Equivariant analytic torsion on hyperbolic Riemann surfaces and the arithmetic Lefschetz trace of an Atkin-Lehner involution on a compact Shimura curve

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Zusammenfassung

In dieser Dissertation berechnen wir die äquivariante analytische Torsion eines Hermiteschen Vektorbündels über einer hyperbolischen Riemannschen Fläche, welches durch einen Automorphiefaktor beliebigen Gewichtes und Ranges gegebenen ist. Wir drücken diese aus durch Werte einer geeigneten äquivarianten Selberg-Zeta-Funktion und Ableitungen der Lerch'schen Phi-Funktion (Theorem 1.3). Für Tensorpotenzen des kanonischen Geradenbündels beweisen wir ein spezielleres Resultat (Korollar 1.4).

Der Beweis von Theorem 1.3 benutzt den Zusammenhang zwischen der Funktionaldeterminante des Laplace-Operators auf automorphen Formen und einer geeignet vervollständigten Selberg-Zeta-Funktion, den wir in Theorem 1.1 für kokompakte Fuchssche Gruppen mit elliptischen Elementen bereitstellen. Des weiteren verwenden wir ein Fouriertransformationsargument.

Als Nebenresultat berechnen wir auch die gewöhnliche analytische Torsion sehr ampler Potenzen des kanonischen Geradenbündels (Korollar 1.12).

Mit Hilfe der Eichlerschen Theorie der indefiniten rationalen Quaternionenalgebren können wir die äquivariante Selberg-Zeta-Funktion bezüglicher einer Atkin-Lehner-Involution berechnen (Proposition 2.10). Mittels Modulinterpretation und verallgemeinerter Chowla-Selberg-Formel (Theorem 2.14) gelingt uns auch die Berechnung der Höhe des Fixpunktschemas einer Atkin-Lehner-Involution (Proposition 2.13).

Setzt man die letzten beiden Resultate in die arithmetische Lefschetz-Fixpunkt-Formel von Köhler und Roessler ein, so ergibt sich eine explizite Formel für die arithmetische Lefschetz-Spur einer Atkin-Lehner-Involution (Theorem 0.1).

Schließlich weisen wir auf eine interessante Identität (Proposition 2.18) hin, die man auf arithmetischen Flächen vom Geschlecht zwei erhält, indem man den arithmetische Lefschetz-Fixpunkt-Satz mit dem arithmetischen Riemann-Roch-Satz von Gillet und Soulé kombiniert.

Alle Ergebnisse über Shimura-Kurven werden anhand des Beispiels der Diskriminante 26 veranschaulicht.

Mathematics Subject Classification (2000): 11M36, 14G40, 58J52, 11G18

Abstract

In this thesis, we compute the equivariant analytic torsion of a Hermitian vector bundle over a hyperbolic Riemann surface given by a factor of automorphy of arbitrary weight and rank in terms of an equivariant Selberg zeta function and derivatives of Lerch's Phi function (Theorem 1.3). We also specialise this result to the case of powers of the canonical bundle (Corollary 1.4).

We accomplish this by comparing the functional determinant of the automorphic Laplacian for a cocompact Fuchsian group with elliptic elements with the completed Selberg zeta function (Theorem 1.1) and employing a Fourier transform argument.

As a byproduct, we also compute the ordinary analytic torsion of very ample powers of the canonical bundle (Corollary 1.12).

Using Eichler's theory of indefinite rational quaternion algebras, we succeed in computing the equivariant Selberg zeta function (Proposition 2.10) with respect to an Atkin-Lehner involution acting on a compact Shimura curve. With the help of the moduli interpretation and the generalised Chowla-Selberg formula (Theorem 2.14), we also manage to compute the height of the fixed point scheme of an Atkin-Lehner involution (Proposition 2.13).

Combined with these two results, the arithmetic Lefschetz fixed point formula of Köhler and Roessler then yields an explicit formula for the arithmetic Lefschetz trace of an Atkin-Lehner involution (Theorem 0.1).

Finally we point out a curious identity on arithmetic surfaces of genus two (Proposition 2.18) that can be obtained from a simultaneous application of the arithmetic Lefschetz fixed point theorem and the arithmetic Riemann-Roch theorem of Gillet and Soulé.

All results about Shimura curves are illustrated by means of the example of discriminant 26.

Mathematics Subject Classification (2000): 11M36, 14G40, 58J52, 11G18

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Introduction

A Hermitian vector bundle $\overline{E}$ over a compact Hermitian manifold X has an invariant $\mathcal{T}(\overline{E})$, its analytic torsion which was introduced by Ray and Singer [34]. In the presence of an automorphism, i.e. a holomorphic isometry, g of $\overline{E}$, one also studies a variant $\mathcal{T}_g(\overline{E})$, the so-called equivariant torsion. As an object in its own right, it was first defined by Köhler [22] although it had already appeared implicitly in Ray's paper [33]. Almost by definition, one has $\mathcal{T} = \mathcal{T}_{\mathrm{id}}$ so the latter concept subsumes the former to which we shall refer as ordinary torsion.

In Chapter 1 of this thesis, we compute equivariant torsion in the case where X has dimension one, i.e. is a compact Riemann surface. We treat a very general class of vector bundles on hyperbolic Riemann surfaces, i.e. on those of genus $h \geq 2$. Equivariant torsion on the projective line has been computed by Köhler [22, Thm. 2], for ordinary analytic torsion on elliptic curves see [34, Thm. 4.1] and lastly, for equivariant torsion of elliptic curves (or more generally Abelian varieties) consult [25, Thm. 4.2].

The main results are Theorem 1.3 and Corollary 1.4 which compute equivariant torsion in terms of an equivariant Selberg zeta function (for this new concept see Definition 5) and derivatives of Lerch's Phi function. While Theorem 1.3 applies to a general vector bundle given by a factor of automorphy of arbitrary weight and rank, Corollary 1.4 is a specialisation to tensor powers of the canonical line bundle.

Generalising results of Sarnak [36], we obtain intermediate results of independent interest about the functional determinant (Theorem 1.1) and the reduced determinant (Corollary 1.2) of the automorphic Laplacian for a cocompact Fuchsian group with elliptic elements.

As a byproduct we also compute the ordinary torsion of the line bundle of k-differentials (Corollary 1.12), a result for which we have found no reference but which is implicit in [11] except for the fine point arising from the fact that the Kodaira Laplacian and the automorphic Laplacian differ by a factor of 2, see Section 1.5.3.

For a more detailed overview of Chapter 1 see Section 1.1.

Chapter 2 contains applications of the results of Chapter 1, first and foremost of Corollary 1.4.

Whereas the setting of Chapter 1 is entirely analytic, our interest in Chapter 2 shifts towards arithmetic. The primary objects of study are no longer Riemann surfaces (and on them Hermitian holomorphic vector bundles) but rather arithmetic surfaces (and on them algebraic vector bundles equipped with a Hermitian structure). The contents of Chapter 1 fit into this broader framework as being considerations at infinity, i.e. on the complex points of the schemes.

Chapter 2 starts with specialised statements of the arithmetic Riemann-Roch theorem of Gillet and Soulé and the arithmetic Lefschetz fixed point formula of Köhler and Roessler. It is in the latter theorem where equivariant torsion makes its appearance.

As we want to apply these two theorems to Shimura curves, Chapter 2 then presents all the necessary material about quaternion algebras, Shimura curves and Atkin-Lehner involutions.

The main original result of Chapter 2 is Proposition 2.10 which computes the equivariant Selberg zeta function of a Shimura curve with respect to an Atkin-Lehner involution. This proposition makes numerical approximations possible whose quality we also discuss (Lemma 2.12).

We also compute the height of the fixed point scheme of an Atkin-Lehner involution (Proposition 2.13). Then the arithmetic Lefschetz fixed point formula yields the following neat result which may well be regarded as the climax of this thesis:

Theorem 0.1 (The arithmetic Lefschetz trace of an Atkin-Lehner involution on a Shimura curve)**.** *Consider a compact Shimura curve $\mathcal{X} = \mathcal{X}(D, N)$ with N square-free, and let $n \mid D$, $n \neq 1$. Then the arithmetic Lefschetz trace of $\overline{\omega}^k$, $k \geq 2$, with respect to the Atkin-Lehner involution w_n is given by*

$$\widehat{\deg}(\overline{H^0(\mathcal{X}, \omega^k)_+}) - \widehat{\deg}(\overline{H^0(\mathcal{X}, \omega^k)_-}) = \sum_{t \in n\mathbb{N}_0} \prod_{p \mid \frac{D}{n}} \left(1 - \left(\frac{D_F}{p}\right)\right) \sum_{\substack{m' \mid m \\ (m', \frac{D}{n})=1}} \prod_{p \mid N} \left(1 + \left\{\frac{\Lambda'}{p}\right\}\right) A(n, t, \Lambda')$$

modulo rational multiples of $\log(2)$*. For every t of the outer sum, denote by D_F the discriminant of the quadratic field $F := \mathbb{Q}(\sqrt{t^2 - 4n})$ and define $m \geq 1$ by $t^2 - 4n = m^2 D_F$. Then for every m' of the inner sum, let Λ' be the order of conductor m' in F and let $\left\{\frac{\Lambda'}{p}\right\}$ denote Eichler's symbol (see Definition 10).*

To define the term $A(n, t, \Lambda')$ in a unified manner, we let for any quadratic order Λ

$$\tilde{L}(\Lambda, s) := |D_\Lambda|^{s/2} L_\Lambda(s)$$

where D_Λ denotes the discriminant of Λ. Furthermore, $L_\Lambda := \zeta_\Lambda/\zeta$ is the L-function associated to Λ, i.e. the quotient of the zeta function of Λ by the Riemann zeta function, the former being defined as

$$\zeta_\Lambda(s) := \sum_{\mathfrak{a} \subset \Lambda} N(\mathfrak{a})^{-s}$$

with the sum extending over all Λ-ideals contained in Λ and $N(\mathfrak{a})$ denoting the index $[\Lambda \colon \mathfrak{a}]$.

Then for $D_{\Lambda'} > 0$ we let

$$A(n, t, \Lambda') := \frac{1}{\log N} \frac{N^{-k}}{1 - \frac{1}{N}} 2\tilde{L}'(\Lambda', 0), \qquad N := (t + \sqrt{t^2 - 4n})^2/(4n)$$

whereas for $D_{\Lambda'} < 0$, we set

$$A(n, t, \Lambda') := h(\Lambda') A_{\text{const}} + \frac{(-1)^k}{2} \left(k - \frac{1}{2}\right) (-\tilde{L}'(\Lambda', 0))$$

with $h(\Lambda')$ the ideal class number of Λ' and A_{const} the constant

$$A_{\mathrm{const}} := \frac{(-1)^k}{4}\Big[\sum_{1\le j\le 2k-2}(-1)^j\log(j) \\ -\log\left(\frac{\pi}{2}\right)-\frac{1}{2}\log(2)+(2k-1)\left(\log(4\pi)+\log(D)\right)\Big].$$

It is tempting to conjecture that mutatis mutandis, a similar formula should give the arithmetic Lefschetz trace of a general Hecke operator, and it should pose no problem to derive it from an arithmetic Lefschetz fixed point theorem for correspondences.

We then conclude Chapter 2 with a curious identity (Proposition 2.18) on arithmetic surfaces of genus two where the arithmetic Riemann-Roch theorem and the Lefschetz fixed point formula happen to be simultaneously applicable.

For a more detailed overview of Chapter 2 see Section 2.1.

Appendix A contains an alternative but less successful approach to calculating the height of the fixed point scheme using a projective model.

Finally, as the reader may want to experiment himself, Appendix B contains the source code of the PARI script used for numerical computations of Selberg zeta values along with some numerical tables.

Contents

Chapter 1

Equivariant analytic torsion on hyperbolic Riemann surfaces

1.1 Overview

The setting of this chapter is purely analytic: We study Hermitian holomorphic vector bundles over a compact Riemann surface of genus $h \geq 2$.

Let us give an overview of the sections of Chapter 1.

In Section 1.2, we outline our approach to computing equivariant torsion on Riemann surfaces in more detail, culminating in the statement of our main results (Theorem 1.3 and Corollary 1.4). This section should serve as a guide through the first chapter.

This is followed by Section 1.3 which contains everything we need about determinants of operators, especially the three notions of zeta-regularised, reduced and functional determinant and how they are interrelated.

Then, Section 1.4 reviews the definitions of ordinary and equivariant torsion, shows the simplifications possible on a Riemann surface (Lemma 1.6) and the basic relation between ordinary and equivariant torsion via finite Fourier transformation (Lemma 1.7).

Next is Section 1.5 supplying all the material about the Selberg zeta function of a cocompact Fuchsian group, especially Fischer's completion factors along with asymptotic formulae for their logarithms (Lemmas 1.10 and 1.11).

Last but not least, Section 1.6 contains all the proofs and some further remarks.

1.2 Outline of approach

1.2.1 The Fourier transform argument

Due to the extremely simple Hodge theory of a Riemann surface, the ordinary as well as the equivariant torsion of $\overline{E}$ depend only on $\Box_0$, the degree 0 part of the Kodaira Laplacian (Lemma 1.6).

By Schwarz's theorem, the automorphism group of a hyperbolic Riemann surface X is finite. Hence the action of g on X is of finite order, say n. For convenience, we shall assume that g acting on E is also of the same finite order n. Then we can employ our key tool (Lemma 1.7) to reduce the problem to computing the reduced determinants of the restrictions of $\Box_0$ to the various g^*-eigenspaces:

$$\mathcal{T}_g(\overline{E}) = -\sum_{\xi^n=1} \xi \log\det{}' \left(\Box_0|_{\mathrm{Eig}(g^*,\xi)}\right). \tag{1.1}$$

Here g^* denotes the action of g on 0-forms with values in E which are of course just the sections of E.

1.2.2 Hyperbolic uniformisation

So next I would like to explain how we can come to grips with the objects occurring in (1.1).

A holomorphic vector bundle E on X can be lifted to the universal cover of X which is the hyperbolic plane in our case. Since this lifted bundle can be globally trivialised, E can actually always be thought of as given by a factor of automorphy (cf. [19, Chap. I, §3]).

So from now on, we shall always think of the Riemann surface X as the quotient of the hyperbolic upper half plane $\mathbb{H}$ by a cocompact Fuchsian group $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ containing -1 and acting without fixed points. The holomorphic vector bundles E which we shall consider are those given by a holomorphic factor of automorphy of weight $2k$ and rank d for Γ, which in particular covers the case of arbitrary line bundles. There is a simple relation between weight and degree:

$$\deg(E) = dk(2h-2), \tag{1.2}$$

generalising the well-known fact that the canonical line bundle ω, i.e. the holomorphic cotangent bundle, has degree $2h-2$ and is given by the canonical weight 2 scalar factor of automorphy. When E is equipped with the classical Petersson metric, an elementary computation shows that its Kodaira Laplacian in degree 0 is related to the usual automorphic Laplacian by

$$2\Box_0 = -\Delta_{2k} - k(1-k). \tag{1.3}$$

Note that we adhere to the tradition of replacing the holomorphic factor of automorphy by the corresponding unitary one. This does not cause any problems because the spaces of square-integrable automorphic forms for either factor are isometric (see Section 1.5.2 for more on this).

As for the automorphism g, its action on the base X can be thought of as an automorphism $\tilde{g}$ of $\mathbb{H}$ which normalises Γ. But what about the action of g on E? Under the assumption that it is of the same order n as the action on the base X, such an action is simply given by an extension of $\mathfrak{j}$ to a factor of automorphy $\tilde{\mathfrak{j}}$ for the group $\tilde{\Gamma}$ generated by $\tilde{g}$ and Γ. Note that while still being cocompact, the enlarged group $\tilde{\Gamma}$ contains elliptic elements whenever $\tilde{g}$ has a fixed point.

Now the ξ eigenspace of g^* occurring in (1.1) is just the subspace of forms automorphic with respect to a suitable factor of automorphy for $\tilde{\Gamma}$. Explicitly, this factor of automorphy is $\overline{\rho}_\xi \otimes \tilde{\mathfrak{j}}$ where ρ_ξ is the unique scalar factor which is trivial on Γ and satisfies $\rho_\xi(\tilde{g}) = \xi$. Here bar denotes complex conjugation.

1.2.3 Generalising Sarnak's method

Thus, we are left with the problem of computing the reduced determinant of the automorphic Laplacian for the cocompact, but not necessarily strictly hyperbolic Fuchsian group $\tilde{\Gamma}$.

For a cocompact Fuchsian group without elliptic elements, this problem has been solved elegantly by Sarnak [36, Cor. 1]. The key idea is to use well-known asymptotic expansions to express the *functional* determinant

$$D(s) := \det(-\Delta_{2k} - \lambda), \qquad \lambda = s(1-s)$$

of the the automorphic Laplacian in terms of the Selberg zeta function Z (which depends on $\mathfrak{j}$), an identity completion factor Ξ_I and an explicit constant e^{C_I}. Then the *reduced* determinant can be computed from the *functional* determinant by differentiating suitably often.

In this thesis, we generalise this approach to the case with elliptic elements. All we need is an appropriate elliptic completion factor Ξ_ell depending on $\mathfrak{j}$ which has been supplied by Fischer [14]. We thus obtain the following generalisation of Sarnak's result [36, Thm. 1]

Theorem 1.1 (Functional determinant of automorphic Laplacian)**.** *Let Γ be a cocompact Fuchsian group (which may contain elliptic elements) and $\mathfrak{j}$ a factor of automorphy of weight $2k$. Then the functional determinant of the Laplacian $-\Delta_{2k}$ on $\mathfrak{j}$ automorphic forms is given by*

$$D(s) = \Xi(s)\, e^{C_\mathrm{I}+C_\mathrm{ell}} \qquad \text{for } s \in \mathbb{C} \tag{1.4}$$

with $\Xi := \Xi_I \cdot \Xi_{ell} \cdot Z$ and explicit constants C_I (independent of $\mathfrak{j}$, cf. (1.17) and C_ell (depending on $\mathfrak{j}$, cf. (1.20).

From this we deduce

Corollary 1.2 (Reduced determinant of automorphic Laplacian)**.** *In the situation of Theorem 1.1, the reduced determinant of $-\Delta_{2k} - k(1-k)$ can be computed as*

$$\det{}'(-\Delta_{2k} - k(1-k)) = \begin{cases} \frac{e^{C_\mathrm{I}+C_\mathrm{ell}}}{N!(2k-1)^N}\, \Xi^{(N)}(k), & k \neq \frac{1}{2}; \\ \frac{e^{C_\mathrm{I}+C_\mathrm{ell}}}{(2N)!}\, \Xi^{(2N)}(\frac{1}{2}), & k = \frac{1}{2} \end{cases}$$

with N its kernel dimension. For $k \neq \frac{1}{2}$ one may alternatively use

$$\det{}'(-\Delta_{2k} - k(1-k)) = \frac{e^{C_\mathrm{I}+C_\mathrm{ell}}}{N!(1-2k)^N}\, \Xi^{(N)}(1-k).$$

Note that the alternative formula comes from the symmetry of Ξ with respect to $s \leftrightarrow 1-s$.

1.2.4 Computing equivariant torsion

Using the above corollary and Fourier transformation, we then obtain our main result

Theorem 1.3 (Equivariant torsion on hyperbolic Riemann surfaces)**.** *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a strictly hyperbolic Fuchsian group containing -1 and $\tilde{g}$ an element of $\mathrm{SL}_2(\mathbb{R})$ normalising Γ. Let $\mathfrak{j}$ be a holomorphic factor of automorphy for Γ of weight $2k$ that can be extended to a factor of automorphy $\tilde{\mathfrak{j}}$ for the group $\tilde{\Gamma}$ generated by $\tilde{g}$ and Γ. Let $\overline{E}$ be the Hermitian holomorphic vector bundle given by $\mathfrak{j}$ together with the Petersson metric over the Riemann surface $X = \Gamma\backslash\mathbb{H}$ and let g be the automorphism of $\overline{E}$ induced by $\tilde{\mathfrak{j}}$. Then the equivariant torsion of $\overline{E}$ can be computed as either of the two limits*

$$\begin{aligned}\mathcal{T}_g(\overline{E}) = &-\lim_{s\to k,1-k}\Bigg\{\log Z_{\tilde{g}\Gamma}(s) - \operatorname{tr}\left(g^*|_{H^0(X,E)}\right)\,\log\big(k(1-k)-s(1-s)\big)\\ &+\sum_{p\in X^g}\frac{i\operatorname{tr}(g|_{E_p})}{2\sin(\theta)}\Big(e^{\theta i}\Phi'(e^{2\theta i},0,s+k) - e^{-\theta i}\Phi'(e^{-2\theta i},0,s-k)\Big)\Bigg\}\\ &+\log(2)\kappa.\end{aligned}$$

Here $Z_{\tilde{g}\Gamma}$ is an appropriate equivariant Selberg zeta function depending on the coset $\tilde{g}\Gamma$ and $\tilde{\mathfrak{j}}$ (see Definition 5). The sum is over the fixed points p of the action of g on X and the angle $\theta = \theta(p)$ is such that $e^{-2\theta i}$ is the derivative of g acting on X at p. Finally Φ' denotes the derivative of Lerch's Phi function $\Phi(z,w,a) := \sum_{j\geq 0} z^j(j+a)^{-w}$ with respect to w and

$$\kappa := \sum_{p\in X^g}\frac{\operatorname{tr}(g|_{E_p})}{|1-T_pg|^2} - \operatorname{tr}\left(g^*|_{H^0(X,E)}\right).$$

For the case where both $\mathfrak{j}$ and $\tilde{\mathfrak{j}}$ are the trivial scalar factor, we can give a slightly more explicit result

Corollary 1.4 (Equivariant torsion of powers of the canonical line bundle on a hyperbolic Riemann surface)**.** *Let g be an isometry of X and consider its induced action on a power ω^k of the canonical line bundle. Then we have*

$$\begin{aligned}\mathcal{T}_g(\overline{\mathcal{O}}) = \mathcal{T}_g(\overline{\omega}) = &-\log Z'_{\tilde{g}\Gamma}(1) - \sum_{p\in X^g}\Big(\mathrm{Re}\Big(F'(e^{2\theta i},0)\Big) - \cot(\theta)R^{\mathrm{rot}}(2\theta)\Big)\\ &+\log(2)\kappa\end{aligned}$$

and for $k\geq 2$

$$\begin{aligned}\mathcal{T}_g(\overline{\omega}^k) = &-\log Z_{\tilde{g}\Gamma}(k)\\ &-\sum_{p\in X^g}\left(\frac{\sin((2k-1)\theta)}{\sin(\theta)}\mathrm{Re}\Big(F'(e^{2\theta i},0)\Big) - \frac{\cos((2k-1)\theta)}{\sin(\theta)}R^{\mathrm{rot}}(2\theta)\right)\\ &-\sum_{p\in X^g}\frac{e^{-2\theta i(k-1)}}{1-e^{2\theta i}}\sum_{1\leq j\leq 2k-2} e^{2\theta i\,j}\log(j)\\ &+\log(2)\kappa\end{aligned}$$

with $R^{\mathrm{rot}}(2\theta)$ being the imaginary part of $F'(e^{2\theta i},0)$ whereby we mean the derivative at $w=0$ of Jonquière's function $F(z,w) := \sum_{j\geq 1} z^j j^{-w}$ and all other notation as in Theorem 1.3. Moreover, the real part of $F'(e^{2\theta i},0)$ can be computed in terms of the digamma function Ψ and Euler's constant γ_{Eul} as

$$\mathrm{Re}\Big(F'(e^{2\theta i},0)\Big) = -\frac{1}{4}\Psi\left(\frac{\theta}{\pi}\right) - \frac{1}{4}\Psi\left(1-\frac{\theta}{\pi}\right) - \frac{1}{2}\log(2\pi) - \frac{1}{2}\gamma_{\mathrm{Eul}}.$$

Remark. Note that Jonquière's function is a specialisation of Lerch's Phi function, namely $F(z,w) = z\Phi(z,w,1)$.

Remark. For a further specialisation to the case of involutive g see Section 2.15.

1.3 Determinants of operators

For proofs and more information about the material of this section see [45]. Let B be a self-adjoint operator bounded from below (think Laplacian!). From the eigenvalues $\lambda_0 \leq \lambda_1 \leq \ldots$ of B (each repeated according to multiplicity) we form the partition function of B

$$\Theta_B(t) := \sum_{j=0}^{\infty} e^{-t\lambda_j}.$$

Henceforth we will make two assumptions: first we assume that the above series converges for all $t > 0$ and second we suppose that there are asymptotic expansions

$$\Theta_B(t) = \sum_{k=-1}^{n} c_k t^k + O\left(t^{n+1}\right), \qquad t \searrow 0 \tag{1.5}$$

for all $n \geq -1$. Then we fix $a > -\lambda_0$ and consider the following sum

$$\sum_{j=0}^{\infty} (\lambda_j + a)^{-w}.$$

By the above assumptions on Θ_B, this sum converges for $\mathrm{Re}(w) \gg 0$ and it can be continued to a meromorphic function $\zeta_B(w,a)$, the shifted zeta function of B, on the whole complex plane which is regular at $w = 0$. We write $\zeta_B(w)$ for the unshifted zeta function $\zeta_B(w,0)$ of B.

Definition 1 (Three kinds of determinants)**.** *Let B be an operator as above.*

1. *If $B > 0$ we define the* (zeta-regularised) determinant *of B to be*

$$\det(B) := \exp(-\zeta_B'(0)),$$

prime denoting the derivative with respect to w.

2. *If $B \geq 0$ we define the* reduced determinant *of B as the determinant of the restriction B' of B to the orthogonal complement of its kernel, i.e.*

$$\det{}'(B) := \det(B'), \qquad B' = B|_{\ker(B)^\perp}.$$

3. *In any case, the* functional determinant *of B is defined as the analytic continuation of*

$$\lambda \mapsto \exp(-\zeta_B'(0,-\lambda)), \qquad \lambda < \lambda_0.$$

We denote it by $\det(B - \lambda)$.

Since $\zeta_B(w, -\lambda)$ is the unshifted zeta function of the operator $B - \lambda$, our notation for the functional determinant makes sense. When we take logarithms of determinants we shall always use the branches occurring in the above definition. We collect the following facts

Proposition 1.5 (Properties of the functional determinant)**.** *Let B be an operator as above.*

1. *The functional determinant of B is an entire function of order 1 with zeroes exactly at the eigenvalues of B, multiplicities agreeing.*
2. *Its logarithm obeys the following asymptotics for $\lambda \to -\infty$:*
$$\log \det(B - \lambda) = c_{-1}\Big(\log(-\lambda) - 1\Big)\lambda + c_0 \log(-\lambda) + o(1).$$
3. *Suppose $B \geq \mu_0$ for some real number μ_0. Then the reduced determinant of the non-negative operator $B - \mu_0$ can be computed from the functional determinant of B as follows*
$$\det{}'(B - \mu_0) = \lim_{\lambda \to \mu_0} \frac{\det(B - \lambda)}{(\mu_0 - \lambda)^N}$$
where N is the multiplicity of μ_0 (which is zero unless $\mu_0 = \lambda_0$). More generally, the logarithms satisfy
$$\log \det{}'(B - \mu_0) = \lim_{\lambda \to \mu_0} \left(\log \det(B - \lambda) - N \log(\mu_0 - \lambda)\right).$$
4. *Moreover let $\alpha > 0$. Then the reduced determinant of the rescaled operator $\alpha(B - \mu_0)$ obeys*
$$\log \det{}'(\alpha(B - \mu_0)) = \log \det{}'(B - \mu_0) + \log(\alpha)(c_0 + \mu_0 c_{-1} - N).$$

Proof. The first two statements are proved in [45, Sect. 4 & 5]. As for the third statement, note that
$$\sum_{j=N}^{\infty} (\lambda_j - \mu_0)^{-w}$$
is the zeta function of the operator $(B - \mu_0)'$. Using this, the logarithmic statement is easily proved, the other version follows by exponentiation.

