{L}})$ and therefore $\mathcal{M}_{\mathcal{L}}$.*

Proof. With Proposition 2.17 we have that the rank of $\mathcal{M}_{\mathcal{L}}$ equals the index of the central division algebra $D(\mathcal{M}_{\mathcal{L}})$. By Remark 1.17, the index of $D(\mathcal{M}_{\mathcal{L}})$ is the smallest among the degrees of finite separable field extension L of k that splits $D(\mathcal{M}_{\mathcal{L}})$. Since $\mathrm{End}(\mathcal{M}_{\mathcal{L}})\otimes_k L=D(\mathcal{M}_{\mathcal{L}})\otimes_k L\simeq D(\mathcal{M}_{\mathcal{L}}\otimes_k L)=\mathrm{End}(\mathcal{M}_{\mathcal{L}}\otimes_k L)$, we conclude that L also splits the locally free sheaf $\mathcal{M}_{\mathcal{L}}$. □

We now give the generalization of Theorem 1.45 and Corollary 1.46 that holds for arbitrary proper k-schemes. We first fix some notation. From Proposition 2.5 we conclude that $\mathrm{Pic}(X)$ is a subgroup of $\mathrm{Pic}(X\otimes_k\bar{k})$. Suppose we have $\mathrm{Pic}(X\otimes_k\bar{k})\simeq\mathbb{Z}$, what therefore implies that $\mathrm{Pic}(X)$ is also isomorphic to $\mathbb{Z}$. Now let $\mathcal{L}$ denote the generator of $\mathrm{Pic}(X\otimes_k\bar{k})$ and $\mathcal{J}$ the generator of $\mathrm{Pic}(X)$. Then we have $\mathcal{J}\otimes_k\bar{k}\simeq\mathcal{L}^{\otimes r}$ for some unique r. This number r can be interpreted as follows: As in the case of Brauer–Severi varieties it is the smallest number such that $\mathcal{L}^{\otimes r}$ descents to a invertible sheaf on X. Without loss of generality, this r can be supposed to be positive. We want to call this integer r the *period* of X. With this notation we have the following result.

Theorem 2.19. *Let X be a proper k-scheme with $\mathrm{Pic}(X\otimes_k\bar{k})\simeq\mathbb{Z}$ and period r. Suppose there is an indecomposable absolutely isotypical sheaf $\mathcal{M}_{\mathcal{L}}$ of type $\mathcal{L}$, where $\mathcal{L}$ is the generator of $\mathrm{Pic}(X\otimes_k\bar{k})$. Denote by $\mathcal{J}$ the generator of $\mathrm{Pic}(X)$. Then all indecomposable AS-bundles are up to isomorphism of the form*

$$\mathcal{M}_{\mathcal{L}^{\otimes j}}\otimes\mathcal{J}^{\otimes a}$$

with unique $a\in\mathbb{Z}$ and unique $0\leq j\leq r-1$.

Proof. We follow the idea of the proof of Theorem 1.45. Let $\mathcal{E}$ be an arbitrary, not necessarily indecomposable AS-bundle and $\pi: X\otimes_k\bar{k}\to X$ the projection. Note that we have shown above right after the conjecture on page 60 that since $\mathcal{M}_{\mathcal{L}}$ is indecomposable absolutely isotypical of type $\mathcal{L}$, there exist unique indecomposable absolutely isotypical sheaves of type $\mathcal{L}^{\otimes j}$ for all $j\in\mathbb{Z}$. We denote these locally free sheaves by $\mathcal{M}_{\mathcal{L}^{\otimes j}}$. Now consider these locally free sheaves $\mathcal{M}_{\mathcal{L}^{\otimes j}}$ only for $j=0,...,r-1$. Set $d=\mathrm{lcm}(\mathrm{rk}(\mathcal{M}_{\mathcal{O}_{X_{\bar{k}}}}),\mathrm{rk}(\mathcal{M}_{\mathcal{L}}),...,\mathrm{rk}(\mathcal{M}_{\mathcal{L}^{\otimes(r-1)}}))$ to be the least common multiple and consider the bundle $\pi^*(\mathcal{E}^{\oplus d})$. Since $\mathcal{E}$ is an

AS-bundle, the direct sum $\mathcal{E}^{\oplus d}$ is an AS-bundle too. Hence we can decompose $\pi^*(\mathcal{E}^{\oplus d})$ into a direct sum of invertible sheaves and find after reordering mod r, that $\pi^*(\mathcal{E}^{\oplus d})$ is isomorphic to

$$\left(\bigoplus_{i=0}^{s_0}(\mathcal{L}^{\otimes a_{i_0}\cdot r})^{\oplus d}\right) \oplus \left(\bigoplus_{i=0}^{s_1}(\mathcal{L}^{\otimes a_{i_1}\cdot r+1})^{\oplus d}\right) \oplus \dots \oplus \left(\bigoplus_{i=0}^{s_{r-1}}((\mathcal{L}^{\otimes a_{i_{r-1}}\cdot r+(r-1)}))^{\oplus d}\right).$$

By definition of d, there are h_j such that $h_j \cdot \mathrm{rk}(\mathcal{M}_{\mathcal{L}^{\otimes j}}) = d$, for $0 \le j \le r-1$. Furthermore, the sheaves $\mathcal{M}_{\mathcal{L}^{\otimes j}}$ have the property that $\pi^*\mathcal{M}_{\mathcal{L}^{\otimes j}} \simeq (\mathcal{L}^{\otimes j})^{\oplus d_j}$, where $d_j = \mathrm{rk}(\mathcal{M}_{\mathcal{L}^{\otimes j}})$. Now for the direct summands $(\mathcal{L}^{\otimes a_{i_j}\cdot r+j})^{\oplus d}$ we have

$$\left(\mathcal{L}^{\otimes a_{i_j}\cdot r+j}\right)^{\oplus d} = \left(\left(\mathcal{L}^{\otimes a_{i_j}\cdot r+j}\right)^{\oplus d_j}\right)^{\oplus h_j}.$$

Now considering the locally free sheaf $(\mathcal{M}_{\mathcal{L}^{\otimes j}} \otimes \mathcal{J}^{\otimes a_{i_j}})^{\oplus h_j}$ on X, we find

$$\pi^*\left(\mathcal{M}_{\mathcal{L}^{\otimes j}} \otimes \mathcal{J}^{\otimes a_{i_j}}\right)^{\oplus h_j} \simeq \left(\mathcal{L}^{\otimes a_{i_j}\cdot r+j}\right)^{\oplus d}.$$

This implies that for the locally free sheaf

$$\mathcal{R} = \left(\bigoplus_{i=0}^{r_0}(\mathcal{J}^{\otimes a_{i_0}})^{\oplus d}\right) \oplus \left(\bigoplus_{i=0}^{r_1}(\mathcal{J}^{\otimes a_{i_1}}) \otimes \mathcal{M}_{\mathcal{L}}^{\oplus h_1}\right) \oplus \dots \oplus \left(\bigoplus_{i=0}^{r_{p-1}}(\mathcal{J}^{\otimes a_{i_{p-1}}}) \otimes \mathcal{M}_{\mathcal{L}^{\otimes j}}^{\oplus h_{r-1}}\right)$$

we have $\pi^*\mathcal{R} \simeq \pi^*(\mathcal{E}^{\oplus d})$. Applying Proposition 2.5 yields that $\mathcal{E}^{\oplus d}$ is isomorphic to $\mathcal{R}$. And because the Krull–Schmidt Theorem holds for locally free sheaves on X, we conclude that $\mathcal{E}$ is isomorphic to the direct sum of these $\mathcal{M}_{\mathcal{L}^{\otimes j}} \otimes \mathcal{J}^{\otimes a}$ with unique $a \in \mathbb{Z}$ and $0 \le j \le r-1$. Furthermore, since all these bundles are indecomposable by definition, we finally get that all the indecomposable AS-bundles are of the form $\mathcal{M}_{\mathcal{L}^{\otimes j}} \otimes \mathcal{J}^{\otimes a}$ with unique $a \in \mathbb{Z}$ and $0 \le j \le r-1$. This completes the proof. ☐

Corollary 2.20. *Let X be a proper k-scheme with $\mathrm{Pic}(X \otimes_k \bar{k}) \simeq \mathbb{Z}$ and period r. Suppose there is an indecomposable absolutely isotypical sheaf $\mathcal{M}_{\mathcal{L}}$ of type $\mathcal{L}$, where $\mathcal{L}$ is the generator of $\mathrm{Pic}(X \otimes_k \bar{k})$. Then all AS-bundles $\mathcal{E}$ are of the form*

$$\mathcal{E} \simeq \bigoplus_{j=0}^{r-1}\left(\bigoplus_{i=0}^{s_j}\mathcal{M}_{\mathcal{L}^{\otimes j}} \otimes \mathcal{J}^{\otimes a_{i_j}}\right),$$

with unique $a_{i_j} \in \mathbb{Z}$ and unique $s_j > 0$, with $0 \le j \le r-1$.

Remark 2.21. In the case the field k is supposed to be perfect, we have that for all $\mathcal{L} \in \mathrm{Pic}^G(X \otimes_k k^{sep})$ there exists an indecomposable absolutely isotypical sheaf of type $\mathcal{L}$ and hence Theorem 2.19 implies Theorem 2.8.

Under the assumption that the Picard group is cyclic we have the following description of the ranks of $\mathcal{M}_{\mathcal{L}^{\otimes j}}$.

Proposition 2.22. *Let X be a proper k-scheme with $H^0(X, \mathcal{O}_X) = k$ and $\mathrm{Pic}(X \otimes_k \bar{k}) \simeq \mathbb{Z}$. Suppose there is an indecomposable absolutely isotypical sheaf $\mathcal{M}_{\mathcal{L}}$ of type $\mathcal{L}$, where $\mathcal{L}$ is the generator of $\mathrm{Pic}(X \otimes_k \bar{k})$. Then one has for all $j \in \mathbb{Z}$*

$$\mathrm{rk}(\mathcal{M}_{\mathcal{L}^{\otimes j}}) = \mathrm{ind}(D(\mathcal{M}_{\mathcal{L}})^{\otimes j}).$$

Proof. For the locally free sheaf $\mathcal{M}_{\mathcal{L}}^{\otimes j}$ we have

$$\pi^* \mathcal{M}_{\mathcal{L}}^{\otimes j} \simeq (\mathcal{L}^{\oplus \mathrm{rk}(\mathcal{M}_{\mathcal{L}})})^{\otimes j},$$

what implies that $\mathcal{M}_{\mathcal{L}}^{\otimes j}$ is absolutely isotypical of type $\mathcal{L}^{\otimes j}$ but in general not indecomposable. By Proposition 1.31, the endomorphism algebra $\mathrm{End}(\mathcal{M}_{\mathcal{L}^{\otimes j}})$ is central simple and, since $\mathcal{M}_{\mathcal{L}^{\otimes j}}$ is indecomposable, it is moreover a division algebra over k according to Proposition 1.32. Applying the Krull–Schmidt Theorem for the locally free sheaf $\mathcal{M}_{\mathcal{L}}^{\otimes j}$ on X we have a decomposition

$$\mathcal{M}_{\mathcal{L}}^{\otimes j} \simeq \bigoplus_{i=1}^{m} \mathcal{E}_i,$$

where all $\mathcal{E}_i$ are indecomposable. After base change to the algebraic closure we have

$$(\mathcal{L}^{\oplus \mathrm{rk}(\mathcal{M}_{\mathcal{L}})})^{\otimes j} \simeq (\mathcal{M}_{\mathcal{L}}^{\otimes j}) \otimes_k \bar{k} \simeq \oplus_{i=1}^{m} \mathcal{E}_i \otimes_k \bar{k}.$$

Applying Krull–Schmidt Theorem for locally free sheaves on $X \otimes_k \bar{k}$ yields that all the $\mathcal{E}_i$ are absolutely isotypical of type $\mathcal{L}^{\otimes j}$. According to Proposition 2.5 all $\mathcal{E}_i$ are isomorphic to $\mathcal{M}_{\mathcal{L}^{\otimes j}}$. This finally implies that $\mathrm{End}(\mathcal{M}_{\mathcal{L}}^{\otimes j})$ is a matrix algebra over the division algebra $\mathrm{End}(\mathcal{M}_{\mathcal{L}^{\otimes j}})$. Note that under the assumption $H^0(X, \mathcal{O}_X) = k$, we have shown in the proof of Proposition 2.17 that $D(\mathcal{M}_{\mathcal{L}^{\otimes j}}) = \mathrm{End}(\mathcal{M}_{\mathcal{L}^{\otimes j}})$. But $\mathrm{End}(\mathcal{M}_{\mathcal{L}}^{\otimes j})$ is isomorphic to $\mathrm{End}(\mathcal{M}_{\mathcal{L}})^{\otimes j} = D(\mathcal{M}_{\mathcal{L}})^{\otimes j}$, what implies that the rank of $\mathcal{M}_{\mathcal{L}^{\otimes j}}$ is equal to the degree of $\mathrm{End}(\mathcal{M}_{\mathcal{L}^{\otimes j}})$ and hence must be equal to the index of $D(\mathcal{M}_{\mathcal{L}})^{\otimes j}$. □

Remark 2.23. The proof of the above proposition shows that one always has $\mathrm{rk}(\mathcal{M}_{\mathcal{L}^{\otimes j}}) = \mathrm{ind}(D(\mathcal{M}_{\mathcal{L}})^{\otimes j})$ for all $\mathcal{L}^{\otimes j}$ in the cyclic subgroup of $\mathrm{Pic}(X \otimes_k \bar{k})$ generated by some invertible sheaf $\mathcal{L}$, provided there exists an indecomposable absolutely isotypical sheaf $\mathcal{M}_{\mathcal{L}}$ of type $\mathcal{L}$.

We now shortly explain how we get back Theorem 1.45 and Theorem 1.52 of Chapter 1. For this, we consider a Brauer–Severi variety X over a field k with corresponding central simple k-algebra A. Since $X \otimes_k \bar{k} \simeq \mathbb{P}^n$, we have $\mathrm{Pic}(\mathbb{P}^n) \simeq \mathbb{Z}$ and hence $\mathrm{Pic}(X) \simeq \mathbb{Z}$ as discussed before. We take $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)$ as the generator of the Picard group of $\mathbb{P}^n$. In chapter 1 we have shown that there exists an up to isomorphism unique locally free sheaf $\mathcal{W}_1$ such that $\mathcal{W}_1 \otimes_k \bar{k} \simeq \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1}$. In this chapter we showed that for all $j \in \mathbb{Z}$, $\mathcal{O}_{\mathbb{P}^n}(j)^{\oplus s_j}$ also descent for suitable $s_j > 0$. Now it is easy to see that $\mathcal{W}_j \simeq \mathcal{M}_{\mathcal{L}^{\otimes j}}$, since both are indecomposable and of type $\mathcal{O}_{\mathbb{P}^n}(j)$ and hence isomorphic according to Proposition 1.35 and Remark 1.38. Now Theorem 2.19 gives back Theorem 1.45. Note that the period of X, as defined in this chapter, is exactly the period of A. Finally, Proposition 2.22 implies Proposition 1.51 and hence Theorem 1.52.

As in Chapter 1, one can define the AS-type of a proper k-scheme as the sequence $(\mathrm{rk}(\mathcal{W}_{\mathcal{L}}))_{\mathcal{L}\in M}$, where M is the set of all invertible sheaves $\mathcal{L}$ such that there exists an indecomposable absolutely isotypical sheaf of type $\mathcal{L}$. Note that under the assumption $H^0(X,\mathcal{O}_X) = k$, $D(\mathcal{M}_{\mathcal{L}})$ is a central simple k-algebra (see proof of Proposition 2.17) and the ranks of $\mathcal{W}_{\mathcal{L}}$ are equal to the index of the central simple algebra $D(\mathcal{W}_{\mathcal{L}})$ (see also Proposition 2.17). These central simple algebras also have a period and it would be interesting to study the sequence $(\mathrm{rk}(\mathcal{W}_{\mathcal{L}}))_{\mathcal{L}\in M}$ and the periods of $D(\mathcal{W}_{\mathcal{L}})$ and relate it to the geometry of X. For Brauer–Severi varieties this was done in Chapter 1 and in the next section this will be done for generalized Brauer–Severi varieties. Furthermore, if $\mathrm{Pic}(X\otimes_k \bar{k})\simeq\mathbb{Z}$, one has the period r of X. As in the Brauer–Severi case one observes a periodicity of the AS-type with respect to r. To show this is the purpose of the next fact.

Proposition 2.24. *Let X be a proper k-scheme of period r with $\mathrm{Pic}(X\otimes_k\bar{k})\simeq\mathbb{Z}$ and $H^0(X,\mathcal{O}_X) = k$. Suppose there is an indecomposable absolutely isotypical sheaf $\mathcal{M}_{\mathcal{L}}$ of type $\mathcal{L}$, where $\mathcal{L}$ is the generator of $\mathrm{Pic}(X\otimes_k\bar{k})$. Then for all $j\in\mathbb{Z}$ the locally free sheaves $\mathcal{W}_{\mathcal{L}^{\otimes j}}$ have the following properties:*

(i) $\mathrm{rk}(\mathcal{W}_{\mathcal{L}^{\otimes j}}) = \mathrm{rk}(\mathcal{W}_{\mathcal{L}^{\otimes(-j)}})$.

(ii) $\mathrm{rk}(\mathcal{W}_{\mathcal{L}^{\otimes(j+ar)}}) = \mathrm{rk}(\mathcal{W}_{\mathcal{L}^{\otimes j}})$.

Proof. For the invertible sheaf $\mathcal{L}$ we have indecomposable locally free sheaves $\mathcal{W}_{\mathcal{L}}$ and $\mathcal{W}_{\mathcal{L}^\vee}$. Since $\mathcal{W}_{\mathcal{L}}$ is an absolutely isotypical sheaf of type $\mathcal{L}$, we conclude that $\mathcal{W}_{\mathcal{L}}^\vee$ is absolutely isotypical of type $\mathcal{L}^\vee$. According to Proposition 2.5, $\mathcal{W}_{\mathcal{L}}^\vee$ is isomorphic to $\mathcal{W}_{\mathcal{L}^\vee}$ and hence both have the same rank. The same argument shows that $\mathrm{rk}(\mathcal{W}_{\mathcal{L}^{\otimes j}}) = \mathrm{rk}(\mathcal{W}_{\mathcal{L}^{\otimes(-j)}})$. This proves (i). To prove (ii), we apply the same argument. For this let $\mathcal{J}$ be the generator of $\mathrm{Pic}(X)$ and note that by the definition of the period of X, we have $\mathcal{J}\otimes_k\bar{k}\simeq\mathcal{L}^{\otimes r}$. Now consider the sheaf $\mathcal{W}_{\mathcal{L}^{\otimes(j+ar)}}$ and note that it is absolutely isotypical of type $\mathcal{L}^{\otimes(j+ar)}$. The locally free sheaf $\mathcal{W}_{\mathcal{L}^{\otimes j}}\otimes\mathcal{J}^{\otimes a}$ is indecomposable and also absolutely isotypical of type $\mathcal{L}^{\otimes(j+ar)}$. Again, by Proposition 2.5 we conclude that $\mathcal{W}_{\mathcal{L}^{\otimes(j+ar)}}\simeq\mathcal{W}_{\mathcal{L}^{\otimes j}}\otimes\mathcal{J}^{\otimes a}$ and hence both have the same rank. This completes the proof. □

Remark 2.25. Under the assumption on X as in the above proposition, we conclude that the AS-type of X (as in the case of Brauer–Severi varieties) is completely determined by the $r+1$-tupel $(1,\mathrm{rk}(\mathcal{W}_{\mathcal{L}}),...,\mathrm{rk}(\mathcal{W}_{\mathcal{L}^{\otimes r-1}}),1)$. Note that $\mathrm{rk}(\mathcal{W}_{\mathcal{O}_{X\otimes_k\bar{k}}}) = \mathrm{rk}(\mathcal{M}_{\mathcal{L}^{\otimes r}}) = 1$, since both the structure sheaf and $\mathcal{L}^{\otimes r}$ descent.

We now want to discuss a potential example where the above theorems apply. We cite the *Noether–Lefschetz Theorem* that can be found among others in [73].

Theorem 2.26. *If S_d is a general surface of degree $d\geq 4$ in $\mathbb{P}^3_{\mathbb{C}}$, then $\mathrm{Pic}(\mathbb{P}^3_{\mathbb{C}})\simeq\mathrm{Pic}(S_d)$.*

The notion *general* in the above theorem has the following meaning (see [73]): Let Y be the space of surfaces of degree d in $\mathbb{P}^3_{\mathbb{C}}$. One can show that Y is isomorphic to $\mathbb{P}^N_{\mathbb{C}}$, for some $N>0$. That S_d is *general* now means that $S_d\in Y\smallsetminus V$, where V is a countable union of proper subvarieties of $Y\simeq\mathbb{P}^N_{\mathbb{C}}$.

Note that in the case where we consider general hypersurfaces S_d of degree d in $\mathbb{P}^n_{\mathbb{C}}$, for $n>3$, the Grothendieck–Lefschetz Theorem implies that $\mathrm{Pic}(S_d)\simeq\mathbb{Z}$.

Precisely, this Theorem states that smooth complete intersections in $\mathbb{P}^n_{\mathbb{C}}$, for $n > 3$ have Picard group isomorphic to $\mathbb{Z}$ (see [SGA] 2, Corollary 3.7). Now let $G = \mathrm{Gal}(\mathbb{C}|\mathbb{R})$ be the Galois group of the extension $\mathbb{R} \subset \mathbb{C}$. Suppose that on the hypersurface (or complete intersection) S_d the involution $\sigma \in G$ acts as an automorphism. Then we can consider the quotient scheme $X_d = S_d//G$, that in this situation is a projective $\mathbb{R}$-scheme, since as a projective $\mathbb{C}$-scheme, S_d has a G-stable affine cover. It has the property of being isomorphic to S_d after base change to $\mathbb{C}$. Assuming furthermore, that the generator of S_d is G-invariant, Theorem 2.19 applies and we can classify all AS-bundles on X_d. In view of the exact sequence (2.1), we have in this special situation:

$$0 \longrightarrow \mathrm{Pic}(X_d) \longrightarrow \mathrm{Pic}^G(S_d) \longrightarrow \mathrm{Br}(\mathbb{R}).$$

But as pointed out in Chapter 1, Example 1.14, $\mathrm{Br}(\mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z}$. This implies that the period of X_d is either one or two. To see this, let $\mathcal{L}$ be the generator of $\mathrm{Pic}(S_d)$ and $\mathcal{J}$ the generator of $\mathrm{Pic}(X_d)$. Now for the generator $\mathcal{L}$ there exists a unique indecomposable absolutely isotypical sheaf $\mathcal{W}_{\mathcal{L}}$ of type $\mathcal{L}$. Since $\mathrm{Br}(\mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z}$, the central division $\mathbb{R}$-algebra $D(\mathcal{W}_{\mathcal{L}})$ has to be of index one or of index two. But the index of $D(\mathcal{W}_{\mathcal{L}})$ is according to Proposition 2.17 the rank of $\mathcal{W}_{\mathcal{L}}$. In the case the index is two, taking the determinant of $\mathcal{W}_{\mathcal{L}}$ yields $\bigwedge^2 \mathcal{W}_{\mathcal{L}} \simeq \mathcal{L}^{\otimes 2}$. But this implies that the period of X_d has to be at most two. In the case the index is one, we do not get anything new and the indecomposable AS-bundles are the invertible sheaves. In the case the index is two, we have that the AS-bundles $\mathcal{E}$ on X_d are of the form

$$\mathcal{E} \simeq \left(\bigoplus_{i=1}^{r} \mathcal{J}^{\otimes a_i} \right) \oplus \left(\bigoplus_{j=1}^{s} \mathcal{J}^{\otimes b_j} \otimes \mathcal{W}_{\mathcal{L}} \right) \tag{2.3}$$

with unique r, s, a_i and b_j. The AS-type of X_d is in this case $(1,2,1)$.

Remark 2.27. The above discussion shows that a Brauer–Severi variety over $\mathbb{R}$ and a twisted $\mathbb{R}$-form of a general hypersurface as above would have the same AS-type.

The hole discussion on AS-bundles on proper k-schemes, especially the investigation of Brauer–Severi varieties and the discussion of general degree d hypersurfaces in $\mathbb{P}^n$ for $n \geq 3$ shows, that the AS-type hardly depends on $\mathrm{Pic}(X)$, $\mathrm{Br}(k)$ and the way how the Galois group acts on the scheme X. Remind that the AS-type also gives a birational invariant in the case X is a Brauer–Severi variety. This shows that the AS-type is closely connected to the geometry of X and as pointed out a few lines up it would be interesting to investigate the relation between the AS-type and the periods of the central simple algebras $D(\mathcal{M}_{\mathcal{L}})$ on one side and the geometry of X on the other side. As a last comment, to provide some more examples where a classification of the AS-type is possible, one can try to generalize Theorem 2.19 in the sense that instead of the assumption on $\mathrm{Pic}(X \otimes_k \bar{k})$ being isomorphic to $\mathbb{Z}$, one can assume that $\mathrm{Pic}(X \otimes_k \bar{k}) \simeq \mathbb{Z}^{\oplus m}$. An understanding of AS-bundles and the AS-type of this schemes would lead to the investigation of projective bundles $\mathbb{P}(\mathcal{E})$ of locally free sheaves $\mathcal{E}$ on X, provided $\mathrm{Pic}(X) \simeq \mathbb{Z}$.

2.2 Generalized Brauer–Severi varieties

In this section we give another application of the classification of AS-bundles on proper k-schemes. We consider generalized Brauer–Severi varieties and classify all AS-bundles and determine the AS-type. We start with some preliminary notations and considerations concerning generalized Brauer–Severi varieties. The main references are [32] and [115].

Let A be a central simple k-algebra of degree n and $1 \leq d \leq n$. Now consider the subset of $\mathrm{Grass}_k(d \cdot n, A)$ consisting of those subspaces of A that are left ideals L of dimension $d{\cdot}n$. This subset of $\mathrm{Grass}_k(d{\cdot}n, A)$ can be given a structure of a projective scheme over k, defined by the relations stating that the L are left ideals (see [32], p.100 for details). We now give the definition of generalized Brauer–Severi varieties. We follow the definition given in [32].

Definition 2.28. Let A be a central simple k-algebra of degree n and $1 \leq d \leq n$. The *generalized Brauer–Severi variety* corresponding to A is the the projective scheme of left ideals of A of dimension $d \cdot n$ and is denoted by $\mathrm{BS}(d, A)$. It is a closed subscheme of the Grassmannian $\mathrm{Grass}_k(dn, A)$.

In what follows we want to state some properties of the generalized Brauer–Severi varieties. They all can be found in [32], where Blanchet for the first time presented a systematic discussion of these properties. We start with Proposition 1 of [32].

Proposition 2.29. *Let A be a central simple k-algebra of degree n and $1 \leq d \leq n$. Then for a field extension $k \subset E$ one has $\mathrm{BS}(d, A \otimes_k E) \simeq \mathrm{BS}(d, A) \otimes_k E$.*

In the case the central simple k-algebra A is split and hence isomorphic to some matrix algebra over k, we have the following result (see [32], Corollary 1).

Proposition 2.30. *If the central simple k-algebra A of degree n is isomorphic to $M_n(k)$, then one has $\mathrm{BS}(d, A) \simeq \mathrm{Grass}_k(d, n)$.*

An immediate consequence of the last two propositions is the following:

Corollary 2.31. *Let A be a central simple k-algebra A of degree n and $1 \leq d \leq n$. Then there is a finite Galois extension $k \subset K$ such that $\mathrm{BS}(d, A) \otimes_k K \simeq \mathrm{Grass}_K(d, n)$.*

Proof. Since the central simple k-algebra A is split by a finite Galois extension $k \subset K$ (see [71], Corollary 2.2.6), we conclude with Proposition 2.29 and 2.30:

$$\mathrm{BS}(d, A) \otimes_k K \simeq \mathrm{BS}(d, A \otimes_k K) \simeq \mathrm{BS}(d, M_n(K)) \simeq \mathrm{Grass}_K(d, n).$$

This completes the proof. □

Remark 2.32. Since $\bar{k}$ is also a splitting field for a central simple k-algebra A, one clearly has $\mathrm{BS}(d, A) \otimes_k \bar{k} \simeq \mathrm{Grass}_{\bar{k}}(d, n)$. Note that Corollary 2.31 also shows that generalized Brauer–Severi varieties are k-forms of the Grassmannians.

Corollary 2.33. *Let A be a central simple k-algebra of degree n and $d = 1$. Then the generalized Brauer–Severi variety $\mathrm{BS}(1, A)$ is isomorphic to the Brauer–Severi variety corresponding to A.*

Proof. Let $k \subset E$ be a finite Galois splitting field for A. Since $\mathrm{BS}(1,A)\otimes_k E \simeq \mathrm{Grass}_E(1,n) = \mathbb{P}^{n-1}$, by Corollary 2.31, we conclude that the scheme $\mathrm{BS}(1,A)$ is a Brauer–Severi variety over k. Now see [10], (1.1) to conclude that this Brauer–Severi variety corresponds to the central simple k-algebra A. □

In Chapter 1, Theorem 1.10 we have seen that there is a one-to-one correspondence between isomorphism classes of Brauer–Severi varieties over k and isomorphism classes of central simple k-algebras. This result does not hold for generalized Brauer–Severi varieties as can be concluded by the next result, originally due to Chow but stated also in [32], Theorem 1.

Theorem 2.34. *Let* $\mathrm{Grass}_k(d,n)$ *be the Grassmannian over a field* k *of characteristic zero. Then for the* k*-automorphism one has* $\mathrm{Aut}_k(\mathrm{Grass}_k(d,n)) = \mathrm{PGL}_n(k)$ *if* $d = 1$, *or* $2d \neq n$ *and* $\mathrm{Aut}_k(\mathrm{Grass}_k(d,n)) = \mathrm{PGL}_n(k)\times \mathbb{Z}/2\mathbb{Z}$ *if* $2d = n$.

As in Chapter 1 in the case of Brauer–Severi varieties, for a Galois extension $k \subset L$ the isomorphism classes of k-forms of the Grassmannian over L are in one-to-one correspondence with elements in $H^1(G,\mathrm{Aut}_L(\mathrm{Grass}_L(d,n)))$, where G is the Galois group. But since the automorphism group of the Grassmannian is not isomorphic to the automorphism group of $M_n(L)$ if $2d = n$, there are in general other forms not coming from k-forms of $M_n(L)$. This is the reason why a definition of a generalized Brauer–Severi variety, corresponding to a central simple k-algebra A, as a k-scheme becoming isomorphic to the Grassmannian after base change is not sensible. Indeed, the generalized Brauer–Severi varieties are forms of the Grassmannian, but there are in general forms that are not associated to a central simple algebra. It is still possible to consider k-forms of Grassmannians, but the interesting feature of generalized Brauer–Severi varieties X is the fact that one can relate the geometry of X to the structure of the associated central simple algebra.
As for the Brauer–Severi varieties in Chapter 1, applying Theorem 2.7.1 in [77] (or [92], Lemma 2.12), one can show that $\mathrm{BS}(d,A)$ is irreducible and smooth. Clearly, Theorem 1.33 also holds for the class of locally free sheaves on $\mathrm{BS}(d,A)$.

Corollary 2.35. *Let* A *be a central simple* k*-algebra of degree* n *and* $1 \le d \le n$. *Then the Krull–Schmidt Theorem holds for locally free sheaves of finite rank on* $\mathrm{BS}(d,A)$.

On the Grassmannian $X = \mathrm{Grass}_k(d,n)$ there is the tautological exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_X^{\oplus n} \longrightarrow \mathcal{Q} \longrightarrow 0$$

with the tautological sheaf $\mathcal{S}$, that is a locally free sheaf of rank d. On the generalized Brauer–Severi variety $\mathrm{BS}(d,A)$ one has also a tautological short exact sequence (see [115], p.114)

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathrm{BS}(d,A)}^{\oplus n^2} \longrightarrow \mathcal{R} \longrightarrow 0.$$

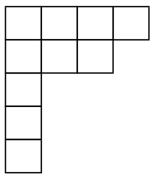
This short exact sequence has the property that after base change to some splitting field L one gets on $\mathrm{BS}(d,A)\otimes_k L \simeq \mathrm{Grass}_L(d,n)$ (see [115], Section 4):

$$0 \longrightarrow \mathcal{S}^{\oplus n} \longrightarrow \mathcal{O}_X^{\oplus n^2} \longrightarrow \mathcal{Q}^{\oplus n} \longrightarrow 0.$$

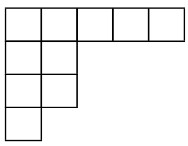
Therefore one has a locally free sheaf $\mathcal{I}$ on $\mathrm{BS}(d,A)$ such that $\mathcal{I} \otimes_k L$ is isomorphic to $\mathcal{S}^{\oplus n}$, where $\mathcal{S}$ is the tautological sheaf on $\mathrm{BS}(d,A) \otimes_k L \simeq \mathrm{Grass}_L(d,n)$. According to Proposition 2.5 the locally free sheaf $\mathcal{I}$ is unique up to isomorphism and motivates the following definition:

Definition 2.36. Let A be a central simple k-algebra of degree n and $1 \leq d \leq n$. Then the locally free sheaf $\mathcal{I}$ on $\mathrm{BS}(d,A)$ from above is called the *tautological sheaf* of the generalized Brauer–Severi variety $\mathrm{BS}(d,A)$.

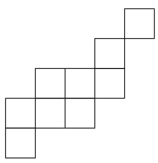
To continue with the investigation of generalized Brauer–Severi varieties we now want to recall some basic facts about the Schur functor that also became important in the next chapter. The main reference is [67] (see also [2] for a broader exposition on Schur functors). We start with Young diagrams. A *Young diagram* is a collection of boxes arranged in rows, with a weakly decreasing number of boxes. This gives a partition of the total number of boxes n. For example, a partition of 10 corresponds to the Young diagram



We denote such a partition by $\lambda = (\lambda_1, ..., \lambda_m)$, where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m$. For the above example the partition of 10 is denoted by $\lambda = (4,3,1,1,1)$. Flipping a Young diagram for a partition λ over its main diagonal gives the *conjugate diagram*. This partition is denoted by λ' and, considering the partition $\lambda = (4,3,1,1,1)$ from above, the Young diagram for the conjugate λ' is given as



A *skew diagram* is the diagram obtained by removing a smaller Young diagram from a lager one that contains it. For example if we take the Young diagram corresponding to $\lambda = (5,4,4,3,1)$ and the Young diagram corresponding to $\mu = (4,3,1)$, then the skew diagram is



and is denoted by λ/μ. Furthermore, for a partition λ we write $|\lambda|$ for $\sum \lambda_i$ and $|\lambda/\mu|$ for $\sum(\lambda_i - \mu_i)$. For any commutative ring R and any R-module M, and any partition λ one can construct an R-module $\Sigma^\lambda(M)$ with the property that for the partition $\mu = (n)$ and $\gamma = (1,1,...,1)$ one has $\Sigma^\mu(M) = \mathrm{Sym}^n(M)$ and $\Sigma^\gamma(M) = \bigwedge^n(M)$. This module is called *Schur module* of M. We shortly explain how one gets the Schur module and refer to [67], § 8.1 for further details. Let λ be a partition of n. We will write $M^{\times\lambda}$ for the cartesian product of n copies of the module M, labeled by the n boxes of the Young diagram corresponding to the partition λ. An element $m \in M^{\times\lambda}$ is given by specifying an element of M for each box of the Young diagram. Now consider maps $\phi : M^{\times\lambda} \to N$ from $M^{\times\lambda}$ to a R-module N, with the following properties:

(1) ϕ is R-multilinear.

(2) ϕ is alternating in the entries of any column of the partition λ. That is, ϕ vanishes whenever two entries in the same column are equal.

(3) ϕ satisfies an exchange condition. For any $m \in M^{\times\lambda}$, $\phi(m) = \sum \phi(v)$, where the sum is taken over all v obtained from m by an exchange between two given columns, with a given subset of boxes in the chosen column.

We define the Schur module $\Sigma^{\lambda}(M)$ to be the universal target module for such maps ϕ. Thus, $\Sigma^{\lambda}(M)$ is a R-module and there is a map $h : M^{\times\lambda} \to \Sigma^{\lambda}(M)$, satisfying (1), (2) and (3), such that for any N and any $\phi : M^{\times\lambda} \to N$, there is a unique map $\tilde{\phi} : \Sigma^{\lambda}(M) \to N$ such that the diagram

$$\begin{array}{ccc} M^{\times\lambda} & \xrightarrow{\phi} & N \\ & {\scriptstyle h}\searrow \quad \nearrow{\scriptstyle \tilde{\phi}} & \\ & \Sigma^{\lambda}(M) & \end{array}$$

commutes. By the universal property, the Schur module $\Sigma^{\lambda}(M)$ is unique up to canonical isomorphism. This construction gives a functor from the category of R-modules to itself. When $\lambda = (n)$, property (2) is empty and (3) says that all entries commute. By the universal property of the Schur module we see that $\Sigma^{\lambda}(M) = \mathrm{Sym}^n(M)$. Similarly, if $\lambda = (1, 1, ..., 1)$ then property (3) is empty and (2) says that all entries are alternating. Hence $\Sigma^{\lambda}(M) = \bigwedge^n(M)$. The above construction yields also a functor from the category of locally free sheaves to itself. Thus, for our purposes, where we are primarily interested in smooth, projective and integral k-schemes, we get a for a locally free sheaf $\mathcal{E}$ and a partition λ a locally free sheaf $\Sigma^{\lambda}(\mathcal{E})$. Finally, we note that the Schur modules became also important in the representation theory of $GL_n(k)$ (see [67]).

We now proceed with the investigation of generalized Brauer–Severi varieties. For this, let A be a central simple k-algebra of degree n and $\mathrm{BS}(d, A)$ the generalized Brauer–Severi variety. Then $\mathrm{BS}(d, A) \otimes_k L \simeq \mathrm{Grass}_L(d, n)$, for an arbitrary splitting field L of A. According to Corollary 2.31 this splitting field L ca be chosen to be finite and Galois. Let Σ^{λ} be the Schur functor associated with the partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_d)$, where $0 \leq \lambda_i \leq n - d$. Then Levine, Srinivas and Weyman [115], Section 4 proved that for the tautological sheaf $\mathcal{S}$ on $\mathrm{Grass}_L(d, n)$ the locally free sheaves $\Sigma^{\lambda}(\mathcal{S})^{\oplus n \cdot |\lambda|}$ descent to locally free sheaves $\mathcal{N}_{\lambda}$ on $\mathrm{BS}(d, A)$. According to Proposition 2.5, these locally free sheaves are unique up to isomorphism. Especially for $\lambda = (1, 1, ..., 1)$ we conclude that the locally free sheaf $\det(\mathcal{S})^{\oplus n \cdot d}$ descents to a locally free sheaf that we want to denote by $\mathcal{N}$. We recall that for the Grassmannian $\mathrm{Grass}(d, n)$ the Picard group coincides with its first Chow group (see [68], Example 15.3.6) and by Proposition

14.6.6 in [68] we have that $\mathrm{Pic}(\mathrm{Grass}(d,n)) \simeq \mathbb{Z}$, where the generator can be taken to be $\mathcal{L} = \det(\mathcal{S}^\vee)$. For a generalized Brauer–Severi variety $\mathrm{BS}(d,A)$ we therefore also have $\mathrm{Pic}(\mathrm{BS}(d,A)) \simeq \mathbb{Z}$. Denote by $\mathcal{M}$ be the generator of $\mathrm{Pic}(\mathrm{BS}(d,A)) \simeq \mathbb{Z}$ and let $\mathcal{L} = \det(\mathcal{S}^\vee)$ be the generator of $\mathrm{Pic}(\mathrm{BS}(d,A)\otimes_k L) \simeq \mathbb{Z}$. Then for the invertible sheaf $\mathcal{M}$ one has $\mathcal{M}\otimes_k L \simeq \mathcal{L}^{\otimes r}$ for a unique $r \in \mathbb{Z}$. The question arises, what is the relation between the period of $\mathrm{BS}(d,A)$ and the period of A. In Chapter 1 we have seen that for the Brauer–Severi varieties $X = \mathrm{BS}(1,A)$ one has that the period of X, as defined in this chapter (see discussion after Corollary 2.18), equals the period of A. To investigate the generalized Brauer–Severi varieties, we first state the following result due to Blanchet [32], Theorem 7.

Theorem 2.37. *Let $X = \mathrm{BS}(d,A)$ be a generalized Brauer–Severi variety over k and $F(X)$ the function field of X. Then the kernel of the restriction map $\mathrm{Br}(k) \to \mathrm{Br}(F(X))$ is a cyclic subgroup generated by the class of $A^{\otimes d}$.*

Now let $k \subset L$ be a finite Galois extension, that splits the central simple k-algebra A. The same arguments as in the proof of [71], Proposition 5.4.4 and Lemma 5.4.6 applied to $X = \mathrm{BS}(d,A)$, since one only needs $\mathrm{Pic}(\mathrm{BS}(d,A)\otimes_k L) \simeq \mathbb{Z}$ for the arguments, provides us with the following exact sequence:

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}^G(X\otimes_k L) \xrightarrow{\delta} \mathrm{Br}(k) \xrightarrow{res} \mathrm{Br}(F(X)).$$

For $X = \mathrm{BS}(d,A)$ and $X\otimes_k L \simeq \mathrm{Grass}_L(d,n)$, with Theorem 2.37 we obtain that the image of δ is equal to the cyclic subgroup of $\mathrm{Br}(k)$ generated by $A^{\otimes d}$. But this implies that the period of X is exactly the period of the central simple k-algebra $A^{\otimes d}$. Note that from the above discussion we know that for the generator $\mathcal{L} = \det(\mathcal{S}^\vee)$ of $\mathrm{Pic}(\mathrm{BS}(d,A)\otimes_k L)$ there is up to isomorphism an unique indecomposable locally free sheaf $\mathcal{W}_{\mathcal{L}}$, such that $\mathcal{W}_{\mathcal{L}}\otimes_k L \simeq \mathcal{L}^{\oplus \mathrm{rk}(\mathcal{W}_{\mathcal{L}})}$. We conclude that $\mathcal{W}_{\mathcal{L}}$ has to be the direct summand of the above sheaf $\mathcal{N}^\vee$. As explained before, for all $\mathcal{L}^{\otimes j}$, with $j \in \mathbb{Z}$, there also exist up to isomorphism unique indecomposable locally free sheaves $\mathcal{W}_{\mathcal{L}^{\otimes j}}$, with $\mathcal{W}_{\mathcal{L}^{\otimes j}}\otimes_k L \simeq (\mathcal{L}^{\otimes j})^{\oplus \mathrm{rk}(\mathcal{W}_{\mathcal{L}^{\otimes j}})}$. Replacing L by $\bar{k}$ the same arguments show that there are unique indecomposable absolutely rank-one-isotypical sheaves for every invertible sheaf $\mathcal{L} \in \mathrm{Pic}(\mathrm{Grass}_{\bar{k}}(d,n))$.

Definition 2.38. Let A be a central simple k-algebra of degree n and $1 \le d \le n$. We define for $j \in \mathbb{Z}$ the locally free sheaves $\mathcal{W}_{\mathcal{L}^{\otimes j}}$ on $\mathrm{BS}(d,A)$ to be the indecomposable absolutely isotypical sheaves of type $\mathcal{L}^{\otimes j}$, where $\mathcal{L} = \det(\mathcal{S}^\vee)$ is the ample generator of $\mathrm{Pic}(\mathrm{Grass}_{\bar{k}}(d,n))$.

For the projection $\pi : \mathrm{BS}(d,A)\otimes_k \bar{k} \to \mathrm{BS}(d,A)$ we immediately hold on to the following fact:

Proposition 2.39. *The locally free sheaves $\mathcal{W}_{\mathcal{L}^{\otimes j}}$ on $\mathrm{BS}(d,A)$ are indecomposable AS-bundles and have the property*

$$\pi^*\mathcal{W}_{\mathcal{L}^{\otimes j}} \simeq ((\det\mathcal{S}^\vee)^{\otimes j})^{\oplus \mathrm{rk}(\mathcal{W}_{\mathcal{L}^{\otimes j}})} \quad for \quad j \ge 0$$
$$\pi^*\mathcal{W}_{\mathcal{L}^{\otimes j}} \simeq ((\det\mathcal{S})^{\otimes j})^{\oplus \mathrm{rk}(\mathcal{W}_{\mathcal{L}^{\otimes j}})} \quad for \quad j < 0,$$

where $\mathcal{S}$ is the tautological sheaf on $\mathrm{BS}(d,A)\otimes_k \bar{k} \simeq \mathrm{Grass}(d,n)$.

Note that $H^0(\mathrm{BS}(d,A)\otimes_k \bar{k}, \mathcal{O}_{\mathrm{BS}(d,A)}\otimes_k \bar{k}) \simeq \bar{k}$ and hence $\mathrm{BS}(d,A)$ is geometrically integral. As an application of Theorem 2.19 we get the following classification of AS-bundles on generalized Brauer–Severi varieties:

Theorem 2.40. *Let $X = \mathrm{BS}(d,A)$ be a generalized Brauer–Severi variety over a field k for the central simple k-algebra A of period r and $\mathcal{M}$ the generator of $\mathrm{Pic}(\mathrm{BS}(d,A))$. Then all indecomposable AS-bundles of finite rank are of the form:*

$$\mathcal{M}^{\otimes a}\otimes \mathcal{W}_{\mathcal{L}^{\otimes j}}$$

with unique $a\in\mathbb{Z}$ and unique $0\leq j\leq r-1$.

With the same notation as above we have:

Corollary 2.41. *Let $X = \mathrm{BS}(d,A)$ be a generalized Brauer–Severi variety over k for the central simple k-algebra A of period r. Then the AS-bundles $\mathcal{E}$ of finite rank are of the form:*

$$\mathcal{E}\simeq\bigoplus_{j=0}^{r-1}\left(\bigoplus_{i=0}^{m_j}\mathcal{M}^{\otimes a_{i_j}}\otimes\mathcal{W}_{\mathcal{L}^{\otimes j}}\right)$$

with unique a_{i_j}, s and m_j and $0\leq j\leq r-1$.

Proof. This is a direct application of Theorem 2.40 and Corollary 2.20. □

As in the case of Brauer–Severi varieties, to have a complete understanding of the AS-bundles on generalized Brauer–Severi varieties we have to determine the ranks of the locally free sheaves $\mathcal{W}_{\mathcal{L}^{\otimes j}}$.

Proposition 2.42. *Let $X = \mathrm{BS}(d,A)$ be a generalized Brauer–Severi variety over k for the central simple k-algebra A of degree n. Then for all $j\in\mathbb{Z}$ one has:*

$$\mathrm{rk}(\mathcal{W}_{\mathcal{L}^{\otimes j}}) = \mathrm{ind}(A^{\otimes j\cdot d}).$$

Proof. Let $\lambda=(\lambda_1,\lambda_2,...,\lambda_d)$ be a partition with $0\leq\lambda_i\leq n-d$ and $\mathcal{S}$ the tautological sheaf on $\mathrm{BS}(d,A)\otimes_k L\simeq \mathrm{Grass}_L(d,n)$, where L is an arbitrary finite Galois splitting field. As mentioned above, in [115], Section 4 it is proved that the locally free sheaves $\Sigma^{\lambda}(\mathcal{S})^{n\cdot|\lambda|}$ descent to locally free sheaves $\mathcal{N}_{\lambda}$ that are unique up to isomorphism according to Proposition 2.5. Especially the sheaf $\det(\mathcal{S})^{\oplus n\cdot d}$ descents to a sheaf that we denote by $\mathcal{N}$. Then for the endomorphism algebra of $\mathcal{N}^{\vee}$ one has $\mathrm{End}(\mathcal{N}^{\vee})\simeq A^{\otimes d}$ (see also [115]). Since $\mathcal{N}^{\vee}$ is absolutely isotypical of type $\det(\mathcal{S}^{\vee})$, we conclude that $A^{\otimes d}$ is a matrix algebra over $\mathrm{End}(\mathcal{W}_{\mathcal{L}}) = D(\mathcal{W}_{\mathcal{L}})$. Applying Proposition 2.22 yields that $\mathrm{rk}(\mathcal{W}_{\mathcal{L}^{\otimes j}}) = \mathrm{ind}(D(\mathcal{W}_{\mathcal{L}})^{\otimes j})$. But the index of $D(\mathcal{W}_{\mathcal{L}})^{\otimes j}$ is the same as the index of $A^{\otimes j\cdot d}$. □

Together with Theorem 2.40 this classifies all AS-bundles on generalized Brauer–Severi varieties. We summarize the previews results in the following:

Theorem 2.43. *Let $X = \mathrm{BS}(d,A)$ be a generalized Brauer–Severi variety over k for the central simple k-algebra A of degree n and period r. Then the AS-bundles $\mathcal{E}$ of finite rank are of the form:*

$$\mathcal{E} \simeq \bigoplus_{j=0}^{r-1} \left(\bigoplus_{i=0}^{m_j} \mathcal{M}^{\otimes a_{i_j}} \otimes \mathcal{W}_{\mathcal{L}^{\otimes j}} \right)$$

with unique a_{i_j}, s, m_j and $0 \leq j \leq r-1$ and $\mathrm{rk}(\mathcal{W}_{\mathcal{L}^{\otimes j}}) = \mathrm{ind}(A^{\otimes j \cdot d})$.

In the case that we are given a central simple k-algebra of degree n and $d = 1$, we have that $\mathrm{BS}(1,A)$ is isomorphic to the Brauer–Severi variety corresponding to A. Clearly, Proposition 2.42 should give back Proposition 1.51. It is the purpose of the next proposition to make this clear.

Proposition 2.44. *Let $X = \mathrm{BS}(1,A)$ be a Brauer–Severi variety over k corresponding to the central simple k-algebra A of degree n. Then for all $j \in \mathbb{Z}$ one has $\mathcal{W}_{\mathcal{L}^{\otimes j}} \simeq \mathcal{W}_j$ and $\mathrm{rk}(\mathcal{W}_{\mathcal{L}^{\otimes j}}) = \mathrm{ind}(A^{\otimes |j| \cdot 1}) = \mathrm{ind}(A^{\otimes |j|}) = \mathrm{rk}(\mathcal{W}_j)$.*

Proof. To prove that $\mathcal{W}_{\mathcal{L}^{\otimes j}}$ is isomorphic to $\mathcal{W}_j$ we have to investigate the short exact sequence of Example 1.30. Since the degree of A is n, the Brauer–Severi variety $\mathrm{BS}(1,A)$ became isomorphic to $\mathbb{P}^{n-1} = \mathrm{Grass}_{\bar{k}}(1,n)$ after base change to the algebraic closure $\bar{k}$. The Euler sequence for the projective space $\mathbb{P}^{n-1}$ is

$$0 \longrightarrow \Omega^1_{\mathbb{P}^{n-1}} \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{\oplus n} \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow 0.$$

Tensoring with $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ yields the tautological sequence for $\mathrm{Grass}_{\bar{k}}(1,n) = \mathbb{P}^{n-1}$

$$0 \longrightarrow \Omega^1_{\mathbb{P}^{n-1}}(1) \longrightarrow \mathcal{O}^{\oplus n}_{\mathbb{P}^{n-1}} \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(1) \longrightarrow 0,$$

where $\Omega^1_{\mathbb{P}^{n-1}}(1)$ is the tautological sheaf on $\mathrm{Grass}_{\bar{k}}(1,n) = \mathbb{P}^{n-1}$. Since we have $\mathcal{L} = \det((\Omega^1_{\mathbb{P}^{n-1}}(1))^{\vee}) \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(1)$, we immediately conclude with Proposition 2.5 that $\mathcal{W}_{\mathcal{L}^{\otimes j}}$ is isomorphic to $\mathcal{W}_j$. Furthermore, since $\mathrm{rk}(\mathcal{W}_{\mathcal{L}^{\otimes j}}) = \mathrm{rk}(\mathcal{W}_j) = \mathrm{ind}(A^{\otimes j \cdot 1})$, we get with $\mathrm{ind}(A^{\otimes |j|}) = \mathrm{ind}(A^{\otimes j})$ the assertion. $\square$

As in Section 3, Theorem 1.50, one can ask for a *AS*-criterion for the generalized Brauer–Severi variety. To get such a criterion, we want to exploit the fact that generalized Brauer–Severi varieties are twisted forms of the Grassmannians. We first state a splitting criterion for the Grassmannians that is due to Ottaviani [127], Theorem 2.1. Remind that $\mathcal{Q}$ is the quotient sheaf sitting in the tautological sequence of $\mathrm{Grass}(d,n)$.

