Coprime Modules and Comodules

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by

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Introduction

From the very beginning the study of coalgebras was motivated by the existing theory of algebras and rings. Many notions and results could be formulated easily by transferring the corresponding knowledge from algebras to coalgebras and from modules to comodules. However, despite of its importance in ring theory, the notion of primeness for rings and modules did not find an adequate counterpart in the coalgebraic setting. This may be due to the finiteness theorem for comodules which says that for a coalgebra C over a field (ring) R, any finitely generated C-comodule is also a finitely generated R-module. So to some extent the behaviour of coalgebras is similar to that of finite dimensional algebras and for those primeness means just simplicity.

In this context one of the questions one may ask is when the dual algebra $C^* = \mathrm{Hom}_R(C, R)$ (with the convolution product) of an R-coalgebra C is a prime algebra. Perhaps the first paper to consider this was by Xu, Lu, and Zhu [41] who observed that this is the case if C is a coalgebra over a field k and $(C^* * f) \rightharpoonup C = C$ for any non-zero element $f \in C^*$. Another approach in this direction can be found in Jara, Merino, Ruiz [17] and Nekooei-Torkzadeh [26] where *coprime* coalgebras (over fields) are defined by using the wedge product and it is shown that these are characterized by the primeness of C^*.

Dualizing primeness condition, *coprimeness* can also be defined for modules and algebras. The purpose of this thesis is to investigate the resulting notions for modules and then transfer them to comodules and coalgebras over commutative rings. Notice that for any algebra A, coprimeness implies that A is simple but for a coalgebra C the condition to be coprime is not so restrictive.

For prime rings localization techniques were developed to construct *rings of quotients*. These were also extended to modules. Thus the question arises if there is anything like *colocalization* to construct a *coalgebra of quotients*. Such theories have been considered in module categories, we apply them to comodules and coalgebras.

A large part of our research is based on the fact that for any coalgebra C over a commutative ring R, comodule categories can be considered as module categories over the ring C^* provided an appropriate (weak) condition is imposed on the R-module structure of C (locally projective, α-condition). In this case the category of right C-comodules is equivalent to the category $\sigma[_{C^*}C]$ of those left C^*-modules which are subgenerated by C.

Thus in the first part of the thesis we reconsider categories of type $\sigma[M]$ where M is any module over a ring R. Already known notions like prime, endo-prime, coprime, fully and strongly prime modules are recalled and we prove new properties of interest for the application to comodules. Strongly and fully coprime modules are defined and studied. If M is prime, then every projective module P in $\sigma[M]$ is prime (1.2.7). If $\mathrm{Soc}(M) \neq 0$ and M is prime, then $\overline{R} := R/\mathrm{Ann}_R(M)$ is a left primitive ring (1.2.12). A result in the dual situation is : If M is coprime with $\mathrm{Rad}(M) \neq M$, then $\overline{R} := R/\mathrm{Ann}_R(M)$ is a left primitive ring (1.3.10). Moreover, if $p : P \to M$ is a small epimorphism in $\sigma[M]$ and M is a coprime and faithful R-module, then P is coprime (1.3.11). If M is fully prime with $\mathrm{Soc}(M) \neq 0$, then M is a homogeneous semisimple module and $\overline{R} := R/\mathrm{Ann}_R(M)$ is a primitive ring (1.6.9). The dual situation of this is : If M is fully coprime with $\mathrm{Rad}(M) \neq M$, then M is generated by a module that is cogenerated by a simple module and $\overline{R} := R/\mathrm{Ann}_R(M)$ is a left primitive ring (1.7.15). If P is a projective hull of M and M is strongly coprime, then $M \simeq P$ (1.8.11). It turns out that for a self-injective self-cogenerator module M, the notions endo-coprime, fully and strongly coprime coincide (1.7.12 and 1.8.12).

These results are then applied to comodules. We observe that primeness conditions on comodules with non-zero socle and coprimeness conditions on comodules with proper radicals lead to trivial situations. For coalgebras over fields this was also seen by Rodrigues [29] and many of her results follow as corollaries from our propositions.

For comodules, coprimeness conditions are more interesting and we provide a series of theorems on these cases. Eventually we consider these conditions on the coalgebra C itself. For example, if C is prime as a right C-comodule, then C is finitely generated as an R-module and C^* is a prime algebra (2.3.6). Moreover, if C is prime as a right C-comodule with $\mathrm{Soc}(C) \neq 0$, then C^* is a simple algebra and finitely generated R-module (2.3.8). If C is coprime as a right C-comodule and $\mathrm{Rad}(C) \neq C$, then C^* is a simple algebra and finitely generated R-module (2.4.4). If C is fully prime as a right C-comodule and $\mathrm{Soc}(C) \neq 0$, then C is semisimple (2.7.6). Let $p : P \to C$ be a small epimorphism in $\mathbf{M}^C$. If P is self-projective as C^*-module, then P is strongly coprime. Moreover, if P is projective in $\mathbf{M}^C$, then $C \simeq P$ (2.9.8).

Over a field, C is a self-injective self-cogenerator as C^*-module and hence the various coprimeness conditions are the same and also equivalent to the primeness of C^* (as mentioned above). Moreover, C is fully coprime as a right C-comodule and coprime as a left (right) C-comodule if and only if $C = (C^* * f) \rightharpoonup C$ for any non-zero element $f \in C^*$ (2.11.7), a characterization found by Xu et.al. [41].

Studies of localization and colocalization of coalgebras over a field have been done, for example, by Năstăsescu and Torrecillas in [24], [25], Gómez-Torrecillas, Năstăsescu and Torrecillas in [12] and Jara, Merino, Navarro and Ruiz in [18]. They are mainly interested in colocalization with respect to coidempotent subcoalgebras of C. Their methods heavily depend on the base ring being a field. To avoid this restriction we first give an outline of colocalization in module categories and then apply it to comodules and coalgebras.

Notice that in abelian categories the existence of a colocalization functor depends on the presence of enough projectives in the category. We transfer the technique of colocalization in the category of R-modules to the comodule situation. In particular we consider the question when the comodule arising from colocalization of C allows for a coalgebra structure.

We show that for cohereditary torsion theories induced by some C-comodule P which is finitely generated and projective as C^*-module, a coalgebra structure can be defined on $P \otimes_S P^*$ and the colocalization $P \otimes_S P^* \to C$ is a coalgebra morphism (2.13.6).

In module categories of type $\sigma[M]$ the torsion theory induced by the injective hull of M is of particular interest and primeness of the module leads to a special structure of the module of quotients. A special case of this theory is the localization of the prime $\mathbb{Z}$-module $\mathbb{Z}$ yielding the quotient module (ring) $\mathbb{Q}$.

Thus the question arises about the role of a projective hull of a subgenerator in the dual case. However, no comparable constructions are possible in this situation. In fact, the existence of a projective hull of a strongly coprime coalgebra implies that $\mathrm{Rad}(P) = 0$ and $C \simeq P$ (2.12.16).

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Chapter 1

Prime and Coprime Modules

In the category of left R-modules there exist various notions of prime objects which generalize the well known notion of a prime associative ring R. In this section some of these notions and their duals are considered : prime and coprime, endo-prime and endo-coprime, fully prime and fully coprime, strongly prime and strongly coprime modules. These notions will be applied to define primeness and coprimeness for comodules and coalgebras.

1.1 Preliminaries

In commutative ring theory, the notion of a prime ring plays an important role, mainly due to the fact that it can be embedded into a field. Generalizing the notions to modules has been done in various ways, for example by Johnson ([19], [20]) and Dauns [8]. Wisbauer in [37] studied the primeness conditions on modules, and moreover in [39] outlined the localization theory in $\sigma[M]$ induced by the singular modules.

Some notions we will use later are given in this section. Unless explicitely stated, throughout this chapter R is an associative ring with unit, M usually will be a left R-module. The category of left R-modules is denoted as ${}_R\mathbf{M}$. The morphisms are written on the right side of the module and if it is needed, we use the $\diamond$ for the composition of mappings written on the right side. The usual composition is denoted by $\circ$ and thus $(u)f \diamond g$ is equal to $g \circ f(u)$ when writing the maps on the left side.

An ideal T in a ring R is said to be *prime* if $T \neq R$ and, for ideals $I, J \subseteq R$, if $IJ \subseteq T$ then $I \subseteq T$ or $J \subseteq T$. The ring R itself is called *prime* if 0 is a prime ideal of R. For this we have the following well-known characterization :

1.1.1 Lemma. *For a ring R the following assertions are equivalent :*

(a) *R is a prime ring.*

(b) *For any left ideals I, J of R, $IJ = 0$ implies $I = 0$ or $J = 0$.*

(c) *For any finitely generated left ideals I, J of R, $IJ = 0$ implies $I = 0$ or $J = 0$.*

(d) *For any $a, b \in R$, $aRb = 0$ implies $a = 0$ or $b = 0$.*

We recall some familiar properties of a prime ring.

1.1.2 Lemma. *Let R be a ring.*

(i) *If R is prime, then for any idempotent element e, eRe is also a prime ring.*

(ii) *If R is commutative and A is a prime R-algebra, then $\overline{R} := R/\text{Ann}_R(A)$ is prime.*

(iii) *If R is commutative and A is a simple algebra that is finitely generated, projective and faithful as R-module, then R is a field.*

Proof. (iii) Let I be a proper ideal in R. Then by 18.9 of [38], $IA \neq A$. However, $IA = 0$, since A is a simple algebra. Thus $I = 0$. □

1.1.3 Generators and cogenerators. Let $\mathcal{U}$ be a set of R-modules. An R-module N is called *$\mathcal{U}$-generated* if there exists an epimorphism $\bigoplus_\Lambda U_\lambda \to N$ where $U_\lambda \in \mathcal{U}$.

For an R-module N, the submodule

$$\text{Tr}(\mathcal{U}, N) = \sum\{\text{Im } h \mid h \in \text{Hom}_R(U, N), U \in \mathcal{U}\} \subseteq N$$

is called the *trace* of $\mathcal{U}$ in N . Thus N is $\mathcal{U}$-generated if and only if $\text{Tr}(\mathcal{U}, N) = N$. An R-module M is called a *self-generator* if M generates all its submodules.

N is called *$\mathcal{U}$-cogenerated* if there exists a monomorphism $N \to \prod_\Lambda U_\lambda$ where $U_\lambda \in \mathcal{U}$. For an R-module N, the submodule

$$\text{Rej}(N, \mathcal{U}) = \bigcap\{\text{Ker } f \mid f \in \text{Hom}_R(N, U), U \in \mathcal{U}\} \subseteq N$$

is called the *reject* of $\mathcal{U}$ in N. Thus N is $\mathcal{U}$-cogenerated if and only if $\text{Rej}(N, \mathcal{U}) = 0$. An R-module is called a *self-cogenerator* if it cogenerates all its factor modules.

1.1.4 The category $\sigma[M]$. For two R-modules M, N, we say N is *subgenerated by* M if N is isomorphic to a submodule of an M-generated module. The full subcategory of ${}_R\mathbf{M}$ whose objects are the modules subgenerated by M is denoted by $\sigma[M]$. For a family $\{N_\lambda\}_\Lambda$ of modules in $\sigma[M]$, the product in $\sigma[M]$ exists and is given by $\prod_\Lambda^M N_\lambda := \mathrm{Tr}(\sigma[M], \prod_\Lambda N_\lambda)$. For more details on these notions see [38].

1.1.5 Annihilators. For a non-empty subset $K \subset M$ denote by

$$\mathrm{Ann}_R(K) := \{r \in R \mid rk = 0, \text{ for all } k \in K\}$$

the *(left) annihilator* of K. Notice that for any R-modules M and N, if $N \in \sigma[M]$, then $\mathrm{Ann}_R(M) \subseteq \mathrm{Ann}_R(N)$.

We observe some properties of annihilators that will be used to analyse the primeness and coprimeness of modules later on.

1.1.6 Lemma. *For any submodule K of M,*

$$\mathrm{Ann}_R(K)\mathrm{Ann}_R(M/K) \subseteq \mathrm{Ann}_R(M).$$

Proof. Take any $x \in \mathrm{Ann}_R(K)$, $y \in \mathrm{Ann}_R(M/K)$. We have $yM \subset K$ and $xyM \subset xK = 0$. □

1.1.7 Proposition. *Let R, S be rings and M be an (R,S)-bimodule. Then the following assertions are equivalent :*

(a) $\overline{R} = R/\mathrm{Ann}_R(M)$ *is a prime ring.*

(b) *For any submodule K of M,* $\mathrm{Ann}_R(K) = \mathrm{Ann}_R(M)$ *or* $\mathrm{Ann}_R(M/K) = \mathrm{Ann}_R(M)$.

(c) *For any (R,S)-subbimodule K of M, we have* $\mathrm{Ann}_R(K) = \mathrm{Ann}_R(M)$ *or* $\mathrm{Ann}_R(M/K) = \mathrm{Ann}_R(M)$.

Proof. (a) $\Longrightarrow$ (b) is obvious by Lemma 1.1.6.

(b) $\Longrightarrow$ (c) is trivial.

(c) $\Longrightarrow$ (a) Take any two ideals I, J of R with $IJ \subset \mathrm{Ann}_R(M)$. Then $IJM = 0$. Since JM is an (R,S)-bimodule, by assumption $\mathrm{Ann}_R(JM) = \mathrm{Ann}_R(M)$ or $\mathrm{Ann}_R(M/JM) = \mathrm{Ann}_R(M)$.

If $JM = M$, then $\mathrm{Ann}_R(M/JM) \neq \mathrm{Ann}_R(M)$, hence $I \subset \mathrm{Ann}_R(JM) = \mathrm{Ann}_R(M)$.