To prove the fourth statement, let $B_1 = B - \mu_0$ and note the elementary fact that
$$\log \det{}'(\alpha B_1) = \log \det{}'(B_1) + \log(\alpha)\zeta_{B_1'}(0)$$
for any non-negative operator B_1. Then we apply the trace identity [45, Eq. (3.3)] to the positive operator B_1' to obtain
$$\zeta_{B_1'}(0) = c_0(B_1').$$
Now N is the dimension of the kernel of B_1, hence $\Theta_{B_1'} = \Theta_{B_1} - N$ as functions and therefore $c_0(B_1') = c_0(B_1) - N$. Finally $c_0(B_1) = c_0 + \mu_0 c_{-1}$ is clear from comparing coefficients of t^0 in
$$\Theta_{B_1}(t) = e^{\mu_0 t}\Theta_B(t) = (1 + \mu_0 t + \dots)(c_1 t^{-1} + c_0 + \dots).$$
□

1.4 Review of ordinary and equivariant analytic torsion

1.4.1 Analytic torsion

Let X be a compact Hermitian manifold and $\overline{E}$ a Hermitian vector bundle on X.

First let us briefly and informally recall the general definition of the *analytic torsion* of $\overline{E}$: We start off by considering differential forms of type $(0,q)$ on X with values in E. In each degree q, the square-integrable forms form a Hilbert space $\mathfrak{H}_q$ with respect to the L^2 product furnished by the Hermitian metrics on E and X. For each q, the Kodaira Laplacian $\Box_q$ is a compact self-adjoint non-negative operator in $\mathfrak{H}_q$ with a spectrum satisfying the requirements of Section 1.3. The *analytic torsion* of $\overline{E}$ is then a peculiarly weighted sum of the logarithms of the reduced determinants of the $\Box_q$'s:

$$\mathcal{T}(\overline{E}) := \sum_{q>0}(-1)^q q \log\det{}'(\Box_q).$$

1.4.2 Equivariant torsion

Now if $\overline{E}$ has an automorphism g, we may also consider equivariant analytic torsion, *equivariant torsion* for short. This hinges on the fact that g induces a linear operator g_q^* in $\mathfrak{H}_q$. We define the following equivariant zeta function

$$\zeta_{q,g}(w) := \sum_{\substack{\lambda\in\operatorname{spec}(\Box_q)\\ \lambda>0}} \operatorname{tr}(g_q^*|\operatorname{Eig}(\Box_q,\lambda))\lambda^{-w}. \tag{1.6}$$

It enjoys the same properties as the zeta function of an operator considered in Section 1.3. Hence we can take the derivative at $w=0$ of its meromorphic continuation and sum as in (1.4.1) to give the definition of the equivariant torsion of $\overline{E}$:

$$\mathcal{T}_g(\overline{E}) := -\sum_{q>0}(-1)^q q\,\zeta'_{q,g}(0). \tag{1.7}$$

Since $\operatorname{tr}(\mathrm{id}\,|\operatorname{Eig}(\Box_q,\lambda))$ is the multiplicity of λ as an eigenvalue of $\Box_q$, we have

$$\mathcal{T}_{\mathrm{id}}(\overline{E}) = \mathcal{T}(\overline{E}).$$

1.4.3 Torsion on Riemann surfaces

As mentioned in the Section 1.1, everything simplifies in dimension one:

Lemma 1.6 (Torsion on Riemann surfaces)**.** *Let X be a Riemann surface with a Hermitian vector bundle $\overline{E}$ having an automorphism g. Then the equivariant torsion of $\overline{E}$ can be computed as*

$$\mathcal{T}_g(\overline{E}) = \frac{d}{dw}\zeta_{0,g}(w)\bigg|_{w=0},$$

in particular

$$\mathcal{T}(\overline{E}) = -\log\det{}'(\Box_0).$$

Proof. By definition

$$\mathcal{T}_g(\overline{E}) = \frac{d}{dw}\zeta_{1,g}(w)\bigg|_{w=0}.$$

But we even have $\zeta_{1,g} = \zeta_{0,g}$ because $\bar{\partial}$ gives an isomorphism $\ker(\Box_0)^\perp \to \ker(\Box_1)^\perp$ intertwining $\Box_0$ with $\Box_1$ as well as g_0^* with g_1^*. The latter statement is true because g is a holomorphic isometry of $\overline{E}$. $\square$

1.4.4 Fourier transformation

For simplicity, we state the following lemma only in the dimension one case:

Lemma 1.7 (Fourier transform)**.** *Let X be a compact Riemann surface with a Hermitian vector bundle $\overline{E}$ having an automorphism g of finite order n. Denote by g^* the action of g on sections of E. Then the equivariant torsion of $\overline{E}$ can be computed as*

$$\mathcal{T}_g(\overline{E}) = -\sum_{\xi^n=1} \xi \log\det{}' \left(\Box_0|_{\mathrm{Eig}(g^*,\xi)}\right)$$

where the sum is over all n-th roots of unity.

Proof. As with Lemma 1.6, the statement holds already on the level of zeta functions. This is true because $\Box_0$ and g^* commute and can therefore be simultaneously diagonalised:

$$\ker(\Box_0)^\perp = \bigoplus_{\xi^n=1} \bigoplus_{\substack{\lambda\in\mathrm{spec}(\Box_0)\\ \lambda>0}} \mathrm{Eig}(g^*,\xi)\cap\mathrm{Eig}(\Box_0,\lambda).$$

$\square$

Remark. When g acts on X without fixed points, Lemma 1.7 yields a neat formula expressing the equivariant torsion of $\overline{E}$ in terms of ordinary torsions of vector bundles over the quotient manifold $\langle g\rangle\backslash X$

$$\mathcal{T}_g(\overline{E}) = \sum_{\xi^n=1} \xi\, \mathcal{T}(\langle g\rangle\backslash\overline{E}\otimes\overline{L}_\xi^*)$$

with suitable line bundles $\overline{L}_\xi$. This formula appears in [24, Proof of Lemma 3.3]. In this thesis, we shall however be interested in the case where g does have fixed points. In this case, a geometric interpretation would require the notion of orbifolds, a subject we shall not delve into. For more on this we refer the reader to [29].

1.5 Completion factors for the Selberg zeta function

The main reference for this section is Fischer's monograph [14]. In particular, we use most of his notation.

1.5.1 Preliminaries

Let us begin with cocompact Fuchsian groups containing -1. Apart from ± 1, such a group Γ contains hyperbolic elements P and elliptic elements R characterised by $|\operatorname{tr} P| > 2$ and $|\operatorname{tr} R| < 2$ respectively. Conjugating by elements of $\mathrm{SL}_2(\mathbb{R})$, we can bring them into normal forms $\pm D_N$ and R_θ with

$$D_N := \begin{pmatrix} N^{1/2} & \\ & N^{-1/2} \end{pmatrix}, \qquad N = N(P) > 1$$

and

$$R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad 0 < \theta = \theta(R) < 2\pi, \quad \theta \neq \pi$$

respectively. Centralisers can represented as

$$Z(P) = \{\pm P_0^m \mid m \in \mathbb{Z}\}$$

and

$$Z(R) = \{R_0^m \mid m = 1, \ldots, 2\nu\}$$

respectively. The elements P_0 and R_0 are the associated primitive elements. The hyperbolic primitive element P_0 is the unique element of the centraliser conjugate to a matrix $+D_{N_0}$ such that P or $-P$ is a positive power of P_0. The elliptic primitive element R_0 is the element of the centraliser conjugative to R_θ with minimal positive θ. Note that an elliptic element belongs to $Z(R)$ if and only if it has the same fixed point in $\mathbb{H}$ as R. Furthermore, we remark that ν is the projective order of $Z(R)$ since we assume $-1 \in \Gamma$.

Following Fischer, we consider unitary factors of automorphy for Γ of the form $\mathfrak{j} = j_{2k}\,\chi$ with

$$j_{2k}(S,\, z) := \left(\frac{cz+d}{|cz+d|}\right)^{2k}, \qquad z \in \mathbb{H}, \quad S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \tag{1.8}$$

and χ a so-called unitary multiplier system for Γ of weight $2k$ and rank d, i.e. a map from Γ to the unitary matrices of rank d satisfying the two conditions

$$\begin{aligned} \chi(-1) &= e^{-\pi i 2k}\mathbb{1}_d, \\ \chi(ST) &= \frac{j_{2k}(S,\, T(z))j_{2k}(T,\, z)}{j_{2k}(ST,\, z)}\chi(S)\chi(T). \end{aligned}$$

We assume that the power in (1.8) is defined by the choice of $-\pi < \arg \leq \pi$. Then the first condition on χ ensures that $\mathfrak{j}(-1) = \mathbb{1}_d$ which is obviously necessary in order for $\mathfrak{j}$ to define a vector bundle over $\Gamma\backslash\mathbb{H}$. The second condition on χ makes sure that $\mathfrak{j}$ is indeed a factor of automorphy, i.e.

$$\mathfrak{j}(ST,\, z) = \mathfrak{j}(S,\, T(z))\mathfrak{j}(T,\, z).$$

Now the Laplacian on $\mathfrak{j}$-automorphic forms has an explicit formula (which we will take as a definition), namely

$$\Delta_{2k} := y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 2kiy\frac{\partial}{\partial x}. \tag{1.9}$$

Note that this formula really only depends on the weight $2k$ of $\mathfrak{j}$. It is a self-adjoint operator on the Hilbert space of square-integrable $\mathfrak{j}$-automorphic functions, i.e. on the measurable functions f from the upper half plane $\mathbb{H}$ to $\mathbb{C}^d$ satisfying $f(\gamma(z)) = \mathfrak{j}(\gamma, z)f(z)$ and $(f, f) < \infty$ with

$$(f, g) := \int_{\mathcal{F}} \langle f(z), g(z)\rangle \ \mathrm{dvol}(z) \tag{1.10}$$

for any choice of fundamental domain $\mathcal{F}$ for Γ. Here $\mathrm{dvol} = y^{-2}dxdy$ is the hyperbolic volume element.

1.5.2 The holomorphic vector bundles

The above factor of automorphy $\mathfrak{j}$ is obviously not holomorphic, hence it does not define a holomorphic bundle in a natural manner. Our real interest is in the holomorphic vector bundle $\overline{E}$ defined by the holomorphic factor of automorphy $(cz + d)^{2k}\chi$ and equipped with the classical Petersson metric

$$(f, g)_{\mathrm{Pet}} := \int_{\mathcal{F}} \langle f(z), g(z)\rangle y^{2k} \ \mathrm{dvol}(z). \tag{1.11}$$

However, comparing (1.11) with (1.10) and remembering $\mathrm{Im}(cz+d) = \mathrm{Im}(z)/|cz+d|^2$, we see that $f \mapsto fy^{-k}$ furnishes an isometry of the respective Hilbert spaces.

What is more, let us see if formula (1.2) for the degree of E makes any sense: The Petersson metric has matrix $H = y^{2k}\mathbb{1}_d$ in the obvious holomorphic trivialisation over a fundamental domain. The first Chern form of the Chern connection is

$$\begin{aligned}\frac{i}{2\pi}\operatorname{tr}\left(\bar\partial\left(\partial H \cdot H^{-1}\right)\right) &= \frac{i}{2\pi}\operatorname{tr}\left(\bar\partial\left(\frac{-i}{2}2ky^{-1}dz\mathbb{1}_d\right)\right)\\ &= \frac{i}{2\pi}\operatorname{tr}\left(\frac{i}{2}\frac{-i}{2}(-2k)y^{-2}d\bar z\wedge dz\mathbb{1}_d\right) = \frac{dk}{2\pi}\,\mathrm{dvol}(z).\end{aligned}$$

Now Gauß-Bonnet says $\int_X \mathrm{dvol} = -2\pi\chi(X) = 2\pi(2h-2)$, which proves formula (1.2).

Remark. By Weil's theorem [46], all indecomposable Hermitian holomorphic vector bundles can be obtained from the above construction.

1.5.3 Comparing the Kodaira Laplacian and the automorphic Laplacian

Let us also verify relation (1.3) between the Kodaira Laplacian $\Box_0$ on $\overline{E}$ and the automorphic Laplacian Δ_{2k}. For degree reasons, the Kodaira Laplacian in degree zero is just $\bar\partial_1^*\bar\partial_0$. Hence, the Hodge $*$ formula for the adjoint of $\bar\partial$ from [48, Chap. V, Prop. 2.4] shows that $-\Box_0$ is the composition

$$A^{0,0}(E) \xrightarrow{\bar\partial_0} A^{0,1}(E) \xrightarrow{\bar *} A^{1,0}(E^*) \xrightarrow{\bar\partial_1} A^{1,1}(E^*) \xrightarrow{\bar *} A^{0,0}(E). \tag{1.12}$$

Since the above trivialisation is holomorphic, the first and third map in (1.12) simply take the form $f \mapsto \frac{\partial f}{\partial \bar z}d\bar z$ and $g^t\,dz \mapsto \frac{\partial g^t}{\partial \bar z}d\bar z\wedge dz$, (^{t} denoting transpose) respectively. As for the second map in (1.12), note that is defined by requiring

$$f_1\,d\bar z \wedge \bar*(f_2\,d\bar z) \stackrel{!}{=} f_1^t\cdot H\cdot \overline{f_2}\,|d\bar z|^2\ \mathrm{dvol}(z) = f_1^t\cdot\overline{f_2}\,y^{2k}\,2y^2\,y^{-2}\frac{-i}{2}d\bar z\wedge dz,$$

from which we can read it off as $f\,d\bar{z} \mapsto -iy^{2k}\overline{f}^t\,dz$. As for the last map in (1.12), it is defined by requiring

$$g_1^t\,d\bar{z}\wedge dz \wedge \bar{*}(g_2^t\,d\bar{z}\wedge dz) \stackrel{!}{=} g_1^t \cdot H^{-1}\cdot \overline{g_2}\,|d\bar{z}\wedge dz|^2\ \mathrm{dvol}(z)$$
$$= g_1^t \cdot \overline{g_2}\,y^{-2k}\,4y^4\,y^{-2}\frac{-i}{2}d\bar{z}\wedge dz$$

from which we read it off as $g^t\,d\bar{z}\wedge dz \mapsto -2iy^{2-2k}\overline{g}$. Putting everything together we get

$$\Box_0 f = 2iy^{2-2k}\overline{\frac{\partial}{\partial\bar{z}}\left(-iy^{2k}\overline{\frac{\partial f}{\partial\bar{z}}}\right)} = -2y^{2-2k}\frac{\partial}{\partial z}\left(y^{2k}\frac{\partial f}{\partial\bar{z}}\right). \tag{1.13}$$

Remark 1.1. For $k=0$, this formula computes the Kodaira Laplacian of the trivial bundle as $-2y^2\partial^2/\partial z\partial\bar{z}$ which is half the well-known formula for the Laplacian on functions – exactly as one expects on a Kähler manifold!

What is more, the above computations are for the most natural choice of metric on the tangent bundle, namely $y^{-2}\left(\mathrm{d}x^2+\mathrm{d}y^2\right)$ which has constant curvature -1. Equation (1.13) shows that if this metric is rescaled by a constant $\alpha>0$, the Laplacian is multiplied by $1/\alpha$, cf. [3, (1.161c)] whereas changing the metric on the vector bundle E by a constant does nothing.

Finally we remark that in their seminal paper [34], Ray and Singer inadvertently use the spectrum of the de Rham Laplacian instead of the Kodaira Laplacian when computing analytic torsion on tori (see the definitions of the eigenvalue $\lambda_{m,n}$ and eigenfunction $\phi_{m,n}$ on p. 166 (loc. cit.) and use that when $\mathbb{C}$ is equipped with the standard metric, the Kodaira Laplacian is $-2\partial^2/\partial z\partial\bar{z}$, cf. [18, p.83]).

Now we complete the verification of (1.3) by computing the operator corresponding to (1.13) under $f\mapsto y^{-k}f$:

$$-2y^{2-k}\frac{\partial}{\partial z}\left(y^{2k}\frac{\partial}{\partial\bar{z}}\left(y^{-k}f\right)\right) = -2y^{2-k}\frac{\partial}{\partial z}\left(\frac{i}{2}(-k)y^{k-1}f + y^k\frac{\partial f}{\partial\bar{z}}\right)$$
$$= \frac{1}{2}k(k-1)f + \underbrace{iky\left(\frac{\partial f}{\partial z}+\frac{\partial f}{\partial\bar{z}}\right) - 2y^2\frac{\partial^2 f}{\partial z\partial\bar{z}}}_{-\frac{1}{2}\Delta_{2k}f}.$$

1.5.4 The Selberg zeta function

Now we define the Selberg zeta function:

Definition 2 (Selberg zeta function)**.** *The Selberg zeta function $Z(s)=Z(s,\jmath)$ of Γ with respect to $\jmath$ is the exponential of*

$$\log Z(s) := \sum_{\{P_0\}_\Gamma}\sum_{m=0}^{\infty}\operatorname{tr}\log\left(\mathbb{1}_d - \chi(P_0)N(P_0)^{-s-m}\right) \tag{1.14}$$

where the sum is over all primitive hyperbolic conjugacy classes.

The above sum is known to converge locally uniformly on $\mathrm{Re}(s)>1$ and exponentiation gives the more familiar product formula for Z. Furthermore, it is also classical that $\log Z(s)=o(1)$ as $s\to\infty$.

Lemma 1.8 (Alternative formula for the logarithm of Selberg zeta)**.** *The logarithm of the Selberg zeta function is also given by*

$$\log Z(s) = -\sum_{\substack{\{P\}_\Gamma \\ \operatorname{tr} P>2}} \operatorname{tr}\chi(P)\frac{\log N(P_0)}{\log N(P)}\,\frac{N(P)^{-s}}{1-N(P)^{-1}}, \qquad \operatorname{Re}(s)>1. \tag{1.15}$$

Proof. Write P as P_0^m and use the standard properties $\chi(P_0^m) = \chi(P_0)^m$ and $N(P_0^m) = N(P_0)^m$ of the multiplier system and the Selberg norm. Then you can easily reduce the formula to (1.14) using the geometric and the logarithm series. □

Remark. The series in the lemma is better suited for Fourier transformation because of the $\operatorname{tr}\chi(P)$ term.

In the next two subsections, we shall recall Fischer's completion factors Ξ_{I} and Ξ_{ell} for the Selberg zeta function. They are invaluable because of

Proposition 1.9 (The completed Selberg zeta function)**.** *The completed Selberg zeta function*

$$\Xi := Z \cdot \Xi_{\mathrm{I}} \cdot \Xi_{\mathrm{ell}}$$

is an entire function of finite order. Furthermore, it has the same divisor as the functional determinant $D(s)$ of the automorphic Laplacian.

Proof. The first statement is [14, Lemma 3.2.3]. For the second statement, we know from [14, Section 3.1] that Ξ is an entire function satisfying

$$\Xi(1-s) = \Xi(s)$$

with zeroes at exactly those s for which $\lambda = s(1-s)$ is an eigenvalue of $-\Delta_{2k}$. Multiplicities agree except for $s=\frac{1}{2}$ where the multiplicity of the zero is twice the multiplicity of the eigenvalue. □

Note that main ingredient in the proof of Theorem 1.1 will be asymptotic formulae for the logarithms of the completion factors.

1.5.5 The identity completion factor

We start with Fischer's identity completion factor.

Definition 3 (Identity completion factor for the Selberg zeta function)**.** *The identity completion factor Ξ_{I} is the exponential of $\log \Xi_{\mathrm{I}}^{+} + \log \Xi_{\mathrm{I}}^{-}$ where*

$$\log \Xi_{\mathrm{I}}^{\pm}(s) := \frac{d\operatorname{vol}(\Gamma)}{2\pi}\Bigg(\tfrac{1}{2}\log(2\pi)s + \tfrac{1}{2}s(1-s) \\ + \left(\tfrac{1}{2}\pm k\right)\log\Gamma(s\pm k) + \log\Gamma_2(s\pm k+1)\Bigg).$$

Here $\operatorname{vol}(\Gamma)$ *is the volume of the quotient $\Gamma\backslash\mathbb{H}$ and Γ_2 is the double Gamma function, i.e. the reciprocal of the Barnes G function which is the unique meromorphic function satisfying*

$$G(z+1) = \Gamma(z)G(z), \qquad G(1)=1, \qquad G \text{ is } \mathcal{C}^\infty \text{ on } \mathbb{R}_{\geq 1} \text{ with } (\log G)''' \geq 0.$$

Note that $\mathrm{vol}(\Gamma)$ is the hyperbolic volume of a fundamental domain in $\mathbb{H}$ for the Fuchsian group Γ which is classically computed from the genus and the orders of the fixed points. In particular, if Γ is strictly hyperbolic it equals -2π times the Euler characteristic, i.e. $\mathrm{vol}(\Gamma)/2\pi = 2h - 2$.

As for the double Gamma function, the Weierstraß canonical product for the Barnes G function is classical, hence we might also take

$$\frac{1}{\Gamma_2(z+1)} = (2\pi)^{z/2} e^{-((1+\gamma)z^2+z)/2} \prod_{k=1}^{\infty} \left(\left(1 + \frac{z}{k}\right)^k e^{-z+\frac{z^2}{2k}} \right)$$

as a definition. For more about this function see [45, Appendix] and [8].

Lemma 1.10 (Asymptotics of identity factor)**.** *For $s \to \infty$ we have*

$$\log \Xi_{\mathrm{I}}(s) = -C_{\mathrm{I}} + \frac{d\,\mathrm{vol}(\Gamma)}{2\pi} \Bigg[\tfrac{1}{2} \Big(\log\big(s(s-1)\big) - 1 \Big) s(1-s) + \Big(\tfrac{k^2}{2} - \tfrac{1}{6} \Big) \log\big(s(s-1)\big) \Bigg] + o(1) \tag{1.16}$$

with an explicit constant

$$C_{\mathrm{I}} := \frac{d\,\mathrm{vol}(\Gamma)}{2\pi} \left(-\tfrac{1}{4} - \tfrac{1}{2}\log(2\pi) + 2\zeta'(-1) \right). \tag{1.17}$$

Proof. One can prove the lemma using asymptotic formulae for Γ and Γ_2, which is not difficult but tedious. Therefore, we prefer to reduce the statement to a lemma in Fischer's book [14, Lemma 3.4.1]. Plugging the relation $\zeta'(-1) = \frac{1}{12} - \log \mathcal{A}$ between the derivative of Riemann zeta and Kinkelin's constant into the definition of the constant C_{I} we get

$$-C_{\mathrm{I}} = \frac{d\,\mathrm{vol}(\Gamma)}{2\pi} \left(\tfrac{1}{12} + \tfrac{1}{2}\log(2\pi) + 2\log\mathcal{A} \right)$$

which certainly agrees with Fischer's result. Now let $s = \sigma + \frac{1}{2}$ and replace $s(s-1)$ by $\sigma^2 - \frac{1}{4}$ in (1.16) and use $\log\left(\sigma^2 - \frac{1}{4}\right) = 2\log(\sigma) - \frac{1}{4}\sigma^{-2} + O(\sigma^{-4})$ to compute

$$\begin{aligned}
&\tfrac{1}{2}\Big(\log\left(\sigma^2 - \tfrac{1}{4}\right) - 1\Big)\left(\tfrac{1}{4} - \sigma^2\right) + \Big(\tfrac{k^2}{2} - \tfrac{1}{6}\Big)\log\left(\sigma^2 - \tfrac{1}{4}\right) \\
&= \Big(\log(\sigma) - \tfrac{1}{8}\sigma^{-2} - \tfrac{1}{2}\Big)\left(\tfrac{1}{4} - \sigma^2\right) + \left(k^2 - \tfrac{1}{3}\right)\log(\sigma) + o(1) \\
&= -\sigma^2\log(\sigma) + \tfrac{1}{2}\sigma^2 + \left(k^2 - \tfrac{1}{12}\right)\log(\sigma) + o(1)
\end{aligned}$$

which again agrees perfectly with Fischer's result. □

Remark. The reader might know the multiple Gamma function defined by $\Gamma_n = G_n^{(-1)^{n-1}}$ where $G_1 = \Gamma$ and for $n \geq 2$, G_n is the unique (cf. [43, Prop. 2.8]) meromorphic function which is smooth on $\mathbb{R}_{\geq 1}$ with non-negative $(n+1)$st logarithmic derivative and satisfies

$$G_n(z+1) = G_{n-1}(z)G_n(z), \qquad G_n(1) = 1.$$

For these multiple Gamma functions, there exist asymptotic formulae generalising Stirling's formula for Γ and the corresponding formula for the Barnes G function, see [8] for more on that.

1.5.6 The elliptic completion factor

Last but not least we introduce Fischer's elliptic completion factor:

Definition 4 (Elliptic completion factor for the Selberg zeta function)**.** *The elliptic completion factor* $\Xi_{\text{ell}}(s) = \Xi_{\text{ell}}(s, \mathfrak{j})$ *of* Γ *with respect to* $\mathfrak{j}$ *is the exponential of* $\log \Xi_{\text{ell}} = \log \Xi_{\text{ell}}^{+} + \log \Xi_{\text{ell}}^{-}$ *where*

$$\log \Xi_{\text{ell}}^{\pm} := \sum_{\substack{\{R\}_\Gamma \\ 0<\theta<\pi}} e^{i\theta 2k} \operatorname{tr} \chi(R) \frac{\pm i e^{\pm\theta i}}{2\sin(\theta)} \frac{1}{\nu} \sum_{\ell=0}^{\nu-1} e^{\pm 2\theta i \ell} \log \Gamma\left(\frac{s \pm k + \ell}{\nu}\right). \tag{1.18}$$

Remark. This is not the actual definition Fischer gives in [14, Cor. 2.3.5] but rather its less refined ancestor concealed in [14, Prop. 2.3.4]. Again, (1.18) is attractive for us because the $\operatorname{tr} \chi(R)$ term makes it ideal for Fourier transformation.

We finish this section with another asymptotic lemma

Lemma 1.11 (Asymptotics of elliptic factor)**.** *For* $s \to \infty$ *we have*

$$\log \Xi_{\text{ell}}(s) = A \log\bigl(s(s-1)\bigr) - C_{\text{ell}} + o(1)$$

with explicit constants A *and* $C_{\text{ell}} = C_{\text{ell}}^{+} + C_{\text{ell}}^{-}$ *given by*

$$A := \sum_{\substack{\{R\}_\Gamma \\ 0<\theta<\pi}} e^{i\theta 2k} \operatorname{tr} \chi(R) \frac{1}{4\nu \sin^2(\theta)}, \tag{1.19}$$

$$C_{\text{ell}}^{\pm} := \sum_{\substack{\{R\}_\Gamma \\ 0<\theta<\pi}} e^{i\theta 2k} \operatorname{tr} \chi(R) \frac{\pm i e^{\pm\theta i}}{2\sin(\theta)} \frac{\log(\nu)}{\nu^2} \sum_{\ell=0}^{\nu-1} e^{\pm 2\theta i \ell} \ell. \tag{1.20}$$

Proof. By Stirling's formula we know that as $s \to \infty$

$$\log \Gamma\left(\frac{s+k+\ell}{\nu}\right) = \log\sqrt{2\pi} + \left(\frac{s+k+\ell}{\nu} - \frac{1}{2}\right) \log\left(\frac{s+k+\ell}{\nu}\right) - \frac{s+k+\ell}{\nu} + o(1).$$

Into this we plug the expansion

$$\log\left(\frac{s+k+\ell}{\nu}\right) = \log\left(s+k+\ell\right) - \log(\nu) = \log(s) + \frac{k+\ell}{s} - \log(\nu) + O(s^{-2})$$

and obtain

$$\sum_{\ell=0}^{\nu-1} e^{2\theta i \ell} \log \Gamma\left(\frac{s+k+\ell}{\nu}\right) = \sum_{\ell=0}^{\nu-1} e^{2\theta i \ell} \left(\frac{\ell}{\nu}\log(s) - \frac{\ell}{\nu}\log(\nu)\right) + o(1)$$

because terms independent of ℓ drop out. Now you can read off $C_{\text{ell}}^{\pm}$. For A keep in mind that $\log(s) = \frac{1}{2}\log\bigl(s(s-1)\bigr) + o(1)$ and

$$\frac{1}{\nu} \sum_{\ell=0}^{\nu-1} e^{2\theta i \ell} \ell = \frac{1}{e^{2\theta i} - 1} = \frac{e^{-\theta i}}{2i \sin(\theta)}. \tag{1.21}$$

□

Remark. The reason why we don't use (1.21) to simplify $C^{\pm}_{\mathrm{ell}}$ will become clear when we prove Theorem 1.3 in Section 1.6.4.