Theorem 2.45. *Let k be a field of characteristic zero and $n \geq 3$. Then a locally free sheaf $\mathcal{E}$ on $X = \mathrm{Grass}_k(d,n)$ splits as a direct sum of invertible sheaves if and only if for $0 < r < \dim(X)$ and all $t \in \mathbb{Z}$ one has*

$$H^r(X, \bigwedge^{i_1}(\mathcal{Q}^{\vee}) \otimes \ldots \otimes \bigwedge^{i_s}(\mathcal{Q}^{\vee}) \otimes \mathcal{E}(t)) = 0$$

for all $i_1, \ldots, i_s$, such that $0 \leq i_1, \ldots, i_s \leq n-d$ and $s \leq d$.

Remark 2.46. In the above Theorem one has to make the assumption on k, since it is proved by applying the Borel–Weil–Bott Theorem that only holds in characteristic zero.

We now can state a AS-criterion for generalized Brauer–Severi varieties over fields of characteristic zero. We denote by $\mathcal{M}$ the generator of $\mathrm{Pic}(\mathrm{BS}(d,A))$. Furthermore, considering the above $i_1,...,i_s$, we note that for a fixed s-tuple $i_1,...,i_s$ the i_s can be ordered in a weakly decreasing way. We denote the reordering by $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_s$. So in this way we get a partition $\lambda = (\lambda_1,...,\lambda_s)$ and we can associate a Young diagram to it with at most d rows and $n-d$ columns. Now let μ' be the conjugate of the partition

$$\mu = \square\square\square\square\cdots\square$$

where we have exactly λ_i boxes. Hence we get $\Sigma^{\mu'}(\mathcal{Q}^\vee) = \bigwedge^{\lambda_i}(\mathcal{Q}^\vee)$ on $\mathrm{BS}(d,A)\otimes_k \bar{k} \simeq \mathrm{Grass}_{\bar{k}}(d,n)$. In [115], Section 4 it is shown that $(\Sigma^{\mu'}(\mathcal{Q}^\vee))^{\oplus n\cdot|\mu'|}$ descents to a locally free sheaf $\mathcal{P}_{\lambda_i}$ on $\mathrm{BS}(d,A)$. With Proposition 2.5 we conclude that these locally free sheaves are unique up to isomorphism. Denoting by $\mathcal{L}$ the generator of $\mathrm{Pic}(\mathrm{Grass}(d,n))$, we write simply $\mathcal{F}(m)$ for $\mathcal{F}\otimes\mathcal{L}^{\otimes m}$. With this notation we have the following result:

Theorem 2.47. *(AS-criterion) Let* $\mathrm{BS}(d,A)$ *be the generalized Brauer–Severi variety of period r for the central simple k-algebra A of degree $n \geq 3$ and $\mathcal{P}_{\lambda_i}$ the locally free sheaves from above. A locally free sheaf $\mathcal{E}$ of finite rank is an AS-bundle if and only if for* $0 < r < \dim(\mathrm{BS}(d,A))$ *and all $t \in \mathbb{Z}$ one has*

$$H^r(\mathrm{BS}(d,A),\mathcal{P}_{\lambda_1}\otimes ... \otimes \mathcal{P}_{\lambda_s}\otimes \mathcal{M}^{\otimes t}\otimes\mathcal{E}) = 0$$

for all $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_s$, such that $0 \leq \lambda_1,...,\lambda_s \leq n-d$ and $s \leq d$, where $\mathcal{M}$ is the generator of $\mathrm{Pic}(\mathrm{BS}(d,A))$.

Proof. Note that for a locally free sheaf $\mathcal{E}$ we have

$$H^r((\mathrm{BS}(d,A),\mathcal{P}_{\lambda_1}\otimes ... \otimes \mathcal{P}_{\lambda_s}\otimes \mathcal{M}^{\otimes t}\otimes\mathcal{E}) = 0$$

if and only if

$$H^r(\mathrm{Grass}_{\bar{k}}(d,n),(\bigwedge^{\lambda_1}(\mathcal{Q}^\vee))^{\oplus n\cdot|\lambda_i|}\otimes ... \otimes(\bigwedge^{\lambda_s}(\mathcal{Q}^\vee))^{\oplus n\cdot|\lambda_s|}\otimes(\mathcal{E}\otimes_k\bar{k})(t\cdot r)) = 0.$$

Applying Theorem 2.45 yields the assertion. □

At the end of this chapter we want to study the relation of the AS-types between birational generalized Brauer–Severi varieties and generalized Brauer–Severi varieties corresponding to Brauer equivalent central simple algebras. It turns out that the results of Chapter 1 naturally generalize to the case of generalized Brauer–Severi varieties. Proposition 2.24 and Remark 2.25 yield that for generalized Brauer–Severi varieties of period r, the AS-type is completely determined by the $r+1$-tupel $(1,\mathrm{rk}(\mathcal{W}_{\mathcal{L}}),...,\mathrm{rk}(\mathcal{W}_{\mathcal{L}^{\otimes r-1}}))$. Recall that the period r of $\mathrm{BS}(d,A)$ equals the period of $A^{\otimes d}$. Keeping this fact in mind we have the following results:

Proposition 2.48. *Let D be a central division algebra over k of degree n and $A = M_m(D)$. Then* $\mathrm{BS}(d,D)$ *and* $\mathrm{BS}(d,A)$ *have the same AS-type.*

Proof. Since D and A are Brauer equivalent, we conclude that $\mathrm{ind}(D^{\otimes i}) = \mathrm{ind}(A^{\otimes i})$ for all $i \in \mathbb{Z}$. Thus $\mathrm{ind}(D^{\otimes d\cdot i}) = \mathrm{ind}(A^{\otimes d\cdot i})$ for all $i\in\mathbb{Z}$. Since D and A are Brauer-equivalent, the period of $D^{\otimes d}$ equals the period of $A^{\otimes d}$. Applying Proposition 2.42 and Theorem 2.43 yields the assertion. □

We note that the converse of the above proposition does not hold. This was proved in Chapter 1, Proposition 1.66. Moreover, we have the following observation.

Proposition 2.49. *Let* $\mathrm{BS}(d,A)$ *and* $\mathrm{BS}(d,B)$ *be two generalized Brauer–Severi varieties such that* A *and* B *have same index and period and that the period equals the index. Suppose furthermore that the period of* $A^{\otimes d}$ *equals the period of* $B^{\otimes d}$*. Then they have the same AS-type.*

Proof. By assumption A and B have same period and index and the in both cases the period equals the index. Now let p be both the period and the index of A and B respectively. Proposition 1.57 now yields that $p/(p,r) = \mathrm{ind}(A^{\otimes r}) = \mathrm{ind}(B^{\otimes r})$, for all $r > 0$. Applying Theorem 2.43 yields that $\mathrm{BS}(d,A)$ and $\mathrm{BS}(d,B)$ have the same AS-type. □

Proposition 2.50. *Let* $\mathrm{BS}(d,A)$ *a generalized Brauer–Severi variety corresponding to a central simple* k*-algebra* A*, such that the period* r *of* $A^{\otimes d}$ *equals the index of* $A^{\otimes d}$*. Then the AS-type is* $(1,r,r/(r,2),...,r/(r,r-2),r,1)$.

Proof. Set $A' = A^{\otimes d}$. Then r is the period and the index of A'. Applying Proposition 1.57 to A' together with Theorem 2.43 then yields the assertion. □

Proposition 2.51. *Let* D *be a central division algebra of degree* n *over a local or global field* k*. Let* r *be the period of* $\mathrm{BS}(d,D)$*. Then the AS-type of* $\mathrm{BS}(d,D)$ *is* $(1,r,r/(r,2),...,r/(r,r-2),r,1)$.

Proof. By Theorem 1.61 the period of $D^{\otimes d}$ equals the index of $D^{\otimes d}$. Proposition 2.50 yields the assertion. □

To see what happens in the case that two generalized Brauer–Severi varieties are birational we first state a consequence of Theorem 2.37.

Corollary 2.52. *Let* $X = \mathrm{BS}(d,A)$ *and* $Y = \mathrm{BS}(d,B)$ *be two birational generalized Brauer–Severi varieties over* k*, then* $A^{\otimes d}$ *and* $B^{\otimes d}$ *generate the same cyclic subgroup in* $\mathrm{Br}(k)$.

Proof. Since X and Y are birational we have $F(X) \simeq F(Y)$. Hence $\mathrm{Br}(F(X))$ is isomorphic to $\mathrm{Br}(F(Y))$ what by Theorem 2.37 implies that $A^{\otimes d}$ and $B^{\otimes d}$ generate the same cyclic subgroup in $Br(k)$. □

Proposition 2.53. *Let* $X = \mathrm{BS}(d,A)$ *and* $Y = \mathrm{BS}(d,B)$ *be two birational generalized Brauer–Severi varieties over* k*, then they have the same AS-type.*

Proof. By Corollary 2.52, the period of $A^{\otimes d}$ equals the period of $B^{\otimes d}$ and hence the period of X equals the period of Y. Now the proof of Proposition 1.67 shows that $\mathrm{ind}(A^{\otimes d\cdot j}) = \mathrm{ind}(B^{\otimes d\cdot j})$ and from Theorem 2.43 we conclude that they have the same AS-type. □

Remark 2.54. Note that Proposition 1.66 also shows that in general two generalized Brauer–Severi varieties can have the same AS-type even if they are not birational. This means that the converse of the Proposition 2.53 does not hold.

Finally, we want to mention that Levine, Srinivas and Weyman [115] investigated the K-theory of generalized Brauer–Severi varieties and showed that, as distinguished from the case of Brauer–Severi varieties, the AS-bundles do not generate the Grothendieck group of $\mathrm{BS}(d, A)$, for $d > 1$. As proved in [115], Theorem 3.4, for a generalized Brauer–Severi variety $\mathrm{BS}(d, A)$, with $\deg(A) = n$, the Grothendieck group is generated by $\mathcal{N}_\lambda$, where $\mathcal{N}_\lambda$ are the locally free sheaves obtained by descent from $\Sigma^\lambda(\mathcal{S})^{\oplus n\cdot|\lambda|}$, where Σ^λ is the Schur functor associated to the partition $\lambda = (\lambda_1, ..., \lambda_d)$ with $0 \leq \lambda_i \leq n - d$ and $\mathcal{S}$ the tautological bundle on $\mathrm{BS}(d, A) \otimes_k \bar{k} \simeq \mathrm{Grass}(d, n)$. Nonetheless, in analogy to the Brauer–Severi varieties of Chapter 1, in the next chapter we will show how the tautological sheaf $\mathcal{S}$ give rise to a tilting object on the general Brauer–Severi variety.

Chapter 3

Tilting theory in geometry

3.1 Derived category of coherent sheaves

In this chapter we want to prove the existence of tilting objects for some schemes and quotient stacks. As an application we provide some further evidence for the dimension conjecture of Orlov.

In this section we recall some basic facts about derived categories of coherent sheaves and important derived functors. The main references are [70] and [89]. Derived categories were originally introduced by Grothendieck and Verdier [151], [152] as a basement for a natural formulation of derived functors and cohomology. It turned out that the derived category of coherent sheaves carries interesting structures itself, what then started the investigation of these categories as interesting invariants of schemes. Derived categories belong to a wider class of categories called triangulated. We briefly explain the notion of a *triangulated category*. For this, let $\mathcal{T}$ denote a category with an autoequivalence $[1] : \mathcal{T} \to \mathcal{T}$ which is called *shift functor*. A *triangle* in $\mathcal{T}$ is a collection of objects A, B and C and morphisms

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1].$$

Such triangles are usually abbreviated by the following diagram:

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & C & \end{array}$$

One can define a morphism between such triangles as a commutative diagram of the form:

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

We say a morphism between triangles is an isomorphism, if f, g and h are isomorphisms.

A *triangulated category* is an additive category $\mathcal{T}$ with a shift functor [1] and triangles, called *distinguished triangles*, satisfying several axioms that we do not want to reproduce here as they can be found in loc.cit.. As stated above, an example of such a triangulated category is the *derived category* of an abelian category $\mathcal{A}$. We briefly recall the basic facts and refer to [70] or [89] for further details.
Let $\mathcal{A}$ be an abelian category and denote by $\mathrm{Kom}(\mathcal{A})$ the category of chain complexes. Two morphisms f and g between complexes are called *homotopically equivalent*, if there exist morphisms $h^i : A^i \to B^{i-1}$ such that $f^i - g^i = h^{i+1} \circ d^i + d^{i-1} \circ h^i$. One can show that this is an equivalence relation and in view of this fact we define the homotopy category of chain complexes $\mathrm{K}(\mathcal{A})$ as $\mathrm{Kom}(\mathcal{A})/\sim$, with objects being complexes and morphisms being morphisms between chain complexes modulo homotopy equivalence. A morphism between two complexes $A^\bullet$ and $B^\bullet$ is called a *quasi-isomorphism* if the induced morphism on cohomology is an isomorphism. The derived category of $\mathcal{A}$, which is denoted by $\mathrm{D}(\mathcal{A})$, is now obtained by localizing the homotopy category $\mathrm{K}(\mathcal{A})$ at $\mathcal{S}$, where $\mathcal{S}$ is the set of quasi-isomorphisms. One has the following Theorem (see [89], Theorem 2.10).

Theorem 3.1. *Let $\mathcal{A}$ be an abelian category. Then the derived category $D(\mathcal{A})$ has the following properties:*

(i) *There is a functor $Q : \mathrm{Kom}(\mathcal{A}) \to D(\mathcal{A})$ that sends quasi-isomorphisms to isomorphisms.*

(ii) *Q is universal with respect to (i) in the following sense: Given any category $\mathcal{D}$ and any functor $F : \mathrm{Kom}(\mathcal{A}) \to D(\mathcal{A})$ which sends quasi-isomorphisms to isomorphisms, then there is a unique functor $G : D(\mathcal{A}) \to \mathcal{D}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Kom}(\mathcal{A}) & \xrightarrow{\quad Q \quad} & D(\mathcal{A}) \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle G} & \\ & \mathcal{D} & \end{array}$$

If one considers the category of chain complexes $\mathrm{Kom}^\star(\mathcal{A})$, with $\star = +, -$, or b as the category of complexes $A^\bullet$ with $A^i = 0$, for $i << 0$, $i >> 0$, $|i| >> 0$ respectively, then one can construct derived categories $D^\star(\mathcal{A})$ by localizing $\mathrm{K}^\star(\mathcal{A})$ at the set of quasi-isomorphisms. One shows that the natural functor $D^\star(\mathcal{A}) \to D(\mathcal{A})$ identifies $D^\star(\mathcal{A})$ with the full subcategory of $D(\mathcal{A})$ consisting of complexes $A^\bullet$ with $H^i(A^\bullet) = 0$ for $i << 0$, $i >> 0$ and $|i| >> 0$ respectively.
While the objects of $D(\mathcal{A})$ are the same as in $\mathrm{Kom}(\mathcal{A})$, the morphisms in $D(\mathcal{A})$ are in general difficult to handle. When $\mathcal{A}$ has enough injectives, which is usually the case in geometric situations (i.e., $\mathcal{A} = \mathrm{Coh}(X)$), one has for complexes $A^\bullet, B^\bullet, I^\bullet$ in $D^+(\mathcal{A})$, where $I^\bullet$ is a complex of injective objects quasi-isomorphic to $B^\bullet$ (see [89], p.40 f.):

$$\mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet) \simeq \mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, I^\bullet).$$

Likewise, when $\mathcal{A}$ has enough projectives, which usually is not the case in geometric situations, one has for a complex $P^\bullet \in D^-(\mathcal{A})$ of projective objects quasi-isomorphic to $A^\bullet \in D^-(\mathcal{A})$ that

$$\mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet) \simeq \mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, B^\bullet).$$

Furthermore, if $\mathcal{A}$ has enough injectives, then given two objects A and B in $\mathcal{A}$ there is a isomorphism (see [89], Proposition 2.56)

$$\mathrm{Hom}_{D(\mathcal{A})}(A, B[p]) \simeq \mathrm{Ext}^p_{\mathcal{A}}(A, B).$$

This gives a possibility to calculate certain morphisms in the derived category. But there are also some important spectral sequences that may help. Again, if $\mathcal{A}$ has enough injectives and $B^\bullet$ is bounded below, we have the following spectral sequences (see [89], Example 2.70):

$$E_2^{p,q} = \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, H^q(B^\bullet)[p]) \Longrightarrow \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet[p+q]).$$

For more important spectral sequences we refer to [89]. Now we briefly want to recall *derived functors*. For this, we are given a left exact functor $\Psi : \mathcal{A} \to \mathcal{B}$ between abelian categories. We would like to extend Ψ to a functor between $D(\mathcal{A}) \to D(\mathcal{B})$. The same we would like to do for right exact functors $\Phi : \mathcal{A} \to \mathcal{B}$. If Ψ or Φ are exact, then we apply them to complexes and get the desired fuctors between $D(\mathcal{A})$ and $D(\mathcal{B})$. The question arises what one can do if the functors in question are not exact. If $\mathcal{A}$ has enough injectives we can consider the full additive subcategory $\mathcal{I} \subset \mathcal{A}$ of all injectives. It is a matter of fact that if $\mathcal{A}$ has enough injectives, $i : \mathrm{K}^+(\mathcal{I}) \to D^+(\mathcal{A})$ is an equivalence (see [89], Proposition 2.40). With this equivalence one can construct the extended functor via the following diagram:

$$\begin{array}{ccccc}
\mathrm{K}^+(\mathcal{I}_{\mathcal{A}}) & \xrightarrow{h} & \mathrm{K}^+(\mathcal{A}) & \xrightarrow{K(\Psi)} & \mathrm{K}^+(\mathcal{B}) \\
& \searrow^{i} & \downarrow{\scriptstyle Q_{\mathcal{A}}} & & \downarrow{\scriptstyle Q_{\mathcal{B}}} \\
& & D^+(\mathcal{A}) & & D^+(\mathcal{B})
\end{array}$$

The *right derived* functor of Ψ is denoted by $\mathbb{R}\Psi$ and is given by the composition $Q_{\mathcal{B}} \circ K(\Psi) \circ h \circ i^{-1}$ and is unique up to isomorphism. Similar constructions can also be made for the right exact functor.

In the following we give examples of derived categories and derived functors that become important for the rest of the work. The first example is the bounded derived category of coherent sheaves on some noetherian scheme X. We denote that derived category simply by $D^b(X)$. The underlying abelian category is $\mathrm{Coh}(X)$, the category of coherent sheaves on X. For the abelian category of quasicoherent sheaves on X we write $\mathrm{Qcoh}(X)$ whereas for the derived category we write $D(\mathrm{Qcoh}(X))$. If A is a ring, then the category of (left) right A-modules will be denoted by $\mathrm{Mod}(A)$ while the category of finitely generated A-modules will be denoted by $\mathrm{mod}(A)$. For their derived categories we write $D(\mathrm{Mod}(A))$ and $D^b(A)$ respectively. The next example arises naturally when studying group actions on schemes. For a reference on group actions on schemes see GIT of Mumford, Fogarty and Kirwan [122]. Now let X be a (noetherian) scheme over a field k and G a finite group acting on X such that the characteristic of k does not divide the order of the group G. Following [122], §3, a *G-linearized* quasicoherent sheaf $\mathcal{F}$ on X is a quasicoherent sheaf together with an isomorphism $\lambda_g : g^*\mathcal{F} \to \mathcal{F}$ for each $g \in G$, such that $\lambda_{gh} \simeq \lambda_h \circ h^*\lambda_g$ for all $g, h \in G$. Note that G-linearized sheaves are also called G-equivariant or simply

G-sheaves in the literature.
Given two quasicoherent G-sheaves $\mathcal{F}$ and $\mathcal{G}$ with *linearized structures* λ and μ, we get an action of G on $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ that can be interpreted as representation of G on $\mathrm{Hom}(\mathcal{F},\mathcal{G})$. Indeed, given $g \in G$ and $\phi \in \mathrm{Hom}(\mathcal{F},\mathcal{G})$ one has $g \cdot \phi = \mu_g^{-1} \circ \phi \circ \lambda_g$ as the group action on $\mathrm{Hom}(\mathcal{F},\mathcal{G})$. The morphisms commuting with the linearized structures are those invariant under this action and hence $\mathrm{Hom}_G(\mathcal{F},\mathcal{G}) = \mathrm{Hom}(\mathcal{F},\mathcal{G})^G$. In the special case where G acts on $\mathrm{End}(\mathcal{F})$, we see that G acts as an automorphism on the ring $\mathrm{End}(\mathcal{F})$. Note that in particular if X is for instance quasiprojective over k, one can show that the category of coherent G-linearized sheaves $\mathrm{Coh}_G(X)$ is an abelian category. If X is assumed to be smooth and quasiprojective over k, every object in $\mathrm{Coh}_G(X)$ has a finite locally free resolution. We denote by $D^b_G(X)$ the bounded derived category of coherent G-linearized sheaves and by $D_G(\mathrm{Qcoh}(X))$ the derived category of G-linearized quasicoherent sheaves.

Now we give some examples of derived functors. When $A^\bullet \in \mathrm{Kom}(\mathrm{Qcoh}_G(X))$ we let $\mathbb{R}\mathrm{Hom}(A^\bullet,-)$ be the derived functor of $\mathrm{Hom}(A^\bullet,-)$ which computes all morphisms and not just the G-linearized ones. Then $\mathbb{R}\mathrm{Hom}(A^\bullet,-)$ takes values in the derived category of representations of G which we denote by $D(\mathrm{Repr}(G))$. Assuming that the characteristic of k does not divide the order of G, taking G-invariants is exact (see Remark 3.83 below) and hence computing $\mathbb{R}\mathrm{Hom}_G(A^\bullet,-)$ can be performed by taking invariants of $\mathbb{R}\mathrm{Hom}(A^\bullet,-)$ term-by-term. For a complex $\mathcal{F}^\bullet \in \mathrm{Kom}(\mathrm{Qcoh}_G(X))$ one has functors $\mathcal{H}om(\mathcal{F}^\bullet,-) : \mathrm{K}(\mathrm{Qcoh}_G(X)) \to \mathrm{K}(\mathrm{Qcoh}_G(X))$ and $\mathcal{F}^\bullet \otimes - : \mathrm{K}(\mathrm{Qcoh}_G(X)) \to \mathrm{K}(\mathrm{Qcoh}_G(X))$ and both have right and left derived functors respectively

$$\begin{aligned} \mathbb{R}\mathcal{H}om(\mathcal{F}^\bullet,-) &: D_G(X) \longrightarrow D_G(X) \\ \mathcal{F}^\bullet \otimes^L - &: D_G(X) \longrightarrow D_G(X). \end{aligned}$$

The object $\mathbb{R}\mathcal{H}om(\mathcal{F}^\bullet,\mathcal{O}_X)$ is called the *derived dual* and is also denoted by $(\mathcal{F}^\bullet)^\vee$. For instance on a smooth quasiprojective and integral k-scheme we can resolve $\mathcal{F}^\bullet$ by a finite complex of G-linearized locally free sheaves $\mathcal{E}^\bullet$. In this case there are isomorphisms of functors:

$$\begin{aligned} \mathcal{F}^\bullet \otimes^L - &\simeq \mathcal{E}^\bullet \otimes - \\ \mathbb{R}\mathcal{H}om(\mathcal{F}^\bullet,-) &\simeq \mathcal{H}om(\mathcal{E}^\bullet,-) \simeq (\mathcal{E}^\bullet)^\vee \otimes -. \end{aligned}$$

Considering objects in $\mathrm{Coh}_G(X)$ one has

$$\begin{aligned} H^i(\mathbb{R}\mathcal{H}om(\mathcal{F},\mathcal{G}) &= \mathcal{E}xt^i(\mathcal{F},\mathcal{G}) \\ H^i(\mathcal{F} \otimes \mathcal{G}) &= \mathcal{T}or_i(\mathcal{F},\mathcal{G}) \end{aligned}$$

both carrying natural G-linearized structures. Now given a morphism $f : X \to Y$ of smooth projective G-schemes, we have functors $f^* : \mathrm{K}(\mathrm{Qcoh}_G(Y)) \to \mathrm{K}(\mathrm{Qcoh}_G(X))$ and $f_* : \mathrm{K}(\mathrm{Qcoh}_G(X)) \to \mathrm{K}(\mathrm{Qcoh}_G(Y))$. This gives the derived functors:

$$\begin{aligned} \mathbb{L}f^* &: D_G(\mathrm{Qcoh}(Y)) \longrightarrow D_G(\mathrm{Qcoh}(X)) \\ \mathbb{R}f_* &: D_G(\mathrm{Qcoh}(X)) \longrightarrow D_G(\mathrm{Qcoh}(Y)) \end{aligned}$$

Now we forget about the action of a finite group and consider two projective k-schemes X and Y. Clearly, all the above functors between the derived categories also exist without considering the schemes with the action of a finite group. Then for a proper morphism $f : X \to Y$ one has the following *projection formula* (see [89], p.83):

$$\mathbb{R}f_*(\mathcal{F}^\bullet) \otimes^L \mathcal{G}^\bullet \simeq \mathbb{R}f_*(\mathcal{F}^\bullet \otimes^L \mathbb{L}f^*(\mathcal{G}^\bullet)).$$

Furthermore the functor $\mathbb{L}f^*$ is left adjoint to $\mathbb{R}f_*$ and one has

$$\mathrm{Hom}(\mathbb{L}f^*\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq \mathrm{Hom}(\mathcal{F}^\bullet, \mathbb{R}f_*\mathcal{G}^\bullet).$$

A very important and deep compatibility between the above mentioned derived functors is the *Grothendieck–Verdier duality.* For this, let $f : X \to Y$ be a morphism of smooth schemes over a field k of relative dimension $n = \dim(X) - \dim(Y)$. The *relative dualizing sheaf* is defined as $\omega_f = \omega_X \otimes f^*\omega_Y^*$. Then one can show that for any $\mathcal{F}^\bullet \in D^b(X)$ and any $\mathcal{G}^\bullet \in D^b(Y)$ one has (see [89], Theorem 3.34):

$$\mathbb{R}\mathcal{H}om(\mathbb{R}f_*\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq \mathbb{R}f_*\mathbb{R}\mathcal{H}om(\mathcal{F}^\bullet, \mathrm{L}f^*(\mathcal{G}^\bullet) \otimes \omega_f[n]).$$

Applying global sections and taking cohomology in degree zero on both sides yields (see [89], proof of Corollary 3.25)

$$\mathrm{Hom}_{D^b(Y)}(\mathbb{R}f_*\mathcal{F}^\bullet, \mathcal{E}^\bullet) \simeq \mathrm{Hom}_{D^b(X)}(\mathcal{F}^\bullet, \mathbb{L}f^*(\mathcal{E}^\bullet) \otimes \omega_f[n]). \tag{3.1}$$

Now for a smooth and projective k scheme X one gets as a consequence from the above duality the *Serre duality for derived categories.* For this, we consider the structure morphism $f : X \to \mathrm{Spec}(k)$ and apply (3.1). If $\mathcal{E}^\bullet, \mathcal{F}^\bullet \in D^b(X)$ then (see [89], p.88)

$$\begin{aligned}\mathrm{Hom}_{D^b(X)}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \omega_X[\dim(X)]) &\simeq \mathrm{Hom}_{D^b(X)}(\mathbb{R}\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet), \omega_X[\dim(X)]) \\ &\simeq \mathrm{Hom}_{D^b(\mathrm{Spec}(k))}(\mathbb{R}\Gamma(\mathbb{R}\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)), k) \\ &\simeq \mathrm{Hom}_{D^b(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)^*.\end{aligned}$$

Hence one has

$$\mathrm{Hom}_{D^b(X)}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \omega_X[\dim(X)]) \simeq \mathrm{Hom}_{D^b(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)^*.$$

For coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ we then have:

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Ext}^{n-i}(\mathcal{G}, \mathcal{F} \otimes \omega_X)^*.$$

For $\mathcal{E}^\bullet \simeq \mathcal{O}_X$ one immediately gets

$$\mathrm{Hom}_{D^b(X)}(\mathcal{F}^\bullet, \omega_X[\dim(X)]) = \mathrm{Hom}_{D^b(\mathrm{Spec}(k))}(\mathbb{R}\Gamma(\mathcal{F}^\bullet), k),$$

what in particular for a coherent sheaf $\mathcal{F}$ implies the *classical Serre duality*

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \simeq H^{n-i}(X, \mathcal{F})^*.$$

In geometric situations the *Fourier–Mukai transforms* are also very important. A comprehensive overview of the theory of *Fourier–Mukai transforms* is developed by Bridgeland [45] (see also [89]). Suppose we are given smooth projective and integral k-schemes X and Y. Then one has two projections:

$$p : X \times Y \longrightarrow X \text{ and } q : X \times Y \longrightarrow Y.$$

Now fix an object $\mathcal{P} \in D^b(X \times Y)$. Then the functor

$$\Phi_{\mathcal{P}} : D^b(X) \longrightarrow D^b(Y), \quad \mathcal{F} \longmapsto \mathbb{R}q_*(\mathbb{L}p^*(\mathcal{F}) \otimes^L \mathcal{P})$$

is called the *Fourier–Mukai transform* with *kernel* $\mathcal{P}$. Since p is flat the $\mathbb{L}$ can be omitted. If $\mathcal{P}$ is flat over X, then with Proposition 4.2.2 in [45] one has $\Phi_{\mathcal{P}}(k(x)) = \mathcal{P}_x$. Here $\mathcal{P}_x$ is the restriction $\mathbb{L}j_x^*(\mathcal{P})$ for $j_x : p^{-1}(x) \to X \times Y$. Furthermore, the composition of Fourier–Mukai transforms is again a Fourier–Mukai transform (see [89], Proposition 5.10). We complete this repetition by giving an elementary but important example.
For a morphism $f : X \to Y$ we consider the graph $\Gamma_f \subset X \times Y$ and the object $\mathcal{O}_{\Gamma_f} \in D^b(X \times Y)$. Then one can show (see for instance [89], Example 5.4 (ii))

$$f_* \simeq \Phi_{\mathcal{O}_{\Gamma_f}} : D^b(X) \longrightarrow D^b(Y)$$
$$f^* \simeq \Phi_{\mathcal{O}_{\Gamma_f}} : D^b(Y) \longrightarrow D^b(X).$$

Since the kernel $\mathcal{O}_{\Gamma_f}$ is supported on the graph $\Gamma_f \subset X \times Y$, one has for the skyscraper sheaf of a closed point $x \in X$, $\Phi_{\mathcal{O}_{\Gamma_f}}(k(x)) = k(f(x))$.

3.2 Tilting Theory

In this section we recall some basic facts of tilting theory. Especially we will be interested in geometric situations where tilting theory can be applied.
In general, tilting theory arises as a method for constructing equivalences between categories. Classically, tilting theory was introduced to understand the module categories coming from representation theory of k-algebras. Central results were obtained by Brenner and Butler [44] and led to the work of Happel [80] and Rickard [135]. At this point we want to mention the 'Handbook of Tilting Theory' [81], which gives an excellent overview over the theory. The geometric tilting theory started with the work of Beilinson [23], where he investigated the derived category of $\mathbb{P}^n$ and proved the existence of tilting objects. We start with the definition of a tilting object in the geometric situation (see [52] or [81]).

Definition 3.2. Let X be a noetherian quasiprojective scheme over a field k and $D(\mathrm{Qcoh}(X))$ the derived category of quasicoherent sheaves on X. An object $\mathcal{T} \in D(\mathrm{Qcoh}(X))$ is called *tilting object for* $D(\mathrm{Qcoh}(X))$ if the following hold:

(i) $\mathrm{Hom}_{D(\mathrm{Qcoh}(X))}(\mathcal{T}, \mathcal{T}[i]) = 0$ for $i \neq 0$.

(ii) If $\mathcal{N} \in D(\mathrm{Qcoh}(X))$ satisfies $\mathbb{R}\mathrm{Hom}_{D(\mathrm{Qcoh}(X))}(\mathcal{T}, \mathcal{N}) = 0$, then $\mathcal{N} = 0$.

(iii) $\mathrm{Hom}_{D(\mathrm{Qcoh}(X))}(\mathcal{T}, -)$ commutes with direct sums.

Remark 3.3. If one has a tilting object $\mathcal{T} \in D(\mathrm{Qcoh}(X))$, one can form the smallest full triangulated subcategory of $D(\mathrm{Qcoh}(X))$ containing $\mathcal{T}$, that is closed under direct sums and direct summands. We denote this category by $\langle \mathcal{T} \rangle$. One can show that condition (ii) of the above definition is equivalent to the fact that $\langle \mathcal{T} \rangle = D(\mathrm{Qcoh}(X))$ (see [52], Remark 1.2). Usually, this condition is referred to as the tilting object is *generating* the derived category.

Remark 3.4. Note that for a projective k-scheme X, one can also define *tilting objects for* $D^b(X)$ as objects $\mathcal{T} \in D^b(X)$, for that (i), (ii) and (iii) from above hold in the derived category $D^b(X)$, i.e.,

(i) $\mathrm{Hom}_{D^b(X)}(\mathcal{T}, \mathcal{T}[i]) = 0$ for $i \neq 0$.

(ii) If $\mathcal{N} \in D^b(X)$ satisfies $\mathbb{R}\mathrm{Hom}_{D^b(X)}(\mathcal{T}, \mathcal{N}) = 0$, then $\mathcal{N} = 0$.

(iii) $\mathrm{Hom}_{D^b(X)}(\mathcal{T}, -)$ commutes with direct sums.

Again condition (ii) is equivalent to the fact that the smallest full triangulated subcategory containing $\mathcal{T}$, that is closed under direct sums and direct summands equals $D^b(X)$ and is also referred to as the object $\mathcal{T}$ is *generating* $D^b(X)$.

When it is clear from the context if we are considering $D(\mathrm{Qcoh}(X))$ or $D^b(X)$, we will simply write $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ for $\mathrm{Hom}_{D(\mathrm{Qcoh}(X))}(\mathcal{A}, \mathcal{B})$ respectively $\mathrm{Hom}_{D^b(X)}(\mathcal{A}, \mathcal{B})$.

Following [124], we call an object $\mathcal{F} \in D(\mathrm{Qcoh}(X))$ *compact* if $\mathrm{Hom}(\mathcal{F}, -)$ commutes with direct sums and $D(\mathrm{Qcoh}(X))$ *compactly generated*, if there exists a set G of compact objects satisfying (ii) of Definition 3.2. If X is an arbitrary quasicompact and separated scheme, [124], Proposition 2.5 states that $D(\mathrm{Qcoh}(X))$ is compactly generated. If $D(\mathrm{Qcoh}(X))$ is compactly generated, then to test that an object $\mathcal{T}$ generates $D(\mathrm{Qcoh}(X))$ it suffices to check that the smallest triangulated subcategory containing $\mathcal{T}$, that is closed under direct sums and direct summands equals the subcategory of compact objects (see [52], 1.4). In fact, if $D(\mathrm{Qcoh}(X))$ is compactly generated, a compact object $\mathcal{T}$ generates $D(\mathrm{Qcoh}(X))$ if and only if it generates the full subcategory of compact objects of $D(\mathrm{Qcoh}(X))$ (see [40], Theorem 2.1.2). Without going into the details, we want to note that in the case of a smooth projective and integral k-scheme X, the subcategory of compact objects is equivalent to the bounded derived category of coherent sheaves $D^b(X)$ (see [40], Theorem 3.1.1, together with the comment after Theorem 3.1.3). In this case the above discussion implies, that if the smallest triangulated subcategory containing $\mathcal{T}$, that is closed under direct sums and direct summands equals $D^b(X)$, then $\mathcal{T}$ generates $D(\mathrm{Qcoh}(X))$.

Remark 3.5. If X is a smooth projective and integral k-scheme and $\mathcal{T}$ a tilting object for $D(\mathrm{Qcoh}(X))$ being an element of the subcategory $D^b(X)$, then $\mathcal{T}$ also generates $D^b(X)$. In fact, as mentioned above, [40], Theorem 2.1.2 states that the generating of $D(\mathrm{Qcoh}(X))$ by $\mathcal{T}$ is equivalent to the generating of $D^b(X)$ by $\mathcal{T}$.

The previous comments now have the following well-known consequence.

Proposition 3.6. *Let X be a smooth projective and integral k-scheme. Then an object $\mathcal{T} \in D^b(X)$ is a tilting object for $D(\mathrm{Qcoh}(X))$ if and only if it is a tilting object for $D^b(X)$.*

Proof. Suppose first, $\mathcal{T}$ is a tilting object for $D(\mathrm{Qcoh}(X))$. Since the natural functor $D^b(X) \to D(\mathrm{Qcoh}(X))$ is fully faithful, we have $\mathrm{Hom}_{D^b(X)}(\mathcal{A}, \mathcal{B}) \simeq \mathrm{Hom}_{D(\mathrm{Qcoh}(X))}(\mathcal{A}, \mathcal{B})$ for all $\mathcal{A}, \mathcal{B} \in D^b(X)$. This directly implies that (i) of Remark 3.4 holds and with Remark 3.5 and the fact that $\mathrm{Hom}_{D^b(X)}(\mathcal{T}, -)$ commutes with direct sums, we conclude that $\mathcal{T}$ is a tilting object for $D^b(X)$

(as defined in Remark 3.4). Now suppose that $\mathcal{T}$ is a tilting object for $D^b(X)$ (as defined in Remark 3.4). Since $D^b(X)$ is a full triangulated subcategory of $D(\mathrm{Qcoh}(X))$, one has

$$\mathrm{Hom}_{D^b(X)}(\mathcal{T}, \mathcal{T}[i]) \simeq \mathrm{Hom}_{D(\mathrm{Qcoh}(X))}(\mathcal{T}, \mathcal{T}[i]) = 0, \text{ for } i \neq 0.$$

Furthermore, Remark 3.5 shows that if $\mathcal{T}$ generates $D^b(X)$, it also generates $D(\mathrm{Qcoh}(X))$. Since by assumption $\mathcal{T} \in D^b(X)$, we conclude that $\mathcal{T}$ is a compact object and hence $\mathrm{Hom}_{D(\mathrm{Qcoh}(X))}(\mathcal{T}, -)$ commutes with direct sums. This finally implies that $\mathcal{T}$ is a tilting object for $D(\mathrm{Qcoh}(X))$. □

Classically, there was a more restricted definition of a tilting object, namely that of a tilting sheaf for $D^b(X)$, where X is a smooth projective and integral k-scheme (see for instance Baer [18]). Recall that the global dimension of a noetherian ring is the supremum of the length of all projective resolutions of R-modules. Now for a smooth projective and integral k-scheme X, a coherent sheaf $\mathcal{T}$ on X is called a *tilting sheaf for* X if it satisfies the following conditions:

(i) $\mathrm{Ext}^l(\mathcal{T}, \mathcal{T}) = 0$ for all $l > 0$.

(ii) The sheaf $\mathcal{T}$ generates $\mathrm{D}^b(X)$ in the sense of Remark 3.4.

(iii) The k-algebra $A = \mathrm{End}(\mathcal{T})$ has finite global dimension.

If the coherent sheaf $\mathcal{T}$ is locally free, it is simply called *tilting bundle*.

Remark 3.7. Tilting sheaves $\mathcal{T}$, or more generally tilting objects as defined in 3.2 (or in Remark 3.4), are nowadays called *classical* to distinguish them from the more general notion where the derived category $D(\mathrm{Qcoh}(X))$ is equivalent to $D(\mathrm{Mod}(A))$, where $A = \mathbb{R}\mathrm{Hom}(\mathcal{T}, \mathcal{T})$ is a dg-algebra and $\mathcal{T}$ a perfect complex (see [81], [85] or see [40], Corollary 3.1.8). In this situation one gets the equivalence $D(\mathrm{Qcoh}(X)) \xrightarrow{\sim} D(\mathrm{Mod}(A))$ without assuming the vanishing of $\mathrm{Hom}(\mathcal{T}, \mathcal{T}[i])$ for $i \neq 0$. If the vanishing of $\mathrm{Hom}(\mathcal{T}, \mathcal{T}[i])$ holds for $i \neq 0$, then the dg-algebra $\mathbb{R}\mathrm{Hom}(\mathcal{E}, \mathcal{E})$ becomes an algebra. See [85] and references therein for further details.

To go further, we clarify some notations. For a ring A, for the rest of the work for we want to denote the category of right A-modules by $\mathrm{Mod}(A)$ and the category of finitely generated right A-modules by $\mathrm{mod}(A)$. Furthermore, the bounded category of $\mathrm{mod}(A)$ will be denoted by $D^b(A)$. With this notation one has the following key-result for geometric tilting theory [81], p.172, Theorem 7.6. (see also [52], Theorem 1.8)

Theorem 3.8. *Let X be projective over a finitely generated k-algebra R. Suppose X admits a tilting object $\mathcal{T}$ and set $A = \mathrm{End}(\mathcal{T})$. Then the following hold:*

(i) *There is an equivalence* $\mathbb{R}\mathrm{Hom}(\mathcal{T}, -) : D(\mathrm{Qcoh}(X)) \xrightarrow{\sim} D(\mathrm{Mod}(A))$.

(ii) *The equivalence of (i) restricts to an equivalence* $D^b(X) \xrightarrow{\sim} D^b(A)$.

(iii) *If X is smooth over R, then A has finite global dimension.*

Remark 3.9. If X is a smooth projective and integral k-scheme and $\mathcal{T}$ a tilting object for $D^b(X)$ that is a coherent sheaf, then it is also tilting sheaf in the sense as mentioned after Proposition 3.6. This is due to the fact that by Theorem 3.8 the smoothness of X implies finiteness of the global dimension of $\mathrm{End}(\mathcal{T})$. Hence for smooth projective and integral k-schemes X, to verify that a coherent sheaf $\mathcal{T}$ is a tilting sheaf for $D^b(X)$, it suffices to verify only Ext-vanishing and the generating property.

For convenience to the reader, we reproduce the proof of the classical tilting correspondence in the special case when X is a smooth projective and integral k-scheme admitting a tilting sheaf in the sense as mentioned after Proposition 3.6. This result was firstly proved by Bondal [37], but was also proved by Baer [18], Theorem 3.1.2.

Proposition 3.10. *Let $\mathcal{T}$ be a tilting sheaf on a smooth, projective and integral k-scheme X, with endomorphism algebra $A = \mathrm{End}(\mathcal{T})$. Then the following functor is an equivalence of triangulated categories*

$$\mathbb{R}\mathrm{Hom}(\mathcal{T}, -) : D^b(X) \longrightarrow D^b(A)$$

with quasi-inverse being the derived functor $- \otimes_A^L \mathcal{T}$.

Proof. We follow exactly the proof of Theorem 3.1.2 in [18]. Let $\mathcal{F}$ be a quasicoherent sheaf on X. The vector space $\mathrm{Hom}(\mathcal{T}, \mathcal{F})$ has a right A-module structure by composition. Indeed, for a $f \in \mathrm{Hom}(\mathcal{T}, \mathcal{T})$ and $g \in \mathrm{Hom}(\mathcal{T}, \mathcal{F})$ we set $g * f = g \circ f$. Since $\mathrm{Hom}(\mathcal{T}, -)$ is a left exact functor and since the category of quasicoherent sheaves has enough injectives, one gets the right derived functor

$$\mathbb{R}\mathrm{Hom}(\mathcal{T}, -) : D^b(\mathrm{Qcoh}(X)) \longrightarrow D(\mathrm{Mod}(A)).$$

Taking cohomology of the image yields

$$H^i(\mathbb{R}\mathrm{Hom}(\mathcal{T}, \mathcal{F})) = \mathbb{R}^i\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = \mathrm{Ext}^i(\mathcal{T}, \mathcal{F}).$$

The smoothness of X implies that $\mathrm{Ext}^i(\mathcal{T}, \mathcal{F}) = 0$ for $i < 0$ and $i >> 0$. This is due to the local-to-global spectral sequence $H^p(X, \mathcal{E}xt^q(-,-)) \Rightarrow \mathrm{Ext}^{p+q}(-,-)$ (see [89], p.86 (3.16)). By Grothendieck's vanishing theorem $H^i(X, -)$ vanishes for $i > \dim(X)$ and hence $\mathrm{Ext}^i(\mathcal{T}, \mathcal{F}) = 0$ for $i < 0$ and $i >> 0$. Thus, the image of $\mathbb{R}\mathrm{Hom}(\mathcal{T}, -)$ lies in the bounder derived category $D^b(\mathrm{Mod}(A))$. Since $D^b(X)$ is equivalent to the full subcategory of $D^b(\mathrm{Qcoh}(X))$ whose objects have coherent cohomology sheaves, we consider the restriction of the functor $\mathbb{R}\mathrm{Hom}(\mathcal{T}, -)$ to $D^b(X)$. But if $\mathcal{F}$ is coherent then $\mathrm{Ext}^i(\mathcal{T}, \mathcal{F})$ is a finite dimensional k-vector space and hence finitely generated as a right A-module. Therefore the image lies in $\mathrm{mod}(A)$. Finally we get the functor

$$\mathbb{R}\mathrm{Hom}(\mathcal{T}, -) : D^b(X) \longrightarrow D^b(A).$$

On the other hand, since the category $\mathrm{Mod}(A)$ has enough projectives and the functor $- \otimes_A \mathcal{T}$ is right exact, one obtains

$$- \otimes_A^L \mathcal{T} : D^b(\mathrm{Mod}(A)) \longrightarrow D(\mathrm{Qcoh}(X)).$$

For a right A-module M, the cohomology sheaves of the image are

$$\mathcal{H}^j(M \otimes_A^L \mathcal{T}) = \mathcal{T}or_{-j}^A(M, \mathcal{T}).$$

But these vanish for $|j| >> 0$ since A has finite global dimension. Now restricting to finitely generated right A-modules yields

$$- \otimes_A^L \mathcal{T} : D^b(A) \longrightarrow D^b(X).$$

Since $\mathcal{T}$ satisfies (i) of the definition, we get

$$(\mathbb{R}\mathrm{Hom}(\mathcal{T}, -) \circ (- \otimes_A^L \mathcal{T}))(A) = \mathbb{R}\mathrm{Hom}(\mathcal{T}, \mathcal{T}) = \mathrm{Hom}(\mathcal{T}, \mathcal{T}) = A.$$

Since A has finite global dimension, every finitely generated right A-module has a finite projective resolution. This implies that A generates $D^b(A)$, i.e., the smallest full triangulated subcategory containing A that is closed under direct sums and direct summands equals $D^b(A)$. Hence $(\mathbb{R}\mathrm{Hom}(\mathcal{T}, -) \circ (- \otimes_A^L \mathcal{T}))$ is the the identity on $D^b(A)$. This implies that the left derived functor $- \otimes_A^L \mathcal{T}$ identifies $D^b(A)$ with the triangulated subcategory of $\mathrm{D}^b(X)$ generated by $A \otimes_A^L \mathcal{T} = \mathcal{T}$. But since $\mathcal{T}$ generates $D^b(X)$, we obtain the desired derived equivalence. □

Example 3.11. Let $X = \mathrm{Spec}(A)$ be an affine scheme for a finitely generated k-algebra A. Then one has $\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X) \simeq H^0(X, \mathcal{O}_X) \simeq A$ and hence an equivalence of categories $\Gamma(X, -) : \mathrm{Coh}(X) \to \mathrm{mod}(A)$. Thus there is a derived equivalence $\mathbb{R}\mathrm{Hom}(\mathcal{O}_X, -) : D^b(X) \to D^b(A)$. Note that $\mathrm{Ext}^i(\mathcal{O}_X, \mathcal{O}_X) \simeq H^i(X, \mathcal{O}_X) = 0$, for $i > 0$ so that $\mathcal{O}_X$ has only trivial self-extensions and (i) of the definition of tilting objects is fulfilled. Clearly, the structure sheaf generates $D^b(X)$ and hence $\mathcal{O}_X$ is a tilting object. Another easy fact is that a finite-dimensional k-algebra A is always a tilting object for $D^b(A)$. So whenever one has an equivalence $F : D^b(X) \to D^b(A)$, the object $F^{-1}(A)$ is a tilting object for $D^b(X)$.

Example 3.12. Beilinson [23] proved that $\mathcal{T} = \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(i)$ is a tilting bundle for $\mathbb{P}^n$. A proof of this result can also be found in [89], 8.3. That $\mathcal{T}$ is a characteristic-free tilting bundle for $\mathbb{P}^n$ can also be deduced from a more general result of Buchweitz, Leuschke and Van den Bergh [53], Theorem 1.3, stating a characteristic-free tilting bundle for the Grassmannian.

The discovery of a tilting bundle $\mathcal{T}$ for the projective space $\mathbb{P}^n$ was the beginning of the geometric tilting theory and had many applications in the study of locally free sheaves on $\mathbb{P}^n$ (see [128]). It suggested once more that derived categories of (quasi-) coherent sheaves are very important invariants for schemes. Furthermore, for the first time there was established a connection between the category of coherent sheaves and the representation theory of the endomorphism algebra $\mathrm{End}(\mathcal{T})$. In what follows we explain this connection very roughly, since it also provides us with a motivation for studying the existence of tilting bundles. We start with the notion of quivers and their representations. The main references are [12] and [16].

A *quiver* Q is a quadruple $Q = (Q_0, Q_1, t, h)$ consisting of a set Q_0, whose elements are called *vertices* and a set Q_1, whose elements are called *arrows*, together with maps $t : Q_1 \to Q_0$ and $h : Q_1 \to Q_0$, specifying the *tail* and the *head* of each arrow. We assume that Q_0 and Q_1 are finite sets and that Q is connected, meaning that the underlying graph of Q is connected. A nontrivial *path* p of Q of length $l \in \mathbb{N}$ from a vertex i to a vertex j is a sequence of arrows $a_1, ..., a_l$ such that $t(a_1) = i$, $h(a_l) = j$ and $h(a_r) = t(a_{r+1})$, for $1 \leq r \leq l - 1$. For such a path p, we write $t(p) = i$ and $h(p) = j$. In addition, for each vertex $i \in Q_0$ one has a *trivial path* e_i, that has length zero and where $h(e_i) = t(e_i) = i$. Furthermore, a *cycle* in Q is a nontrivial path p with $h(p) = t(p)$. Mentioning a quiver $Q = (Q_1, Q_0, t, h)$, we omit the maps t and h and write simply $Q = (Q_0, Q_1)$ for a quiver. For an arrow $a \in Q_1$, with $t(a) = i$ and $h(a) = j$, we will write $a : i \to j$. Now consider a path p consisting of arrows $a_1, ..., a_l$ and a path q consisting of arrows $b_1, ..., b_n$. If $h(a_l) = t(b_1)$ we write for the path $a_1, ..., a_l, b_1, ..., b_n$ simply pq.

Example 3.13. Let $Q_0 = \{1, 2\}$ and $Q_1 = \{a, b\}$. Then the quiver $Q = (Q_0, Q_1)$ is

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

If one takes $Q_0 = \{1\}$ and $Q_1 = \{a\}$, then one has the quiver $Q = (Q_0, Q_1)$

$$1 \circlearrowleft$$

Now let k be a field. The *path algebra* kQ of a quiver Q is the k-algebra whose underlying k-vector space has as basis the set of paths of Q and where the product of paths p and q is defined as follows:

$$p \star q = \begin{cases} pq & \text{if } h(p) = t(q) \\ p & \text{if } q = e_{h(p)} \\ q & \text{if } p = e_{t(q)} \\ 0 & \text{otherwise} \end{cases}$$

This yields the k-algebra kQ. It is a graded k-algebra, where the grading is given by the length of the paths.