If $JM \neq M$ and $I \not\subset \mathrm{Ann}_R(M)$, then $I \subset \mathrm{Ann}_R(JM) \not\subset \mathrm{Ann}_R(M)$. Thus $J \subset \mathrm{Ann}_R(M/JM) = \mathrm{Ann}_R(M)$. □

1.1.8 Corollary. *Let R, S be rings and M be an (R,S)-bimodule. If ${}_RM$ is faithful, then the following assertions are equivalent :*

(a) *R is a prime ring.*

(b) *For any submodule K of M, $\mathrm{Ann}_R(K) = 0$ or $\mathrm{Ann}_R(M/K) = 0$.*

(c) *For any (R,S)-subbimodule K of M, $\mathrm{Ann}_R(K) = 0$ or $\mathrm{Ann}_R(M/K) = 0$.*

Let $S := \mathrm{End}_R(M)$, $K \subset M$ a submodule and $I \subset S$ a left ideal. Denoting by $\pi_K : M \to M/K$ the canonical projection, we put

$$\begin{aligned} \mathrm{Ann}_S(K) &:= \{f \in S \mid (K)f = 0\} = \pi_K \diamond \mathrm{Hom}_R(M/K, M), \\ \mathrm{Ker}\ I &:= \bigcap \{\mathrm{Ker}\ f \mid f \in I\}. \end{aligned}$$

It always holds : $K \subseteq \mathrm{Ker}\ \mathrm{Ann}_S(K)$ and $I \subseteq \mathrm{Ann}_S(\mathrm{Ker}\ I)$. For equality, injectivity or cogenerator properties of a module M are needed. Recall that M is called a *self-injective* module if it is M-injective and M is called a *self-projective* module if it is M-projective.

From 28.1 part (2) and (4) of [38] we recall :

1.1.9 Lemma. *Let M be a module and $S = \mathrm{End}_R(M)$.*

(i) *For any submodule $K \subseteq M$,*

$$\mathrm{Ker}\ \mathrm{Ann}_S(K) = \mathrm{Ker}\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) = K$$

if and only if M is a self-cogenerator module.

(ii) *If M is self-injective, then for every finitely generated right ideal $I \subseteq S$,*

$$\mathrm{Hom}_R(M/\mathrm{Ker}\ I, M) = I.$$

The following definition is helpful for our investigations.

1.1.10 Dual orthogonal. Let M be an R-module, $M^* = \mathrm{Hom}_R(M, R)$. For any submodule $K \subset M$ and any subset $I \subset M^*$, put

$$\begin{aligned} K^{\perp M^*} &:= \{f \in M^* \mid (K)f = 0\} = \pi_K \diamond \mathrm{Hom}_R(M/K, R) \subset M^*, \\ I^{\perp M} &:= \bigcap \{\mathrm{Ker}\ f \mid f \in I\} \subset M, \end{aligned}$$

where $\pi_K : M \to M/K$ is the canonical projection.

Let K be a submodule of M. If for any $f \in \mathrm{End}_R(M)$, $(K)f \subseteq K$, K is called a *fully invariant* submodule of M.

1.1.11 $(*)$ and $(**)$-conditions. Consider the following conditions for an R-module M:

- $(*)$ For any non-zero submodule K of M, $\mathrm{Ann}_R(M/K) \not\subset \mathrm{Ann}_R(M)$.
- $(*fi)$ For any non-zero fully invariant submodule K of M, $\mathrm{Ann}_R(M/K) \not\subset \mathrm{Ann}_R(M)$.
- $(**)$ For any proper (fully invariant) submodule K of M, $\mathrm{Ann}_R(K) \not\subset \mathrm{Ann}_R(M)$.

The conditions $(*)$ and $(*fi)$ are not necessarily equivalent, since there are R-modules M which satisfy $(*fi)$, but do not satisfy $(*)$:

Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. Notice that $\mathrm{End}_{\mathbb{Z}}(\mathbb{Q}) \simeq \mathbb{Q}$ and hence $\mathbb{Q}$ has no non-zero fully invariant submodules. Thus $(*fi)$ holds trivially. However, for $\mathbb{Z} \subset \mathbb{Q}$, $0 = \mathrm{Ann}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) = \mathrm{Ann}_{\mathbb{Z}}(\mathbb{Q})$, i.e., $(*)$ does not hold.

1.1.12 Retractable and coretractable. M is called *retractable* if for any non-zero submodule K of M and $S = \mathrm{End}_R(M)$,

$$\mathrm{Hom}_R(M, K) = \{f \in S \mid (M)f \subseteq K\} \neq 0.$$

A module M is called *fi-retractable* if for any non-zero fully invariant submodule K of M,

$$\mathrm{Hom}_R(M, K) = \{f \in S \mid (M)f \subseteq K\} = \mathrm{Ann}_S(M/K) \neq 0.$$

Dually, M is called *coretractable* if for any proper submodule K of M,

$$\pi_K \diamond \mathrm{Hom}_R(M/K, M) = \{f \in S \mid (K)f = 0\} \neq 0.$$

M is called *fi-coretractable* if for any proper fully invariant submodule K of M,

$$\pi_K \diamond \mathrm{Hom}_R(M/K, M) = \{f \in S \mid (K)f = 0\} = \mathrm{Ann}_S(K) \neq 0.$$

Let $S = \mathrm{End}_R(M)$ and consider the module M as a right S-module. If M_S satisfies $(*)$ as an S-module, then it is fi-retractable as an R-module. If M_S satisfies $(**)$ as an S-module, then it is fi-coretractable as an R-module.

Any self-generator module is retractable, and any self-cogenerator module is coretractable.

1.1.13 Proposition. *Let M be a coretractable R-module and $S = \mathrm{End}_R(M)$. The following are equivalent :*

(a) *S has no zero-divisor.*

(b) *For any proper submodule K of M,* $\mathrm{Hom}_R(M, K) = 0$.

(c) *For any $0 \neq f \in S$, $(M)f = M$ (any non-zero endomorphism is an epimorphism).*

Proof. (a) $\Longrightarrow$ (b) For any proper submodule K of M, it holds

$$\mathrm{Hom}_R(M, K) \diamond \pi_K \diamond \mathrm{Hom}_R(M/K, M) = 0,$$

where $\pi_K : M \to M/K$ is the canonical projection. By coretractibility we have $\mathrm{Hom}_R(M/K, M) \neq 0$. By (a), $\mathrm{Hom}_R(M, K)$ has to be zero.

(b) $\Longrightarrow$ (a) Take any $f, g \in S$ such that $fg = 0$. Assume $g \neq 0$. Then $\mathrm{Im}\ f \subset \mathrm{Ker}\ g \neq M$ and $f \in \mathrm{Hom}_R(M, \mathrm{Ker}\ g) = 0$. Thus $f = 0$.

(b) $\Longleftrightarrow$ (c) Assume there exists $0 \neq f \in S$ such that $(M)f = K \neq M$. But $f \in \mathrm{Hom}_R(M, K) = 0$, a contradiction. □

1.1.14 Proposition. *Let M be a fi-coretractable R-module and denote $S = \mathrm{End}_R(M)$. The following are equivalent :*

(a) *S is a prime ring.*

(b) *For any proper fully invariant submodule K of M,* $\mathrm{Hom}_R(M, K) = 0$.

(c) *For any $0 \neq f \in S$, $(M)fS = M$.*

(d) *For any ideal $0 \neq I \subset S$, $MI = M$.*

Proof. (a) $\Longrightarrow$ (b) It is similar to the proof of Proposition 1.1.13.

(b) $\Longrightarrow$ (c) Assume $U = (M)fS \neq M$ for some $f \in S$. Clearly, U is a fully invariant submodule of M, $Mf \subset U$ and $(M/U)f = 0$. Clearly $fS \subset \mathrm{Hom}_R(M, U)$ and hence by (b), $\mathrm{Hom}_R(M, U) = 0$, thus $f = 0$.

(c) $\Longrightarrow$ (d) Assume $MI \neq M$ for some non-zero ideal $I \subset S$. Then MI is a fully invariant submodule of M, since $MIS \subseteq MI$. By (c), $MI = M$.

(d) $\Longrightarrow$ (a) Take any ideals I, J of S such that $IJ = 0$. If $I \neq 0$, then by (d), $MI = M$ and $MIJ = MJ = 0$. Thus $J = 0$. □

1.1.15 Proposition. *Let M be a retractable R-module and $S = \mathrm{End}_R(M)$. The following are equivalent :*

(a) *S has no zero-divisor.*

(b) *For any non-zero submodule K of M, $\mathrm{Hom}_R(M/K, M) = 0$.*

(c) *For any $0 \neq f \in S$, $\mathrm{Ker}\ f = 0$ (any non-zero endomorphism is a monomorphism).*

Proof. Dual to the proof of Proposition 1.1.13. □

1.1.16 Proposition. *Let M be a fi-retractable R-module and $S = \mathrm{End}_R(M)$. The following are equivalent :*

(a) *S is a prime ring.*

(b) *For any non-zero fully invariant submodule K of M, $\mathrm{Hom}_R(M/K, M) = 0$.*

(c) *For any $0 \neq f \in S$, $\mathrm{Ker}\ Sf = 0$.*

(d) *For any ideal $0 \neq I \subset S$, $\mathrm{Ker}\ I = 0$.*

Proof. For any left ideal $0 \neq I \subset S$, Ker I is a fully invariant submodule of M. Hence the dual of the proof of Proposition 1.1.14 applies here. □

As a consequence we obtain :

1.1.17 Corollary. *Let M be a retractable and coretractable R-module, $S = \mathrm{End}_R(M)$. The following are equivalent :*

(a) *S has no zero-divisor.*

(b) *M is a simple module.*

(c) *S is a division ring.*

Proof. (a) $\Longrightarrow$ (b) By Proposition 1.1.13, for any proper submodule K of M, $\mathrm{Hom}_R(M, K) = 0$. But M is retractable and hence there is no non-zero proper submodule K of M.

(b) $\Longrightarrow$ (c) By Schur's Lemma, M is simple implies S is a division ring.

(c) $\Longrightarrow$ (a) It is obvious. □

A small submodule K of M is denoted by $K \ll M$. We observe some relation between projectivity and retractibility.

1.1.18 Pseudo-projective modules. An R-module P is called *pseudo-projective in* $\sigma[M]$ if any diagram in $\sigma[M]$ with exact bottom line

$$\begin{array}{ccccc} & & P & & \\ & & \big\downarrow{\scriptstyle f} & & \\ N & \xrightarrow{p} & L & \longrightarrow & 0 \end{array}$$

can be extended non trivially by some $s \in \mathrm{End}_R(P)$ and $g : P \to N$ to the commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{s} & P & & \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} & & \\ N & \xrightarrow{p} & L & \longrightarrow & 0, \end{array}$$

that is, $gp = sf \neq 0$.

An epimorphism $p : P \to M$ is a *pseudo-projective hull* of M in $\sigma[M]$ if P is pseudo-projective in $\sigma[M]$ and $\mathrm{Ker}\ p \ll M$.

1.1.19 Lemma. *Let P be a module in $\sigma[M]$.*

(i) *If P is pseudo-projective in $\sigma[M]$, then for any non-small submodule $U \subset P$,* $\mathrm{Hom}_R(P, U) \neq 0$.

(ii) *If P is pseudo-projective in $\sigma[M]$ with* $\mathrm{Rad}(P) = 0$, *then P is retractable.*

Proof. (i) Consider any proper non-small submodule $U \subset P$ and let $V \subset P$ be a proper submodule such that $U + V = P$. Then there is an epimorphism $f : U \to P/V$ by the composition $U \to (U + V)/V \to P/V$. There is a commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{t} & P & & \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle h} & & \\ U & \xrightarrow{f} & P/V & \longrightarrow & 0, \end{array}$$

where $h : P \to P/V$ is the canonical projection and $gf = th \neq 0$. Hence $0 \neq g : P \to U$ and $\mathrm{Hom}_R(P, U) \neq 0$.

(ii) is clear from (i). □

Notice that Lemma 1.1.19 is also true for self-projective modules with a similar proof.

The following observation from ([27], Lemma 17) will be useful for our investigations.

1.1.20 Lemma. *Let M, N be R-modules and $f \in \mathrm{Hom}_R(M, N)$ an epimorphism.*

(i) *If* $\mathrm{Ker}\, f$ *is fully-invariant and L is a fully-invariant submodule of N, then $(L)f^{-1}$ is a fully-invariant submodule of M.*

(ii) *If M is self-projective and U is a fully-invariant submodule of M, then $(U)f$ is a fully-invariant submodule of N.*

Proof. (i) Let L be a fully-invariant submodule of N, $K := \mathrm{Ker}\, f$ be fully-invariant and $g \in \mathrm{End}_R(M)$. Since f is an epimorphism and $Kg \subseteq K$, there is $h \in \mathrm{End}_R(N)$ such that $gf = fh$ and $Lh \subseteq L$. Therefore $(L)f^{-1}fh \subseteq L$ and hence $(L)f^{-1}gf \subseteq L$. Thus $(L)f^{-1}g \subseteq (L)f^{-1}$.

(ii) Let U be a fully-invariant submodule of M and $h \in \mathrm{End}_R(N)$. Then there is $g \in \mathrm{End}_R(M)$ such that $gf = fh$ and hence $(U)fh = (U)gf \subseteq (U)f$. $\square$

1.1.21 Corollary. *Let M be an R-module, $K \subset U \subset M$ submodules of M, $K \subset M$ fully invariant. If U/K is a proper fully-invariant submodule of the factor module M/K, then U is a proper fully invariant submodule of M.*

1.2 Prime modules

We begin our investigations with an overview of primeness and coprimeness conditions of modules which will be discussed in the next sections. The following list shows the definitions we use here. Let M be an R-module and $S = \text{End}_R(M)$.