1.6 Proofs and remarks

1.6.1 Proof of Theorem 1.1

We want to apply the first and second part of Proposition 1.5 to the case $B = -\Delta_{2k}$, $\lambda = s(1-s)$ in order to conclude that the functional determinant $D(s)$ is an entire function of finite order with

$$\log D(s) = c_{-1}\Big(\log(s(s-1)) - 1\Big)s(1-s) + c_0 \log(s(s-1)) + o(1), \qquad s \to \infty. \tag{1.22}$$

For then by Proposition 1.9, $D(s)$ and the completed Selberg zeta function $\Xi(s)$ are entire functions of finite order with the same zeroes of the same multiplicities, hence their logarithms only differ by a polynomial in s. We remark that this polynomial is actually a polynomial in $s(1-s)$ since both functions are symmetric under $s \leftrightarrow 1-s$. Anyway, Theorem 1.1 is then obvious from comparing (1.22) with the asymptotic expansions for $\log Z$ (classical), $\log \Xi_{\mathrm{I}}$ (Lemma 1.10) and $\log \Xi_{\mathrm{ell}}$ (Lemma 1.11).

Remark 1.2. The above proof actually gives the equality

$$\log D = \log \Xi\, + C_{\mathrm{I}} + C_{\mathrm{ell}}.$$

of the logarithms we agreed to choose.

For this we need to prove that the heat kernel of $-\Delta_{2k}$ has an asymptotic expansion of the form (1.5). If Γ is strictly hyperbolic, this is a classical statement about heat kernels on the manifold $X = \Gamma\backslash\mathbb{H}$. If there are elliptic elements, there are two ways to reduce the proof to the strictly hyperbolic situation.

First we may examine the elliptic contribution to the Selberg trace formula [37, Eq. (3.2), p. 74] when we plug in $h(r) = e^{-t(r^2+1/4)}$ to compute the heat trace. For example, if $\mathfrak{j}$ is flat, i.e. $k = 0$, the elliptic contribution is a finite linear combination of integrals of the form

$$\int_{-\infty}^{\infty} \frac{e^{-2\theta r}}{1+e^{-2\pi r}} e^{-t(r^2+1/4)}\, dr, \qquad 0 < \theta < \pi.$$

Using the exponential series, we see that this actually has a expansion as required in (1.5) with vanishing c_{-1}.

A more elegant alternative is to use the classical theorem that Γ has a strictly hyperbolic subgroup Γ_0 of finite index. If $\mathfrak{j}_0$ denotes the restriction of $\mathfrak{j}$ to this subgroup, the domain of the $\mathfrak{j}$-automorphic Laplacian injects into the domain of the $\mathfrak{j}_0$-automorphic Laplacian. Why is this so? First of all, any function f on $\mathbb{H}$ automorphic with respect $\mathfrak{j}$ is trivially automorphic with respect to the restriction $\mathfrak{j}_0$. Furthermore, from a fundamental domain $\mathcal{F}$ for Γ one obtains a fundamental domain (up to a null set) for Γ_0 in the usual fashion, namely by choosing representatives $\gamma_1, \ldots, \gamma_r$ for the Γ_0 cosets in Γ and letting

$$\mathcal{F}_0 = \bigcup_{j=1}^{r} \gamma_j \mathcal{F}.$$

Now if f is square integrable over $\mathcal{F}$ and automorphic with respect to $\mathfrak{j}$, a straightforward computation shows that it is also square-integrable over $\mathcal{F}_0$:

$$\begin{aligned}\int_{\mathcal{F}_0} |f(z)|^2 \,\mathrm{dvol}(z) &= \sum_{j=1}^{r} \int_{\mathcal{F}} |f(\gamma_j(z))|^2 \,\mathrm{dvol}(z) \\ &= \sum_{j=1}^{r} \int_{\mathcal{F}} |\mathfrak{j}(\gamma_j, z) f(z)|^2 \,\mathrm{dvol}(z) \\ &= \sum_{j=1}^{r} \int_{\mathcal{F}} |f(z)|^2 \,\mathrm{dvol}(z)\end{aligned}$$

where dvol denotes the hyperbolic measure on $\mathbb{H}$ and the last step uses the fact that $\mathfrak{j}$ is unitary. The above injection of domains obviously respects the Laplace eigenspace decomposition. Hence we see, that the $\mathfrak{j}$-automorphic Laplacian has the same eigenvalues as the $\mathfrak{j}_0$-automorphic Laplacian just with possibly smaller multiplicities and we've again reduced the problem to the strictly hyperbolic case.

1.6.2 Proof of Corollary 1.2

Now Corollary 1.2 about the reduced determinant of the automorphic Laplacian is proved easily applying the third formula of Proposition 1.5 to the case $B = -\Delta_{2k}$ and $\mu_0 = k(1-k)$. This is possible because the Kodaira Laplacian $\Box_0$ is known to be non-negative so that the basic identity (1.3) implies

$$-\Delta_{2k} \geq k(1-k).$$

We obtain

$$\det{}'(-\Delta_{2k} - k(1-k)) = \lim_{s \to k, 1-k} \frac{D(s)}{(k(1-k) - s(1-s))^N}.$$

For $k \neq \frac{1}{2}$ all derivatives of the denominator at $s = k$ or $s = 1-k$ up order $N-1$ vanish and the Nth derivative is $N!(2k-1)^N$ or $N!(1-2k)^N$, respectively. For $k = \frac{1}{2}$ the denominator factors as $\left(s - \frac{1}{2}\right)^{2N}$. If we now express $D(s)$ in terms of $\Xi(s)$ via Theorem 1.1, the corollary follows from L'Hôpital's rule.

Remark. Combining the logarithmic version of the third part of Proposition 1.5 with Remark 1.2, we get the slightly stronger result

$$\begin{aligned}\log \det{}'(-\Delta_{2k} - k(1-k)) &= C_{\mathrm{I}} + C_{\mathrm{ell}} \\ &\quad + \lim_{s \to k, 1-k} \left\{\log \Xi(s) - N \log\bigl(k(1-k) - s(1-s)\bigr)\right\}. \end{aligned} \tag{1.23}$$

1.6.3 A byproduct: Computing ordinary torsion

Note that for torsion-free Γ, equation (1.23) almost computes the ordinary torsion of the vector bundle $\overline{E}$ over the Riemann surface $X = \Gamma \backslash \mathbb{H}$ given by $\mathfrak{j}$. The only little mistake one makes is that the operator $-\Delta_{2k} - k(1-k)$ is actually twice the Kodaira Laplacian $\Box_0$. But this is easily amended: By the second statement of Proposition 1.5, the expansion coefficients of $\Theta_{-\Delta_{2k}}$ can also be read off from the expansion of the functional determinant. In the torsion-free case, the only relevant contribution comes from $\log \Xi_{\mathrm{I}}$. Thus Lemma 1.10 tells us that

$$c_{-1}(-\Delta_{2k}) = \frac{d\,\mathrm{vol}(X)}{4\pi}, \qquad c_0(-\Delta_{2k}) = \frac{d\,\mathrm{vol}(X)}{2\pi}\left(\frac{k^2}{2} - \frac{1}{6}\right).$$

Now we use the fourth part of Proposition 1.5 to compute the torsion of $\overline{E}$ as follows

$$\begin{aligned}\mathcal{T}(\overline{E}) &= -\log\det{}'\left(\tfrac{1}{2}\left(-\Delta_{2k} - k(1-k)\right)\right)\\ &= -\log\det{}'\left(-\Delta_{2k} - k(1-k)\right) + \log(2)(c_0 + k(1-k)c_{-1} - N)\\ &= -\log\det{}'\left(-\Delta_{2k} - k(1-k)\right) + \log(2)\left(\frac{d\,\mathrm{vol}(X)}{2\pi}\left(\frac{k}{2} - \frac{1}{6}\right) - N\right)\end{aligned} \tag{1.24}$$

The terms on the right all have a simple topological interpretation: By Hodge theory we may view N as the dimension of the cohomology $H^0(X, E)$. Plugging in $\mathrm{vol}(X)/2\pi = 2h-2 = -\chi(X)$ as well as $d = \mathrm{rk}(E)$ and $dk(2h-2) = \deg(E)$, we ultimately arrive at

$$\begin{aligned}\mathcal{T}(\overline{E}) = &-\log\det{}'\left(-\Delta_{2k} - k(1-k)\right)\\ &+ \log(2)\left(\mathrm{rk}(E)\frac{\chi(X)}{6} + \frac{\deg(E)}{2} - \dim(H^0(X,E))\right)\end{aligned}$$

So we conclude that equation (1.23) indeed computes the torsion of $\overline{E}$ up to a topological correction term. Evaluating the limit in (1.23) in the case $\overline{E} = \overline{\omega}^k$, $k \geq 2$, we obtain the following explicit formula for the ordinary torsion of k-differentials:

Corollary 1.12 (Analytic torsion of very ample powers of the canonical bundle)**.** *Let Γ be a strictly hyperbolic Fuchsian group. Then the ordinary torsion of the line bundle of k-differentials, $k \geq 2$, on the Riemann surface $X = \Gamma\backslash\mathbb{H}$ can be computed as follows*

$$\begin{aligned}\mathcal{T}(\overline{\omega}^k) = &-\log Z(k) - V\left(2\zeta'(-1) - \left(k - \frac{1}{2}\right)^2 + \left(k - \frac{1}{2}\right)\log(2\pi)\right)\\ &- V\sum_{j=1}^{2k-2}\left(j - \left(k - \frac{1}{2}\right)\right)\log(j)\\ &+ \log(2)\left(-V\left(\frac{k}{2} - \frac{1}{3}\right)\right)\end{aligned}$$

with $V := \mathrm{vol}(X)/(2\pi) = -\chi(X)$ and the last term on the right being the metric correction term.

Remark. In Section 2.6 we apply this formula to quadratic differentials, i.e. the case $k = 2$.

Proof. When we combine (1.24) with (1.23), choose the limit $s \to k$ and plug in the definitions of C_{I} (see Lemma 1.10) and $\log \Xi_{\mathrm{I}}$ (see Definition 3), we get

$$\begin{aligned} \mathcal{T}(\overline{\omega}^k) = &- \log Z(k) - \underbrace{V\left(-\frac{1}{4} - \frac{1}{2}\log(2\pi) + 2\zeta'(-1)\right)}_{C_{\mathrm{I}}} \\ &- V\left(k\log(2\pi) + k(1-k) + \left(\frac{1}{2} + k\right)\log\Gamma(2k) + \log\Gamma_2(2k+1)\right) \\ &- \lim_{s\to k}\left\{V\left(\frac{1}{2} - k\right)\log\Gamma(s-k) - N\log\bigl(k(1-k) - s(1-s)\bigr)\right\} \\ &+ \log(2)\underbrace{\left(V\left(\frac{k}{2} - \frac{1}{6}\right) - N\right)}_{\text{metric correction term}} \end{aligned} \tag{1.25}$$

Since there are no elliptic or parabolic elements, we know $V = -\chi(X) = 2h-2$, h being the genus of X. Moreover, recall that in that case by Riemann-Roch, $N = \dim(H^0(X, \omega^k)) = \deg(\omega^k) - h + 1 = k(2h-2) - h + 1 = \left(k - \frac{1}{2}\right)V$ so that the third line simplifies to give $V\left(k - \frac{1}{2}\right)$ times

$$\begin{aligned} &\lim_{s\to k}\left\{\log\Gamma(s-k) + \log\bigl(k(1-k) - s(1-s)\bigr)\right\} \\ &= \lim_{s\to k}\log\left(\frac{k(1-k) - s(1-s)}{s-k}\right) = \log(2k-1) \end{aligned}$$

where we have used $\Gamma(s) \sim \frac{1}{s}$ as $s \to 0$ and L'Hôpital's rule. Next we tackle the two Gamma terms from the second line in (1.25) using the rule $\log\Gamma_2(n+1) = \sum_{j=1}^{n} j\log(j) - n\log(n!)$:

$$\begin{aligned} &\left(\frac{1}{2} + k\right)\log\Gamma(2k) + \log\Gamma_2(2k+1) \\ &= \left(\frac{1}{2} + k\right)\log((2k-1)!) + \sum_{j=1}^{2k}(j-2k)\log(j) \\ &= \sum_{j=1}^{2k-1}\left(j - \left(k - \frac{1}{2}\right)\right)\log(j) \end{aligned}$$

Then we finish the proof by noting that the summand for $j = 2k-1$ cancels with the contribution of the limit term from the third line. $\square$

1.6.4 Proof of Theorem 1.3

The eigenspace interpretation

First we verify the eigenspace interpretation given at the end of Section 1.2.2. We take an element in the domain of the Laplacian Δ_{2k}, i.e. a square-integrable section of the vector bundle E over the Riemann surface $X = \Gamma\backslash\mathbb{H}$. In our interpretation, this is a measurable function f on the upper half plane $\mathbb{H}$ with values in $\mathbb{C}^d$, square-integrable over any fundamental domain $\mathcal{F}$ for Γ, such that

$$f(\gamma(z)) = \mathfrak{j}(\gamma, z)f(z), \qquad \gamma \in \Gamma.$$

Next we ask: Exactly when is f an eigenvector with eigenvalue ξ for the action g^* on the sections of E? By the very definition of this action, this means in our automorphic interpretation that

$$\tilde{\mathfrak{j}}(\tilde{g},\tilde{g}^{-1}(z))f(\tilde{g}^{-1}(z)) = \xi\, f(z),$$

which is equivalent to

$$f(\tilde{g}(z)) = \overline{\xi}\,\tilde{\mathfrak{j}}(\tilde{g},z)f(z).$$

So $g^*f = \xi f$ iff f is automorphic with respect to the factor $\overline{\rho_\xi}\otimes\tilde{\mathfrak{j}}$ defined on the extended group $\tilde{\Gamma}$ in Section 1.2.2. Since Γ is of finite index n in $\tilde{\Gamma}$, the same proof as in Section 1.6.1 shows that for such an f the square-integrability conditions with respect to Γ and $\tilde{\Gamma}$ are equivalent.

Plan of attack

In Section 1.6.3 we have seen what correction in computing torsion is necessary because $-\Delta_{2k} - k(1-k)$ is twice the Kodaira Laplacian. We combine this with the Fourier transform formula of Lemma 1.7 and the above eigenspace interpretation to conclude that the equivariant torsion of $\overline{E}$ is given by

$$\begin{aligned}\mathcal{T}_g(\overline{E}) = -\sum_{\xi^n=1}\xi\log\det{}'\big(-\Delta_{2k}(\xi)-k(1-k)\big)&\\ \underbrace{+\log(2)\sum_{\xi^n=1}\xi\big(c_0(\xi)+k(1-k)c_{-1}(\xi)-N(\xi)\big)}_{\text{correction term}}&\end{aligned}\tag{1.26}$$

where $\Delta_{2k}(\xi)$ denotes the Laplacian on $\overline{\rho_\xi}\otimes\tilde{\mathfrak{j}}$-automorphic forms and $c_j(\xi)$ and $N(\xi)$ its heat kernel expansion constants and kernel dimension respectively. The hard part is the first sum that we shall tackle first. By (1.23), i.e. the logarithmic version of Corollary 1.2, we know that it can be computed as minus the limit of

$$\begin{aligned}&\sum_{\xi^n=1}\xi\Bigg(C_{\mathrm{I}}+C_{\mathrm{ell}}(\xi)\\ &+\log Z(\xi)(s)+\log\Xi_{\mathrm{I}}(s)+\log\Xi_{\mathrm{ell}}(\xi)(s)-N(\xi)\log\big(k(1-k)-s(1-s)\big)\Bigg)\end{aligned}\tag{1.27}$$

as $s\to k$ or $s\to 1-k$. We immediately see that the two identity terms drop out because they don't depend on ξ. It is also clear that the Fourier transform of $N(\xi)$ is the trace of the action of the automorphism g on the kernel of $\Box_0$, i.e. by Hodge theory

$$\sum_{\xi^n=1}\xi N(\xi) = \operatorname{tr}\left(g^*|_{H^0(X,E)}\right).\tag{1.28}$$

The other terms, the hyperbolic and the elliptic contribution, need a bit more explaining. This is done in the next two subsections. The proof of Theorem 1.3 is then completed in the last subsection where we compute the correction term.

The hyperbolic contribution

So let us consider the hyperbolic part of (1.27). Since $\mathfrak{j} = j_{2k}\chi$ extends to a factor of automorphy $\tilde{\mathfrak{j}}$ on the extended group $\tilde{\Gamma}$ iff χ extends to a multiplier $\tilde{\chi}$ for this group such that $\tilde{\mathfrak{j}} = j_{2k}\tilde{\chi}$, it makes sense to form the Selberg zeta function for $\tilde{\chi}$. Using Lemma 1.8, we compute the hyperbolic part of (1.27) as

$$-\sum_{\xi^n=1} \xi \sum_{\substack{\{\tilde{P}\}_{\tilde{\Gamma}} \\ \operatorname{tr}\tilde{P}>2}} \overline{\rho_\xi}(\tilde{P}) \operatorname{tr}\tilde{\chi}(\tilde{P}) \frac{\log N(\tilde{P}_0)}{\log N(\tilde{P})} \frac{N(\tilde{P})^{-s}}{1-N(\tilde{P})^{-1}}. \tag{1.29}$$

Since $\tilde{g}$ normalises Γ, every element $\tilde{P}$ of the enlarged group $\tilde{\Gamma}$ (along with all its $\tilde{\Gamma}$ conjugates) belongs to a unique coset $\tilde{g}^\ell\Gamma$, $0 \le \ell \le n-1$. We call this ℓ the g-exponent of $\tilde{P}$. By the definition of ρ_ξ, we have

$$\overline{\rho_\xi}(\tilde{P}) = \overline{\xi}^\ell.$$

Hence in the above Fourier transform only those $\tilde{P}$ with g-exponent 1 survive since

$$\sum_{\xi^n=1} \xi\overline{\xi}^\ell = \begin{cases} n, & \ell = 1 \\ 0, & \text{otherwise.} \end{cases} \tag{1.30}$$

So suppose from now on that $\tilde{P}$ has g-exponent 1, i.e. $\tilde{P} \in \tilde{g}\Gamma$. The first thing to note is that its conjugacy class with respect to $\tilde{\Gamma}$ is actually not larger than the one with respect to the smaller group Γ. This follows from the fact that conjugation by $\tilde{g}$ can just as well be realised using the element $\tilde{P}^{-1}\tilde{g}$ of Γ:

$$\tilde{g}^{-1}\tilde{P}\tilde{g} = \left(\tilde{P}^{-1}\tilde{g}\right)^{-1} \tilde{P} \left(\tilde{P}^{-1}\tilde{g}\right).$$

Next consider $\tilde{P}_0 \in \tilde{\Gamma}$, the primitive hyperbolic element associated to $\tilde{P} \in \tilde{\Gamma}$. Since $\tilde{P}$ is a power of $\tilde{P}_0$, the g-exponent of $\tilde{P}_0$ is necessarily prime to n. If P_0 denotes the smallest positive power of $\tilde{P}_0$ that belongs to the smaller group Γ, we thus have $P_0 = \tilde{P}_0^n$. Hence the centraliser of $\tilde{P}$ in the smaller group Γ is given by

$$Z_\Gamma(\tilde{P}) = \{\pm P_0^m \mid m \in \mathbb{Z}\}$$

and it has index n in $Z_{\tilde{\Gamma}}(\tilde{P})$. In particular, we have for the Selberg norms

$$N(\tilde{P}_0) = \frac{1}{n} N(P_0).$$

This cancels nicely with (1.30) so that (1.29) equals $\log Z_{\tilde{g}\Gamma}(s)$ which we define as follows:

Definition 5 (Equivariant Selberg zeta function)**.** *The equivariant Selberg zeta function with respect to $\tilde{\mathfrak{j}} = j_{2k}\tilde{\chi}$ and $\tilde{g}$ is defined as the exponential of*

$$\log Z_{\tilde{g}\Gamma}(s) := -\sum_{\substack{\{\tilde{P}\}_\Gamma \subset \tilde{g}\Gamma \\ \operatorname{tr}\tilde{P}>2}} \operatorname{tr}\tilde{\chi}(\tilde{P}) \frac{\log N(P_0)}{\log N(\tilde{P})} \frac{N(\tilde{P})^{-s}}{1-N(\tilde{P})^{-1}}. \tag{1.31}$$

The elliptic contribution

This section contains the most interesting part of the proof. We use the definition of $\log \Xi^{\pm}_{\mathrm{ell}}$ from (1.18) and the explicit formula for $C^{\pm}_{\mathrm{ell}}$ from Lemma 1.11 to obtain by the same reasoning about g-exponents as before that the total elliptic contribution equals $E^{+}(s) + E^{-}(s)$ with

$$E^{\pm}(s) := n \sum_{\substack{\{\tilde{R}\}_\Gamma \subset \tilde{g}\Gamma \\ 0<\theta<\pi}} e^{i\theta 2k} \operatorname{tr} \tilde{\chi}(\tilde{R}) \frac{\pm i e^{\pm\theta i}}{2\sin(\theta)} \times$$
$$\times \frac{1}{n} \sum_{\ell=0}^{n-1} e^{\pm 2\theta i \ell} \left(\log \Gamma \left(\frac{s \pm k + \ell}{n} \right) + \frac{\ell \log(n)}{n} \right).$$

The only delicate point one has to note is that $\nu(\tilde{R}) = n$ because Γ is assumed strictly hyperbolic. The inner sum can now be expressed via the derivative at 0 of Lerch's Phi function:

Lemma 1.13. *For an n-th root of unity $e^{2\theta i}$ and $a \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$ we have*

$$\Phi'(e^{2\theta i}, 0, a) = \sum_{\ell=0}^{n-1} e^{2\theta i \ell} \left(\log \Gamma \left(\frac{a + \ell}{n} \right) + \frac{\ell}{n} \log(n) \right).$$

Proof. First of all, recall that the value and derivative of the slightly simpler Hurwitz zeta function $\zeta(w, a) = \sum_{j \geq 0} (j + a)^{-w}$ at $w = 0$ are classically known:

$$\zeta(0, a) = \tfrac{1}{2} - a, \qquad \zeta'(0, a) = \log \Gamma(a) - \log \left(\sqrt{2\pi} \right). \tag{1.32}$$

So all we need to do is express Lerch's Phi in terms of Hurwitz' zeta:

$$\begin{aligned} \Phi(e^{2\theta i}, w, a) &= \sum_{p=0}^{\infty} \sum_{\ell=0}^{n-1} \frac{e^{2\theta i (pn+\ell)}}{(pn + \ell + a)^w} \\ &= \sum_{\ell=0}^{n-1} e^{2\theta i \ell} \frac{1}{n^w} \sum_{p=0}^{\infty} \frac{1}{\left(p + \frac{\ell + a}{n}\right)^w} \\ &= \sum_{\ell=0}^{n-1} e^{2\theta i \ell} \frac{1}{n^w} \zeta \left(w, \frac{\ell + a}{n} \right). \end{aligned}$$

Taking the derivative at $w = 0$ we obtain

$$\Phi'(e^{2\theta i}, 0, a) = \sum_{\ell=0}^{n-1} e^{2\theta i \ell} \left(\zeta' \left(0, \frac{\ell + a}{n} \right) - \log(n)\, \zeta \left(0, \frac{\ell + a}{n} \right) \right).$$

Now we plug in the classical facts from (1.32) and watch the terms that don't depend on ℓ disappear. $\square$

This leaves us with the much more compact formula

$$E^{\pm}(s) = \sum_{\substack{\{\tilde{R}\}_\Gamma \subset \tilde{g}\Gamma \\ 0<\theta<\pi}} e^{i\theta 2k} \operatorname{tr} \tilde{\chi}(\tilde{R}) \frac{\pm i e^{\pm\theta i}}{2\sin(\theta)} \Phi'(e^{\pm 2\theta i}, 0, s \pm k) \tag{1.33}$$

which already looks a lot like the corresponding term in the statement of Theorem 1.3 and completes the computational part of its proof.

The rest of this section is devoted to checking interpretations. Let us first explain why the sum in (1.33) really ranges over the set X^g of fixed points of the action of g on the Riemann surface X. Consider the map which sends an elliptic element $\tilde{R}$ in the coset $\tilde{g}\Gamma$ to its unique fixed point $z_0(\tilde{R})$ in the upper half plane $\mathbb{H}$. For any $T \in \Gamma$ the conjugate $T\tilde{R}T^{-1}$ has a fixed point which is equivalent mod Γ. To be precise, we have

$$z_0\left(T\tilde{R}T^{-1}\right) = T(z_0(\tilde{R})).$$

Hence we get a well-defined map from the Γ conjugacy classes in the coset $\tilde{g}\Gamma$ to X. The image is exactly X^g because a point z mod Γ of $X = \Gamma\backslash\mathbb{H}$ is a fixed point of the action induced by $\tilde{g}$ iff there exists $S \in \Gamma$ such that

$$\tilde{g}(z) = S(z),$$

i.e. iff $z = z_0(S^{-1}\tilde{g})$. Now what about injectivity? Suppose

$$z_0(\tilde{R}_1) \equiv z_0(\tilde{R}_2) \mod \Gamma.$$

Can we deduce that $\{\tilde{R}_1\}_\Gamma = \{\tilde{R}_2\}_\Gamma$? Not quite but almost. If z_1 and z_2 are the fixed points in $\mathbb{H}$ of $\tilde{R}_1$ and $\tilde{R}_2$ respectively, chose $S \in \Gamma$ with $S(z_1) = z_2$. Then z_1 is also the fixed point of $S^{-1}\tilde{R}_2S$. But then $\tilde{R}_1$ and $S^{-1}\tilde{R}_2S$ are two elliptic elements of the Fuchsian group $\tilde{\Gamma}$ with the same fixed point. This means that they are powers $\neq \pm 1$ of the same primitive elliptic element. But exactly two of these powers have g-exponent 1 and they are the negatives of each other whence

$$\{\tilde{R}_1\}_\Gamma = \pm\{\tilde{R}_2\}_\Gamma.$$

The restriction $0 < \theta < \pi$ in (1.33) removes this ambiguity making it really a sum over the fixed point set X^g.