Example 3.14. If we consider the quiver Q, with $Q_0 = \{1\}$ and $Q_1 = \{a\}$ from above, the path algebra is isomorphic to $k[X]$. If we take the quiver Q, with $Q_0 = \{1\}$ and $Q_1 = \{a, b\}$ it looks like

$$\circlearrowright 1 \circlearrowleft$$

and the path algebra is isomorphic to $k[X, Y]$.

In most contexts one also has certain relations on the paths, so that one gets a quiver with relations. A *relation* σ on a quiver Q over a field k is a k-linear combination of paths $r_1 p_1 + ... + r_n p_n$, with $r_i \in k$ and $h(p_1) = ... = h(p_n)$ and $t(p_1) = ... = t(p_n)$. We here assume that the length of each p_i is at least 2. If $R = \{\sigma_s\}_{s \in T}$ is a set of relations on Q, the pair (Q, R) is called *quiver with relations*. For this quiver with relations we also have a k-algebra $kQ/\langle R \rangle$, where $\langle R \rangle$ denotes the ideal in kQ generated by the relations.

Example 3.15. Let k be a field and Q the quiver $1 \circlearrowleft$. The relations are given by $R = \{a^3 - a^2\}$, where a is the arrow of the quiver Q. Then $kQ/\langle R \rangle \simeq kQ/\langle a^3 - a^2 \rangle \simeq k[X]/(X^3 - X^2)$.

For a quiver Q, a *representation* (V, f) over k is a set of finite dimensional k-vector spaces $\{V_i | i \in Q_0\}$ together with k-linear maps $f_a : V_i \to V_j$ for each arrow $a : i \to j$. A morphism $g : (V, f) \to (V', f')$ between two representations is a collection $\{g_i\}_{i\in Q_0}$ of k-linear maps between V_i and V_i', such that for each arrow $a : i \to j$ the diagram

$$\begin{array}{ccc} V_i & \xrightarrow{g_i} & V_i' \\ {\scriptstyle f_a}\downarrow & & \downarrow{\scriptstyle f_a'} \\ V_j & \xrightarrow{g_j} & V_j' \end{array}$$

is commutative. This gives the category of finite dimensional representations $\mathrm{Repr}(Q)$ of the quiver Q. It is easy to verify that this category is an abelian category (see [12], III, Lemma 1.3) with sum of two morphisms $\{f_i\}_{i\in Q_0}$ and $\{g_i\}_{i\in Q_0}$ between representations $\{V_i | i \in Q_0\}$ and $\{V_i' | i \in Q_0\}$ being $\{f_i + g_i\}_{i\in Q_0}$. For a quiver with relations (Q,R), the category of finite dimensional representations $\mathrm{Repr}((Q,R))$ is defined as follows: Let (V,f) be a finite-dimensional representation of Q. For a non-trivial path p from a vertex m to a vertex n, consisting of arrows $a_1,...,a_l$, we define f_p to be the k-linear map $f_p = f_{a_l} \circ f_{a_{l-1}} \circ ... \circ f_{a_1}$. This definition of f_p naturally extends to k-linear combinations of paths with common tail and head. Now the category of finite dimensional representations $\mathrm{Repr}((Q,R))$ of a quiver with relations (Q,R) is defined to be the full subcategory of $\mathrm{Repr}(Q)$ whose objects are (V,f), with $f_\sigma = 0$ for each relation $\sigma \in R$. One can show that there is an equivalence between the category $\mathrm{Repr}((Q,R))$ and the category of finitely generated $kQ/\langle R\rangle$-modules (see [12], III, Theorem 1.6).

With this short review of quivers and their representations, we are now able to explain briefly the connection between tilting objects and representations of quivers. Assuming the existence of a tilting object $\mathcal{T}$, Theorem 3.6 yields an equivalence

$$\mathbb{R}\mathrm{Hom}(\mathcal{T},-) : D^b(X) \longrightarrow D^b(A),$$

where $A = \mathrm{End}(\mathcal{T})$ is a finite dimensional k-algebra. If the field k is supposed to be algebraically closed, any finite-dimensional k-algebra A admits a *complete set of primitive orthogonal idempotents* $e_1,...,e_n$ (see [12], I.4). Idempotents $e_1,...,e_n$ are called *orthogonal* if $e_ie_j = e_je_i = 0$ for $i \neq j$ and *complete* if $e_1 + ... + e_n = 1$. Furthermore, an idempotent e is called *primitive* if it cannot be written as a sum of two non-zero orthogonal idempotents. Now let $e_1,...,e_n$ be the complete set of primitive orthogonal idempotents of the above endomorphism algebra A. Associated to A, there is a finite-dimensional k-algebra A' with a complete set of primitive orthogonal idempotents $e_1',...,e_r'$ such that $e_i'A' = e_j'A'$ as right A-modules only if $i = j$ (see [12], I.6, Definition 6.3). Furthermore, one has an equivalence of categories between $\mathrm{mod}(A)$ and $\mathrm{mod}(A')$ (see [12], I.6, Corollary 6.10). Now the key point is, that to every such algebra A', with $e_i'A' = e_j'A'$ as right A-modules only if $i = j$, one can associate a quiver with relations (Q,R) as follows: The set Q_0 is given by the set $e_1',...,e_r'$ and the number of arrows from e_i' to e_j' is given by $\mathrm{Ext}^1_{A'}(S_i,S_j)$, where $S_l = e_l'A'/e_l'\mathrm{rad}(A')$ (see [12], II.3, Definition 3.1, see also [16], p.52 and Proposition 1.14). Note that this quiver does not depend on the choice of the set complete idempotents (see [12], II.3, Lemma 3.2). Moreover, the quiver (Q,R) is uniquely determined up to isomorphism by A' and the path algebra of (Q,R) is isomorphic to A'

(see [16], III Theorem 1.9, Corollary 1.10). This yields an equivalence between $\mathrm{mod}(A)$ and $\mathrm{mod}(kQ/\langle R\rangle)$ and hence between $D^b(X)$ and $\mathrm{D}^b(\mathrm{mod}(\mathrm{kQ}/\langle R\rangle))$. Under this equivalence, every direct summand $e_i'A'$ of A' is mapped to a direct summand $\mathcal{E}_i$ of the tilting object $\mathcal{T}$ and the direct sum $\mathcal{T}' = \bigoplus_{i=1}^r \mathcal{E}_i$ is again a tilting object for $D^b(X)$. The difference between $\mathcal{T}$ and $\mathcal{T}'$ is that $\mathcal{T}$ may contain several copies of $\mathcal{E}_i$. This equivalence between $D^b(X)$ and $\mathrm{D}^b(\mathrm{mod}(\mathrm{kQ}/\langle R\rangle))$ now enables one to apply representation-theoretical techniques to investigate the derived category of coherent sheaves on X. As a classical example consider the tilting bundle $\mathcal{T} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ on the projective line $\mathbb{P}^1$. The corresponding quiver consists of two vertices and two arrows from the first vertex to the second $1 \rightrightarrows 2$ and the representations were studied by Kronecker and are well-known. For details and further examples we refer to [18], [37], [50], [58], [59], [120] and [132].
One last comment on the connection between geometric tilting theory and the representation theory of quivers: Studying quivers and its representations, it is a matter of fact that most of the quivers are wild and a classification of their representation is extremely difficult. Now it is possible to construct moduli spaces of representations of such quivers (see [134] for an excellent survey) and one can try to understand these moduli spaces with geometric techniques. Interestingly, it turns out that some of these moduli spaces admit tilting objects (see for instance [59], [66] and [103]).

In the literature, instead of the tilting object $\mathcal{T}$ one often studied the set $\mathcal{E}_1, ..., \mathcal{E}_n$ of its indecomposable, pairwise non-isomorphic direct summands. There is a special case where all the summands form a so-called full strongly exceptional collection. Closely related to the notion of a full strongly exceptional collection is that of a semiorthogonal decomposition. We recall the definition of an exceptional collection and a semiorthogonal decomposition respectively. We will follow the definition given in [89] and refer to the work of Bondal and Orlov [38] for further details.

Definition 3.16. Let X be a noetherian quasiprojective k-scheme. An object $\mathcal{E} \in D^b(X)$ is called *exceptional* if $\mathrm{End}(\mathcal{E}) = k$ and $\mathrm{Hom}(\mathcal{E}, \mathcal{E}[l]) = 0$ for all $l \neq 0$. An *exceptional collection* is a collection of exceptional objects $\mathcal{E}_1$,...,$\mathcal{E}_n$ such that $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j[l]) = k$ for $l = 0$ and $i = j$, and $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j[l]) = 0$ if $i > j$ or if $l \neq 0$ and $i = j$. An exceptional collection is called *full* if the collection generates $D^b(X)$, i.e., if the smallest full triangulated subcategory containing $\mathcal{E}_1, ..., \mathcal{E}_n$ that is closed under direct summands and direct sums equals $D^b(X)$. If in addition $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j[l]) = 0$ for all i, j and $l \neq 0$ the collection is called *strongly exceptional*.

As a generalization one has the notion of a semiorthogonal decomposition of $D^b(X)$ (see [38] or [89]).

Definition 3.17. Let X be as above. A collection $\mathcal{D}_1, ..., \mathcal{D}_r$ of full triangulated subcategories is called a *semiorthogonal decomposition* for $D^b(X)$ if the following properties hold:

(i) The inclusion $\mathcal{D}_i \subset D^b(X)$ has a right adjoint $p : D^b(X) \to \mathcal{D}_i$.

(ii) $\mathcal{D}_j \subset \mathcal{D}_i^{\perp} = \{B \in D^b(X) | \mathrm{Hom}(A, B[l]) = 0,\ \forall l \in \mathbb{Z} \text{ and } \forall A \in \mathcal{D}_i\}$ for $i > j$.

(iii) The collection $\mathcal{D}_i$ generates $D^b(X)$, i.e., the smallest full triangulated subcategory containing all $\mathcal{D}_i$ that is closed under direct summands and direct sums equals $D^b(X)$.

For the semiorthogonal decomposition of $D^b(X)$ we write $D^b(X) = \langle \mathcal{D}_1, ..., \mathcal{D}_r \rangle$.

Remark 3.18. We want to note that Definition 3.2, 3.16 and 3.17 can all be formulated in the more general setting of k-linear triangulated categories (see [52] and [89]).

Example 3.19. If we have a full exceptional collection $\mathcal{E}_1, ..., \mathcal{E}_n$ in $D^b(X)$, then by Lemma 1.58 in [89] we have that the inclusion $\langle \mathcal{E}_i \rangle \to D^b(X)$ has a right adjoint. Furthermore, condition (ii) is fulfilled for $\mathcal{D}_i = \langle \mathcal{E}_i \rangle$ and since the collection $\mathcal{E}_1, ..., \mathcal{E}_n$ is full, the collection $\langle \mathcal{E}_1 \rangle, ..., \langle \mathcal{E}_n \rangle$ generates $D^b(X)$. Hence a full exceptional collection $\mathcal{E}_1, ...\mathcal{E}_n$ give rise to a semiorthogonal decomposition $D^b(X) = \langle \langle \mathcal{E}_1 \rangle, ..., \langle \mathcal{E}_n \rangle \rangle$.

The above described representation-theoretical approach to understand the derived category of coherent sheaves is not the only motivation to study derived categories with exceptional collections or tilting objects. Another motivation comes from Kontsevich's Homological Mirror Symmetry conjecture [106], see also [89], 13.2. Very roughly, this conjecture states that for a Calabi–Yau variety X, there exists an object Y, carrying a symplectic structure, such that the bounded derived category of coherent sheaves $D^b(X)$ is equivalent to the derived Fukaya-category of Y. Moreover, a conjecture of Dubrovin [62] states that the quantum cohomology of a smooth projective variety X is generically semisimple if and only if there exists a full exceptional collection in $D^b(X)$ and the validity of this conjecture would also provide evidence for the Homological Mirror Symmetry conjecture. We do not want to recall all the details of these conjectures and refer to [106] and [62] for further information and deeper understanding. Moreover, motivated by the Mirror Symmetry, in the recent past full strongly exceptional collections have also been considered in physics in the context of string theory, concretely in studying so-called D-branes (see for instance [11], [27]). Particular interest in exceptional collections also comes from local Calabi–Yau varieties. Consider the total space $\pi : \mathrm{Tot}(\omega_X) \to X$ for the canonical bundle ω_X. This is a local Calabi-Yau variety and it follows from results of Bridgeland [49], that a full strongly exceptional collection $\mathcal{E}_i$ on X can be extended to a cyclic strongly exceptional collection if and only if the pullbacks $\pi^*\mathcal{E}_i$ give rise to a tilting object on $\mathrm{Tot}(\omega_X)$. It is also an interesting observation that the endomorphism algebra of this titling object for $\mathrm{Tot}(\omega_X)$ gives an example of noncommutative resolutions in the sense of [150]. Other examples where Theorem 3.8 applies include some crepant resolutions of quotient singularities (see [85], 7 ff.). Finally, we want to note that the work of Bridgeland [48], Kuznetsov [110], Orlov [129] and others indicates that birational schemes should have bounded derived categories of coherent sheaves related by semiorthogonal decompositions. This philosophy will be the driving idea in the last chapter of the present work, where we want to study birational Brauer–Severi varieties in terms of their semiorthogonal decompositions.

We now state simple well-known observations concerning the relation between exceptional collections and tilting objects.

Proposition 3.20. *Let X be a smooth projective and integral k-scheme and $\mathcal{E}_1, \ldots, \mathcal{E}_n$ a full strongly exceptional collection in $D^b(X)$, then $\mathcal{T} = \bigoplus_{i=1}^n \mathcal{E}_i$ is a tilting object for $D^b(X)$.*

Proof. We first prove that $\mathcal{T}$ has trivial self-extension. For this, we have to consider $\mathrm{Hom}(\mathcal{T}, \mathcal{T}[l])$. By the definition of $\mathcal{T}$ we have

$$\mathrm{Hom}(\mathcal{T}, \mathcal{T}[l]) \simeq \bigoplus \bigoplus \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j[l]) = 0,$$

for $l \neq 0$, since $\mathcal{E}_1, \ldots, \mathcal{E}_n$ is a strongly exceptional collection. The generating property of $\mathcal{T}$ is fulfilled because the collection of the $\mathcal{E}_i$ is assumed to be full. Hence $\mathcal{T}$ generates $D^b(X)$ what finally implies that $\mathcal{T}$ is a tilting object for $D^b(X)$. Note that according to Theorem 3.8, $\mathrm{End}(\mathcal{T}$ has finite global dimension, since X is supposed to be smooth. □

Note that if the collection $\mathcal{E}_1, \ldots, \mathcal{E}_n$ is only a full exceptional collection, one has a semiorthogonal decomposition (see Example 3.19) whereas it is not guaranteed that $D^b(X)$ is of the form $D^b(A)$ for some algebra A. If the full strongly exceptional collection from above consists of coherent sheaves, the obtained tilting object $\mathcal{T}$ is a tilting sheaf in the more restricted sense as defined after Proposition 3.6. One only has to prove that the global dimension of $\mathrm{End}(\mathcal{T})$ is finite. Indeed, since X is assumed to be smooth, we conclude with Theorem 3.8, that $\mathrm{End}(\mathcal{T})$ has finite global dimension. Alternatively, from the fact that the collection is exceptional, we conclude that $\mathrm{End}(\mathcal{T})$ is a upper triangular matrix algebra with entries on the diagonal being k. Then [16], Proposition 2.7 (or Corollary 3.102 below) yields that $\mathrm{End}(\mathcal{T})$ has finite global dimension. As we have seen, a full strongly exceptional collection gives rise to a tilting object, but the direct summands of a tilting object in general do not give rise to a full strongly exceptional collection. The next proposition is also well-known and will clarify when this however is true.

Proposition 3.21. *Let X be a smooth projective and integral k-scheme and $\mathcal{T}$ a tilting sheaf for $D^b(X)$. Consider the Krull–Schmidt decomposition $\mathcal{T} = \bigoplus_{i=1}^r \mathcal{E}_i^{\oplus n_i}$ and suppose that all $\mathcal{E}_i$ are invertible, with* $\mathrm{End}(\mathcal{E}_i) = k$*. Then, after possibly reordering, the sheaves $\mathcal{E}_i$ form a full strongly exceptional collection.*

Proof. Since $\mathcal{T}$ is a tilting sheaf, we have

$$\mathrm{Ext}^l(\mathcal{T}, \mathcal{T}) = \bigoplus \bigoplus \mathrm{Ext}^l(\mathcal{E}_i, \mathcal{E}_j) = 0$$

for $l \neq 0$. This together with the assumption $\mathrm{End}(\mathcal{E}_i) = k$ implies that all $\mathcal{E}_i$ are exceptional. We now explain why we can list the sheaves $\mathcal{E}_i$ such that $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j) = 0$ for $i > j$, as claimed in the definition of an exceptional collection. Since all the $\mathcal{E}_i$ are supposed to be invertible, one of the finite-dimensional vector spaces $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j)$ or $\mathrm{Hom}(\mathcal{E}_j, \mathcal{E}_i)$, for $i \neq j$, has to be trivial, otherwise $\mathcal{E}_i \simeq \mathcal{E}_j$, what contradicts the fact that $\mathcal{E}_i$ are pairwise non-isomorphic. Reordering the sheaves in the way that $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j) = 0$, whenever $i > j$ yields a strongly exceptional collection. Since $\mathcal{T}$ generates $D^b(X)$, the collection is furthermore full. This completes the proof. □

Remark 3.22. Note that the above assumption $\mathrm{End}(\mathcal{E}_i) = k$ in general does not imply that $\mathcal{E}_i$ is an invertible sheaf. Actually, if one has a tilting sheaf $\mathcal{T}$ such that the direct summands are not invertible sheaves, it is still possible that the summands form a full strongly exceptional collection. Later on we will see that this is the case for Grassmannians as proved by Kapranov [95]. But if all the direct summands $\mathcal{E}_i$ of $\mathcal{T}$ are invertible we clearly have $\mathrm{Hom}(\mathcal{E}_i) = k$, provided X is for instance geometrically integral, and hence the above proposition shows that the collection $\mathcal{E}_1, ..., \mathcal{E}_n$ forms a full strongly exceptional collection.

Remark 3.23. Moreover, we want to mention that if the direct summands $\mathcal{E}_i$ of the above tilting sheaf $\mathcal{T}$ are not assumed to be invertible, it can happen that they cannot be rearranged in such a way that they form a full strongly exceptional collection. Even if we assume that there is a tilting object $\mathcal{T}$ with direct summands $\mathcal{E}_i$ (assumed to be pairwise non-isomorphic) and $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_i) = k$, in general, the direct summands do not form a full strongly exceptional collection. This fact follows from the work of Hille and Perling [87], where they proved the existence of tilting objects on smooth projective rational surfaces where in general these tilting objects are not obtained as a direct sum of objects forming a full strongly exceptional collection. To my best knowledge, it is hardly non-trivial to find schemes X, such that they may have tilting objects but no full strongly exceptional collections. Perling [131], Theorem 4.8.2 showed that a smooth complete toric surface S admits a full strongly exceptional collection of invertible sheaves if and only if $S \neq \mathbb{P}^2$ can be obtained from a Hirzebruch surface in at most two steps by blowing up torus fixed points. In view of this results it seems very naturally to ask for examples of schemes X, admitting a tilting object $\mathcal{T}$ with direct summands $\mathcal{E}_i$ and $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_i) = k$, such that the collection $\mathcal{E}_i$ does not form a full strongly exceptional collection. More generally, one can ask for examples of schemes X, admitting a tilting object $\mathcal{T}$ but not admitting a full strongly exceptional collection consisting of arbitrary objects of $D^b(X)$.

In the present work we will be interested in tilting objects (sheaves or bundles) for the derived category $D^b(X)$ as defined in Remark 3.4. The guiding problem in geometric tilting theory is the following:

Problem. *Classify schemes X (or more generally algebraic stacks) that have tilting objects for $D^b(X)$.*

It is a widely open question whether on a given smooth projective and integral k-scheme X a tilting object exits. In general the above problem is far from being solved and the existence of tilting objects is only known in some cases. These cases include the following examples:

- Projective space $\mathbb{P}^n$, by Beilinson [23]
- flag varieties of type A_n, by Kapranov [97]
- Grassmannians and quadrics over $\mathbb{C}$, by Kapranov [95] and [96]
- Grassmannians over arbitrary fields, by Buchweitz, Leuschke and Van den Bergh [53]
- rational surfaces, by Hille and Perling [87]
- Fano surfaces, various toric varieties and fibrations, by Costa, Di Rocco and Miró-Roig [56] and [57]

- generalized Brauer–Severi varieties over fields of characteristic zero, by Blunk [34]
- certain Fano 3-folds, by Ciolli [55]
- weighted projective lines. See Meltzer [120].
- The moduli space of n-pointed stable curves of genus zero $\bar{\mathcal{M}}_{0,n}$, by Manin and Smirnov [117]
- 2-dimensional smooth toric weak Fano stacks, by Ishii and Ueda [90]

We note that tilting objects are known to exist in some more cases (see for instance [18], [49], [50], [59], [60], [85]). Furthermore, semiorthogonal decompositions and full exceptional collections (in general not strong) are also known in some more cases (see [28], [34], [35], [36], [37], [38], [43], [51], [53], [58], [63], [64], [65], [69], [72], [83], [84], [86], [100], [103], [109], [110], [111], [112], [129], [132], [140] and [144]) and in special cases these semiorthogonal decompositions give rise to full strongly exceptional collections and hence to tilting objects. But in general this is not the case and indeed there are schemes admitting a semiorthogonal decomposition but not a tilting object or a full strongly exceptional collection. In the present work we are interested in the case where the tilting bundle is not obtained as the direct sum of objects that from a full strongly exceptional collection. In what follows we give some well-known general results concerning tilting objects that will be needed later on.

We recall that in the category of coherent sheaves on some proper k-scheme X the Krull–Schmidt theorem holds and every coherent sheaf has a direct sum decomposition, where the direct summands are unique up to isomorphism and the decomposition is unique up to permutation (see Theorem 1.33). If for a projective and integral k-scheme X, one considers $D^b(X)$ instead of $\mathrm{Coh}(X)$, one can also show that every object in $D^b(X)$ admits a direct sum decomposition into indecomposable objects that are unique up to isomorphism and the decomposition being unique up to permutation (see [6], Theorem 2.2 and Proposition 2.3). Such decompositions are also called *Krull–Schmidt decomposition* and the category $D^b(X)$ a *Krull–Schmidt category.* For further details see [6] and references therein.

Proposition 3.24. *Let X be a projective and integral k-scheme. Suppose $\mathcal{T} = \bigoplus_{i=1}^n \mathcal{T}_i$ is a tilting object for $D^b(X)$. Then for integers $r_i > 0$ the object $\bigoplus_{i=1}^n \mathcal{T}_i^{\oplus r_i}$ is a tilting object too.*

Proof. This is obtained directly from the definition of tilting objects for the derived category $D^b(X)$ given in Remark 3.4. □

Remark 3.25. Clearly, the above proposition also holds for tilting objects $\mathcal{T} = \bigoplus_{i=1}^n \mathcal{T}_i$ of $D(\mathrm{Qcoh}(X))$.

Proposition 3.26. *Let X be a projective and integral k-scheme and $\mathcal{T}$ a tilting object for $D^b(X)$. Then for all invertible sheaves $\mathcal{L}$ on X, the object $\mathcal{T} \otimes \mathcal{L}$ is a tilting object too.*

Proof. Clearly, one has $\mathrm{Hom}(\mathcal{T}\otimes\mathcal{L},\mathcal{T}\otimes\mathcal{L}[i]) \simeq \mathrm{Hom}(\mathcal{T},\mathcal{T}[i]) = 0$ for $i \neq 0$, since $\mathcal{T}$ is by assumption a tilting object. Furthermore, we have $\mathbb{R}\mathrm{Hom}(\mathcal{T}\otimes\mathcal{L},\mathcal{N}) \simeq \mathbb{R}\mathrm{Hom}(\mathcal{T},\mathcal{L}^\vee\otimes\mathcal{N}) = 0$, what implies that $\mathcal{L}^\vee\otimes\mathcal{N} = 0$ and hence $\mathcal{N} = 0$. Finally, $\mathcal{T}\otimes\mathcal{L}$ is a compact object and hence $\mathrm{Hom}(\mathcal{T}\otimes\mathcal{L},-)$ commutes with direct sums. This completes the proof. □

The next two results are certainly folklore and well-known. The relative versions of these results are proved in [52]. We start with the proposition stating that if X and Y have tilting objects, $X\times_k Y$ admits a tilting object too. As mentioned a few lines up, this result can be found in its relative version in [52], Proposition 2.6. For the result in the absolute case, we refer to [35], Proposition 2.1.18. Considering the projections $p: X\times_k Y\to X$ and $q: X\times_k Y\to Y$, we write for an object $p^*\mathcal{F}\otimes^L q^*\mathcal{G}$ simply $\mathcal{F}\boxtimes\mathcal{G}$. With this notation one has the following result.

Proposition 3.27. *Let X and Y be smooth projective and integral k-schemes and $\mathcal{T}_X$ and $\mathcal{T}_Y$ are tilting objects for $D^b(X)$ and $D^b(Y)$ respectively. Then $\mathcal{T}=\mathcal{T}_X\boxtimes\mathcal{T}_Y$ is a tilting object for $D^b(X\times_k Y)$.*

The relative version of the next result is also well-known (see [52], Proposition 2.9). Below we will state the result only in its absolute version, since this will suffice for our purpose.

Proposition 3.28. *Let X be a smooth projective k-scheme admitting a tilting object $\mathcal{T}$ and $k\subset L$ an arbitrary field extension. Then $\mathcal{T}\otimes_k L$ is a tilting object for $D^b(X\otimes_k L)$.*

Proof. Consider the projection $v: X\otimes_k L\to X$. We claim that $\mathcal{T}'=v^*\mathcal{T}\simeq \mathcal{T}\otimes_k L$ is a tilting object for $D^b(X\otimes_k L)$. To prove the claim, we first calculate $\mathrm{Hom}(\mathcal{T}',\mathcal{T}'[l])$. Consider the following base change diagram

$$\begin{array}{ccc} X\otimes_k L & \xrightarrow{v} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Spec}(L) & \xrightarrow{u} & \mathrm{Spec}(k) \end{array}$$

Under the above assumption on X, an object $\mathcal{F}\in D^b(X)$ is quasi-isomorphic to a complex of locally free sheaves. For two arbitrary complexes of locally free sheaves $\mathcal{F},\mathcal{G}\in D^b(X)$, flat base change (see [89], p.85 (3.18)) yields the isomorphism of functors

$$\begin{aligned} u^*(\mathbb{R}\mathrm{Hom}(\mathcal{F},\mathcal{G})) &\simeq u^*\mathbb{R}p_*\mathbb{R}\mathcal{H}om(\mathcal{F},\mathcal{G}) \\ &\simeq \mathbb{R}q_*v^*\mathbb{R}\mathcal{H}om(\mathcal{F},\mathcal{G}) \\ &\simeq \mathbb{R}q_*v^*(\mathcal{F}^\vee\otimes^L\mathcal{G}) \\ &\simeq \mathbb{R}q^*\mathbb{R}\mathcal{H}om(v^*\mathcal{F},v^*\mathcal{G}) \\ &\simeq \mathbb{R}\mathrm{Hom}(v^*\mathcal{F},v^*\mathcal{G}). \end{aligned}$$

This now implies

$$\mathrm{Hom}(v^*\mathcal{T},v^*\mathcal{T}[l]) \simeq \mathrm{Hom}(\mathcal{T}',\mathcal{T}'[l]) \simeq \mathrm{Hom}(\mathcal{T},\mathcal{T}[l])\otimes_k L = 0$$

for $l \neq 0$, since $\mathcal{T}$ is a tilting object for $D^b(X)$. For the generation property, we take an object $\mathcal{F} \in D^b(X \otimes_k L)$ and assume $\mathbb{R}\mathrm{Hom}(\mathcal{T}', \mathcal{F}) = 0$. This implies $\mathrm{Hom}(v^*\mathcal{T}, \mathcal{F}[l]) = 0$ for all $l \in \mathbb{Z}$. Since $\mathcal{T}$ generates $D^b(X)$, we conclude by adjunction of v^* and v_* that $v_*\mathcal{F} = 0$. But this forces $\mathcal{F} = 0$. This completes the proof. □

The next result should also be well-known. Since we have not found any proof of it in the literature, we give a proof below. Let X be as above and suppose there is an object $\mathcal{R} \in D^b(X)$, such that for an arbitrary field extension $k \subset L$, $\mathcal{R} \otimes_k L$ is a tilting object for $D^b(X \otimes_k L)$. In this situation one obtains the following:

Proposition 3.29. *Let X be a smooth, projective and integral k-scheme and $k \subset L$ an arbitrary field extension. Now given an object $\mathcal{R} \in D^b(X)$, suppose that $\mathcal{R} \otimes_k L$ is a tilting object for $D^b(X \otimes_k L)$, then $\mathcal{R}$ is a tilting object for $D^b(X)$.*

Proof. Consider the projection $v : X \otimes_k L \to X$. By assumption $\mathcal{T} = v^*\mathcal{R}$ is a tilting object for $D^b(X \otimes_k L)$. We claim that $\mathcal{R}$ is a tilting object for $D^b(X)$. To prove the claim, we first calculate $\mathrm{Hom}(\mathcal{R}, \mathcal{R}[l])$. Consider the following base change diagram

$$\begin{array}{ccc} X \otimes_k L & \xrightarrow{v} & X \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ \mathrm{Spec}(L) & \xrightarrow{u} & \mathrm{Spec}(k) \end{array}$$

Under the above assumption on X an object $\mathcal{F} \in D^b(X)$ is quasi-isomorphic to a complex of locally free sheaves. For two arbitrary complexes of locally free sheaves $\mathcal{F}, \mathcal{G} \in D^b(X)$, flat base change (see [89], p.85 (3.18)) yields the isomorphism of functors

$$\begin{aligned} u^*(\mathbb{R}\mathrm{Hom}(\mathcal{F}, \mathcal{G})) &\simeq u^*\mathbb{R}p_*\mathbb{R}\mathcal{H}om(\mathcal{F}, \mathcal{G}) \\ &\simeq \mathbb{R}q_*v^*\mathbb{R}\mathcal{H}om(\mathcal{F}, \mathcal{G}) \\ &\simeq \mathbb{R}q_*v^*(\mathcal{F}^\vee \otimes^L \mathcal{G}) \\ &\simeq \mathbb{R}q^*\mathbb{R}\mathcal{H}om(v^*\mathcal{F}, v^*\mathcal{G}) \\ &\simeq \mathbb{R}\mathrm{Hom}(v^*\mathcal{F}, v^*\mathcal{G}) \end{aligned}$$

This now implies

$$\mathrm{Hom}(v^*\mathcal{R}, v^*\mathcal{R}[l]) \simeq \mathrm{Hom}(\mathcal{T}, \mathcal{T}[l]) \simeq \mathrm{Hom}(\mathcal{R}, \mathcal{R}[l]) \otimes_k L = 0$$

for $l \neq 0$, since $\mathcal{T}$ is a tilting object for $D^b(X \otimes_k L)$. Hence $\mathrm{Hom}(\mathcal{R}, \mathcal{R}[l]) = 0$, for $l \neq 0$ and Ext-vanishing holds. For the generation property, we take an object $\mathcal{F} \in D^b(X)$ and assume $\mathbb{R}\mathrm{Hom}(\mathcal{R}, \mathcal{F}) = 0$. The above equivalences obtained from flat base change yield

$$\begin{aligned} 0 = u^*(\mathbb{R}\mathrm{Hom}(\mathcal{R}, \mathcal{F})) &\simeq u^*\mathbb{R}p_*\mathbb{R}\mathcal{H}om(\mathcal{R}, \mathcal{F}) \\ &\simeq \mathbb{R}q_*v^*\mathbb{R}\mathcal{H}om(\mathcal{R}, \mathcal{F}) \\ &\simeq \mathbb{R}q_*v^*(\mathcal{R}^\vee \otimes^L \mathcal{F}) \\ &\simeq \mathbb{R}q^*\mathbb{R}\mathcal{H}om(v^*\mathcal{R}, v^*\mathcal{F}) \\ &\simeq \mathbb{R}\mathrm{Hom}(v^*\mathcal{R}, v^*\mathcal{F}). \end{aligned}$$

Since $v^*\mathcal{R} \simeq \mathcal{T}$ is a tilting object for $D^b(X \otimes_k L)$, we get $v^*\mathcal{F} = 0$. Since v is a flat morphism, this implies $\mathcal{F} = 0$ and hence $\mathcal{R}$ generates $D^b(X)$. Finally, since X is smooth, the global dimension of $\mathrm{End}(\mathcal{R})$ is finite. This completes the proof. □

At the end of this section we want to state two necessary conditions for a scheme X to admit a tilting object. We cite both results and refer to the literature for the proofs. We start with the fact that the Grothendieck group has to be a free abelian group of finite rank (see for instance [18], Proposition 3.2.3).

Proposition 3.30. *Let X be a smooth projective and integral k-scheme such that $D^b(X)$ admits a tilting object $\mathcal{T}$. Then $K_0(X)$ is a free abelian group of finite rank.*

Proof. We sketch the idea of the proof. Since X is projective, $A = \mathrm{End}(\mathcal{T})$ is a finite-dimensional k-algebra. The Grothendieck group $K_0(A)$ is thus a free abelian group with rank equal to the number of simple A-modules (see [16], Theorem 1.7). The equivalence $\mathbb{R}\mathrm{Hom}(\mathcal{T}, -) : D^b(X) \xrightarrow{\sim} D^b(A)$ now implies that $K_0(X)$ is also a free abelian group of the same rank. Under this equivalence the number of simple A-modules translates to the number of pairwise non-isomorphic direct summands of $\mathcal{T}$. □

Additionally, there is also a well-known compatibility between K_0 and a semiorthogonal decomposition of $D^b(X)$. It is the following fact (see [26], Lemma 3.4) that will be used also in one of the next sections:

Proposition 3.31. *Let X be a smooth projective and integral k-scheme and $D^b(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ a semiorthogonal decomposition. Then $K_0(X) \simeq K_0(\mathcal{A}) \oplus K_0(\mathcal{B})$.*

The next results are due to Buchweitz and Hille [52] and are summarized in the following theorem (see [52], Corollary 4.2 and Theorem 4.3).

Theorem 3.32. *Let X be a smooth projective scheme over a field k admitting a tilting object. Then $H^i(X, \mathcal{O}_X) = 0$, for $i \neq 0$ and if $\mathrm{char}(k) = 0$, then $H^q(X, \Omega^p_X) = 0$, for $q \neq p$.*

Especially if X is a smooth projective and integral $\mathbb{C}$-scheme, one has the following result (see [52], Theorem 5.1):

Theorem 3.33. *Let X be a smooth projective and integral $\mathbb{C}$-scheme admitting a tilting object, then $H^q(X, \Omega^p_X) = 0$, for $q \neq p$.*

Remark 3.34. The above Theorems 3.32 and 3.33 are consequences of the deep fact that if X admits a tilting object $\mathcal{T}$, the Hochschild cohomology of X and $A = \mathrm{End}(\mathcal{T})$ are isomorphic (see [52], Theorem 4.1). We also want to mention that $H^q(X, \Omega^p_X) = 0$, for $q \neq p$ holds without the assumption $k = \mathbb{C}$, provided the scheme X admits a full strongly exceptional collection (see [118], p.12). This follows from the fact that in this case the Chow motive decomposes as a direct sum of tensor powers of the Lefschetz motive (see [118], Theorem 1.3).

The above results exclude the possibility of the existence of tilting objects for many schemes. For instance, if X is a scheme with $\omega_X = \mathcal{O}_X$, it follows from Serre duality that there is no exceptional sheaf. In particular in dimension one when X is an elliptic curve there is no tilting object for X, since $\dim H^0(X, \Omega^1_X) = 1 \neq 0$. The same holds for all other curves C of genus $g > 1$. Recall, by the Enriques classification of complex algebraic surfaces (see [24] and [25]), one has the following possible minimal surfaces:

$\kappa = -\infty$	$\kappa = 0$		$\kappa = 1$	$\kappa = 2$
rational	abelian	K3	properly elliptic	surfaces of general type
ruled of genus $g \geq 1$	bielliptic	Enriques		

For rational surfaces S, Hille and Perling [87] proved that S always admits a tilting object. Considering the second class of Kodaira dimension $\kappa = -\infty$, the class of ruled surfaces $\pi : S \to C$ over smooth complex curves C of genus $g \geq 1$, it is a matter of fact that $h^{1,0} = h^{0,1} = g$ and hence there is no tilting object on these surfaces according to Theorem 3.33. For $\kappa = 0$ and $\kappa = 1$ one can show that ω_S is a torsion element in $\mathrm{Pic}(S)$ (see [24]). Since $\mathrm{Pic}(S)$ is a subgroup of $K_0(S)$, the Grothendieck group is not free and hence there is no tilting object for $D^b(S)$ according to Proposition 3.30. Finally, one has to consider the minimal surfaces of general type. For some of them one can also exclude the existence of tilting objects (see Section 5 below), but in general it is not clear why all of them fail to have tilting objects. We should mention here that it is conjectured that the existence of tilting objects on smooth projective surfaces S is equivalent to S being rational. We do not want to discuss this here in detail and refer to Section 5.

At the end of this section we want to give some more examples of schemes that cannot have a tilting object. We start with the following.

Proposition 3.35. *Let X be a d-dimensional intersection of two even dimensional quadrics over an algebraically closed field of characteristic zero. Then X does not have a tilting object.*

Proof. Note that Bondal and Orlov [38], Theorem 2.9 proved that X has a semiorthogonal decomposition

$$D^b(X) = \langle \mathcal{O}_X(-d+3), ..., \mathcal{O}_X, D^b(C) \rangle,$$

where C is a hyperelliptic curve. Proposition 3.31 yields $K_0(X) \simeq \mathbb{Z}^{\oplus d-2} \oplus K_0(D^b(C))$. Note that $K_0(D^b(C)) = K_0(C)$. But the Grothendieck group $K_0(C)$ of a hyperelliptic curce C is not free of finite rank. Thus $K_0(X)$ is not free of finite rank and according to Proposition 3.30 X cannot have a tilting object. □

A very natural source for producing examples of schemes that cannot have a tilting object is to blow up a given smooth projective scheme X along certain smooth closed subschemes $Y \subset X$. That the blow up $\widetilde{X}$ very often cannot have a tilting object is a consequence of a well-known result of Orlov [129] stating a semiorthogonal decomposition for such blow ups. We recall Orlov's result. For this, consider a smooth projective and integral k-scheme X and a smooth closed subscheme $Y \subset X$ of codimension r. Now if $\widetilde{X}$ is the blow up of X along Y, Orlov [129], Theorem 4.3 proved that one has a semiorthogonal decomposition

$$D^b(\widetilde{X}) = \langle D^b(Y)_{-r+1}, ..., D^b(Y)_{-1}, D^b(X)\rangle$$

where $D^b(Y)_j$ is the full triangulated subcategory of $D^b(\widetilde{X})$ obtained as the image of the fully faithful functor $j_* \circ (\mathcal{O}_{\mathcal{N}}(j) \otimes -) \circ p^* : D^b(Y) \to D^b(\widetilde{X})$ induced from the commutative blow up diagram

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{j} & \widetilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow[i]{} & X \end{array}$$

where $\mathcal{N} = \mathcal{N}_{Y/X}$ is the normal bundle of rank r and $\mathbb{P}(\mathcal{N}) = \widetilde{Y}$. Now the strategy is very simple. Since $D^b(Y)_j$ is equivalent to $D^b(Y)$, blowing up the scheme X along a smooth closed subscheme Y with Grothendieck group $K_0(Y)$ being not free of finite rank yields a blow up $\widetilde{X}$ that cannot have a tilting object according to Proposition 3.31. This is for instance the case whenever one considers a smooth projective $\mathbb{C}$-scheme X on which there exists an elliptic curve C. Blowing up X along C yields a smooth projective scheme that cannot have a tilting object. As far as the author knows it is an unsolved problem if in the above generality the blow up $\widetilde{X}$ admits a tilting object if X and Y admit such.

3.3 Tilting objects for Brauer–Severi varieties

Beilinson [23] proved the existence of tilting objects on the projective space $\mathbb{P}^n$ and started the geometric tilting theory. His theorem represented a breakthrough and his technique was applied to study locally free sheaves on $\mathbb{P}^n$ [128]. In this section we show how the AS-bundles on Brauer–Severi varieties very naturally give rise to tilting objects. Moreover, we show that the generalized Brauer–Severi varieties $\mathrm{BS}(d, A)$ also admit tilting objects, obtained from the tautological sheaf on $\mathrm{BS}(d, A) \otimes_k \bar{k} \simeq \mathrm{Grass}(d, n)$. We first study Brauer–Severi varieties. This was done by Blunk [34] in the case the ground field k is of characteristic zero. We found our tilting object independently during the work on Brauer–Severi varieties and it turns out that our tilting object is characteristic-free and a direct summand of the tilting object given in [34].

Kapranov [97] investigated if exceptional collections exist on projective homogenous varieties X, that is, X is of the form G/P, where G is a semisimple linear algebraic group and P a parabolic subgroup. He proved that some homogenous varieties have full strongly exceptional collections and hence a tilting object. In a recent work of Kuznetsov and Polishchuk [111] it is conjectured that every G/P should posses a full exceptional collection and several cases are proved in favor of this conjecture (see [111], p.3 and [140]). Furthermore, it is conjectured by Catanese that G/P should posses full strongly exceptional collection with a bijection of the elements of the collection with the Schubert varieties of G/P (see [35]). In general it is also open if every such projective homogenous variety X admits a tilting object. Since Brauer–Severi and generalized Brauer–Severi varieties are k-forms of projective spaces and Grassmannians respectively, we therefore investigate in this section some k-forms of homogenous varieties.

Blunk [34] proved that for a n-dimensional Brauer–Severi variety X the locally free sheaf $\mathcal{R} = \bigoplus_{i=0}^n \mathcal{W}_{-1}^{\otimes i}$ is a tilting bundle. Here $\mathcal{W}_{-1}$ is the locally free sheaf of Definition 1.39. The proof uses the Borel–Weil–Bott Theorem that only holds in characteristic zero. We want to note here, that the proof of Blunk also works in arbitrary characteristic if one uses cohomology on the projective space after base change instead of the Borel–Weil–Bott Theorem. The tilting bundle we found has the property that all indecomposable direct summands are chosen to be optimal in some sense (see Remark 3.39). Furthermore, as mentioned above, our tilting bundle is a direct summand of $\mathcal{R}^{\vee} = \bigoplus_{i=0}^n \mathcal{W}_1^{\otimes i}$.

Theorem 3.36. *Let X be a n-dimensional Brauer–Severi variety over an arbitrary field k and let A be the corresponding central simple k-algebra. Let $\mathcal{W}_i$ be the locally free sheaves of Definition 1.39. Then the locally free sheaf $\mathcal{T} = \bigoplus_{i=0}^n \mathcal{W}_i$ is a tilting bundle for $D^b(X)$.*

Proof. Let $\pi : X \otimes_k L \to X$ be the projection, where L is an arbitrary splitting field. According to Proposition 1.40, we have $\pi^*\mathcal{W}_i \simeq \mathcal{O}_{\mathbb{P}^n}(i)^{\oplus \mathrm{rk}(\mathcal{W}_i)}$. With Proposition 1.51 we conclude that $\mathrm{rk}(\mathcal{W}_i) = \mathrm{ind}(A^{\otimes i}) > 0$. Applying Proposition 3.24 and Example 3.12 yields that $\pi^*\mathcal{T} \simeq \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(i)^{\oplus \mathrm{rk}(\mathcal{W}_i)}$ is a tilting bundle for $D^b(\mathbb{P}^n)$. Now Proposition 3.29 shows that $\mathcal{T} = \bigoplus_{i=0}^n \mathcal{W}_i$ is a tilting bundle for $D^b(X)$. Note that since X is smooth, the endomorphism algebra $\mathrm{End}(\mathcal{T})$ has finite global dimension according to Theorem 3.8 (iii). □

Remark 3.37. As mentioned at the end of the proof of Theorem 3.36, the endomorphism algebra $\mathrm{End}(\mathcal{T})$ has finite global dimension according to Theorem 3.8. Therefore, $\mathcal{T}$ is also a tilting sheaf in the more restricted sense as stated in the definition right after Proposition 3.6 above.

We now show that $\mathcal{T}$ is a direct summand of the locally free sheaf $\mathcal{R}^{\vee} = \bigoplus_{i=0}^n \mathcal{W}_1^{\otimes i}$. Note that since $\mathcal{R}$ is a tilting bundle, $\mathcal{R}^{\vee}$ is a tilting bundle too (see [52], Proposition 2.6 (1)).

Proposition 3.38. *Let X be a n-dimensional Brauer–Severi variety and $\mathcal{T}$ the tilting bundle from Theorem 3.36. Then $\mathcal{T}$ is a direct summand of the tilting bundle $\mathcal{R}^{\vee} = \bigoplus_{i=0}^n \mathcal{W}_1^{\otimes i}$.*

Proof. By definition, the locally free sheaves $\mathcal{W}_j$ are indecomposable direct summands of $\mathcal{W}_1^{\otimes j}$. Hence $\mathcal{T}$ is a direct summand of $\mathcal{R}^{\vee}$. □

Remark 3.39. By definition, the locally free sheaves $\mathcal{W}_j$ have the property of being indecomposable. Furthermore, by Proposition 1.41 they are up to isomorphism the only indecomposable locally free sheaves such that $\mathcal{W}_j \otimes_k \bar{k} \simeq \mathcal{O}_{\mathbb{P}^n}(j)^{\oplus \mathrm{rk}(\mathcal{W}_j)}$. In view of this fact, we have chosen the tilting bundle $\mathcal{T} = \bigoplus_{i=0}^n \mathcal{W}_i$ somehow optimal in the sense that all direct summands are indecomposable and the rank of $\mathcal{T}$ is minimal with respect to the property that $\mathcal{T} \otimes_k \bar{k}$ is a tilting bundle for $D^b(\mathbb{P}^n)$ obtained as the direct sum of invertible sheaves on $\mathbb{P}^n$. Such kind of "optimal" choices among tilting objects were also considered by Hille and Perling [87], where they chose tilting bundles for rational surfaces such that the ranks of the direct summands are minimal. This is very useful since it simplifies the calculation of the representations of the endomorphism algebra. Moreover, they proved that the tilting object $\mathcal{T}$ can be chosen such that the endomorphism algebra $\mathrm{End}(\mathcal{T})$ is quasi-hereditary and hence belongs to a class of algebras that is well-understood (see [87] for details).

As a consequence of the Theorem 3.36, one immediately gets a result of Bernardara [28], stating a semiorthogonal decomposition for Brauer–Severi varieties. For this, we recall the definition of Morita equivalence. Two associative unital rings R and S are called *Morita equivalent* if there is an equivalence $\mathrm{Mod}(R) \xrightarrow{\sim} \mathrm{Mod}(S)$. One can show that R and $S = M_n(R)$ are Morita equivalent for all n (see [7], Corollary 22.6). Furthermore, if R and S are Morita equivalent one has $\mathrm{mod}(R) \simeq \mathrm{mod}(S)$ (see [7], p.266) and hence $D^b(R) \xrightarrow{\sim} D^b(S)$. The next lemma is a special case of [37], Theorem 3.2 and is also proved in [89], Lemma 1.58.

Lemma 3.40. *Let X be a smooth projective and integral k-scheme and $\mathcal{E}$ an exceptional object in $D^b(X)$. Then the inclusion functor $\langle \mathcal{E} \rangle \subset D^b(X)$ has a right adjoint.*

With this fact we obtain the following observation.

Proposition 3.41. *Let X be a n-dimensional Brauer–Severi variety and $\mathcal{T} = \bigoplus_{i=0}^n \mathcal{W}_i$ the tilting bundle from Theorem 3.36. Then the inclusion functor $\langle \mathcal{W}_i \rangle \subset D^b(X)$, for $0 \le i \le n$, has a right adjoint.*

Proof. By [37], Lemma 3.1, to show that $\langle \mathcal{W}_i \rangle \subset D^b(X)$ has a right adjoint is equivalent to prove that for all $\mathcal{F} \in D^b(X)$ there exists a distinguished triangle

$$\mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{E}[1],$$

such that $\mathcal{E} \in \langle \mathcal{W}_i \rangle$ and $\mathcal{G} \in \{\mathcal{H} \in D^b(X) | \mathrm{Hom}(\mathcal{M}, \mathcal{H}[l]) = 0,\ \forall l \in \mathbb{Z},\ \mathcal{M} \in \langle \mathcal{W}_i \rangle\}$. In order to prove that, we consider the canonical morphism

$$\bigoplus \mathrm{Hom}(\mathcal{W}_i, \mathcal{F}[l]) \otimes \mathcal{W}_i[-l] \longrightarrow \mathcal{F}.$$

This can be completed to a distinguished triangle

$$\bigoplus \mathrm{Hom}(\mathcal{W}_i, \mathcal{F}[l]) \otimes \mathcal{W}_i[-l] \longrightarrow \mathcal{F} \longrightarrow \mathcal{A}. \tag{3.2}$$

Now $\bigoplus \mathrm{Hom}(\mathcal{W}_i, \mathcal{F}[l])$ is a finite-dimensional vector space, what implies that $\bigoplus \mathrm{Hom}(\mathcal{W}_i, \mathcal{F}[l]) \otimes \mathcal{W}_i[-l]$ is an element of $\langle \mathcal{W}_i \rangle$. It remains to show that $\mathrm{Hom}(\mathcal{M}, \mathcal{A}[l]) = 0$, $\forall l \in \mathbb{Z}$ and all $\mathcal{M} \in \langle \mathcal{W}_i \rangle$. After base change to some splitting field L, we have

$$\mathrm{Hom}(\mathcal{M}, \mathcal{A}[l]) \otimes_k L \simeq \mathrm{Hom}(\mathcal{M} \otimes_k L, \mathcal{A} \otimes_k L[l]).$$

The distinguished triangle (3.2) becomes after base change to L

$$(\bigoplus \mathrm{Hom}(\mathcal{W}_i, \mathcal{F}[l]) \otimes \mathcal{W}_i[-l]) \otimes_k L \longrightarrow \mathcal{F} \otimes_k L \longrightarrow \mathcal{A} \otimes_k L.$$

Again by flat base change we have

$$(\bigoplus \mathrm{Hom}(\mathcal{W}_i, \mathcal{F}[l]) \otimes \mathcal{W}_i[-l]) \otimes_k L \simeq \bigoplus \mathrm{Hom}(\mathcal{W}_i \otimes_k L, \mathcal{F} \otimes_k L[l]) \otimes \mathcal{W}_i \otimes_k L[-l].$$

Using the fact that $\mathcal{W}_i \otimes_k L \simeq \mathcal{O}_{\mathbb{P}^n}(i)^{\oplus \mathrm{rk}(\mathcal{W}_i)}$ and that $\mathcal{O}_{\mathbb{P}^n}(i)$ is exceptional, we conclude that $\mathrm{Hom}(\mathcal{M}\otimes_k L, \mathcal{A}\otimes_k L[l]) = 0$, for all $l \in \mathbb{Z}$, since $\mathcal{M}\otimes_k L \in \langle \mathcal{O}_{\mathbb{P}^n}(i)\rangle$. Hence $\mathrm{Hom}(\mathcal{M}, \mathcal{A}[l]) = 0$, $\forall l \in \mathbb{Z}$ and all $\mathcal{M} \in \langle \mathcal{W}_i \rangle$, what implies that $\mathcal{A} \in \{\mathcal{H} \in D^b(X) | \mathrm{Hom}(\mathcal{M}, \mathcal{H}[l]) = 0, \ \forall l \in \mathbb{Z}, \ \mathcal{M} \in \langle \mathcal{W}_i \rangle\}$ and hence $\langle \mathcal{W}_i \rangle \subset D^b(X)$ has a right adjoint. □

With this proposition and the above notation we have the following result, originally due to Bernardara [28], Corollary 5.8.