If for any fully invariant submodule K of M,	then we call M
$\text{Ann}_R(K) = \text{Ann}_R(M)$, $K \neq 0$	prime
$\text{Ann}_R(M/K) = \text{Ann}_R(M)$, $K \neq M$	coprime
$\text{Ann}_S(K) = 0$, $K \neq 0$	endo-prime
$\text{Ann}_S(M/K) = 0$, $K \neq M$	endo-coprime
M is K-cogenerated, $K \neq 0$	fully prime
M is M/K-generated, $K \neq M$	fully coprime
$M \in \sigma[K]$, $K \neq 0$	strongly prime
$M \in \sigma[M/K]$, $K \neq M$	strongly coprime
$M \in \sigma[K]$ or $M \in \sigma[M/K]$	duprime

Let us point out some elementary relations between the prime and coprime notions : If M is fully prime or strongly prime, then it is prime. If M is fully coprime or strongly coprime, then it is coprime. If M is strongly prime or strongly coprime, then it is duprime.

We will consider all the cases mentioned above and repeat the definitions explicitely.

1.2.1 Definition. M is called *prime* if for every non-zero fully-invariant submodule K of M, $\text{Ann}_R(K) = \text{Ann}_R(M)$.

The following properties characterize the primeness of a module M.

1.2.2 Prime modules. *For a module M the following are equivalent :*

(a) *M is a prime module.*

(b) $\text{Ann}_R(K) = \text{Ann}_R(M)$ *for any non-zero submodule K of M.*

(c) $R/\text{Ann}_R(M)$ *is cogenerated by K for any non-zero submodule K of M.*

(d) $R/\text{Ann}_R(M)$ *is cogenerated by K for any non-zero fully-invariant submodule K of M.*

Proof. (a) $\iff$ (b) We have a trivial implication from (b) to (a). Conversely, let K be a submodule of M. Then KS is a fully-invariant submodule of M and $\text{Ann}_R(K) = \text{Ann}_R(KS) = \text{Ann}_R(M)$.

(b) $\iff$ (c) For any R-module M, $R/\text{Ann}_R(M)$ is M-cogenerated. By assumption in (b) we have $R/\text{Ann}_R(M)$ is cogenerated by K for every $0 \neq K \subseteq M$. Conversely, by assumption in (c) we know that K is a faithful $R/\text{Ann}_R(M)$-module, thus $\text{Ann}_R(K) = \text{Ann}_R(M)$ for any submodule K of M.

(a) $\iff$ (d) is similar to the previous proof. □

The following proposition is a slight modification of 13.1 of [39].

1.2.3 Proposition. *Let M be an R-module, $S = \text{End}_R(M)$ and we denote $\overline{R} := R/\text{Ann}_R(M)$.*

(i) *If M is prime, then $\overline{R}$ is a prime ring.*

(ii) *If $\overline{R}$ is a prime ring and M satisfies $(*fi)$, then M is prime.*

(iii) *If ${}_RM$ is prime and satisfies $(*fi)$, then M_S is prime (and S is a prime ring).*

Proof. (i) Assume M to be prime, i.e., $\text{Ann}_R(K) = \text{Ann}_R(M)$ for every non-zero fully-invariant submodule $K \subset M$. Then by Proposition 1.1.7, $\overline{R}$ is prime.

(ii) Assume $\overline{R}$ to be prime. By assumption and Proposition 1.1.7, for any non-zero fully invariant submodule K of M holds $\text{Ann}_{\overline{R}}(K) = 0$. It is equivalent to $\text{Ann}_R(K) = \text{Ann}_R(M)$. Thus M is prime.

(iii) Let ${}_RM$ be prime and K be an S-submodule of M. For any $f \in \text{Ann}_S(K) = \text{Ann}_S(RK)$, $(\text{Ann}_{\overline{R}}(M/RK))Mf \subset (RK)f = 0$. Since ${}_RM$ is prime and $\text{Ann}_{\overline{R}}(M/RK) \neq 0$, we have $Mf = 0$ and f has to be zero. □

A faithful module over a prime ring need not be prime :

1.2.4 Example. Consider the $\mathbb{Z}$-module $T := \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$. It is a faithful $\mathbb{Z}$-module. For the proper submodule $\mathbb{Z}/3\mathbb{Z}$, $\text{Ann}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}) \neq 0$. Thus T is not prime.

The product of faithful prime modules is again prime.

1.2.5 Lemma. *Let $\{M_\lambda\}_\Lambda$ be a family of faithful modules, Λ an index set. Then $\prod_\Lambda M_\lambda$ is prime if and only if each M_λ is prime.*

Proof. ($\Longrightarrow$) Let $\prod_\Lambda M_\lambda$ be prime. Consider any submodule $U \subset M_\mu \subset \prod_\Lambda M_\lambda$. Then $\mathrm{Ann}_R(U) = \mathrm{Ann}_R(\prod_\Lambda M_\lambda) = 0$.

($\Longleftarrow$) Let each M_λ be prime and $V \subset \prod_\Lambda M_\lambda$. There is a non-zero canonical projection $(V)\pi_{\lambda_0} \subset M_{\lambda_0}$, for some $\lambda_0 \in \Lambda$. Since $(V)\pi_{\lambda_0}$ is faithful, V is also faithful. □

1.2.6 Corollary. *If the ring R is prime, then submodules of an R-cogenerated module are prime. In particular, every projective module is prime.*

Proof. Every R-cogenerated module is a submodule of some R^Λ. According to Lemma 1.2.5, if R is prime, then R^Λ is prime.

Every projective module P is a direct summand of some $R^{(\Lambda)} \subset R^\Lambda$. If R is prime, then P is prime. □

If M is prime, then it implies the primeness of projective modules in $\sigma[M]$.

1.2.7 Proposition. *Let M be prime. Then*

(i) *every M-cogenerated module is prime;*

(ii) *every projective module P in $\sigma[M]$ is prime.*

Proof. (i) According to Lemma 1.2.5, M^Λ is prime and hence any M-cogenerated module is prime.

(ii) Any projective module P in $\sigma[M]$ is isomorphic to a submodule of some $M^{(\Lambda)}$, Λ index set, hence P is prime. □

1.2.8 Proposition. *Let M be a projective module in $\sigma[M]$. If every non-zero submodule of M cogenerates M, then $\mathrm{End}_R(M)$ is a prime ring.*

Proof. Let $J \subset S := \mathrm{End}_R(M)$ be a finitely generated proper left ideal. By assumption, M is MJ-cogenerated, i.e., there is a short exact sequence $0 \to M \to (MJ)^\Lambda$. Applying $\mathrm{Hom}_R(P, -)$ to this exact sequence yields the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{Hom}_R(P,P) & \longrightarrow & \mathrm{Hom}_R(P,(PJ)^\Lambda) \\ & & \downarrow = & & \downarrow \simeq \\ 0 & \longrightarrow & S & \longrightarrow & J^\Lambda, \end{array}$$

since $\mathrm{Hom}_R(P,(PJ)^\Lambda) = \mathrm{Hom}_R(P,(PJ))^\Lambda$ and, by projectivity, $\mathrm{Hom}_R(P, PJ) = J$ (see 18.4 part 3i [38]). Thus J is a faithful left S-module, i.e., $\mathrm{End}_R(P)$ is a (left) prime ring. □

As a consequence, we obtain Proposition 1.3 of [15] :

1.2.9 Corollary. *Let P be a projective R-module. If the ring R is prime, then* $\mathrm{End}_R(P)$ *is prime.*

Proof. Since R is prime, R^Λ is also prime for some index set Λ and hence $P \subset R^\Lambda$ is prime and faithful. Let K be a non-zero submodule of P. Then K cogenerates R (by 1.2.2 part (c)). But R cogenerates P, thus K cogenerates P.
Now we apply Proposition 1.2.8 to prove $\mathrm{End}_R(P)$ is prime. □

An essential submodule N of M is denoted by $N \trianglelefteq M$. Primeness of the module M extends to essential extensions.

1.2.10 Lemma. *Let M be a faithful R-module. The following assertions are equivalent:*

(a) *M is a prime module.*

(b) *Every essential extension of M is a prime module.*

(c) *The injective hull $\widehat{M}$ of M in $\sigma[M]$ is a prime module.*

(d) *The injective hull $E(M)$ of M in ${}_R\mathbf{M}$ is a prime module.*

Proof. (a) $\Longrightarrow$ (b) Let M' be an essential extension of M. Assume that M is prime, i.e., $\mathrm{Ann}_R(K) = 0$ for any $0 \neq K \subset M$, and let $0 \neq L \subset M'$. Then $L \cap M \neq 0$ since M is essential in M' and

$$\mathrm{Ann}_R(M') \subseteq \mathrm{Ann}_R(L) \subseteq \mathrm{Ann}_R(M \cap L) = \mathrm{Ann}_R(M) = 0.$$

Thus $\mathrm{Ann}_R(M') = \mathrm{Ann}_R(L)$, M' is prime.
(b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) Follow from $M \trianglelefteq \widehat{M} \trianglelefteq E(M)$.
(d) $\Longrightarrow$ (a) If $E(M)$ is prime, then the submodules of $E(M)$ are also prime. In particular, M is prime. □

1.2.11 Lemma. *Let M be a faithful self-generator R-module, $S = \mathrm{End}_R(M)$ and $B = \mathrm{End}(M_S)$. If ${}_BM$ is prime, then ${}_RM$ is prime.*

Proof. By 15.6 of [38], every R-submodule of M is a B-submodule. We have an injective ring homomorphism $\varphi : R \to B$, $r \mapsto [\varphi(r)(m) := rm]$, and for any R-submodule K of M, $\mathrm{Ann}_R(K) \subset \mathrm{Ann}_B(K) = 0$. □

1.2.12 Proposition. *Let M be a module with* $\mathrm{Soc}(M) \neq 0$. *If M is prime, then $\overline{R} := R/\mathrm{Ann}_R(M)$ is a left primitive ring. If, in addition, R is commutative, then $\overline{R}$ is a field.*

Proof. By assumption, there is a simple $\overline{R}$-submodule K of M which is faithful, thus $\overline{R}$ is left primitive. If $\overline{R}$ is commutative and primitive, then $\overline{R}$ is a field. □

1.3 Coprime modules

Dual to prime modules, we define coprime modules in the following way.

1.3.1 Definition. M is called *coprime* if for every proper fully-invariant submodule K of M, $\mathrm{Ann}_R(M/K) = \mathrm{Ann}_R(M)$.

We characterize the coprimeness of a module M in

1.3.2 Coprime modules. *For a module M the following are equivalent :*

(a) *M is a coprime module.*

(b) $\mathrm{Ann}_R(M/K) = \mathrm{Ann}_R(M)$ *for any proper submodule K of M.*

(c) *$R/\mathrm{Ann}_R(M)$ is cogenerated by M/K for any proper submodule K of M.*

(d) *$R/\mathrm{Ann}_R(M)$ is cogenerated by M/K for any proper fully-invariant submodule K of M.*

Proof. (a) $\Longleftrightarrow$ (b) We only need to prove one direction. Let K be a submodule of M and assume $I := \mathrm{Ann}_R(M/K) \not\subset \mathrm{Ann}_R(M)$. Then $0 \neq IM \subset K$, and IM is fully invariant since $(IM)S = I(MS) = IM$. By (a), $I \subset \mathrm{Ann}_R(M/IM) = \mathrm{Ann}_R(M)$, a contradiction.

(b) $\Longleftrightarrow$ (c) and (a) $\Longleftrightarrow$ (d) are obvious. □

The condition (b) of 1.3.2 is used to define coprime modules in Annin [2]. Notice that any module which has no proper fully invariant submodule is coprime.

1.3.3 Lemma. *Let M be an R-module.*

(i) *If M is coprime, then $\overline{R}$ is prime.*

(ii) *If $\overline{R}$ is prime and M satisfies $(**)$, then M is coprime.*

Proof. (i) Assume M to be coprime, i.e., $\mathrm{Ann}_R(M/K) = \mathrm{Ann}_R(M)$ for every proper (fully-invariant) submodule $K \subset M$. Then by Proposition 1.1.7, $\overline{R}$ is prime.

(ii) By $(**)$, $\mathrm{Ann}_R(K) \neq \mathrm{Ann}_R(M)$ for any proper (fully-invariant) submodule $K \subset M$. Now $\mathrm{Ann}_R(K)\mathrm{Ann}_R(M/K) \subseteq \mathrm{Ann}_R(M)$ and $\overline{R}$ is prime implies $\mathrm{Ann}_R(M/K) = \mathrm{Ann}_R(M)$, hence M is coprime. □

The example $M = {}_{\mathbb{Z}}\mathbb{Z}$ illustrates that without condition $(**)$, assertion (ii) in the lemma above does not hold.

1.3.4 Example. As a $\mathbb{Z}$-module, $\mathbb{Z}$ is faithful and $\mathbb{Z}$ is a prime ring. For any $n \in \mathbb{N}$, $\mathbb{Z}/n\mathbb{Z}$ is a $\mathbb{Z}$-torsion module with $\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}) \neq 0$, hence $\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}) \neq \mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z})$. Thus $\mathbb{Z}$ is not a coprime module over $\mathbb{Z}$.

1.3.5 Example. Prüfer group. Recall that for any prime number p, the p-component of $\mathbb{Q}/\mathbb{Z}$ is the Prüfer group $\mathbb{Z}_{p^\infty}$. Any non-zero factor module $\mathbb{Z}_{p^\infty}/K$ of $\mathbb{Z}_{p^\infty}$ is isomorphic to $\mathbb{Z}_{p^\infty}$ itself. Thus $\mathbb{Z}_{p^\infty}$ is coprime. Moreover, a proper submodule $K \subset \mathbb{Z}_{p^\infty}$ is of the form $K = \mathbb{Z}\{\frac{1}{p^k} + \mathbb{Z}\}$ for some $k \in \mathbb{N}$, thus it is not faithful. Hence $\mathbb{Z}_{p^\infty}$ is not prime.

1.3.6 Example. For any ring R, $\mathrm{Ann}_R(R/I) = I$ for any ideal I of R. Thus ${}_RR$ is a coprime module if and only if R is a simple ring.

1.3.7 Example. $\mathbb{Q}$ is a prime $\mathbb{Z}$-module and is a coprime $\mathbb{Z}$-module, since it does not have any non-trivial fully-invariant submodule.

The coprimeness of a module is preserved by some factor module.