Next we check the interpretation of $e^{-2\theta i}$ as the derivative at the fixed point. This is a straightforward calculation: By lifting, the derivative of the automorphism at $p = z_0 \mod \Gamma$ on X, $z_0 = z_0(\tilde{R})$, is the derivative of $\tilde{R}$ at z in $\mathbb{H}$. Choosing $S \in \mathrm{SL}_2(\mathbb{R})$ with $S(i) = z$ we get $S^{-1}\tilde{R}S = R_\theta$, $\theta = \theta(\tilde{R})$. So we may assume $z_0 = i$ and $\tilde{R} = R_\theta$. But then

$$\frac{d}{dz}\left(\frac{\cos\theta z - \sin\theta}{\sin\theta z + \cos\theta}\right)_{z=i} = (\sin\theta i + \cos\theta)^{-2} = e^{-2\theta i}.$$

At last, we need to check that $e^{i\theta 2k}\operatorname{tr}\tilde{\chi}(\tilde{R})$ is indeed the trace of the action of g restricted to the fibre of E_p over the fixed point $p = z_0 \mod \Gamma$. In our automorphic interpretation, this is action is given by the endomorphism $\tilde{\mathrm{j}}(\tilde{R}, z_0) = j_{2k}(\tilde{R}, z_0)\tilde{\chi}(\tilde{R})$ of $\mathbb{C}^d$. Therefore the proof is complete if we can show that $j_{2k}(\tilde{R}, z_0)$ equals $j_{2k}(R_\theta, i)$ because the latter can be computed from the definition as $e^{i\theta 2k}$. To that end, recall Petersson's additive factor system $w(M, S)$ by which we can express the extent to which j_{2k} fails to be a factor of automorphy:

$$\frac{j_{2k}(M, S(z))j_{2k}(S, z)}{j_{2k}(MS, z)} = \exp(2\pi i\, 2k\, w(M, S)), \qquad M, S \in \mathrm{SL}_2(\mathbb{R}).$$

From [14, Formula (1.3.8)a on p. 18] we know that it satisfies

$$w(\tilde{R}, S) = w(S, R_\theta), \qquad \tilde{R} = SR_\theta S^{-1}.$$

Using this in the previous formula at $z = i$ we get

$$\frac{j_{2k}(\tilde{R}, S(i))j_{2k}(S, i)}{j_{2k}(\tilde{R}S, i)} = \frac{j_{2k}(S, R_\theta(i))j_{2k}(R_\theta, i)}{j_{2k}(SR_\theta, i)},$$

which after cancellation leaves us with the desired equality.

The correction term

We can now easily compute the correction term in (1.26). We use Lemmas 1.10 and 1.11 to determine $c_{-1}(\xi)$ and $c_0(\xi)$. In doing so, we see that the identity contributions don't depend on ξ and therefore get killed when summing over all n-th roots of unity ξ, which leaves us with $A = A(\xi)$ from Lemma 1.11 as the only relevant constant. Hence the correction term reduces to

$$\log(2) \sum_{\xi^n=1} \xi\big(A(\xi) - N(\xi)\big).$$

Using the explicit formula (1.19) for the constant A from Lemma 1.10 and the usual g-exponent considerations, we get

$$\sum_{\xi^n=1} \xi A(\xi) = \sum_{\substack{\{\tilde{R}\}_\Gamma \\ 0<\theta<\pi}} e^{i\theta 2k} \operatorname{tr} \tilde{\chi}(\tilde{R}) \frac{1}{4\sin^2(\theta)}.$$

Remembering (1.28) and the identifications proved in the previous section, the correction term is clearly

$$\log(2) \left(\sum_{p\in X^g} \frac{\operatorname{tr}(g|_{E_p})}{|1 - T_p g|^2} - \operatorname{tr}\left(g^*|_{H^0(X,E)}\right) \right) \tag{1.34}$$

because $4\sin^2(\theta) = |1 - e^{\pm 2\theta i}|^2$.

1.6.5 Proof of Corollary 1.4

Let us write $\overline{L}$ for $\overline{\omega}^k$ in this proof. To compute $\mathcal{T}_g(\overline{L})$, we apply Theorem 1.3 in the case where $k \geq 0$ is an integer and both χ and $\tilde{\chi}$ are the trivial scalar multiplier. The elliptic contribution to $\mathcal{T}_g(\overline{L})$ is then simply minus the limit of

$$\sum_{p\in X^g} \frac{ie^{\theta i 2k}}{2\sin(\theta)} \left(e^{\theta i}\Phi'(e^{2\theta i}, 0, s+k) - e^{-\theta i}\Phi'(e^{-2\theta i}, 0, s-k)\right) \tag{1.35}$$

as $s \to k$ or as $s \to 1 - k$.

Let us begin with the case $k = 0$ in which L is simply the trivial line bundle on X. Note that by Lemma 1.6, this covers the case $k = 1$ as well. In the case

$k = 0$ it is most convenient to take the limit $s \to 1$ of (1.35) because this simply gives the finite term

$$\sum_{p \in X^g} \frac{i}{2\sin(\theta)} \left(e^{-\theta i} F'(e^{2\theta i}, 0) - e^{\theta i} F'(e^{-2\theta i}, 0)\right)$$

By the fact that $F'(e^{-2\theta i}, 0)$ is the complex conjugate of $F'(e^{2\theta i}, 0)$, this equals

$$\begin{aligned} &\sum_{p \in X^g} \frac{i}{2\sin(\theta)} 2i\mathrm{Im}\Big(e^{-\theta i} F'(e^{2\theta i}, 0)\Big) \\ &= \sum_{p \in X^g} \Big(\mathrm{Re}\Big(F'(e^{2\theta i}, 0)\Big) - \cot(\theta)\mathrm{Im}\Big(F'(e^{2\theta i}, 0)\Big)\Big). \end{aligned}$$

When L is the trivial line bundle, $H^0(X, L)$ is just the one-dimensional space of constant functions on X on which our automorphism clearly acts with eigenvalue 1. Hence the other two terms in Theorem 1.3 are just

$$\log Z_{\tilde{g}\Gamma}(s) - \log\big(s(s-1)\big)$$

whose limit as $s \to 1$ is $\log Z'_{\tilde{g}\Gamma}(1)$ by L'Hôpital's rule.

Now we proceed to the case $k \geq 2$. This time, the obvious limit $s \to k$ will do. Then we know that $\log Z_{\tilde{g}\Gamma}(s)$ has the finite limit $\log Z_{\tilde{g}\Gamma}(k)$ since its series representation (1.31) converges for $\mathrm{Re}(s) > 1$. As for the elliptic terms, we start with an elementary lemma

Lemma 1.14. *Let $a \geq 0$ be an integer. Then as $s \to a$*

$$\Phi'(z, 0, s) \sim \begin{cases} -\log(s) + F'(z, 0), & a = 0 \\ z^{-a}\big(F'(z, 0) + \sum_{1 \leq j \leq a-1} z^j \log(j)\big), & a \geq 1. \end{cases}$$

Proof. For $a = 0$ take the derivative at $w = 0$ of the trivial identity

$$\Phi(z, w, s) = s^{-w} + z\Phi(z, w, s+1)$$

and remember that $z\Phi(z, w, 1) = F(z, w)$. For $a \geq 1$ do the same to the equally trivial equation

$$\Phi(z, w, a) = z^{-a} \sum_{j \geq a} z^j j^{-w} = z^{-a} \left(F(z, w) - \sum_{1 \leq j \leq a-1} z^j j^{-w}\right).$$

□

By this lemma, (1.35) equals

$$\begin{aligned} \sum_{p \in X^g} \frac{ie^{\theta i 2k}}{2\sin(\theta)} \Bigg(& e^{\theta i} e^{2\theta i(-2k)} \Big(F'(e^{2\theta i}, 0) + \sum_{1 \leq j \leq 2k-1} e^{2\theta i\, j} \log(j)\Big) \\ & - e^{-\theta i}\Big(-\log(s-k) + F'(e^{-2\theta i}, 0)\Big)\Bigg) + o(1) \end{aligned}$$

as $s \to k$. By $ie^{\theta i}/2\sin(\theta) = e^{2\theta i}/(1-e^{2\theta i})$ and $ie^{-\theta i}/2\sin(\theta) = 1/(1-e^{2\theta i})$, this in turn becomes

$$\sum_{p\in X^g} \frac{i}{2\sin(\theta)} 2i\mathrm{Im}\Big(e^{\theta i(1-2k)}F'(e^{2\theta i},0)\Big)$$
$$+\sum_{p\in X^g} \frac{e^{-2\theta i(k-1)}}{1-e^{2\theta i}} \sum_{1\le j\le 2k-2} e^{2\theta i\, j}\log(j)$$
$$+\sum_{p\in X^g} \frac{e^{\theta i 2k}}{1-e^{2\theta i}}\big(\log(s-k)+\log(2k-1)\big)+o(1).$$

At a fixed point, $e^{-2\theta i}$ and $e^{\theta i2k}$ are the actions of the automorphism on the tangent bundle and on L respectively. From the Lefschetz trace formula we deduce that

$$\sum_{p\in X^g} \frac{e^{\theta i2k}}{1-e^{2\theta i}} = \mathrm{tr}\left(g^*|_{H^0(X,L)}\right),$$

using the fact that L is very ample. Putting everything together, we get that as $s \to k$, (1.35) equals

$$\sum_{p\in X^g}\left(\frac{\sin((2k-1)\theta)}{\sin(\theta)}\mathrm{Re}\Big(F'(e^{2\theta i},0)\Big)-\frac{\cos((2k-1)\theta)}{\sin(\theta)}\mathrm{Im}\Big(F'(e^{2\theta i},0)\Big)\right)$$
$$+\sum_{p\in X^g}\frac{e^{-2\theta i(k-1)}}{1-e^{2\theta i}}\sum_{1\le j\le 2k-2} e^{2\theta i\, j}\log(j)$$
$$+\mathrm{tr}\left(g^*|_{H^0(X,L)}\right)\big(\log(s-k)+\log(2k-1)\big)+o(1).$$

Again we see by L'Hôpital's rule that as $s \to k$, the last line cancels nicely with the singular term

$$-\,\mathrm{tr}\left(g^*|_{H^0(X,L)}\right)\,\log\big(k(1-k)-s(1-s)\big)$$

from Theorem 1.3.

Now all that is left to prove is the assertion about the real part of $F'(e^{2\theta i},0)$. For this we apply [23, Lemma 13] to see that for $0<\theta<\pi$, the value of the meromorphic continuation of

$$\sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}} \frac{e^{2\theta ij}\log\left(j^2\right)}{|j|^{2w}}, \qquad \mathrm{Re}(w) > \frac{1}{2}$$

at $w=0$ equals

$$2\log(2\pi)+2\gamma_{\mathrm{Euler}}+\Psi\left(\frac{\theta}{\pi}\right)+\Psi\left(1-\frac{\theta}{\pi}\right).$$

Then we finish by noting that the above series equals $-4\mathrm{Re}\Big(F'(e^{2\theta i},2w)\Big)$.

Chapter 2

Arithmetic applications

2.1 Overview

The setting of this chapter is now arithmetic: We study Hermitian algebraic vector bundles over an arithmetic surface.

Let us give an overview of the sections of Chapter 2.

Section 2.2 is devoted to the exposition of two general arithmetic theorems: The arithmetic Riemann-Roch theorem of Gillet and Soulé (Theorem 2.1) and the arithmetic Lefschetz fixed point formula of Köhler and Roessler (Theorem 2.2). While both theorems hold in very general settings, we shall only need them for powers $\overline{\omega}^k$ of the canonical bundle on arithmetic surfaces over the integers. Modesty commands that we only outline how to obtain the specialisations we need from the general theorems.

The first theorem links the arithmetic degree of the cohomology (cf. Definition 6) of $\overline{\omega}^k$ to its ordinary torsion and the height of the arithmetic surface (cf. Definition 7) while the second, provided an action of the group scheme of n-th roots unity, establishes a relation between the arithmetic Lefschetz trace (cf. Definition 8) of $\overline{\omega}^k$ on the one hand and its equivariant torsion and the arithmetic height of the fixed point scheme (cf. Definition 9) on the other hand.

We also explain how a weaker result can be obtained if we are only given an action of the constant group scheme $\mathbb{Z}/(n)$ (Remark 2.1) and state this weaker result in the $n = 2$ case (Corollary 2.16) as it will be needed in the final Section 2.6.

The following Section 2.3 is but an interlude. It contains all the classical definitions and facts from the theory of indefinite rational quaternion algebras needed for our treatment of Shimura curves, especially the notion of an Eichler order along with Eichler's formula for the number of optimal embeddings of a quadratic order into a given Eichler order (Proposition 2.3). It is in this section where we start illustrating all the formulae and facts by computing a concrete example (henceforth referred to as the Main Example), namely that of discriminant $D = 26$ and level $N = 1$.

Then ensues the rather long Section 2.4 covering compact Shimura curves $\mathcal{X}(D, N)$. To emphasise the importance of this class of examples, we start off by giving a fascinating formula for their arithmetic height (Theorem 2.4) in the case $N = 1$.

Then using the terminology of the preceding Section 2.3, we can easily define Shimura curves as the quotient of the upper half plane by the Fuchsian group obtained by embedding the norm 1 units of an Eichler order into the real 2×2 matrices. This is followed by the well-known formulae for the genus (Theorem 2.5) and the number of elliptic cycles (Theorem 2.6) of a Shimura curve, which enables us to compute its volume by the Riemann-Hurwitz formula (2.9).

We then discuss the automorphisms of a Shimura curve, concentrating on Atkin-Lehner involutions, especially on those corresponding to divisors n of the discriminant D. The most important observation is that such an Atkin-Lehner involution is induced by the action of any element of the Eichler order having norm n (Remark 2.2).

Next we show how to count conjugacy classes of given norm and trace using Eichler's optimal embeddings formula (Corollary 2.7). This result result can be further simplified when the norm divides the discriminant D (Corollary 2.8), which in turn enables us to explicitly evaluate Selberg zeta functions, both classical and equivariant (Proposition 2.10). We do this numerically in our Main Example and discuss the quality of such approximations in general (Lemma 2.12).

Then we finish Section 2.4 by computing the height of the fixed point scheme of an Atkin-Lehner involution (Proposition 2.13) which reduces the problem to an application of the generalised Chowla-Selberg formula (Theorem 2.14).

In Section 2.5 we specialise our computation of equivariant torsion of k-differentials (Corollary 2.15) and the arithmetic Lefschetz fixed point formula (Corollary 2.16) to the case of an involution. Then we apply these results to Atkin-Lehner involutions on Shimura curves, using our computations of the equivariant Selberg zeta function and of the height of the fixed point scheme to obtain the explicit formula for the arithmetic Lefschetz trace of k-differentials stated in Theorem 0.1.

The last Section 2.6 contains the *finale grande*: We consider quadratic differentials on an arithmetic surface of genus two and the hyperelliptic involution and observe that in this case, the arithmetic degree and the arithmetic Lefschetz trace, i.e. the left hand sides of the two arithmetic theorems, are equal. This gives a curious relationship between ordinary torsion and the height of the surface on the one hand and equivariant torsion and the height of the fixed point scheme on the other (Proposition 2.18). All terms involved in this curious identity have been computed explicitly in our Main Example.

2.2 Two arithmetic theorems

2.2.1 Introduction to the setting

The main references for this section are [39] and [24]. For a readable survey see [40]. Let us now suppose that we are given an arithmetic surface $\mathcal{X}$ over a regular arithmetic ring D. By arithmetic surface we mean a regular scheme $\mathcal{X}$ of absolute Krull dimension 2 whose map of definition $f\colon \mathcal{X} \to \operatorname{Spec}(D)$ is projective and flat. For an explanation of regular arithmetic rings see [24, Sect. 4]. For simplicity we shall henceforth assume $D = \mathbb{Z}$ although analogous results can be obtained in the more general case. We shall write $X = \mathcal{X}(\mathbb{C})$ for the Riemann surface consisting of the complex points of $\mathcal{X}$. Furthermore, if $\overline{\mathcal{E}}$ is a

Hermitian vector bundle over $\mathcal{X}$, we denote by $\overline{E}$ the corresponding Hermitian bundle over X. A typical example for $\mathcal{X}$ will be an integral model $\mathcal{X}(D,N)$ of a Shimura curve (for details see Section 2.4), for $\overline{\mathcal{E}}$ we will usually take integral powers of the canonical bundle $\overline{\omega}$, the latter being equipped with the usual Petersson metric.

From now on, denote by $\widehat{\deg}\colon \widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathbb{Z})) \xrightarrow{\cong} \mathbb{R}$ the *arithmetic degree* map. A Hermitian structure on a finitely generated $\mathbb{Z}$-module V is by definition a Hermitian metric on $V \otimes \mathbb{C}$. Such a Hermitian $\mathbb{Z}$-module $\overline{V}$ defines an element of $\widehat{K_0}(\mathrm{Spec}(\mathbb{Z}))$, its *first arithmetic Chern class* $\widehat{c_1}(\overline{V})$ is an element of $\widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathbb{Z}))$ and its degree can be easily calculated as

$$\widehat{\deg}\big(\widehat{c_1}(\overline{V})\big) = -\log\left(\frac{\mathrm{covol}(V_{\mathrm{free}})}{\# V_{\mathrm{tors}}}\right)$$

where V_{tors} and V_{free} denote the torsion and free part of V, respectively. By the choice of a Kähler form on X, we can equip the harmonic q-forms on X with values in E with an L^2 metric. Via Hodge theory this gives a metric on $H^q(X,E)$, hence we have a Hermitian structure on the sheaf cohomology $H^q(\mathcal{X},\mathcal{E})$ and denote the resulting Hermitian $\mathbb{Z}$-module by $\overline{H^q(\mathcal{X},\mathcal{E})}$. We make the following

Definition 6 (Arithmetic degree of cohomology)**.** *The arithmetic degree of the cohomology of $\overline{\mathcal{E}}$ is the alternating sum*

$$\widehat{\deg}(\overline{H^\cdot(\mathcal{X},\mathcal{E})}) := \sum_{q\geq 0}(-1)^q \widehat{\deg}\left(\widehat{c_1}\left(\overline{H^q(\mathcal{X},\mathcal{E})}\right)\right).$$

That said, since $\mathcal{X}$ is of relative dimension 1 over $\mathrm{Spec}(\mathbb{Z})$, the induced map f_* of arithmetic Chow groups takes $\widehat{\mathrm{CH}}^2(\mathcal{X})$ to $\widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathbb{Z}))$. We apply this map to the square of the first Chern class of the canonical bundle $\overline{\omega}$ of $\mathcal{X}$. Up to taking the arithmetic degree, this is the height of the surface $\mathcal{X}$:

Definition 7 (Height of arithmetic surface)**.** *Let $\mathcal{X}$ be an arithmetic surface as above. Then its height $h_{\overline{\omega}}(\mathcal{X})$ with respect to the canonical bundle is defined as*

$$h_{\overline{\omega}}(\mathcal{X}) := \widehat{\deg}\left(f_*\widehat{c_1}(\overline{\omega})^2\right).$$

2.2.2 The arithmetic Riemann-Roch theorem

Now assume that $\overline{\omega} = \overline{Tf}^*$ and consider $\overline{\mathcal{E}} = \overline{\omega}^k$, $k \in \mathbb{Z}$. Then the arithmetic degree of the cohomology is related to the height of the surface and the ordinary torsion via the arithmetic Riemann-Roch theorem (see [15] and compare also [13]):

Theorem 2.1 (Gillet-Soulé)**.** *We have the following equality of real numbers*

$$\widehat{\deg}(\overline{H^\cdot(\mathcal{X},\omega^k)}) = \frac{1}{2}\mathcal{T}(\overline{\omega}^k) - \frac{1}{2}\left(2\zeta'(-1) - \frac{1}{12}\right)\chi(X) + \frac{1}{12}\left(6k^2 - 6k + 1\right)h_{\overline{\omega}}(\mathcal{X});$$

here $\chi(X) := \int_X c_1(TX)$ is the Euler characteristic of X, by ζ' we denote the derivative of Riemann's zeta function and $\mathcal{T}$ is the ordinary analytic torsion of a Hermitian vector bundle (cf. Section 1.4.1).

Proof. From [15, 4.2.3] we take

$$\widehat{\deg}(\overline{H^\cdot(\mathcal{X},\mathcal{E})}) = \frac{1}{2}\mathcal{T}(\overline{E}) - \frac{1}{2}\int_X \mathrm{Td}(TX)R(TX)\mathrm{ch}(E) + \widehat{\deg}\left(f_*\left(\widehat{\mathrm{Td}}(\overline{Tf})\widehat{\mathrm{ch}}(\overline{\mathcal{E}})\right)\right) \tag{2.1}$$

which is valid for any vector bundle $\overline{\mathcal{E}}$. Since the characteristic classes in the integrand are

$$\begin{aligned} \mathrm{Td}(TX) &= 1 + \dots \\ R(TX) &= \big(2\zeta'(-1) + \zeta(-1)\big)c_1(TX) + \dots \\ \mathrm{ch}(E) &= \mathrm{rk}(E) + \dots, \end{aligned}$$

the middle term on the right is just $-\frac{1}{2}\int_X \big(2\zeta'(-1) + \zeta(-1)\big)c_1(TX)$ whenever E is a line bundle.

As for the last term on the right, it suffices to show that the degree 2 part of the product $\widehat{\mathrm{Td}}(\overline{Tf})\widehat{\mathrm{ch}}(\overline{\omega}^k)$ equals $\left(\frac{k^2}{2} - \frac{k}{2} + \frac{1}{12}\right)\widehat{c_1}(\overline{\omega})^2$. But this is clear from the well-known expansions

$$\begin{aligned} \widehat{\mathrm{Td}}(\overline{Tf}) &= 1 + \frac{1}{2}\widehat{c_1}(\overline{Tf}) + \frac{1}{12}\widehat{c_1}(\overline{Tf})^2 + \dots \\ &= 1 - \frac{1}{2}\widehat{c_1}(\overline{\omega}) + \frac{1}{12}\widehat{c_1}(\overline{\omega})^2 + \dots \\ \widehat{\mathrm{ch}}(\overline{\omega}^k) &= 1 + k\widehat{c_1}(\overline{\omega}) + \frac{k^2}{2}\widehat{c_1}(\overline{\omega})^2 + \dots \end{aligned}$$

□

2.2.3 The arithmetic Lefschetz fixed point formula

The right arithmetic setting of an equivariant situation is the following: Suppose that the group scheme μ_n of n-th roots of unity acts on the scheme $\mathcal{X}$ in such a fashion that over $\mathbb{C}$, the automorphism g corresponds to a primitive root of unity ζ. Then the cohomology decomposes with respect to μ_n as

$$H^\cdot(\mathcal{X},\mathcal{E}) = \bigoplus_{\ell\in\mathbb{Z}/(n)} H^\cdot(\mathcal{X},\mathcal{E})_\ell \tag{2.2}$$

and we make the following

Definition 8 (Arithmetic Lefschetz trace)**.** *We define the arithmetic Lefschetz trace of* $\overline{\mathcal{E}}$ *as*

$$\sum_{\ell\in\mathbb{Z}/(n)} \zeta^\ell \widehat{\deg}(\overline{H^\cdot(\mathcal{X},\mathcal{E})_\ell}).$$

Clearly for $n = 1$, the arithmetic Lefschetz trace reduces to the the arithmetic degree of cohomology. Next we decompose the restriction of $\mathcal{E}$ to the fixed point scheme $\mathcal{X}_{\mu_n}$ as

$$\mathcal{E}|_{\mathcal{X}_{\mu_n}} = \bigoplus_{\ell\in\mathbb{Z}/(n)} \mathcal{E}_\ell.$$

Now we make the following

Definition 9 (Height of the fixed point scheme)**.** *The height of the fixed point scheme with respect to the ℓ-part of the canonical bundle is given by*

$$h_{\overline{\omega}_\ell}(\mathcal{X}_{\mu_n}) := \widehat{\deg}\,(f_*^{\mu_n}\hat{c}_1(\overline{\omega}_\ell))$$

where $f^{\mu_n}\colon \mathcal{X}_{\mu_n} \to \mathbb{Z}$ is the defining map of the fixed point scheme.

Morally, this is the height of the sum of the fixed point schemes at which g acts as ζ^ℓ on $\overline{\omega}$. Now as before, we specialise to the case $\overline{\mathcal{E}} = \overline{\omega}^k$, $k \in \mathbb{Z}$. Then these "fixed point heights" are related to the arithmetic Lefschetz trace via the equivariant torsion just as the height of the entire surface is related to the arithmetic degree of the cohomology via ordinary torsion:

Theorem 2.2 (Köhler-Roessler, [24, Thm. 7.14])**.** *We have the following equality of real numbers*

$$\begin{aligned}\sum_{\ell \in \mathbb{Z}/(n)} \zeta^\ell \widehat{\deg}(\overline{H^\cdot(\mathcal{X}, \omega^k)_\ell}) &= \frac{1}{2}\mathcal{T}_g(\overline{\omega}^k) \\ &+ \sum_{p \in X^g} \frac{e^{2\theta i k}}{1 - e^{2\theta i}} i R^{\mathrm{rot}}(2\theta) + \sum_{\substack{\ell \in \mathbb{Z}/(n) \\ \ell \neq 0}} \frac{\zeta^{k\ell}}{1-\zeta^\ell}\left(\frac{\zeta^\ell}{1-\zeta^\ell} + k\right) h_{\overline{\omega}_\ell}(\mathcal{X}_{\mu_n});\end{aligned}$$

here $R^{\mathrm{rot}}(2\theta)$ is the imaginary part of $F'(e^{2\theta i}, 0)$ whereby we mean the derivative at $w = 0$ of Jonquière's function $F(z, w) := \sum_{j \geq 1} z^j j^{-w}$ and $\mathcal{T}_g$ denotes the equivariant analytic torsion of a Hermitian vector bundle (cf. Section 1.4.2).

Proof. In [24, Thm. 7.14] we find a statement completely analogous to (2.1), namely

$$\begin{aligned}\sum_{\ell \in \mathbb{Z}/(n)} \zeta^\ell \widehat{\deg}(\overline{H^\cdot(\mathcal{X}, \mathcal{E})_\ell}) &= \frac{1}{2}\mathcal{T}_g(\overline{E}) \\ -\frac{1}{2}\int_{X^g} \mathrm{Td}_g(TX) R_g(TX) \mathrm{ch}_g(E) &+ \widehat{\deg}\left(f_*^{\mu_n}(\widehat{\mathrm{Td}}_{\mu_n}(\overline{Tf})\widehat{\mathrm{ch}}_{\mu_n}(\overline{\mathcal{E}}))\right). \qquad (2.3)\end{aligned}$$

with appropriate characteristic classes: The ordinary classes Td_g, R_g, ch_g live on the fixed point manifold $X^g = \mathcal{X}_{\mu_n}(\mathbb{C})$ whereas the arithmetic classes $\widehat{\mathrm{Td}}_{\mu_n}$, $\widehat{\mathrm{ch}}_{\mu_n}$ are objects on the fixed point scheme $\mathcal{X}_{\mu_n}$. On the right hand side, the integral over the 0-dimensional fixed point set X^g is simply the sum over the fixed points of the product of the degree 0-parts of the characteristic classes. From [24, p. 348] we know that

$$\mathrm{ch}_g(E)|_p = \mathrm{tr}(g|_{E_p}) + \dots \qquad \text{and} \qquad \mathrm{Td}_g(TX)|_p = \frac{1}{1 - (T_p g)^{-1}} + \dots$$

where $T_p g = e^{-2\theta(p)i}$ is the differential at p of the action of g on X. Straight from its definition [24, Def. 3.5], we see that the Bismut equivariant R-class satisfies

$$\begin{aligned}R_g(TX)|_p &= F'(T_p g, 0) - F'(1/T_p g, 0) + \dots \\ &= 2iR^{\mathrm{rot}}(-2\theta(p)) + \dots\end{aligned}$$

Now use that R^{rot} is an odd function.