Corollary 3.42. *Let X be a Brauer–Severi scheme of dimension n over k. Then there is a semiorthogonal decomposition*

$$D^b(X) = \langle \mathcal{D}_0, ..., \mathcal{D}_n \rangle,$$

with $\mathcal{D}_i \simeq D^b(A^{\otimes i})$ for $0 \le i \le n$.

Proof. Theorem 3.36 tells us that $\mathcal{T} = \bigoplus_{i=0}^{n} \mathcal{W}_i$ is a tilting bundle for $D^b(X)$. By the definition of $\mathcal{W}_i$, (ii) of Definition 3.17 is fulfilled for $\mathcal{D}_i = \langle \mathcal{W}_i \rangle$, as can be verified after base change to $\bar{k}$. By Proposition 3.41, the inclusion functors $\langle \mathcal{W}_i \rangle \subset D^b(X)$ have right adjoints and hence the triangulated subcategories $\langle \mathcal{W}_i \rangle$ from a semiorthogonal decomposition. Therefore we have

$$D^b(X) = \langle \langle \mathcal{W}_0 \rangle, ..., \langle \mathcal{W}_n \rangle \rangle.$$

By a result of Keller [102], Theorem 8.5, we get a triangulated equivalence $\langle \mathcal{W}_i \rangle \simeq D^b(\mathrm{End}(\mathcal{W}_i))$. In the proof of Proposition 1.51 we have seen that $\mathrm{End}(\mathcal{W}_i)$ is Brauer-equivalent to $A^{\otimes i}$ and hence they are Morita equivalent. This finally implies $\langle \mathcal{W}_i \rangle \simeq D^b(A^{\otimes i})$. This completes the proof. □

Remark 3.43. Note that the above semiorthogonal decomposition for a Brauer–Severi variety can also be obtained from more general results as stated by Elagin [65], where a descent theory for semiorthogonal decomposition is formulated, and by Sosna [144], where scalar extensions of derived categories were treated.

We now want to study the generalized Brauer–Severi varieties $\mathrm{BS}(d, A)$ associated to a central simple k-algebra A of degree n. To apply the same arguments as in Theorem 3.36 for generalized Brauer–Severi varieties, one needs a characteristic-free tilting bundle for $\mathrm{BS}(d, A) \otimes_k \bar{k} \simeq \mathrm{Grass}(d, n)$. Kapranov [95] investigated Grassmannians in characteristic zero and proved the existence of a tilting bundle, making use of the Borel–Weil–Bott Theorem. More precise, he proved that on a Grassmannian $\mathrm{Grass}(d, n)$ over a field k of characteristic zero the locally free sheaf $\mathcal{T} = \bigoplus_{\lambda} \Sigma^{\lambda}(\mathcal{S})$ is a tilting bundle and the direct summands form a full strongly exceptional collection. Here $\mathcal{S}$ is the tautological sheaf on $\mathrm{Grass}(d, n)$, Σ the Schur functor (see p.69) and the direct sum is taken over all partitions λ with at most d rows and at most $n-d$ columns. Now the main problem in arbitrary characteristic is that first, there is no Borel–Weil–Bott theorem, and second, that Kaneda [94] showed that the above bundle $\mathcal{T}$ remains a tilting bundle as long as $\mathrm{char}(k) \ge n-1$. The bundle $\mathcal{T}$ fails to be a tilting bundle in very small characteristic as shown by Buchweitz, Leuschke and Van den Bergh [53], 3.3. To be more precise, they showed that in $\mathrm{char}(k) = 2$ the bundle from above cannot be a tilting bundle on Grass(2, 4), since $\mathrm{Ext}^1(\mathrm{Sym}^2(\mathcal{S}), \bigwedge^2(\mathcal{S})) \neq 0$ and hence $\mathcal{T}$ has non-trivial self extension. But instead of taking the above bundle $\mathcal{T}$, the authors proved that $\mathcal{T}' = \bigoplus_\lambda \bigwedge^{\lambda'}(\mathcal{S})$ is a characteristic-free tilting bundle (see [53], Theorem 1.3). Here λ' is the conjugate partition of λ and $\bigwedge^\alpha(\mathcal{S}) = \bigwedge^{\alpha_1}(\mathcal{S}) \otimes ... \otimes \bigwedge^{\alpha_s}(\mathcal{S})$ for an arbitrary partition $\alpha = (\alpha_1, ..., \alpha_s)$. As in characteristic zero, the sum is taken over all partitions λ with at most d rows and at most $n - d$ columns.

Now to guarantee that the same argument as in the proof of Theorem 3.36 works, we have to consider the direct summands $\bigwedge^{\lambda'}(\mathcal{S})$ and investigate if they descent. Let $\lambda = (\lambda_1, ..., \lambda_d)$, where $0 \leq \lambda_i \leq n - d$, be a partition with at most d rows and at most $n - d$ columns, then the conjugate partition $\lambda' = (\lambda'_1, ..., \lambda'_{n-d})$ has at most $n - d$ rows and at most d columns. Now we consider $\bigwedge^{\lambda'}(\mathcal{S}) = \bigwedge^{\lambda'_1}(\mathcal{S}) \otimes ... \otimes \bigwedge^{\lambda'_{n-d}}(\mathcal{S})$ and investigate if $\bigwedge^{\lambda'_i}(\mathcal{S})$ descents. Note that $0 \leq \lambda'_i \leq d$. By the definition of the Schur functor, $\bigwedge^{\lambda'_i}(\mathcal{S}) = \Sigma^{\alpha_i}(\mathcal{S})$, for the partition $\alpha_i = (1, 1, ..., 1)$ with the ones in the partition occurring λ'_i-times. Since $0 \leq \lambda'_i \leq d$, the partition α_i is a partition belonging to the set of partitions of at most d rows and at most $n - d$ columns. As mentioned in the last chapter, the sheaves $\Sigma^{\alpha_i}(\mathcal{S})$ do not descent but $\Sigma^{\alpha_i}(\mathcal{S})^{\oplus n\cdot\lambda'_i}$ do (see [115], Section 4, p.114). Let $\mathcal{N}_{\lambda'_i}$ be the locally free sheaf with the property that $\mathcal{N}_{\lambda'_i} \otimes_k L \simeq \Sigma^{\alpha_i}(\mathcal{S})^{\oplus n\cdot\lambda'_i}$ for a Galois splitting field $k \subset L$ (note that $\mathrm{BS}(d, A) \otimes_k L \simeq \mathrm{Grass}(d, n)$ and $\mathcal{S}$ is the tautological sheaf on Grass(d, n)). By Proposition 2.5, this locally free sheaf is unique up to isomorphism. We set $\mathcal{N}_{\lambda'} = \mathcal{N}_{\lambda'_1} \otimes ... \otimes \mathcal{N}_{\lambda'_{n-d}}$ and consider the locally free sheaf $\mathcal{T} = \bigoplus_\lambda \mathcal{N}_{\lambda'}$, where the sum is taken over all partitions λ with at most d rows and at most $n - d$ columns. With this notation one has the following result:

Theorem 3.44. *Let $X = \mathrm{BS}(d, A)$ be a generalized Brauer–Severi variety over a field k associated to the degree n central simple k-algebra A. Then the locally free sheaf $\mathcal{T} = \bigoplus_\lambda \mathcal{N}_{\lambda'}$ from above is a tilting bundle for $D^b(X)$.*

Proof. Let $\pi : \mathrm{BS}(d, A) \otimes_k \bar{k} \to \mathrm{BS}(d, A)$ be the projection and $\lambda = (\lambda_1, ..., \lambda_d)$ a partition with at most d rows and $n - d$ columns. By the above discussion we have for $\lambda' = (\lambda'_1, ..., \lambda'_{n-d})$, $\pi^*\mathcal{N}_{\lambda'_i} \simeq \Sigma^{\alpha_i}(\mathcal{S})^{\oplus n\cdot\lambda'_i} \simeq \bigwedge^{\lambda'_i}(\mathcal{S})^{\oplus n\cdot\lambda'_i}$, where $\mathcal{S}$ is the tautological sheaf on $\mathrm{BS}(d, A) \otimes_k \bar{k} \simeq \mathrm{Grass}(d, n)$. Hence

$$\pi^*\mathcal{N}_{\lambda'} \simeq \pi^*\mathcal{N}_{\lambda'_1} \otimes ... \otimes \pi^*\mathcal{N}_{\lambda'_{n-d}} \simeq \bigwedge^{\lambda'_1}(\mathcal{S})^{\oplus n\cdot\lambda'_1} \otimes ... \otimes \bigwedge^{\lambda'_{n-d}}(\mathcal{S})^{\oplus n\cdot\lambda'_{n-d}}.$$

By Proposition 3.24 and [53], Theorem 1.3, the object

$$\pi^*\mathcal{T} \simeq \bigoplus_\lambda \pi^*\mathcal{N}_{\lambda'} \simeq \bigoplus_\lambda (\bigwedge^{\lambda'_1}(\mathcal{S})^{\oplus n\cdot\lambda'_1} \otimes ... \otimes \bigwedge^{\lambda'_{n-d}}(\mathcal{S})^{\oplus n\cdot\lambda'_{n-d}})$$

is a characteristic-free tilting bundle for $D^b(\mathrm{BS}(d, A) \otimes_k \bar{k})$. Again Proposition 3.29 shows that $\mathcal{T}$ is a tilting bundle for $D^b(\mathrm{BS}(d, A))$. Since $\mathrm{BS}(d, A)$ is smooth, the endomorphism algebra $\mathrm{End}(\mathcal{T})$ has finite global dimension according to Theorem 3.8. □

Remark 3.45. Note that the last theorem implies Theorem 3.36, since a Brauer–Severi variety is a generalized Brauer–Severi variety with $d = 1$.

With Theorem 3.44 and Proposition 3.27 one immediately has:

Corollary 3.46. *Let $X = X_1 \times ... \times X_r$ be a finite product of generalized Brauer–Severi varieties, then $D^b(X)$ admits a tilting object.*

For a Brauer–Severi variety X corresponding to a central simple k-algebra A, we consider the product of X and $Y = \mathrm{BS}(d, A^{\otimes m})$, where $m > 0$. Karpenko [98], Corollary 6.4 proved that $Y \times X$ is a Grassmann bundle over X and Corollary 3.46 implies that this Grassmann bundle has a tilting object. In the next section we will see that this holds in general, provided the base scheme admits a tilting object.
The tilting bundle for the n-dimensional Brauer–Severi variety X corresponding to a central simple k-algebra $A \simeq M_m(D)$ consists of indecomposable direct summands $\mathcal{W}_i$ with the property that $\mathrm{End}(\mathcal{W}_i)$ is a central simple k-algebra (see Chapter 1). Especially for $\mathcal{W}_1$ we have $\mathrm{End}(\mathcal{W}_1) \simeq D$, what implies that the sheaves $\mathcal{W}_i$ are not exceptional and hence they do not form a full strongly exceptional collection. In fact, it turns out that in general, Brauer–Severi varieties, and hence generalized Brauer–Severi varieties, do not admit full strongly exceptional collections consisting of coherent sheaves, although they always have a tilting object. To show this is the content of the next result.

Theorem 3.47. *Let X be a non-split 1-dimensional Brauer–Severi variety. Then X does not admit a full strongly exceptional collection of coherent sheaves.*

Proof. Suppose there is a full strongly exceptional collection $\mathcal{E}_1, ..., \mathcal{E}_m$ on X. First of all, Proposition 3.30 yields that $K_0(X)$ is free abelian of rank two, since it is generated by the indecomposable locally free sheaves $\mathcal{O}_X$ and $\mathcal{W}_1$ that form a tilting bundle $\mathcal{T} = \mathcal{O}_X \oplus \mathcal{W}_1$, according to Theorem 3.36. Thus, assuming the existence of a full strongly exceptional collection $\mathcal{E}_1, ..., \mathcal{E}_m$, we conclude that it also forms a basis of $K_0(X)$ and hence there has to be only of two them, say $\mathcal{E}_1$ and $\mathcal{E}_2$. Now by assumption, we have $\mathrm{End}(\mathcal{E}_i) = k$ and hence they have to be simple sheaves. But these sheaves remain simple after base change to some finite Galois splitting field L, since $\mathrm{End}(\mathcal{E}_i \otimes_k L) \simeq L$. Simple coherent sheaves on the projective space $X \otimes_k L \simeq \mathbb{P}^1$ are known to be invertible sheaves or skyscraper sheaves supported on a closed point. Thus, the simple sheaves $\mathcal{E}_i \otimes_k L$ have to be isomorphic to either $\mathcal{O}_{\mathbb{P}^1}(n)$ or $L(x)$. Since the two exceptional sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ form a full strongly exceptional collection, they give rise to a tilting sheaf $\mathcal{E}_1 \oplus \mathcal{E}_2$, according to Proposition 3.20. This tilting sheaf remains a tilting sheaf after base change to L by Proposition 3.28 and hence we have that $(\mathcal{E}_1 \otimes_k L) \oplus (\mathcal{E}_2 \otimes_k L)$ is a tilting sheaf for $D^b(\mathbb{P}^1)$. Furthermore, since the period of X is two, every invertible sheaf on $X \otimes_k L \simeq \mathbb{P}^1$ coming from an invertible sheaf on the Brauer–Severi variety X is by Theorem 1.45 of the form $\mathcal{O}_{\mathbb{P}^1}(2n)$. We now show that $\mathcal{E}_1 \otimes_k L \oplus \mathcal{E}_2 \otimes_k L$ cannot be a tilting object for $\mathbb{P}^1$. There are three cases that have to be considered:

(1) Firstly we consider the case where $\mathcal{E}_1 \otimes_k L$ and $\mathcal{E}_2 \otimes_k L$ are both invertible sheaves. In this case $\mathcal{E}_1 \otimes_k L = \mathcal{O}_{\mathbb{P}^1}(2n)$ and $\mathcal{E}_2 \otimes_k L = \mathcal{O}_{\mathbb{P}^1}(2m)$. Without loss of generality, we can assume $\mathcal{E}_1 \otimes_k L = \mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{E}_2 \otimes_k L = \mathcal{O}_{\mathbb{P}^1}(2)$. But then we have
$$\mathrm{Ext}^l(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2), \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \neq 0$$
for $l > 0$, since $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(2), \mathcal{O}_{\mathbb{P}^1}) = H^1(X, \mathcal{O}_{\mathbb{P}^1}(-2)) = L$. Hence the vanishing of Ext fails to hold.

(2) For the second case, we consider $\mathcal{E}_1 \otimes_k L = L(x)$ and $\mathcal{E}_2 \otimes_k L = L(y)$. Again, considering $\mathrm{Ext}^l(L(x) \oplus L(y), L(x) \oplus L(y))$ we find

$$\mathrm{Ext}^1(L(x), L(x)) \simeq T_x,$$

where T_x is the tangent space in x (see [89], Example 11.9) that is non-zero. Again the Ext vanishing fails to hold.

(3) It remains the case $\mathcal{E}_1 \otimes_k L = L(x)$ and $\mathcal{E}_2 \otimes_k L = \mathcal{O}_{\mathbb{P}^1}(2n)$. But then, for $\mathcal{O}_{\mathbb{P}^1}(2n) \oplus L(x)$ we find

$$\mathrm{Ext}^l(\mathcal{O}_{\mathbb{P}^1}(2n) \oplus L(x), \mathcal{O}_{\mathbb{P}^1}(2n) \oplus L(x)) \neq 0$$

for $l > 0$, since $\mathrm{Ext}^1(L(x), L(x)) \simeq T_x$ as in (2) that is non-zero.

This completes the proof and we see that there is no such collection. □

Summarizing the results in this section one can state that generalized Brauer–Severi varieties admit tilting bundles although they in general do not admit a full strongly exceptional collection consisting of coherent sheaves. Theorem 3.47 treated the case where the Brauer–Severi variety is non-split and one-dimensional, but conjecturally the same result should hold for arbitrary non-split Brauer–Severi varieties.

Conjecture. *Let X be a non-split n-dimensional Brauer–Severi variety. Then X does not admit a full strongly exceptional collection of coherent sheaves.*

Note that Theorem 3.47 shows that k-forms of projective homogenous varieties may not posses a full strongly exceptional collection, at least if the collection is supposed to consist of coherent sheaves. This suggests that the conjecture of Catanese [35] concerning the existence of full strongly exceptional collections on homogeneous varieties may not hold for twisted forms of them. Nonetheless, they may always admit a tilting object. It is therefore reasonable to presume that any k-form of a projective homogenous variety should posses a tilting object. In view of this reflection, we formulate the problem of finding (more) examples of schemes X admitting a tilting object, but not a full strongly exceptional collection consisting of arbitrary objects.

3.4 Tilting objects for relative flags

In this section we investigate some relative versions of flag varieties, namely the projective bundle, the Grassmann bundle and relative flag varieties of type A_n. Furthermore, we study if quadric bundles (or quadric fibrations) have tilting objects. The motivation for considering the relative version of flag varieties is not only to generalize the classical results obtained by Beilinson [23] and Kapranov [95], [96], [97], but also to investigate some projective homogenous varieties occurring as successive iteration of these relative flags. As mentioned in the last section, it is still an open problem if projective homogenous varieties admit tilting objects. For the projective and Grassmann bundle Orlov [129] proved the existence of semiorthogonal decompositions. Böhning [35] furthermore showed that relative flags of type A_n also have a semiorthogonal decomposition. In some following papers Costa, Di Rocco and Miró–Roig [56], [57] investigated the projective bundle and showed, that $\mathbb{P}(\mathcal{E})$ admits a full strongly exceptional collection consisting of invertible sheaves, provided the base scheme X admits such. As we have seen above, Brauer–Severi varieties in general do not have full strongly exceptional collections of invertible sheaves, but they always admit a tilting bundle. So in view of this phenomenon, that also appears for some smooth projective surfaces (see [131]), we want to investigate the situation where in general the base scheme X admits a tilting object, not necessarily consisting of direct summands that form a full strongly exceptional collection. Then, taking for instance X to be a generalized Brauer–Severi variety, we will produce some new examples of schemes for that tilting objects exist.

We start with the investigation of the projective bundle. For this, let X be a smooth projective and integral k-scheme and $\mathcal{E}$ a locally free sheaf of rank r on X. Then one has the projective bundle $\mathbb{P}(\mathcal{E}) = \mathrm{Proj}(S^\bullet(\mathcal{E}))$, where $S^\bullet(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$, together with the projective structure morphism $\pi : \mathbb{P}(\mathcal{E}) \to X$. Defining $D^b(X)_j$ to be the triangulated subcategory of $D^b(\mathbb{P}(\mathcal{E}))$ consisting of objects of the form $\pi^*\mathcal{H}\otimes\mathcal{O}_{\mathcal{E}}(j)$, where $\mathcal{H}$ is an object of $D^b(X)$, Orlov [129], Theorem 2.6 proved that $\langle D^b(X)_{-r+1}, ..., D^b(X)_0\rangle$ is a semiorthogonal decomposition for $D^b(\mathbb{P}(\mathcal{E}))$. Note that Orlov proved that result for $k = \mathbb{C}$ but the theorem still holds for arbitrary fields k (see [89], Section 8). We start our investigation with a lemma.

Lemma 3.48. *Let X and $\mathcal{E}$ be as above and suppose that a coherent sheaf $\mathcal{A}$ generates $D^b(X)$. Then $\bigoplus_{i=0}^{r-1}\pi^*\mathcal{A}\otimes\mathcal{O}_{\mathcal{E}}(-i)$ generates $D^b(\mathbb{P}(\mathcal{E}))$.*

Proof. First we note that $D^b(X)$ is derived equivalent to $D^b(X)_j$ via the functor $\mathcal{F}\mapsto\pi^*\mathcal{F}\otimes\mathcal{O}_{\mathcal{E}}(j)$. Indeed, by adjunction of π^* and π_* we have

$$\mathrm{Hom}(\pi^*\mathcal{F}\otimes\mathcal{O}_{\mathcal{E}}(j),\pi^*\mathcal{G}\otimes\mathcal{O}_{\mathcal{E}}(j)) \simeq \mathrm{Hom}(\mathcal{F},\mathbb{R}\pi_*\pi^*(\mathcal{G})) \simeq \mathrm{Hom}(\mathcal{F},\mathcal{G}),$$

since $\mathbb{R}\pi_*\pi^*(\mathcal{G}) \simeq \mathbb{R}\pi_*\mathcal{O}_{\mathcal{E}}\otimes\mathcal{G}$ according to the projection formula and due to the fact that $\mathbb{R}\pi_*\mathcal{O}_{\mathcal{E}} \simeq \mathcal{O}_X$. Since $\mathcal{A}$ generates $D^b(X)$ and $D^b(X)$ is derived equivalent to $D^b(X)_j$, we conclude that $\pi^*\mathcal{A}\otimes\mathcal{O}_{\mathcal{E}}(j)$ generates $D^b(X)_j$. Hence $\bigoplus_{i=0}^{r-1}\pi^*\mathcal{A}\otimes\mathcal{O}_{\mathcal{E}}(-i)$ generates $D^b(\mathbb{P}(\mathcal{E}))$ because $\langle D^b(X)_{-r+1}, ..., D^b(X)_0\rangle$ is a semiorthogonal decomposition for $D^b(\mathbb{P}(\mathcal{E}))$. □

With this lemma we make the following observation:

Proposition 3.49. *Let X be as above and $\mathcal{E}$ a locally free sheaf of rank r on X. Suppose X admits a tilting bundle $\mathcal{T}_X$ and that $H^m(X,\mathcal{T}^\vee{}_X\otimes\mathcal{T}_X\otimes S^l(\mathcal{E})) = 0$, for $m > 0$ and all $0 \leq l \leq r-1$. Then $\mathcal{T} = \bigoplus_{i=0}^{r-1}\pi^*\mathcal{T}_X\otimes\mathcal{O}_{\mathcal{E}}(-i)$ is a tilting bundle for $\mathbb{P}(\mathcal{E})$.*

Proof. To prove that $\mathcal{T} = \bigoplus_{i=0}^{r-1}\pi^*\mathcal{T}_X\otimes\mathcal{O}_{\mathcal{E}}(-i)$ is a tilting bundle for $\mathbb{P}(\mathcal{E})$, we have to check $\mathrm{Ext}^m(\mathcal{T},\mathcal{T}) = 0$ for $m > 0$. By the definition of $\mathcal{T}$ it is enough to check this for $\mathrm{Ext}^m(\pi^*\mathcal{T}_X\otimes\mathcal{O}_{\mathcal{E}}(-r_1),\pi^*\mathcal{T}_X\otimes\mathcal{O}_{\mathcal{E}}(-r_2))$, with $0 \leq r_1, r_2 \leq r-1$. Adjunction of π^* and π_* and projection formula yields

$$\mathrm{Hom}(\pi^*\mathcal{T}_X\otimes\mathcal{O}_{\mathcal{E}}(-r_1),\pi^*\mathcal{T}_X\otimes\mathcal{O}_{\mathcal{E}}(-r_2)[m]) \simeq \mathrm{Hom}(\mathcal{T}_X,\mathcal{T}_X\otimes\mathbb{R}\pi_*\mathcal{O}_{\mathcal{E}}(r_1-r_2)[m]).$$

The crucial object here is clearly $\mathbb{R}\pi_*\mathcal{O}_{\mathcal{E}}(r_1-r_2)$ whose cohomology is well-known. There are three cases that have to be considered:

(1) $r_1 = r_2$; In this case we have $\mathbb{R}\pi_*\mathcal{O}_{\mathcal{E}}(r_1 - r_2) = \mathbb{R}\pi_*\mathcal{O}_{\mathcal{E}} \simeq \mathcal{O}_X$, what implies

$$\begin{aligned}\mathrm{Ext}^m(\pi^*\mathcal{T}_X \otimes \mathcal{O}_{\mathcal{E}}(-r_1), \pi^*\mathcal{T}_X \otimes \mathcal{O}_{\mathcal{E}}(-r_2)) &\simeq \mathrm{Ext}^m(\mathcal{T}_X, \mathcal{T}_X \otimes \mathbb{R}\pi_*\mathcal{O}_{\mathcal{E}}) \\ &\simeq \mathrm{Ext}^m(\mathcal{T}_X, \mathcal{T}_X) = 0,\end{aligned}$$

for $m > 0$, since $\mathcal{T}_X$ is a tilting bundle by assumption.

(2) $0 \leq r_1 < r_2 \leq r-1$; In this case we have $\mathbb{R}\pi_*\mathcal{O}_{\mathcal{E}}(r_1 - r_2) = 0$, since $r_1 - r_2 > -r$ and hence we find

$$\mathrm{Ext}^m(\mathcal{T}_X, \mathcal{T}_X \otimes \mathbb{R}\pi_*\mathcal{O}_{\mathcal{E}}(r_1 - r_2)) \simeq \mathrm{Ext}^m(\mathcal{T}_X^i, 0) = 0,$$

for $m \geq 0$.

(3) $0 \leq r_2 < r_1 \leq r - 1$; We set $r_1 - r_2 = l$ and the only case we have to consider is $\mathbb{R}^0\pi_*\mathcal{O}_{\mathcal{E}}(l)$, since $\mathbb{R}^i\pi_*\mathcal{O}_{\mathcal{E}}(l) = 0$ for $i \neq 0$ and all $l > 0$. Note that $\mathbb{R}^0\pi_*\mathcal{O}_{\mathcal{E}}(l) \simeq S^l(\mathcal{E})$ for $l \geq 0$ (see [82]). But by assumption we have

$$H^m(X, \mathcal{T}_X^\vee \otimes \mathcal{T}_X \otimes S^l(\mathcal{E})) = 0,$$

for $m > 0$ and all $0 \leq l \leq r - 1$. This now implies that

$$\begin{aligned}\mathrm{Ext}^m(\pi^*\mathcal{T}_X \otimes \mathcal{O}_{\mathcal{E}}(-r_1), \pi^*\mathcal{T}_X \otimes \mathcal{O}_{\mathcal{E}}(-r_2)) &\simeq \mathrm{Ext}^m(\mathcal{T}_X, \mathcal{T}_X \otimes \mathbb{R}\pi_*\mathcal{O}_{\mathcal{E}}(l)) \\ &\simeq \mathrm{Ext}^m(\mathcal{T}_X, \mathcal{T}_X \otimes S^l(\mathcal{E})) \\ &\simeq H^m(X, \mathcal{T}_X^\vee \otimes \mathcal{T}_X \otimes S^l(\mathcal{E})) \\ &= 0,\end{aligned}$$

for $m > 0$ and all $0 \leq l \leq r - 1$. Summarizing (1), (2) and (3) we finally get

$$\mathrm{Ext}^m(\mathcal{T}, \mathcal{T}) = 0, \text{ for } m > 0.$$

The generating property of $\bigoplus_{i=0}^{r-1} \pi^*\mathcal{T}_X \otimes \mathcal{O}_{\mathcal{E}}(-i)$ now follows from Lemma 3.48, since $\mathcal{T}_X$ is a tilting bundle for $D^b(X)$ and hence generates $D^b(X)$. Thus, $\mathcal{T}$ is a tilting bundle for $D^b(\mathbb{P}(\mathcal{E}))$ and, since $\mathbb{P}(\mathcal{E})$ is smooth over k, the global dimension of $\mathrm{End}(\mathcal{T})$ is finite by Theorem 3.8. □

With this proposition, we now obtain the following result.

Theorem 3.50. *Let X be as above and $\mathcal{E}$ a locally free of rank r and suppose that $D^b(X)$ admits a tilting bundle. Then $D^b(\mathbb{P}(\mathcal{E}))$ admits a tilting object.*

Proof. Let $\mathcal{T}_X$ be the tilting bundle for X. Proposition 3.49 tells us that we have to verify $H^m(X, \mathcal{T}^\vee{}_X \otimes \mathcal{T}_X \otimes S^l(\mathcal{E})) = 0$ for $m > 0$ and all $0 \leq l \leq r - 1$. The case $l = 0$ was proved in Proposition 3.49 (1) so that we only have to consider $0 < l \leq r - 1$. Note that for every invertible sheaf $\mathcal{L}$ on X one has $S^l(\mathcal{E} \otimes \mathcal{L}) \simeq S^l(\mathcal{E}) \otimes \mathcal{L}^{\otimes l}$. Since X is projective, there is for a fixed $l > 0$ an ample invertible sheaf $\mathcal{L}$ and an integer $n_l > 0$ such that

$$\begin{aligned}H^m(X, \mathcal{T}^\vee{}_X \otimes \mathcal{T}_X \otimes S^l(\mathcal{E} \otimes \mathcal{L}^{\otimes n_l})) &\simeq H^m(X, \mathcal{T}^\vee{}_X \otimes \mathcal{T}_X \otimes S^l(\mathcal{E}) \otimes \mathcal{L}^{\otimes n_l \cdot l}) \\ &= 0,\end{aligned}$$

for $m > 0$. Since $0 < l \leq r-1$, we have only finitely many $l > 0$ and we can choose $n > \max\{n_l | 0 \leq l \leq r-1\}$, so that for $\mathcal{L}^{\otimes n}$ we have

$$\begin{aligned} H^m(X, \mathcal{T}^\vee{}_X \otimes \mathcal{T}_X \otimes S^l(\mathcal{E} \otimes \mathcal{L}^{\otimes n})) &\simeq H^m(X, \mathcal{T}^\vee{}_X \otimes \mathcal{T}_X \otimes S^l(\mathcal{E}) \otimes \mathcal{L}^{\otimes n \cdot l}) \\ &= 0, \end{aligned}$$

for $m > 0$ and all $0 < l \leq r-1$. Proposition 3.49 now yields that for all $N \geq n$, for $\mathcal{E}' = \mathcal{E} \otimes \mathcal{L}^{\otimes N}$, the locally free sheaf $\mathcal{T} = \bigoplus_{i=0}^{r-1} \pi^* \mathcal{T}_X \otimes \mathcal{O}_{\mathcal{E}'}(-i)$ is a tilting bundle for $\mathbb{P}(\mathcal{E} \otimes \mathcal{L}^{\otimes N})$. Since $\mathbb{P}(\mathcal{E} \otimes \mathcal{L}^{\otimes N})$ is isomorphic to $\mathbb{P}(\mathcal{E})$, we finally conclude that $\mathbb{P}(\mathcal{E})$ admits a tilting bundle (see Remark 3.51 for an explanation). Note that the global dimension of the endomorphism algebra of the tilting bundle is finite according to Theorem 3.8, since $\mathbb{P}(\mathcal{E})$ is a smooth k-scheme by assumption. This completes the proof. □

Remark 3.51. In general, if one has an equivalence $F : D^b(X) \to D^b(Y)$ for two smooth projective and integral k-schemes X and Y, a tilting object $\mathcal{T}$ for $D^b(X)$ yields a tilting object $F(\mathcal{T})$ for $D^b(Y)$. This is clear, since the properties (i), (ii) and (iii) in the definition of a tilting object are preserved under the equivalence. In the situation of Theorem 3.50, we therefore would argue as follows: Since $\mathbb{P}(\mathcal{E} \otimes \mathcal{L}^{\otimes N})$ is isomorphic to $\mathbb{P}(\mathcal{E})$, one has an induced equivalence $F : D^b(\mathbb{P}(\mathcal{E} \otimes \mathcal{L}^{\otimes N})) \xrightarrow{\sim} D^b(\mathbb{P}(\mathcal{E}))$. Then the tilting object $\mathcal{T}$ for $\mathbb{P}(\mathcal{E} \otimes \mathcal{L}^{\otimes N})$ provides us with the tilting object $F(\mathcal{T})$ for $\mathbb{P}(\mathcal{E})$.

Example 3.52. Let X be a generalized Brauer–Severi variety and $\mathcal{E}$ a locally free sheaf of finite rank. Then $\mathbb{P}(\mathcal{E})$ admits a tilting bundle.

Example 3.53. Since a Hirzebruch surface S is a projective bundle over $\mathbb{P}^1$, we conclude that S admits a tilting bundle. This is well-known and follows from the fact that all rational surfaces admit tilting objects (see [87]). By a result of Nagata [123], Theorem 7, a smooth surface X over $k = \bar{k}$ of degree d in $\mathbb{P}^{d+1}$, that is not contained in any hyperplane, is either a projective bundle over $\mathbb{P}^1$, or $\mathbb{P}^2$, or the Veronese surface in $\mathbb{P}^5$. In all three cases the surfaces admit a tilting bundle and hence X admits a tilting bundle. As well, this fact is well-known since X is rational and therefore admits a tilting bundle.

As mentioned at the beginning of this section, Costa and Miró–Roig [56] investigated the projective bundle and showed that $\mathbb{P}(\mathcal{E})$ admits a full strongly exceptional collection of invertible sheaves (and hence a tilting bundle), provided the base scheme X admits such a collection. If the base scheme does not admit a full strongly exceptional collection of invertible sheaves, their results do not apply. Hille and Perling [87], Theorem 1.1 showed that any smooth projective rational surface admits a tilting bundle. This tilting bundle is not obtained as a direct sum of invertible sheaves forming a full strongly exceptional collection. Moreover, Perling [131], Theorem 4.8.2, proved that a smooth complete toric surface $X \neq \mathbb{P}^2$ has a full strongly exceptional collection of invertible sheaves if and only if X can be obtained from a Hirzebruch surface in at most two steps by blowing up torus fixed points. Hence toric surfaces not obtained in this way do not admit a full strongly exceptional collection of invertible sheaves, but they admit a tilting bundle. Although the result of Costa and Miró–Roig does not apply to this situation, Theorem 3.50 applies and we conclude:

Corollary 3.54. *Let X be a smooth projective rational surface. Suppose Y is obtained by iteratively taking projective bundles of locally free sheaves of finite rank (finitely many times) started from X, then $D^b(Y)$ admits a tilting object.*

As another application of Theorem 3.50, we consider a vector space V of dimension $2n$ over an algebraically closed field of characteristic zero together with a non-degenerate skew-symmetric bilinear form $\omega \in \bigwedge^2 V^*$, i.e. a symplectic vector space V. For $1 \leq k_1 < ... < k_t \leq n$ one can consider the *variety of partial isotropic flags* in V, that is, the space of flags $L_{k_1} \subset L_{k_2} \subset ... \subset L_{k_t} \subset V$ of isotropic subspaces of V with $\dim(L_{k_i}) = k_i$, that we want to denote by IF_V. One can show that a complete isotropic flag variety is obtained as an iteration of projective bundles over $\mathbb{P}^{2n-1}$ (see [35], p.50 or [140], proof of Theorem 4.1) and is isomorphic to the homogenous varieties of the symplectic group Sp_{2n}. This fact, Theorem 3.50 and the fact that the projective space admits a tilting bundle has the following consequence:

Corollary 3.55. *Let $X = \mathrm{IF}_V$ be the partial isotropic flag variety from above. Then X admits a tilting bundle.*

As a slight generalization of the above theorem, we want to consider locally trivial fibration. This was done by Costa, Di Rocco and Miró–Roig [57] and by Samokhin [140] for arbitrary fibrations. Suppose we are given a smooth projective and integral k-scheme Z. A *locally trivial fibration over Z with typical fiber F* is a smooth projective and integral k-scheme X together with a flat morphism $\pi : X \to Z$, with the property that for the open sets in the Zariski topology $U \subset Z$ one has $\pi^{-1}(U) \simeq U \times F$. Here F is also supposed to be a smooth projective and integral k-scheme. Now Costa, Di Rocco and Miró–Roig proved that X has a full strongly exceptional collection of invertible sheaves if Z and F have one (see [57], Theorem 1.3). By contrast, Samokhin [140], Theorem 3.1 constructed a semiorthogonal decomposition for an arbitrary fibration $X \to Z$. We want to generalize the result of Costa, Di Rocco and Miró–Roig as assuming Z to have a tilting bundle such that in general the indecomposable direct summands are not invertible sheaves and do not form a full strongly exceptional collection. In view of a result of Perling [131], Theorem 4.8.2 and Theorem 3.47 above, this is a reasonable generalization. Notice, that any invertible sheaf $\mathcal{L}$ on F can be lifted to an invertible sheaf $\tilde{\mathcal{L}}$ on X (see [57], Remark 2.6).

Remark 3.56. Warning: The notation in [57] is the other way around, i.e., $\tilde{\mathcal{L}}$ denotes an invertible sheaf on F and the lifted invertible sheaf on X is denoted by $\mathcal{L}$.

We start with a lemma (see [57], p.10006 (4)) that also holds for arbitrary fibrations and if the sheaves $\tilde{\mathcal{L}}_i$ are arbitrary locally free sheaves of finite rank such that the restriction to any fiber gives a full strongly exceptional collection (see [140], p.6).

Lemma 3.57. *Let $\pi : X \to Z$ be a locally trivial fibration with typical fiber F over an algebraically closed field of characteristic zero and $\mathcal{L}_1, ..., \mathcal{L}_n$ a full strongly exceptional collection of invertible sheaves for $D^b(F)$. Consider the lifted collection $\tilde{\mathcal{L}}_1, ..., \tilde{\mathcal{L}}_n$ on X (see [57], Remark 2.6). Then the following* *holds:*

$$\mathbb{R}^s\pi_*(\tilde{\mathcal{L}}_q \otimes \tilde{\mathcal{L}}_p^\vee) = \begin{cases} 0 & \textit{for } s > 0 \\ 0 & \textit{for } s = 0 \textit{ and } q < p \\ \pi_*(\tilde{\mathcal{L}}_q \otimes \tilde{\mathcal{L}}_p^\vee) & \textit{for } s = 0 \textit{ and } q \geq p \end{cases}$$

To prove the next theorem we need a further previous observation. It is the following fact that can be found in [57] or [140].

Lemma 3.58. *Let $\pi : X \to Z$ be a locally trivial fibration with typical fiber F and $\mathcal{L}_1, ..., \mathcal{L}_n$ a full strongly exceptional collection of invertible sheaves (or locally free sheaves of finite rank) for $D^b(F)$. Suppose that $D^b(Z)$ is generated by some object $\mathcal{A}$, then $D^b(X)$ is generated by $\mathcal{R} = \bigoplus_{i=1}^n \pi^*(\mathcal{A}) \otimes \tilde{\mathcal{L}}_i$.*

Proof. The proof follows the lines of the proof of Lemma 3.48. Note that $\tilde{\mathcal{L}}_{i|F} = \mathcal{L}_i$ (see [57], Remark 2.6). With this fact and under the assumption from above, Samokhin [140], Theorem 3.1 proved that the functor $\pi^*(-) \otimes \tilde{\mathcal{L}}_i : D^b(Z) \to D^b(X)$ is fully faithful and that $D^b(X) = \langle \pi^* D^b(Z) \otimes \tilde{\mathcal{L}}_1, ..., \pi^* D^b(Z) \otimes \tilde{\mathcal{L}}_n \rangle$ is a semiorthogonal decomposition. The full subcategories $\pi^* D^b(Z) \otimes \tilde{\mathcal{L}}_i$ consist of objects of the form $\pi^*\mathcal{M} \otimes \tilde{\mathcal{L}}_i$, where $\mathcal{M} \in D^b(Z)$. Therefore, the functor $\pi^*(-) \otimes \tilde{\mathcal{L}}_i$ from above induces an equivalence between $D^b(Z)$ and $\pi^* D^b(Z) \otimes \tilde{\mathcal{L}}_i$. Now since $\mathcal{A}$ generates $D^b(Z)$, $\pi^*(\mathcal{A}) \otimes \tilde{\mathcal{L}}_i$ generates $\pi^* D^b(Z) \otimes \tilde{\mathcal{L}}_i$ and hence $\mathcal{R} = \bigoplus_{i=1}^n \pi^*(\mathcal{A}) \otimes \tilde{\mathcal{L}}_i$ generates $D^b(X)$. □

Remark 3.59. Clearly, if one assumes the sheaves $\tilde{\mathcal{L}}_i$ to be arbitrary locally free sheaves of finite rank giving a full strongly exceptional collection on any fiber, Lemma 3.58 still holds, since it is not necessary to assume the $\tilde{\mathcal{L}}_i$ to be invertible (see [140]). One can also prove a slight modification of Lemma 3.58. For this, consider the above fibration $\pi : X \to Z$ and the semiorthogonal decomposition $D^b(X) = \langle \pi^* D^b(Z) \otimes \tilde{\mathcal{L}}_1, ..., \pi^* D^b(Z) \otimes \tilde{\mathcal{L}}_n \rangle$. Suppose there are objects $\mathcal{A}_1, ..., \mathcal{A}_n$, such that for all $1 \leq i \leq n$, $\mathcal{A}_i$ generates $D^b(Z)$. Then clearly, the object $\bigoplus_{i=1}^n \pi^*(\mathcal{A}_i) \otimes \tilde{\mathcal{L}}_i$ also generates $D^b(X)$. The same holds for arbitrary fibrations as in [140] and arbitrary locally free sheaves $\tilde{\mathcal{L}}_i$ such that the restriction to any fiber gives a full strongly exceptional collection.

Theorem 3.60. *Let $\pi : X \to Z$ be a locally trivial fibration with typical fiber F over an algebraically closed field k of characteristic zero. Suppose $D^b(Z)$ admits a tilting bundle $\mathcal{T}$ and that $\tilde{\mathcal{L}}_1, ..., \tilde{\mathcal{L}}_n$ is a collection of locally free sheaves of finite rank on X such that $\pi_*(\tilde{\mathcal{L}}_i \otimes \tilde{\mathcal{L}}_i^\vee) \simeq \mathcal{O}_Z$. Denote by $\mathcal{L}_1, ..., \mathcal{L}_n$ the restriction of the collection $\tilde{\mathcal{L}}_i$ to the fiber F and suppose that $\mathcal{L}_1, ..., \mathcal{L}_n$ is a full strongly exceptional collection in $D^b(F)$. Then there exists an ample invertible sheaf $\mathcal{M}$ on Z, such that $\mathcal{R} = \bigoplus_{i=1}^n \pi^*(\mathcal{T} \otimes \mathcal{M}^{\otimes i}) \otimes \tilde{\mathcal{L}}_i$ is a tilting object for $D^b(X)$.*

Proof. We will show that there is an ample invertible sheaf $\mathcal{M}$ on Z such that $\mathcal{R} = \bigoplus_{i=1}^n \pi^*(\mathcal{T} \otimes \mathcal{M}^{\otimes i}) \otimes \tilde{\mathcal{L}}_i$ is a tilting object for $D^b(X)$. For the vanishing of Ext, we therefore have to find the ample invertible sheaf $\mathcal{M}$ such that

$$\mathrm{Ext}^l(\pi^*(\mathcal{T} \otimes \mathcal{M}^{\otimes i}) \otimes \tilde{\mathcal{L}}_i, \pi^*(\mathcal{T} \otimes \mathcal{M}^{\otimes j}) \otimes \tilde{\mathcal{L}}_j) = 0, \text{ for } l > 0.$$

But this is equivalent to

$$H^l(X, \pi^*(\mathcal{T} \otimes \mathcal{T}^\vee \otimes \mathcal{M}^{\otimes(j-i)}) \otimes \tilde{\mathcal{L}}_j \otimes \tilde{\mathcal{L}}_i^\vee) = 0, \text{ for } l > 0.$$

Applying the Leray spectral sequence for the morphism π, one gets

$$H^r(Z, \mathbb{R}^s\pi_*(\pi^*(\mathcal{T} \otimes \mathcal{T}^\vee \otimes \mathcal{M}^{\otimes(j-i)}) \otimes \tilde{\mathcal{L}}_j \otimes \tilde{\mathcal{L}}_i^\vee)) \Longrightarrow H^{r+s}(X, \pi^*(\mathcal{T} \otimes \mathcal{T}^\vee \otimes \mathcal{M}^{\otimes(j-i)}) \otimes \tilde{\mathcal{L}}_j \otimes \tilde{\mathcal{L}}_i^\vee).$$

With the projection formula we find

$$\mathbb{R}^s\pi_*(\pi^*(\mathcal{T} \otimes \mathcal{T}^\vee \otimes \mathcal{M}^{\otimes(j-i)}) \otimes \tilde{\mathcal{L}}_j \otimes \tilde{\mathcal{L}}_i^\vee) \simeq \mathcal{T} \otimes \mathcal{T}^\vee \otimes \mathcal{M}^{\otimes(j-i)} \otimes \mathbb{R}^s\pi_*(\tilde{\mathcal{L}}_j \otimes \tilde{\mathcal{L}}_i^\vee).$$

Now by Lemma 3.57 we know that $\mathbb{R}^s\pi_*(\tilde{\mathcal{L}}_j \otimes \tilde{\mathcal{L}}_i^\vee)$ is non-vanishing only for $s = 0$ and $j \geq i$ and that in this case one has $\mathbb{R}^s\pi_*(\tilde{\mathcal{L}}_j \otimes \tilde{\mathcal{L}}_i^\vee) \simeq \pi_*(\tilde{\mathcal{L}}_j \otimes \tilde{\mathcal{L}}_i^\vee)$. Thus for $j < i$ we have $\mathbb{R}^s\pi_*(\tilde{\mathcal{L}}_j \otimes \tilde{\mathcal{L}}_i^\vee) = 0$ and therefore

$$H^r(Z, \mathcal{T} \otimes \mathcal{T}^\vee \otimes \mathcal{M}^{\otimes(j-i)} \otimes \mathbb{R}^s\pi_*(\tilde{\mathcal{L}}_j \otimes \tilde{\mathcal{L}}_i^\vee)) = 0.$$

Hence we find

$$H^l(X, \pi^*(\mathcal{T} \otimes \mathcal{T}^\vee \otimes \mathcal{M}^{\otimes(j-i)}) \otimes \tilde{\mathcal{L}}_j \otimes \tilde{\mathcal{L}}_i^\vee) = 0,$$

for $l > 0$ by above spectral sequence. It remains the case $j \geq i$. For $j = i$ we have $\mathbb{R}^s\pi_*(\tilde{\mathcal{L}}_i \otimes \tilde{\mathcal{L}}_i^\vee) \simeq \pi_*(\tilde{\mathcal{L}}_i \otimes \tilde{\mathcal{L}}_i^\vee) \simeq \mathcal{O}_Z$ by assumption. From this we get

$$\begin{aligned} H^r(Z, \mathcal{T} \otimes \mathcal{T}^\vee \otimes \mathcal{M}^{\otimes(i-i)} \otimes \mathbb{R}^s\pi_*(\tilde{\mathcal{L}}_i \otimes \tilde{\mathcal{L}}_i^\vee)) &\simeq H^r(Z, \mathcal{T} \otimes \mathcal{T}^\vee \otimes \mathcal{O}_Z) \\ &\simeq \mathrm{Ext}^r(\mathcal{T}, \mathcal{T}) = 0, \end{aligned}$$

for $r > 0$, since $\mathcal{T}$ is by assumption a tilting bundle for $D^b(X)$. Again by the above spectral sequence we conclude

$$H^l(X, \pi^*(\mathcal{T} \otimes \mathcal{T}^\vee \otimes \mathcal{M}^{\otimes(i-i)}) \otimes \tilde{\mathcal{L}}_i \otimes \tilde{\mathcal{L}}_i^\vee) = 0,$$

for $l > 0$. Finally we have to consider the case $j > i$. For this, we again consider the above spectral sequence and see that it becomes

$$H^r(Z, \mathcal{T} \otimes \mathcal{T}^\vee \otimes \mathcal{M}^{\otimes(j-i)} \otimes \pi_*(\tilde{\mathcal{L}}_j \otimes \tilde{\mathcal{L}}_i^\vee)) \Longrightarrow H^r(X, \pi^*(\mathcal{T} \otimes \mathcal{T}^\vee \otimes \mathcal{M}^{\otimes(j-i)}) \otimes \tilde{\mathcal{L}}_j \otimes \tilde{\mathcal{L}}_i^\vee).$$

Now, since there are only finitely many $\tilde{\mathcal{L}}_i$ and Z is projective, we can choose an ample invertible sheaf $\mathcal{N}$ on Z and an integer $m >> 0$, such that for $\mathcal{M} = \mathcal{N}^{\otimes m}$ we get

$$H^r(Z, \mathcal{T} \otimes \mathcal{T}^\vee \otimes \mathcal{M}^{\otimes(j-i)} \otimes \pi_*(\tilde{\mathcal{L}}_j \otimes \tilde{\mathcal{L}}_i^\vee)) = 0 \text{ for } r > 0.$$

This finally yields

$$H^l(X, \pi^*(\mathcal{T} \otimes \mathcal{T}^\vee \otimes \mathcal{M}^{\otimes(j-i)}) \otimes \tilde{\mathcal{L}}_j \otimes \tilde{\mathcal{L}}_i^\vee) = 0, \text{ for } l > 0$$

and therefore

$$\mathrm{Ext}^l(\pi^*(\mathcal{T} \otimes \mathcal{M}^{\otimes i}) \otimes \tilde{\mathcal{L}}_i, \pi^*(\mathcal{T} \otimes \mathcal{M}^{\otimes j}) \otimes \tilde{\mathcal{L}}_j) = 0, \text{ for } l > 0.$$

For the generating property we notice that $D^b(Z)$ is generated by $\mathcal{T}$, since $\mathcal{T}$ is assumed to be a tilting bundle for $D^b(X)$. By Proposition 3.26, for all $1 \leq i \leq n$, the object $\mathcal{T} \otimes \mathcal{M}^{\otimes i}$ is also a tilting object for $D^b(Z)$ and thus generates $D^b(Z)$. Lemma 3.58 together with Remark 3.59 then guarantees that $\mathcal{R} = \bigoplus_{i=1}^n \pi^*(\mathcal{T} \otimes \mathcal{M}^{\otimes i}) \otimes \tilde{\mathcal{L}}_i$ generates $D^b(X)$ and therefore, $\mathcal{R}$ is a tilting object for $D^b(X)$. Note that the global dimension of $\mathrm{End}(\mathcal{R})$ is finite, since X is supposed to be smooth over k. □

As in the case of projective bundles, if one considers fibrations over smooth projective surfaces S that do not admit a full strongly exceptional collection the result of Costa, Di Rocco and Miró–Roig does not apply. But in this situation Theorem 3.60 applies and hence we conclude that fibrations over S with fibers admitting full strongly exceptional collections of invertible sheaves have tilting objects. Below we will see a more important application of Theorem 3.60 as it provides us with tilting objects on some homogeneous varieties.

To generalize Kapranov's result [95], stating the existence of a tilting bundle for Grassmannians over fields of characteristic zero, we want to investigate the relative version of the Grassmannian, the Grassmann bundle. For this, we take a smooth projective and integral k-scheme X and a locally free sheaf $\mathcal{E}$ of rank $r+1$. We denote by $\mathrm{Grass}_X(l,\mathcal{E})$ the relative Grassmannian and by $\pi : \mathrm{Grass}_X(l,\mathcal{E}) \to X$ the projective structure morphism. Furthermore, one has the tautological subbundle $\mathcal{R}$ of rank l in $\pi^*\mathcal{E}$ and the tautological short exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \pi^*\mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0. \tag{3.3}$$

Remark 3.61. As in the case of projective bundles, for an invertible sheaf $\mathcal{L}$ on X one has $\mathrm{Grass}_X(l,\mathcal{E}) \simeq \mathrm{Grass}_X(l,\mathcal{E}\otimes\mathcal{L})$. This can be seen as follows: One can define the Grassmann bundle over X as the X-scheme that represents the functor $\mathrm{Grass}(l,\mathcal{E})(-) : (\mathrm{Sch}/X) \to \mathrm{Set}$, from the category of X-schemes to the category of sets (see [77]). For a X-scheme T with structure morphism $f : T \to X$, $\mathrm{Grass}(l,\mathcal{E})(T)$ is defined to be the set of isomorphism classes of pairs $(\mathcal{F}, g)$, where $\mathcal{F}$ is locally free of rank l on T and $g : f^*(\mathcal{E}) \to \mathcal{F}$ is an epimorphism of quasicoherent modules on T. Now if $\mathcal{L}$ is an invertible sheaf on X, we get an induced epimorphism $g' : f^*(\mathcal{E}\otimes\mathcal{L}) \to \mathcal{F}\otimes f^*\mathcal{L}$. This yields a map $\mathrm{Grass}(l,\mathcal{E})(T) \to \mathrm{Grass}(l,\mathcal{E}\otimes\mathcal{L})(T)$, assigning to the epimorphism g the epimorphism g'. This map is clearly invertible, where the inverse map is obtained by tensoring an element of $\mathrm{Grass}(l,\mathcal{E}\otimes\mathcal{L})(T)$ with $f^*(\mathcal{L}^\vee)$. Now the Yoneda Lemma yields the desired isomorphism of X-schemes $\mathrm{Grass}_X(l,\mathcal{E}) \simeq \mathrm{Grass}_X(l,\mathcal{E}\otimes\mathcal{L})$.