1.3.8 Proposition. *If M is coprime and K is a proper fully invariant submodule of M, then M/K is coprime.*

Proof. Take any proper fully invariant submodule U/K of M/K, where $K \subset U \subset M$ are proper submodules and K is fully invariant. By Corollary 1.1.21, U is fully invariant in M. Thus $(M/K)/(U/K) \simeq M/U$ is faithful, i.e.

$$\mathrm{Ann}_R((M/K)/(U/K)) = \mathrm{Ann}_R(M/U) = \mathrm{Ann}_R(M) = \mathrm{Ann}_R(M/K). \quad \square$$

The direct sum of a family of faithful coprime modules is again coprime.

1.3.9 Lemma. *Let $\{M_\lambda\}_\Lambda$ be a family of faithful modules, Λ an index set. Then $\bigoplus_\Lambda M_\lambda$ is coprime if and only if each M_λ is coprime.*

Proof. ($\Longrightarrow$) There is an epimorphism $f : \bigoplus_\Lambda M_\lambda \to M_\mu/L$ for any proper submodule $L \subset M_\mu$. Hence $M_\mu/L \simeq (\bigoplus_\Lambda M_\lambda)/K$ for some submodule $K \subset \bigoplus_\Lambda M_\lambda$ and

$$\mathrm{Ann}_R(M_\mu/L) = \mathrm{Ann}_R((\bigoplus_\Lambda M_\lambda)/K) = 0.$$

($\Longleftarrow$) Let each M_λ be coprime and denote by e_λ the canonical idempotents in $\mathrm{End}_R(\bigoplus_\Lambda M_\lambda)$. For any proper fully-invariant submodule $U \subset \bigoplus_\Lambda M_\lambda$, $U = \sum U e_\lambda$ and $U e_\lambda \neq M_\lambda$ for some λ. Thus $\bigoplus_\Lambda M_\lambda/U = \bigoplus_\Lambda (M_\lambda/U e_\lambda) \neq 0$ and

$$\mathrm{Ann}_R((\bigoplus_\Lambda M_\lambda)/U) = \mathrm{Ann}_R(\bigoplus_\Lambda (M_\lambda/U e_\lambda)) = \bigcap \mathrm{Ann}_R(M_\lambda/U e_\lambda) = 0. \quad \square$$

The following observation is dual to an observation for prime modules (see Proposition 1.2.12).

1.3.10 Proposition. *If M is coprime and* $\mathrm{Rad}(M) \neq M$*, then :*

(i) $\overline{R} := R/\mathrm{Ann}_R(M)$ *is a left primitive ring.*

(ii) *If R is commutative, then $\overline{R}$ is a field.*

Proof. (i) By assumption there is a maximal submodule K in M. Consider the fully invariant submodule $\mathrm{Rej}(M, M/K) \subset K \neq M$. By coprimeness of M, $M/\mathrm{Rej}(M, M/K)$ is a faithful $\overline{R}$-module and is cogenerated by the simple module M/K (see 14.5 of [38]). Thus M/K is also a faithful $\overline{R}$-module. Thus $\overline{R}$ has a faithful simple left module, i.e., $\overline{R}$ is a left primitive ring.

(ii) If $\overline{R}$ is commutative and primitive, then $\overline{R}$ is a field. $\square$

Without the condition $\mathrm{Rad}(M) \neq M$, Proposition 1.3.10 does not hold. For example, the Prüfer group $\mathbb{Z}_{p^\infty}$ is a coprime $\mathbb{Z}$-module which has no maximal submodule.

1.3.11 Proposition. *Let $p : P \to M$ be a small epimorphism in $\sigma[M]$. If M is a faithful R-module and coprime, then P is coprime.*

Proof. Consider a proper fully-invariant submodule $L \subset P$ and $K := \text{Ker}\, p$. Since $K \ll P$, $L + K \neq P$ and $(L)p \subset M$ is proper. Consider the following commutative diagram with exact rows :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & P & \xrightarrow{p} & M \simeq P/K & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & P/L & \longrightarrow & P/(K+L) & \longrightarrow & 0.
\end{array}$$

$P/(K+L) \neq 0$ and by the coprimeness of M, $\text{Ann}_R(P/(K+L)) = 0$ and hence $\text{Ann}_R(P/L) = 0$. □

1.4 Endo-prime modules

The primeness condition for the ring R can be expressed in the following way : any non-zero fully invariant submodule of ${}_RR$ is faithful over the endomorphism ring $\text{End}_R(R) \simeq R$. Haghany and Vedadi [13] generalized this property to modules.

1.4.1 Definition. A module M with $S = \text{End}_R(M)$ is called *endo-prime* if for any non-zero fully invariant R-submodule K, $\text{Ann}_S(K) = 0$.

An endo-prime left R-module M can be considered as a prime module over its endomorphism ring $\text{End}_R(M)$. An interesting question is to look at the interdependence of endo-primeness of modules and primeness of the ring of endomorphism. The following is a modification of Proposition 1.3 of [13].

1.4.2 Endo-prime modules. *Let M be a left R-module with $S = \text{End}_R(M)$.*

(i) *${}_RM$ is endo-prime if and only if M_S is prime.*

(ii) *If ${}_RM$ is endo-prime, then S is a prime ring.*

(iii) *If ${}_RM$ is fi-retractable and S is a prime ring, then ${}_RM$ is endo-prime.*

(iv) *If N is simple, then $N^{(\Lambda)}$ is endo-prime for any set Λ.*

(v) *If ${}_RM$ is endo-prime and fi-retractable, then* $\text{Ann}_R(M)$ *is a prime ideal of* R.

Proof. (i) If M_S is prime, then for any non-zero fully invariant submodule K of M holds $\text{Ann}_S(K) = 0$. Thus ${}_RM$ is endo-prime.

Now assume ${}_RM$ is endo-prime, and let K be a non-zero S-submodule of M. Then RK is a fully-invariant R-submodule and $\mathrm{Ann}_S(K) = \mathrm{Ann}_S(RK) = 0$ showing that M_S is prime.

(ii) ${}_RM$ is endo-prime means $\mathrm{Ann}_S(K) = 0$ for any non-zero fully invariant submodule K. If we consider M as a right S-module, Corollary 1.1.8 applied to M_S yields that S is a prime ring.

(iii) Equivalent to the primeness of S is that for any fully-invariant submodule K of ${}_RM$ holds $\mathrm{Ann}_S(K) = 0$ or $\mathrm{Ann}_S(M/K) = 0$ (see Corollary 1.1.8). But ${}_RM$ is fi-retractable, i.e. $\mathrm{Ann}_S(M/K) \neq 0$, hence $\mathrm{Ann}_S(K) = 0$.

(iv) If N is simple, then $N^{(\Lambda)}$ is fi-retractable and $S = \mathrm{End}_R(N^{(\Lambda)})$ is a primitive ring, hence prime. Then we apply (iii).

(v) Let I, J be ideals of R with $I \not\subseteq \mathrm{Ann}_R(M)$, $J \not\subseteq \mathrm{Ann}_R(M)$ but $IJ \subseteq \mathrm{Ann}_R(M)$. It means $IM \neq 0$, $JM \neq 0$ and $IJM = 0$. Since ${}_RM$ is fi-retractable, there is a non-zero $f \in S$ such that $Mf \subseteq JM$ and hence $IMf = 0$. By endo-primeness of M we conclude that $f = 0$, a contradiction. □

With some additional conditions we get some relationships between primeness and endo-primeness of a module.

1.4.3 Corollary. *Let M be an R-module.*

(i) *If ${}_RM$ is prime and satisfies $(*fi)$, then it is endo-prime.*

(ii) *Let R be a commutative ring. If M is a faithful R-module, endo-prime and fi-retractable, then ${}_RM$ is prime.*

(iii) *If M is a faithful self-generator R-module and endo-prime, then ${}_RM$ is prime.*

Proof. (i) This follows by Proposition 1.2.3.

(ii) Let K be a fully-invariant submodule of M. ${}_RM$ is endo-prime implies S is prime (see 1.4.2). Since M is fi-retractable and $\mathrm{Ann}_S(M/K) \neq 0$, $\mathrm{Ann}_S(K) = 0$. Now $R \subset S$ implies $\mathrm{Ann}_R(K) = 0$.

(iii) As a self-generator M is fi-retractable, i.e., satisfies $(*fi)$ as an S-module. By assertion (i) and assumption, ${}_BM$ is prime, where $B = \mathrm{End}(M_S)$. Now by Lemma 1.2.11, ${}_RM$ is prime. □

1.4.4 Example. Let P be a projective R-module satisfying $(*fi)$. If the ring R is prime, then P is prime (see Corollary 1.2.6). By Corollary 1.4.3 part (i), P is endo-prime.

For $M = R$ we obtain Proposition 1.4 of [13].

1.4.5 Proposition. *The following statements are equivalent for a ring R.*

(a) *R is prime.*

(b) *There exists a faithful retractable endo-prime left (right) R-module.*

Proof. Consider R as a left R-module and apply Corollary 1.4.3. □

As an immediate consequence we extend Corollary 1.6 of [13].

1.4.6 Corollary. *For ${}_RM$ suppose that at least one of the following conditions holds :*

(i) *${}_RM$ is retractable and satisfies the $(*)$ condition,*

(ii) *${}_RM$ satisfies the $(*)$ condition and R is commutative,*

(iii) *${}_RM$ is finitely generated free (i.e. $M \simeq {}_RR^n$ for some n).*

Then ${}_RM$ is prime if and only if ${}_RM$ is endo-prime if and only if $\mathrm{Ann}_R(M)$ is a prime ideal.

1.5 Endo-coprime modules

Dual to endo-prime modules, we define :

1.5.1 Definition. A module M with $S = \mathrm{End}_R(M)$ is called *endo-coprime* if for any proper fully invariant R-submodule K, $\mathrm{Ann}_S(M/K) = 0$.

In 1.4.2 properties of endo-prime modules are considered. Here we give the corresponding properties for

1.5.2 Endo-coprime modules. *Let M be a left R-module and $S = \mathrm{End}_R(M)$.*

(i) *${}_RM$ is endo-coprime if and only if M_S is coprime.*

(ii) *If ${}_RM$ is endo-coprime, then S is prime.*

(iii) *If ${}_RM$ is fi-coretractable and S is prime, then ${}_RM$ is endo-coprime.*

(iv) *If R is commutative and ${}_RM$ is endo-coprime, then $R/\mathrm{Ann}_R(M)$ is prime.*

Proof. (i) Let K be a proper (R,S)-submodule of M. Then $\mathrm{Ann}_S(M/K)=0$ since M_S is coprime and thus ${}_RM$ is endo-coprime.

Assume now ${}_RM$ to be endo-coprime and let K be a proper right S-submodule of M. Assume $\mathrm{Ann}_S(M/K)=J\neq 0$. Then $MJ\subset K$ is an (R,S)-submodule and $J\subset \mathrm{Ann}_S(M/MJ)=0$, a contradiction.

(ii) Assume ${}_RM$ to be endo-coprime and hence M_S is coprime by (i). Then we use Corollary 1.1.8 to conclude that S is prime.

(iii) M is fi-coretractable means $\mathrm{Ann}_S(K)\neq 0$ for any fully-invariant submodule K of M. Since S is prime, according to Corollary 1.1.8, $\mathrm{Ann}_S(M/K)=0$. Thus ${}_RM$ is endo-coprime.

(iv) Since R is commutative and S is prime, by Lemma 1.1.2 part (ii), $R/\mathrm{Ann}_R(M)$ is prime. □

The following lemma is dual to Lemma 1.8 of [13].

1.5.3 Lemma. *Let M be an R-module and K be a proper fully-invariant submodule of M. Consider the group homomorphism $\psi:\mathrm{End}_R(M)\to\mathrm{End}_R(M/K)$ with $f\mapsto\psi(f)=\overline{f}$, where $f\pi_K=\pi_K\overline{f}$, π_K is the canonical projection $M\to M/K$.*

(i) *If M is endo-coprime, then ψ is injective.*

(ii) *If M is self-projective, then ψ is surjective.*

Proof. (i) Let $f\in\mathrm{End}_R(M)=S$, $0\neq K\subset M$ be a submodule and consider the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ {\scriptstyle\pi_K}\downarrow & & \downarrow{\scriptstyle\pi_K} \\ M/K & & M/K. \end{array}$$

There exists $\psi(f)=\overline{f}\in\mathrm{End}_R(M/K)$ such that $f\pi_K=\pi_K\overline{f}$ and $(m)f\pi_K=(m)\pi_K\overline{f}$ for any $m\in M$, hence $(m+K)\overline{f}=(m)f+K$.

Assume M is endo-coprime, i.e., $\mathrm{Ann}_S(M/K)=0$ for any proper fully-invariant submodule K of M. Let $f\in\mathrm{Ker}\ \psi$, $\psi(f)=\overline{f}=0$, $(M/K)\overline{f}=0$, $(M)f\subset K$. Hence $f\in\mathrm{Ann}_S(M/K)=0$ and thus $f=0$.

(ii) Let $\alpha\in\mathrm{End}_R(M/K)$. We have the diagram

$$\begin{array}{ccccc} M & \xrightarrow{\pi_K} & M/K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \\ M & \xrightarrow[\pi_K]{} & M/K & \longrightarrow & 0. \end{array}$$

Since M is self-projective, there is a homomorphism $g \in \mathrm{End}_R(M)$ such that $\pi_K \alpha = g \pi_K$. In other words, $\psi(g) = \alpha$, i.e. ψ is surjective. □

1.6 Fully prime modules

1.6.1 Definition. A module M is called *fully prime* if for any non-zero fully invariant submodule K of M, M is K-cogenerated.

In order to characterize fully prime modules, we recall a notion of product of fully invariant submodules of M studied by Raggi, Ríos, Rincón, Fernández-Alonso and Signoret [27].