As for the last term on the right of (2.3), note that $\mathcal{X}_{\mu_n}$ is of relative dimension 0 over $\mathrm{Spec}(\mathbb{Z})$, hence we only need to show that the degree 1 part of the product $\widehat{\mathrm{Td}}_{\mu_n}(\overline{Tf})\widehat{\mathrm{ch}}_{\mu_n}(\overline{\omega}^k)$ equals $\sum_{\substack{\ell\in\mathbb{Z}/(n)\\ \ell\neq 0}} \frac{\zeta^{k\ell}}{1-\zeta^\ell}\left(\frac{\zeta^\ell}{1-\zeta^\ell}+k\right)\widehat{c}_1(\overline{\omega}_\ell)$. But this is clear from the expansions [24, p. 348]

$$\begin{aligned}
\widehat{\mathrm{Td}}_{\mu_n}(\overline{Tf}) &= \prod_{\substack{\ell\in\mathbb{Z}/(n)\\ \ell\neq 0}} \frac{1}{(1-\zeta^{-\ell})^{\mathrm{rk}(Tf_\ell)}} - \sum_{\substack{\ell\in\mathbb{Z}/(n)\\ \ell\neq 0}} \frac{1}{1-\zeta^{-\ell}}\frac{\widehat{c}_1(\overline{Tf}_\ell)}{\zeta^\ell-1}+\dots\\
&= \prod_{\substack{\ell\in\mathbb{Z}/(n)\\ \ell\neq 0}} \frac{1}{(1-\zeta^{\ell})^{\mathrm{rk}(\overline{\omega})}} + \sum_{\substack{\ell\in\mathbb{Z}/(n)\\ \ell\neq 0}} \frac{\zeta^\ell}{(1-\zeta^\ell)^2}\widehat{c}_1(\overline{\omega})+\dots,\\
\widehat{\mathrm{ch}}_{\mu_n}(\overline{\mathcal{L}}^k) &= \sum_{\ell\in\mathbb{Z}/(n)} \zeta^{k\ell}\left(\mathrm{rk}((\overline{\mathcal{L}}^k)_{k\ell})+\widehat{c}_1((\overline{\mathcal{L}}^k)_{k\ell})+\dots\right)\\
&= \sum_{\ell\in\mathbb{Z}/(n)} \zeta^{k\ell}\left(\mathrm{rk}(\overline{\mathcal{L}}_\ell)+k\,\widehat{c}_1(\overline{\mathcal{L}}_\ell)+\dots\right),
\end{aligned}$$

the latter of which holds for any Hermitian line bundle $\overline{\mathcal{L}}$. □

Remark 2.1. If we only have an action of the constant group scheme $\mathbb{Z}/(n)$, all hope is not lost: By restricting to a suitable subset U of $\mathrm{Spec}(D)$ as in [25, Ch. 2], we can still get a μ_n action. The price we have to pay is that Theorem 2.2 only holds in a certain quotient of $\mathbb{R}$. For example, if $D=\mathbb{Z}$ and $p_1,\dots,p_r$ are the primes dividing n, we may choose $U=\mathrm{Spec}(\mathbb{Z})\setminus\{p_1,\dots,p_r\}$ because over U, a μ_n action is the same thing as a $\mathbb{Z}/(n)$ action (see [25, Lem. 2.2]). But then $\widehat{\deg}$ identifies $\widehat{CH}^1(U)$ with $\mathbb{R}/\Big(\mathbb{Q}\log(p_1)+\dots+\mathbb{Q}\log(p_r)\Big)$ and Theorem 2.2 holds in the quotient $\mathbb{R}/\Big(\mathbb{Q}(\zeta)\log(p_1)+\dots+\mathbb{Q}(\zeta)\log(p_r)\Big)$.

2.3 Indefinite rational quaternion algebras

This section contains all the theory about quaternion algebras that we shall need to define and work with Shimura curves and their Atkin-Lehner involutions. Our main reference is the monograph [1] to whose notation we shall adhere as closely as possible.

2.3.1 Basic definitions

We start with an integer $D>1$ which we assume to be square-free with an even number of prime factors. Then, up to isomorphism, there exists a unique quaternion algebra H over $\mathbb{Q}$ with discriminant $D_H=D$. This means that H is a central simple $\mathbb{Q}$-algebra of dimension 4 and D is the product of those rational primes p which ramify in H, i.e. $H\otimes\mathbb{Q}_p$ is a division algebra, and H does not ramify at the place ∞, i.e. $H\otimes\mathbb{R}$ is isomorphic to the matrix algebra $M_2(\mathbb{R})$. This last condition says that H is *indefinite* which refers to the character of its norm which we shall define below.

For the sake of concreteness, we may think of H as given by a $\mathbb{Q}$-basis $\{1,i,j,ij\}$ and relations $i^2=a$, $j^2=b$ and $ij=-ji$ with rational a, $b\neq 0$.

Then we write $H = \left(\frac{a,b}{\mathbb{Q}}\right)$ and D_H is related to a and b via the Hilbert symbol, more precisely, we have for the Hilbert symbol $(a,b)_p = -1$ iff p ramifies in $\left(\frac{a,b}{\mathbb{Q}}\right)$.

Main Example. Let us verify that the quaternion algebra $\left(\frac{2,13}{\mathbb{Q}}\right)$ has discriminant 26. To see how the Hilbert symbol may be computed see [5, Thm. 1.6.7]. First of all, we have $(2,13)_p = 1$ for any prime $p \neq 2, 13$. Thus ramification is only possible at the places 2, 13 and ∞. Now $(2,13)_2 = (-1)^{(13^2-1)/8} = -1$ and $(2,13)_{13}$ is Legendre's symbol $\left(\frac{2}{13}\right)$ which equals -1. Now we are done because the total number of ramified places (finite and infinite) is always even.

We can be even more concrete: Supposing wlog $a > 0$, we can embed $\left(\frac{a,b}{\mathbb{Q}}\right)$ into $M_2(\mathbb{Q}(\sqrt{a}))$ by

$$1 \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad i \mapsto \begin{pmatrix} \sqrt{a} & \\ & -\sqrt{a} \end{pmatrix}, \quad j \mapsto \begin{pmatrix} & 1 \\ b & \end{pmatrix},$$

which in particular yields an isomorphism $\Phi\colon H \otimes \mathbb{R} \to M_2(\mathbb{R})$.

Lastly, if $H = \left(\frac{a,b}{\mathbb{Q}}\right)$, we can define an involutive anti-automorphism called conjugation by the formula familiar from Hamilton's quaternions, i.e. $\bar{1} = 1$, $\bar{i} = -i$ and $\bar{j} = -j$. Then, the reduced norm and trace are given by $\mathrm{nm}(\alpha) = \alpha\bar{\alpha}$ and $\mathrm{tr}(\alpha) = \alpha + \bar{\alpha}$ such that via Φ, they simply correspond to the determinant and trace.

Notation. For an integer n we let

$$\mathcal{O}^{(n)} := \{\alpha \in \mathcal{O} \mid \mathrm{nm}(\alpha) = n\}.$$

Main Example. In the above example, we have

$$\mathrm{nm}(x + yi + zj + tij) = x^2 - 2y^2 - 13z^2 + 26t^2$$

which is indeed an indefinite quadratic form.

2.3.2 Eichler orders

An *order* $\mathcal{O}$ in H is a full $\mathbb{Z}$-lattice in H which is also a subring. Choosing a $\mathbb{Z}$-basis $\{v_1, v_2, v_3, v_4\}$ of $\mathcal{O}$, we can define or compute its discriminant by

$$D_{\mathcal{O}} = \sqrt{|\det(\mathrm{tr}(v_i v_j))|}.$$

Then $D_H \mid D_{\mathcal{O}}$ with equality iff $\mathcal{O}$ is maximal. An *Eichler order* is by definition the intersection of two maximal orders. Its level $N_{\mathcal{O}}$ is simply given by

$$D_{\mathcal{O}} = D_H N_{\mathcal{O}}.$$

It turns out that $\gcd(D_H, N_{\mathcal{O}}) = 1$ for Eichler orders and conversely, given an N coprime to D_H, there exists an Eichler order of level N. Furthermore, there is an easy-to-check sufficient condition for $\mathcal{O}$ to be Eichler, namely the square-freeness of the quotient $D_{\mathcal{O}}/D_H$.

In our situation over $\mathbb{Q}$, any two Eichler orders of the same level (in particular any two maximal orders) are H-conjugate so that given coprime numbers D and N with D square-free, we simply write $\mathcal{O}(D, N)$ for any Eichler order of level N in the quaternion algebra with discriminant D.

Example. Consider $\mathcal{O} = \mathbb{Z}[1, i, (1+j)/2, (i+ij)/2]$ in $H = \left(\frac{a,b}{\mathbb{Q}}\right)$ with coprime integers a and b. By elementary means, one checks that $\mathcal{O}$ is closed under multiplication, hence an order. To compute its reduced discriminant we form the matrix

$$\begin{pmatrix} \mathrm{tr}(1) & \mathrm{tr}(i) & \mathrm{tr}((1+j)/2) & \mathrm{tr}((i+ij)/2) \\ \mathrm{tr}(i) & \mathrm{tr}(i^2) & \mathrm{tr}(i(1+j)/2) & \mathrm{tr}(i(i+ij)/2) \\ \mathrm{tr}((1+j)/2) & \mathrm{tr}((1+j)i/2) & \mathrm{tr}((1+j)^2/4) & \mathrm{tr}((1+j)(i+ij)/4) \\ \mathrm{tr}((i+ij)/2) & \mathrm{tr}((i+ij)i/2) & \mathrm{tr}((i+ij)(1+j)/4) & \mathrm{tr}((i+ij)^2/4) \end{pmatrix}$$

Its determinant is

$$\begin{vmatrix} 2 & 0 & 2 & 0 \\ 0 & 2a & 0 & a \\ 1 & 0 & \frac{b+1}{2} & 0 \\ 0 & a & 0 & \frac{a(1-b)}{2} \end{vmatrix} = -a^2b^2$$

so that $\mathcal{O}$ is maximal iff the discriminant of $H = \left(\frac{a,b}{\mathbb{Q}}\right)$ equals ab.

Main Example. For $D = 26$ we now have an explicit maximal order $\mathcal{O} = \mathbb{Z}[1, i, (1+j)/2, (i+ij)/2]$ in $H = \left(\frac{2,13}{\mathbb{Q}}\right)$.

2.3.3 Counting optimal embeddings à la Eichler

Let F be a maximal commutative subalgebra of H. Then F is of dimension 2 over $\mathbb{Q}$ and by our assumption $D > 1$ we know that H is a skew-field, hence F is actually a field. Conversely given a quadratic field F, it can be embedded into H iff none of the primes which ramify in H split in F. If D_F denotes the discriminant of F, this condition can be expressed using the (extended) Legendre symbol

$$\left(\frac{D_F}{p}\right) \neq 1 \qquad \text{for all } p \mid D_H. \tag{2.4}$$

Recall that if $F = \mathbb{Q}(\sqrt{d})$ with integer square-free $d \neq 0$, 1, the discriminant of F is given by

$$D_F = \begin{cases} d, & d \equiv 1 \mod 4, \\ 4d, & d \equiv 2, 3 \mod 4, \end{cases} \tag{2.5}$$

which by definition equals the discriminant of the maximal order $\Lambda_F = \mathbb{Z}[1, \omega_d]$ where

$$\omega_d = \begin{cases} \frac{1+\sqrt{d}}{2}, & d \equiv 1 \mod 4 \\ \sqrt{d}, & d \equiv 2, 3 \mod 4 \end{cases}.$$

More generally, recall that any order Λ of F is of the form $\Lambda(d, m) := \mathbb{Z}[1, m\,\omega_d]$ for a unique integer $m \geq 1$ called its *conductor*. Furthermore, the discriminant of this order is $D_\Lambda = m^2 D_F$.

Now given an embedding $\varphi\colon F \to H$, the preimage $\Lambda = \varphi^{-1}(\mathcal{O})$ of the Eichler order $\mathcal{O}$ is an order in F and we say that φ is an *optimal embedding* of Λ into $\mathcal{O}$. Conversely, given $\Lambda = \Lambda(d, m) \subset F$ (and assuming that F can be embedded into H), one may ask whether Λ can be optimally embedded into $\mathcal{O}$. There is a necessary condition, namely that its conductor and the discriminant of the quaternion algebra be coprime, i.e.

$$(m, D_H) = 1. \tag{2.6}$$

For $\mathcal{O}$ maximal, i.e. $N = 1$, this condition is also sufficient.

But there is an even better statement: One can precisely count the number of optimal embeddings up to conjugation by the norm 1 elements $\mathcal{O}^{(1)}$. The theorem can be stated elegantly for square-free N using Eichler's symbol which we define now:

Definition 10 (Eichler's symbol over the rationals)**.** *Let Λ be the quadratic order of discriminant D_Λ. Then Eichler's symbol $\left\{\frac{\Lambda}{p}\right\}$ is set to 1 if Λ is not maximal at p. Otherwise it is defined to equal the (extended) Legendre symbol $\left(\frac{D_\Lambda}{p}\right)$.*

Remark. Note that $\Lambda(d,m)$ is maximal at p iff p does not divide its conductor m and in this case $\left(\frac{D_\Lambda}{p}\right) = \left(\frac{m^2 D_F}{p}\right) = \left(\frac{D_F}{p}\right)$. Hence the fully explicit formula

$$\left\{\frac{\Lambda(d,m)}{p}\right\} = \begin{cases} 1, & p \mid m, \\ \left(\frac{D_F}{p}\right), & \text{otherwise.} \end{cases} \tag{2.7}$$

Proposition 2.3 (Counting optimal embeddings)**.** *Let $\mathcal{O}$ be an Eichler order of square-free level N in an indefinite rational quaternion algebra H of discriminant D_H. Let Λ be a quadratic order, denote its class number by $h(\Lambda)$ and define*

$$(2)_\Lambda := \begin{cases} 1, & \Lambda \textit{ contains a unit of norm } -1, \\ 2, & \textit{otherwise.} \end{cases}$$

Then the number of optimal embeddings of Λ into $\mathcal{O}$ up to conjugation by $\mathcal{O}^{(1)}$ is given by

$$(2)_\Lambda h(\Lambda) \prod_{p \mid D_H} \left(1 - \left\{\frac{\Lambda}{p}\right\}\right) \prod_{p \mid N} \left(1 + \left\{\frac{\Lambda}{p}\right\}\right).$$

Proof. We apply the first formula of [44, Cor. 5.12] to the ground field $\mathbb{Q}$ (which has class number $h = 1$) and obtain the number of optimal embeddings modulo all units $\mathcal{O}^\times$ as the last three factors. Then apply [44, Cor. 5.13] with $G = \mathcal{O}^{(1)}$ to get the extra factor

$$[\mathrm{nm}(\mathcal{O}^\times) \colon \mathrm{nm}(\Lambda^\times)].$$

This index equals $(2)_\Lambda$ because it is classically known that $\mathcal{O}$ always contains a unit of norm -1. $\square$

Remark. From this proposition, we recover the above conditions (2.4) and (2.6) whose sufficiency in the case of maximal $\mathcal{O}$ is also clear.

Remark. Proposition 4.23 of [1] seems to get the relation between the embeddings modulo $\mathcal{O}^{(1)}$ and $\mathcal{O}^\times$ wrong.

Remark. The ideal class number of $\Lambda = \Lambda(d,m)$ can be computed from the class number $h(F)$ of the quadratic field F by means of Dirichlet's formula

$$h(\Lambda) = \frac{h(F)}{[\Lambda_F^\times \colon \Lambda^\times]} m \prod_{p \mid m} \left(1 - \left(\frac{D_F}{p}\right)\frac{1}{p}\right).$$

Example 2.1. Let us take a maximal order $\mathcal{O}$ and consider the two imaginary orders containing non-trivial roots of unity, namely $\Lambda(-1,1)$ and $\Lambda(-3,1)$. It is well-known that

$$\Lambda(-1,1)^\times = \{\pm 1, \pm\sqrt{-1}\},$$
$$\Lambda(-3,1)^\times = \{\pm 1, \pm\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}\}.$$

They are both maximal (i.e. $m = 1$) and their discriminants are -4 and -3 respectively. Their class numbers both equal 1. Since the norm is a positive definite binary quadratic form on any imaginary order, we have $(2)_\Lambda = 2$ so that the number of embeddings modulo $\mathcal{O}^{(1)}$ is

$$2\prod_{p|D_H}\left(1-\left(\frac{-4}{p}\right)\right) \qquad \text{and} \qquad 2\prod_{p|D_H}\left(1-\left(\frac{-3}{p}\right)\right), \tag{2.8}$$

respectively.

Main Example. For $D_H = 26$, there are no optimal embeddings of either order because $\left(\frac{-4}{13}\right) = \left(\frac{-3}{13}\right) = 1$.

2.4 Compact Shimura curves

In this section, we recall the compact Shimura curves $X(D,N)$ of discriminant D and level N. As is well-known, the Shimura curve is compact iff $D > 1$ and within that class of curves, our primary interest shall be in the level 1 case ($N = 1$). The original reference is [38] from which the existence of a canonical model $\mathcal{X}(D,N)$ is known, cf. also [27, Prop 1.1(ii)]. The advantage of Shimura curves is that all terms in Theorems 2.1 and 2.2 can be computed explicitly. For example, the height of the arithmetic surface $\mathcal{X}(D,1)$ is known, see [27, Thm. 0.5 & Cor. 11.2] for an unconditional result and [7, Eq. (6.8) and Thm. 6.9] for the case when the Shimura curve can be embedded into a Hilbert modular surface.

Theorem 2.4 (Kudla-Rapoport-Yang, Bruinier-Burgos-Kühn)**.** *The height of the arithmetic surface associated to the Shimura curve $\mathcal{X}(D,1)$ with respect to the canonical bundle $\hat{\omega}$ equipped with the renormalised Petersson metric $|dz|^2 = e^{-2C}(4\pi y)^2$, is*

$$h_{\hat{\omega}}(\mathcal{X}(D,1)) = -2\zeta_D(-1)\left(2C - 2\frac{\zeta'(-1)}{\zeta(-1)} - 1 + \frac{1}{2}\sum_{p|D}\frac{p+1}{p-1}\log(p)\right).$$

where ζ denotes Riemann's zeta function and ζ_D is Riemann's zeta function with the Euler factors for $p \mid D$ removed, i.e. $\zeta_D(s) := \zeta(s)\prod_{p|D}(1-p^{-s})$.

Remark. Explicitly, we get

$$\zeta_D(-1) = \zeta(-1)\prod_{p|D}(1-p) = -\frac{1}{12}\prod_{p|D}(p-1),$$

the last equality holding because D has an even number of prime divisors.

To match [7] with [27], one needs to take into account that modular forms of weight k on a Hilbert modular surface restrict to modular forms of weight $k+k=2k$ on an embedded Shimura curve and, of course, the fact that the Hodge bundle appearing in [27] is isomorphic with the canonical bundle, i.e. the bundle of modular forms of weight 2 (see [27, Sec. III.3] and also [26, Sec. 3]).

What is more, Brunier et al. use the normalisation $\mathrm{d}x\mathrm{d}y/(4\pi y^2)$ for the volume form, [7, Cor. 3.9,ii)], so that their Shimura curve volume is $-\zeta_D(-1)$, note the sign mistake in [27, Chap. I, (0.20)] and compare Section 2.4.2 below.

It might be helpful to remark that setting $C=0$ in the above theorem produces the renormalisation of Petersson's metric used by Bruinier et al., compare [7, Def. 2.4] and [27, Chap. III, (3.6)]. In the latter equation, note that a factor of 4 has been forgotten.

Main Example. For $D=26$, we get

$$h_{\hat{\omega}}(\mathcal{X}(26,1)) = 2\left(2C + 24\zeta'(-1) - 1 + \frac{3}{2}\log(2) + \frac{7}{12}\log(13)\right).$$

Let us now recall the definition of the Shimura curve $X(D,N)$.

2.4.1 The Fuchsian groups defining Shimura curves

Now given an Eichler order $\mathcal{O}=\mathcal{O}(D,N)$, the Shimura curve $X=X(D,N)$ can be defined in a straightforward manner as the quotient $X=\Gamma\backslash\mathbb{H}$ of the hyperbolic plane $\mathbb{H}=\{x+iy \mid y>0\}$ by the subgroup

$$\Gamma = \Gamma(D,N) := \Phi(\mathcal{O}^{(1)})$$

of $\mathrm{SL}_2(\mathbb{R})$ obtained by embedding the norm 1 units of $\mathcal{O}$ via Φ. It is well-known that this Γ is discrete and has finite covolume, i.e. is a Fuchsian group of the first kind.

Furthermore, it is equally classical that Γ is cocompact iff $D>1$. For some concrete fundamental domains for the above Fuchsian groups see the beautiful paper [21].

2.4.2 The volume and the genus of a Shimura curve

First of all, there is an explicit arithmetic formula for the (suitably normalised) volume of the Shimura curve

Theorem 2.5 (Arithmetic volume of Shimura curve)**.** *The volume $V(D,N)$ of the Shimura curve $X(D,N)$ with respect to the metric* $\mathrm{d}x\mathrm{d}y/(2\pi y^2)$ *can be computed as*

$$V(D,N) = \frac{N}{6}\prod_{p\mid D}(p-1)\prod_{p\mid N}\left(1+\frac{1}{p}\right).$$

Proof. Use the formula from [44, p. 120 bottom] with $N=N_0$. ∎

Main Example. In our favourite example $D_H=26$, the formula gives $V(26,1)=2$.

Then secondly, for any cocompact Fuchsian group Γ of the first kind we have the Riemann-Hurwitz formula which allows to compute the genus h of the quotient $X = \Gamma\backslash\mathbb{H}$ from the arithmetic volume V and the numbers of elliptic cycles e_q of order q, $q = 2, \ldots$

$$V = 2h - 2 + \sum_{q\geq 2}(1 - 1/q)e_q \quad \Leftrightarrow \quad h = \frac{1}{2}\left(V + 2 - \sum_{q\geq 2}(1-1/q)e_q\right). \quad (2.9)$$

For a proof of this formula, see [30, Theorem 2.4.3].

In order for this formula to be useful, we need to get a hand on the numbers e_q:

Theorem 2.6 (Elliptic cycles of Shimura curves)**.** *To begin with,* $e_q = 0$ *for* $q > 3$*. Then* $e_2 = 0$ *if* $4 \mid N$ *and* $e_3 = 0$ *if* $9 \mid N$*. Otherwise*

$$e_2 = \prod_{p|D_H}\left(1 - \left(\frac{-4}{p}\right)\right)\prod_{p|N}\left(1 + \left(\frac{-4}{p}\right)\right)$$
$$e_3 = \prod_{p|D_H}\left(1 - \left(\frac{-3}{p}\right)\right)\prod_{p|N}\left(1 + \left(\frac{-3}{p}\right)\right).$$

Proof. Apply the formula given in [44, Chap. IV, Sec. 3A] to the case $N = N_0$. □

Remark. Note the similarity with the formulae (2.8) for the number of embeddings of $\Lambda(-1,1)$ and $\Lambda(-3,1)$. This is not accidental as will become clearer soon.

Main Example. The genus of $X(26,1)$ equals $\frac{1}{2}(2+2) = 2$ as there are no elliptic cycles at all (cf. Example 2.1). In particular, like any curve of genus 2, $X(26,1)$ is automatically hyperelliptic. An amusing way to convince oneself of the last fact is to use the inequality $2h+2 \leq n \leq h^3 - h$ for the number n of Weierstraß points together with the fact that a Riemann surface is hyperelliptic iff n is minimal.

2.4.3 Automorphisms of Shimura curves: The normaliser and the Atkin-Lehner group

For a very readable account of the following, see [17, Sec. 1.2] for the classical modular case. For genuine Shimura curves cf. [35].

We are interested in the group A of automorphisms of the Shimura curve $X = X(D,N)$.

Clearly, elements of the normaliser $\mathcal{N}(\Gamma)$ of $\Gamma = \Gamma(D,N)$ in $\mathrm{SL}_2(\mathbb{R})$ operate on the coset space X as automorphisms so $B := \mathcal{N}(\Gamma)/\Gamma$ is in general a subgroup of A.

However, if Γ is strictly hyperbolic so that $\mathbb{H} \to X$ is the (unramified!) universal cover and Γ is in fact the fundamental group of the Shimura curve, then it is equally obvious that in fact $A = B$.

Next, for every exact divisor n of DN we can find an element w_n of $\mathcal{N}(\Gamma)$ called the *Atkin-Lehner involution* associated to n. Modulo Γ, these satisfy $w_n = 1$ iff $n = 1$ and $w_n w_{n'} = w_{nn'/(n,n')}$. Hence we obtain a subgroup W of

B called the *Atkin-Lehner group* which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$ where r is the number of prime divisors of DN.

As before, W is in general only a subgroup of B. For example, if $D_H = 1$ and Γ is the classical modular group $\Gamma_0(N)$ and $4 \mid N$ or $9 \mid N$ there are extra elements

$$S_q = \begin{pmatrix} q & 1 \\ & q \end{pmatrix}, \qquad q \text{ a suitable power of 2 or 3,}$$

of the normaliser $\mathcal{N}(\Gamma_0(N))$ which are not of Atkin-Lehner type.

So the whole picture is $A \supseteq B \supseteq W$, and all inclusions may be strict.

Now, D being square-free, $n \mid D$ implies $n \parallel D$ and to give the corresponding Atkin-Lehner involution is particularly easy: Just pick any quaternion α in $\mathcal{O}$ which is of norm n. Then

$$w_n := n^{-1/2}\Phi(\alpha)$$

will do.

Remark 2.2. This is equivalent to saying that

$$w_n\Gamma = n^{-1/2}\Phi(\mathcal{O}^{(n)}),$$

so we may think of the coset $w_n\Gamma$ as just the elements of $\mathcal{O}$ of norm n. In particular, Γ-conjugacy classes in $w_n\Gamma$ correspond bijectively to $\mathcal{O}^{(1)}$-conjugacy classes in $\mathcal{O}^{(n)}$.

Remark. By its very definition, for $p \mid D$ the Atkin-Lehner involution w_n is nothing but the Hecke operator T_p introduced by Eichler, e.g. see [12, p.93 bottom]. Hecke operators in general are not maps but only correspondences, i.e. one-to-many, but interestingly, these T_p for $p \mid D$ are one-to-one.

Main Example. In our primary example, $D_H = 26$, $N = 1$, with $\mathcal{O}$ as above, let us find an element α of norm 26, i.e.

$$\alpha = x1 + yi + z(1+j)/2 + t(i+ij)/2 = (x+z/2)1 + (y+t/2)i + (z/2)j + (t/2)ij$$

with $(x+z/2)^2 - 2(y+t/2)^2 - 13(z/2)^2 + 26(t/2)^2 = 26$. An obvious solution is $t = 2$, $y = -1$, $x = z = 0$, i.e. $\alpha = ij$. Hence we get

$$w_{26} = \frac{1}{\sqrt{26}}\Phi(ij) = \frac{1}{\sqrt{26}}\begin{pmatrix} \sqrt{2} & \\ & -\sqrt{2} \end{pmatrix}\begin{pmatrix} & 1 \\ 13 & \end{pmatrix} = \begin{pmatrix} & 1/\sqrt{13} \\ -\sqrt{13} & \end{pmatrix}.$$

Note that incidentally, $w_{26}^2 = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} = 1$ exactly, not just modulo Γ. It was proved by Ogg [32, Thm. 7] that w_{26} is in fact the hyperelliptic involution of $X(26,1)$. We shall reprove this fact by showing that w_{26} has $2g+2 = 6$ fixed points (which are then – more or less by definition – the Weierstraß points).

2.4.4 Counting elliptic and hyperbolic Γ-conjugacy classes

Next we want to count Γ-conjugacy classes in $w_n\Gamma$ when $n \mid D_H$. Since $w_1\Gamma = \Gamma$, this subsumes the problem of counting Γ-conjugacy classes in Γ itself. From Remark 2.2, we know that we can equally well count $\mathcal{O}^{(1)}$-conjugacy classes in $\mathcal{O}^{(n)}$. The solution to the problem is to enumerate these conjugacy classes by their traces and to prove that for any fixed trace $t \in \mathbb{Z}$, there are only a finite number $m = m(n,t)$ of them, i.e. setting $\mathcal{O}^{(n,t)} := \{\alpha \in \mathcal{O}^{(n)} \mid \mathrm{tr}(\alpha) = t\}$ we have decompositions

$$\mathcal{O}^{(n)} = \coprod_{t\in\mathbb{Z}} \mathcal{O}^{(n,t)} = \coprod_{t\in\mathbb{Z}} \coprod_{j=1}^{m(n,t)} \{\alpha_j\}_{\mathcal{O}^{(1)}}.$$

We will see that the formulae of Section 2.3.3 can be used to compute $m(n,t)$.