To prove that the Grassmann bundle admits a tilting bundle, provided the base scheme X admits one, we first have to state two lemmas. Denote by $P(l, r+1-l)$ the set of partitions $\lambda = (\lambda_1, ..., \lambda_l)$ with $0 \le \lambda_l \le ... \le \lambda_1 \le r+1-l$. For $\lambda \in P(l, r+1-l)$ we have the Schur functor Σ^λ and locally free sheaves $\Sigma^\lambda(\mathcal{R})$. Furthermore, one can choose a total order $\prec$ on the set $P(l, r+1-l)$ such that for two partitions λ and μ, $\lambda \prec \mu$ means that the Young diagram of λ is not contained in that of μ, i.e., $\exists i : \mu_i < \lambda_i$. Let P' be the above set of partitions equipped with this order. Suppose that X is a smooth projective and integral k-scheme, where k is an algebraically closed field of characteristic zero. Orlov [129] proved that one has a semiorthogonal decomposition

$$D^b(\mathrm{Grass}_X(l,\mathcal{E})) = \langle ..., D^b(X)\otimes\Sigma^\lambda(\mathcal{R}), ..., D^b(X)\otimes\Sigma^\mu(\mathcal{R}), ...\rangle, \tag{3.4}$$

with $\lambda \prec \mu$ as explained above. Here $D^b(X)\otimes\Sigma^\lambda(\mathcal{R})$ is the full triangulated subcategory of $D^b(\mathrm{Grass}_X(l,\mathcal{E}))$ consisting of elements of the form $\pi^*\mathcal{M}\otimes\Sigma^\lambda(\mathcal{R})$, where $\mathcal{M} \in D^b(X)$. Furthermore, for all partitions λ one has an equivalence between $D^b(X)$ and $D^b(X)\otimes\Sigma^\lambda(\mathcal{R})$, given by the functor $\pi^*(-)\otimes\Sigma^\lambda(\mathcal{R})$ (see [129], §3).

Lemma 3.62. *Let k be an algebraically closed field of characteristic zero, X a smooth projective and integral k-scheme, P' the above ordered set of partitions and $\mathcal{E}$ a locally free sheaf of rank $r+1$ on X. Suppose the object $\mathcal{A} \in D^b(X)$ generates the category $D^b(X)$. Then the object $\mathcal{N} = \bigoplus_{\lambda\in P'} \pi^*\mathcal{A} \otimes \Sigma^{\lambda}(\mathcal{R})$ generates $D^b(\mathrm{Grass}_X(l,\mathcal{E}))$.*

Proof. In view of the equivalence $\pi^*(-) \otimes \Sigma^{\lambda}(\mathcal{R}) : D^b(X) \xrightarrow{\sim} D^b(X) \otimes \Sigma^{\lambda}(\mathcal{R})$ and with the assumption that $D^b(X)$ is generated by the object $\mathcal{A}$, we conclude that $D^b(X) \otimes \Sigma^{\lambda}(\mathcal{R})$ is generated by the object $\pi^*\mathcal{A} \otimes \Sigma^{\lambda}(\mathcal{R})$. From the semiorthogonal decomposition (3.4) of $D^b(\mathrm{Grass}_X(l,\mathcal{E}))$ we finally get that the object $\mathcal{N} = \bigoplus_{\lambda\in P'} \pi^*\mathcal{A} \otimes \Sigma^{\lambda}(\mathcal{R})$ generates $D^b(\mathrm{Grass}_X(l,\mathcal{E}))$. □

We now want to consider the higher direct images of $\Sigma^{\lambda}(\mathcal{R}^{\vee})$ under the Grassmann bundle $\pi : \mathrm{Grass}(l,\mathcal{E}) \to X$. Recall that we are considering partitions λ with at most l rows and at most $r+1-l$ columns. One can extend the Schur functors and define them for a non-increasing sequence of integers $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_l$ by $\Sigma^{\lambda}(\mathcal{F}) := \Sigma^{\lambda+m}(\mathcal{F}) \otimes \det(\mathcal{F})^{-m}$, where $\mathcal{F}$ is locally free of finite rank, $m \in \mathbb{N}$, $\lambda = (\lambda_1,...,\lambda_l)$ and $\lambda + m = (\lambda_1 + m,...,\lambda_l + m)$. Note that $\Sigma^{\lambda}(\mathcal{F})^{\vee} \simeq \Sigma^{\lambda}(\mathcal{F}^{\vee}) \simeq \Sigma^{-\lambda}(\mathcal{F})$, where $-\lambda = (-\lambda_l, -\lambda_{l-1},...,-\lambda_1)$. In the following we consider this extended Schur functors. The next lemma is well-known (see for instance [60], Lemma 4.1).

Lemma 3.63. *Let k, X and $\mathcal{E}$ be as above. Then for every partition λ, the higher direct images of $\Sigma^{\lambda}(\mathcal{R}^{\vee})$ under the Grassmann bundle $\pi : \mathrm{Grass}(l,\mathcal{E}) \to X$ satisfy*

$$\mathbb{R}^s\pi_*(\Sigma^{\lambda}(\mathcal{R}^{\vee})) = \begin{cases} \Sigma^{\lambda}(\mathcal{E}^{\vee}) & \textit{if } s = 0 \textit{ and if } \lambda_1 \geq \lambda_2 ... \geq \lambda_l \geq 0 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. In the case the base scheme X is a point, this was done by Kapranov [95], Lemma 2.2 (a) (see also [97], Lemma 3.2 a)). Kapranov proved the following: Let $Z = \mathrm{Grass}(l,n)$ be the Grassmannian for some n-dimensional vector space V over a field of characteristic zero and $\Sigma^{\lambda}(\mathcal{S}^{\vee})$ the locally free sheaf obtained by applying the Schur functor to the dual of the tautological sheaf $\mathcal{S}$ of $\mathrm{Grass}(l,n)$, where λ is a non-increasing collection of integers $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_l \geq -(n-l)$. Then one has

$$H^s(Z, \Sigma^{\lambda}(\mathcal{S}^{\vee})) = \begin{cases} \Sigma^{\lambda}(V^{\vee}) & \text{if s = 0 and if } \lambda_1 \geq \lambda_2 ... \geq \lambda_l \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Now for $x \in X$ we have $\pi^{-1}(x) \simeq \mathrm{Grass}(l, E_x)$, where E_x is the fiber of $\mathcal{E}$ over x. The fiber of $\mathbb{R}^s\pi_*(\Sigma^{\lambda}(\mathcal{R}^{\vee}))$ over x is $H^s(\mathrm{Grass}(l,E_x), \Sigma^{\lambda}(\mathcal{R}^{\vee})_{|\mathrm{Grass}(l,E_x)})$. By the definition of the tautological bundle $\mathcal{R}$, the restriction of $\mathcal{R}$ to $\mathrm{Grass}(l,E_x)$ is exactly the tautological rank l bundle on $\mathrm{Grass}(l,E_x)$. We denote it by S_x. Hence the restriction of $\Sigma^{\lambda}(\mathcal{R}^{\vee})$ to the fiber over x is $\Sigma^{\lambda}(S_x^{\vee})$. The above result then follows from the result of Kapranov and by varying the point $x \in X$. This completes the proof. □

With this two lemmas we obtain the following result:

Theorem 3.64. *Let k, X and $\mathcal{E}$ be as above and suppose X has a tilting bundle $\mathcal{T}_X$. Then the Grassmann bundle $\mathrm{Grass}_X(l,\mathcal{E})$ admits a tilting bundle too.*

Proof. To prove that there exists a tilting bundle for $D^b(\mathrm{Grass}_X(l,\mathcal{E}))$ we consider the object $\mathcal{T} = \bigoplus_{\lambda\in P'} \pi^*\mathcal{T}_X \otimes \Sigma^\lambda(\mathcal{R})$. We follow the idea of the proof of Theorem 3.50 and investigate when

$$\mathrm{Ext}^l(\bigoplus_\lambda \pi^*\mathcal{T}_X \otimes \Sigma^\lambda(\mathcal{R}), \bigoplus_\lambda \pi^*\mathcal{T}_X \otimes \Sigma^\lambda(\mathcal{R}))$$

vanishes for $l > 0$. Since $\mathcal{T} = \bigoplus_\lambda \pi^*\mathcal{T}_X \otimes \Sigma^\lambda(\mathcal{R})$, it is enough to calculate

$$\mathrm{Ext}^l(\pi^*\mathcal{T}_X \otimes \Sigma^\lambda(\mathcal{R}), \pi^*\mathcal{T}_X \otimes \Sigma^\mu(\mathcal{R})),$$

for two partitions $\lambda, \mu \in P'$. Adjunction of π^* and π_* and the projection formula yields

$$\mathrm{Hom}(\pi^*\mathcal{T}_X \otimes \Sigma^\lambda(\mathcal{R}), \pi^*\mathcal{T}_X \otimes \Sigma^\mu(\mathcal{R})[l]) \simeq$$
$$\mathrm{Hom}(\mathcal{T}_X, \mathcal{T}_X \otimes \mathbb{R}\pi_*(\Sigma^\lambda(\mathcal{R})^\vee \otimes \Sigma^\mu(\mathcal{R}))[l]).$$

Hence we have to calculate $\mathbb{R}\pi_*(\Sigma^\lambda(\mathcal{R})^\vee \otimes \Sigma^\mu(\mathcal{R})) \simeq \mathbb{R}\pi_*\mathcal{H}om(\Sigma^\lambda(\mathcal{R}), \Sigma^\mu(\mathcal{R}))$. From the Littlewood–Richardson rule (see [2], Theorem IV.2.1), it follows that we can decompose $\mathcal{H}om(\Sigma^\lambda(-), \Sigma^\mu(-))$ into a direct sum of irreducible summands $\Sigma^\gamma(-)$. Since λ and μ are partitions with at most l rows and at most $r+1-l$ columns, it follows that γ is a non-increasing sequence of integers $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_l \geq -(r+1-l)$ (see [97] 3.3). Now with Lemma 3.63 for each irreducible summand $\Sigma^\gamma(\mathcal{R}) \simeq \Sigma^{-\gamma}(\mathcal{R}^\vee)$ of $\mathcal{H}om(\Sigma^\lambda(\mathcal{R}), \Sigma^\mu(\mathcal{R}))$ we have

$$\mathbb{R}\pi_*(\Sigma^\gamma(\mathcal{R})) \simeq \mathbb{R}\pi_*(\Sigma^{-\gamma}(\mathcal{R}^\vee)) \simeq \Sigma^{-\gamma}(\mathcal{E}^\vee)$$

for $-\gamma \geq 0$, i.e., $-\gamma_l \geq -\gamma_{l-1} \geq \ldots \geq -\gamma_1 \geq 0$, otherwise $\mathbb{R}\pi_*(\Sigma^\gamma(\mathcal{R})) = 0$. Since we have only finitely many partitions in P' and finitely many irreducible summands $\Sigma^\gamma(\mathcal{R})$ of $\mathcal{H}om(\Sigma^\lambda(\mathcal{R}), \Sigma^\mu(\mathcal{R}))$, we hold on to the fact that is is enough to prove the vanishing of

$$\mathrm{Ext}^l(\mathcal{T}_X, \mathcal{T}_X \otimes \Sigma^{\gamma'}(\mathcal{E}^\vee)),$$

for $l > 0$ and $\gamma' \geq 0$. For the case $\gamma' = 0$ one has $\Sigma^\gamma(\mathcal{R}^\vee) = \mathcal{O}_\mathcal{E}$ and hence $\mathbb{R}\pi_*\mathcal{O}_\mathcal{E} \simeq \mathcal{O}_X$. Hence the Ext-vanishing in the case $\gamma' = 0$ follows from the fact that $\mathcal{T}_X$ is by assumption a tilting bundle on X. So we can restrict ourselves to the case $\gamma' > 0$. Note that for an arbitrary locally free sheaf $\mathcal{F}$ and an arbitrary invertible sheaf $\mathcal{L}$, it is a fact that applying the Schur functor to $\mathcal{F} \otimes \mathcal{L}$ yields $\Sigma^\gamma(\mathcal{F} \otimes \mathcal{L}) \simeq \Sigma^\gamma(\mathcal{F}) \otimes \mathcal{L}^{(\Sigma\gamma_i)}$, provided $\gamma \geq 0$. Since there are only finitely many summands $\Sigma^\gamma(\mathcal{R})$ of $\mathcal{H}om(\Sigma^\lambda(\mathcal{R}), \Sigma^\mu(\mathcal{R}))$ and X is projective, we can choose for a fixed $\gamma > 0$ an ample invertible sheaf $\mathcal{L}$ and an integer $n_\gamma >> 0$ such that

$$\mathrm{Ext}^l(\mathcal{T}_X, \mathcal{T}_X \otimes \Sigma^\gamma((\mathcal{E} \otimes \mathcal{L}^{\otimes(-n_\gamma)})^\vee)) \simeq \mathrm{Ext}^l(\mathcal{T}_X, \mathcal{T}_X \otimes \Sigma^\gamma(\mathcal{E}^\vee) \otimes \mathcal{L}^{\otimes(n_\gamma \cdot \Sigma\gamma_i)}) = 0,$$

for $l > 0$. As mentioned above, there are only finitely many irreducible summands $\Sigma^\gamma(-)$ of $\mathcal{H}om(\Sigma^\lambda(-), \Sigma^\mu(-))$, so that we can consider an integer $n > \max\{n_\gamma | \mathrm{Ext}^l(\mathcal{T}_X, \mathcal{T}_X \otimes \Sigma^\gamma(\mathcal{E}^\vee) \otimes \mathcal{L}^{\otimes(n_\gamma \cdot \Sigma\gamma_i)}) = 0, \text{ for } l > 0\}$. For this $n >> 0$ we consider the invertible $\mathcal{L}^{\otimes(-n)}$ and the Grassmann bundle $\mathrm{Grass}(l, \mathcal{E} \otimes \mathcal{L}^{\otimes(-n)})$, with tautological bundle $\mathcal{R}'$. On this Grassmann bundle we have for all $\gamma > 0$ with $\mathbb{R}\pi_*(\Sigma^\gamma(\mathcal{R}')) = \Sigma^\gamma((\mathcal{E} \otimes \mathcal{L}^{\otimes(-n)})^\vee)$:

$$\mathrm{Ext}^l(\mathcal{T}_X, \mathcal{T}_X \otimes \mathbb{R}\pi_*(\Sigma^\gamma(\mathcal{R}))) \simeq \mathrm{Ext}^l(\mathcal{T}_X, \mathcal{T}_X \otimes \Sigma^\gamma(\mathcal{E}^\vee) \otimes \mathcal{L}^{(n \cdot \Sigma\gamma_i)}) = 0,$$

for $l > 0$. This yields Ext-vanishing for $\mathcal{T}' = \bigoplus_{\lambda \in P'} \pi^* \mathcal{T}_X \otimes \Sigma^\lambda(\mathcal{R}')$ on $\mathrm{Grass}(l, \mathcal{E} \otimes \mathcal{L}^{\otimes(-n)})$. Since $\mathcal{T}_X$ is a tilting bundle for $D^b(X)$, it generates $D^b(X)$ and according to Lemma 3.62, the object $\mathcal{T}'$ generates $D^b(\mathrm{Grass}(l, \mathcal{E} \otimes \mathcal{L}^{\otimes(-n)}))$. This gives us a tilting object $\mathcal{T}'$ for $\mathrm{Grass}(l, \mathcal{E} \otimes \mathcal{L}^{-n})$. By Remark 3.61 we have an isomorphism $\mathrm{Grass}(l, \mathcal{E} \otimes \mathcal{L}^{\otimes(-n)}) \simeq \mathrm{Grass}(l, \mathcal{E})$ and hence we get a tilting object $\tilde{\mathcal{T}}$ for $D^b(\mathrm{Grass}(l, \mathcal{E}))$, as explained in Remark 3.51. Finally, since $\mathrm{Grass}(l, \mathcal{E})$ is by assumption smooth over k, Theorem 3.8 implies that $\mathrm{End}(\tilde{\mathcal{T}})$ has finite global dimension. This completes the proof. □

More generally, consider a locally free sheaf $\mathcal{E}$ of rank $r+1$ on some smooth projective and integral k-scheme. For $1 \leq l_1 < ... < l_t \leq r+1$ consider the relative flag variety $\mathrm{Flag}_X(l_1, ..., l_t, \mathcal{E})$ of type $(l_1, ..., l_t)$ in the fibers of $\mathcal{E}$, with projection $\pi : \mathrm{Flag}_X(l_1, ..., l_t, \mathcal{E}) \to X$. One has the tautological subbundles $\mathcal{R}_1 \subset \mathcal{R}_2 \subset ... \subset \mathcal{R}_t \subset \pi^*\mathcal{E}$ and by construction $\mathrm{Flag}_X(l_1, ..., l_t, \mathcal{E})$ is obtained as the successive iteration of Grassmann bundles

$$\mathrm{Flag}_X(l_1, ..., l_t, \mathcal{E}) = \mathrm{Grass}_{\mathrm{Flag}_X(l_2,...,l_t,\mathcal{E})}(l_1, \mathcal{R}_2) \longrightarrow \mathrm{Flag}_X(l_2, ..., l_t, \mathcal{E}) = \mathrm{Grass}_{\mathrm{Flag}_X(l_3,...,l_t,\mathcal{E})}(l_2, \mathcal{R}_3) \longrightarrow ... \longrightarrow X.$$

With these facts we now obtain the following consequence of Theorem 3.64.

Theorem 3.65. *Let k be an algebraically closed field of characteristic zero, X a smooth projective k-scheme and $\mathcal{E}$ a locally free sheaf of rank $r+1$ on X. Suppose $\mathcal{T}$ is a tilting bundle for X. Then the relative flag $\mathrm{Flag}_X(l_1, ..., l_t, \mathcal{E})$ admits a tilting bundle too.*

Proof. Apply the last Theorem iteratively to the following succession of relative Grassmannians

$$\mathrm{Flag}_X(l_1, ..., l_t, \mathcal{E}) = \mathrm{Grass}_{\mathrm{Flag}_X(l_2,...,l_t,\mathcal{E})}(l_1, \mathcal{R}_2) \longrightarrow \mathrm{Flag}_X(l_2, ..., l_t, \mathcal{E}) = \mathrm{Grass}_{\mathrm{Flag}_X(l_3,...,l_t,\mathcal{E})}(l_2, \mathcal{R}_3) \longrightarrow ... \longrightarrow X.$$

This finally gives a tilting bundle for $\mathrm{Flag}_X(l_1, ..., l_t, \mathcal{E})$. □

Example 3.66. Let S be a rational surface over $k = \mathbb{C}$ and $\mathcal{E}$ a locally free sheaf of finite rank on S. As mentioned earlier, Hille and Perling proved that S always admits a tilting bundle $\mathcal{T}_S$. Now Theorem 3.65 implies that $\mathrm{Flag}_S(l_1, ..., l_t, \mathcal{E})$ admits a tilting bundle too.

At this point we want to mention some problems that occur considering Grassmann bundles over a scheme X defined over a field k of characteristic $p > 0$. The first step in the proof of Theorem 3.64 was to establish the vanishing of Ext. For this, we needed a result of Kapranov on the cohomology of the Schur modules that was proved using the Borel–Weil–Bott Theorem. But in arbitrary characteristic one has only the Kempf vanishing Theorem (see [93], p.227 ff.). Furthermore we make use of Littlewood–Richardson rule, that also holds only in characteristic zero. Following the ideas in avoiding these problems for the case where X is a point, where Buchweitz, Leuschke and Van den Bergh [53] constructed a characteristic-free tilting bundle, one can try to apply their ideas to get some characteristic-free tilting object in the relative version. This is planed to be done by the author in some following work.

At the end of this section, we want to give a relative version of Kapranov's tilting bundle on quadrics [96]. Böhning [35] constructed a semiorthogonal decomposition for quadric bundles under certain assumptions. In some following work Kuznetsov [109] considered quadric fibrations and intersections of quadrics and proved the existence of semiorthogonal decompositions. We want to follow the ideas of [35] and [109] to construct a tilting object for quadric bundles (quadric fibrations). For this, we take a smooth projective and integral scheme over $\mathbb{C}$. Let $\mathcal{E}$ be a locally free sheaf of rank $r+1$ and q a symmetric quadratic form $q \in \Gamma(X, \mathrm{Sym}^2(\mathcal{E}^\vee))$ which is non-degenerate on each fiber. We denote by $\mathcal{Q} = \{q = 0\} \subset \mathbb{P}(\mathcal{E})$ the quadric bundle and by π the projection $\pi : \mathcal{Q} \to X$. Under some technical assumptions stated below, Böhning [35] established two ordered sets of locally free sheaves on $\mathcal{Q}$

$$\mathcal{V} = \{\Sigma(-r+1) \prec \mathcal{O}_{\mathcal{Q}}(-r+2) \prec \ldots \prec \mathcal{O}_{\mathcal{Q}}(-1) \prec \mathcal{O}_{\mathcal{Q}}\} \tag{3.5}$$
$$\mathcal{V}' = \{\Sigma^+(-r+1) \prec \Sigma^-(-r+1) \prec \ldots \prec \mathcal{O}_{\mathcal{Q}}(-1) \prec \mathcal{O}_{\mathcal{Q}}\}. \tag{3.6}$$

We refer to [35] for all the details on the twisted spinor bundles $\Sigma(-r+1)$, $\Sigma^+(-r+1)$ and $\Sigma^-(-r+1)$ in this relative setting and when they exist. Now Böhning proved the following [35], Theorem 3.2.7.

Theorem 3.67. *Let X be as above, $\mathcal{E}$ an orthogonal locally free sheaf of rank $r+1$ on X and $\mathcal{Q}$ the quadric bundle. Suppose $H^1(X, \mathbb{Z}/2\mathbb{Z}) = 0$ and that $\mathcal{E}$ carries a spin structure. Then there is a semiorthogonal decomposition*

$$D^b(\mathcal{Q}) = \langle D^b(X) \otimes \Sigma(-r+1), D^b(X) \otimes \mathcal{O}_{\mathcal{Q}}(-r+2), \\ \ldots, D^b(X) \otimes \mathcal{O}_{\mathcal{Q}}(-1), D^b(X)\rangle$$

for $r+1$ odd and

$$D^b(\mathcal{Q}) = \langle D^b(X) \otimes \Sigma^+(-r+1), D^b(X) \otimes \Sigma^-(-r+1), \\ \ldots, D^b(X) \otimes \mathcal{O}_{\mathcal{Q}}(-1), D^b(X)\rangle$$

for $r+1$ even.

With this result we try to find some tilting object for $D^b(\mathcal{Q})$. The proof of Theorem 3.67 needs the following for our purposes also very useful result (see [35], Lemma 3.2.5).

Lemma 3.68. *Consider the two ordered sets (3.5) and (3.6) from above. If $\mathcal{W}, \mathcal{V}_1, \mathcal{V}_2 \in \mathcal{V}$ (resp. $\in \mathcal{V}'$) with $\mathcal{V}_1 \prec \mathcal{V}_2$, $\mathcal{V}_1 \neq \mathcal{V}_2$, then one has*

- **(i)** $\mathbb{R}^i\pi_*(\mathcal{W} \otimes \mathcal{W}^\vee) = 0$, $\forall i \neq 0$
- **(ii)** $\mathbb{R}^i\pi_*(\mathcal{V}_1 \otimes \mathcal{V}_2^\vee) = 0$, $\forall i \in \mathbb{Z}$
- **(iii)** $\mathbb{R}^i\pi_*(\mathcal{V}_2 \otimes \mathcal{V}_1^\vee) = 0$, $\forall i \neq 0$

and the canonical morphism $\pi_*(\mathcal{W} \otimes \mathcal{W}^\vee) \to \mathcal{O}_X$ is an isomorphism.

Lemma 3.69. *Let X and $\mathcal{E}$ be as above with all the assumptions on X and $\mathcal{E}$ of Theorem 3.67 being fulfilled. Suppose the object $\mathcal{A}$ generates the category $D^b(X)$. Then the object*

$$\mathcal{N} = \bigoplus_{i=0}^{r} \pi^*\mathcal{A} \otimes \mathcal{V}_i,$$

where $\mathcal{V}_i$ are the elements of the set (3.5), generates $D^b(\mathcal{Q})$ for $r+1$ odd and the object

$$\mathcal{N}' = \bigoplus_{i=0}^{r+1} \pi^*\mathcal{A} \otimes \mathcal{V}_i',$$

where $\mathcal{V}_i'$ are the elements of the set (3.6), generates $D^b(\mathcal{Q})$ for $r+1$ even.

Proof. We prove only the odd case, since the even case is analogous. Note that Lemma 3.68 (i), together with the isomorphsim $\pi_*(\mathcal{W}\otimes\mathcal{W}^\vee) \to \mathcal{O}_X$, adjunction of π^* and π_* and projection formula yields that the functor $\pi^*(-)\otimes\mathcal{W} : D^b(X) \to D^b(X)\otimes\mathcal{W}$ is an equivalence for any $\mathcal{W}$ of the set (3.5). Now, since the object $\mathcal{A}$ is supposed to generate $D^b(X)$, we conclude that $\pi^*\mathcal{A}\otimes\mathcal{W}$ generates $D^b(X)\otimes\mathcal{W}$. The semiorthogonal decomposition given in Theorem 3.67 now yields that $\mathcal{N} = \bigoplus_{i=0}^{r} \pi^*\mathcal{A}\otimes\mathcal{V}_i$, with $\mathcal{V}_i$ being the elements of the set (3.5), generates $D^b(\mathcal{Q})$ for $r+1$ odd. □

With Lemma 3.68 and 3.69, we now obtain the following:

Theorem 3.70. *Let X be as above, $\mathcal{E}$ an orthogonal locally free sheaf of rank $r+1$ on X and $\mathcal{Q}$ a smooth quadric bundle. Suppose $H^1(X,\mathbb{Z}/2\mathbb{Z}) = 0$ and that $\mathcal{E}$ carries a spin structure. Suppose furthermore that $\mathcal{T}_X$ is a tilting bundle for $D^b(X)$ and that $\mathrm{Ext}^l(\mathcal{T}_X, \mathcal{T}_X \otimes \pi_*(\mathcal{V}_j \otimes \mathcal{V}_i^\vee)) = 0$ for $l \neq 0$ and $\mathcal{V}_i \prec \mathcal{V}_j$, $\mathcal{V}_i \neq \mathcal{V}_j$, where $\mathcal{V}_i, \mathcal{V}_j \in \mathcal{V}$ (resp. $\in \mathcal{V}'$). Then $\mathcal{Q} \subset \mathbb{P}(\mathcal{E})$ admits a tilting bundle.*

Proof. We claim that $\mathcal{T} = \bigoplus_i \pi^*\mathcal{T}_X \otimes \mathcal{V}_i$, with $\mathcal{V}_i$ being elements of the set (3.5), is a tilting bundle in the odd case and $\mathcal{T} = \bigoplus_i \pi^*\mathcal{T}_X \otimes \mathcal{V}_i'$, with $\mathcal{V}_i'$ being elements of the set (3.6), in the even case. We give the proof only for the odd case and note that the proof for the even case is analogous. We start with the vanishing of Ext and consider $\mathrm{Ext}^l(\pi^*\mathcal{T}_X \otimes \mathcal{V}_i, \pi^*\mathcal{T}_X \otimes \mathcal{V}_j)$. By adjunction of π^* and π_* and the projection formula we obtain

$$\mathrm{Hom}(\pi^*\mathcal{T}_X \otimes \mathcal{V}_i, \pi^*\mathcal{T}_X \otimes \mathcal{V}_j[l]) \simeq \mathrm{Hom}(\mathcal{T}_X, \mathcal{T}_X \otimes \mathbb{R}\pi_*(\mathcal{V}_i^\vee \otimes \mathcal{V}_j)[l]).$$

Lemma 3.68 together with the assumption yields

$$\mathrm{Ext}^l(\pi^*\mathcal{T}_X \otimes \mathcal{V}_i, \pi^*\mathcal{T}_X \otimes \mathcal{V}_j) = 0, \text{ for } l > 0.$$

Note that for $i = j$ we have with Lemma 3.68

$$\mathrm{Hom}(\mathcal{T}_X, \mathcal{T}_X \otimes \mathbb{R}\pi_*(\mathcal{V}_i^\vee \otimes \mathcal{V}_i)[l]) \simeq \mathrm{Ext}^l(\mathcal{T}_X, \mathcal{T}_X) = 0 \text{ for } i \neq 0,$$

since $\mathcal{T}_X$ is a tilting bundle by assumption. The generating property of $\mathcal{T} = \bigoplus_i \pi^*\mathcal{T}_X\otimes\mathcal{V}_i$ is now guaranteed by Lemma 3.69, since $\mathcal{T}_X$ is supposed to generate $D^b(X)$. Since $\mathcal{Q}$ is smooth over $\mathbb{C}$, Theorem 3.8 implies that $\mathrm{End}(\mathcal{T})$ has finite global dimension. This completes the proof. □

In what follows we want to discuss when the assumption $\mathrm{Ext}^l(\mathcal{T}_X, \mathcal{T}_X \otimes \pi_*(\mathcal{V}_j \otimes \mathcal{V}_i^\vee)) = 0$ for $l \neq 0$ holds. We also recall the result given by Kuznetsov [109]. For this let X be as above and $\mathcal{E}$ a locally free sheaf of rank r. Let $\mathcal{L}$ be an invertible sheaf and suppose $q : \mathcal{L} \to S^2(\mathcal{E}^\vee)$ is an embedding of locally free sheaves, i.e., $q \in \Gamma(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathcal{E}}(2) \otimes \mathcal{L}^\vee)$. Set $\mathcal{Q} \subset \mathbb{P}(\mathcal{E})$ to be the zero locus of q on $\mathbb{P}(\mathcal{E})$ and denote by $\pi : \mathcal{Q} \to X$ the structure morphisms. This quadric fibration is line-bundle valued with line bundle being $\mathcal{L}$. Böhning [35] therefore considered the case $\mathcal{L} = \mathcal{O}_X$. Note that the relative dimension of $\mathcal{Q}$ is $r-2$. With this notation Kuznetsov [109], Theorem 4.2 showed that the flat quadric fibration $\pi : \mathcal{Q} \to X$ has a semiorthogonal decomposition

$$D^b(\mathcal{Q}) = \langle D^b(X, \mathcal{B}_0), \pi^* D^b(X) \otimes \mathcal{O}_{\mathcal{Q}}(1), ..., \pi^* D^b(X) \otimes \mathcal{O}_{\mathcal{Q}}(r-2) \rangle$$

where $D^b(X, \mathcal{B}_0)$ is the derived category of coherent sheaves of $\mathcal{B}_0$-modules on X. Here $\mathcal{B}_0$ is the sheaf of even parts of the Clifford algebra (see [109] for details). To prove the above semiorthogonal decomposition, Kuznetsov first proved a fact concerning the direct images of the invertible sheaves $\mathcal{O}_{\mathcal{Q}}(m)$ (see [109], Lemma 4.3). To prove this lemma, one considers the short exact sequence (see [109], p.14)

$$0 \longrightarrow \mathcal{O}_{\mathcal{E}}(m-2) \otimes \mathcal{L} \longrightarrow \mathcal{O}_{\mathcal{E}}(m) \longrightarrow i_* \mathcal{O}_{\mathcal{Q}}(m) \longrightarrow 0$$

Applying the functor $\mathbb{R}\pi_*$ to the exact sequence yields a distinguished triangle

$$S^{m-2}(\mathcal{E}^\vee) \otimes \mathcal{L} \longrightarrow S^m(\mathcal{E}^\vee) \longrightarrow \mathbb{R}\pi_* \mathcal{O}_{\mathcal{Q}}(m)$$

for $m > 2$. For $m = 0$ one has $\mathbb{R}\pi_* \mathcal{O}_{\mathcal{Q}} \simeq \mathcal{O}_X$ and for $m = 1$, $\mathbb{R}\pi_* \mathcal{O}_{\mathcal{Q}}(1) \simeq \mathcal{E}^\vee$. Finally, for $m = 2$ one obtains the distinguished triangle

$$\mathcal{L} \longrightarrow S^2(\mathcal{E}^\vee) \longrightarrow \mathbb{R}\pi_* \mathcal{O}_{\mathcal{Q}}(2).$$

Note that in view of Lemma 3.68 the assumption $\mathrm{Ext}^l(\mathcal{T}_X, \mathcal{T}_X \otimes \pi_*(\mathcal{V}_j \otimes \mathcal{V}_i^\vee)) = 0$ for $l \neq 0$ exactly treats the case $\pi_* \mathcal{O}_{\mathcal{Q}}(m)$ with $m \geq 0$. For $\mathcal{L} = \mathcal{O}_X$ one can consider the quadric bundle $\mathcal{Q}$ as a subscheme of $\mathbb{P}(\mathcal{E} \otimes \mathcal{M}) \simeq \mathbb{P}(\mathcal{E})$ for some suitable ample invertible sheaf $\mathcal{M}$ on X. The same arguments showing that the projective bundle has a tilting object yield the vanishing of $\mathrm{Ext}^l(\mathcal{T}_X, \mathcal{T}_X \otimes \pi_*(\mathcal{V}_j \otimes \mathcal{V}_i^\vee))$ for the quadric bundle $\mathcal{Q}$, considered as quadric in $\mathbb{P}(\mathcal{E} \otimes \mathcal{M})$ for some suitable ample invertible sheaf $\mathcal{M}$. Hence the assumption form above can be omitted and the quadric bundle of Theorem 3.70 admits a tiltting bundle.
Remind that the classical semisimple algebraic groups over an algebraically closed field of characteristic zero are given by $\mathrm{SL}_k(n)$, $\mathrm{SO}_k(n)$ and $\mathrm{Sp}_k(n)$. The Dynkin diagram A_n corresponds to $\mathrm{SL}_k(n+1)$, B_n to $\mathrm{SO}_k(2n+1)$, C_n to $\mathrm{Sp}_k(2n)$ and D_n to $\mathrm{SO}_k(2n)$. Finally, as a consequence of the results obtained in this section, especially of Theorem 3.50 and 3.60, we get the following result.

Theorem. *Let G be a semisimple algebraic group of classical type, B a Borel subgroup and G/B the flag variety of G. Then $D^b(G/B)$ admits a tilting object.*

Proof. The strategy of the proof is simply by considering every possible case. The homogeneous varieties of the groups of type A_n were treated by Kapranov [97], Theorem 3.10. These homogeneous varieties admit full strongly exceptional collections and hence tilting bundles. Note that this case also follows from Theorem 3.65 of the present work. We now consider the homogeneous varieties of the groups of type C_n. As pointed out by Samokhin [140] these correspond to the group $\mathrm{Sp}_k(2n)$ and are partial isotropic flags in a symplectic vector space. Corollary 3.55 now yields that these homogeneous varieties admit tilting objects. We proceed with the homogeneous varieties of the groups of type B_n and D_n. We restrict ourselves to the case of the orthogonal group of type B_n, the case of D_n being similar. The arguments of the proof of Theorem 4.1 in [140] show that the homogeneous varieties corresponding to B_n are obtained as a successive iteration of smooth quadric fibrations over a smooth quadric $Q_{2n-1} \subset \mathbb{P}^{2n}$. By a result of Kapranov [96], a smooth quadric admits a full strongly exceptional collection of locally free sheaves and hence a tilting bundle. Furthermore, since all the fibers of a smooth quadric fibration $\pi : \mathcal{Q} \to X$ are quadrics, the fibers also admit a full strongly exceptional collection. As pointed out by Samokhin [140], p.9, in this particular case the Spinor bundles exist in the relative setting (see also [35], 3.2). Therefore, one has a collection of locally free sheaves $\mathcal{E}_i$ on the smooth quadric fibration $\mathcal{Q}$ whose restriction to any fiber gives a full strongly exceptional collection on the fiber. Furthermore, in this special case one has $\pi_*(\mathcal{E}_i \otimes \mathcal{E}_i^{\vee}) \simeq \mathcal{O}_X$ (see Lemma 3.68). Now copying the arguments of the proof of Theorem 3.60 yields that the successive iteration of smooth quadric fibrations over a smooth quadric $Q_{2n-1} \subset \mathbb{P}^{2n}$ admits a tilting object. This completes the proof. □

A step further in favor of the conjecture that any homogeneous variety X has a tilting object would be to investigate the case of parabolic subgroups. As for the problem of finding exceptional collections on G/P, to prove the existence of tilting objects on these varieties is essentially more difficult than for the Borel subgroups above (see [111], [140]). In several cases such homogeneous varieties occur as iteration of partial isotropic flags (see [35], [111]) and therefore Theorem 3.65 and Corollary 3.55 yield the existence of tilting objects in these cases.

3.5 Tilting objects for curves and surfaces

In this section we discuss the case of curves and surfaces. We prove that a curve admits a tilting bundle if and only if it is a curve of genus one. Furthermore, we discuss some facts concerning the existence of tilting objects on surfaces.

In what follows we classify all curves admitting a tilting object.

Theorem 3.71. *Let X be a smooth integral and proper curve over an arbitrary field k. Then the following are equivalent:*

- **(i)** *X admits a tilting object.*
- **(ii)** *$H^1(X, \mathcal{O}_X) = 0$.*
- **(iii)** *X is a Brauer–Severi variety.*

Proof. First suppose that X admits a tilting object $\mathcal{T}$. Theorem 3.32 now implies that $H^1(X, \mathcal{O}_X) = 0$ and therefore $H^1(X \otimes_k \bar{k}, \mathcal{O}_{X \otimes_k \bar{k}}) = 0$ what implies that the genus of of the curve $X \otimes_k \bar{k}$ is zero. Thus $X \otimes_k \bar{k} \simeq \mathbb{P}^1_{\bar{k}}$. But this means X is a Brauer–Severi variety. Thus we proved that (i) implies (ii) and (ii) implies (iii). Now if X is a Brauer–Severi variety, Theorem 3.36 yields that X admits a tilting object. Summarizing, we have proved the equivalence of (i), (ii) and (iii). This completes the proof. □

Considering curves over an algebraically closed field k of characteristic zero (or $k = \mathbb{C}$), we have the following well-known consequence of the equivalence between (i) and (iii) of the theorem above.

Corollary 3.72. *Let C be a smooth, integral and proper curve over $\mathbb{C}$. Then $D^b(C)$ admits a tilting object if and only if C is rational.*

Theorem 3.71 classifies all curves over arbitrary fields admitting a tilting object. The next step would be to classify all smooth projective surfaces over an algebraically closed field of characteristic zero that admit a tilting object. As mentioned at the end of Section 2, p.95, by the Enriques classification of complex algebraic surfaces, minimal surfaces of Kodaira dimension $\kappa = 0$ and $\kappa = 1$ do not admit tilting objects. In Kodaira dimension $\kappa = 0$ only the rational surfaces have tilting objects as proved by Hille and Perling [87]. Now in the class of the surfaces of general type there is one class of surfaces containing candidates that have all the strong properties we need. It is the class of surfaces of general type with $q(X) = p_g(X) = 0$. Note that for Kodaira dimension $\kappa = 2$ all other surfaces with either $q(X) = h^{0,1} \neq 0$ or $p_g(X) = h^{2,0} \neq 0$ cannot have a tilting object according to Theorem 3.33. We are therefore left with the class of surfaces of general type with $q(X) = p_g(X) = 0$. In this class it is also possible to exclude many surfaces that cannot have a tilting object, since the first homology $H_1(X, \mathbb{Z})$ is not vanishing and hence the Picard group has torsion elements (see [22] and [24] for a list of surfaces with $H_1(X, \mathbb{Z}) \neq 0$). For instance if X is a Burniat, Godeaux or a Beauville surface one can show that X does not admit a tilting object. Note that all three surfaces satisfy $q(X) = p_g(X) = 0$ (see [24], § VII, 11 for a detailed construction of these surfaces). In all three cases the derived categories were investigated and semiorthogonal decompositions established. This was done in the context of finding so-called *geometric phantom categories*. An admissible triangulated subcategory $\mathcal{A}$ in $D^b(X)$, where X is a smooth projective and integral k-scheme, is called *quasiphantom* if the Hochschild homology vanishes and $K_0(\mathcal{A})$ is a finite abelian group. It is called *phantom* if in addition $K_0(\mathcal{A}) = 0$. There was an opinion among experts that the Hochschild homology and K_0 see an admissible subcategory of a semiorthogonal decomposition, but it turns out that this is not always the case. Indeed, Gorchinskiy and Orlov [72], Theorem 1.12 proved the existence of a phantom on the product of two surfaces S and S' for which the Bloch conjecture for 0-cycles holds. For more details we refer to the work of Gorchinsky and Orlov [72] and references therein. The existence of phantoms would also give a negative answer to the question if the generators of a Grothendieck group $K_0(X) \simeq \mathbb{Z}^{\oplus n}$ give rise to a full strongly exceptional collection. Quasiphantom categories have been found in several cases, as will be stated below, but in all these cases the surfaces considered where of general type with $q(X) = p_g(X) = 0$ and it is for instance an open question if there are phantoms in $D^b(\mathbb{P}^2)$. In what follows we cite some results and refer to the literature for further details. We start with the derived category of the classical Godeaux surface. This was done by Böhning, Graf von Bothmer and Sosna [36], Theorem 8.2.

Theorem 3.73. *Let X be the classical Godeaux surface. Then there exist a semiorthogonal decomposition*

$$D^b(X) = \langle \mathcal{L}_1, ..., \mathcal{L}_{11}, \mathcal{A} \rangle,$$

where $\mathcal{L}_1, ..., \mathcal{L}_{11}$ is an exceptional collection of invertible sheaves of maximal length and $\mathcal{A}$ a quasiphantom with $K_0(\mathcal{A}) \simeq \mathbb{Z}/5\mathbb{Z}$.

The next result was obtained by Alexeev and Orlov [4], Theorem 4.12 and states a semiorthogonal decomposition for Burniat surfaces.

Theorem 3.74. *For any Burniat surface X one has a semiorthogonal decomposition*

$$D^b(X) = \langle \mathcal{L}_1, ..., \mathcal{L}_6, \mathcal{A} \rangle,$$

where $\mathcal{L}_1, ..., \mathcal{L}_6$ is an exceptional collection of invertible sheaves and $\mathcal{A}$ a quasiphantom with $K_0(\mathcal{A}) \simeq (\mathbb{Z}/2\mathbb{Z})^6$.

Finally, Galkin and Shinder [69], Theorem 3.5 and Proposition 3.10 obtained the following:

Theorem 3.75. *For any Beauville surface X one has a semiorthogonal decomposition*

$$D^b(X) = \langle \mathcal{L}_1, ..., \mathcal{L}_4, \mathcal{A} \rangle,$$

where $\mathcal{L}_1, ..., \mathcal{L}_4$ is an exceptional collection of invertible sheaves and $\mathcal{A}$ a quasiphantom with $K_0(\mathcal{A}) \simeq (\mathbb{Z}/5\mathbb{Z})^2$.

A consequence of the above theorems is the following fact mentioned above.

Corollary 3.76. *Let X be a Burniat, classical Godeaux or a Beauville surface. Then X does not admit a tilting object.*

Proof. In all three cases one has a semiorthogonal decomposition

$$D^b(X) = \langle \mathcal{B}, \mathcal{A} \rangle$$

such that $K_0(\mathcal{A})$ is $(\mathbb{Z}/2\mathbb{Z})^6$, $\mathbb{Z}/5\mathbb{Z}$ and $(\mathbb{Z}/5\mathbb{Z})^2$ respectively. Proposition 3.31 now yields $K_0(X) \simeq K_0(\mathcal{B}) \oplus K_0(\mathcal{A})$ and hence $K_0(X)$ is not free. According to Proposition 3.30, this excludes the possibility that X admits a tilting object. □

It is still an open question if the existence of a tilting object on a smooth complex algebraic surface X implies that X is rational. The existence of exceptional collections consisting of invertible sheaves on complete rational surfaces was studied intensively by Perling [131] and by Hille and Perling [86]. Among others, they gave a counterexample to King's conjecture claiming the existence of exceptional collections on smooth complete toric varieties. Concretely, King [103] conjectured that if X is a smooth complete toric variety then $D^b(X)$ admits a tilting bundle which is a direct sum of invertible sheaves. There are a lot of positive examples in favor of this conjecture (see [56], [58], [83], [100]), but in general it turns out that this conjecture is false. Indeed, Hille and Perling [86], Theorem 8.2 proved that a smooth complete toric surface X admits a full strongly exceptional collection of invertible sheaves if and only if $X \neq \mathbb{P}^2$ is obtained by blowing up a Hirzebruch surface in at most two times (in possibly several points in each step). So all smooth complete toric surfaces not obtained in this way cannot admit such a collection. Nonetheless, since such toric surfaces are rational, they admit a tilting object as proved by Hille and Perling in [87], Theorem 1.1. They used so-called universal (co)extensions to produce a tilting bundle from a given full exceptional collection of sheaves (see [87], Theorem 4.2). By construction, the direct summands of this tilting bundle do not form a full strongly exceptional collection. As in the case of Brauer–Severi varieties, smooth projective surfaces in general do not admit a full strongly exceptional collection of invertible sheaves, but always admit tilting objects. Beside the exclusion of surfaces not satisfying the necessary conditions of Proposition 3.30 and Theorem 3.33, Brown and Shipman [51], Theorem 4.3 proved the following result that provides some further evidence for fact that the existence of a tilting object on a smooth projective surface implies that the surface has to be rational.

Theorem 3.77. *Let k be an algebraically closed field of characteristic zero and X a smooth projective surface over k admitting a tilting bundle $\mathcal{T}$ that is a direct sum of invertible sheaves. Then X is rational.*

The above discussion together with the above theorems leads us to the to experts well-known suggestion:

Conjecture. *Let X be a smooth projective complex surface. Then the following are equivalent:*

(i) *X admits a tilting object*

(ii) *X is rational*

Proving this conjecture would provide us with a 2-dimensional version of Corollary 3.72 and would may lead to some structural insight into the geometry of X, responsible for the existence of a tilting object. For more details we refer to [86].

3.6 G-equivariant tilting objects

In this section we want to prove the existence of tilting objects on some stacks. Exceptional collections are known to exist in some cases (see [90], [100]) and the number of examples where tilting objects are known to exist is even smaller then in the case of schemes. The class of stacks we want to study is obtained by group actions on schemes and is called the class of global quotient stacks. In particular, we are interested in the question if the derived category of equivariant coherent sheaves admits a tilting object. We start our investigation considering finite groups G acting on a noetherian scheme X. The derived category $D^b_G(X)$ of G-linearized (or G-equivariant) coherent sheaves were considered by Brav [43] and Elagin [63], [64] and [65]. Especially Elagin [63], [64] proved that under certain assumptions there is a semiorthogonal decomposition for $D^b_G(X)$. In some special cases these semiorthogonal decompositions give rise to full strongly exceptional collections and hence to tilting objects for $D^b_G(X)$. As in the previous sections we are interested in the more general situation where the tilting object is not obtained as a direct sum of sheaves forming a full strongly exceptional collection.

We briefly recall some basic facts about algebraic groups. We refer to [41], [93] and [145] for details.
An *algebraic group G* is a separated scheme of finite type over a field k whose set of closed points is endowed with the structure of a group such that

(i) the group multiplication $m : G \times G \to G$, $(g, h) \mapsto gh$

(ii) the inverse $\iota : G \to G$, $g \mapsto g^{-1}$

are morphisms of schemes. An important class of algebraic groups is the class of affine algebraic groups, called *linear* groups. By the structure theorem, every linear algebraic groups G has to be isomorphic to a closed subgroup of some $\mathrm{GL}_n(k)$. Examples of these are $\mathrm{GL}_n(k)$ itself, the multiplicative group $\mathbb{G}_m$, the algebraic torus $\mathbb{T}^m$, the group $\mathbb{D}_n$ of non-singular diagonal matrices or the group of upper triangular matrices $\mathbb{T}_n$. Furthermore, in an algebraic group there is a unique irreducible component containing the unity element e. It is denoted by G^0 and is called the *identity component* of G. One can show that G^0 is a closed normal subgroup of G (see [145], Proposition 2.2.1). The commutator $[-,-] : G \times G \to G$ is defined as $[g,h] = ghg^{-1}h^{-1}$. Furthermore, the *derived series* of G is defined inductively as $D^0G = G$, $D^{i+1}G = [D^iG, D^iG]$ and the algebraic group G is called *solvable* if its derived series terminates in $\{e\}$. Now the largest connected solvable normal subgroup of G is called *radical* of G and is denoted by $R(G)$ (see [145], 6.4.14). Similarly, there is also a maximal closed, connected unipotent subgroup of G, called *unipotent radical* $R_u(G)$. If the unipotent radical $R_u(G)$ is trivial, the group G is called *reductive* (see [145], 6.4.14). If furthermore the group is linear it is called *linearly reductive.* Note that in characteristic zero an algebraic group is reductive if and only if it is linearly reductive (see [122], Appendix A). Examples of such linearly reductive groups include finite groups, provided the characteristic of k does not divide the order of G, and $\mathrm{GL}_n(k)$ or $\mathrm{SL}_n(k)$, at least in characteristic zero.

To continue, we first recall the definition of a G-linearized sheaf in the general setting where an arbitrary algebraic group G acts on a scheme X. The main reference is GIT [122]. Let k be a field, X an integral k-scheme and G an algebraic group acting on X. Now let $m : G \times G \to G$ be the multiplication morphism and $\sigma : G \times X \to X$ the action of G on X. Denote the projections of $G \times G$, $G \times X$ or $G \times G \times X$ onto the i-th factor by p_i and the projections of $G \times G \times X$ or $G \times G \times G$ on the product of the first two (last two) factors by p_{12} (or p_{23}) respectively. A *G-linearized sheaf* $\mathcal{F}$ on X is a sheaf $\mathcal{F}$, together with an isomorphism $\theta : p_2^*\mathcal{F} \to \sigma^*\mathcal{F}$ of sheaves on $G \times X$, satisfying the following condition:

$$(m \times id_X)^*\theta = p_{23}^*\theta \circ (id_G \times \sigma)^*\theta. \tag{3.7}$$

The isomorphism θ is called *linearized structure* of $\mathcal{F}$. Note that in the literature such sheaves are also called *G-equivariant* and θ the *G-equivariant structure.* In the present work, we will use the notion G-equivariant. If the G-equivariant sheaf $\mathcal{F}$ is quasicoherent (coherent or locally free), is is called G-equivariant quasicoherent (coherent or locally free) sheaf. A morphism of G-equivariant sheaves is defined to be compatible with the G-equivariant structure and the group of G-equivariant homomorphisms will be denoted by $\mathrm{Hom}_G(\mathcal{F},\mathcal{G})$. Note that one has a natural action of G on $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ and taking invariants yields $\mathrm{Hom}_G(\mathcal{F},\mathcal{G}) = \mathrm{Hom}(\mathcal{F},\mathcal{G})^G$. Considering G-equivariant (quasi-) coherent sheaves on X one has abelian categories denoted by $\mathrm{Qcoh}_G(X)$ and $\mathrm{Coh}_G(X)$ respectively. We write $D_G(\mathrm{Qcoh}(X))$ for the derived category of G-equivariant quasicoherent sheaves on X and $D^b_G(X)$ for the bounded derived category of G-equivariant coherent sheaves. To give another interpretation of the categories $\mathrm{Qcoh}_G(X)$ and $\mathrm{Coh}_G(X)$ respectively, we briefly recall the definition of a *stack* and refer to [114] for all the technical details. Moreover, we use the Appendix of [153], since it gives a nice and comprehensive summary of the main definitions needed in this section.