For any fully invariant submodules K, L of M, consider the product

$$K *_M L := K\mathrm{Hom}_R(M, L).$$

1.6.2 Remark. In the language of torsion theory, for any fully invariant submodule K of M, and for any module N, define a preradical

$$\alpha_K^M(N) := \sum \{(K)f \mid f : M \to N\}.$$

Then for any fully invariant submodules K and L, we have $K *_M L = \alpha_K^M(L)$.

Notice that such a product is defined in Bican et.al. [4] for every pair of submodules $K, L \subset M$ (not necessary fully-invariant). That product is used to define "prime module" and the condition is more restrictive than the one we consider here. For example, $\mathbb{Q}$ has no non-trivial fully invariant submodule, hence it is a fully prime module. However, it is not necessary a "prime module" in the sense of Bican, since $\mathbb{Q}$ is not cogenerated by the submodule $\mathbb{Z}$ of $\mathbb{Q}$.

Some characterizations are given in Proposition 2.3 of [4] and we have similar characterizations here.

1.6.3 Fully prime modules. *The following are equivalent for an R-module M :*

(a) *M is a fully prime module.*

(b) $\mathrm{Rej}(M, K) = 0$ *for any non-zero fully-invariant submodule $K \subset M$.*

(c) $K *_M L \neq 0$ *for any non-zero fully-invariant submodules $K, L \subset M$.*

(d) $\mathrm{Rej}(-, M) = \mathrm{Rej}(-, K)$ *for any non-zero fully-invariant submodule K of M, i.e., any M-cogenerated module is also K-cogenerated.*

Proof. (a) $\iff$ (b) $\iff$ (d) are obvious by the definition of cogenerating.

(b) $\Longrightarrow$ (c) For any nonzero fully-invariant submodules $K, L \subset M$ assume $K\mathrm{Hom}_R(M, L) = 0$. Then $0 \neq K \subset \mathrm{Rej}(M, L)$.

(c) $\Longrightarrow$ (b) Assume $\mathrm{Rej}(M, K) = U \neq 0$ for some non-zero fully-invariant submodule $K \subset M$. Then $U\mathrm{Hom}_R(M, K) = U *_M K = 0$. $\square$

Based on the $*_M$-product we define

1.6.4 Fully prime submodules. A fully invariant submodule N of M is *fully prime* in M if for any fully invariant submodules K, L of M, the relation $K *_M L \subseteq N$ implies $K \subseteq N$ or $L \subseteq N$.

Thus the module M is fully prime if the zero submodule is fully prime in M.

Proposition 18 of [27] provides a relationship between a fully prime submodule N of M and the factor module M/N.

1.6.5 Proposition. *Let N be a proper fully-invariant submodule of M.*

(i) *If N is fully prime in M, then M/N is a fully prime module.*

(ii) *If M is self-projective and M/N is fully prime, then N is a fully prime submodule in M.*

Proof. (i) Let $K/N, L/N$ be fully-invariant submodules of M/N such that $K/N *_{M/N} L/N = 0$. By Corollary 1.1.21, K and L are fully-invariant submodules of M with $K *_M L \subseteq N$ and hence $K \subseteq N$ or $L \subseteq N$, that is $K/N = 0$ or $L/N = 0$.

(ii) Let K, L be fully-invariant submodules of M such that $K *_M L \subseteq N$. By Lemma 1.1.20 part (ii), $K' = (K+N)/N$ and $L' = (L+N)/N$ are fully-invariant submodules of M/N such that $K' *_{M/N} L' = 0$. Hence $K' = 0$ or $L' = 0$, that is $K \subseteq N$ or $L \subseteq N$. $\square$

Consider R as a left R-module and let I, J be ideals of R. Then $I *_R J = IJ$. Since every ideal of R is a fully invariant R-submodule, we get :

1.6.6 Proposition. *The following are equivalent for a two-sided ideal I :*

(a) *R/I is a prime ring.*

(b) *I is a fully prime submodule in R.*

(c) *I is a prime ideal.*

In general prime modules need not be fully prime. For the following relationship we adopt the proof of [39], Proposition 13.2.

1.6.7 Proposition. *For an R-module M with $(*fi)$, the following are equivalent :*

(a) *M is prime and fi-retractable.*

(b) *M is fully prime.*

Proof. (a) $\Longrightarrow$ (b) By assumption M satisfies $(*fi)$, hence following Proposition 1.2.3 part (iii), S is prime. Hence, for non-zero fully invariant submodules K, L of M, by fi-retractibility we have $\mathrm{Hom}_R(M, L) \neq 0$, $\mathrm{Hom}_R(M, K) \neq 0$ and

$$L\mathrm{Hom}_R(M, K) \supset M\mathrm{Hom}_R(M, L)\mathrm{Hom}_R(M, K) \neq 0,$$

thus $L *_M K \neq 0$. By 1.6.3, M is K-cogenerated, i.e., M is fully prime.

(b) $\Longrightarrow$ (a) By assumption, $\mathrm{Rej}(M, K) = 0$ for all non-zero fully-invariant submodule $K \subset M$. Thus $\mathrm{Hom}_R(M, K) \neq 0$. Moreover, K is a faithful $R/\mathrm{Ann}_R(M)$-module, i.e. M is prime. □

Notice that for any ring R, $\mathrm{End}_R(R) \simeq R$ and as a left R-module, R satisfies $(*fi)$ and is fi-retractable. If $M = R$, Proposition 1.6.7 yields

1.6.8 Corollary. *For the ring R the following assertions are equivalent :*

(a) *R is a prime ring.*

(b) *${}_RR$ is a prime module.*

(c) *${}_RR$ is a fully prime module.*

1.6.9 Proposition. *Let M be a module with $\mathrm{Soc}(M) \neq 0$. If M is fully prime, then*

(i) *M is cogenerated by a simple module.*

(ii) *$\overline{R} := R/\mathrm{Ann}_R(M)$ is a left primitive ring.*

Proof. (i) Let K be a simple submodule of M. Then $\mathrm{Tr}(K, M)$ is a fully invariant submodule and hence M is $\mathrm{Tr}(K, M)$-cogenerated. $\mathrm{Tr}(K, M)$ is K-cogenerated, and hence M is K-cogenerated.

(ii) $\overline{R}$ is cogenerated by M and hence by the simple module K (from (i)). □

Notice that (ii) is also a consequence of Proposition 1.2.12, since M is fully prime implies M is prime.

1.7 Fully coprime modules

1.7.1 Definition. A module M is called *fully coprime* if for any proper fully invariant submodule K of M, M is M/K-generated.

An inner coproduct of fully invariant submodules of M can be defined in the following way. For any fully invariant submodules $K, L \subset M$, put

$$
\begin{aligned}
K :_M L &:= \bigcap\{(L)f^{-1} \mid f \in \mathrm{End}_R(M), K \subseteq \mathrm{Ker} f\} && (1.1)\\
&= \mathrm{Ker}\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) \diamond \pi_L, && (1.2)
\end{aligned}
$$

where $\pi_K : M \to M/K$ and $\pi_L : M \to M/L$ denote the canonical projections. $K :_M L$ is also a fully invariant submodule with the properties (see Raggi et.al. [28]):

(i) $K + L \subset K :_M L$.

(ii) If $N \subset M$ is a fully invariant submodule with $K, L \subset N$, then

$$K :_N L \subset K :_M L$$

Notice that such a coproduct is considered in Bican et.al. [4] for any pair of submodules $K, L \subset M$ (not necessary fully invariant) and then a definition of "coprime modules" is derived from this coproduct.

The difference with our definition is, for example, shown by the $\mathbb{Z}$-module $\mathbb{Q}$. $\mathbb{Q}$ has no non-trivial fully-invariant submodules, hence it is trivially fully coprime. Indeed, it is not necessarily a "coprime module" in the sense of Bican, since $\mathbb{Q}$ is not generated by $\mathbb{Q}/\mathbb{Z}$.

The Prüfer group $\mathbb{Z}_{p^\infty}$ is a fully coprime module which has many fully invariant submodules. A fully coprime ring is nothing but a simple ring.

1.7.2 Remark. In the language of torsion theory, for any fully invariant submodule K of M, and for any module $N \in \sigma[M]$, define a preradical

$$\omega_K^M(N) = \bigcap\{(K)g^{-1} \mid g : N \to M\} \subset N.$$

For any two preradicals τ, ρ there is a coproduct defined by

$$(\tau : \rho)(M)/\tau(M) = \rho(M/\tau(M)).$$

With this notation, for any fully invariant submodules K, L of M, (see Raggi et.al. [28])

$$K :_M L = (\omega_K^M : \omega_L^M)(M).$$

We characterize fully coprime modules in the proposition below. This is similar to Proposition 4.3 of [4] but here we consider proper fully invariant submodules.

1.7.3 Fully coprime modules. *The following are equivalent for an R-module M :*

(a) *M is a fully coprime module.*

(b) *If $K :_M L = M$, then $K = M$ or $L = M$, for any fully invariant submodules K, L of M.*

(c) *$K :_M L \neq M$ for any proper fully invariant submodules K, L of M;*

(d) *$\mathrm{Tr}(M/K, -) = \mathrm{Tr}(M, -)$ for any proper fully invariant submodules K of M, i.e. any M-generated module is also M/K-generated.*

Proof. (a) $\Longleftrightarrow$ (d) and (b) $\Longleftrightarrow$ (c) are trivial.

(c) $\Longrightarrow$ (a) Let $K \subset M$ be a proper fully invariant submodule such that

$$N = \mathrm{Tr}(M/K, M) = (M)\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) \neq M.$$

Then $0 = (M)\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) \diamond \pi_N$ and $K :_M N = M$.

(d) $\Longrightarrow$ (c) Let K, L be any proper fully invariant submodules of M and assume $(M)\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) \diamond \pi_L = 0$. Then

$$M = \mathrm{Tr}(M, M) = \mathrm{Tr}(M/K, M) \subset L. \quad \square$$

1.7.4 Fully coprime rings. *For the ring R the following are equivalent :*

(a) *${}_RR$ is coprime.*

(b) *${}_RR$ is fully coprime.*

(c) *R is a simple ring.*

1.7.5 Lemma. *Let M be fully coprime, $S = \mathrm{End}_R(M)$. Then M is indecomposable as (R, S)-bimodule.*

Proof. Assume $M = U \oplus V$ where U, V are (R, S)-subbimodules of M. Then $\mathrm{Hom}_R(U, V) = 0$. Since M is fully coprime, M is generated by $M/U \simeq V$. It means V also generates U, thus contradicts $\mathrm{Hom}_R(U, V) = 0$. $\square$

A module M is called *semilocal* if $M/\mathrm{Rad}(M)$ is semisimple. Obviously, a semisimple module is semilocal.

1.7.6 Corollary. *Let M be a fully coprime module. If M is semilocal, then M is homogeneous semisimple.*

Proof. $\mathrm{Rad}(M)$ is a fully invariant submodule of M, hence M is generated by $M/\mathrm{Rad}(M)$ which is semisimple. Thus M is semisimple and now apply Lemma 1.7.5. □

Similar to fully prime submodules we define

1.7.7 Fully coprime submodules. A fully invariant submodule $N \subset M$ is called *fully coprime in M* if for any fully invariant submodules $K, L \subset M$, $N \subseteq K :_M L$ implies $N \subset K$ or $N \subset L$.

By 1.7.3, M is fully coprime if and only if M is fully coprime in M. An immediate consequence of the definition is (compare with Proposition 1.6.7)

1.7.8 Proposition. *If a module M is fully coprime, then M is coprime and fi-coretractable.*

Proof. If M is M/K-generated, it is clear that $\mathrm{Ann}_R(M/K) = \mathrm{Ann}_R(M)$ and $\mathrm{Hom}_R(M/K, M) \neq 0$. □

In view of later use for comodules and coalgebras (wedge product), we consider another coproduct of two proper fully invariant submodules $K, L \subset M$. Put

$$K \ \wedge^M \ L \ := \ \mathrm{Ker}\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) \diamond \pi_L \diamond \mathrm{Hom}_R(M/L, M) \quad (1.3)$$

$$= \ \mathrm{Ker}\ (\mathrm{Ann}_S(K) \diamond \mathrm{Ann}_S(L)), \quad (1.4)$$

a fully invariant submodule of M. The relation between the coproducts (1.1) and (1.3) is obviously,

$$K :_M L \subseteq K \ \wedge^M \ L.$$

The next result shows when equality holds.

1.7.9 Proposition. *Consider the following assertions for a module M :*

(a) *M is a fully coprime module.*

(b) *If $K \ \wedge^M \ L = M$, then $K = M$ or $L = M$, for any fully invariant submodules K, L of M.*

(c) $K \wedge^M L \neq M$ *for any proper fully invariant submodules* K, L *of* M.

(d) $\mathrm{Tr}(M/K, -) = \mathrm{Tr}(M, -)$ *for any proper fully invariant submodules* K *of* M*, i.e. an* M*-generated module is also an* M/K*-generated module.*

Then we have (a) $\Longleftrightarrow$ *(d), (b)* $\Longleftrightarrow$ *(c) and (c)* $\Longrightarrow$ *(a).*

If M *is a self-cogenerator, then (d)* $\Longrightarrow$ *(c) and*

$$K :_M L = K \wedge^M L.$$

Proof. (c) $\Longleftrightarrow$ (b) is trivial and (a) $\Longleftrightarrow$ (d) is known (see 1.7.3).

(c) $\Longrightarrow$ (a) If $K :_M L = M$, then $K \wedge^M L = M$.