So let $\alpha \in \mathcal{O}$ be an element of norm n and trace t, i.e. $\alpha \in \mathcal{O}^{(n,t)}$. Since it is a root of its minimal polynomial $P = P_\alpha$

$$P(X) = X^2 - tX + n, \tag{2.10}$$

the map $X \mapsto \alpha$ furnishes an embedding φ into H of the algebra $\mathbb{Q}[X]/(P(X))$. According to the discriminant $D_P = t^2 - 4n$ of P, we can distinguish four cases:

$$\mathbb{Q}[X]/(P(X)) = \begin{cases} \mathbb{Q}, & D_P \text{ a perfect square} \\ \mathbb{Q} \oplus \mathbb{Q}, & D_P = 0 \\ \mathbb{Q}(\sqrt{D_P}), & D_P > 0 \text{ not a perfect square} \\ \mathbb{Q}(\sqrt{D_P}), & D_P < 0 \end{cases}$$

Since the square of the trace of $n^{-1/2}\Phi(\alpha)$ is t^2/n, these cases correspond to the latter element being equal to $\pm I$, parabolic, hyperbolic and elliptic in $\mathrm{SL}_2(\mathbb{R})$, respectively. Since we are only interested in the hyperbolic and the elliptic case, assume that $\sqrt{t^2 - 4n} \notin \mathbb{Q}$ so that $\mathbb{Q}[X]/(P(X))$ is a quadratic field $F = \mathbb{Q}(\sqrt{d})$, d a square-free integer $\neq 0$, 1, and write ξ (instead of X) for the element of F with $\varphi(\xi) = \alpha$.

Next, we decompose the polynomial discriminant as $D_P = m^2 D_F$ and d is related to D_F via (2.5). Since φ preserves norm and trace, it is clear that ξ is integral in F. Furthermore, it generates the order $\Lambda = \Lambda(d,m)$ of conductor m in F. Now consider the order $\Lambda' = \varphi^{-1}(\mathcal{O})$ of F optimally embedded by φ. Then obviously $\Lambda' \supseteq \Lambda$ which is the case iff the conductor m' of Λ' divides the conductor m of Λ.

To cut a long story short, if $\sqrt{t^2 - 4n} \notin \mathbb{Q}$, then to every element α of $\mathcal{O}^{(n,t)}$, there corresponds an embedding of the splitting field F of $P = P_\alpha$ which is optimal for some order Λ' containing the roots of P.

Furthermore, this correspondence is one-to-one and equivariant with respect to $\mathcal{O}^{(1)}$ which acts by conjugation both on $\mathcal{O}^{(n,t)}$ and on the embeddings.

Hence from Proposition 2.3 we obtain

Corollary 2.7 (Conjugacy classes of given norm and trace)**.** *Let $\mathcal{O}$ be an Eichler order of square-free level N. Suppose $\sqrt{t^2 - 4n} \notin \mathbb{Q}$ and write $t^2 - 4n = m^2 D_F$ with $F = \mathbb{Q}(\sqrt{t^2 - 4n})$. Then the number of $\mathcal{O}^{(1)}$-conjugacy classes in $\mathcal{O}$ of norm n and trace t is given by*

$$m(n,t) = \sum_{\substack{m'|m \\ (m',D_H)=1 \\ \Lambda'=\Lambda(d,m')}} (2)_{\Lambda'} h(\Lambda') \prod_{p|D_H} \left(1 - \left(\frac{D_F}{p}\right)\right) \prod_{p|N} \left(1 + \left\{\frac{\Lambda'}{p}\right\}\right). \tag{2.11}$$

Proof. Use formula (2.7) to see that $1 - \left\{\frac{\Lambda'}{p}\right\}$ gives 0 for some $p \mid D_H$ unless $(m', D_H) = 1$, which proves formula (2.11). $\square$

Remark. In the hyperbolic case, $(2)_{\Lambda'} h(\Lambda')$ is simply the narrow class number of Λ'.

Remark. The calculation of conjugacy classes in $\mathrm{SL}_2(\mathbb{Z})$ (i.e. the case $D_H = N = n = 1$ of Corollary 2.7) in [44, p. 96] seems to contain a mistake: The $(2)_{\Lambda'}$ factor is pulled out of the sum over all orders containing a given one, suggesting $(2)_{\Lambda(d,m')} = (2)_{\Lambda(d,1)}$ in general. But this is wrong. For instance, take $d = 5$ and $m' = 3$. The maximal order $\Lambda(5,1)$ is $\mathbb{Z}[1, \omega_5]$ with $\omega_5 = \frac{1+\sqrt{5}}{2}$. Its fundamental unit is in fact ω_5 which has norm -1. However, the fundamental unit of $\Lambda(5,3)$ equals $2 + 3\omega_5$ which has norm 1.

Remark. Note that we can now easily deduce Theorem 2.6 for maximal $\mathcal{O}$. Just apply the corollary to the case $N = n = 1$ and $t^2 - 4 < 0$. This leaves the two possibilities $t = 0$ and $t = \pm 1$. In the first case we have $t^2 - 4 = -4 = D_{\mathbb{Q}(\sqrt{-1})}$, in the second $t^2 - 4 = -3 = D_{\mathbb{Q}(\sqrt{-3})}$. So in both cases we have $m = 1$ and hence only one summand (namely $m' = 1$) in formula (2.11). As we have seen in Example 2.1, both above fields have class number one and as imaginary fields, they never contain units of norm -1, i.e. $(2)_{\Lambda'} = 2$. So we get

$$m(1,0) = 2 \prod_{p \mid D_H} \left(1 - \left(\frac{-4}{p}\right)\right)$$
$$m(1,\pm 1) = 2 \prod_{p \mid D_H} \left(1 - \left(\frac{-3}{p}\right)\right).$$

But we know that to every elliptic cycle of order 2 (resp. 3), there correspond two (resp. four) elliptic conjugacy classes in $\Gamma \cong \mathcal{O}^{(1)}$. Hence $e_2 = m(1,0)/2$ and $e_3 = (m(1,1) + m(1,-1))/4$, which is exactly what Theorem 2.6 says.

If the norm divides the quaternion algebra discriminant, we can evaluate some Legendre symbols to get an even simpler formula

Corollary 2.8. *Suppose in the situation of Corollary 2.7 that $n \mid D_H$. Then $m(n,t) = 0$ if $n \nmid t$. If $n \mid t$, we have*

$$m(n,t) = \sum_{\substack{m' \mid m \\ (m', \frac{D_H}{n}) = 1 \\ \Lambda' = \Lambda(d,m')}} (2)_{\Lambda'} h(\Lambda') \prod_{p \mid \frac{D_H}{n}} \left(1 - \left(\frac{D_F}{p}\right)\right) \prod_{p \mid N} \left(1 + \left\{\frac{\Lambda'}{p}\right\}\right)$$

and $t^2 - 4n$ was not a perfect square.

Proof. The whole simplification hinges on the fact that $(m,n) = 1$ for the conductor m defined by $t^2 - 4n = m^2 D_F$, which follows from the elementary Lemma 2.9 below. For then, the relaxation from the condition $(m', D_H) = 1$ on m' to the milder condition $(m', \frac{D_H}{n}) = 1$ is clear and we can compute for $p \mid n$

$$\left(\frac{D_F}{p}\right) = \left(\frac{m^2 D_F}{p}\right) = \left(\frac{t^2 - 4n}{p}\right) = \left(\frac{t^2}{p}\right) = \begin{cases} 0, & p \mid t \\ 1, & p \nmid t. \end{cases}$$

Since n is square-free, we know that $n \nmid t$ iff there is a prime dividing n but not t and we obtain

$$\prod_{p \mid n} \left(1 - \left(\frac{D_F}{p}\right)\right) = \begin{cases} 1, & n \mid t \\ 0, & n \nmid t. \end{cases}$$

The last statement is clear if a prime $p > 2$ divides n because then letting $t = kn$ shows that $(kn)^2 - 4n$ is divisible by p but not by p^2. For the remaining cases $n = 1$ and $n = 2$, suppose $t^2 - 4n = m^2$ with $m \geq 1$. Then one discusses $(t+m)(t-m) = 4n$ by elementary means. □

Lemma 2.9. *Let t, n, m and D be integers with $t^2 - 4n = m^2 D$ and $D \equiv 0, 1 \mod 4$. Then $p \parallel n$ implies $p \nmid m$.*

Proof. Let us do $p > 2$ first. Suppose p divides both m and n. Then $t^2 = 4n + m^2 D \equiv 0 \mod p$, hence $t^2 \equiv 0 \mod p^2$. But then $4n = t^2 - m^2 D \equiv 0 \mod p^2$, which can't happen if p divides n exactly.

For $p = 2$, it is clear that m must be odd if t is odd because in that case, $t^2 - 4n$ is odd. So suppose t and m are even and divide the whole equation by 4 to get $t'^2 - 2n' = D'$ where $t' = t/2$, $n' = n/2$ and $D' = D(m/2)^2$. Then n' is odd, hence $2n' \equiv 2 \mod 4$. But $t'^2, D' \equiv 0, 1 \mod 4$, so we have a contradiction modulo 4. □

Main Example. As an application of Corollary 2.8, let us go back to our favourite Shimura curve $X(26, 1)$ and compute the numbers of all elliptic conjugacy classes of norms $n \mid D_H$:

n	t with $n \mid t$, $\lvert t\rvert < 2\sqrt{n}$	D_P	m	$h(D_F)$	$\left(\frac{D_F}{2}\right)$	$\left(\frac{D_F}{13}\right)$	$m(n,t)$
1	0	-4	1	1	0	1	0
	± 1	-3	1	1	-1	1	0
2	0	-8	1	1	0	-1	4
	± 2	-4	1	1	0	1	0
13	0	-52	1	2	0	0	4
26	0	-104	1	6	0	0	12

Since two conjugacy classes correspond to one fixed point, we conclude that w_2 and w_{13} both have two fixed points whereas w_{26} has six fixed points. Since $6 = 2h + 2$, we have seen that w_{26} is indeed the hyperelliptic involution.

2.4.5 Equivariant Selberg zeta functions for Atkin-Lehner involutions

To make Corollary 2.15 fully explicit for an Atkin-Lehner involution w_n acting on k-differentials, $k \geq 2$, on a Shimura curve $X(D, 1)$ we use Corollary 2.8 to evaluate the equivariant Selberg zeta function:

Proposition 2.10 (Equivariant Selberg zeta functions for Atkin-Lehner involutions)**.** *Let $\Gamma = \Gamma(D, N)$ with N square-free, and let w_n, $n \mid D$, be an*

Atkin-Lehner involution. Then for $\mathrm{Re}(s) > 1$*, we have*

$$Z_{w_n\Gamma}(s) = -\sum_{\substack{t\in n\mathbb{N}_0\\ t^2-4n>0}} \prod_{p\mid \frac{D}{n}} \left(1 - \left(\frac{D_F}{p}\right)\right) \frac{4}{\log N} \frac{N^{-s}}{1-\frac{1}{N}} \times$$
$$\times \sum_{\substack{m'\mid m\\ (m',\frac{D}{n})=1}} \prod_{p\mid N} \left(1 + \left\{\frac{\Lambda'}{p}\right\}\right) h(\Lambda') \log(\varepsilon_{\Lambda'}).$$

For every t *of the outer sum, we set* $N := (t+\sqrt{t^2-4n})^2/(4n)$*, denote by* D_F *the discriminant of the quadratic field* $F := \mathbb{Q}(\sqrt{t^2-4n})$ *and define* $m \geq 1$ *by* $t^2 - 4n = m^2 D_F$*. Then for every* m' *of the inner sum, let* Λ' *be the order of conductor* m' *in* F *with class number* $h(\Lambda')$ *and fundamental unit* $\varepsilon_{\Lambda'}$*. For Eichler's symbol* $\left\{\frac{\Lambda'}{p}\right\}$ *see (2.7).*

Remark. Note that the case $n = 1$ covers the problem of evaluating the ordinary Selberg zeta function because $w_1\Gamma = \Gamma$.

Proof. By Definition 5, the equivariant Selberg zeta function is given as

$$Z_{w_n\Gamma}(s) = -\sum \frac{\log N(P_0)}{\log N(\tilde{P})} \frac{N(\tilde{P})^{-s}}{1-N(\tilde{P})^{-1}},$$

the sum extending over all hyperbolic conjugacy classes $\{\tilde{P}\}_\Gamma$ in the coset $w_n\Gamma$ of positive trace and N denoting the Selberg norm of $\tilde{P}$ and the corresponding primitive element P_0, respectively.

We now apply Proposition 2.3 to make the above sum completely explicit.

Any conjugacy class $\{\tilde{P}\}_\Gamma$ in $w_n\Gamma$ corresponds to a unique conjugacy class $\{\alpha\}_{\mathcal{O}^{(1)}}$ in $\mathcal{O}^{(n)}$ via $\tilde{P} = n^{-1/2}\Phi(\alpha)$ and we can enumerate the latter according to their traces.

Then the Selberg norm $N(\tilde{P})$ only depends on the trace t of α. Since $\Phi(\alpha)$ and α have the same characteristic polynomial, namely $X^2 - tX + n$, $\Phi(\alpha)$ has eigenvalues $(t \pm \sqrt{t^2-4n})/2$ so that the eigenvalues of $\tilde{P}$ are clearly $(t \pm \sqrt{t^2-4n})/(2\sqrt{n})$. For $t > 0$, the eigenvalue of greater of modulus is the one with the plus sign, therefore

$$N(\tilde{P}) = \left(\frac{t+\sqrt{t^2-4n}}{2\sqrt{n}}\right)^2.$$

The Selberg norm $N(P_0)$ is slightly more delicate. Suppose $\{\alpha\}_{\mathcal{O}^{(1)}}$ corresponds to an embedding $\varphi\colon F \to H$ and let $\Lambda' = \varphi^{-1}(\mathcal{O})$ be the order of F for which this embedding is optimal. Then $N(P_0)$ depends on Λ' as follows:

Since $\alpha \notin \mathbb{Q}$, its centraliser in H is the quadratic subfield it generates, i.e. $\varphi(F)$. Hence its centraliser in $\mathcal{O}$ equals $\varphi(F) \cap \mathcal{O}$ which is isomorphic to Λ' by φ. Furthermore since φ preserves norms, it identifies the norm 1 units $\Lambda'^{(1)}$ with $\varphi(F) \cap \mathcal{O}^{(1)}$ which is the centraliser of α in $\mathcal{O}^{(1)}$. Now the structure of the former is explicitly known in terms of the fundamental unit $\varepsilon = \varepsilon_{\Lambda'}$:

$$\Lambda'^{(1)} = \{\pm\eta^m \mid m \in \mathbb{Z}\}$$

where

$$\eta = \eta_{\Lambda'} = \begin{cases} \varepsilon^2, & \text{if } \operatorname{nm}(\varepsilon) = -1; \\ \varepsilon, & \text{if } \operatorname{nm}(\varepsilon) = 1. \end{cases}$$

Moreover, η (which is sometimes called the *proper fundamental unit*) is usually computed from the minimal solution of Pell's equation $x^2 - D'y^2 = 4$ with D' the discriminant of Λ': If (x_0, y_0) is the solution with $x_0, y_0 > 0$ and y_0 as small as possible, then $\eta = (x_0 + y_0\sqrt{D'})/2$.

Now η has norm 1 and trace $x_0 > 0$. Since clearly $P_0 = \Phi(\varphi(\eta))$, the same is true for P_0. Therefore, the same reasoning as for $N(\tilde{P})$ above shows

$$N(P_0) = \left(\frac{x_0 + \sqrt{x_0^2 - 4}}{2} \right)^2 .$$

By Pell's equation, $x_0^2 - 4 = y_0^2 D'$, which finally gives $N(P_0) = \eta^2$.

Noting that $(2)_{\Lambda'} \log(\eta_{\Lambda'}^2) = 4 \log(\varepsilon_{\Lambda'})$, we are done. □

Remark. Similar computations can be found in [2] and [41].

Remark. For several ways to express the product $h(\Lambda') \log(\varepsilon_{\Lambda'})$ in terms of L-function values see Lemma 2.11 below. But beware that all these formulae, though aesthetically pleasing, are unfortunately useless for computational purposes if D' is large.

Lemma 2.11. *Let Λ be the order of conductor m in a real quadratic field F. Then the product of its class number and its regulator satisfies*

$$\begin{aligned} h(\Lambda) \log \varepsilon_\Lambda &= \frac{1}{2} \sqrt{D_\Lambda} L(D_\Lambda, 1) \\ &= \frac{1}{2} m \sqrt{D_F} \prod_{p|m} \left(1 - \left(\frac{D_F}{p} \right) \frac{1}{p} \right) L(D_F, 1) \\ &= m \prod_{p|m} \left(1 - \left(\frac{D_F}{p} \right) \frac{1}{p} \right) \sum_{\ell=1}^{D_F - 1} \left(\frac{D_F}{\ell} \right) \log \Gamma \left(\frac{\ell}{D_F} \right) \end{aligned}$$

where $D_\Lambda = m^2 D_F$ is the discriminant of Λ and $L(D, s) = \sum_{n=1}^{\infty} \left(\frac{D}{n}\right) n^{-s}$ for any discriminant D.

Proof. All these identities can already be found in Landau's classical monograph [28]. As for the first equality, we have

$$2h(\Lambda) \log \varepsilon_\Lambda = h_{\text{nar}}(\Lambda) \log \eta_\Lambda = \sqrt{D_\Lambda} L(D_\Lambda, 1)$$

by [28, Satz 209]. The second equality is [28, Satz 214].

For the third equality, we take from [28, Satz 217] the formula

$$L(D_F, 1) = -\frac{1}{\sqrt{D_F}} \sum_{\ell=1}^{D_F - 1} \left(\frac{D_F}{\ell} \right) \log \sin \left(\frac{\pi \ell}{D_F} \right)$$

and apply the multiplication formula for the Gamma function. □

2.4.6 Numerical aspects

The preceding corollary can be used to approximate $Z_{w_n}(k)$ numerically. To make an educated guess for the quality of approximation, we give an easy lemma

Lemma 2.12. *In the situation of Corollary 2.10, let*

$$H := \sum_{m'} \prod_{p|N} \left(1 + \left\{\frac{\Lambda'}{p}\right\}\right) h(\Lambda') \log(\varepsilon_{\Lambda'})$$

and

$$M := \sup_t \frac{\sum_{m'} h(\Lambda') \log(\varepsilon_{\Lambda'})/\log(N)}{t/n}.$$

Then the following tail estimate holds

$$\left| -4 \sum_{t>T} \prod_{p|\frac{D}{n}} \left(1 - \left(\frac{D_F}{p}\right)\right) \frac{H}{\log N} \frac{N^{-k}}{1-\frac{1}{N}} \right| < \frac{4 \cdot 2^r MC}{n^k} \frac{1}{(2k-2)L^{2k-2}}$$

where r is the number of primes dividing DN/n, $L := \lfloor T/n \rfloor$ and

$$C := \frac{N(T)^{-k}/(1-1/N(T))}{(t^2/n)^{-k}}.$$

Remark. For $T \gg 0$, the second constant C is ≈ 1.0, hence negligible.

Proof. The modulus of the tail equals

$$\begin{aligned} &4 \sum_{t>T} \prod_{p|\frac{D}{n}} \left(1 - \left(\frac{D_F}{p}\right)\right) \frac{H}{\log N} \frac{N^{-k}}{1-\frac{1}{N}} \\ &\leq 4 \cdot 2^r MC \sum_{t>T} \frac{t}{n} \left(\frac{t^2}{n}\right)^{-k} \end{aligned}$$

where we use the fact that $N^{-k}/(1-1/N) \searrow (t^2/n)^{-k}$ as $t \to \infty$. By letting $\ell = t/n$ and $L = \lfloor T/n \rfloor$, this is certainly less than or equal to

$$\begin{aligned} 4 \cdot 2^r MC \sum_{\ell=L+1}^{\infty} \ell \left(\ell^2 n\right)^{-k} &= \frac{4 \cdot 2^r MC}{n^k} \sum_{\ell=L+1}^{\infty} \frac{1}{\ell^{2k-1}} \\ &\leq \frac{4 \cdot 2^r MC}{n^k} \int_L^{\infty} \frac{1}{x^{2k-1}} \,\mathrm{d}x. \end{aligned}$$

□

Main Example. In our main example $X(26, 1)$, we sum over $t \leq T = 520\,000\,000$ using a simple PARI [42] script (see Appendix B for the source code) to get the approximation

$$Z_{w_{26}\Gamma}(2) \approx -4 \sum_{t \leq T} \frac{H}{\log N} \frac{N^{-2}}{1-\frac{1}{N}} = -0.017\,760\,240\,104\,545\,08\ldots$$

and observe numerically that the bound $M = 4$ is sharp (and attained very early at the third term $t = 78$). So by Lemma 2.12, we get an error bound of

$$\frac{16}{26^2}\frac{1}{2 \cdot 20\,000\,000^2} = \frac{1}{338} \cdot 10^{-14} \approx 3 \cdot 10^{-17}.$$

telling us that all but the last two digits are correct.

As for the ordinary case, summing over $t \leq T = 80\,000\,000$, we obtain for $Z = Z_{w_1\Gamma}$

$$Z(2) \approx -0.125\,879\,797\,776\,281\,50$$

with $M = 1/8$ (attained again early at $t = 8$). So by Lemma 2.12, we get an error bound of

$$\frac{4 \cdot 2^2}{8}\frac{1}{2 \cdot 80\,000\,000^2} = 1.5625 \cdot 10^{-16},$$

and again, all but the last two digits are correct.

2.4.7 The height of the fixed point scheme of an Atkin-Lehner involution

In this section, we set about computing the height of the fixed point scheme $\mathcal{X}^{w_n}$ of an Atkin-Lehner involution w_n, $n \mid D = D_H$, $n \neq 1$, on a Shimura curve $\mathcal{X} = \mathcal{X}(D, N)$, N square-free. The key observation – which is probably perfectly obvious to the specialist – is that in the moduli interpretation, this fixed point scheme consists of Abelian surfaces with extra complex multiplication (CM).

Let us explain this in more detail for the case $N = 1$ (fixing a level structure would only obscure things): The Shimura curve is then the coarse moduli space of Abelian surfaces together with a fixed action of $\mathcal{O}$. It consists of pairs $(\mathcal{A}, i)$ where $\mathcal{A}$ is an Abelian variety of dimension 2 and i a fixed embedding of $\mathcal{O}$ into $\mathrm{End}(\mathcal{A})$.

More concretely, $\mathcal{X}(\mathbb{C})$ parametrises complex Abelian surfaces with a fixed action by $\mathcal{O}$: To every $z \in \mathbb{H}$ we associate A_z which is the quotient of $\mathbb{C}^2$ by a lattice L_z associated to z as follows

$$A_z := \mathbb{C}^2/L_z, \qquad L_z := \{\Phi(u) \cdot \begin{pmatrix} z \\ 1 \end{pmatrix} \mid u \in \mathcal{O}\}.$$

Then the definition of i_z is the obvious one ($u \in \mathcal{O}$ acts as left multiplication by $\Phi(u)$), and it is classical that (A_z, i_z) is isomorphic to $(A_{z'}, i_{z'})$ iff z and z' are equivalent modulo the Fuchsian group $\Gamma = \Gamma(D, 1)$.

Now it is clear what the complex points $\mathcal{X}^{w_n}(\mathbb{C})$ are. They are the Γ-orbits of points $z_0 \in \mathbb{H}$ that are fixed by $\Phi(\alpha)$ for an element α of $\mathcal{O}$ of norm n and trace t such that $t^2 - 4n < 0$.

Next we observe that A_{z_0} has an extra endomorphism x commuting with $\mathcal{O}$ and satisfying $x^2 - tx + n = 0$, namely we can take the $\mathbb{R}$-linear map $M \mapsto M \cdot \Phi(\alpha)$ from $M_2(\mathbb{R})$ to itself given by right multiplication and transport it to $\mathbb{C}^2$ via the $\mathbb{R}$-linear isomorphism $M \mapsto M \cdot \begin{pmatrix} z_0 \\ 1 \end{pmatrix}$ to get x. In short: x maps $M \cdot \begin{pmatrix} z_0 \\ 1 \end{pmatrix} \mapsto M \cdot \Phi(\alpha) \cdot \begin{pmatrix} z_0 \\ 1 \end{pmatrix}$. Then x is clearly an $\mathbb{R}$-linear endomorphism of $\mathbb{C}^2$ mapping the lattice L_{z_0} into itself while commuting with the fixed action

i_{z_0} of $\mathcal{O}$ because the latter action is by left multiplication. When $\Phi(\alpha)$ fixes z_0 as a Möbius transformation, x is in fact multiplication by the non-zero complex number ξ for which $\Phi(\alpha) \cdot \begin{pmatrix} z_0 \\ 1 \end{pmatrix} = \xi \begin{pmatrix} z_0 \\ 1 \end{pmatrix}$ as can be easily verified:

$$M \cdot \Phi(\alpha) \cdot \begin{pmatrix} z_0 \\ 1 \end{pmatrix} = M \cdot \begin{pmatrix} \xi z_0 \\ \xi \end{pmatrix} = \xi M \cdot \begin{pmatrix} z_0 \\ 1 \end{pmatrix}.$$

Now since α has minimal polynomial $X^2 - tX + n = 0$ (cf. (2.10)), we have $x^2 - tx + n = \xi^2 - t\xi + n = 0$.

Let us immediately check this in our

Main Example. For $\alpha = ij$ we get

$$\Phi(\alpha) = \begin{pmatrix} & \sqrt{2} \\ -13\sqrt{2} & \end{pmatrix}$$

which clearly has determinant 26, trace 0 and fixed point $z_0 = i/\sqrt{13}$ in the upper half plane. Then

$$\begin{pmatrix} & \sqrt{2} \\ -13\sqrt{2} & \end{pmatrix} \cdot \begin{pmatrix} i/\sqrt{13} \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ -i\sqrt{26} \end{pmatrix},$$

and we read off $\xi = -i\sqrt{26}$ which clearly satisfies $\xi^2 = -26$.

Now let Λ' be the maximal quadratic suborder of $\mathrm{End}(A_{z_0})$ containing x. Then A_{z_0} has CM exactly by Λ' and this order is isomorphic with the maximal quadratic suborder of $\mathcal{O}$ containing α. By Section 2.4.4, we know exactly how many fixed points w_n has and which order they have CM by: Namely take t divisible by n such that $t^2 - 4n < 0$ and set $t^2 - 4n = m^2 D_F$ with $F = \mathbb{Q}(\sqrt{t^2 - 4n})$. Then for each $m' \mid m$ with $(m', \frac{D_H}{n}) = 1$ there are

$$h(\Lambda') \prod_{p \mid \frac{D_H}{n}} \left(1 - \left(\frac{D_F}{p}\right)\right)$$

many fixed points z_0 (modulo Γ) with CM exactly by the order Λ' of conductor m' in F, and all fixed points are obtained this way. Recall that two conjugacy classes correspond to one fixed point, cf. Section 2.4.4. We also remark that a level structure $N > 1$ would simply give rise to extra factors with Eichler symbols.