Let S be a scheme and $\mathcal{Z}$ a category over S, fibered in groupoids. One then has a functor $p : \mathcal{Z} \to (\mathrm{Sch}/S)$, called *projection.* Given any S-scheme T, we denote by $\mathcal{Z}(T)$ the category whose objects are objects $a \in \mathcal{Z}$ such that $p(a) = T$ and whose arrows are arrows f in $\mathcal{Z}$ with $p(f) = id$. We want to call this category the *fiber of* $\mathcal{Z}$ over T. Now $\mathcal{Z}$ is called a *stack* over S if the following hold:

(i) For any $U \in (\mathrm{Sch}/S)$ and any two objects $a, b \in \mathcal{Z}(U)$ the functor $\mathcal{I}so_U(a,b) : (\mathrm{Sch}/U) \to \mathrm{Set}$, $V \mapsto \{a_{|V} \xrightarrow{\sim} b_{|V} \in \mathcal{Z}(V)\}$ is a sheaf in the étale topology. Precisely, this means the following: For all $U \in (\mathrm{Sch}/S)$ and all $a, b \in \mathcal{Z}(U)$ and for all open covers $\{U_i \to U\}$ in the étale topology and all isomorphisms $\alpha_i : a_{|U_i} \xrightarrow{\sim} b_{|U_i}$ such that $\alpha_i|_{U_{ij}} = \alpha_j|_{U_{ij}}$, there is a unique isomorphism $\alpha : a \xrightarrow{\sim} b$, such that $\alpha|_{U_i} = \alpha_i$.

(ii) For all open covers $\{U_i \to U\}$ the descent datum is effective. This means the following: For all open covers $\{U_i \to U\}$ in the étale topology and all $a_i \in \mathcal{Z}(U_i)$ and all isomorphisms $\alpha_{ij} : \alpha_i|_{U_{ij}} \xrightarrow{\sim} \alpha_j|_{U_{ij}}$ in $\mathcal{Z}(U_i \times_U U_j)$ such that $\alpha_{ik} = \alpha_{jk} \circ \alpha_{ij}$, there is a $a \in \mathcal{Z}(U)$ and $\alpha_i : a_{|U_i} \xrightarrow{\sim} a_i$ such that $\alpha_{ij} = \alpha_j|_{U_{ij}} \circ (\alpha_i|_{U_{ij}})^{-1}$.

A *morphism of stacks* is a functor $F : \mathcal{Z}_1 \to \mathcal{Z}_2$, such that for the projections $p_{\mathcal{Z}_1} : \mathcal{Z}_1 \to (\mathrm{Sch}/S)$ and $p_{\mathcal{Z}_2} : \mathcal{Z}_2 \to (\mathrm{Sch}/S)$ one has $p_{\mathcal{Z}_1} = p_{\mathcal{Z}_2} \circ F$.

Example 3.78. A scheme X can be considered as a stack via its functor of points (see [153], Example 7.2). To be more precise, consider a S-scheme X with structure morphism $\pi : X \to S$. Then consider the functor $p : (\mathrm{Sch}/X) \to (\mathrm{Sch}/S)$ assigning to a X-scheme T the S-scheme $T \to X \to S$. One can show that (Sch/X) is fibered in groupoids over S. The fibers $\mathcal{Z}(U)$ over a S-scheme U are just $\mathrm{Hom}_S(U, X)$ as a set. Both conditions (i) and (ii) from above can be verified to hold true. Therefore we can consider the S-scheme X as a stack via its functor of points. To every S-scheme T the corresponding fiber of the stack is given by $\mathrm{Hom}_S(T, X)$.

Note that one can also form the fiber product $\mathcal{Z}_1 \times_{\mathcal{Z}} \mathcal{Z}_2$ of two morphism of stacks $\mathcal{Z}_1 \to \mathcal{Z}$ and $\mathcal{Z}_2 \to \mathcal{Z}$ (see [153], Definition 7.9). The diagonal morphism $\Delta_{\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z} \times_S \mathcal{Z}$ is given by the two identity morphisms. A morphism $F : \mathcal{Z}_1 \to \mathcal{Z}_2$ of stacks is called *representable* if for any S-scheme T and any morphism $T \to \mathcal{Z}_2$ the fiber product $\mathcal{Z}_1 \times_{\mathcal{Z}_2} T$ is a scheme (see [153], Definition 7.11). Let $\mathcal{Z}$ be an stack and T a S-scheme considered as a stack via its functor of points. An étale surjective morphism $T \to \mathcal{Z}$ is called *atlas.* A for our further investigations important class of stacks is the class of *Deligne–Mumford stacks.*

Definition 3.79. A *Deligne–Mumford stack* is a stack $\mathcal{Z}$ satisfying the following conditions:

(i) The diagonal morphism $\Delta_{\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z} \times_S \mathcal{Z}$ is representable, quasicompact and separated.

(ii) There is a scheme T and an étale surjective morphism $T \to \mathcal{Z}$.

Remark 3.80. A stack $\mathcal{Z}$ is called *Artin stack* if it satisfies (i) from above and if one instead of the existence of a scheme T with an étale surjective morphism $T \to \mathcal{Z}$ claims the existence of a scheme T with a smooth surjective morphism $T \to \mathcal{Z}$.

Example 3.81. Let G be a smooth linear algebraic group acting on a k-scheme X. Denote by $[X/G]$ the category fibered over the category of k-schemes (Sch/k), the fibers over a k-scheme T being defined as the set of principal G-bundles $P \to T$ together with G-equivariant map $P \to X$. One can show that $[X/G]$ is an Artin stack and if the stabilizers of the geometric points of X are finite and reduced, $[X/G]$ is a Deligne–Mumford stack (see [153], Example 7.17). In particular, if G is a finite group acting on X such that the characteristic of k does not divide the order of G, the stack $[X/G]$ is a Deligne–Mumford stack. Stacks of the form $[X/G]$ are called *global quotient stacks*.

Now let $\mathcal{Z}$ be a stack. A *quasicoherent sheaf* $\mathcal{F}$ on $\mathcal{Z}$ consists of the following data (see [153], Definition 7.18).

(i) For each atlas $T \to \mathcal{Z}$ one has a quasicoherent sheaf $\mathcal{F}_T$ on T.

(ii) For each morphism $\phi : T \to U$ of atlases one has an isomorphism $\alpha_\phi : \mathcal{F}_T \to \phi^*\mathcal{F}_U$.

The isomorphisms α_ϕ are required to satisfy the cocycle condition that we do not want to reproduce here, referring to the literature. We now can give another interpretation of the category of G-equivariant coherent sheaves. Let X be a smooth projective and integral k-scheme X and G a smooth linearly reductive group acting on X. Consider the quotient stack $[X/G]$. The atlas is given by the projection $X \to [X/G]$ (see [153], Example 7.17) and one has an isomorphism of groupoids between $X \times_{[X/G]} X \rightrightarrows X$ and $X \times_S G \rightrightarrows X$ (see [153], Example 7.21). This isomorphism directly implies that a coherent sheaf $\mathcal{F}$ on the stack $[X/G]$ is by definition a G-equivariant coherent sheaf $\mathcal{F}$ on X. Note that the cocycle condition translates to condition (3.7) from above and hence the categories $\mathrm{Coh}_G(X)$ and $\mathrm{Coh}([X/G])$ are equivalent. This implies $D^b_G(X) \simeq D^b([X/G])$. Summarizing, we hold on to the fact that G-equivariant (quasi-) coherent sheaves on X are the same as (quasi-) coherent sheaves on the quotient stack $[X/G]$.

Example 3.82. If X is a smooth projective and integral k-scheme together with a group action of a finite group G, such that the characteristic of k does not divide the order of G, the quotient stack $[X/G]$ is a smooth, proper, tame and connected Deligne–Mumford stack with coarse projective moduli space. To see this, we first recall the notion of a *tame* stack. Recall, the *inertia stack* of a stack $\mathcal{Z}$ over k is defined to be $\mathcal{IZ} := \mathcal{Z} \times_{\mathcal{Z}\times_k\mathcal{Z}} \mathcal{Z}$. If $\mathcal{IZ} \to \mathcal{Z}$ is finite, there exists a coarse moduli space $\rho : \mathcal{Z} \to M$ and if furthermore $\rho_* : \mathrm{Qcoh}(\mathcal{Z}) \to \mathrm{Qcoh}(M)$ is exact, then $\mathcal{Z}$ is called *tame* (see [1] for details). Note that $X \to [X/G]$ is the atlas of $[X/G]$. Now [153], Proposition 2.11 implies that the quotient scheme $X//G$ is a moduli space for $[X/G]$. Since X is a smooth projective and integral k-scheme, the quotient $X//G$ is a projective scheme. Furthermore, since X is smooth and projective, and hence proper, and since the characteristic of k does not divide the order of G, the quotient stack $[X/G]$ is a smooth, proper, connected and tame Deligne–Mumford stack. To see why $[X/G]$ is tame we refer to [1], Theorem 3.2. At this point one needs that the characteristic of k does not divide the order of G so that G is linearly reductive.

Remark 3.83. Taking a point and considering the action of an algebraic group G on that point provides us with the stack $[pt/G]$, also denoted by BG and called the *classifying stack* of G. Quasicoherent sheaves on this stack can be thought of as representations of the group G. We want to note that there is also a very useful characterization of linearly reductive groups in terms of the category of quasicoherent sheaves on $[pt/G]$. A linear algebraic group G is called linearly reductive if the functor $(-)^G : \mathrm{Qcoh}([pt/G]) \to \mathrm{Qcoh}(pt)$, $V \mapsto V^G$ is exact (see [1], Definition 2.4). Moreover, since we are considering algebraic groups over fields, G is linearly reductive if and only if the functor $\mathrm{Coh}([pt/G]) \to \mathrm{Coh}(pt)$, $V \mapsto V^G$ is exact (see [1], Proposition 2.5).

We now state a very useful observation that will be needed later on quite frequently. It is well-known and can be found for instance in [20], Lemma 2.2.8. Although formulated for projective k-schemes below it also holds for quasiprojective k-schemes (see [20]).

Lemma 3.84. *Let X be smooth projective and integral k-scheme and G a linearly reductive group acting on X. Then for arbitrary complexes of quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$ the following holds:*

$$\mathrm{Hom}_G(\mathcal{F}, \mathcal{G}[i]) \simeq \mathrm{Hom}_X(\mathcal{F}, \mathcal{G}[i])^G.$$

Proof. We sketch the proof since it can be found in [20]. Recall that there is an isomorphism of functors $\mathrm{Hom}_G(-,-) \simeq \mathrm{Hom}(-,-)^G$. Grothendieck spectral sequence for the composition of two functors applied to the two functors $\mathrm{Hom}(-,-)$ and $(-)^G$ yields the desired isomorphism. Note that under the assumption on G being linearly reductive, taking G-invariants is exact (see Remark 3.83). This yields the above isomorphism. □

We now give a definition of tilting objects for G-equivariant derived categories. As mentioned in Remark 3.18, the notion of tilting objects can be defined for arbitrary triangulated categories, so that we just have to adapt the definition given in [52] to our geometric situation.

Definition 3.85. Let k be a field, X a noetherian quasiprojective k-scheme and G an algebraic group acting on X. Denote by $D_G(\mathrm{Qcoh}(X))$ the derived category of G-equivariant quasicoherent sheaves on X. An object $\mathcal{T} \in D_G(\mathrm{Qcoh}(X))$ is called tilting object for $D_G(\mathrm{Qcoh}(X))$ if the following hold:

(i) $\mathrm{Hom}_G(\mathcal{T}, \mathcal{T}[i]) = 0$ for $i \neq 0$.

(ii) If $\mathcal{N} \in D_G(\mathrm{Qcoh}(X))$ satisfies $\mathbb{R}\mathrm{Hom}_G(\mathcal{T}, \mathcal{N}) = 0$, then $\mathcal{N} = 0$.

(iii) $\mathrm{Hom}_G(\mathcal{T}, -)$ commutes with direct sums.

Remark 3.86. As in Remark 3.3, if one has a titling object $\mathcal{T}$ for $D_G(\mathrm{Qcoh}(X))$ one can form the smallest full triangulated subcategory containing $\mathcal{T}$, that is closed under direct sums and direct summands. Again we denote this category by $\langle \mathcal{T} \rangle$. One can show that condition (ii) from above is equivalent to $\langle \mathcal{T} \rangle = D_G(\mathrm{Qcoh}(X))$ (see [52], Remark 1.2). Again we say $\mathcal{T}$ is *generating* the derived category $D_G(\mathrm{Qcoh}(X))$. Furthermore, if $D_G(\mathrm{Qcoh}(X))$ is compactly generated and the compact objects are exactly $D^b_G(X)$, then to show that an object $\mathcal{T}$ generates $D_G(\mathrm{Qcoh}(X))$ is equivalent to show that it generates $D^b_G(X)$, i.e., that the smallest full triangulated subcategory containing $\mathcal{T}$ that is closed under direct sums and direct summands equals $D^b_G(X)$ (see [40], Theorem 2.2.1).

One has the following G-equivariant tilting correspondence proved in [43], Theorem 3.1.1. It is a direct application of a result of Keller [102], Theorem 8.5. Notice that $\mathrm{Mod}(A)$ below is the category of right A-modules.

Theorem 3.87. *Let X be a noetherian quasiprojective k-scheme and G a finite group acting on X, such that the characteristic of k does not divide the order of G. Suppose we are given a tilting object $\mathcal{T}$ for $D_G(\mathrm{Qcoh}(X))$ and let $A = \mathrm{End}_G(\mathcal{T})$. Then the following hold:*

- **(i)** *The functor $\mathbb{R}\mathrm{Hom}_G(\mathcal{T},-) : D_G(\mathrm{Qcoh}(X)) \to D(\mathrm{Mod}(A))$ is an equivalence.*
- **(ii)** *If X is smooth, then the equivalence of (i) restricts to an equivalence*
$$D^b_G(X) \xrightarrow{\sim} \mathrm{perf}(A).$$
- **(iii)** *If the global dimension of A is finite, then $\mathrm{perf}(A) \simeq D^b(A)$.*

Proof. We reproduce the proof given by Brav [43]. Note that $D(\mathrm{Qcoh}(X))$ is an *algebraic triangulated category* in the sense of [102] and hence $D_G(\mathrm{Qcoh}(X))$ is. The equivalence of (i) is just an application of [102], Theorem 8.5. To prove (ii), we note that the equivalence $\psi = \mathbb{R}\mathrm{Hom}_G(\mathcal{T},-)$ from (i) restricts to an equivalence of compact objects. This can be seen as follows: Since ψ is an equivalence, the right adjoint is given by the inverse ψ^{-1} (see [89], Proposition 1.26). Hence for a compact object $\mathcal{C} \in D_G(\mathrm{Qcoh}(X))$ we have

$$\begin{aligned} \mathrm{Hom}_A(\psi(\mathcal{C}), \bigoplus_i \mathcal{F}_i) &\simeq \mathrm{Hom}_G(\mathcal{C}, \bigoplus_i \psi^{-1}(\mathcal{F}_i)) \\ &\simeq \bigoplus_i \mathrm{Hom}_G(\mathcal{C}, \psi^{-1}(\mathcal{F}_i)) \simeq \bigoplus_i \mathrm{Hom}_A(\psi(\mathcal{C}), \mathcal{F}_i). \end{aligned}$$

The compact objects of $D(\mathrm{Mod}(A))$ are all of $\mathrm{perf}(A)$ (see [102], Theorem 8.2). Denoting by $D^c_G(\mathrm{Qcoh}(X))$ the subcategory of compact objects, the equivalence of (i) restricts to an equivalence

$$D^c_G(\mathrm{Qcoh}(X)) \xrightarrow{\sim} \mathrm{perf}(A).$$

Since X is smooth and projective, the compact objects $D^c_G(\mathrm{Qcoh}(X))$ are exactly the perfect objects (see [79], Section 3, Section 4, Example 4.9). The smoothness and the fact that G is linearly reductive yields with a result of Thomason [146], Theorem 2.18 that every G-equivariant coherent sheaf has a finite resolution by G-equivariant locally free sheaves and hence the perfect objects are all of $D^b_G(X)$. This gives the equivalence

$$D^b_G(X) \xrightarrow{\sim} \mathrm{perf}(A).$$

Finally, (iii) is clear. This completes the proof. □

The next result shows that in the above theorem the smoothness of X implies already the finiteness of the global dimension of A. This result, together with Theorem 3.87, gives a G-equivariant version of Theorem 3.8. Below we will also give an alternative proof for the following theorem.

Theorem 3.88. *Let X, G and $\mathcal{T}$ be as in Theorem 3.87. If X is smooth projective and integral, then $A = \mathrm{End}_G(\mathcal{T})$ has finite global dimension and the equivalence (i) of Theorem 3.87 restricts to an equivalence $D^b_G(X) \xrightarrow{\sim} D^b(A)$.*

Proof. Imitating the proof of Theorem 3.8, that can be found in [81], p.172, Theorem 7.6, one can argue as follows: For arbitrary finitely generated right A-modules M and N we conclude with the equivalence $\psi : D^b_G(X) \to \mathrm{perf}(A)$ (see Theorem 3.87 (ii)):

$$\mathrm{Ext}^i_A(M,N) \simeq \mathrm{Hom}_G(\psi^{-1}(M), \psi^{-1}(N)[i]) \simeq \mathrm{Hom}(\psi^{-1}(M), \psi^{-1}(N)[i])^G = 0$$

for $i >> 0$, since X is by assumption smooth projective and integral. Again, this is obtained by local-to-global spectral sequence, Grothendieck vanishing Theorem and Lemma 3.84. Furthermore, since X is projective, $A = \mathrm{End}_G(\mathcal{T})$ is a finite-dimensional k-algebra and hence a noetherian ring. But for noetherian rings the vanishing of $\mathrm{Ext}^i_A(M,N)$ for $i >> 0$ for any two finitely generated A-modules M and N suffices to conclude that the global dimension of A has to be finite. □

For global quotient stacks obtained from a finite group action we now have the following result.

Theorem 3.89. *Let X be a smooth projective and integral k-scheme and G a finite group acting on X such that $\mathrm{char}(k)$ does not divide $\mathrm{ord}(G)$. Suppose that $\mathcal{T}$ is a tilting sheaf for $D^b(X)$ and suppose furthermore, that $\mathcal{T}$ is G-equivariant. Let W_i be the irreducible representations of G. Then $\mathcal{T}_G = \bigoplus_i \mathcal{T} \otimes W_i$ is a tilting object for $D^b([X/G])$ and one has an equivalence*

$$\mathbb{R}\mathrm{Hom}_G(\mathcal{T}_G, -) : D^b([X/G]) \xrightarrow{\sim} D^b(\mathrm{End}_G(\mathcal{T}_G)).$$

Proof. Recall that $D^b_G(X) \simeq D^b([X/G])$. Note that for every i one has the canonical isomorphisms on X,

$$\mathrm{Ext}^i(\mathcal{T} \otimes W_l, \mathcal{T} \otimes W_m) \simeq \mathrm{Ext}^i(\mathcal{T}, \mathcal{T}) \otimes \mathrm{Hom}(W_l, W_m). \tag{3.8}$$

Since $\mathcal{T}$ is supposed to be G-equivariant, the coherent sheaf $\mathcal{T} \otimes W_i$ is also G-equivariant. By assumption, the characteristic of k does not divide the order of G and hence G is linearly reductive. In this situation Lemma 3.84 applies and we have with (3.8)

$$\mathrm{Ext}^i_G(\mathcal{T} \otimes W_l, \mathcal{T} \otimes W_m) \simeq (\mathrm{Ext}^i(\mathcal{T}, \mathcal{T}) \otimes \mathrm{Hom}(W_l, W_m))^G.$$

Since $\mathcal{T}$ is by assumption a tilting sheaf for $D^b(X)$, we have $\mathrm{Ext}^i(\mathcal{T}, \mathcal{T}) = 0$ for $i \neq 0$. Hence

$$\mathrm{Ext}^i_G(\mathcal{T} \otimes W_l, \mathcal{T} \otimes W_m) = 0 \text{ for } i \neq 0.$$

This implies $\mathrm{Ext}^i_G(\mathcal{T}_G, \mathcal{T}_G) = 0$ for $i \neq 0$ and hence the Ext vanishing holds true. To see that $\mathcal{T}_G$ generates $D_G(\mathrm{Qcoh}(X))$, we note that the quotient stack $[X/G]$ is a quasicompact and separated Deligne–Mumford stack with coarse moduli space being the quotient scheme $X//G$ (in the sense of GIT). Hence by [147], Corollary 4.2 the derived category $D_G(\mathrm{Qcoh}(X))$ is compactly generated. So by Remark 3.86 it suffices to prove that $\mathcal{T}_G$ generates the subcategory of compact objects of $D_G(\mathrm{Qcoh}(X))$. But as pointed out in the proof of Theorem 3.87 these are all of $D^b_G(X)$. So we assume $\mathcal{F} \in D^b_G(X)$ and $\mathbb{R}\mathrm{Hom}_G(\mathcal{T}_G, \mathcal{F}) = 0$. This implies

$$\mathrm{Hom}_G(\mathcal{T}_G, \mathcal{F}[i]) = 0 \text{ for all } i \in \mathbb{Z}.$$

With Lemma 3.84 we get

$$\mathrm{Hom}(\mathcal{T}_G, \mathcal{F}[i])^G = 0 \text{ for all } i \in \mathbb{Z}.$$

By the construction of $\mathcal{T}_G$ we conclude that

$$\mathrm{Hom}(\mathcal{T} \otimes W_m, \mathcal{F}[i])^G = 0,$$

for all $i \in \mathbb{Z}$ and all irreducible representations W_m. But then $\mathrm{Hom}(\mathcal{T}, \mathcal{F}[i])$ contains no copy of any irreducible representation W_m and so must be zero. Since $\mathcal{T}$ is a tilting sheaf for $D^b(X)$ and therefore generates $D^b(X)$, $\mathrm{Hom}(\mathcal{T}, \mathcal{F}[i]) = 0$ for all $i \in \mathbb{Z}$ implies $\mathcal{F} = 0$. This shows that $\mathcal{T}_G$ generates $D^b(X)$ and hence $D_G(\mathrm{Qcoh}(X))$. To finish the proof, we apply Theorem 3.87, together with Theorem 3.88 to get that $\mathcal{T}_G$ is a tilting object and that $D^b_G(X) \to D^b(\mathrm{End}_G(\mathcal{T}_G))$ is an equivalence. □

We want to note that in the case the ground field k is supposed to be perfect, we can also give an alternative proof of Theorem 3.88. In order to give this alternative proof, we have to give some definitions becoming important later on as well. The main references are [19] and [137]. Let T be a triangulated category. For a subcategory M, we denote by $\langle M \rangle$ the full triangulated subcategory of T whose objects are isomorphic to summands of finite coproducts of shifts of objects in M. Concretely, this means that $\langle M \rangle$ is the smallest full triangulated subcategory containing M that is closed under isomorphisms, shifting and taking finite coproducts and summands (see [137]). Furthermore, for two triangulated subcategories M and N of T, we want to denote by $M \star N$ the smallest full subcategory of objects R, such that there exists a distinguished triangles of the form

$$X_1 \longrightarrow R \longrightarrow X_2 \longrightarrow X_1[1],$$

where $X_1 \in M$ and $X_2 \in N$. Then set $M \diamond N = \langle M \star N \rangle$ and define $\langle M \rangle_0$ to be $\langle M \rangle$. We define inductively $\langle M \rangle_i = \langle M \rangle_{i-1} \diamond \langle M \rangle$.

Definition 3.90. Let A be an object of an triangulated category T. If there is an $n \geq 0$ with $\langle A \rangle_n = T$, we define the *generation time* $\mathrm{gt}(A)$ of A to be $\min\{n | \langle A \rangle_n = T\}$. Otherwise, we set $\mathrm{gt}(A) = \infty$. If $\mathrm{gt}(A)$ is finite, we say A is a *strong generator*, otherwise A is called *generator*. The *dimension of* the triangulated category T is now defined to be the minimal generation time among strong generators and is denoted by $\mathrm{dim}T$. It is set to be ∞ if there are no strong generators.

Remark 3.91. Notice that if the triangulated category T is the derived category $D^b(X)$ of some projective k-scheme X, the above definition of generating coincides with the definition given in Definition 3.2, Remark 3.3 and 3.4.

Recall, the *Hochschild dimension* of a k-algebra A is defined as follows: Let A be a k-algebra and A^{op} the opposite algebra. One can consider A as an $A \otimes_k A^{op}$-module and ask for the projective dimension of A as a $A \otimes_k A^{op}$-module. This projective dimension is called *Hochschild dimension* of A and is denoted by $\mathrm{hd}(A)$. For the connection between Hochschild cohomology and tilting objects see [52]. The next two results will also be very important in the next section and are due to Ballard and Favero [19]. Their results show the deep connection between the dimension of the derived category of certain stacks (always of finite type over k) and the Hochschild dimension (resp. global dimension) of the endomorphism k-algebra of a tilting object. It is the following result (see [19], Theorem 3.2)

Theorem 3.92. *Let $\mathcal{X}$ be a smooth, proper, tame and connected Deligne–Mumford stack with a projective coarse moduli space. Suppose $\mathcal{T}$ is a generator for $D^b(\mathcal{X})$ satisfying $\mathrm{Hom}(\mathcal{T},\mathcal{T}[i]) = 0$ for $i \neq 0$ and let i_0 be the largest i for which $\mathrm{Hom}(\mathcal{T},\mathcal{T}\otimes\omega^{\vee}_{\mathcal{X}}[i])$ is non-zero. Then the generation time $\mathrm{gt}(\mathcal{T})$ is bounded from above by $\dim(\mathcal{X})+i_0$. Moreover, the Hochschild dimension $\mathrm{hd}(A)$ of $A = \mathrm{End}(\mathcal{T})$ equals $\dim(\mathcal{X}) + i_0$. If furthermore k is a perfect field then the generation time $\mathrm{gt}(\mathcal{T})$ equals $\dim(\mathcal{X})+i_0$ in particular, $\dim D^b(\mathcal{X}) \leq \dim(\mathcal{X})+i_0$.*

From this result Ballard and Favero obtained the following (see [19], Theorem 3.4).

Theorem 3.93. *Let X be a smooth projective and integral k-scheme and $\mathcal{T}$ a tilting object for $D^b(X)$. Let i_0 be the largest i for which $\mathrm{Hom}(\mathcal{T},\mathcal{T}\otimes\omega^{\vee}_{X}[i])$ is non-zero. Then the generation time $\mathrm{gt}(\mathcal{T})$ is bounded from above by $\dim(X)+i_0$. Moreover, the Hochschild dimension $\mathrm{hd}(A)$ of $A = \mathrm{End}(\mathcal{T})$ equals $\dim(X)+i_0$. If furthermore k is a perfect field then the generation time $\mathrm{gt}(\mathcal{T})$ equals $\dim(\mathcal{X})+i_0$ in particular, $\dim D^b(X) \leq \dim(X) + i_0$.*

Thus, if k is perfect one has $\dim(\mathcal{X}) \leq \dim D^b(\mathcal{X}) \leq \dim(\mathcal{X}) + i_0$, where the first inequality is [19], Lemma 2.17. Now if $i_0 = 0$, then

$$\dim(\mathcal{X}) = \dim D^b(\mathcal{X}).$$

The same holds if X is a smooth projective and integral k-scheme. This fact will be needed in the next section to provide some further evidence for the dimension conjecture of Orlov (see Section 7).

Remark 3.94. We want to make two comments. The first comment is that in the above theorems it is impossible that $\mathrm{Hom}(\mathcal{T},\mathcal{T}\otimes\omega^{\vee}_{\mathcal{X}}[i]) = 0$ for all $i \in \mathbb{Z}$, since this would imply $\mathcal{T}\otimes\omega^{\vee}_{\mathcal{X}} = 0$ (note that $\mathcal{T}$ is a compact generator) what is absurd. Hence there really exist i such that $\mathrm{Hom}(\mathcal{T},\mathcal{T}\otimes\omega^{\vee}_{\mathcal{X}}[i]) \neq 0$. Furthermore, since $\mathcal{X}$ (resp. X) is supposed to be smooth one has $\mathrm{Hom}(\mathcal{T},\mathcal{T}\otimes\omega^{\vee}_{\mathcal{X}}[i]) = 0$ for $i >> 0$ and hence the set of all i with $\mathrm{Hom}(\mathcal{T},\mathcal{T}\otimes\omega^{\vee}_{\mathcal{X}}[i]) \neq 0$ is bounded from above. Thus it is reasonable to speak about the largest i with this property. The second comment is the following: Note that [19], Lemma 2.13 states that for finite-dimensional k-algebras A over perfect fields k, the Hochschild dimension of A equals the global dimension of A. This is a consequence of [137], Proposition 7.4. Therefore, in Theorem 3.93 we have $\mathrm{hd}(\mathrm{End}(\mathcal{T})) = \mathrm{gldim}(\mathrm{End}(\mathcal{T}))$.

We now give an alternative proof of Theorem 3.88 under the additional assumption that k is perfect.

Proof. Since the quotient stack $[X/G]$ is a smooth, proper, tame and connected Deligne–Mumford stack with coarse projective moduli space (see Example 3.82), Theorem 3.92 applies and we have to determine the largest i such that

$$\mathrm{Hom}_G(\mathcal{T}, \mathcal{T} \otimes \omega^{\vee}_{[X/G]}[i])$$

is non-zero. By [19], Definition 2.27 the object $\omega^{\vee}_{[X/G]}$ is an invertible sheaf. Since both $\mathcal{T}$ and $\omega^{\vee}_{[X/G]}$ are G-equivariant, Lemma 3.84 applies and we find

$$\mathrm{Hom}_G(\mathcal{T}, \mathcal{T} \otimes \omega^{\vee}_{[X/G]}[i]) \simeq \mathrm{Hom}(\mathcal{T}, \mathcal{T} \otimes \omega^{\vee}_{[X/G]}[i]))^G.$$

Again, smoothness of X and local-to-global spectral sequence yields

$$\mathrm{Ext}^i(\mathcal{T}, \mathcal{T} \otimes \omega^{\vee}_{[X/G]})) = 0 \text{ for } i >> 0,$$

what therefore implies

$$\mathrm{Hom}(\mathcal{T}, \mathcal{T} \otimes \omega^{\vee}_{[X/G]}[i]))^G = 0 \text{ for } i >> 0.$$

Theorem 3.92 now implies that the Hochschild dimension of $\mathrm{End}_G(\mathcal{T})$ is finite. Since $\mathrm{End}_G(\mathcal{T})$ is a finite-dimensional k-algebra over a perfect field, Remark 3.94 now implies that the global dimension of $\mathrm{gldim}(\mathrm{End}_G(\mathcal{T}))$ is finite too. □

Remark 3.95. Note that if the tilting object $\mathcal{T}$ from above is supposed to be a locally free sheaf, $\mathrm{Ext}^i(\mathcal{T}, \mathcal{T} \otimes \omega^{\vee}_{[X/G]})) \simeq H^i(X, \mathcal{T}^{\vee} \otimes \mathcal{T} \otimes \omega^{\vee}_{[X/G]})) = 0$ for $i >> 0$ by the Grothendieck vanishing Theorem.

In what follows, we want to generalize a result of Brav [43], Theorem 4.2.1 and of Bridgeland and Stern [50], Theorem 3.6. In loc.cit. the authors investigated certain total spaces and proved the existence of tilting objects for their bounded derived categories. The existence of tilting objects on certain total spaces also led Weyman and Zhao [155] to a construction of non-commutative desigularizations. In the spirit of this section we want to study total spaces with finite group actions. For this, let G be a finite group acting on some smooth projective and integral k-scheme X, such that the characteristic of k does not divide the order of G. Furthermore, let $\mathcal{E}$ be a G-equivariant locally free sheaf of finite rank. Now consider the total space $\mathrm{Tot}(\mathcal{E}) = \mathcal{S}pec(S^{\bullet}(\mathcal{E}))$, where $S^{\bullet}(\mathcal{E}) = \mathrm{Sym}(\mathcal{E})$. Since $\mathcal{E}$ is G-equivariant, the group G acts on $\mathrm{Tot}(\mathcal{E})$ in the natural way. Note that the total space comes equipped with an affine structure morphism $\pi : \mathrm{Tot}(\mathcal{E}) \to X$ that is compatible with the action of G. Assuming $D^b(X)$ to have a tilting object $\mathcal{T}$ with G-equivariant structure, the question arises if the stack $[\mathrm{Tot}(\mathcal{E})/G]$ admits a tilting object. There is a natural candidate for a tilting object on $[\mathrm{Tot}(\mathcal{E})/G]$. Take the tilting object $\mathcal{T}$ for $D^b(X)$. Then, according to Theorem 3.89, the stack $[X/G]$ admits a tilting object $\mathcal{T}_G$. This is a G-equivariant coherent sheaf on X and pulling it back to $\mathrm{Tot}(\mathcal{E})$ yields the object $\pi^*\mathcal{T}_{\mathcal{G}}$. This is a coherent sheaf on $\mathrm{Tot}(\mathcal{E})$ that has a natural G-equivariant structure. In order to prove that $\pi^*\mathcal{T}_{\mathcal{G}}$ is a tilting object, there is one problem that occurs. Since $\mathrm{Tot}(\mathcal{E})$ is not projective over k, Theorem 3.88 cannot be applied. So we first have to investigate what happens to the global dimension of $\mathrm{End}_G(\pi^*\mathcal{T}_{\mathcal{G}})$ in this case. Note that by adjointness of

π^* and π_* (see [63] for a proof of G-equivariant adjontness of π^* and π_*) one gets

$$\mathrm{End}_G(\pi^*\mathcal{T}_{\mathcal{G}}) = \mathrm{Hom}_G(\pi^*\mathcal{T}_{\mathcal{G}}, \pi^*\mathcal{T}_{\mathcal{G}}) \simeq \mathrm{Hom}_G(\mathcal{T}_G, \mathcal{T}_G \otimes S^\bullet(\mathcal{E})),$$

since $\pi_*\pi^*\mathcal{T}_G \simeq S^\bullet(\mathcal{E}) \otimes \mathcal{T}_G$. Hence $A = \mathrm{End}_G(\pi^*\mathcal{T}_{\mathcal{G}})$ is a graded k-algebra which of course is infinite-dimensional. Note that the proof of Theorem 3.88 works also if the endomorphism algebra is supposed to be a noetherian ring. In fact, the algebra A is noetherian so that the arguments of the proof of Theorem 3.88 can be applied. To see why A is noetherian, note that $\mathrm{Tot}(\mathcal{E})$ is a noetherian scheme, since X is noetherian. In this situation the pullback $\pi^*\mathcal{T}_G$ is a coherent sheaf on $\mathrm{Tot}(\mathcal{E})$. Now it is a fact that the endomorphism algebra $\mathrm{End}(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ on a noetherian scheme Y is again noetherian. This can be seen by considering the algebra $\mathrm{End}(\mathcal{F})$ as finitely generated module over the global sections $B = \Gamma(Y, \mathcal{O}_Y)$. Note that since Y is noetherian, B is a noetherian algebra. If a finite group G is acting on Y, it is well-known that B is a finitely generated module over B^G. Therefore $\mathrm{End}_G(\mathcal{F})$ is noetherian for a G-equivariant coherent sheaf $\mathcal{F}$ on X. This implies that the endomorphism algebra $\mathrm{End}_G(\pi^*\mathcal{T}_G)$ is a noetherian ring.

With the above discussion we obtain the following result.

Theorem 3.96. *Let X and G be as above and $\mathcal{E}$ a G-equivariant locally free sheaf of finite rank. Suppose $\mathcal{T}$ is a tilting bundle for $D^b(X)$ and suppose furthermore, that $\mathcal{T}$ is G-equivariant. If $H^i(X, \mathcal{T}^\vee \otimes \mathcal{T} \otimes S^l(\mathcal{E})) = 0$ for all $i \neq 0$ and all $l > 0$, then one has an equivalence*

$$D^b([\mathrm{Tot}(\mathcal{E})/G]) \xrightarrow{\sim} D^b(\mathrm{End}_G(\pi^*\mathcal{T}_G)).$$

Proof. First, we want to show that $\mathrm{Hom}_G(\pi^*\mathcal{T}_G, \pi^*\mathcal{T}_G[i]) = 0$ for $i \neq 0$. By adjointness of π^* and π_* (see [63], Proposition 3.3 for G-equivariant adjointness), we have

$$\mathrm{Hom}_G(\pi^*\mathcal{T}_G, \pi^*\mathcal{T}_G[i]) \simeq \mathrm{Hom}_G(\mathcal{T}_G, \pi_*\pi^*\mathcal{T}_G[i]).$$

Projection formula for G-equivariant morphisms now implies $\pi_*\pi^*\mathcal{T}_G \simeq \mathcal{T}_G \otimes S^\bullet(\mathcal{E})$ and hence

$$\mathrm{Hom}_G(\pi^*\mathcal{T}_G, \pi^*\mathcal{T}_G[i]) \simeq \mathrm{Hom}_G(\mathcal{T}_G, S^\bullet(\mathcal{E}) \otimes \mathcal{T}_G[i]).$$

Since X is a smooth projective and integral k-scheme and $S^\bullet(\mathcal{E})$ quasicoherent on X, Lemma 3.84 can be applied and we obtain

$$\mathrm{Hom}_G(\mathcal{T}_G, S^\bullet(\mathcal{E}) \otimes \mathcal{T}_G[i]) \simeq \mathrm{Hom}(\mathcal{T}_G, S^\bullet(\mathcal{E}) \otimes \mathcal{T}_G[i])^G.$$

Recall that $\mathcal{T}_G = \bigoplus_i \mathcal{T} \otimes W_i$, where W_i are the irreducible representations of G. Now for a fixed $l > 0$ one has for irreducible representations W_r and W_s canonical isomorphisms on X

$$\mathrm{Ext}^i(\mathcal{T} \otimes W_r, S^l(\mathcal{E}) \otimes \mathcal{T} \otimes W_s) \simeq \mathrm{Ext}^i(\mathcal{T}, S^l(\mathcal{E}) \otimes \mathcal{T}) \otimes \mathrm{Hom}(W_r, W_s).$$

By assumption $\mathrm{Ext}^i(\mathcal{T}, S^l(\mathcal{E}) \otimes \mathcal{T}) \simeq H^i(X, \mathcal{T}^\vee \otimes \mathcal{T} \otimes S^l(\mathcal{E})) = 0$ for all $i \neq 0$ and all $l > 0$, what therefore implies that

$$\mathrm{Hom}(\mathcal{T}_G, S^\bullet(\mathcal{E}) \otimes \mathcal{T}_G[i])^G = 0$$

for $i \neq 0$. Thus $\mathrm{Hom}_G(\pi^*\mathcal{T}_G, \pi^*\mathcal{T}_G[i]) = 0$ for $i \neq 0$. We now want to prove that $\mathcal{R} = \pi^*\mathcal{T}_G$ generates $D_G(\mathrm{Qcoh}(\mathrm{Tot}(\mathcal{E}))$. The argument is exactly the same as in Theorem 3.89. First note that the stack $[\mathrm{Tot}(\mathcal{E})/G]$ is a quasicompact and separated Deligne–Mumford stack with coarse moduli being the quotient scheme $\mathrm{Tot}(\mathcal{E})//G$. Thus [147], Corollary 4.2 implies that $D_G(\mathrm{Qcoh}(\mathrm{Tot}(\mathcal{E}))$ is compactly generated. The compact objects are all of $D^b_G(\mathrm{Tot}(\mathcal{E}))$ and hence it suffices to prove that $\mathcal{R}$ generates $D^b_G(\mathrm{Tot}(\mathcal{E}))$. So we take an object $\mathcal{F} \in D^b_G(\mathrm{Tot}(\mathcal{E}))$ and assume $\mathbb{R}\mathrm{Hom}_G(\pi^*\mathcal{T}_G, \mathcal{F}) = 0$. Adjoint property of π^* and π_* implies $\mathbb{R}\mathrm{Hom}_G(\mathcal{T}_G, \pi_*\mathcal{F}) = 0$. The same argument that shows the generating property in the proof of Theorem 3.89 now implies that $\pi_*\mathcal{F} = 0$. Since π is affine, we conclude $\mathcal{F} = 0$ and hence $\pi^*\mathcal{T}_G$ generates $D^b_G(\mathrm{Tot}(\mathcal{E}))$ and therefore it generates $D_G(\mathrm{Qcoh}(\mathrm{Tot}(\mathcal{E}))$. Now since $\mathrm{End}_G(\pi^*\mathcal{T}_G)$ is noetherian, the argument in the proof of Theorem 3.88 shows that the global dimension of $\mathrm{End}_G(\pi^*\mathcal{T}_G)$ is finite (notice that the noetherian property in the arguments of the proof of Theorem 3.88 is enough). This establishes the equivalence $D^b([\mathrm{Tot}(\mathcal{E})/G]) \xrightarrow{\sim} D^b(\mathrm{End}_G(\pi^*\mathcal{T}_G))$ and completes the proof. □

Note that for X being a Fano variety and $G = 1$ one obtains the result of Bridgeland and Stern and if $X = \mathrm{Spec}(\mathbb{C})$ the result of Brav. In both cases the assumption $H^i(X, \mathcal{T}^\vee \otimes \mathcal{T} \otimes S^l(\mathcal{E})) = 0$ for all $i \neq 0$ and all $l > 0$ can be shown to be fulfilled.
It is also very natural to consider projective bundles with group actions. A semiorthogonal decomposition for the equivariant derived category of the projective bundles was proved to exist by Elagin [63]. Below we will prove that if the base scheme X admits a G-equivariant tilting bundle, the stack $[\mathbb{P}(\mathcal{E})/G]$ admits a tilting object too. We start with some preliminary notations and observations. Let X be a smooth projective and integral k-scheme and G a finite group acting on X. Suppose the characteristic of k does not divide the order of G. Let $\mathcal{E}$ be a G-equivariant locally free sheaf of rank r on X. This provides us with a projective bundle $\mathbb{P}(\mathcal{E})$ on which G acts naturally. The structural morphism $\pi : \mathbb{P}(\mathcal{E}) \to X$ is a G-equivariant morphism and Elagin [63], Theorem 4.3 proved that one has a semiorthogonal decomposition

$$D^b_G(\mathbb{P}(\mathcal{E})) = \langle \pi^* D^b_G(X), \pi^* D^b_G(X) \otimes \mathcal{O}_{\mathcal{E}}(1), ..., \pi^* D^b_G(X) \otimes \mathcal{O}_{\mathcal{E}}(r-1) \rangle.$$

Suppose that $\mathcal{T}$ is a tilting bundle for $D^b(X)$ admitting a G-equivariant structure, then Theorem 3.89 yields that $\mathcal{T}_G = \bigoplus_j \mathcal{T} \otimes W_j$ is a tilting bundle for $D^b_G(X)$. In view of the above semiorthogonal decomposition, one easily verifies that $\mathcal{R} = \bigoplus_{i=0}^{r-1} \pi^*\mathcal{T}_G \otimes \mathcal{O}_{\mathcal{E}}(i)$ generates $D^b_G(\mathbb{P}(\mathcal{E}))$. We now obtain the following result.

Theorem 3.97. *Let X, G and $\mathcal{E}$ be as above and assume that $D^b(X)$ admits a tilting bundle $\mathcal{T}$ with G-equivariant structure. Then the quotient stack $[\mathbb{P}(\mathcal{E})/G]$ admits a tilting object.*

Proof. The proof is exactly the same as in Proposition 3.49 and Theorem 3.50 with the difference of taking G-equivariant cohomology. We investigate the object $\mathcal{R} = \bigoplus_{i=0}^{r-1} \pi^*\mathcal{T}_G \otimes \mathcal{O}_{\mathcal{E}}(i)$. G-equivariant adjuction of π^* and π_* and projection formula yields for $0 \leq r_1, r_2 \leq r-1$

$$\mathrm{Hom}_G(\pi^*\mathcal{T}_G \otimes \mathcal{O}_{\mathcal{E}}(r_1), \pi^*\mathcal{T}_G \otimes \mathcal{O}_{\mathcal{E}}(r_2)[m]) \simeq$$
$$\mathrm{Hom}_G(\mathcal{T}_G, \mathcal{T}_G \otimes \mathbb{R}\pi_*\mathcal{O}_{\mathcal{E}}(r_2 - r_1)[m]).$$

If $r_1 = r_2$ we have $\mathbb{R}\pi_*\mathcal{O}_{\mathcal{E}}(r_2 - r_1) \simeq \mathcal{O}_X$ and hence

$$\mathrm{Hom}_G(\pi^*\mathcal{T}_G \otimes \mathcal{O}_{\mathcal{E}}(r_1), \pi^*\mathcal{T}_G \otimes \mathcal{O}_{\mathcal{E}}(r_2)[m]) \simeq \mathrm{Ext}^m_G(\mathcal{T}_G, \mathcal{T}_G) = 0$$

for $m > 0$ since $\mathcal{T}_G$ is a tilting object for $D^b_G(X)$ according to Theorem 3.89. If $0 \leq r_2 < r_1 \leq r-1$ we have $r_2 - r_1 > -r$ and hence $\mathbb{R}\pi_*\mathcal{O}_{\mathcal{E}}(r_2 - r_1) = 0$ what implies

$$\mathrm{Hom}_G(\pi^*\mathcal{T}_G \otimes \mathcal{O}_{\mathcal{E}}(r_1), \pi^*\mathcal{T}_G \otimes \mathcal{O}_{\mathcal{E}}(r_2)[m]) \simeq \mathrm{Ext}^m_G(\mathcal{T}_G, 0) = 0$$

for all $m \geq 0$. It remains the case $0 \leq r_1 < r_2 \leq r-1$. In this case we have for $l = r_2 - r_1$, $\mathbb{R}\pi_*\mathcal{O}_{\mathcal{E}}(r_2 - r_1) \simeq S^l(\mathcal{E})$ and hence

$$\begin{aligned}\mathrm{Hom}_G(\pi^*\mathcal{T}_G \otimes \mathcal{O}_{\mathcal{E}}(r_1), \pi^*\mathcal{T}_G \otimes \mathcal{O}_{\mathcal{E}}(r_2)[m]) &\simeq \mathrm{Ext}^m_G(\mathcal{T}_G, \mathcal{T}_G \otimes S^l(\mathcal{E})) \\ &\simeq H^m(X, \mathcal{T}^\vee_G \otimes \mathcal{T}_G \otimes S^l(\mathcal{E}))^G.\end{aligned}$$

To achieve the vanishing of the above cohomology we proceed as in Theorem 3.50 and take a G-equivariant ample invertible sheaf $\mathcal{L}$ on X. Such an $\mathcal{L}$ always exist by the following argument: Take a projective embedding $X \to \mathbb{P}^N$ and construct a G-equivariant embedding $X \to \mathbb{P}^N \times \ldots \times \mathbb{P}^N$, where we take $\mathrm{ord}(G)$ copies of $\mathbb{P}^N$ and the embedding is obtained by permuting the factors in the product. Following this by a Segre embedding leads to a projective embedding $\iota : X \to \mathbb{P}^N \times \ldots \times \mathbb{P}^N \to \mathbb{P}^M$ that is compatible with a linear action of G on $\mathbb{P}^M$. Then take $\iota^*\mathcal{O}_{\mathbb{P}^M}(1)$ to be the G-equivariant ample invertible sheaf $\mathcal{L}$ on X. Since X is projective, there is for a fixed $l > 0$ an $n_l >> 0$ such that

$$\begin{aligned}H^m(X, \mathcal{T}^\vee_G \otimes \mathcal{T}_G \otimes S^l(\mathcal{E} \otimes \mathcal{L}^{\otimes n_l})) &\simeq H^m(X, \mathcal{T}^\vee_G \otimes \mathcal{T}_G \otimes S^l(\mathcal{E}) \otimes \mathcal{L}^{\otimes n_l \cdot l}) \\ &= 0\end{aligned}$$

for $m > 0$. Since $0 < l \leq r-1$, we have only finitely many $l > 0$ and we can choose $n > \max\{n_l | 0 < l \leq r-1\}$ so that for $\mathcal{L}^{\otimes n}$ we have

$$\begin{aligned}H^m(X, \mathcal{T}^\vee_G \otimes \mathcal{T}_G \otimes S^l(\mathcal{E} \otimes \mathcal{L}^{\otimes n})) &\simeq H^m(X, \mathcal{T}^\vee_G \otimes \mathcal{T}_G \otimes S^l(\mathcal{E}) \otimes \mathcal{L}^{\otimes n \cdot l}) \\ &= 0\end{aligned}$$

for $m > 0$ and all $0 < l \leq r-1$. This implies that $\mathcal{R}' = \bigoplus_{i=0}^{r-1} \pi^*\mathcal{T}_G \otimes \mathcal{O}_{\mathcal{E}'}(i)$ is a tilting object for $\mathbb{P}(\mathcal{E}')$ with $\mathcal{E}' = \mathcal{E} \otimes \mathcal{L}^{\otimes n}$. Note that the generating property of $\mathcal{R}'$ follows from the observations given directly before the present theorem and the finiteness of the global dimension of $\mathrm{End}_G(\mathcal{R}')$ is obtained in the proof of Theorem 3.88. Finally, since $\mathbb{P}(\mathcal{E}') \simeq \mathbb{P}(\mathcal{E})$ as G-schemes we conclude that $D^b_G(\mathbb{P}(\mathcal{E}))$ admits a tilting object. □

The above arguments should also work in the case of Grassmann bundles obtained from G-equivariant locally free sheaves and therefore the stack $[\mathrm{Grass}(l, \mathcal{E})/G]$ should have a tilting object if X admits a G-equivariant one. Before investigating the case where on X acts an arbitrary algebraic group, we first want to study more closely the endomorphism algebras of tilting objects occurring in the situation of Theorem 3.89. Considering X and G as in Theorem 3.89, we also want to look at the more special case where X admits a full strongly exceptional collection. As expected, in this case we find that the representation theory for $[X/G]$ is the G-invariant representation theory of X.

Proposition 3.98. *Let X be a smooth projective and integral k-scheme. Suppose G is a finite group acting on X such that the characteristic of k does not divide the order of G. Suppose $\mathcal{T}$ is a tilting bundle for $D^b(X)$, that furthermore is G-equivariant. Then $D^b_G(X)$ admits a tilting object $\mathcal{T}_G = \bigoplus_i \mathcal{T} \otimes W_i$ and one has*

$$(\bigoplus_{i=1}^{n} \mathrm{End}(\mathcal{T}) \otimes \mathrm{End}(W_i))^G \simeq \mathrm{End}_G(\mathcal{T}_G),$$

where n is the number of conjugacy classes of G and W_i the irreducible representations.

Proof. That $\mathcal{T}_G$ is a tilting object was proved in Theorem 3.89. Now for the endomorphism algebra of $\mathcal{T}_G$ one has

$$\mathrm{End}_G(\mathcal{T}_G) = \mathrm{End}_G(\bigoplus_i \mathcal{T} \otimes W_i) \simeq \bigoplus_i \mathrm{End}_G(\mathcal{T} \otimes W_i)$$

and therefore

$$\mathrm{End}_G(\mathcal{T}_G) \simeq (\bigoplus_{i=1}^{n} \mathrm{End}(\mathcal{T}) \otimes \mathrm{End}(W_i))^G.$$

Note that by Theorem 3.89 the endomorphism algebra $\mathrm{End}_G(\mathcal{T}_G)$ has finite global dimension. □

Remark 3.99. The endomorphism algebra $\mathrm{End}_G(\mathcal{T}_G)$ of Proposition 3.98 can be interpreted as a matrix algebra. For this, simply consider the k-algebra $A = \mathrm{End}(\mathcal{T}_G) = \bigoplus_{i=1}^{n} \mathrm{End}(\mathcal{T}) \otimes \mathrm{End}(W_i)$. One then has $A^G = \mathrm{End}_G(\mathcal{T}_G)$. But the k-algebra A can also be written as the following matrix algebra:

$$\begin{pmatrix} \mathrm{End}(\mathcal{T}) \otimes \mathrm{End}(W_1) & 0 & \dots & 0 \\ 0 & \mathrm{End}(\mathcal{T}) \otimes \mathrm{End}(W_2) & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & \mathrm{End}(\mathcal{T}) \otimes \mathrm{End}(W_n) \end{pmatrix}$$

Therefore, the derived category $D^b([X/G])$ can be understood via the representation theory of the invariants of the above matrix algebra.