(d) $\Longrightarrow$ (c) Assume M is a self-cogenerator. Let K, L be proper fully invariant submodules of M and assume

$$(M)\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) \diamond \pi_L \diamond \mathrm{Hom}_R(M/L, M) = 0.$$

Since M is a self-cogenerator, we obtain by Lemma 1.1.9,

$$(M)\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) \subset \mathrm{Ker}\ \pi_L \diamond \mathrm{Hom}_R(M/L, M) = L,$$

and $M = \mathrm{Tr}(M, M) = \mathrm{Tr}(M/K, M) \subseteq L$, a contradiction. □

1.7.10 Proposition. *Let* M *be a self-cogenerator and* $S = \mathrm{End}_R(M)$*. If* S *is prime, then* M *is fully coprime.*

Proof. Let K, L be proper fully invariant submodules of M which satisfy $M = K :_M L$. By assumption we have

$$(M)\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) \diamond \pi_L \diamond \mathrm{Hom}_R(M/L, M) = 0.$$

Since S is prime, $\pi_K \diamond \mathrm{Hom}_R(M/K, M) = 0$ or $\pi_L \diamond \mathrm{Hom}_R(M/L, M) = 0$. Hence $K = M$ or $L = M$ since M is a self-cogenerator. □

Let I, J be ideals in $\mathrm{End}_R(M)$ and put $\mathrm{Ker}\ I = K$, $\mathrm{Ker}\ J = L$. Then

$$I \subseteq \mathrm{Hom}\ (M/K, M), \quad J \subseteq \mathrm{Hom}\ (M/L, M). \tag{1.5}$$

For the converse of Proposition 1.7.10 the equalities in (1.5) are of interest.

1.7.11 Proposition. *Let M be a self-cogenerator and $S = \text{End}_R(M)$.*

(i) *If M is self-injective and fully coprime, then S is prime and M is endo-coprime.*

(ii) *If M is endo-coprime, then M is fully coprime.*

Proof. (i) Let M be a fully coprime module and I, J finitely generated right ideals in S with $IJ = 0$. Put $K = \text{Ker } I$ and $L = \text{Ker } J$. Then by Lemma 1.1.9, $\text{Hom}(M/K, M) = I$ and $\text{Hom}(M/L, M) = J$ and $K :_M L = M$. Then $M = K$ or $M = L$, thus $I = 0$ or $J = 0$. Hence the ring S is prime. Since ${}_RM$ is fi-coretractable, M is endo-coprime (see 1.5.2).

(ii) We assume that M_S is coprime, hence S is prime. Then the assertion follows from Proposition 1.7.10. □

1.7.12 Corollary. *If M is a self-injective self-cogenerator and $S = \text{End}_R(M)$, then the following assertions are equivalent :*

(a) *M is fully coprime.*

(b) *M is endo-coprime.*

(c) *S is a prime ring.*

Proof. The equivalence holds by Proposition 1.7.10 and Proposition 1.7.11. □

1.7.13 Proposition. ([28], Proposition 4.9) *For a module M, let $K \subseteq H \subseteq M$ be submodules such that K is fully invariant in H and H is fully invariant in M.*

(i) *If K is fully coprime in M, then K is fully coprime in H.*

(ii) *If K is fully coprime in M, then K is a fully coprime module.*

In Lemma 1.2.10 relations between primeness and injectivity are considered. A partial converse of Proposition 1.7.13 yields a relationship between coprimeness and injectivity.

1.7.14 Proposition. ([28], Theorem 4.10) *Let $M \subseteq Q$ be a fully invariant submodule of a self-injective module Q. Then M is a fully coprime module if and only if M is fully coprime in Q.*

1.7.15 Proposition. *Let M be a fully coprime module with* $\mathrm{Rad}(M) \neq M$. *Then :*

(i) *M is generated by a module that is cogenerated by a simple module.*

(ii) *For any projective module P in $\sigma[M]$,* $\mathrm{Rad}(P) = 0$.

(iii) $\overline{R} := R/\mathrm{Ann}_R(M)$ *is a left primitive ring.*

Proof. (i) By assumption there is a maximal submodule K in M. Consider the fully invariant submodule $\mathrm{Rej}(M, M/K) \subset K \neq M$. By assumption M is $M/\mathrm{Rej}(M, M/K)$-generated, where $M/\mathrm{Rej}(M, M/K)$ is cogenerated by the simple module M/K.

(ii) By (i), P is subgenerated by $M/\mathrm{Rej}(M, M/K)$ which is cogenerated by M/K. Hence P is subgenerated by a product Q of copies of (M/K), and $P \subset Q^{(\Lambda)}$, for some index Λ (see 18.4 of [38]). Thus P is M/K-cogenerated and $\mathrm{Rad}(P) = 0$.

(iii) It is a consequence of Proposition 1.3.10, since M fully coprime implies that M is coprime. □

By Lemma 1.2.10, a module M is prime if and only if its M-injective hull is prime. Thus the question arises : If M is fully coprime, when is its projective hull (if it exists) also fully coprime?

1.7.16 Proposition. *Let $p : P \to M$ be a small epimorphism in $\sigma[M]$. Assume that P is M-generated and M is fully coprime.*

(i) *If P is self-projective, then P is fully coprime.*

(ii) *If P is projective in $\sigma[M]$, then M is projective in $\sigma[M]$, i.e. $M \simeq P$.*

Proof. (i) (Compare with Proposition 1.3.11) Consider a proper fully invariant $L \subset P$ and $K := \mathrm{Ker}\ p$. Since $K \ll P$, $L + K \neq P$ and $(L)p \subset M$ is proper. The morphism p induces an epimorphism $\widehat{p}$ such that the following diagram is commutative.

$$\begin{array}{ccc} P & \xrightarrow{p} & M \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ P/L & \xrightarrow{\widehat{p}} & M/(L)p \end{array}$$

By Lemma 1.1.20 part (ii), $(L)p$ is fully invariant. Since M is fully coprime, M is generated by $M/(L)p$ and hence P is P/L-generated. Thus P is fully coprime.

(ii) By the projectivity of P, $\mathrm{Rad}(P) \neq P$ (see [38], 22.3). By (i), P is fully coprime. By Proposition 1.7.15, $\mathrm{Rad}(P) = 0$ and hence $\mathrm{Ker}\ p = 0$. We obtain $M \simeq P$. In particular M is projective. □

1.8 Strongly prime and coprime modules

A ring R is left strongly prime if for every left ideal $I \subset R$ there is a monomorphism $R \to I^k$ for some $k \in \mathbb{N}$ (Handelman-Lawrence [14]). This notion was extended to left modules in Beidar-Wisbauer [3].

1.8.1 Definition. A module M is called *strongly prime* if for any non-zero fully invariant submodule $K \subseteq M$, $M \in \sigma[K]$.

We recall some characterizations of these modules (see [39], 13.3).

1.8.2 Strongly prime modules. *For an R-module M with M-injective hull $\widehat{M}$, the following are equivalent :*

(a) *M is a strongly prime module.*

(b) *M is subgenerated by each of its non-zero submodules.*

(c) *$\widehat{M}$ is generated by each of its non-zero (fully invariant) submodules.*

(d) *M is contained in every non-zero fully invariant submodule of $\widehat{M}$.*

(e) *$\widehat{M}$ has no non-trivial fully invariant submodules.*

If these conditions are satisfied, then $\widehat{M}$ is fully coprime (trivially).

1.8.3 Lemma. *Let M^Λ be the product of Λ copies of M in $\sigma[M]$ (see 1.1.4), for any index set Λ. M^Λ is strongly prime if and only if M is strongly prime.*

Proof. ($\Longrightarrow$) Let M^Λ be strongly prime. Consider any submodule $U \subset M \subset M^\Lambda$. Then $M \subset M^\Lambda \in \sigma[U]$ and hence M is strongly prime.

($\Longleftarrow$) Let M be strongly prime and $V \subset M^\Lambda$. There is some non-zero canonical projection $(V)\pi_{\lambda_0} \subset M$. Thus $M \in \sigma[(V)\pi_{\lambda_0}]$ which implies $M \in \sigma[V]$ and hence $M^\Lambda \in \sigma[V]$. □

1.8.4 Proposition. *Let M be strongly prime. Then*

(i) *every M-cogenerated module in $\sigma[M]$ is strongly prime;*

(ii) *every projective module in $\sigma[M]$ is strongly prime;*

(iii) *for every finitely generated projective module P in $\sigma[M]$, $\mathrm{End}_R(P)$ is strongly prime.*

Proof. (i) According to Lemma 1.8.3, M^{Λ} is strongly prime. Let U be a non-zero (fully invariant) submodule of an M-cogenerated module N. Then $N \subset M^{\Lambda} \in \sigma[U]$.

(ii) Consider a projective module P in $\sigma[M]$. Then $P \simeq X \subseteq M^{(\Lambda)}$ for some Λ. P is strongly prime.

(iii) Let $J \subset \mathrm{End}_R(P)$ be a proper finitely generated left ideal. Since P is strongly prime (by ii), PJ is a subgenerator in $\sigma[P]$ and hence $P \subset (PJ)^k$, $k \in \mathbb{N}$. Since P is finitely generated projective, $\mathrm{Hom}_R(P, PJ) = J$ (see [38], 18.4) and

$$\mathrm{End}_R(P) \subset \mathrm{Hom}_R(P, (PJ)^k) \simeq \mathrm{Hom}_R(P, (PJ))^k = J^k$$

showing that $\mathrm{End}_R(P)$ is (left) strongly prime. □

1.8.5 Proposition. *Let M be a module with* $\mathrm{Soc}(M) \neq 0$. *If M is strongly prime, then*

(i) *M is homogeneous semisimple.*

(ii) *$\overline{R} := R/\mathrm{Ann}_R(M)$ is primitive.*

Proof. (i) Let K be a simple submodule of M. Then $\mathrm{Tr}(K, M)$ is a fully invariant submodule and hence $M \in \sigma[\mathrm{Tr}(K, M)]$. Thus M is K-generated, i.e., homogeneous semisimple.

(ii) It is a consequence of Proposition 1.2.12, since M is strongly prime implies M is prime. □

For example, locally artinian modules (i.e., modules whose finitely generated submodules are artinian) have non-zero socle. Thus a locally artinian and strongly prime module is homogeneous semisimple.

Dual to strongly prime, we define

1.8.6 Definition. A module M is called *strongly coprime* if for any proper fully invariant submodule $K \subset M$, $M \in \sigma[M/K]$.

1.8.7 Proposition. *If M is a strongly coprime module and K is a proper fully invariant submodule of M, then M/K is strongly coprime.*

Proof. Let K be a fully invariant submodule of M. We take any proper fully invariant submodule U/K of M/K, where $K \subset U \subset M$. By Corollary 1.1.21, U is fully invariant. Thus $M \in \sigma[M/U]$ and $M/K \in \sigma[M/U] = \sigma[(M/K)/(U/K)]$. □

1.8.8 Proposition. *M is strongly coprime if and only if $M^{(\Lambda)}$ is strongly coprime for any index set Λ.*

Proof. ($\Longrightarrow$) Let $V \subset M^{(\Lambda)}$ be a proper fully invariant submodule. For some $\lambda_0 \in \Lambda$, $\pi_{\lambda_0} : M^{(\Lambda)} \to M_{\lambda_0}$ such that $(V)\pi_{\lambda_0} \subset M_{\lambda_0} = M$. Denote $\varepsilon_{\lambda_0} : M_{\lambda_0} \to M^{(\Lambda)}$. Then

$$(V)\pi_{\lambda_0}\varepsilon_{\lambda_0}\mathrm{End}_R(M^{(\Lambda)})\pi_{\lambda_0} \subset (V)\pi_{\lambda_0} \subset M_{\lambda_0},$$

hence $(V)\pi_{\lambda_0}$ is a fully invariant submodule of M_{λ_0}. Thus

$$M^{(\Lambda)} \in \sigma[M/(V)\pi_{\lambda_0}] \subset \sigma[M^{(\Lambda)}/V].$$

($\Longleftarrow$) Let U be a proper fully invariant submodule of M. Then $U^{(\Lambda)}$ is a proper fully-invariant submodule of $M^{(\Lambda)}$ and $M^{(\Lambda)}/U^{(\Lambda)} = (M/U)^{(\Lambda)}$. Thus $M \subset M^{(\Lambda)} \in \sigma[(M/U)^{(\Lambda)}] = \sigma[M/U]$. □

1.8.9 Proposition. *Let M be a strongly coprime and semilocal module. Then M is homogeneous semisimple.*

Proof. $\mathrm{Rad}(M)$ is a fully invariant submodule of M, hence M is subgenerated by $M/\mathrm{Rad}(M)$ which is semisimple. Thus M is semisimple and hence every module in $\sigma[M]$ is injective. Now a proof similar to the proof of Lemma 1.7.5 shows that M is homogeneous semisimple. □

1.8.10 Proposition. *Let M be a module with $\mathrm{Rad}(M) \neq M$. If M is strongly coprime, then :*

(i) *M is subgenerated by a product of copies of some simple module.*

(ii) *For any projective module P in $\sigma[M]$, $\mathrm{Rad}(P) = 0$.*

(iii) *$\overline{R} := R/\mathrm{Ann}_R(M)$ is primitive.*

Proof. (i) By assumption there is a maximal submodule K in M. Consider the fully invariant submodule $\mathrm{Rej}(M, M/K) \subset K \neq M$. By assumption M is $M/\mathrm{Rej}(M, M/K)$-subgenerated, where $M/\mathrm{Rej}(M, M/K)$ is cogenerated by the simple module M/K (product in $\sigma[M]$). Thus M is $(M/K)^{\Lambda}$-subgenerated for some index set Λ.

(ii) Since M is strongly coprime and $\mathrm{Rad}(M)$ is a fully invariant submodule of M, $M \in \sigma[M/\mathrm{Rad}(M)]$. By definition of the radical, $M/\mathrm{Rad}(M)$ is cogenerated by simple modules (see 14.5 of [38]). The projectivity of P implies that $P \subset (M/\mathrm{Rad}(M))^{(\Lambda)}$ for some index set Λ, and hence $\mathrm{Rad}(P) = 0$.

(iii) It is a consequence of Proposition 1.3.10, since M strongly coprime implies that M is coprime. □

The strong coprimeness of modules transfers to small epimorphisms.