Now we claim that A_{z_0} has to split as a product of elliptic curves $E_1 \times E_2$ because

$$\mathrm{End}(A_{z_0}) \supset \mathcal{O} \times \Lambda'.$$

This is clear because the endomorphism ring of a simple Abelian variety is an order in a skew-field, hence without zero divisors. Furthermore, it is clear that E_1 and E_2 have CM by the order Λ' as well. Again, we check this by pedestrian calculations in our

Main Example. Using the basis 1, i, $(j+1)/2$ and $i(j+1)/2$ of $\mathcal{O}$ we compute the period matrix $\Pi = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{pmatrix}$ of $A_{i/\sqrt{13}}$ with respect to the standard basis of $\mathbb{C}^2$:

$$\lambda_1 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} i/\sqrt{13} \\ 1 \end{pmatrix} = \begin{pmatrix} i/\sqrt{13} \\ 1 \end{pmatrix}$$
$$\lambda_2 = \begin{pmatrix} \sqrt{2} & \\ & -\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} i/\sqrt{13} \\ 1 \end{pmatrix} = \begin{pmatrix} i\sqrt{2}/\sqrt{13} \\ -\sqrt{2} \end{pmatrix}$$
$$\lambda_3 = \begin{pmatrix} 1/2 & 1/2 \\ 13/2 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} i/\sqrt{13} \\ 1 \end{pmatrix} = \begin{pmatrix} (i/\sqrt{13}+1)/2 \\ (i\sqrt{13}+1)/2 \end{pmatrix}$$
$$\lambda_4 = \begin{pmatrix} \sqrt{2} & \\ & -\sqrt{2} \end{pmatrix} \cdot \lambda_3 = \begin{pmatrix} (i/\sqrt{13}+1)/\sqrt{2} \\ -(i\sqrt{13}+1)/\sqrt{2} \end{pmatrix}$$

Now we set $\lambda_2' = \lambda_2 - 2\lambda_4 = \begin{pmatrix} -\sqrt{2} \\ i\sqrt{26} \end{pmatrix}$, $\lambda_3' = \lambda_3 - 7\lambda_1 = \begin{pmatrix} (-i\sqrt{13}+1)/2 \\ (i\sqrt{13}-13)/2 \end{pmatrix}$ and observe that $\lambda_2' = i\sqrt{26}\lambda_1$, $\lambda_4 = i\sqrt{2}/\sqrt{13}\lambda_3'$ so that the period matrix of $A_{i/\sqrt{13}}$ with respect to the basis λ_1, λ_3' of $\mathbb{C}^2$ is

$$\begin{pmatrix} 1 & i\sqrt{26} & 0 & 0 \\ 0 & 0 & 1 & i\sqrt{2}/\sqrt{13} \end{pmatrix}.$$

If the reader finds these computations a bit too ad-hoc we refer him to [4, Sect. 10.6] for a systematic treatment. In any case, we now know that $A_{i/\sqrt{13}}$ is isomorphic to $E(i\sqrt{26}) \times E(i\sqrt{2}/\sqrt{13})$. Here we use the standard notation $E(\tau) = \mathbb{C}/\Lambda_\tau$, $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$. It is trivial to check (again) with one's bare hands that multiplication by $\sqrt{-26}$ preserves $\Lambda_{i\sqrt{26}}$ and $\Lambda_{i\sqrt{2}/\sqrt{13}}$ which again shows that both elliptic curves have CM by the maximal order $\mathbb{Z} + \sqrt{-26}\mathbb{Z}$ of $\mathbb{Q}(\sqrt{-26})$. However they are not isomorphic as one checks by computing their j-invariants.

Now it is a simple matter to prove

Proposition 2.13 (Height of fixed point scheme of Atkin-Lehner involution)**.** *For $n \mid D_H$, $n \neq 1$, the height of the fixed point scheme $\mathcal{X}^{w_n}$ of the Atkin-Lehner involution w_n acting on $\mathcal{X}(D,N)$ is given by*

$$h_{\hat{\omega}}(\mathcal{X}^{w_n}) = \sum_{\substack{t \in n\mathbb{N}_0 \\ t^2 - 4n < 0}} \prod_{p \mid \frac{D_H}{n}} \left(1 - \left(\frac{D_F}{p}\right)\right) \sum_{\substack{m' \mid m \\ (m', \frac{D_H}{n}) = 1}} \prod_{p \mid N} \left(1 + \left\{\frac{\Lambda'}{p}\right\}\right) h(\Lambda') \times \\ \times \left(C + \log(D_H) + 2h_{\mathrm{geom}}(\mathcal{E}_{\Lambda'})\right). \quad (2.12)$$

For every t of the outer sum, denote by D_F the discriminant of the quadratic field $F := \mathbb{Q}(\sqrt{t^2 - 4n})$ and define $m \geq 1$ by $t^2 - 4n = m^2 D_F$. Then for every m' of the inner sum, let Λ' be the order of conductor m' in F with class number $h(\Lambda')$. For Eichler's symbol $\left\{\frac{\Lambda'}{p}\right\}$ see (2.7). Furthermore $\hat{\omega}$ is the canonical bundle equipped with the renormalised Petersson metric $|dz|^2 = e^{-2C}(4\pi y)^2$ as in Theorem 2.4 and $h_{\mathrm{geom}}(\mathcal{E}_{\Lambda'})$ is the geometric height of an elliptic curve with CM by Λ' as defined in the proof below.

Proof. By base change to a suitable number field K we have as divisors

$$\mathcal{X}^{w_n} = \mathcal{P}_1 + \cdots + \mathcal{P}_M$$

such that $\mathcal{P}_1(\mathbb{C}), \dots, \mathcal{P}_M(\mathbb{C})$ are the complex fixed points $\mathcal{X}^{w_n}(\mathbb{C})$. Thus we have to compute $h_{\hat{\omega}}(\mathcal{P}_0)$ for a summand $\mathcal{P}_0$ of the right hand side. By definition

$$\begin{aligned} h_{(\omega, e^{-2C}(4\pi y)^2)}(\mathcal{P}_0) &:= -\frac{1}{2[K\colon \mathbb{Q}]} \sum_{\sigma\colon K\to\mathbb{C}} \log |f_\sigma(z_0)\mathrm{d}z|^2 \\ &= C - \frac{1}{2[K\colon \mathbb{Q}]} \sum_{\sigma\colon K\to\mathbb{C}} \log\left((4\pi y_0)^2 |f_\sigma(z_0)|^2\right) \end{aligned}$$

with $f_\sigma(z_0)\mathrm{d}z$ the image under σ of a generator of $\omega_{\mathcal{P}_0}$ and y_0 the imaginary part of z_0, $\mathcal{P}_0(\mathbb{C}) = \Gamma z_0$. We want to compare this to the geometric height of the Abelian surface $\mathcal{A}_0$ corresponding to $\mathcal{P}_0$, cf. [10, Sec. 1.2]. By enlarging K if necessary we may assume $\mathcal{A}_0$ has semi-stable reduction over K. Then by definition

$$h_{\mathrm{geom}}(\mathcal{A}_0) := -\frac{1}{2[K\colon \mathbb{Q}]} \sum_{\sigma\colon K\to\mathbb{C}} \log\left(\frac{1}{(2\pi)^2} \int_{\mathcal{A}_0(\mathbb{C})} |\alpha_\sigma \wedge \bar{\alpha}_\sigma|\right)$$

where α_σ is the image under σ of a Neron differential α, i.e. a generator of $\Lambda^2(\mathrm{Lie}\mathcal{A}_0)^*$. Since over $\mathbb{C}$, the Kodaira-Spencer isomorphism $\omega_{\mathcal{P}_0} \cong \Lambda^2(\mathrm{Lie}\mathcal{A}_0)^*$ takes the form $\mathrm{d}z \mapsto 2\pi i \mathrm{d}w_1 \wedge 2\pi i \mathrm{d}w_2$ with w_1, w_2 the standard coordinates of $\mathbb{C}^2 \cong \mathrm{Lie}A_0(\mathbb{C})$, we can use $\alpha_\sigma = -(2\pi)^2 f_\sigma(z_0)\mathrm{d}w_1 \wedge \mathrm{d}w_2$ and compute the integral on the right hand side as

$$(2\pi)^4 |f_\sigma(z_0)|^2 \int_{\mathbb{C}^2/L_{z_0}} |\mathrm{d}w_1 \wedge \mathrm{d}w_2 \wedge \mathrm{d}\bar{w}_1 \wedge \mathrm{d}\bar{w}_2|.$$

Letting $w_j = u_j + iv_j$, $j = 1, 2$, the integrand equals $4\mathrm{d}u_1 \wedge \mathrm{d}v_1 \wedge \mathrm{d}u_2 \wedge \mathrm{d}v_2$. Pulling it back to $M_2(\mathbb{R})$ along $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} z_0 \\ 1 \end{pmatrix}$, we get $(2y_0)^2 \mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}c \wedge \mathrm{d}d$. Hence

$$\int_{\mathbb{C}^2/L_{z_0}} |\mathrm{d}w_1 \wedge \mathrm{d}w_2 \wedge \mathrm{d}\bar{w}_1 \wedge \mathrm{d}\bar{w}_2| = (2y_0)^2 \operatorname{vol}(M_2(\mathbb{R})/\Phi(\mathcal{O})) = (2y_0 D_H)^2$$

as $\mathcal{O}$ has discriminant D_H. Putting everything together we obtain

$$h_{\mathrm{geom}}(\mathcal{A}_0) = -\frac{1}{2[K\colon \mathbb{Q}]} \sum_{\sigma\colon K\to\mathbb{C}} \log\left((4\pi y_0 D_H)^2 |f_\sigma(z_0)|^2\right)$$

and thus

$$h_{(\omega, e^{-2C}(4\pi y)^2)}(\mathcal{P}_0) = C + \log(D_H) + h_{\mathrm{geom}}(\mathcal{A}_0).$$

Now we have seen that $\mathcal{A}_0$ is a product $\mathcal{E}_1 \times \mathcal{E}_2$ of elliptic curves with CM by some Λ' (depending on the parameters t and m' of the fixed point). Obviously, we can also construct α as $\alpha_1 \wedge \alpha_2$ from Neron differentials α_1 on $\mathcal{E}_1$ and α_2 on $\mathcal{E}_2$, which proves $h_{\mathrm{geom}}(\mathcal{A}_0) = h_{\mathrm{geom}}(\mathcal{E}_1) + h_{\mathrm{geom}}(\mathcal{E}_2)$.

The generalised Chowla-Selberg formula stated below shows that the geometric heights of elliptic curves with the same CM agree. □

The above proposition is best combined with the following result due to Kaneko [20] and Nakkajima, Taguchi [31]:

Theorem 2.14 (Generalised Chowla-Selberg formula)**.** *Let $\mathcal{E}_\Lambda$ be an elliptic curve with CM exactly by the order Λ of conductor m (and discriminant $D_\Lambda = m^2 D_F$) in an imaginary quadratic number field F. Then its geometric height is given by*

$$h_{\mathrm{geom}}(\mathcal{E}_\Lambda) = -\frac{1}{2}\left(\log(\sqrt{|D_\Lambda|}) + \frac{L'_\Lambda(0)}{L_\Lambda(0)}\right). \tag{2.13}$$

Here, $L_\Lambda := \zeta_\Lambda/\zeta$ is the L-function associated to Λ, i.e. the quotient of the zeta function of Λ by the Riemann zeta function, the former being defined as

$$\zeta_\Lambda(s) := \sum_{\mathfrak{a}\subset\Lambda} N(\mathfrak{a})^{-s}$$

with the sum extending over all Λ-ideals contained in Λ and $N(\mathfrak{a})$ denoting the index $[\Lambda\colon \mathfrak{a}]$.

Furthermore, we have explicitly

$$\begin{aligned}\frac{L'_\Lambda(0)}{L_\Lambda(0)} &= -\log|D_F| + \frac{w(F)}{2h(F)}\sum_{\ell=1}^{|D_F|-1}\left(\frac{D_F}{\ell}\right)\log\Gamma\left(\frac{\ell}{|D_F|}\right) \\ &\quad - 2\log(m) + \sum_{p\mid m} e(p)\log(p);\end{aligned} \tag{2.14}$$

here $w(F)$ denotes as usual the number of roots of unity of F and

$$e(p) := \frac{(1-p^{-n})(1-\left(\frac{D_F}{p}\right))}{(1-p^{-1})(p-\left(\frac{D_F}{p}\right))}, \qquad p^n \parallel m.$$

Remark. For the case of maximal Λ, i.e. $m=1$, ζ_Λ is the Dedekind zeta function ζ_F of F and L_Λ is the usual L-function $L(D_F,s) := \sum_{n=1}^\infty \left(\frac{D_F}{n}\right)n^{-s}$ associated to the primitive Dirichlet character $\left(\frac{D_F}{\cdot}\right)$.

But be warned: For non-maximal Λ, L_Λ is *neither* the L function associated to the non-primitive Dirichlet character $\left(\frac{D_\Lambda}{\cdot}\right)$ *nor* the L-function associated to a discriminant as in [49, Eq. (7)]!

What is more for maximal Λ, the second line on the right hand side of (2.14) vanishes and the connoisseur will easily recognise the resulting formula as the classical Chowla-Selberg formula, stated for example in [10, Rem. 1.5].

Sketch of proof. The proof of the generalised formula proceeds in three main steps: First one has to make the arithmetic observation that (see [31, Lemma 3])

$$-2h_{\mathrm{geom}}(\mathcal{E}_\Lambda) = \log\left(\frac{\sqrt{|D_\Lambda|}}{2\pi}\right) + \frac{1}{12h(\Lambda)}\sum_{\mathfrak{a}\in\mathrm{Cl}(\Lambda)}\log(F(\mathfrak{a}))$$

where the sum is over the ideal class group of Λ and $F := \Delta(\mathfrak{a})\Delta(\mathfrak{a}^{-1})$.

The second step is an application of Kronecker's Grenzformel: Arguing as in [47, Chap. IX, §4], one sees that $\zeta_\Lambda(0) = -h(\Lambda)/w(\Lambda)$ and $\zeta'_\Lambda(0) = -1/12w(\Lambda)\sum_{\mathfrak{a}\in\mathrm{Cl}(\Lambda)}\log(F(\mathfrak{a}))$. These identities prove

$$-2h_{\mathrm{geom}}(\mathcal{E}_\Lambda) = \log\left(\frac{\sqrt{|D_\Lambda|}}{2\pi}\right) + \frac{\zeta'_\Lambda(0)}{\zeta_\Lambda(0)},$$

whence (2.13).

The third step is to apply the proposition in [20] which states that $\zeta_\Lambda = \zeta_F \prod_{p|m} \varepsilon_{m,p}$ with

$$\varepsilon_{m,p}(s) := \frac{(1-u)(1-\left(\frac{D_F}{p}\right)u) - p^{n-1}u^{2n}(1-pu)(\left(\frac{D_F}{p}\right) - pu)}{1-pu^2}, \qquad u = p^{-s}$$

where $p^n \parallel m$. Using this and the classical fact that $\zeta_F(s) = \zeta(s)L(D_F, s)$, we can compute

$$\frac{L'_\Lambda(0)}{L_\Lambda(0)} = \frac{L'(D_F,0)}{L(D_F,0)} + \sum_{p|m} \frac{\varepsilon'_{m,p}(0)}{\varepsilon_{m,p}(0)}.$$

Then by the classical Chowla-Selberg formula, the logarithmic derivative of $L(D_F, s)$ at $s = 0$ gives the first term on the right hand side of (2.14) and pedestrian computations show that

$$\frac{\varepsilon'_{m,p}(0)}{\varepsilon_{m,p}(0)} = e(p)\log(p) - 2n\log(p).$$

$\square$

Remark. Recall that $w(F) = 2$ unless $D_F = -3$, -4, see Example 2.1.

Remark. When $\mathcal{O}$ contains no elements of norm n and non-zero trace, the combination of Proposition 2.13 and Theorem 2.14 can be obtained from a result of Kudla, Rapoport and Yang about the height of CM divisors on a Shimura curve [26, Cor. 10.12]. A minor point is that their degree of the fixed point divisor is twice ours, which is due to the fact that they work on a stack. A major point arises when their result is transferred from the Hodge bundle to the canonical bundle via the Kodaira-Spencer isomorphism: This transfer causes the appearance of the term $\log(D_H)$ on the right hand side of (2.12).

Main Example. For the height of the fixed point scheme of the Atkin-Lehner involution w_{26} acting on $\mathcal{X}(26,1)$ we get

$$h_{\hat{\omega}}(\mathcal{X}^{w_{26}}) = 6\left(C + \log(26) + \frac{1}{2}\log(104) - \frac{1}{6}\sum_{\ell=1}^{103}\left(\frac{-104}{\ell}\right)\log\Gamma\left(\frac{\ell}{104}\right)\right)$$

since $\mathbb{Q}(\sqrt{-104})$ contains $w = 2$ roots of unity and class number $h = 6$.

Remark. For an alternative approach to calculating the height of the fixed point scheme see Appendix A where we indicate how to compute heights using a projective model of the arithmetic surface. However let us be frank and admit that for this alternative to be numerically useful, one would need to compute explicitly the uniformisation map from the upper half plane to the Riemann surface $\mathcal{X}(\mathbb{C})$.

2.5 Specialising to involutions

2.5.1 Equivariant torsion of k-differentials with respect to an involution

As a further specialisation of Corollary 1.4, let us compute the equivariant torsion of the line bundle $\overline{\omega}^k$ of k-differentials with respect to an involution, e.g. an Atkin-Lehner involution on a Shimura curve.

Corollary 2.15 (Equivariant torsion of k-differentials with respect to an involution)**.** *Let $k \geq 2$ be an integer and g an involution on $X = \Gamma \backslash \mathbb{H}$. Then*

$$\begin{aligned}\mathcal{T}_g(\overline{\omega}^k) = &- \log Z_{\tilde{g}\Gamma}(k) \\ &- \# X^g \frac{(-1)^k}{2} \left(\log\left(\frac{\pi}{2}\right) - \sum_{1 \leq j \leq 2k-2} (-1)^j \log(j) \right) \\ &+ \log(2) \left(- \# X^g \frac{(-1)^k}{4} \right),\end{aligned}$$

the last term on the right being the metric correction term.

Proof. An involution acts as -1 on ω at all fixed points, i.e. all θ's are equal to $\pi/2$.

Clearly $\sin\big((2k-1)\pi/2\big)/\sin(\pi/2) = \cos\big((k-1)\pi\big) = (-1)^{k-1}$. Furthermore $F'(-1,0)$ is real since $F(-1,w)$ is for $w > 0$. Therefore $R^{\mathrm{rot}}(\pi) = 0$. The easiest way to compute the derivative of $F(-1,w)$ is to express it in terms of Riemann's zeta function as $\zeta(w)(2^{1-w} - 1)$. We thus obtain $F'(-1,0) = -\frac{1}{2}\log(\pi/2)$.

The next term in Corollary 1.4 is simply

$$\# X^g \frac{(-1)^{k-1}}{2} \sum_{1 \leq j \leq 2k-2} (-1)^j \log(j).$$

And finally, the metric correction term from (1.34) gives

$$\begin{aligned}&\log(2) \left(\sum_{p \in X^g} \frac{\operatorname{tr}(g|_{\omega_p^k})}{|1 - T_p g|^2} - \operatorname{tr}\left(g^*|_{H^0(X,\omega^k)}\right) \right) \\ &= \log(2)\, \# X^g \left(\frac{(-1)^k}{4} - \frac{(-1)^k}{2} \right).\end{aligned}$$

The last equation uses the Lefschetz trace formula for the very ample line bundle ω^k. □

2.5.2 The arithmetic Lefschetz fixed point formula for an involution

Combining Theorem 2.2 with Remark 2.1, we get the following

Corollary 2.16 (The arithmetic Lefschetz trace formula for an involution)**.** *Let the arithmetic surface $\mathcal{X}$ carry an action by the constant group scheme $\mathbb{Z}/(2)$ with fixed point scheme $\mathcal{X}^g$ and denote by g the induced involution on $\mathcal{X}(\mathbb{C})$. As usual, denote degree 0 by $+$ and degree 1 by $-$. Then*

$$\widehat{\deg}(\overline{H^\cdot(\mathcal{X}, \omega^k)_+}) - \widehat{\deg}(\overline{H^\cdot(\mathcal{X}, \omega^k)_-}) = \frac{1}{2}\mathcal{T}_g(\overline{\omega}^k) + \frac{(-1)^k}{2}\left(k - \frac{1}{2}\right) h_{\overline{\omega}}(\mathcal{X}^g)$$

modulo rational multiples of $\log(2)$.

2.5.3 Computing the arithmetic Lefschetz trace for a general involution

Putting together Corollary 2.16 and Corollary 2.15 we obtain

Proposition 2.17 (The arithmetic Lefschetz trace formula for an involution)**.** *Let the arithmetic surface $\mathcal{X}$ carry an action by the constant group scheme $\mathbb{Z}/(2)$ with fixed point scheme $\mathcal{X}^g$ and denote by g the induced involution on $\mathcal{X}(\mathbb{C})$. As usual, denote degree* 0 *by* $+$ *and degree* 1 *by* $-$*. Then we have for* $k \geq 2$

$$\begin{aligned}
\widehat{\deg}(\overline{H^0(\mathcal{X},\omega^k)_+}) - \widehat{\deg}(\overline{H^0(\mathcal{X},\omega^k)_-}) &= -\frac{1}{2}\log Z_{\tilde{g}\Gamma}(k) \\
&- \frac{1}{2}\#X^g\frac{(-1)^k}{2}\left(\log\left(\frac{\pi}{2}\right) - \sum_{1\leq j\leq 2k-2}(-1)^j\log(j)\right) \\
&+ \frac{1}{2}\log(2)\left(-\#X^g\frac{(-1)^k}{4}\right) \\
&+ \frac{(-1)^k}{2}\left(k-\frac{1}{2}\right)h_{\overline{\omega}}(\mathcal{X}^g)
\end{aligned}$$

modulo rational multiples of $\log(2)$.

2.5.4 Proof of Theorem 0.1

Now to prove Theorem 0.1, there is not much left to do: We apply the above Proposition 2.17 to the case of an Atkin-Lehner involution acting on a Shimura curve, i.e. $\mathcal{X} = \mathcal{X}(D_H, 1)$ and $g = w_n$. Then the Selberg zeta value $Z_{w_n\Gamma}(k)$ has been computed in Proposition 2.10. Furthermore, the height $h_{\overline{\omega}}(\mathcal{X}^{w_n})$ of the fixed point scheme of w_n is found via Proposition 2.13 and the generalised Chowla-Selberg formula (Theorem 2.14).

Then we only need to compute the derivative at $s = 0$ of the function $\tilde{L}(\Lambda, s)$ in the real and in the imaginary case:

For $D_\Lambda > 0$ we have $L_\Lambda(0) = 0$, $L'_\Lambda(0) = \frac{1}{2}L_\Lambda(1) = h(\Lambda)\log(\varepsilon_\Lambda)$ and, hence in this case

$$\left.\frac{d}{ds}\right|_{s=0}\left(|D_\Lambda|^{s/2}L_\Lambda(s)\right) = h(\Lambda)\log(\varepsilon_\Lambda).$$

On the other hand, if $D_\Lambda < 0$ we have $L_\Lambda(0) = 2h(\Lambda)/w(\Lambda)$ with $w(\Lambda)$ the number of roots of unity contained in Λ. But since $n \neq 1$, we know D_Λ can't equal -3 or -4 so that $w(\Lambda) = 2$. Hence in this case

$$\begin{aligned}
\left.\frac{d}{ds}\right|_{s=0}\left(|D_\Lambda|^{s/2}L_\Lambda(s)\right) &= \left(\log\left(\sqrt{|D_\Lambda|}\right)L_\Lambda(0) + L'_\Lambda(0)\right) \\
&= h(\Lambda)\left(\log\left(\sqrt{|D_\Lambda|}\right) + \frac{L'_\Lambda(0)}{L_\Lambda(0)}\right).
\end{aligned}$$

Thus, the proof of Theorem 0.1 is complete.

2.6 Combining the theorems of Gillet-Soulé and Köhler-Roessler

Now suppose we are in the extraordinary situation that the cohomology decomposition (2.2) consists of one summand only. Loosely speaking, this means that the action of the automorphism on the cohomology is constant.

Then the left hand sides of the arithmetic Riemann-Roch theorem (Theorem 2.1) and the arithmetic Lefschetz fixed point formula (Theorem 2.2) are equal, which gives an interesting relationship between the height of the arithmetic surface and the heights of the fixed point schemes involving both equivariant and ordinary torsion.

2.6.1 A curious identity on arithmetic surfaces of genus two

Let us explicate this for the case of an involution and a very ample power of the canonical bundle. We make the following observation:

Observation. *Let $\mathcal{X}$ be an arithmetic surface equipped with an involution. Suppose for $k \geq 2$*

$$H^0(\mathcal{X}, \omega^k) = H^0(\mathcal{X}, \omega^k)_+ \quad \textit{or} \quad H^0(\mathcal{X}, \omega^k) = H^0(\mathcal{X}, \omega^k)_-. \tag{2.15}$$

Then the first alternative holds, $\mathcal{X}$ has genus two, $k = 2$ and the involution is the hyperelliptic involution.

Thus in this rare case, we obtain the following curious identity

Proposition 2.18 (Quadratic differentials on arithmetic surface of genus two)**.** *Consider an arithmetic surface $\mathcal{X}$ of genus two with its hyperelliptic involution. Let $\mathcal{W}$ be the divisor of Weierstraß points and denote by g the hyperelliptic involution on $\mathcal{X}(\mathbb{C})$. Then*

$$\frac{1}{2}\mathcal{T}(\overline{\omega}^2) + 2\zeta'(-1) - \frac{1}{12} + \frac{13}{12}h_{\overline{\omega}}(\mathcal{X}) = \frac{1}{2}\mathcal{T}_g(\overline{\omega}^2) + \frac{3}{4}h_{\overline{\omega}}(\mathcal{W}) \tag{2.16}$$

modulo rational multiples of $\log(2)$*. Here ζ' is the derivative of Riemann's zeta function and $\mathcal{T}$, $\mathcal{T}_g$ denote ordinary and equivariant torsion, respectively. If the Fuchsian group Γ with $\mathcal{X}(\mathbb{C}) = \Gamma\backslash\mathbb{H}$ is strictly hyperbolic, we can write more concretely*

$$-\frac{1}{2}\log Z(2)+\frac{9}{4}-\frac{8}{3}\log(2)-\frac{1}{12}+\frac{13}{12}h_{\overline{\omega}}(\mathcal{X}) = -\frac{1}{2}\log Z_{\tilde{g}\Gamma}(2)+\frac{9}{4}\log(2)+\frac{3}{4}h_{\overline{\omega}}(\mathcal{W}) \tag{2.17}$$

modulo rational multiples of $\log(2)$ *where Z, $Z_{\tilde{g}\Gamma}$ are the ordinary and equivariant Selberg zeta function, respectively.*

Proof of Proposition. It is clear that the fixed point scheme of the hyperelliptic involution is $\mathcal{W}$ so the right hand side of (2.16) is just the right hand side of Corollary 2.16. As the Euler characteristic of $X = \mathcal{X}(\mathbb{C})$ equals -2, the left hand side of (2.16) is exactly the right hand side of Theorem 2.1.