Corollary 3.100. *Let k be algebraically closed and X a smooth projective and integral k-scheme. Suppose G is a finite group acting on X, such that the characteristic of k does not divide the order of G. Suppose furthermore that $D^b(X)$ admits a tilting bundle $\mathcal{T}$, that is G-equivariant. Then $D^b_G(X)$ admits a tilting bundle $\mathcal{T}_G$ and we have*

$$(\bigoplus_{i=1}^{n} \mathrm{End}(\mathcal{T}))^G \simeq \mathrm{End}_G(\mathcal{T}_G),$$

where n is the number of conjugacy classes of G.

Proof. Since k is algebraically closed, Schur's Lemma implies $\mathrm{End}(W_i) \simeq k$ Proposition 3.98 yields the assertion. □

Therefore, if k is algebraically closed, the derived category of the quotient stack $[X/G]$ can be understood via the representation theory of the invariants of the following matrix algebra:

$$\begin{pmatrix} \mathrm{End}(\mathcal{T}) & 0 & \dots & 0 \\ 0 & \mathrm{End}(\mathcal{T}) & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & \mathrm{End}(\mathcal{T}) \end{pmatrix},$$

Note that the endomorphism algebra $\mathrm{End}(\mathcal{T})$ occurs n times.

To state the next result, that is slightly stronger of that given by Elagin [64], Theorem 2.1, we first need some previous considerations. The next fact is contained in [16], Proposition 2.7

Proposition 3.101. *Let R and S be artinian k-algebras and M an R-S-bimodule finitely generated over k. If S is semisimple then*

$$\mathrm{gldim}\begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \max\{\mathrm{pdim}_R M + 1, \mathrm{gldim} R\}$$

An easy consequence of the above proposition is the following:

Corollary 3.102. *Suppose we are given finite dimensional semisimple k-algebras $S_1, ..., S_r$ with S_i-S_j-bimodules $M_{i,j}$ finitely generated over k. Then*

$$\mathrm{gldim}\begin{pmatrix} S_1 & M_{1,2} & \dots & M_{1,r} \\ 0 & S_2 & \dots & M_{2,r} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & S_r \end{pmatrix} < \infty$$

Proof. We prove the assertion by induction on r. For $r = 1$ this is clear, since a semisimple k-algebra has finite global dimension. Now assume the global dimension is finite for $r - 1$. Set R and S to be the matrices

$$R = \begin{pmatrix} S_1 & M_{1,2} & \dots & M_{1,r-1} \\ 0 & S_2 & \dots & M_{2,r-1} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & S_{r-1} \end{pmatrix} \text{ and } S = S_r$$

Furthermore, set M to be the R-S-bimodule

$$M = \begin{pmatrix} M_{1,r} \\ M_{2,r} \\ \vdots \\ M_{r-1,r} \end{pmatrix}$$

Since by induction hypothesis $\mathrm{gldim}(R)$ is finite, we have $\mathrm{pdim}_R(M)$ is finite. Then Proposition 3.101 yields

$$\mathrm{gldim}\begin{pmatrix} R & M \\ 0 & S \end{pmatrix} < \infty.$$

This completes the proof. □

Theorem 3.103. *Let k be algebraically closed and X a smooth projective and integral k-scheme. Suppose G is a finite group acting on X, such that the characteristic of k does not divide the order of G. Suppose furthermore that $D^b(X)$ admits a full strongly exceptional collection $\mathcal{E}_1, ..., \mathcal{E}_r$ such that all $\mathcal{E}_i$*

are G-equivariant. Then $D^b_G(X)$ admits a full strongly exceptional collection $\mathcal{E}'_1, ..., \mathcal{E}'_s$ too and one has

$$(\bigoplus_{k=1}^{n} \mathrm{End}(\bigoplus \mathcal{E}_i))^G \simeq \mathrm{End}_G(\bigoplus \mathcal{E}'_j),$$

where n is the number of conjugacy classes of G.

Proof. That under the above assumptions the derived category $D^b_G(X)$ admits a full exceptional collection was proved by Elagin [64], Theorem 2.1. To prove that it also admits a full strongly exceptional collection, we consider the collection $\mathcal{E}_1 \otimes W_1, \mathcal{E}_2 \otimes W_1, ..., \mathcal{E}_r \otimes W_1, \mathcal{E}_1 \otimes W_2, \mathcal{E}_2 \otimes W_2, ..., \mathcal{E}_r \otimes W_2, ..., \mathcal{E}_1 \otimes W_n, \mathcal{E}_2 \otimes W_n, ..., \mathcal{E}_r \otimes W_n$, where W_i are the irreducible representations and n is the number of conjugacy classes that is equal to the number of irreducible representations of G. We set $\mathcal{E}'_1 = \mathcal{E}_1 \otimes W_1, \mathcal{E}'_2 = \mathcal{E}_2 \otimes W_1, ..., \mathcal{E}'_s = \mathcal{E}_r \otimes W_n$, just by running through the above list. Recall that one has the canonical isomorphism

$$\mathrm{Ext}^l(\mathcal{E}_i \otimes W_p, \mathcal{E}_j \otimes W_q) \simeq \mathrm{Ext}^l(\mathcal{E}_i, \mathcal{E}_j) \otimes \mathrm{Hom}(W_p, W_q).$$

With this isomorphism and Lemma 3.84 one easily verifies that the collection $\mathcal{E}'_1, ..., \mathcal{E}'_s$ is a strongly exceptional collection. Note that $\mathrm{Hom}(W_p, W_p) \simeq k$ by Schur's lemma, since k is assumed to be algebraically closed. Since $\mathcal{E}_1, ..., \mathcal{E}_r$ is a full strongly exceptional collection for $D^b(X)$, the direct sum $\bigoplus_{i=1}^r \mathcal{E}_i$ is a tilting object that by assumption is G-equivariant. Hence Theorem 3.89 applies and yields that $\bigoplus_{i=1}^s \mathcal{E}'_i$ is a tilting object for $D^b_G(X)$. Thus the collection $\mathcal{E}'_1, ..., \mathcal{E}'_s$ generates $D^b_G(X)$ and we conclude that this collection is a full strongly exceptional collection. Corollary 3.100 now provides us with the isomorphism

$$(\bigoplus_{k=1}^{n} \mathrm{End}(\bigoplus \mathcal{E}_i))^G \simeq \mathrm{End}_G(\bigoplus \mathcal{E}'_j),$$

where n is the number of conjugacy classes. Note that the endomorphism algebra $A = \mathrm{End}_G(\bigoplus \mathcal{E}'_j)$ is a upper triangular matrix algebra with diagonal entries being $\mathrm{End}_G(\mathcal{E}_i \otimes W_p) \simeq (\mathrm{End}(\mathcal{E}_i) \otimes \mathrm{Hom}(W_p, W_p))^G \simeq (k \otimes_k k)^G \simeq k$. Thus Corollary 3.102 applies and gives us an alternative proof for the fact that $\mathrm{End}_G(\bigoplus_{i=1}^s \mathcal{E}'_i)$ has finite global dimension. This completes the proof. □

Remark 3.104. The endomorphism algebra $A = \mathrm{End}(\bigoplus \mathcal{E}'_j)$ from above is given by the following matrix algebra:

$$\begin{pmatrix} \mathrm{End}(\bigoplus \mathcal{E}_i) \otimes \mathrm{End}(W_1) & 0 & \dots & 0 \\ 0 & \mathrm{End}(\bigoplus \mathcal{E}_i) \otimes \mathrm{End}(W_2) & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & \mathrm{End}(\bigoplus \mathcal{E}_i) \otimes \mathrm{End}(W_n) \end{pmatrix}$$

By the chosen ordering of the $\mathcal{E}'_j$, we obtain a upper triangular matrix with diagonal entries being $\mathrm{End}(\mathcal{E}_i) \otimes \mathrm{Hom}_G(W_p, W_p)$. By the Lemma of Schur $\mathrm{Hom}(W_p, W_p) \simeq k$ and due to the fact that $\mathcal{E}_i$ is exceptional, we have $\mathrm{End}(\mathcal{E}_i) \simeq k$. Therefore, the k-algebra $\mathrm{End}(\bigoplus \mathcal{E}_i) \otimes \mathrm{End}(W_p) \simeq \mathrm{End}(\bigoplus \mathcal{E}_i)$ for $0 \leq p \leq n$ has the following form:

$$B = \mathrm{End}(\bigoplus \mathcal{E}_i) = \begin{pmatrix} k & \mathrm{Hom}(\mathcal{E}_2, \mathcal{E}_1) & \dots & \mathrm{Hom}(\mathcal{E}_r, \mathcal{E}_1) \\ 0 & k & \dots & \mathrm{Hom}(\mathcal{E}_r, \mathcal{E}_2) \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & k \end{pmatrix}$$

Understanding the representation theory of B would lead to an understanding of the representation theory of A^G and hence to an understanding of $D^b_G(X)$.

Example 3.105. The following examples are well-known but will be of interest in the next section, so we give them. Let $X = \mathbb{P}^n$ and $\mathcal{T} = \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(i)$ the tilting bundle obtained by Beilinson. Obviously, the invertible sheaves $\mathcal{O}_{\mathbb{P}^n}(i)$, for $0 \le i \le n$ are preserved by automorphisms of X. Take a finite subgroup G of $\mathrm{Aut}_k(X) \simeq \mathrm{PGL}_{n+1}(k)$. Then $\mathcal{T}$ is a tilting object for $D^b(X)$ that is G-equivariant. Theorem 3.89 applies and yields a tilting object for the stack $[\mathbb{P}^n/G]$ (see also [43], Theorem 3.2.1). Next, we consider the Grassmannian $\mathrm{Grass}_k(d,n)$ over an field k of characteristic zero. Theorem 2.34 states that for $2d \neq n$ the automorphisms of $\mathrm{Grass}_k(d,n)$ are exactly $\mathrm{PGL}_n(k)$ and the exceptional collection obtained by Kapranov [95] are preserved under the action of a group G induced by a homomorphism $G \to \mathrm{PGL}_n(k)$ (see [64], 3.4). If G is finite, Theorem 3.89 applies and we get a tilting object for the stack $[\mathrm{Grass}_k(d,n)/G]$.

We now want to investigate what happens if the group G is a non-finite algebraic group. Note that any finite-dimensional representation V of G has a composition series, also called Jordan–Hölder series (see [93], 2.14(5)). Explicitly, V has a composition series of subrepresentations

$$0 = W_0 \subset W_1 \subset \dots \subset W_n = V,$$

where all inclusions are strict and the W_i are a maximal subrepresentations of W_{i+1}. The composition factors W_{i+1}/W_i are irreducible representations and any two composition series are equivalent.

We now state a well-known simple, but for our considerations important result. Remind that we denoted by $D^b(\mathrm{rep}(G))$ the bounded derived category of finite-dimensional representations of G.

Proposition 3.106. *Suppose G is an algebraic group over k. Then the derived category $D^b(\mathrm{rep}(G))$ has a generating object V (in the sense of Definition 3.90) if and only if G has finitely many isomorphism classes of irreducible representations.*

Proof. Suppose G has finitely many irreducible representations $V_1, \dots, V_r$. Then take the direct sum $V = \bigoplus_{i=1}^r V_i$ as the generating object. Indeed, every finite dimensional representation can be obtained by this object since it can be constructed by iterated extensions of the irreducible ones by applying the Jordan–Hölder Theorem. Conversely, suppose that $D^b(\mathrm{rep}(G))$ has a generating object V. By definition, V is a complex of finite-dimensional representations

$$0 \longrightarrow V_{-n} \longrightarrow V_{-n+1} \longrightarrow \dots \longrightarrow V_0 \longrightarrow V_1 \longrightarrow \dots \longrightarrow V_m \longrightarrow 0.$$

Then take all the irreducible components occurring as composition factors in the Jordan–Hölder series of all V_i in the above complex. These are finitely many, since V is an object in $D^b(\mathrm{rep}(G))$ and the smallest triangulated subcategory closed under direct sums and summands cannot contain objects with other irreducible representations. Hence these irreducible ones generate $D^b(\mathrm{rep}(G))$. □

To give a consequence of Proposition 3.106, we first recall a fact about the dimension of derived categories (in the sense of Definition 3.90). It is the following lemma contained in [19], Lemma 2.5 (see also [137], Lemma 3.3)

Lemma 3.107. *Let T and R be two k-linear triangulated categories and $F : T \to R$ a triangulated functor with dense image, then* $\dim(T) \geq \dim(R)$. *In particular, if $\mathcal{A}$ generates T, then $F(\mathcal{A})$ generates R.*

The next result is [64], Lemma 2.4.

Lemma 3.108. *Let X and G be as above and $\mathcal{E}$ an exceptional object of $D^b(X)$, that is also G-equivariant. Then the functor $\mathcal{E} \otimes - : D^b(\mathrm{rep}(G)) \to D^b_G(X)$ is fully faithful and $\mathbb{R}\mathrm{Hom}(\mathcal{E}, -) : D^b_G(X) \to D^b(\mathrm{rep}(G))$ is the right adjoint.*

With these two lemmas we obtain the following result.

Theorem 3.109. *Let k be an algebraically closed field and X a smooth projective and integral k-scheme. Suppose G is an algebraic group acting on X. Suppose furthermore that $D^b(X)$ admits an exceptional object $\mathcal{E}$ that is G-equivariant and that G does not admit finitely many irreducible representations. Then $D^b_G(X)$ does not admit a tilting object.*

Proof. Suppose there is a tilting object $\mathcal{T}$ for $D^b_G(X)$. In particular $\mathcal{T}$ generates $D^b_G(X)$. Lemma 3.108 implies that the functor $\mathcal{E}\otimes - : D^b(\mathrm{rep}(G)) \to D^b_G(X)$ has the right adjoint $\mathbb{R}\mathrm{Hom}(\mathcal{E}, -)$. Thus, for every object $V \in D^b(\mathrm{rep}(G))$ we have $\mathbb{R}\mathrm{Hom}(\mathcal{E}, \mathcal{E}\otimes V) \simeq V$ and hence the functor $\mathbb{R}\mathrm{Hom}(\mathcal{E}, -) : D^b_G(X) \to D^b(\mathrm{rep}(G))$ has dense image. Lemma 3.107 yields that $\mathbb{R}\mathrm{Hom}(\mathcal{E}, \mathcal{T})$ generates $D^b(\mathrm{rep}(G))$. By Proposition 3.106 this implies that G has finitely many irreducible representations, contradicting the assumption. Hence $D^b_G(X)$ cannot have a tilting object. □

Example 3.110. Consider for instance $G = \mathrm{PGL}_2(\mathbb{C})$. We have seen that on $\mathbb{P}^1$ the exceptional collection $\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)$ is preserved by $\mathrm{PGL}_2(\mathbb{C})$. Hence the derived category $D^b(\mathbb{P}^1)$ has an exceptional object, say $\mathcal{O}_{\mathbb{P}^1}$, admitting a G-equivariant structure. Since $\mathrm{PGL}_2(\mathbb{C})$ does not have finitely many irreducible representations, we conclude with the above Theorem that $D^b_G(\mathbb{P}^1)$ cannot have a tilting object.

For instance the Deligne–Mumford stack $[(\mathrm{Spec}[x_0, x_1, x_2] \smallsetminus 0)/\mathbb{G}_m]$, obtained by the $\mathbb{G}_m$ action of weight $(1, 1, n)$, admits $\mathcal{T} = \bigoplus_{i=0}^{n+2} \mathcal{O}(i)$ as tilting bundle. So there are quotient stacks obtained by non-finite group actions that awfully well admit tilting objects. Although many quotient stacks $[X/G]$ cannot have tilting objects, their derived categories have semiorthogonal decompositions. We want to note here that Elagin [63] and [64] considered these cases and established semiorthogonal decompositions for schemes with actions of arbitrary algebraic groups. It is not clear to the author what kind of quotient stacks (or more generally "nice" Deligne–Mumford stacks) may possess tilting objects. One firstly should investigate for what kind of quotient stacks (Deligne–Mumford stacks) the derived categories of (quasi-) coherent sheaves admits a compact generator. But this is a rather delicate issue (see for instance [79] and [147] and references therein for discussions and results in this direction). In general, tilting objects for quotient stacks cannot exist. For instance the category $D^b([pt/G])$ has a generating object if the group has finitely many irreducible representations. For a list of groups such that $D(\mathrm{Qcoh}([pt/G]))$ admits compact generators we refer to [79] and references therein. Moreover, Hall and Rydh [79] proved the existence of compact generators for certain stacks including the class of quasicompact tame Deligne–Mumford stacks with affine diagonal. This includes a huge class of global quotient stacks and it would be of interest to investigate if the existence of a tilting object for $D(\mathrm{Qcoh}([X/G]))$ (resp. $D^b([X/G])$) would imply that G is finite (here X is supposed to be smooth projective and integral).

For arbitrary algebraic groups we have the following tilting correspondence:

Theorem 3.111. *Let X be a projective and integral k-scheme and G an algebraic group acting on X. Suppose we are given a compact generator $\mathcal{T}$ for the derived category $D_G(\mathrm{Qcoh}(X))$ and let $A = \mathbb{R}\mathrm{Hom}_G(\mathcal{T},\mathcal{T})$. Denote by $D^c_G(\mathrm{Qcoh}(X))$ the subcategory of compact objects. Then the following hold:*

(i) *The functor $\mathbb{R}\mathrm{Hom}_G(\mathcal{T},-) : D_G(\mathrm{Qcoh}(X)) \to D(\mathrm{Mod}(A))$ is an equivalence.*

(ii) *The equivalence of (i) restricts to an equivalence*

$$D^c_G(\mathrm{Qcoh}(X)) \xrightarrow{\sim} \mathrm{perf}(A).$$

(iii) *If the global dimension of A is finite then $\mathrm{perf}(A) \simeq D^b(A)$.*

Proof. The equivalence of (i) is just an application of [102], Theorem 8.5. To prove (ii), we note that the equivalence $\psi = \mathbb{R}\mathrm{Hom}_G(\mathcal{T},-)$ from (i) restricts to an equivalence of compact objects. This can be seen as follows: Since ψ is an equivalence it admits as a right adjoint the inverse ψ^{-1} (see [89], Proposition 1.26). Hence for compact object $\mathcal{C} \in D_G(\mathrm{Qcoh}(X))$ we have

$$\begin{aligned}\mathrm{Hom}_A(\psi(\mathcal{C}), \bigoplus_i \mathcal{F}_i) &\simeq \mathrm{Hom}_G(\mathcal{C}, \bigoplus_i \psi^{-1}(\mathcal{F}_i))\\ &\simeq \bigoplus_i \mathrm{Hom}_G(\mathcal{C},\mathcal{F}_i) \simeq \bigoplus_i \mathrm{Hom}_A(\psi(\mathcal{C}),\mathcal{F}_i).\end{aligned}$$

The compact objects of $D(\mathrm{Mod}(A))$ are all of $\mathrm{perf}(A)$ (see [102], Theorem 8.2). This establishes the equivalence

$$D^c_G(\mathrm{Qcoh}(X)) \xrightarrow{\sim} \mathrm{perf}(A).$$

Finally, if the global dimension of A is finite, one has $\mathrm{perf}(A) \simeq D^b(A)$ (see for instance [102], Theorem 8.6). This completes the proof. □

Remark 3.112. Note that in the above situation $\mathbb{R}\mathrm{Hom}_G(\mathcal{T},\mathcal{T})$ is a dg-algebra (see [101] for details on dg-categories). If one furthermore assumes $\mathrm{Hom}_G(\mathcal{T},\mathcal{T}[i]) = 0$ for $i \neq 0$, the dg-algebra becomes an algebra.

To get an equivalence $D^b_G(X) \xrightarrow{\sim} D^b(A)$ in the above situation, one has to prove that $D^c_G(\mathrm{Qcoh}(X))$ is all of $D^b_G(X)$ and that the global dimension of A is finite. Usually, the equivalence between $D^c_G(\mathrm{Qcoh}(X))$ and $D^b_G(X)$ is proved as follows: Firstly, one has to guarantee that the compact objects are the same as the bounded complexes of G-equivariant locally free sheaves of finite rank. But exactly this is in general not possible (see [79] for stacks where this holds). Secondly, provided X is smooth, one has to show that any coherent G-equivariant sheaf $\mathcal{F}$ admits a finite G-equivariant resolution of locally free sheaves of finite rank. This is usually proved by applying results of Thomason [146] that also do not hold for arbitrary algebraic groups. By the work of Totaro [148], resolutions of coherent sheaves by locally free ones somehow imply that the group has to be affine. So in general it is not clear to me how to get the equivalence between $D^c_G(\mathrm{Qcoh}(X))$ and $D^b_G(X)$ and furthermore, how to prove the finiteness of the global dimension of A. Note that for quotient stacks $[X/G]$ obtained by actions of affine algebraic groups G on smooth quasiprojective and integral k-schemes X, the derived category $D^c_G(\mathrm{Qcoh}(X))$ should be all of $D^b_G(X)$ (this should follow from [79] and [146]). It is sensible to believe that in this geometric situation tilting object should exist for certain quotient stacks. But there is also another problem we are faced with. There is a quite interesting phenomenon that can occur: Loosely speaking, Theorem 3.89 states that the quotient stack $[X/G]$, induced by an action of a finite linear reductive group, admits a tilting object if X admits one. The question arises, if it is possible that a quotient stack $[X/G]$ has a tilting object although the scheme X does not admit one. It turns out that this is possible (see Proposition 3.114 below). So it is not only the problem of how to establish an equivalence between $D^c_G(\mathrm{Qcoh}(X))$ and $D^b_G(X)$, but also that the quotient stack $[X/G]$ may admits a tilting object not coming from a tilting object of X. To prove Proposition 3.114 below and to give some more examples of schemes admitting a tilting object, we roughly recall the *derived McKay correspondence* and refer to the celebrated work of Bridgeland, King and Reid [46] for details.
For this, let k be an algebraically closed field of characteristic zero and X a quasiprojective k-scheme Furthermore, let G be a finite subgroup of $\mathrm{Aut}(X)$ acting on X. Note that the quotient scheme $X//G$ is usually singular. The main idea of McKay correspondence is to find a certain "nice" resolution of $X//G$ and to relate the geometry of the resolution to that of $X//G$. Recall, a resolution of singularities $\tilde{X} \to X$ of a given non-singular X is called *crepant* if $\omega_{\tilde{X}}$ is the pullback of ω_X. Whether such resolutions exist is a difficult question and closely related to the minimal model program. In the situation described above, the G-Hilbert scheme of X exists and we denote it by $\mathrm{Hilb}_G(X)$. This G-Hilbert scheme is a projective k-scheme. Very roughly, it classifies all quotients of G-equivariant sheaves under some technical assumptions. See [33] for a very detailed treatment of this topic. Now take $Y \subset \mathrm{Hilb}_G(X)$ to be the irreducible component containing the free orbits. Suppose that G acts on X such that ω_X is locally trivial as a G-equivariant sheaf and write $Z \subset X \times Y$ for the universal closed subscheme. Then there is a commutative diagram of schemes

$$\begin{array}{ccc} Z & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\tau} & X//G \end{array}$$

such that q an τ are birational and p and π finite. Moreover p is flat. Under the assumptions on X and G made above, one then has the *derived McKay-correspondence* proved by Bridgeland, King and Reid (see [46], Theorem 1.1):

Theorem 3.113. *Let X, G and Y be as above and suppose that ω_X is locally trivial as a G-equivariant sheaf. Suppose furthermore, that $\dim(Y \times_{X//G} Y) < \dim(X)+1$, then the functor $\mathbb{R}q_* \circ p^*$ is an equivalence*

$$\mathbb{R}q_* \circ p^* : D^b(Y) \xrightarrow{\sim} D^b_G(X)$$

and $\tau : Y \to X//G$ is a crepant resolution.

Moreover, if $\dim(X) \leq 3$, Bridgeland, King and Reid proved that $\mathrm{Hilb}_G(X)$ is irreducible and hence one has an equivalence $\mathbb{R}q_* \circ p^* : D^b(\mathrm{Hilb}_G(X)) \xrightarrow{\sim} D^b_G(X)$ and in this case $\mathrm{Hilb}_G(X) \to X//G$ is a crepant resolution (see [46], Theorem 1.2). As far as the author knows, there is no generalization of that result for arbitrary fields. Blume [33] generalized the classical McKay correspondence to arbitrary fields of characteristic zero (not necessarily algebraic closed) and gives a complete discussion of G-Hilbert schemes in this general situation.

We now give the following interesting observation.

Proposition 3.114. *Let C be an elliptic curve over an algebraically closed field of characteristic zero with $j \neq 0$ and $j \neq 1728$ and let $G = \{id, -id\} = \mathrm{Aut}(C)$ act on C. Furthermore, consider the induced action of $G' = G \times G$ on $C \times C$. Then both $D^b_G(C)$ and $D^b_{G'}(C \times C)$ admit a tilting object.*

Proof. By assumption on C and G, the quotient scheme $C//G$ exists and is isomorphic to $\mathbb{P}^1$. Hence $(C \times C)//G' \simeq (C//G) \times (C//G) \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Note that $\omega_C = \mathcal{O}_C$ and $\omega_{C \times C} = \mathcal{O}_{C \times C}$ and hence they are locally trivial as G-equivariant sheaf and G'-equivariant sheaf respectively. Since $\dim(C) = 1$, we get a crepant resolution $\mathrm{Hilb}_G(C) \to C//G$ and hence $\mathrm{Hilb}_G(C) \simeq \mathbb{P}^1$. By the discussion above, one has the derived McKay correspondence

$$D^b(\mathbb{P}^1) \xrightarrow{\sim} D^b_G(C)$$

and since $\mathbb{P}^1$ admits a tilting object, $D^b_G(C)$ admits a tilting object too. For $D^b_{G'}(C \times C)$ the argument is the same. Since $\dim(C \times C) = 2$, we get a crepant resolution $\mathrm{Hilb}_{G'}(C \times C) \to (C \times C)//G'$ and hence $\mathrm{Hilb}_{G'}(C \times C)$ is a rational surface. Again we have McKay correspondence

$$D^b(\mathrm{Hilb}_{G'}(C \times C)) \xrightarrow{\sim} D^b_G{}'(C \times C)$$

and since smooth projective rational surfaces admit tilting objects (see [87], Theorem 1.1) we conclude that $D^b_{G'}(C \times C)$ admits a tilting object too. This completes the proof. □

This proposition shows that $D^b_G(C)$ and $D^b_G(C \times C)$ admit a tilting object although C and $C \times C$ do not admit one. Following the idea of exploiting derived McKay correspondence to construct tilting objects, we give the following useful observation.

Theorem 3.115. *Let X, G and Y be as above and suppose that ω_X is locally trivial as a G-equivariant sheaf. Suppose furthermore, that* $\dim(Y \times_{X//G} Y) < \dim(X) + 1$ *and that $D^b(X)$ admits a tilting bundle $\mathcal{T}$ that has a G-equivariant structure. Then $D^b(Y)$ admits a tilting object.*

Proof. Since $\mathcal{T}$ is a titling bundle for $D^b(X)$ admitting a G-equivariant structure, we conclude with Theorem 3.89, that $D^b_G(X)$ admits a tilting object. Under the above assumptions on X and G and since $\dim(Y \times_{X//G} Y) < \dim(X) + 1$, derived McKay correspondence yields an equivalence

$$D^b(Y) \xrightarrow{\sim} D^b_G(X).$$

Since $D^b_G(X)$ admits a tilting object, $D^b(Y)$ admits one too . This completes the proof. □

We now give some applications of Theorem 3.115. For this, we consider the schemes given in Example 3.105 and apply the above theorem.

Corollary 3.116. *For $n \leq 3$, let $G \subset \mathrm{Aut}(\mathbb{P}^n)$ be a finite subgroup such that $\omega_{\mathbb{P}^n}$ is locally trivial as a G-equivariant sheaf. Then $D^b(\mathrm{Hilb}_G(\mathbb{P}^n))$ admits a tilting object.*

Proof. As mentioned in Example 3.105, the full strongly exceptional collection $\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), ..., \mathcal{O}_{\mathbb{P}^n}(n)$ is preserved under automorphisms of $\mathbb{P}^n$. Theorem 3.115 applies and yields, together with the discussion right after Theorem 3.113, that $D^b(\mathrm{Hilb}_G(\mathbb{P}^n))$ admits a tilting object. □

Remark 3.117. Let X be a quadric of dimension ≤ 3 and $G \subset \mathrm{Aut}(X)$ a finite subgroup such that ω_X is locally trivial as a G-equivariant sheaf. Then it is not clear if $D^b(\mathrm{Hilb}_G(X))$ admits a tilting object. As pointed out by Elagin [64], 3.2, the spinor bundles are in general not G-equivariant. This excludes the possibility to apply Theorem 3.113.

For del Pezzo surfaces S it is well-known that $D^b(S)$ admits a full strongly exceptional collection (see for instance [17], Theorem 2.5). Elagin [64], 3.3 discussed del Pezzo surfaces and showed that for del Pezzo's of degree $d \geq 8$, the full strongly exceptional collection is preserved by finite group actions induced from automorphisms. As a direct application of Theorem 3.115 we obtain the following

Corollary 3.118. *Let S be a del Pezzo surface of degree $d \geq 8$ and $G \subset \mathrm{Aut}(S)$ a finite group such that ω_S is locally trivial as a G-equivariant sheaf. Then $D^b(\mathrm{Hilb}_G(S))$ admits a tilting object.*

3.7 Application: Orlov's dimension conjecture

In this section we want to apply the results of the previous sections to provide some further evidence for the dimension conjecture of Orlov. We recall the main definition given in the last section (see Definition 3.90).

Definition 3.119. Let T be a triangulated category. The *dimension of T*, denoted by $\dim(T)$ is defined to be the minimal generation time among strong generators. It is set to be ∞ if there are no strong generators.

Orlov [130] investigated the dimension of triangulated categories coming from geometry and conjectured the following:

Conjecture. *If X is a smooth integral and separated scheme of finite type over k, then* $\dim D^b(X) = \dim(X)$.

Orlov [130] proved that the conjecture holds for smooth projective curves C of genus $g \geq 1$ and therefore showed that in this case $\dim D^b(C) = 1$. For curves of genus $g = 0$ this is clear and we give an argument for this fact below. Some more cases are known where the above conjecture is known to hold true.

- Rouquier [137] showed that the conjecture is true for affine scheme of finite type over k, certain flags and quadrics.
- Ballard and Favero [19] proved that the conjecture holds true for del Pezzo surfaces, certain Fano three-folds, Hirzebruch surfaces, toric surfaces with nef anti-canonical divisor and certain toric Deligne–Mumford stacks over $\mathbb{C}$.

There are also some bounds for the dimension of the bounded derived category of coherent sheaves of X. Rouquier [137], who originally introduced the notion of dimension of triangulated categories, proved the following results:

- Let X be a reduced separated scheme of finite type over k, then $\dim D^b(X) \geq \dim(X)$ (see [137], Proposition 7.17).
- Let X be a smooth quasiprojective k-scheme, then $\dim D^b(X) \leq 2\dim(X)$ (see [137], Proposition 7.9)

Remark 3.120. Note that the above conjecture also makes sense for certain Deligne–Mumford stacks over a field k (see [19]). This is due to the fact that for smooth and tame Deligne–Mumford stacks $\mathcal{X}$ with coarse moduli space being quasiprojective one has the same bound for $\dim D^b(\mathcal{X})$ as in the case of schemes above. Concretely, $\dim D^b(\mathcal{X}) \leq 2\dim(\mathcal{X})$ (see [19], Lemma 2.20). In the same manner as above for reduced separated schemes, for tame Deligne–Mumford stack with coarse moduli being reduced and separated one has $\dim D^b(\mathcal{X}) \geq \dim(\mathcal{X})$ (see [19], Lemma 2.17).

In view of Remark 3.120 we follow Ballard and Favero [19] and extend the above conjecture of Orlov and formulate:

Conjecture. *Let $\mathcal{X}$ be a smooth and tame Deligne–Mumford stack of finite type over k with quasiprojective coarse moduli space, then* $\dim D^b(\mathcal{X}) = \dim(\mathcal{X})$.

We want to apply the Theorems 3.92 and 3.93 of Ballard and Favero to produce some more examples where the above conjectures hold true. We start with the quotient stack $[X/G]$ obtained by an action of a finite linearly reductive group (i.e. characteristic of k does not divide the order of G).

Proposition 3.121. *Let k be a perfect field and X and G as in Theorem 3.89. Suppose that $\mathcal{T}$ is a tilting sheaf for $D^b(X)$ admitting a G-equivariant structure. Suppose furthermore that* $\mathrm{Ext}^i(\mathcal{T}, \mathcal{T} \otimes \omega_X^\vee) = 0$ *for $i > 0$. Then the quotient stack $[X/G]$ satisfies* $\dim([X/G]) = \dim D^b_G(X)$.

Proof. According to Theorem 3.89, $\mathcal{T}_G = \bigoplus_i \mathcal{T} \otimes W_i$ with W_i being the irreducible representations of G, is a tilting object for $D^b_G(X)$. As mentioned in Example 3.82, $[X/G]$ is a smooth, proper, tame and connected Deligne–Mumford stack with projective coarse moduli and hence by Theorem 3.92 we only have to determine the largest i for which

$$\mathrm{Hom}_G(\mathcal{T}_G, \mathcal{T}_G \otimes \omega^\vee_{[X/G]}[i]) \neq 0.$$

Since ω_X has a G-equivariant structure and as a sheaf on X gives rise to the Serre functor $- \otimes \omega_X[\dim(X)]$, we conclude by [19], Lemma 2.26 f. that $\omega_{[X/G]} \simeq \omega_X$ on $[X/G]$, since the Serre functor is unique up to isomorphism (see [89], p.10). Therefore we have

$$\mathrm{Hom}_G(\mathcal{T}_G, \mathcal{T}_G \otimes \omega^\vee_{[X/G]}[i]) \simeq \mathrm{Hom}(\mathcal{T}_G, \mathcal{T}_G \otimes \omega^\vee_X[i])^G$$

by Lemma 3.84. Since we are dealing with coherent sheaves, we get

$$\mathrm{Ext}^i(\mathcal{T}_G, \mathcal{T}_G \otimes \omega^\vee_X) \simeq (\textstyle\bigoplus_j \mathrm{Ext}^i(\mathcal{T}, \mathcal{T} \otimes \omega^\vee_X) \otimes \mathrm{Hom}(W_j, W_j)).$$

But by assumption we have

$$\mathrm{Ext}^i(\mathcal{T}, \mathcal{T} \otimes \omega^\vee_X) = 0 \text{ for } i > 0.$$

Now this implies

$$\mathrm{Hom}_G(\mathcal{T}_G, \mathcal{T}_G \otimes \omega^\vee_{[X/G]}[i]) = 0 \text{ for } i > 0$$

and hence Theorem 3.92 yields $\dim([X/G]) = \dim D^b_G(X)$. □

Corollary 3.122. *Let k be a perfect field and G a finite subgroup of $\mathrm{PGL}_{n+1}(k)$ acting on $\mathbb{P}^n$ such that the characteristic of k does not divide the order of G. Then $\dim([\mathbb{P}^n/G]) = \dim D^b([\mathbb{P}^n/G])$.*

Proof. First of all recall that $\mathcal{T} = \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(i)$ is a tilting bundle for $\mathbb{P}^n$. Since $\mathcal{O}_{\mathbb{P}^n}(i)$ are preserved by automorphisms, the tilting bundle $\mathcal{T}$ is G-equivariant. Furthermore, we have

$$\mathrm{Ext}^i(\mathcal{T}, \mathcal{T} \otimes \mathcal{O}_{\mathbb{P}^n}(n+1)) = 0 \text{ for } i > 0.$$

The rest follows from Proposition 3.121. □

Corollary 3.123. *Let k be an algebraically closed field of characteristic zero and G a finite group. Suppose $2d \neq n$ and that we are given an action of G on $X = \mathrm{Grass}(d,n)$ induced by a homomorphism $G \to \mathrm{PGL}_n(k)$. Then $\dim([X/G]) = \dim D^b([X/G])$.*

Proof. We mentioned in Example 3.105 that under the above assumptions Kapranov's tilting bundle $\mathcal{T} = \bigoplus_\lambda \Sigma^\lambda(\mathcal{S})$, where $\mathcal{S}$ is the tautological sheaf of X, is G-equivariant. Note that $\bigwedge^d(\mathcal{S}) \simeq \mathcal{O}_X(-1)$ and $\omega_X = \mathcal{O}_X(-n)$. To apply Proposition 3.121, we just have to verify $\mathrm{Ext}^i(\mathcal{T}, \mathcal{T} \otimes \mathcal{O}_X(n)) = 0$ for $i > 0$. By the construction of $\mathcal{T}$, it is enough to verify

$$\mathrm{Ext}^i(\Sigma^\lambda(\mathcal{S}), \Sigma^\mu(\mathcal{S}) \otimes \mathcal{O}_X(n)) \simeq H^i(X, \Sigma^\lambda(\mathcal{S}^\vee) \otimes \Sigma^\mu(\mathcal{S}) \otimes \mathcal{O}_X(n)) = 0$$

for $i > 0$. By the Littlewood–Richardson rule the partitions γ of irreducible summands $\Sigma^\gamma(\mathcal{S})$ of $\mathcal{H}om(\Sigma^\lambda(\mathcal{S}), \Sigma^\mu(\mathcal{S})) \simeq \Sigma^\lambda(\mathcal{S}^\vee) \otimes \Sigma^\mu(\mathcal{S})$ satisfy $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_d \geq -(n-d)$ (see [97], 3.3). So we can restrict ourselves to show

$$H^i(X, \Sigma^\gamma(\mathcal{S}) \otimes \mathcal{O}_X(n)) = 0 \text{ for } i > 0.$$

Since $\Sigma^\gamma(\mathcal{S}) \otimes \mathcal{O}_X(n) \simeq \Sigma^{\gamma+n}(\mathcal{S}) \simeq \Sigma^{-\gamma-n}(\mathcal{S}^\vee)$, where $\gamma + n = (\gamma_1 + n, ..., \gamma_d + n)$, we have $\gamma_1 + n \geq \gamma_2 + n \geq ... \geq \gamma_d + n \geq d$ and by Kapranov's calculation of the cohomology of $\Sigma^\alpha(\mathcal{S}^\vee)$ (see [95] or [97]) we conclude:

$$H^i(X, \Sigma^{\gamma+n}(\mathcal{S})) = 0 \text{ for } i > 0.$$

Now Proposition 3.121 applies and we get the desired equality $\dim([X/G]) = \dim D^b([X/G])$. □

As an further application of Proposition 3.121 we obtain the next result.

Proposition 3.124. *Let k, X, G and $Y \subset \mathrm{Hilb}_G(X)$ be as in Theorem 3.115 such that ω_X is locally trivial as a G-equivariant sheaf. Suppose that $D^b(X)$ has a tilting sheaf $\mathcal{T}$ admitting a G-equivariant structure. Furthermore, suppose that* $\dim(Y \times_{X//G} Y) < \dim(X) + 1$ *and that* $\mathrm{Ext}^i(\mathcal{T}, \mathcal{T} \otimes \omega_X^\vee) = 0$ *for $i > 0$. Then one has* $\dim(Y) = \dim D^b(Y)$.

Proof. Note that by assumption k is an algebraically closed field of characteristic zero and hence perfect. Under the above assumptions Proposition 3.121 applies and we have $\dim([X/G]) = \dim D^b_G(X)$. Furthermore, derived McKay correspondence applies and provides us with a birational morphism $Y \to X//G$ and the McKay equivalence $D^b(Y) \xrightarrow{\sim} D^b_G(X)$. Since $\dim(Y) = \dim(X//G) = \dim(X) = \dim([X/G]) = \dim D^b_G(X)$, we conclude with the McKay equivalence that $\dim(Y) = \dim D^b(Y)$. This completes the proof. □

Next we cite a result due to Sosna [144], Proposition 5.4

Proposition 3.125. *Let X be a smooth projective and integral k-scheme and suppose $k \subset L$ is a finite Galois extension. If* $\dim D^b(X)$ *is finite, then one has* $\dim D^b(X) = \dim D^b(X \otimes_k L)$.

Rouquier [137], Example 7.7 showed that the projective space $\mathbb{P}^n$ satisfies $\dim D^b(\mathbb{P}^n) = n$. With this fact we can proof the following:

Corollary 3.126. *Let X be a n-dimensional Brauer–Severi variety. Then* $\dim D^b(X) = n$.

Proof. In the first chapter we have seen that there is always a finite Galois extension $k \subset L$ such that $X \otimes_k L \simeq \mathbb{P}^n$. Note that since X is a smooth projective and integral k-scheme, $\dim D^b(X) \leq 2n$ according to [137], Proposition 7.9 and hence $\dim D^b(X)$ is finite. Now Proposition 3.125 applies and yields $\dim D^b(X) = \dim D^b(\mathbb{P}^n) = n$ and hence $\dim D^b(X) = n$. □

With the last corollary one immediately gets the following well-known fact.

Corollary 3.127. *Let C be a smooth projective and integral curve of genus $g = 0$. Then* $\dim D^b(C) = 1$.

Proof. This is clear since C is a one-dimensional Brauer–Severi variety. □

For the Grassmannian $\mathrm{Grass}(d,n)$ over an algebraically closed field of characteristic zero one has a resolution of the structure sheaf of the diagonal $\Delta \subset \mathrm{Grass}(d,n) \times_k \mathrm{Grass}(d,n)$ obtained as a Koszul complex (see [95]):

$$0 \longrightarrow \bigwedge^{d(n-d)}(\mathcal{S} \boxtimes \mathcal{Q}^\vee) \longrightarrow \ldots \longrightarrow \bigwedge^{2}(\mathcal{S} \boxtimes \mathcal{Q}^\vee) \longrightarrow \mathcal{S} \boxtimes \mathcal{Q}^\vee \longrightarrow \mathcal{O}_\Delta$$

Now by [137], Proposition 7.6 this implies $\mathrm{dim} D^b(\mathrm{Grass}(d,n)) \le d(n-d) = \mathrm{dim}(\mathrm{Grass}(d,n))$. Since $\mathrm{dim}(\mathrm{Grass}(d,n)) \le \mathrm{dim} D^b(\mathrm{Grass}(d,n))$ (see p.138) we have $\mathrm{dim}(\mathrm{Grass}(d,n)) = \mathrm{dim} D^b(\mathrm{Grass}(d,n))$. This of course is well-known but implies the following:

Corollary 3.128. *Let k be a field of characteristic zero, A a degree n central simple k-algebra and $\mathrm{BS}(d,A)$ a generalized Brauer–Severi variety. Then $\mathrm{dim}(\mathrm{BS}(d,A)) = \mathrm{dim} D^b(\mathrm{BS}(d,A))$.*

Proof. In the first chapter we have seen that there is always a finite Galois extension $k \subset L$ such that $\mathrm{BS}(d,A) \otimes_k L \simeq \mathrm{Grass}(d,n)$. Note that since $\mathrm{BS}(d,A)$ is a smooth projective and integral k-scheme, $\mathrm{dim} D^b(\mathrm{BS}(d,A)) \le 2\mathrm{dim}(\mathrm{BS}(d,A))$ according to [137], Proposition 7.9 and hence $\mathrm{dim} D^b(\mathrm{BS}(d,A))$ is finite. Now Proposition 3.125 applies and yields $\mathrm{dim} D^b(\mathrm{BS}(d,A)) = \mathrm{dim} D^b(\mathrm{Grass}(d,n)) = \mathrm{dim}(\mathrm{Grass}(d,n)) = \mathrm{dim}(\mathrm{BS}(d,A))$. □

Corollary 3.122 and Proposition 3.124 have the following consequence.

Corollary 3.129. *Let $n \le 3$ and $G \subset \mathrm{Aut}(\mathbb{P}^n)$ be a finite subgroup such that $\omega_{\mathbb{P}^n}$ is locally trivial as a G-equivariant sheaf, then $\mathrm{dim} D^b(\mathrm{Hilb}_G(\mathbb{P}^n)) = \mathrm{dim}(\mathrm{Hilb}_G(\mathbb{P}^n))$*

Remark 3.130. If one omits the restriction on n being ≤ 3, we get $\mathrm{dim}(Y) = \mathrm{dim} D^b(Y)$ for the crepant resolution Y of $\mathbb{P}^n//G$.

We continue investigating products of schemes. One has the following simple but useful observation.

Proposition 3.131. *Let k be a perfect field and X and Y smooth projective and integral k-schemes admitting tilting bundles $\mathcal{T}_X$ and $\mathcal{T}_Y$ respectively. Suppose that $\mathrm{Ext}^i(\mathcal{T}_X, \mathcal{T}_X \otimes \omega_X^\vee) = 0$ and $\mathrm{Ext}^i(\mathcal{T}_Y, \mathcal{T}_Y \otimes \omega_Y^\vee) = 0$ for $i > 0$. Then one has $\mathrm{dim}(X \times Y) = \mathrm{dim} D^b(X \times Y)$.*

Proof. We apply [19], Corollary 3.5. By assumption we have

$$\mathrm{Ext}^i(\mathcal{T}_X, \mathcal{T}_X \otimes \omega_X^\vee) = 0 \quad for \quad i > 0 \tag{3.9}$$

$$\mathrm{Ext}^i(\mathcal{T}_Y, \mathcal{T}_Y \otimes \omega_Y^\vee) = 0 \quad for \quad i > 0. \tag{3.10}$$

By Proposition 3.27 the locally free sheaf $\mathcal{T}_X \boxtimes \mathcal{T}_Y$ is a tilting bundle for $X \times Y$. The canonical bundle on $X \times Y$ is given by $\omega_{X\times Y} = \omega_X \boxtimes \omega_Y$ and by the Künneth formula we have

$$\mathrm{Ext}^i(\mathcal{T}_X \boxtimes \mathcal{T}_Y, \mathcal{T}_X \boxtimes \mathcal{T}_Y \otimes \omega_{X\times Y}^\vee) \simeq \bigoplus_{p+q=i} \mathrm{Ext}^p(\mathcal{T}_X, \mathcal{T}_X \otimes \omega_X^\vee) \otimes \mathrm{Ext}^q(\mathcal{T}_Y, \mathcal{T}_Y \otimes \omega_Y^\vee).$$

Finally (3.9) and (3.10) imply

$$\mathrm{Ext}^i(\mathcal{T}_X \boxtimes \mathcal{T}_Y, \mathcal{T}_X \boxtimes \mathcal{T}_Y \otimes \omega_{X\times Y}^\vee) = 0 \text{ for } i > 0$$

and hence [19], Corollary 3.5 yields $\dim(X \times Y) = \dim D^b(X \times Y)$. □

Corollary 3.132. *Let $X_1 \times ... X_r$ is the product of Brauer–Severi varieties over a perfect field k. Then* $\dim(X_1 \times ... \times X_r) = \dim D^b(X_1 \times ... \times X_r)$.

Proof. According to Theorem 3.36 a n-dimensional Brauer–Severi variety X has a tilting bundle $\mathcal{T} = \bigoplus_{i=0}^n \mathcal{W}_i$, where $\mathcal{W}_i$ are the locally free sheaves of Definition 1.39. By Proposition 1.40 we have $\mathcal{W}_i \otimes_k \bar{k} \simeq \mathcal{O}_{\mathbb{P}^n}(i)^{\oplus d_i}$, where $d_i = \mathrm{rk}(\mathcal{W}_i)$. Note that $\omega_X^\vee \simeq \mathcal{O}_X(n+1)$. By flat base change we obtain

$$\mathrm{Ext}^l(\mathcal{T}, \mathcal{T} \otimes \mathcal{O}_X(n+1)) \otimes_k \bar{k} \simeq$$
$$\mathrm{Ext}^l(\bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(i)^{\oplus d_i}, (\bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(i)^{\oplus d_i}) \otimes \mathcal{O}_{\mathbb{P}^n}(n+1)) = 0$$

for $l > 0$ as we have seen in the proof of Corollary 3.122. This implies

$$\mathrm{Ext}^l(\mathcal{T}, \mathcal{T} \otimes \mathcal{O}_X(n+1)) = 0 \text{ for } l > 0.$$

Proposition 3.131 yields the equality $\dim(X_1 \times ... \times X_r) = \dim D^b(X_1 \times ... \times X_r)$. □

Before we give some further example by applying Proposition 3.131, we first return to the generalized Brauer–Severi varieties $\mathrm{BS}(d, A)$ over a field k of characteristic zero. In Section 3 we have seen that such generalized Brauer–Severi varieties have tilting bundles obtained by descent of locally free sheaves of the form $\bigwedge^\alpha(\mathcal{S})^{\oplus n_\alpha}$ on $\mathrm{Grass}(d, n)$. In characteristic zero one has Kapranov's tilting bundle on $\mathrm{Grass}(d, n)$, namely $\mathcal{T} = \bigoplus_\lambda \Sigma^\lambda(\mathcal{S})$. As mentioned in Chapter 2, p.69, the locally free sheaves $\Sigma^\lambda(\mathcal{S})^{\oplus n \cdot |\lambda|}$ descent to locally free sheaves $\mathcal{N}_\lambda$ on $\mathrm{BS}(d, A)$. These locally free sheaves $\mathcal{N}_\lambda$ are unique up to isomorphism according to Proposition 2.5. Applying the same arguments as in the proof of Theorem 3.44 one can show that $\mathcal{R} = \bigoplus_\lambda \mathcal{N}_\lambda$ is a tilting bundle for $\mathrm{BS}(d, A)$ in characteristic zero. Considering $\mathrm{Ext}^i(\mathcal{R}, \mathcal{R} \otimes \omega_{\mathrm{BS}(d,A)}^\vee)$, we obtain after field extension to some splitting field L

$$\mathrm{Ext}^i(\mathcal{R}, \mathcal{R} \otimes \omega_{\mathrm{BS}(d,A)}^\vee) \otimes_k L \simeq$$
$$\mathrm{Ext}^i(\bigoplus_\lambda \Sigma^\lambda(\mathcal{S})^{\oplus n \cdot |\lambda|}, (\bigoplus_\lambda \Sigma^\lambda(\mathcal{S})^{\oplus n \cdot |\lambda|}) \otimes \mathcal{O}_{\mathrm{Grass}(d,n)}(n+1)) = 0$$

for $i > 0$ (see proof of Corollary 1.123). This observation has the following consequence.

Corollary 3.133. *Let $X_1 \times ... \times X_r$ be the finite product of generalized Brauer–Severi varieties over a field k of characteristic zero. Then* $\dim(X_1 \times ... \times X_r) = \dim D^b(X_1 \times ... \times X_r)$.