1.8.11 Proposition. *Let $p : P \to M$ be a small epimorphism in $\sigma[M]$ and M strongly coprime.*

(i) *If P is self-projective, then P is strongly coprime.*

(ii) *If P is projective in $\sigma[M]$, then $M \simeq P$.*

Proof. (i) Consider any proper fully-invariant submodule $L \subset P$ and $K :=$ Ker p. Since $K \ll P$, $L + K \neq P$ and $(L)p \subset M$ is a fully-invariant submodule by Lemma 1.1.20 part (ii). By hypothesis, $M \in \sigma[M/(L)p] \subseteq \sigma[P/L]$. Since $P \in \sigma[M]$ and $\sigma[M] \subseteq \sigma[P/L]$, we have $P \in \sigma[P/L]$.

(ii) By the projectivity of P, $\mathrm{Rad}(P) \neq P$ (see [38], 22.3). By (i), P is strongly coprime and by Proposition 1.8.10, $\mathrm{Rad}(P) = 0$. Thus Ker $p = 0$ and $M \simeq P$. □

M is called *duprime* if for any fully-invariant submodule K of M, $M \in \sigma[K]$ or $M \in \sigma[M/K]$ (see [34] and [35]). By definition it is clear that any strongly coprime module is duprime. The convers is true for self-injective modules.

1.8.12 Proposition. *If M is a self-injective R-module, then :*

(i) *M is duprime if and only if it is strongly coprime.*

(ii) *The following are equivalent :*

   (a) *M is fully coprime;*

   (b) *M is strongly coprime;*

   (c) *M is duprime.*

Proof. (i) Let K be a proper fully invariant submodule of M. By assumption, $M \in \sigma[K]$ or $M \in \sigma[M/K]$. If $M \in \sigma[K]$ then M is K-generated, since M is self-injective. But $\mathrm{Tr}(K, M) = K\mathrm{End}_R(M) = K \subset M$, hence M is not K-generated, i.e., $M \notin \sigma[K]$. Thus $M \in \sigma[M/K]$.

(ii) (a) $\Longleftrightarrow$ (b) It is clear that if M/K generates M then $M \in \sigma[M/K]$. Now let M/K be a subgenerator of M. Since M is injective, it is M/K-generated. This is based on the fact that any injective module is generated by subgenerators in $\sigma[M]$.

(ii) (b) $\Longrightarrow$ (c) by definition, (c) $\Longrightarrow$ (b) by (i). □

Compare the proposition above with Corollary 1.7.12. Notice that if M is projective in $\sigma[M]$ or M is polyform, then M is duprime if and only if M is strongly prime (see [34], Theorem 3.3).

1.8.13 $\sigma[M] = {}_R\mathbf{M}$. *Assume* $\sigma[M] = {}_R\mathbf{M}$. *Then :*

(i) *If M is prime, then R is prime.*

(ii) *If M is coprime, then R is prime.*

(iii) *If M is endo-prime and fi-retractable, then R is prime.*

(iv) *Let R be a commutative ring. If M is endo-coprime, then R is prime.*

(v) *If M is fully prime, then R is prime and for any non-zero fully invariant submodule $K \subseteq M$, R is K-cogenerated.*

(vi) *If M is fully coprime, then R is prime.*

(vii) *If M is strongly prime, then R is a left strongly prime ring.*

(viii) *If M is strongly prime and* $\mathrm{Soc}(M) \neq 0$, *then R is a simple ring.*

(ix) *If M is strongly coprime, then R is a submodule of a strongly coprime module.*

Proof. Recall that $\sigma[M] = {}_R\mathbf{M}$ is equivalent to $R \subset M^k$ for some $k \in \mathbb{N}$ and implies that M is a faithful R-module.

(i) By Proposition 1.2.7.

(ii) By Lemma 1.3.3, R is prime.

(iii) By Proposition 1.4.2.

(iv) By Proposition 1.5.2.

(v) By definition, M is fully prime implies M is prime. Hence according to (i), R is prime. Since R is M-cogenerated, then for any non-zero fully invariant submodule $K \subseteq M$, R is K-cogenerated.

(vi) By definition, M is fully coprime implies M is coprime. To prove the assertion, see (ii).

(vii) is clear from Proposition 1.8.4.

(viii) is clear from Proposition 1.8.5

(ix) By Lemma 1.8.8, M^k is strongly coprime for any $k \in \mathbb{N}$. □

1.9 Colocalization in $\sigma[M]$

Let R be a ring with unit and let τ be a preradical in ${}_R\mathbf{M}$ (or in a category $\sigma[M]$ for some $M \in {}_R\mathbf{M}$). The class of modules N with $\tau(N) = N$ is called the *torsion class* $\mathcal{T}_\tau$ of τ, and the modules X with $\mathrm{Hom}(N, X) = 0$ for all $N \in T_\tau$ form the *torsion free class* $\mathcal{F}_\tau$ of τ. If τ is a hereditary torsion radical then $\mathcal{T}_\tau$ is closed under submodules, direct sums, quotients and extensions. Hereditary torsion theories allow to define a *localization functor* (quotient modules).

Dually, for any projective module P in ${}_R\mathbf{M}$ the class

$$\mathcal{F}_P = \{Y \in {}_R\mathbf{M} \,|\, \mathrm{Hom}(P, Y) = 0\}$$

can be taken as the torsion free class of a *cohereditary torsion theory.* In this case the torsion theory is determined by the *trace ideal* $T = \mathrm{Tr}(P, R)$, an idempotent ideal in R, and the torsion class consists of the P-generated modules. Such theories can be applied to define a *colocalization functor* ([16],[22]).

Localization theories in categories of type $\sigma[M]$, M an R-module, is outlined in Wisbauer [36]. In this section we recall the basic facts about colocalization in categories of type $\sigma[M]$ which is developed in [6]. For elementary notions of these theories we refer to [30], [39] and [6].

If a preradical τ preserves epimorphisms, then it is said to be *cohereditary.* Pseudo-projectivity of a module (see Definition 1.1.18) has an influence on the related trace functor (see [6], 6.12).

1.9.1 Lemma. *For $P \in \sigma[M]$ the following assertions are equivalent :*

(a) *P is pseudo-projective in $\sigma[M]$.*

(b) *The trace functor $\mathrm{Tr}(P, -) : \sigma[M] \to \sigma[M]$ preserves epimorphisms.*

(c) $\mathrm{Tr}(P/\mathrm{Tr}(P, N)) = 0$ *for all $N \in \sigma[M]$ and the class*

$$\{X \in \sigma[M] \mid \mathrm{Tr}(P, X) = 0\}$$

is closed under factor module.

By observation 6.13 of [6] we have

1.9.2 Lemma. *For a preradical τ for $\sigma[M]$ the following are equivalent :*

(a) *τ is cohereditary.*

(b) *There is a pseudo-projective module $P \in \sigma[M]$ such that $\mathcal{T}_\tau = \mathrm{Gen}\ (P)$.*

In ${}_R\mathbf{M}$, R is a generator and hence any cohereditary preradical τ for ${}_R\mathbf{M}$ is generated by the ideal $\tau(R)$. Moreover (see [6], 6.14) :

1.9.3 Lemma. *Let τ be a preradical for ${}_R\mathbf{M}$. τ is cohereditary if and only if for any $N \in {}_R\mathbf{M}$, $\tau(N) = \tau(R)N$.*

In particular, pseudo-projective modules P in ${}_R\mathbf{M}$ can be characterized by their trace ideal $\mathrm{Tr}(P, R)$ (see [6], 6.15).

1.9.4 Lemma. *For an R-module P with trace ideal $T = \mathrm{Tr}(P, R)$, the following are equivalent :*

(a) *P is pseudo-projective in ${}_R\mathbf{M}$.*

(b) *The trace functor $\mathrm{Tr}(P, -) : {}_R\mathbf{M} \to {}_R\mathbf{M}$ preserves epimorphisms.*

(c) *For every R-module L, $\mathrm{Tr}(P, L) = TL$.*

(d) *$P = TP$.*

If this conditions hold, then $T = T^2$ and $\mathrm{Gen}(P) = \mathrm{Gen}(T)$.

1.9.5 Proposition. *Assume that $\mathcal{T}^M := \mathrm{Tr}(\sigma[M], -)$ is exact and let $P \in \sigma[M]$, then :*

(i) *$\sigma[M]$ is closed under small epimorphisms in ${}_R\mathbf{M}$;*

(ii) *if P is projective in $\sigma[M]$, then P is projective in ${}_R\mathbf{M}$;*

(iii) *if P is pseudo-projective in $\sigma[M]$, then P is pseudo-projective in ${}_R\mathbf{M}$.*

Proof. (i) and (ii) are from 42.17 part (1) and (2) of [5].

(iii) Let $P \in \sigma[M]$ be pseudo-projective in $\sigma[M]$. From the diagram with exact sequence in ${}_R\mathbf{M}$,

$$\begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & & \downarrow f & & \\ 0 & \longrightarrow & K & \longrightarrow & N & \xrightarrow{p} & L & \longrightarrow & 0. \end{array}$$

we construct the following commutative diagram by applying the exact functor $\mathcal{T}^M$:

$$\begin{array}{ccccccccc}
 & & & & P & \xrightarrow{s} & P & & \\
 & & & & {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle f'} & & \\
0 & \longrightarrow & \mathcal{T}^M(K) & \longrightarrow & \mathcal{T}^M(N) & \xrightarrow{p'} & \mathcal{T}^M(L) & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle i_N}\downarrow & & \downarrow{\scriptstyle i_L} & & \\
0 & \longrightarrow & K & \longrightarrow & N & \xrightarrow{p} & L & \longrightarrow & 0.
\end{array}$$

The morphism $f' : P \to \mathcal{T}^M(L)$ exists since P is in $\sigma[M]$. By pseudo-projectivity of P in $\sigma[M]$, there exist $s : P \to P$ and $t : P \to \mathcal{T}^M(N)$ such that $sf' = tp' \neq 0$, hence $sf = sf'i_L = tp'i_L = ti_Np \neq 0$. Thus P is pseudo-projective in ${}_R\mathbf{M}$. □

1.9.6 $(\mathcal{F}_\tau, N)$**-projective.** Let τ be an idempotent preradical for $\sigma[M]$ and $P \in \sigma[M]$. P is called $(\mathcal{F}_\tau, N)$*-projective* if $\mathrm{Hom}(P, -)$ is exact on the exact sequences in $\sigma[M]$

$$0 \to K \to N \to L \to 0,$$

where $K \in \mathcal{F}_\tau$. P is called $\mathcal{F}_\tau$*-projective in* $\sigma[M]$ if it is $(\mathcal{F}_\tau, N)$-projective for all $N \in \sigma[M]$.

1.9.7 τ**-colocalization.** Let τ be an idempotent preradical with associated classes $(\mathcal{T}_\tau, \mathcal{F}_\tau)$. A homomorphism $g : P \to N$ in $\sigma[M]$ is called *a* τ*-colocalization of* N if $P \in \mathcal{T}_\tau$, P is $\mathcal{F}_\tau$-projective and Ker g and Coker g belong to $\mathcal{F}_\tau$.

Such a τ-colocalization is unique up to isomorphism. The existence of τ-colocalizations depend on the presence of enough projectives (see 6.25 of [6]).

1.9.8 Proposition. *Suppose that there is a projective generator in* $\sigma[M]$. *Then for an idempotent preradical* τ *for* $\sigma[M]$, *the following assertions are equivalent :*

(a) *Every module in* $\sigma[M]$ *has a* τ*-colocalization, i.e. there is a* τ*-colocalization functor from* $\sigma[M] \to \sigma[M]$.

(b) τ *is cohereditary, i.e.,* $\mathcal{F}_\tau$ *is cohereditary.*

The following property (see [6], Proposition 6.28) is needed to describe τ-colocalization.

1.9.9 Lemma. *Let P be an R-module and $S = \mathrm{End}_R(P)$. Consider $I = \mathrm{Tr}(P, R)$. For any $N \in {}_R\mathbf{M}$ consider the evaluation map*

$$\psi_N : P \otimes_S \mathrm{Hom}(P, N) \to N, \quad p \otimes f \mapsto (p)f.$$

Then $I \cdot \mathrm{Ker}\ \psi_N = 0$ and $I \cdot \mathrm{Coker}\ \psi_N = 0$.

The next theorem summarizes colocalization in ${}_R\mathbf{M}$ (see Proposition 6.29 of [6]).

1.9.10 Theorem. *Let τ be a cohereditary radical for ${}_R\mathbf{M}$. Then $I = \tau(R)$ is an idempotent ideal and*

$$\mathcal{T}_\tau = \{N \in {}_R\mathbf{M} \mid IN = N\}, \quad \mathcal{F}_\tau = \{N \in {}_R\mathbf{M} \mid IN = 0\}.$$

Let $P \in \mathcal{T}_\tau$ and $\mathcal{F}_\tau$-projective with $\mathrm{Tr}(P, R) = I$ and put $S = \mathrm{End}_R(P)$.

(i) *For any $N \in {}_R\mathbf{M}$,*

$$\psi_N : P \otimes_S \mathrm{Hom}(P, N) \to N, \; p \otimes f \mapsto (p)f,$$

is a τ-colocalization of N.

(ii) *$\psi_R : P \otimes_S P^* \to R$ is a τ-colocalization of R and is an (R, R)-bimodule morphism.*

(iii) *$\Lambda = P \otimes_S P^*$ has a ring structure (without unit) such that ψ_R is a ring morphism, and $\Lambda = \Lambda^2$.*

Notice that the τ-colocalization Λ of R can be used to describe the τ-colocalization in ${}_R\mathbf{M}$ (see [6], Proposition 6.30).

1.9.11 Lemma. *Let τ be a cohereditary radical for ${}_R\mathbf{M}$, $I = \tau(R)$ and $\mu : \Lambda \to R$ a τ-colocalization of R. Then for any $N \in {}_R\mathbf{M}$, the R-linear map*

$$\varphi : \Lambda \otimes_R N \to N \;, \; \sum \lambda_i \otimes n_i \mapsto \sum (\lambda_i)\mu n_i,$$

is a τ-colocalization of N.