The torsion terms can be computed in terms of Selberg zeta functions with the aid of Corollary 1.12 (applicable as stated only when the Fuchsian group Γ is cocompact without elliptic elements!) as

$$
\begin{aligned}
\frac{1}{2}\mathcal{T}(\overline{\omega}^2) =& -\frac{1}{2}\log Z(2) - \frac{1}{2}\cdot 2\left(2\zeta'(-1) - \left(\frac{3}{2}\right)^2 + \frac{3}{2}\log(2\pi) + \frac{1}{2}\log(2)\right) \\
&+ \frac{1}{2}\log(2)\left(-2\cdot\frac{2}{3}\right) \\
=& -\frac{1}{2}\log Z(2) - 2\zeta'(-1) - \frac{3}{2}\log(\pi) + \frac{9}{4} - \frac{8}{3}\log(2)
\end{aligned}
$$

and by Corollary 2.15 as

$$
\begin{aligned}
\frac{1}{2}\mathcal{T}_g(\overline{\omega}^2) &= -\frac{1}{2}\log Z_{\tilde{g}\Gamma}(2) - \frac{1}{2}\cdot 6\cdot\frac{1}{2}\left(\log\left(\frac{\pi}{2}\right) - \log(2)\right) + \frac{1}{2}\log(2)\left(-6\cdot\frac{1}{4}\right) \\
&= -\frac{1}{2}\log Z_{\tilde{g}\Gamma}(2) - \frac{3}{2}\log(\pi) + \frac{9}{4}\log(2).
\end{aligned}
$$

Plugging this into (2.16), the $\zeta'(-1)$ and $\log(\pi)$ terms cancel nicely and (2.17) is proved. $\square$

Proof of Observation. Obviously for (2.15) to hold, the action of the involution on the holomorphic 1-differentials must be constant because two holomorphic differentials $\eta_\pm$ with $g^*\eta_\pm = \pm\eta_\pm$ would give rise to holomorphic k-differentials η_+^k and $\eta_+^{k-1}\eta_-$ on which the involution acts as ± 1 respectively. The next point is: if the action on $H^0(X,\omega)$ was constant and equal to $+1$, this would contradict the Lefschetz trace formula by which we know

$$
\operatorname{tr}\left(g^*|_{H^0(X,\omega)}\right) = \sum_{p\in X^g}\frac{-1}{1-(-1)} + \operatorname{tr}\left(g^*|_{H^0(X,\mathcal{O})}\right) = -\frac{\#X^g}{2} + 1 \leq 0.
$$

So the involution g has to act as -1 on $H^0(X,\omega)$. But then since $H^0(X,\omega)$ has dimension equal to the genus h of X, the above computation shows that g has $2h+2$ fixed points, i.e. g is hyperelliptic. Now we invoke the Lefschetz formula again, to see that for $k\geq 2$

$$
\operatorname{tr}\left(g^*|_{H^0(X,\omega^k)}\right) = \sum_{p\in X^g}\frac{(-1)^k}{1-(-1)} = \frac{(-1)^k(2h+2)}{2}.
$$

If this is to equal $(-1)^k\dim(H^0(X,\omega^k)) = (-1)^k(2k-1)(h-1)$, we have to solve $h+1 = (2k-1)(h-1)$ with integers $h,\ k\geq 2$. This is equivalent to $1/(h-1) = k-1$, whence we deduce $h = k = 2$.

So (2.15) can only hold for the hyperelliptic involution acting on quadratic differentials over a genus two curve and then it has the form $H^0(X,\omega^2) = H^0(X,\omega^2)_+$. $\square$

Main Example. In our main example, we have computed all terms occurring in

(2.17):

$$\begin{aligned}
\log Z(2) &= -0.125\,879\,797\,776\,281\ldots \\
\log Z_{w_{26}\Gamma}(2) &= -0.017\,760\,240\,104\,545\ldots \\
h_{\overline{\omega}}(\mathcal{X}(26,1)) &= 2\left(2\log(4\pi) + 24\zeta'(-1) - 1 + \frac{3}{2}\log(2) + \frac{7}{12}\log(13)\right) \\
h_{\overline{\omega}}(\mathcal{X}^{w_{26}}) &= 6\left(\log(4\pi) + \log(26) + \frac{1}{2}\log(104) - \frac{1}{6}\sum_{\ell=1}^{103}\left(\frac{-104}{\ell}\right)\log\Gamma\left(\frac{\ell}{104}\right)\right)
\end{aligned}$$

However, it seems hard to determine the rational multiple of log(2) that makes (2.17) exact.

2.6.2 A plausibility check

To check whether (2.16) is plausible at all, we check how both sides transform when the metric on ω is multiplied by e^{-2C}. This means that the metric on the tangent bundle is multiplied by e^{2C} so that the Laplacian is divided by e^{2C}, see Remark 1.1 in Section 1.5.3. Then use the metric correction term stated in Corollary 1.12 and Lemma 2.19 below to see that the left hand side picks up

$$\frac{1}{2}(2C)\left(-V\left(\frac{k}{2}-\frac{1}{3}\right)\right) + \frac{13}{12}(2C)(-\chi(X)) = C\left(-\frac{4}{3}+\frac{13}{3}\right) = 3C.$$

Here we used $V = -\chi(X) = 2$. As for the right hand side, employ Corollary 2.15 and Lemma 2.19 below again to see that it changes by

$$\frac{1}{2}(2C)\left(-\#X^g\frac{(-1)^k}{4}\right) + \frac{3}{4}C\#X^g = C\left(-\frac{6}{4}\right) + C\frac{3}{4}\cdot 6 = 3C,$$

which is rather reassuring. Here we used $\#X^g = 6$.

Lemma 2.19 (How heights transform under a constant rescaling of the metric)**.** *Let ω be the canonical bundle on an arithmetic surface $\mathcal{X}$ equipped with some metric h and let $\mathcal{Z}$ be a divisor. If the metric is rescaled by a constant factor e^{-2C}, heights transform as follows:*

$$\begin{aligned}
h_{(\omega,e^{-2C}h)}(\mathcal{Z}) &= h_{(\omega,h)}(\mathcal{Z}) + C\deg(\mathcal{Z}(\mathbb{C})) \\
h_{(\omega,e^{-2C}h)}(\mathcal{X}) &= h_{(\omega,h)}(\mathcal{X}) - 2C\,\chi(\mathcal{X}(\mathbb{C})).
\end{aligned}$$

Proof. First we state three basic formulae for arithmetic Chow groups: The first and most fundamental formula shows how the arithmetic first Chern class of a Hermitian line bundle $(\mathcal{L}, h)$ over an arithmetic variety $\mathcal{Y}$ changes when the metric is rescaled

$$\widehat{c}_1(\mathcal{L}, e^{-\varphi}h) = \widehat{c}_1(\mathcal{L}, h) + \widetilde{c}_1(e^{-\varphi}h, h).$$

and the Bott-Chern class appearing can be calculated explicitly as

$$\widetilde{c}_1(e^{-\varphi}h, h) = [0, \log\left(\frac{h}{e^{-\varphi}h}\right)] = [0, \varphi],$$

cf. [39, IV.3.3].

Next suppose that $\mathcal{Y}$ is of relative dimension δ over $\mathrm{Spec}(\mathbb{Z})$. Then f_* is a map from $\widehat{CH}^{\delta+1}(\mathcal{Y})$ to $\widehat{CH}^{1}(\mathcal{X})$ and for a smooth form ϕ of type (δ,δ), we have

$$f_*([0,\phi]) = [0, \int_{\mathcal{Y}(\mathbb{C})} \phi].$$

The last fact we need is that the isomorphism $\widehat{\deg}$ from $\widehat{CH}^{1}\big(\mathrm{Spec}(\mathbb{Z})\big)$ to $\mathbb{R}$ satisfies

$$\widehat{\deg}([0,x]) = \frac{1}{2}x.$$

Now denote by $f^{\mathcal{X}}\colon \mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ and $f^{\mathcal{Z}}\colon \mathcal{Z} \to \mathrm{Spec}(\mathbb{Z})$ the defining maps. The transformation of the divisor height is straightforward

$$\begin{aligned} h_{(\omega, e^{-2C}h)}(\mathcal{Z}) &= \widehat{\deg}\left(f^{\mathcal{Z}}_* \widehat{c}_1(\omega, e^{-2C}h)\right) \\ &= h_{(\omega,h)}(\mathcal{Z}) + \widehat{\deg}\left(f^{\mathcal{Z}}_*([0,2C])\right) \\ &= h_{(\omega,h)}(\mathcal{Z}) + \frac{1}{2}\int_{\mathcal{Z}(\mathbb{C})} 2C. \end{aligned}$$

As for the height of the arithmetic surface, we have

$$\begin{aligned} h_{(\omega, e^{-2C}h)}(\mathcal{X}) &= \widehat{\deg}\left(f_* \widehat{c}_1(\omega, e^{-2C}h)^2\right) \\ &= h_{(\omega,h)}(\mathcal{X}) + 2\widehat{\deg}\left(f_*(\widehat{c}_1(\omega,h)\cdot[0,2C])\right) + \widehat{\deg}\left(f_*([0,2C]^2)\right). \end{aligned}$$

By definition of the intersection pairing $\widehat{CH^1}(\mathcal{X}) \times \widehat{CH^1}(\mathcal{X}) \to \widehat{CH^2}(\mathcal{X})$

$$\widehat{c}_1(\omega,h)\cdot[0,2C] = [0, c_1(\omega,h)2C] \qquad \text{and} \qquad [0,2C]^2 = 0.$$

Now recall that $\int_{\mathcal{X}(\mathbb{C})} c_1(\omega,h)$ gives minus the Euler characteristic of $\mathcal{X}(\mathbb{C})$ for any choice of metric h. $\square$

Appendix A

Projective heights of fixed point schemes

For an involution, we indicate how to compute the height of the fixed point scheme

$$h_{\overline{\omega}}(\mathcal{X}^g) = \widehat{\deg}\left(f_* \widehat{c_1}(\overline{\omega}|_{\mathcal{X}^g})\right)$$

when given a model of $\mathcal{X}$ as a plane curve $\mathcal{C}$ of degree d whose singularities do not meet the fixed point scheme.

In particular, letting $X = \mathcal{X}(\mathbb{C})$ and $C = \mathcal{C}(\mathbb{C})$, we then have a uniformising map $\iota\colon X \to \mathbb{P}^2(\mathbb{C})$ and $\iota(X) = C = \{F = 0\}$ with F a homogeneous polynomial of degree d with integer coefficients and F is regular over $\iota(X^g)$.

Then applying the adjunction formula (see [18, p. 280]) locally, we get

$$\omega_{\mathcal{X}}|_{\mathcal{X}^g} \cong \iota^*\left(\mathcal{O}(d-3)|_{\mathcal{C}^g}\right).$$

However, it is clear that we can't expect the Fubini-Study metric on $\mathcal{O}(d-3)$ to pull back onto the Petersson metric on ω. Therefore, we would have to compute

$$\begin{aligned}\widehat{c_1}(\omega|_{\mathcal{X}^g}, h_{\mathrm{Pet}}) &= \widehat{c_1}(\omega|_{\mathcal{X}^g}, \iota^* h_{\mathrm{FS}}) + \tilde{c_1}(h_{\mathrm{Pet}}, \iota^* h_{\mathrm{FS}}) \\ &= \widehat{c_1}(\mathcal{O}(d-3)|_{\mathcal{C}^g}, h_{\mathrm{FS}}) + \sum_{p \in X^g} \log(\iota^* h_{\mathrm{FS}}/h_{\mathrm{Pet}}).\end{aligned}$$

Remark. To carry out this computation numerically, one would need to compute the uniformising map, or at least its derivative at the fixed points.

Henceforth, we shall always assume that the line bundles on projective space are equipped with the Fubini-Study metric. By the linearity of the first arithmetic Chern class, it suffices to compute $\widehat{c_1}(\overline{\mathcal{O}(1)}|_{\mathcal{C}^g})$. By definition, this is the height $h_{\overline{\mathcal{O}(1)}}(C^g)$ of the fixed point scheme considered as a cycle of dimension 1 on $\mathbb{P}^2$ (with respect to the line bundle $\overline{\mathcal{O}(1)}$).

Next, we compute the fixed points $P_1, \ldots P_M$ in $\mathbb{P}^2(\mathbb{C})$ and choose homogeneous coordinates $P_m = (X_m, Y_m, Z_m)$ which are algebraic. Let K be the number field generated by these $3M$ coordinates and $\mathcal{O}_K$ its ring of integers. Changing the base to $S = \mathrm{Spec}(\mathcal{O}_K)$, we get

$$\mathcal{C}^g = P_1 + \cdots + P_M$$

as cycles of dimension 1 on $\mathbb{P}^2$ over S.

Therefore, the problem is reduced to computing the height (over S) of a cycle attached to a rational point $P = (X, Y, Z)$ in $\mathbb{P}^2(K)$ for which there is a classical recipe: Let I be the fractional ideal in K generated by X, Y and Z. Then (see [6, (3.1.6)])

$$h^S_{\overline{\mathcal{O}(1)}}(P) = -\log(\mathcal{N}(I)) + \sum_{\sigma\colon K\hookrightarrow\mathbb{C}} \log\sqrt{|\sigma(X)|^2 + |\sigma(Y)|^2 + |\sigma(Z)|^2}$$

with σ running over all embeddings of K into $\mathbb{C}$.

Lastly, there is the simple relation (see [6, (3.1.8)])

$$h_{\overline{\mathcal{O}(1)}}(\mathcal{C}^g) = \frac{1}{[K\colon\mathbb{Q}]} h^S_{\overline{\mathcal{O}(1)}}(\mathcal{C}^g).$$

So putting everything together and writing I_m for the fractional ideal generated by the coordinates of P_m, we obtain

$$\begin{aligned} h_{\overline{\mathcal{O}(1)}}(\mathcal{C}^g) = \frac{1}{[K\colon\mathbb{Q}]} \sum_{m=1}^{M} \Big\{ & -\log\mathcal{N}(I_m) \\ & + \sum_{\sigma} \log\sqrt{|\sigma(X_m)|^2 + |\sigma(Y_m)|^2 + |\sigma(Z_m)|^2}\Big\}. \end{aligned} \tag{A.1}$$

When $\mathcal{C}$ is hyperelliptic, formula (A.1) can be simplified even further:

Lemma A.1. *Consider a hyperelliptic plane curve of genus h*

$$\mathcal{C}\colon y^2 = p(x)$$

where $p(x) = a_d x^d + \cdots + a_0$, $d = 2h + 2$, is irreducible over $\mathbb{Q}$ with integer coefficients and distinct complex roots $x_1, \ldots, x_d$. Then the height of the divisor of Weierstraß points $\mathcal{W}$ is given by

$$h_{\overline{\mathcal{O}(1)}}(\mathcal{W}) = \log(|a_d|) + \sum_{m=1}^{d} \log\sqrt{|x_m|^2 + 1}.$$

Proof. As is well-known, $(x, y) \mapsto (x, -y)$ is the hyperelliptic involution and its fixed points are the Weierstraß points.

Since $p(x)$ has distinct roots $x_1, \ldots, x_d$, $\mathcal{C}$ is singular only at infinity which is luckily not a fixed point. Hence the above procedure works.

We choose $P_m := (x_m, 0, 1)$ as algebraic coordinates for the $M = d$ fixed points. Then the field K generated by these coordinates is simply the splitting field $\mathbb{Q}(x_1, \ldots, x_d)$ of the polynomial p.

Next we compute the norm of the principal ideal generated by x_m

$$\mathcal{N}(x_m\mathcal{O}_K) = \left|\mathcal{N}_{K/\mathbb{Q}}(x_m)\right| = |\mathcal{N}(x_m)|^{n/d} = \left|\frac{a_0}{a_d}\right|^{n/d}.$$

Here, we have set $n := [K\colon\mathbb{Q}]$, and $\mathcal{N}(x)$ denotes the norm of an algebraic number x which is (up to sign) the quotient of the lowest by the highest coefficient of its minimal polynomial. We have also used the following relationship between norms

$$\mathcal{N}_{K/\mathbb{Q}}(x) = \mathcal{N}(x)^{[K\colon\mathbb{Q}]/[K(x)\colon\mathbb{Q}]}, \qquad x \in K.$$

For more on these facts, see [9, Props. 4.3.2, 4.6.15].

Hence we get

$$-\log \mathcal{N}(I_m) = \frac{n}{d}\log(|a_d|)$$

for the fractional ideal I_m generated by the coordinates x_m, 0 and 1 of the fixed point P_m. Plugging this into (A.1), we get

$$h_{\overline{\mathcal{O}(1)}}(\mathcal{W}) = \frac{1}{n}\sum_{m=1}^{d}\Big\{\frac{n}{d}\log(|a_d|) + \sum_{\sigma}\log\sqrt{|\sigma(x_m)|^2+1}\Big\}.$$

Now remember that the Galois group of K acts transitively on $x_1, \ldots, x_d$. $\square$

Main Example. From [16, Thm. 3.1] we know that $\mathcal{X}(26,1)$ has a model as a hyperelliptic plane curve of degree $d = 2h+2 = 6$, namely

$$\mathcal{C}\colon y^2 = -2x^6 + 19x^4 - 24x^2 - 169. \tag{A.2}$$

The right hand side polynomial is irreducible over the rationals (as may be checked using PARI [42]) and has distinct complex roots which we shall compute explicitly by hand. By Lemma A.1 we know

$$\widehat{c_1}(\overline{\mathcal{O}(3)}|_{\mathcal{W}}) = 3\Big(\log(2) + \log(|z_1|+1) + \log(|z_2|+1) + \log(|z_3|+1)\Big)$$

where z_1, z_2, z_3 are the complex roots of $-2z^3 + 19z^2 - 24z - 169$ (so that $x_{1,2} = \pm\sqrt{z_1}$, $x_{3,4} = \pm\sqrt{z_2}$ and $x_{5,6} = \pm\sqrt{z_3}$ are the roots of p). We want to compute them by means of Cardano's formula which we recall for the reader's convenience:

$$az^3 + bz^2 + cz + d = 0$$

has the three solutions

$$z = -\frac{b}{3a} + S - \frac{Q}{S}$$

where $Q := (3ac - b^2)/(9a^2)$, $R := (9abc - 27a^2d - 2b^3)/(54a^3)$, $D := R^2 + Q^3$ and S is any of the three choices for $\sqrt[3]{R+\sqrt{D}}$.

We compute $Q = -217/6^2$, $R = -6371/6^3$ and $D = 624^2 78/6^6$. Then we let $\alpha = 6S = \sqrt[3]{-6371 + 624\sqrt{78}} = -\sqrt[3]{6371 - 624\sqrt{78}}$ be the real root and take $\rho = (-1 + i\sqrt{3})/2$ the standard primitive third root of unity to obtain three distinct non-zero roots

$$z_1 = \frac{1}{6}\left(19 + \alpha + \frac{217}{\alpha}\right),\ z_2 = \frac{1}{6}\left(19 + \rho\alpha + \frac{217}{\rho\alpha}\right),\ z_3 = \frac{1}{6}\left(19 + \bar{\rho}\alpha + \frac{217}{\bar{\rho}\alpha}\right).$$

Appendix B

Numerical computations using PARI

The following PARI script was used to compute $Z_{w_n\Gamma(D,1)}(k)$ for $k = 2, 3, 4$ and $n \mid D := p_1 p_2$ with $\{p_1, p_2\} \subset \{2, 3, 5, 7, 11, 13\}$.

```
do_one( p, q ) =
{
  P = matsize( p )[2];               \\ p = vector of primes dividing n
  n = prod( j = 1, P, p[j] );

  Q = matsize( q )[2];               \\ q = vector of primes
  D_H = n * prod( j = 1, Q, q[j] );  \\     dividing D_H but not n

  k = vector( K, X, X + 1.0 );       \\ vector of abscissae
  logZ = vector( K, X, 0.0 );        \\ vector for Selberg zeta values

  M = M_max = 0.0; t = t_max = 0; top = 0;

  while( (t+n)^2 - 4*n <= 0, t += n );\\ skip non-hyperbolic classes

  err = 1.0;                         \\ big enough to get loop started
  until( log(err) / log(10) < - min_correct_digits - 2,
    t += n;
    D_P = t^2 - 4*n;

    X = coredisc( D_P, 1 );          \\ flag 1 -> compute also conductor
    D_F = X[1];                      \\ D_F = fundamental discriminant
    m = X[2];                        \\ m = conductor

    L = 1;                           \\ L <- product of Legendre factors
    for( j = 1, Q,
      L *= 1 - kronecker( D_F, q[j] );
      if( L == 0, next(2) );         \\ loop to save time
      while( m % q[j] == 0, m /= q[j] ); \\ only need conductors prime to D_H
      );
```

```
    H = 0.0;                            \\ H <- sum of class numbers * regulators
    fordiv( m, f,                       \\ f is called m prime elsewhere
      Y = quadclassunit( f^2 * D_F,, [0.2,6] );
      H += Y.no * Y.reg;    \\ these ^^^^^^^ params -> correct under GRH
      );

    N = ( t + sqrt( D_P ) )^2 / ( 4*n ); \\ Selberg norm

    logZ += - 4 * L * H * N^(-k) / ( (1 - 1/N) * log (N) ); \\ vector assignmt!

    M = (H/log(N)) / (t/n);             \\ M crucial for error estimate

    if( M > M_max, M_max = M; t_max = t; top++ ); \\ check for new max of M

    err = 4 * 2^Q * M_max / (n^k[1] * (2*k[1]-2) * (t/n)^(2*k[1]-2) );
    ); \\ end of until loop

  print( "D_H = " D_H ", n = " n );
  print( "Summing traces <= T = " t ", we get" );
  for( j = 1, K,
    err = 4 * 2^Q * M_max / (n^k[j] * (2*k[j]-2) * (t/n)^(2*k[j]-2) );
    correct_digits = - truncate( log(err) / log(10) ) - 2;
    print1( "log Z(" truncate( k[j] ) ") = " logZ[j] ", " );
    print( correct_digits " corr. digits after dec. pt." );
    );
  print( "M = " M_max " at t = " t_max " (" top " times topped)\n" );
} \\ end of do_one routine

do_all( p_1, p_2 ) =
{
  do_one( [], [p_1,p_2] ); do_one( [p_1], [p_2] );
  do_one( [p_2], [p_1] ); do_one( [p_1,p_2], [] );
  \\ divisors of D_H = p_1*p_2 are n = 1, p_1, p_2, p_1*p_2
}

\\ MAIN ROUTINE
{
  min_correct_digits = 9;
  K = 3;                                \\ evaluate at 2, 3, 4
  do_all(2, 3); do_all(2, 5); do_all(2, 7); do_all(2,11); do_all(2,13);
  do_all(3, 5); do_all(3, 7); do_all(3,11); do_all(3,13); do_all(5, 7);
  do_all(5,11); do_all(5,13); do_all(7,11); do_all(7,13); do_all(11,13);
}
```

Remark. The routine quadclassunit is *a lot* faster than the combination of qfbclassno and quadunit that the ignorant author used first. But one has to careful to use the right technical parameters as the default settings may produce *wrong* results.

But maybe there is an even faster routine computing the product $h \log(\varepsilon)$

directly, possibly using fast evaluation of L-functions?

What is more, the above script makes the error smaller than 10^{-11} which means that in general only the eleventh digit after the decimal point may be off by one. Thus the ninth digit is only in danger if both the tenth and the eleventh digit are 9 but we have checked by hand that this is never the case.

The results have been collected in the following tables. They are self-explanatory except for the column T which contains the maximum trace up to which we have summed.

D	n	T	$k=2$	$k=3$	$k=4$
6	1	316228	−0.125591678	−0.01580274908183	−0.0021968380692160191
6	2	316228	−0.157322438	−0.02478459418572	−0.0041960271625143010
6	3	447216	−0.062665336	−0.00515865179171	−0.0004852610848540636
6	6	447216	−0.415681214	−0.10599448105826	−0.0282037470519063222
10	1	316228	−0.080629596	−0.00901487984413	−0.0011786821522725556
10	2	365150	−0.181681775	−0.02658537272650	−0.0043180682256177927
10	5	516400	−0.960519119	−0.36115595782341	−0.1377861085500183505
10	10	632460	−0.158867150	−0.01902170279127	−0.0023913044828101134
14	1	316228	−0.137384163	−0.01660674687864	−0.0022539021914759397
14	2	365150	−0.026809042	−0.00127017891551	−0.0000720929015866234
14	7	632457	−0.457333130	−0.09244349483095	−0.0192033316359781992
14	14	632464	−0.066772774	−0.00522834062482	−0.0004342062404373176
22	1	316228	−0.021080660	−0.00077025539949	−0.0000319095099675158
22	2	365150	−0.178154642	−0.02650807737643	−0.0043164679854720146
22	11	730301	−0.122974269	−0.01298901636078	−0.0014478363170754283
22	22	1000010	−0.056084243	−0.00268051380998	−0.0001332335771642118
26	1	316228	−0.125879797	−0.01543159387586	−0.0021564087519445203
26	2	365150	−0.180943801	−0.02655489933059	−0.0043171723894725197
26	13	774605	−0.116537671	−0.01025722656461	−0.0009344663045081664
26	26	894452	−0.017760240	−0.00063053624578	−0.0000255431526260756
15	1	330291	−0.087118725	−0.00923275825023	−0.0011855304158093427
15	3	447222	−0.011407861	−0.00020441901168	−0.0000040790401864853
15	5	577355	−0.992237114	−0.36316473007915	−0.1379040869674464517
15	15	707115	−0.058238404	−0.00410809435009	−0.0003124107475408027

D	n	T	$k=2$	$k=3$	$k=4$
21	1	346411	−0.130235211	−0.01635951480301	−0.0022447400978942404
21	3	447222	−0.114313669	−0.00990143674124	−0.0009522081879005435
21	7	707112	−0.236206157	−0.04647713986081	−0.0096111115453776933
21	21	894432	−0.029397204	−0.00130411460409	−0.0000662656115154705
33	1	346411	−0.022030960	−0.00068966529210	−0.0000246256098691480
33	3	447216	−0.095973896	−0.00926325369765	−0.0009286388676461442
33	11	774598	−0.125683790	−0.01307321357863	−0.0014501623031264408
33	33	1183248	−0.012696344	−0.00030594771322	−0.0000091924053404918
39	1	346413	−0.123483933	−0.01527454561957	−0.0021474649720382621
39	3	447219	−0.102511771	−0.00937841261919	−0.0009308395901712527
39	13	1095471	−0.012068410	−0.00020993050387	−0.0000040290229896053
39	39	1000038	−0.013287298	−0.00033088605576	−0.0000088314722455151
35	1	316228	−0.099837601	−0.01053403323049	−0.0012912724161243961
35	5	632460	−0.990820894	−0.36313810060870	−0.1379036927276806893
35	7	632464	−0.028344353	−0.00099284840539	−0.0000372500625910883
35	35	1083215	−0.009276100	−0.00023823094941	−0.0000070320122936089
55	1	330291	−0.013611373	−0.00033678553706	−0.0000094815254444691
55	5	632460	−0.012982837	−0.00024665975434	−0.0000052705057681940
55	11	894432	−0.122659689	−0.01297965575904	−0.0014477029895060156
55	55	1449140	−0.007468709	−0.00011626202587	−0.0000020988573976575
65	1	320846	−0.066379255	−0.00767724479387	−0.0010719291971478364
65	5	596285	−0.989234711	−0.36312920377268	−0.1379036436163598385
65	13	859352	−0.112554478	−0.01018342050202	−0.0009330756119956216
65	65	1460615	−0.005398929	−0.00006922347133	−0.0000010482355239827
77	1	346411	−0.012746620	−0.00038736248759	−0.0000154596596382693
77	7	730317	−0.446815628	−0.09198713012512	−0.0191852065595551182
77	11	894432	−0.067008072	−0.00664355587004	−0.0007276968225404603
77	77	1861937	−0.005120157	−0.00005963649281	−0.0000007751377959098
91	1	346413	−0.116013814	−0.01502020975190	−0.0021394690297008962
91	7	730303	−0.444851527	−0.09195266065587	−0.0191846252269337173
91	13	852813	−0.113556841	−0.01019319612530	−0.0009331651167337526
91	91	1600053	−0.002970482	−0.00002512517533	−0.0000002637636105598
143	1	346413	−0.030373961	−0.00103833534235	−0.0000397018593047755
143	11	846725	−0.014516256	−0.00026634543614	−0.0000056975250762962
143	13	1095484	−0.132143742	−0.01058122625262	−0.0009409952820360813
143	143	2277275	−0.001588078	−0.00000912655615	−0.0000000619742049972

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Düsseldorf, den 11. Dezember 2006

Tobias Ebel