Proof. As we have seen above $\mathcal{R}$ is a tilting bundle for a generalized Brauer–Severi variety $\mathrm{BS}(d, A)$ and $\mathrm{Ext}^i(\mathcal{R}, \mathcal{R} \otimes \omega_{\mathrm{BS}(d,A)}^\vee) = 0$ for $i > 0$. Now Proposition 1.131 yields the assertion. □

Ballard and Favero [19] investigated the projective bundle $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^n$ with locally free sheaf $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-m)$ for $m > 0$. They proved that the dimension conjecture holds for this projective bundle (see [19], Proposition 3.37). Since $\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^n}(m))$, the dimension conjecture holds also for $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(m))$ with $m > 0$. The classification of toric Fano 3-folds and toric Fano 4-folds by

Batyrev [21] gives us some examples of schemes where the dimension conjecture holds. The classification list of toric Fano 3-folds (resp. 4-folds) provides us with the following toric Fano 3-folds (resp. 4-folds) for that the dimension conjecture holds (see [21], p.19 (resp. p.39 f.)) for a list of all toric Fano 3-folds (resp. 4-folds)):

Corollary 3.134. *For the following toric Fano 3- and 4-folds the dimension conjecture holds.*

- **(i)** *projective bundle* $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(i)) \to \mathbb{P}^2$ *for* $i = 1, 2$
- **(ii)** $\mathbb{P}^2 \times \mathbb{P}^1$ *and* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
- **(iii)** $\mathbb{P}^4$ *and projective bundles* $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(i)) \to \mathbb{P}^3$ *for* $i = 1, 2, 3$
- **(iv)** $\mathbb{P}^1 \times \mathbb{P}^3$ *and* $\mathbb{P}^2 \times \mathbb{P}^2$

Proof. This is a direct application of [19], Proposition 3.37 for all the projective bundles (see above discussion) and of Corollary 3.132 for the products. □

Corollary 3.135. *Let* $\pi : \mathrm{Bl}_x(\mathbb{P}^n) \to \mathbb{P}^n$ *be the blow up of* $\mathbb{P}^n$ *at a closed point* x. *Then* $\dim(\mathrm{Bl}_x(\mathbb{P}^n)) = \dim D^b(\mathrm{Bl}_\wedge(\mathbb{P}^n))$.

Proof. This is a direct application of [19] Proposition 3.37 since for linear subspaces $\wedge \subset \mathbb{P}^n$ of dimension k with $0 \leq k \leq n-2$, $\mathrm{Bl}_\wedge(\mathbb{P}^n)$ is nothing but the projective bundle $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^{n-k-1}$ with the locally free sheaf $\mathcal{E} = \mathcal{O}(1)^{\oplus k+1} \oplus \mathcal{O}(2)$. For $k = 0$ we therefore have that $\mathrm{Bl}_x(\mathbb{P}^n)$ is the projective bundle of $\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(2)$. But $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(2))$ is isomorphic to $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$ and hence the dimension conjecture holds. □

We collect the results of this section in the following theorem.

Theorem 3.136. *The dimension conjecture holds in the following cases:*

- **(i)** *Brauer–Severi varieties over arbitrary fields* k.
- **(ii)** *generalized Brauer–Severi varieties over fields of characteristic zero.*
- **(iii)** *finite products of the schemes in (i) and (ii) over perfect fields.*
- **(iv)** *toric Fano 3-and 4-folds obtained as projective bundles of rank two locally free sheaves or as the product of projective spaces.*
- **(v)** *quotient stacks of the form* $[\mathbb{P}^n/G]$ *where* G *is a finite linearly reductive group.*
- **(vi)** *quotient stacks of the form* $[\mathrm{Grass}(d,n)/G]$, *provided* $2d \neq n$ *and the action of the finite group* G *is induced by a homomorphism* $G \to \mathrm{PGL}_n(k)$.
- **(vii)** G*-Hilbert schemes* $\mathrm{Hilb}_G(\mathbb{P}^n)$ *for* $n \leq 3$ *and* G *a finite subgroup of* $\mathrm{PGL}_{n+1}(k)$ *such that* $\omega_{\mathbb{P}^n}$ *is locally trivial as a* G*-equivariant sheaf.*

Chapter 4

Further prospects: Amitsur conjecture

4.1 The Amitsur conjecture

As mentioned in Chapter 3, p.88 the work of Bridgeland [48], Kuznetsov [110], Orlov [129] and others suggests that the derived categories of birational schemes should be related in the presence of semiorthogonal decompositions. Having this in mind, one can try to study birational Brauer–Severi varieties in terms of their semiorthogonal decompositions. The main motivation for this is to tackle the Amitsur conjecture for Brauer–Severi varieties from a new point of view. In the hole chapter we want to collect some ideas how to attack this conjecture involving the derived geometry.

We start with recalling the Amitsur conjecture. As mentioned in Chapter 1 Amitsur [5] proved the following (see also [71], Theorem 5.4.1):

Theorem 4.1. *Let X be a Brauer–Severi variety and A the corresponding central simple k-algebra. Denote by $F(X)$ the function field of X. Then the kernel of the restriction map $\mathrm{Br}(k) \to \mathrm{Br}(F(X))$ is a cyclic group generated by A.*

This theorem immediately has the following consequence.

Corollary 4.2. *Let X and Y be Brauer–Severi varieties and A and B the corresponding central simple k-algebras. If X and Y are birational, then A and B generated the same subgroup in $\mathrm{Br}(k)$.*

Proof. Since X and Y are birational we have $F(X) \simeq F(Y)$. The rest follows directly from the last theorem. □

Amitsur [5] asked if the converse of the corollary holds and formulated the following conjecture, referred to as *the Amitsur conjecture* for Brauer–Severi varieties:

Conjecture. *Let X and Y be two Brauer–Severi varieties of same dimension and A and B the corresponding central simple k-algebras. If A and B generate the same cyclic subgroup of $\mathrm{Br}(k)$, then X is birational to Y.*

Note that a weaker result is quite easy to prove. Recall, X and Y are called *stably birational* if $X \times_k \mathbb{P}^n$ is birational to $Y \times_k \mathbb{P}^n$. Now if A and B generate the same cyclic subgroup of $\mathrm{Br}(k)$, then $A \otimes_k F(Y)$ and $B \otimes_k F(Y)$ generate the same subgroup in $\mathrm{Br}(F(Y))$. But $B \otimes_k F(Y)$ corresponds to the Brauer–Severi variety $Y \otimes_k F(Y)$ that has a $F(Y)$-rational point coming from the generic point. Hence $Y \otimes_k F(Y) \simeq \mathbb{P}^n$ and the subgroup generated by $B \otimes_k F(Y)$ is trivial. Since $A \otimes_k F(Y)$ generates the same subgroup as $B \otimes_k F(Y)$ it has to be trivial too. But this implies $X \otimes_k F(Y) \simeq \mathbb{P}^n \otimes_k F(Y)$. In particular both schemes have the same function field $F(X \times_k Y)$. Thus $X \times_k Y$ is birational to $\mathbb{P}^n \times_k Y$ and by changing the role of X and Y we obtain (see [71], Remark 5.4.3):

Proposition 4.3. *Let X and Y be two Brauer–Severi varieties of same dimension and A and B the corresponding central simple k-algebras. If A and B generate the same cyclic subgroup of* $\mathrm{Br}(k)$*, then X is stably birational to Y.*

This weaker result shows that the above conjecture is plausible. The Amitsur conjecture is still open in general and seems fascinating as it involves connections between the algebraic structure of A and the geometry of X. In some special cases the conjecture is known to be true. These special cases are the following:

- The Amitsure conjecture holds if the Brauer–Severi variety X has a cyclic splitting field. This was proved by Amitsur [5] and is always true if k is a local or global field (see Theorem 1.61 above)
- Let A be a central simple k-algebra and X the corresponding Brauer–Severi variety. The conjecture holds if $\mathrm{ind}(A) < \deg(A)$ as proved by Roquette [136].
- Let A and B be central simple k-algebras and X and Y the corresponding Brauer–Severi varieties. The conjecture holds if $[A] = -[B]$ in $\mathrm{Br}(k)$. This was also proved by Roquette [136].
- Let A and B be central simple k-algebras and X and Y the corresponding Brauer–Severi varieties. The conjecture holds if $[A] = 2[B]$ in $\mathrm{Br}(k)$ as proved by Tregub [149].
- Let A be a central simple k-algebra and X the corresponding Brauer–Severi variety. If $\mathrm{ind}(A) = 2^i \cdot \prod p_i^{n_i}$ with certain assumptions on i and n_i, then the Amitsur conjecture holds. This was proved by Krashen [108].
- Let A be a central simple k-algebra and X the corresponding Brauer–Severi variety. If A had odd degree n and has a dihedral splitting field of degree $2n$, the the Amitsur conjecture holds. This was also proved by Krashen [107].

In view of Theorem 2.37, the conjecture can also be formulated for generalized Brauer–Severi varieties (see [108], Generalized Conjecture 1.2). In this case one has also the following weaker result (see [108], discussion right after Generalized Conjecture 1.2):

Proposition 4.4. *Let* $\mathrm{BS}(d, A)$ *and* $\mathrm{BS}(d, B)$ *be two generalized Brauer–Severi varieties of same dimension. If A and B generates the same subgroup in* $\mathrm{Br}(k)$ *then* $\mathrm{BS}(d, A)$ *is stably birational to* $\mathrm{BS}(d, B)$.

In the present work we will focus on the conjecture given by Amitsur. Note that the result of Roquette [136] shows that we can restrict ourselves to the case where the Brauer–Severi variety corresponds to a central division algebra D, since for all other central simple algebras A one has $\mathrm{ind}(A) < \deg(A)$. We also want to mention an approach due to Saltman [138] that is captured by Meth [119]. Roughly, to each central simple k-algebra A one has a norm hypersurface $V(A)$ defined by a norm polynomial. Saltman [139] proved a variant of the Amitsur conjecture, namely, that two central simple k-algebras A and B generate the same cyclic subgroup in $\mathrm{Br}(k)$ if and only if $V(A)$ is birational to $V(B)$. Furthermore, Saltman constructed rational embeddings of the Brauer–Severi variety corresponding to A into the norm hypersurface $V(A)$. The idea now is to exploit this fact and to construct rational embeddings of X into $V(A)$ and of Y into $V(B)$ such that the birational map between $V(A)$ and $V(B)$ induces a birational map between X and Y. Meth [119] enlarged the set of rational embeddings and refined some ideas. We also want to mention the work of Kollár [105] and Hogadi [88], where the authors considered products of conics (Brauer–Severi varieties of dimension one) and products of Brauer–Severi surfaces and proved the following: Let P_i and Q_j be finite collections of conics (resp. Brauer–Severi surfaces) and suppose the subgroup in $\mathrm{Br}(k)$ generated by the set P_i equals the subgroup generated by Q_j, then $\prod_i P_i$ is birational to $\prod_j Q_j$. So it is also plausible to extend the Amitsur conjecture to such products. Finally, notice that by the reconstruction theorem of Bondal and Orlov [39] one has for Brauer–Severi varieties X and Y that $D^b(X) \simeq D^b(Y)$ if and only if $X \simeq Y$. This is due to the fact that the anticanonical sheaf of Brauer–Severi varieties is ample. Hence the isomorphism class of Brauer–Severi varieties is completely understood in terms of their derived categories and it would be of interest to understand their birational behavior in terms of their derived categories too. The aim of this chapter is to consider more closely the semiorthogonal decompostion of the derived category of a Brauer–Severi variety and to present an idea how to tackle the Amitsur conjecture from this point of view.

4.2 Autoequivalences and birationality

In this section we investigate the group of autoequivalences of the bounded derived category of coherent sheaves of some Brauer–Severi variety and relate it to the group of autoequivalence of another Brauer–Severi variety that is birational to the first one.

We start with a well-known result of Bondal and Orlov [39], Theorem 3.1.

Theorem 4.5. *Let X be a smooth projective and integral k-scheme with ample (anti-) canonical sheaf. Then one has a canonical isomorphism*

$$\mathrm{Aut}(D^b(X)) \simeq \mathrm{Aut}(X) \ltimes (\mathbb{Z} \oplus \mathrm{Pic}(X)).$$

For a Brauer–Severi variety of period p one therefore has:

Corollary 4.6. *Let X be a Brauer–Severi scheme of period p and A the corresponding central simple k-algebra. Then*

$$\mathrm{Aut}(D^b(X)) \simeq A^\times/k^\times \ltimes (\mathbb{Z} \oplus p\mathbb{Z})$$

Proof. By Theorem 4.5 we have to determine $\text{Pic}(X)$ and $\text{Aut}(X)$. But in Chapter 1 we have seen that $\text{Pic}(X) = p\mathbb{Z} \subset \mathbb{Z} = \text{Pic}(X \otimes_k \bar{k})$, provided X has period p. Since $\text{Aut}(X) = \text{Aut}(A)$ and $\text{Aut}(A) = A^\times/k^\times$ by the Skolem–Noether Theorem (Theorem 1.8 above), we get the assertion. □

For the next proposition we first need a lemma.

Lemma 4.7. *Let $U \rtimes_\phi H$ be a semidirect product and $G \subset H$ a subgroup of H. Then $U \rtimes_\phi G$ is a subgroup of $U \rtimes_\phi H$.*

Proof. The homomorphism $\phi : H \to \text{Aut}(U)$ can be restricted to G. This directly implies that $U \rtimes_\phi G$ is a subgroup of $U \rtimes_\phi H$. □

We have the following observation.

Proposition 4.8. *Let X and Y be Brauer–Severi varieties such that $\dim(X) \leq \dim(Y)$. If X and Y represent the same element in the Brauer group, then $\text{Aut}(D^b(X))$ can be regarded as a subgroup of $\text{Aut}(D^b(Y))$.*

Proof. Let A be the central simple k-algebra corresponding to X and B that to Y. By Theorem 1.7 we have $A \simeq M_n(D)$ and $B \simeq M_m(D)$, since A is Brauer-equivalent to B. The assumption $\dim(X) \leq \dim(Y)$ implies $n \leq m$. Since A and B have the same period, Corollary 4.6 yields

$$\text{Aut}(D^b(X)) \simeq \text{Aut}(A) \ltimes (\mathbb{Z} \oplus p\mathbb{Z}) \tag{4.1}$$
$$\text{Aut}(D^b(Y)) \simeq \text{Aut}(B) \ltimes (\mathbb{Z} \oplus p\mathbb{Z}). \tag{4.2}$$

In view of Lemma 4.7, we only have to show that the automorphism group $\text{Aut}(A)$ can be regarded as a subgroup of $\text{Aut}(B)$. This then implies that $\text{Aut}(D^b(X))$ is a subgroup of $\text{Aut}(D^b(Y))$. Since $M_m(D)$ and $M_m(D)$ are central simple k-algebras, the Skolem–Noether Theorem implies that every automorphism ϕ of $M_n(D)$ has to be inner. Precisely, there is a matrix U in $M_n(D)^\times$ such that $\phi(A) = U^{-1}AU$ for the automorphism $\phi : M_n(D) \to M_n(D)$. To this ϕ we assign $\Phi(-) = V^{-1}(-)V$, where V is the matrix given by

$$V = \left(\begin{array}{c|c} U & 0 \\ \hline 0 & I_{m-n} \end{array}\right)$$

This V is invertible by construction and hence an element of $M_m(D)^\times$. Again the Skolem–Noether Theorem yields that Φ is an automorphism of $M_m(D)$. By construction the map $\phi \mapsto \Phi$ is an injective group homomorphism and hence $\text{Aut}(M_n(D))$ can be regarded as a subgroup of $\text{Aut}(M_m(D))$. This completes the proof. □

This observation has the following consequence.

Proposition 4.9. *Let X and Y be Brauer–Severi varieties of same dimension n and same period p corresponding to central simple k-algebras A and B respectively. Let X_i and Y_j be the Brauer–Severi varieties corresponding to $A^{\otimes i}$ and $B^{\otimes j}$ respectively. If A and B generate the same cyclic subgroup in $\text{Br}(k)$, then there is a bijective map of sets $\phi : \{0, 1, ..., p-1\} \to \{0, 1, ..., p-1\}$ such that $\text{Aut}(D^b(X_i))$ is isomorphic to a subgroup of $\text{Aut}(D^b(Y_{\phi(i)}))$ for $i \leq \phi(i)$ and $\text{Aut}(D^b(Y_{\phi(i)}))$ isomorphic to a subgroup of $\text{Aut}(D^b(X_i))$ else.*

Proof. By assumption, A and B generate the same cyclic subgroup in $\mathrm{Br}(k)$ and hence the generator A of this subgroup is Brauer-equivalent to some $B^{\otimes l}$ with $l < p$. Since B generates the same cyclic subgroup as A, for every element $[A^{\otimes i}] \in \mathrm{Br}(k)$ with $0 \leq i \leq p-1$ there exist a $B^{\otimes j}$ with $0 \leq j \leq p-1$ such that $A^{\otimes i}$ is Brauer-equivalent to $B^{\otimes j}$. We define the map $\phi : \{0,1,...,p-1\} \to \{0,1,...,p-1\}$ as assigning to such an $i \in \{0,1,...,p-1\}$ the element $j \in \{0,1,...,p-1\}$. This map is well defined and since A and B generate the same subgroup it is also bijective. By assumption, the dimension of X equals the dimension of Y and hence $\deg(A) = \deg(B)$. Thus for $i \leq \phi(i)$ we have $\deg(A^{\otimes i}) \leq \deg(B^{\otimes \phi(i)})$ and Proposition 4.8 therefore yields that $\mathrm{Aut}(D^b(X_i))$ is isomorphic to a subgroup of $\mathrm{Aut}(D^b(Y_{\phi(i)}))$. In the other case we have $\phi(i) < i$ so that $\deg(B^{\otimes \phi(i)}) < \deg(A^{\otimes i})$ and again by Proposition 4.8 we have that $\mathrm{Aut}(D^b(Y_{\phi(i)}))$ is isomorphic to a subgroup of $\mathrm{Aut}(D^b(X_i))$. This completes the proof. □

As mentioned in Chapter 1, according to [71], Corollary 5.4.2 the central simple k-algebras A and B corresponding to birational Brauer–Severi varieties generate the same cyclic subgroup in $\mathrm{Br}(k)$. Therefore, Proposition 4.9 has the following consequence.

Corollary 4.10. *Let X and Y be Brauer–Severi varieties of same dimension n and same period p corresponding to central simple k-algebras A and B respectively. Let X_i and Y_j be the Brauer–Severi varieties corresponding to $A^{\otimes i}$ and $B^{\otimes j}$ respectively. If X and Y are birational, then there is a bijective map of sets $\phi : \{0,1,...,p-1\} \to \{0,1,...,p-1\}$ such that $\mathrm{Aut}(D^b(X_i))$ is isomorphic to a subgroup of $\mathrm{Aut}(D^b(Y_{\phi(i)}))$ for $i \leq \phi(i)$ and $\mathrm{Aut}(D^b(Y_{\phi(i)}))$ isomorphic to a subgroup of $\mathrm{Aut}(D^b(X_i))$ else.*

We want to end up this section explaining why the converse of Proposition 4.9 in general cannot hold.

Let X, Y, X_i and Y_j be as above and suppose we are given a bijective map $\phi : \{0,1,...,p-1\} \to \{0,1,...,p-1\}$ such that $\mathrm{Aut}(D^b(X_i))$ is a subgroup of $\mathrm{Aut}(D^b(Y_{\phi(i)}))$ for $i \leq \phi(i)$ and $\mathrm{Aut}(D^b(Y_{\phi(i)}))$ a subgroup of $\mathrm{Aut}(D^b(X_i))$ else. Consider for $i < \phi(i)$ the subgroup $\mathrm{Aut}(D^b(X_i))$ of $\mathrm{Aut}(D^b(Y_{\phi(i)}))$. Corollary 4.6 now implies that we have a subgroup

$$(A^{\otimes i})^\times/k^\times \ltimes (\mathbb{Z} \oplus p\mathbb{Z}) \subset (B^{\otimes \phi(i)})^\times/k^\times \ltimes (\mathbb{Z} \oplus p\mathbb{Z}).$$

By the definition of the semidirect product, we conclude that $(A^{\otimes i})^\times/k^\times$ has to be a subgroup of $(B^{\otimes \phi(i)})^\times/k^\times$. In general it is not clear why this should imply that $A^{\otimes i}$ is a central simple subalgebra of $B^{\otimes \phi(i)}$. In the case the ground field is local or global, it is possible to show this. Now suppose for a while that we can prove that $A^{\otimes i}$ is a central simple subalgebra of $B^{\otimes \phi(i)}$. This however does not imply that $A^{\otimes i}$ and $B^{\otimes \phi(i)}$ are Brauer-equivalent. Indeed, it is possible that a central simple k-algebra R is a subalgebra of another central simple k-algebra S even if they are not Brauer-equivalent. As an example we refer to [71], Theorem 1.5.5, to the Merkurjev–Suslin theorem (see [71], Theorem 2.5.7) or to [71], Proposition 2.5.16. In all three cases a central simple k-algebra S is shown to be isomorphic to a finite tensor product of central division algebras D_l. In this situation the division algebras D_l are central simple subalgebras of S but in general not Brauer-equivalent to S. This excludes the possibility that in general a central simple subalgebra $A^{\otimes i}$ of $B^{\otimes \phi(i)}$ can be Brauer-equivalent to $B^{\otimes \phi(i)}$. Therefore the converse of Proposition 4.9 in general does not hold. This however is not a surprise since the autoequivalences of a Brauer–Severi variety X are determined by the central simple k-algebra A corresponding X and the integers. So the main information lies in A and nothing new comes into play. The next step would be to investigate more closely the semiorthogonal decomposition of $D^b(X)$. This will be the purpose of the next section.

4.3 Semiorthogonal decomposition and birationality

In this section we make some observations concerning the relation between the semiorthogonal decompositions of birational Brauer–Severi varieties. Furthermore, we present some first idea to tackle the Amitsur conjecture.

Let X be a n-dimensional Brauer–Severi variety and A the corresponding central simple k-algebra. Bernardara [28], Corollary 5.8 obtained the following semiorthogonal decomposition for $D^b(X)$:

$$D^b(X) = \langle D^b(k), D^b(A), D^b(A^{\otimes 2}), ..., D^b(A^{\otimes n})\rangle. \tag{4.3}$$

In Chapter 3, Corollary 3.42 we have shown that [102], Theorem 8.5 implies that the full triangulated subcategory $\langle \mathcal{W}_i \rangle$ is equivalent to $D^b(A^{\otimes i})$, where $\mathcal{W}_i$ are the indecomposable locally free sheaves of Definition 1.39. Therefore, we can also write

$$D^b(X) = \langle \mathcal{O}_X, \mathcal{W}_1, ..., \mathcal{W}_{n-1}, \mathcal{W}_n\rangle. \tag{4.4}$$

Taking another n-dimensional Brauer–Severi variety Y corresponding to B we have a semiorthogonal decomposition

$$D^b(Y) = \langle D^b(k), D^b(B), D^b(B^{\otimes 2}), ..., D^b(B^{\otimes n})\rangle \tag{4.5}$$

for that we also write

$$D^b(Y) = \langle \mathcal{O}_X, \mathcal{W}'_1, ..., \mathcal{W}'_{n-1}, \mathcal{W}'_n\rangle. \tag{4.6}$$

To prove the next proposition, we first cite an result that is due to Antieau and a part of Theorem 3.2 in [8].

Theorem 4.11. *Let A and B be central simple k-algebras. Then A is Brauer-equivalent to B if and only if one has a k-linear triangulated equivalence between $D^b(A)$ and $D^b(B)$.*

With the above notation we make the following interesting observation.

Proposition 4.12. *Let X, Y, A and B be as above and let furthermore p be the period of A. Then A and B generate the same cyclic subgroup in $\mathrm{Br}(k)$ if and only if there exists a bijection of sets $\phi : \{0, 1, ..., p-1\} \to \{0, 1, ..., p-1\}$ such that one has k-linear triangulated equivalences between $D^b(A^{\otimes i})$ and $D^b(B^{\otimes \phi(i)})$ and hence between $\langle \mathcal{W}_i \rangle$ and $\langle \mathcal{W}'_{\phi(i)} \rangle$.*

Proof. Suppose that A and B generate the same subgroup. The proof of Proposition 4.9 shows the existence of a bijective map $\phi : \{0,1,...,p-1\} \to \{0,1,...,p-1\}$ such that for all $0 \leq i \leq p-1$ the central simple k-algebra $A^{\otimes i}$ is Brauer-equivalent to $B^{\otimes \phi(i)}$. Theorem 4.11 yields a k-linear triangulated equivalence $D^b(A^{\otimes i}) \tilde{\to} B^{\otimes \phi(i)}$. Suppose conversely that we are given a bijective map $\phi : \{0,1,...,p-1\} \to \{0,1,...,p-1\}$ such that for all $0 \leq i \leq p-1$ we have k-linear triangulated equivalences $D^b(A^{\otimes i}) \tilde{\to} B^{\otimes \phi(i)}$. By Theorem 4.11 we conclude that for all $0 \leq i \leq p-1$ the central simple k-algebra $A^{\otimes i}$ is Brauer-equivalent to $B^{\otimes \phi(i)}$. Note that this implies that B has also period p. Thus, A and B generate the same cyclic subgroup in $\mathrm{Br}(k)$. □

Corollary 4.13. *Let X and Y be Brauer–Severi varieties and A and B the corresponding central simple k-algebras. Furthermore, let p be the period of A. In all the above cases where the Amitsur conjecture holds the following are equivalent.*

- **(i)** *X and Y are birational.*
- **(ii)** *A and B generate the same cyclic subgroup in* $\mathrm{Br}(k)$.
- **(iii)** *There is a bijection of sets $\phi : \{0,1,...,p-1\} \to \{0,1,...,p-1\}$ such that one has k-linear triangulated equivalences $D^b(A^{\otimes i}) \tilde{\to} D^b(B^{\otimes \phi(i)})$ (resp. $\langle \mathcal{W}_i \rangle \tilde{\to} \langle \mathcal{W}'_{\phi(i)} \rangle$).*

For birational Brauer–Severi varieties X and Y the above observation shows the following: The admissible subcategories occurring in the semiorthogonal decomposition of $D^b(X)$ occur in interchanged position as admissible subcategories in the semiorthogonal decomposition of $D^b(Y)$. In the cases where the Amistsur conjecture holds (for instance if k is a local or global field) this interchanging of the admissible subcategories of the semiorthogonal decomposition is actually equivalent to X being birational to Y. So conjecturally this interchanging equivalences $D^b(A^{\otimes i}) \tilde{\to} D^b(B^{\otimes \phi(i)})$ should reflect the birationality between X and Y.

The idea now how to tackle the Amitsur conjecture is the following: Let X and Y be as above and let A and B the corresponding central simple k-algebras generating the same subgroup in $\mathrm{Br}(k)$. Suppose we are able to understand the "glueing behavior" (see next section) between the admissible subcategories of the semiorthogonal decomposition of $D^b(X)$ and $D^b(Y)$ respectively. Then one can try to exploit the interchanging equivalences $D^b(A^{\otimes i}) \tilde{\to} D^b(B^{\otimes \phi(i)})$ to construct a functor $\Phi : D^b(X) \to D^b(Y)$ compatible with the interchanging and respecting the glueing behavior. If it is possible to construct the functor Φ in that way, that $\mathrm{Hom}(\Phi(\mathcal{F}), \Phi(\mathcal{G})[i]) = 0$ for $i < 0$ for all coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, [54], Theorem 1.1 then implies that Φ is a Fourier–Mukai transform. This Fourier–Mukai transform can then be used to investigate if there is a birational map between X and Y. Before explaining in the next section where the glueing comes into play, we first roughly want to explain how a Fourier–Mukai functor can be used to get a birational map.

We start with well-known lemmas contained in [45] or [89]. Note that for a Fourier–Mukai kernel $\mathcal{P}$ one has $\mathrm{supp}(\mathcal{P}) \subset X \times Y$ is a closed subset. The first lemma is inspired by [89], Lemma 6.4

Lemma 4.14. *Let $\mathcal{P}$ be the kernel of a Fourier–Mukai transform $\Phi_{\mathcal{P}}$. Suppose that $\Phi_{\mathcal{P}}(k(x)) \neq 0$ for all closed points x in some open set $U \subset X$. Then the natural projection $p : \mathrm{supp}(\mathcal{P}) \to X$ is surjective over U.*

Proof. We use the spectral sequence (see [89], p.80 (3.9))

$$E_2^{r,s} = \mathcal{T}or_{-r}(\mathcal{H}^s(\mathcal{P}), p^*(k(x))) \Rightarrow \mathcal{T}or_{-(r+s)}(\mathcal{P}, p^*(k(x)))$$

and the fact that $\mathcal{T}or_i$ is trivial for objects with disjoint supports. Thus for a closed point x in U that is not in the image we have that $\mathcal{P} \otimes p^*(k(x))$ is trivial. Hence $\Phi_{\mathcal{P}}(k(x)) \simeq 0$ what contradicts the assumption that $\Phi_{\mathcal{P}}(k(x)) \neq 0$ for all $x \in U$. ☐

The next lemma is a variant of [89], Lemma 6.11.

Lemma 4.15. *Let $\mathcal{P}$ be the kernel of a Fourier–Mukai transform $\Phi_{\mathcal{P}}$ flat over X. Suppose that $\Phi_{\mathcal{P}}(k(x))$ is simple for all closed points x in some open set $U \subset X$. Then the fibers of the projection $\mathrm{supp}(\mathcal{P}) \to X$ are connected over U.*

Proof. Take a closed point $x \in U$ and consider $\mathrm{supp}(\mathcal{P}) \cap (\{x\} \times Y)$. By [89], Lemma 3.29 we have $\mathrm{supp}(\mathcal{P}) \cap (\{x\} \times Y) = \mathrm{supp}(\mathcal{P}_{\{x\}\times Y})$. Suppose the fibers over U are disconnected then $\mathrm{supp}(\mathcal{P}) \cap (\{x\} \times Y) = \mathrm{supp}(\mathcal{P}_{\{x\}\times Y})$ is disconnected. Since $\mathcal{P}$ is flat over X we have $\Phi_{\mathcal{P}}(k(x)) = \mathcal{P}_{\{x\}\times Y}$ (see [89], p.115). In particular, since $\mathcal{P}_{\{x\}\times Y}$ is disconnected, $\mathrm{End}(\Phi_{\mathcal{P}}(k(x))) = \mathrm{End}(\mathcal{F} \oplus \mathcal{G})$ (see [89], Lemma 3.9). But this contradicts the fact that $\Phi_{\mathcal{P}}(k(x))$ is simple for all $x \in U$ since $\mathrm{End}(\mathcal{F} \oplus \mathcal{G})$ cannot be a field. ☐

For the next well-known fact we only sketch the proof, since all arguments are standard arguments contained in [45] or [89].

Proposition 4.16. *Let k be a field of characteristic zero and X and Y smooth projective and integral k-schemes and $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$ a Fourier–Mukai transform with kernel $\mathcal{P}$ being flat over X. Suppose $\Phi_{\mathcal{P}}$ has the following properties:*

- **(i)** *$\Phi_{\mathcal{P}}(k(x_0)) = k(y_0)$ for some closed point $x_0 \in X$.*
- **(ii)** *There is an open $U \subset X$ such that $\Phi_{\mathcal{P}}(k(x))$ is simple for all $x \in U$.*
- **(iii)** *For all $x, y \in U$ with $x \neq y$ one has $\mathrm{Hom}(\Phi_{\mathcal{P}}(k(x)), \Phi_{\mathcal{P}}(k(y))[i]) = 0$ for $i \leq 0$ and if $x = y$ then $\mathrm{Hom}(\Phi_{\mathcal{P}}(k(x)), \Phi_{\mathcal{P}}(k(y))[i]) = 0$ for $i < 0$.*

Then X is birational to Y.

Proof. Assumption (i) $\Phi_{\mathcal{P}}(k(x_0)) = k(y_0)$ implies that the fiber of $p : \mathrm{supp}(\mathcal{P}) \to X$ over x_0 is zero-dimensional. This clearly holds true for all points in an open neighbourhood V of x_0 by Lemma 4.14 and 4.15. This implies that for all x in V the object $\Phi_{\mathcal{P}}(k(x))$ has zero-dimensional support. Now since $\Phi_{\mathcal{P}}(k(x))$ is simple for all $x \in U$, we find that $\Phi_{\mathcal{P}}(k(x))$ is simple for all $x \in U \cap V$. Now the assumption (iii) with [89], Lemma 4.5 implies that $\Phi_{\mathcal{P}}(k(x))$ is of the form $k(z)[m]$ for some close point $z \in Y$. Since $\mathcal{P}$ is flat over $U \cap V$ we conclude by semi-continuity (or with [45], Proposition 4.2.2.) that m must by constant around x_0 so that without loss of generality we can assume $m = 0$. Furthermore, assumption (iii) also implies that for $x \neq y$ in $U \cap V$ with $\Phi_{\mathcal{P}}(k(x)) \simeq k(z)$ and $\Phi_{\mathcal{P}}(k(y)) = k(w)$ we have

$$\mathrm{Hom}(\Phi_{\mathcal{P}}(k(x)), \Phi_{\mathcal{P}}(k(y))) = 0$$

and hence $k(z) \neq k(w)$ on Y. The kernel $\mathcal{P}$ restricted to $U \cap V$ is supported on the graph f assigning to each x from above the z from above. Choosing local sections of $\mathcal{P}$ gives a morphism $f : U \cap V \to Y$ such that on closed points we have $f(x) = z$. This morphism is injective on closed points and over a field of characteristic zero this suffices to conclude that we get a birational map $f : X \dashrightarrow Y$. □

If we assume the existence of a Fourier–Mukai transform $\Psi_{\mathcal{R}} : D^b(Y) \to D^b(X)$ with properties (i), (ii) and (iii) from above and with the additional assumption that $\Psi_{\mathcal{R}} \circ \Phi_{\mathcal{P}}(k(x)) = k(x)$ for some $x \in U$, then one can show that the birationality holds without the assumption on k being of characteristic zero. Moreover, it turns out that the converse of the above proposition also holds.

Proposition 4.17. *Let X and Y be smooth projective and integral k-schemes and let X be birational to Y. Then there is a Fourier–Mukai functor $F : D^b(X) \to D^b(Y)$ with properties (i), (ii) and (iii) of Proposition 4.16.*

Proof. Denote by $f : X \dashrightarrow Y$ the birational map between X and Y and by g the birational inverse $g : Y \dashrightarrow X$. Hence we have $U \simeq V$ for $U \subset X$and $V \subset Y$ induced by the map f. Furthermore, we are given the two graphs $\Gamma_f \subset X \times Y$ and $\Gamma_g \subset Y \times X$ as the closure of the graphs induced by f and g. Now let $i : \Gamma_f \to X \times Y$ be the inclusion and $\mathrm{pr}_X : \Gamma_f \to X$ the projection. Analogously, we have $j : \Gamma_g \to Y \times X$ and $\mathrm{pr}_Y : \Gamma_g \to Y$. With these morphisms we obtain a Fourier–Mukai transform as the composition of the Fourier–Mukai transforms occurring in the following diagram:

$$\begin{array}{ccccccc} & & D^b(X \times Y) & \xrightarrow{\simeq} & D^b(Y \times X) & & \\ & & \uparrow{\scriptstyle i_*} & & \downarrow{\scriptstyle j^*} & & \\ D^b(X) & \xrightarrow{\mathrm{pr}_X^*} & D^b(\Gamma_f) & & D^b(\Gamma_g) & \xrightarrow{\mathrm{pr}_{Y*}} & D^b(Y) \end{array}$$

This composition gives a Fourier–Mukai transform which we denote simply by $F : D^b(X) \to D^b(Y)$. Now we have to verify that F has the desired properties. Since every arrow in the above diagram is given by a Fourier–Mukai transform associated to a morphism of schemes, we have for $x \in U$ (see Chapter 3, p.80):

$$i^* \circ \mathrm{pr}_X^*(k(x)) = k((x, f(x)))$$

and hence with $f(x) \in V$

$$(\mathrm{pr}_Y)_* \circ j_*((f(x), x)) = k(f(x)).$$

One easily verifies that the properties (i), (ii) and (iii) are satisfied. □

Now we come back to Brauer–Severi varieties. If X and Y are birational Brauer–Severi varieties, then there must exist a Fourier–Mukai transform $F : D^b(X) \to D^b(Y)$ satisfying (i), (ii) and (iii) of Proposition 4.16. On the other hand, the existence of a Fourier–Mukai transform $F : D^b(X) \to D^b(Y)$ satisfying (i), (ii) and (iii) would provide us with a rational map $f : X \dashrightarrow Y$. By symmetry reasons if such a Fourier–Mukai transform F can be constructed, there should also be one in the other direction. But the main problem remains. How to construct such Fourier–Mukai transforms from the semiorthogonal decompositions of $D^b(X)$ and $D^b(Y)$ exploiting the interchanging of the admissible subcategories. One idea is to investigate more closely the semiorthogonal decompositions together with the glueing behavior between their admissible pieces. We want to discuss some of that aspects in the next section.

4.4 Derived approach to the Amitsur conjecture

As mentioned in the last section, the next step in following the idea to construct a functor between $D^b(X)$ and $D^b(Y)$ is to focus on the semiorthogonal decompositions together with the glueing behavior of their parts. This is the purpose of this section that will also be the last in the previous work. We roughly recall dg-categories, enhancements and the glueing of dg-categories and refer to [101], [113] and [116] for the details.
Note that we only recall a few points that will suffice for our purposes and introduce some notation. We start with the main definition.

Definition 4.18. A *differential graded category* $\mathcal{D}$ over a field k, or *dg-category* for short, is a k-linear category such that the following hold:

(i) For all objects $X_1, X_2 \in \mathcal{D}$ the space $\mathrm{Hom}_{\mathcal{D}}(X_1, X_2)$ is equipped with a structure of a complex of k-vector spaces.

(ii) The multiplication map

$$\mathrm{Hom}_{\mathcal{D}}(X_2, X_3) \otimes_k \mathrm{Hom}_{\mathcal{D}}(X_1, X_2) \to \mathrm{Hom}_{\mathcal{D}}(X_1, X_3),$$

induced by the composition, is a morphism of complexes.

Remark 4.19. Note that by definition the homomorphism space of two objects in $X, Y \in \mathcal{D}$, $\mathrm{Hom}_{\mathcal{D}}(X,Y) = \bigoplus_{l \in \mathbb{Z}} \mathrm{Hom}^l_{\mathcal{D}}(X,Y)$, is a graded vector space with differential d defined via $d_l : \mathrm{Hom}^l_{\mathcal{D}}(X,Y) \to \mathrm{Hom}^{l+1}_{\mathcal{D}}(X,Y)$. The elements of $\mathrm{Hom}^l_{\mathcal{D}}(X,Y)$ are called *homogenous morphisms* of degree l and we write $\deg(f) = l$. The second part of the above definition is just the *Leibniz rule* for the composition of homogenous morphisms $d(f \circ g) = df \circ g + (-1)^{\deg(f)} f \circ (dg)$.

Example 4.20. Any k-linear category can be considered as a dg-category with same Hom-spaces with zero differential and trivial grading. A second example is given by considering the category of complexes of k-vector spaces with

$$\mathrm{Hom}^l(V,W) = \prod_{i \in \mathbb{Z}} \mathrm{Hom}(V_i, W_{i+l})$$

and differential being $d(f) = d_W \circ f - (-1)^{\deg(f)} f \circ d_V$. This gives a dg-category that we denote by k-dgm.

We now state the second for our considerations important definition.

Definition 4.21. The *homotopy category* $H^0(\mathcal{D})$ of a dg-category $\mathcal{D}$ is defined as follows:

(i) $\mathrm{Ob} H^0(\mathcal{D}) = \mathrm{Ob}\mathcal{D}$

(ii) $\mathrm{Hom}_{H^0(\mathcal{D})}(X,Y) = H^0(\mathrm{Hom}_{\mathcal{D}}(X,Y))$

Note that $H^0(\mathcal{D})$ is a k-linear category.

A k-linear functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ between dg-categories $\mathcal{D}_1$ and $\mathcal{D}_2$ is a *dg-functor* if for any two objects $X_1, Y_1 \in \mathcal{D}_1$ the morphism

$$F : \mathrm{Hom}_{\mathcal{D}_1}(X_1, Y_1) \longrightarrow \mathrm{Hom}_{\mathcal{D}_2}(F(X_1), F(Y_1))$$

is a morphism of complexes, i.e., preserve the grading and commute with the differentials. Each dg-functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ between dg-categories $\mathcal{D}_1$ and $\mathcal{D}_2$ induces a functor $H^0(F) : H^0(\mathcal{D}_1) \to H^0(\mathcal{D}_2)$ on homotopy categories. Furthermore, a dg-functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ is called *quasiequivalence* if for any two $X_1, Y_1 \in \mathcal{D}_1$ the morphism

$$F : \mathrm{Hom}_{\mathcal{D}_1}(X_1, Y_1) \longrightarrow \mathrm{Hom}_{\mathcal{D}_2}(F(X_1), F(Y_1))$$

is a quasi-isomorphism of complexes and if for any $Z \in \mathcal{D}_2$ there is an objects $X \in \mathcal{D}_1$ such that $F(X)$ is homotopy equivalent to Z i.e. if $F(X)$ and Z are isomorphic as objects in $H^0(\mathcal{D}_2)$. In particular, this means that $H^0(F)$ is an equivalence between $H^0(\mathcal{D}_1)$ and $H^0(\mathcal{D}_2)$. Without reproducing the definition, we want to note that the notion of a triangulated category lifts to the dg world and is called *pretriangulated* (see [101] for details). A pretriangulated dg-category guarantees that the homotopy category is triangulated. Many problems in triangulated categories, coming from the fact that the cone of a morphism is not functorial or that the tensor product of such categories cannot be performed without extra data, do not occur in the dg world since the cone of a morphism in pretriangulated dg-categories are functorial and the tensor product can be defined very naturally. So one would like to consider triangulated categories $\mathcal{T}$ together with a dg-category $\mathcal{D}$ such that one recovers $\mathcal{T}$ by taking H^0. Such dg-categories are very important and one has the following definition.

Definition 4.22. An *enhancement* for a triangulated category $\mathcal{T}$ is a pretriangulated dg-category $\mathcal{D}$ with an equivalence $\epsilon : H^0(\mathcal{D}) \xrightarrow{\sim} \mathcal{T}$. We write $(\mathcal{D}, \epsilon)$ for the enhancement and omit the ϵ when it is clear from the context that we are dealing with enhancements.

Two natural questions arise. Does there always exists such a dg-enhancement for a given triangulated category $\mathcal{T}$ and if such an enhancement exists, is it unique? We say that an enhancement $(\mathcal{D}, \epsilon)$ is unique if for any other enhancement $(\mathcal{D}', \epsilon')$ there exists a quasiequivalence $F : \mathcal{D} \to \mathcal{D}'$. Lunts and Orlov [116] investigated several triangulated categories and proved the existence and the uniqueness of enhancements for triangulated categories coming from geometry. Without going into details, we note that they proved among others the following result (see [116], Theorem 2.13):

Theorem 4.23. *Let X be quasiprojective k-scheme. Then $D^b(X)$ has a unique enhancement.*

To state now the main idea how to construct functors between the derived categories of Brauer–Severi varieties, we describe the glueing of dg-categories and refer to [113] for details. Recall, the tensor product $\mathcal{D}_1 \otimes_k \mathcal{D}_2$ of two dg-categories is defined as the dg-category with objects being the same as in $\mathcal{D}_1 \times \mathcal{D}_2$ and morphisms being defined by

$$\mathrm{Hom}_{\mathcal{D}_1 \otimes_k \mathcal{D}_2}((X_1, Y_1), (X_2, Y_2)) = \mathrm{Hom}_{\mathcal{D}_1}(X_1, X_2) \otimes_k \mathrm{Hom}_{\mathcal{D}_2}(Y_1, Y_2).$$

A *left dg-module* M over a dg-category $\mathcal{D}$ is a dg-functor $M : \mathcal{D} \to k$-dgm. Analogously, one defines *right dg-modules* as dg-functors from $\mathcal{D}^{op}$ to k-dgm. Now let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two dg-categories and consider a dg-bimodule $\phi : \mathcal{D}_2^{op} \otimes \mathcal{D}_1 \to k$-dgm. The dg-category $\mathcal{D} = \mathcal{D}_1 \times_\phi \mathcal{D}_2$, called the *glueing* of $\mathcal{D}_1$ and $\mathcal{D}_2$ along ϕ, is defined as follows:

(i) The objects of the dg-category $\mathcal{D}_1 \times_\phi \mathcal{D}_2$ are triples $M = (M_1, M_2, \mu)$ where $M_i \in \mathcal{D}_i$ and $\mu \in \phi(M_2, M_1)$ is of degree zero.

(ii) Let $M = (M_1, M_2, \mu)$ and $N = (N_1, N_2, \nu)$ be two objects of $\mathcal{D}_1 \times_\phi \mathcal{D}_2$. Then the morphism complexes are defined to be sums of

$$\mathrm{Hom}^k_{\mathcal{D}}(M,N) = \mathrm{Hom}^k_{\mathcal{D}_1}(M_1,N_1) \oplus \mathrm{Hom}^k_{\mathcal{D}_2}(M_2,N_2) \oplus \phi^{k-1}(N_2,M_1)$$

with differentials given by

$$d(f_1, f_2, f_{21}) = (d(f_1), d(f_2), -d(f_{21}) - f_2 \circ \mu + \nu \circ f_1).$$

(iii) Let $f \in \mathrm{Hom}_{\mathcal{D}}(M,N)$ and $g \in \mathrm{Hom}_{\mathcal{D}}(L,M)$ be two morphisms. The the multiplication is defined by

$$(f_1, f_2, f_{21}) \circ (g_1, g_2, g_{21}) = (f_1 \circ g_1, f_2 \circ g_2, f_{21} \circ g_1 + (-1)^{\deg(f_2)} f_2 \circ g_{21}).$$

It is a matter of fact that the glueing of two pretriangulated dg-categories is again a pretriangulated dg-category (see [113], Lemma 4.3). Now it turns out that any enhancement of a triangulated category with a semiorthogonal decomposition can be obtained as a glueing of enhancements of the summands (see [113], Proposition 4.10)

Proposition 4.24. *Let $\mathcal{D}$ be a pretriangulated dg-category. Suppose that we have a semiorthogonal decomposition $H^0(\mathcal{D}) = \langle A, B \rangle$. Then the dg-category is quasiequivalent to a glueing of two pretriangulated dg-categories $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $H^0(\mathcal{D}_1) = A$ and $H^0(\mathcal{D}_2) = B$.*

Now let X be a n-dimensional Brauer–Severi variety corresponding to a central simple k-algebra A. The the derived category $D^b(X)$ has a unique enhancement $\mathcal{D}(X)$ according to Theorem 4.23. Considering the semiorthogonal decomposition

$$D^b(X) = \langle D^b(k), D^b(A), ..., D^b(A^{\otimes n}) \rangle$$

Proposition 4.24 provides us with the fact that $\mathcal{D}(X)$ is obtained as a glueing of pretriangulated dg-categories $\mathcal{D}_1, ..., \mathcal{D}_n$ such that $H^0(\mathcal{D}_i) = D^b(A^{\otimes i})$. Let now Y be another n-dimensional Brauer–Severi variety corresponding to a central simple k-algebra B. The derived category $D^b(Y)$ has also a unique enhancement $\mathcal{D}(Y)$ that is obtained as a glueing of pretriangulated dg-categories $\mathcal{D}'_1, ..., \mathcal{D}'_n$ such that $H^0(\mathcal{D}'_i) = D^b(B^{\otimes i})$. Now suppose the period of A is p and that A and B generate the same cyclic subgroup in $\mathrm{Br}(k)$. According to Proposition 4.13 there exists a bijection of sets $\phi : \{0, 1, ..., p-1\} \to \{0, 1, ..., p-1\}$ such that we have equivalences $D^b(A^{\otimes i}) \xrightarrow{\sim} D^b(B^{\otimes \phi(i)})$. Any of the derived categories $D^b(A^{\otimes i})$ has a unique enhancement (see [116], Proposition 2.6) and thus we get for the bijection $\phi : \{0, 1, ..., p-1\} \to \{0, 1, ..., p-1\}$ quasiequivalences $\mathcal{D}_i \to \mathcal{D}'_{\phi(i)}$.

The main problem in order to construct a functor $F : D^b(X) \to D^b(Y)$ that exploits and respects the interchanging equivalences $D^b(A^{\otimes i}) \tilde{\to} D^b(B^{\otimes \phi(i)})$ is, that in fact taking a cone of a morphism is functorial, but this functor is not well-behaved for our purpose. Note that any object $\mathcal{F}$ of $D^b(X)$ can be obtained as a chain

$$0 = \mathcal{F}_n \longrightarrow \mathcal{F}_{n-1} \longrightarrow \mathcal{F}_{n-2} \longrightarrow ... \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{F}$$

such that the cone of $\mathcal{F}_i \to \mathcal{F}_{i-1}$ is in $D^b(A^{\otimes i})$. Now the interchanging equivalences $D^b(A^{\otimes i}) \tilde{\to} D^b(B^{\otimes \phi(i)})$ give us objects in $D^b(B^{\otimes \phi(i)})$ but it is not clear how to construct a chain

$$0 = \mathcal{G}_n \longrightarrow \mathcal{G}_{n-1} \longrightarrow \mathcal{G}_{n-2} \longrightarrow ... \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{G}$$

from these data, such that the cone of $\mathcal{G}_i \to \mathcal{G}_{i-1}$ is in $D^b(B^{\otimes i})$. The idea now is to lift the problem to the dg world and consider the enhancements $\mathcal{D}(X) = \mathcal{D}_1 \times ... \times \mathcal{D}_n$ and $\mathcal{D}(Y) = \mathcal{D}'_1 \times ... \times \mathcal{D}'_n$ obtained by glueing. Since we are more flexible in the dg world because we can consider different enhancements of $D^b(X)$ and $D^b(Y)$, the hope is to construct dg-functors between these enhancements respecting the interchanging quasiequivalences induced by the semiorthogonal decompositions. At the very and such a dg-functor should be constructed in this way, that the induced functor between $D^b(X)$ and $D^b(Y)$ has properties (i), (ii) and (iii) of Proposition 4.16. To work this out, is the authors plan for further articles.

To end up this section and thereby the previous work, we collect some results concerning the birationality of Brauer–Severi varieties in the following result.

Theorem 4.25. *Let X and Y be Brauer–Severi varieties of same dimension corresponding to A and B respectively. Then the following are equivalent*

- **(i)** *A and B generate the same cyclic subgroup in $\mathrm{Br}(k)$*
- **(ii)** *$X \times \mathbb{P}^n$ is birational to $Y \times \mathbb{P}^n$ for some n*
- **(iii)** *There exists a bijective map $\phi : \{0, 1, ..., p-1\} \to \{0, 1, ..., p-1\}$ such that we have equivalences $D^b(A^{\otimes i}) \tilde{\to} D^b(B^{\otimes \phi(i)})$.*
- **(iv)** *There exists a bijective map $\phi : \{0, 1, ..., p-1\} \to \{0, 1, ..., p-1\}$ such that we have quasiequivalences $\mathcal{D}_i \to \mathcal{D}_{\phi(i)}$*
- **(v)** *$\mathrm{ind}(A \otimes_k L) = \mathrm{ind}(B \otimes_k L)$ for any field extension $k \subset L$.*
- **(vi)** *The motive of X is a direct summand of the motive of $\mathrm{BS}(d, B)$ for $[A] = \pm d[B]$ in $\mathrm{Br}(k)$*

Proof. That (i) implies (ii) is Proposition 4.3. Now suppose that $X \times \mathbb{P}^n$ is birational to $Y \times \mathbb{P}^n$. Then their function fields $F(X \times \mathbb{P}^n)$ and $F(Y \times \mathbb{P}^n)$ are isomorphic. In view of the isomorphisms $F(X \times \mathbb{P}^n) \simeq F(X \otimes_k F(\mathbb{P}^n))$ and $F(Y \times \mathbb{P}^n) \simeq F(Y \otimes_k F(\mathbb{P}^n))$ we conclude that the kernel of the restriction map $\mathrm{Br}(k) \to \mathrm{Br}(F(\mathbb{P}^n))$ is the cyclic subgroup generated by A or by B. Hence A and B generate the same subgroup. The equivalence of (i) and (iii) was proved in Proposition 4.12 and the equivalence of (i) and (iv) follows from the discussion above. Karpenko [99], Lemma 7.13 proved the equivalence of (i) and (v) and the equivalence of (i) and (vi) was proved by Zainoulline [157], Theorem 1.1. □

Conjecturally, all the above statements are equivalent to the fact that X and Y are birational. In particular, in all the cases where the Amitsur conjecture was proved to hold, the statements of the above theorem are equivalent to X being birational to Y.

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