The hereditary torsion theory in $\sigma[M]$, whose torsion free class is cogenerated by the injective hull $\widehat{M}$ of M in $\sigma[M]$, is called a *Lambek torsion theory*. The torsion class is

$$\mathcal{T}_{\tau_M} = \{N \in \sigma[M] \mid \mathrm{Hom}_R(N, \widehat{M}) = 0\},$$

that is the largest torsion class for which M is torsion free.

U is *M-rational in N* if and only if for any $U \subset V \subset N$, $\mathrm{Hom}_R(V/U, M) = 0$. M is called *polyform* if any essential submodule is rational in M. The dual notions are

1.9.12 M-corational modules. A module $X \in \sigma[M]$ is called *M-corational* if for all $Y \subset X$, $\mathrm{Hom}_R(M, X/Y) = 0$. M is called *copolyform* if any small submodule of M is M-corational. An epimorphism $f : M \to N$ is called an *M-corational cover* if Ker f is an M-corational module.

If Ker f is an M-corational module then Ker $f \ll M$. If M is pseudo-projective in $\sigma[M]$, then any $N \in \sigma[M]$ with $\mathrm{Hom}_R(M, N) = 0$ is an M-small module (see [6], 8.14).

The dual of the Lambek torsion theory has been studied, for example, by Lomp [21] and Talebi-Vanaja [32], [33].

We denote the class of M-corational modules in $\sigma[M]$ by

$$\mathcal{C}r_M = \{X \in \sigma[M] \mid \mathrm{Hom}_R(M, X/Y) = 0, \text{ for all } Y \subset X\}$$

and define the corresponding torsion class

$$\mathcal{C}r_M^\circ = \{N \in \sigma[M] \mid \mathrm{Hom}_R(N, X) = 0, \text{ for all } X \in \mathcal{C}r_M\}.$$

The classes $(\mathcal{C}r_M^\circ, \mathcal{C}r_M)$ can be characterized in the following way (compare with [6], Proposition 9.3).

1.9.13 Proposition. *Assume there is a small epimorphism $f : P \to M$ where $P \in \sigma[M]$ is pseudo-projective. Then:*

(i) *$(\mathcal{C}r_M^\circ, \mathcal{C}r_M)$ is a cohereditary torsion theory and $\mathcal{C}r_M$ is closed under products in $\sigma[M]$.*

(ii) *$\mathcal{C}r_M = \{X \in \sigma[M] \mid \mathrm{Hom}_R(P, X) = 0\}$, that is, for any module in $\sigma[M]$, M-corational is the same as P-corational.*

For any pseudo-projective module P in ${}_R\mathbf{M}$ and $I = \mathrm{Tr}(P, R)$, using Lemma 1.2 and Proposition 1.3 of [22], we can characterize the cohereditary torsion theory induced by $\mathrm{Tr}(P, -)$ as

$$\mathcal{C}r_P^\circ = \{N \in \sigma[M] \mid IN = N\}, \quad \mathcal{C}r_P = \{N \in \sigma[M] \mid IN = 0\}.$$

By Proposition 9.5 of [6], if $f : P \to M$ is a pseudo-projective cover in $\sigma[M]$, then $P/\mathrm{Tr}(P, \mathrm{Ker}\ f) \to M$ is a corational cover. Therefore, related to the cohereditary torsion theory $(\mathcal{C}r_M^\circ, \mathcal{C}r_M)$, we colocalize the module M in the following way (see [6], 9.16).

1.9.14 Proposition. *Let $f : P \to M$ be a projective cover in $\sigma[M]$. Then* $\mathrm{Tr}(P, -)$ *induces the M-corational torsion theory, $P/\mathrm{Tr}(P, \mathrm{Ker}\ f)$ is a $\mathcal{C}r_M$-projective module, and $P/\mathrm{Tr}(P, \mathrm{Ker}\ f) \to M$ is the* $\mathrm{Tr}(P, -)$*-colocalization of M.*

The following observation relates coprimeness with copolyform modules.

1.9.15 Proposition. *If the module M is endo-coprime and pseudo-projective, then M is copolyform.*

Proof. Assume M is endo-coprime, that is, for any fully-invariant submodule K, $\mathrm{Ann}_S(M/K) = 0$. Equivalently, $\mathrm{Hom}_R(M, K) = 0$. Since $\mathrm{Rad}(M)$ is a fully-invariant submodule of M, $\mathrm{Hom}_R(M, \mathrm{Rad}(M)) = 0$.

For any small submodule $L \ll M$, $L \subset \mathrm{Rad}(M)$. Thus $\mathrm{Hom}_R(M, L) = 0$ and by pseudo-projectivity of M, $\mathrm{Hom}_R(M, L/L') = 0$ for any $L' \subset L$, i.e., L is M-corational. □

1.9.16 Corollary. *Let M be self-projective and $S = \mathrm{End}_R(M)$.*

(i) *If M is fi-coretractable and S is prime, then M is copolyform.*

(ii) *If M is a self-injective self-cogenerator and fully coprime, then M is copolyform.*

(iii) *If M is a self-injective self-cogenerator and strongly coprime, then M is copolyform.*

Proof. (i) By 1.5.2 part (iii), M is endo-coprime. Now apply Proposition 1.9.15.

(ii) If M is a self-injective self-cogenerator, then endo-coprimeness is equivalent to fully coprimeness (see Corollary 1.7.12).

(iii) By Proposition 1.8.12, if M is self-injective and strongly coprime, then it is fully coprime. We apply the result (ii). □

For any modules M and N, we define :

$$\nabla(M, N) := \{f \in \mathrm{Hom}_R(M, N) \mid (M)f \ll N\}.$$

1.9.17 Proposition. *Let P be a projective module in $\sigma[M]$ and assume that* $\mathrm{Rad}(M) \neq M$. *If the module M is fully coprime or strongly coprime, then P is copolyform.*

Proof. By Proposition 1.7.15 and Proposition 1.8.10, $\mathrm{Rad}(P) = 0$. By 9.26 of [6], $\mathrm{Jac}(\mathrm{End}_R(P)) = \nabla(P, P) = 0$. Therefore P is copolyform. □

1.9.18 Proposition. *Let $p : P \to M$ be a projective hull in $\sigma[M]$.*

(i) *If M is endo-coprime and pseudo-projective, then P is copolyform.*

(ii) *If M is strongly coprime, then $M \simeq P$.*

(iii) *If P is M-generated and M is fully coprime, then $R/\mathrm{Ann}_R(P)$ is a left primitive ring.*

Proof. (i) By Proposition 1.9.15, M is copolyform. Now apply 9.26 of [6] to get P is copolyform.

(ii) By Proposition 1.8.11 part (ii).

(iii) By Proposition 1.7.16 P is also fully coprime (e.g. coprime). Then we apply Proposition 1.3.10. □

Chapter 2

Prime and Coprime Comodules

In this chapter R will always denote a commutative ring. Let C be an R-coalgebra and $C^* = \mathrm{Hom}_R(C, R)$. Any right C-comodule M can be considered as a left C^*-module and there is a faithful functor from the category of right C-comodules $\mathbf{M}^C$ to the category of left C^*-modules ${}_{C^*}\mathbf{M}$. We need the α-condition on C to make $\mathbf{M}^C = \sigma[{}_{C^*}C]$ a full subcategory of ${}_{C^*}\mathbf{M}$. This gives us the possibility to apply the results of primeness and coprimeness in the category $\sigma[M]$ to the category $\mathbf{M}^C$.

We transfer the various notions of primeness and coprimeness for modules to comodules. These definitions extend the definitions of primeness of comodules and coalgebras which have been studied by several authors (for example Rodrigues [29] and Ferrero-Rodrigues[11]). In particular we study the coprimeness of the coalgebra itself. It is a generalization of the study of coprime coalgebras over fields in Xu et.al [41], Nekooei-Torkzadeh [26] and Jara et.al. [17].

The basic definitions and properties of comodules and coalgebras will be given in the first part of this section. The theory of coalgebras over fields and their comodules is discussed in various text books (see for example Abe [1], Montgomery [23], Sweedler [31] and Dăscălescu et.al. [7]). The study of coalgebras over commutative rings is presented in Brzeziński-Wisbauer [5].

2.1 Preliminaries

2.1.1 Coalgebras. Throughout R denotes a commutative and associative ring with unit. An *R-coalgebra* is an R-module C with R-linear maps

$$\Delta : C \to C \otimes_R C \quad \text{and} \quad \varepsilon : C \to R$$

called (*coassosiative*) *coproduct* and *counit* respectively, with the properties

$$(I_C \otimes \Delta) \circ \Delta = (\Delta \otimes I_C) \circ \Delta \quad \text{and} \quad (I_C \otimes \varepsilon) \circ \Delta = I_C = (\varepsilon \otimes I_C) \circ \Delta.$$

We use Sweedler's notation, that is for any $c \in C$, we write $\Delta(c) = \sum c_{\underline{1}} \otimes c_{\underline{2}}$.

A coalgebra (C, Δ, ε) is said to be *cocommutative* if $\Delta = \mathrm{tw} \circ \Delta$, where

$$\mathrm{tw} : C \otimes_R C \to C \otimes_R C \ , \ a \otimes b \mapsto b \otimes a,$$

is the twist map.

For any coalgebra C and R-algebra A, $\mathrm{Hom}_R(C, A)$ is an R-algebra by the convolution product, that is for any $f, g \in \mathrm{Hom}_R(C, A)$ and $c \in C$ define

$$f * g = \mu \circ (f \otimes g) \circ \Delta, \quad \text{i.e.} \quad f * g(c) = \sum f(c_{\underline{1}}) g(c_{\underline{2}}).$$

In particular, $C^* = \mathrm{Hom}_R(C, R)$ is called the *dual algebra* of C.

2.1.2 Example. Let (A, μ, ι) be an R-algebra and assume ${}_RA$ to be finitely generated and projective. Then there is an isomorphism

$$A^* \otimes_R A^* \to (A \otimes_R A)^*, \quad f \otimes g \mapsto [a \otimes b \mapsto f(a)g(b)]$$

and the functor $\mathrm{Hom}_R(-, R) = (-)^*$ yields a coproduct

$$\mu^* : A^* \to (A \otimes_R A)^* \simeq A^* \otimes_R A^*,$$

and a counit (as the dual of the unit of A)

$$\varepsilon := \iota^* : A^* \to R \ , \ f \mapsto f(1_A).$$

This makes A^* an R-coalgebra which is cocommutative provided μ is commutative. Notice that $(A^*)^* = A$. If A is prime and a faithful R-module, then the ground ring R is prime (see Lemma 1.1.2).

2.1.3 Coalgebra morphisms. Let (C, Δ, ε) and $(C', \Delta', \varepsilon')$ be R-coalgebras. An R-linear map $f : C \to C'$ is said to be a *coalgebra morphism* provided

$$\Delta' \circ f = (f \otimes f) \circ \Delta \quad \text{and} \quad \varepsilon' \circ f = \varepsilon,$$

i.e. for all $c \in C$,

$$\sum f(c_{\underline{1}}) \otimes f(c_{\underline{2}}) = \sum f(c)_{\underline{1}} \otimes f(c)_{\underline{2}} \quad \text{and} \quad \varepsilon' \circ f(c) = \varepsilon(c).$$

2.1.4 Kernel of coalgebra morphisms. The kernel of a coalgebra map $f : C \to C'$ is related to the kernel of $f \otimes f$ in the category of R-modules $\mathbf{M}_R$. If f is surjective,

$$\mathrm{Ker}\ (f \otimes f) = \mathrm{Ker}\ f \otimes_R C + C \otimes_R \mathrm{Ker}\ f \subset C \otimes_R C.$$

The kernel of a surjective coalgebra morphism $f : C \to C'$ is called a *coideal* of C.

2.1.5 Comodules over coalgebras. A *right C-comodule* is an R-module M with an R-linear map $\varrho^M : M \to M \otimes_R C$ called a *right C-coaction* , with the properties

$$(I_M \otimes \Delta) \circ \varrho^M = (\varrho^M \otimes I_C) \circ \varrho^M \text{ and } (I_M \otimes \varepsilon) \circ \varrho^M = I_M.$$

To denote the action of ϱ^M on elements of M we write $\varrho^M(m) = \sum m_{\underline{0}} \otimes m_{\underline{1}}$.

2.1.6 Comodule morphisms. Let M and N be right C-comodules. An R-linear map $f : M \to N$ is called a *comodule morphism* or a *morphism of right C-comodules* if

$$\varrho^M \circ f = (f \otimes I_C) \circ \varrho^M.$$

Denote by $\mathrm{Hom}^C(M, N)$ the set of C-comodule morphisms from M to N.

The class of right comodules over C together with the comodule morphisms form an additive category which is denoted by $\mathbf{M}^C$.

2.1.7 Subcomodules. An R-submodule K of a right C-comodule M is called a *C-subcomodule* of M provided K has a right comodule structure such that the inclusion is a comodule morphism. In general, the fact that K is an R-submodule of M does not imply that $K \otimes_R C$ is a submodule of $M \otimes_R C$, since the tensor functor is not left exact. However, if K is a C-pure R-submodule of M then $K \otimes_R C \subset M \otimes_R C$, and K is a subcomodule of M provided $\varrho^M(K) \subset K \otimes_R C \subset M \otimes_R C$.

2.1.8 Subcoalgebras. An R-submodule $D \subset C$ is called a *subcoalgebra* provided D has a coalgebra structure and the inclusion map is a coalgebra morphism. Notice that a pure R-submodule $D \subset C$ is a sub-coalgebra provided

$$\Delta(D) \subset D \otimes_R D \subset C \otimes_R C.$$

The restriction of the counit ε to D becomes a counit for D.

A pure submodule D of C is a subcoalgebra if and only if it is a left and right C-subcomodule, since in this case (see Intersection property 40.16 of [5])

$$\Delta(D) \subset D \otimes_R C \cap C \otimes_R D = D \otimes_R D \subset C \otimes_R C,$$

so that D has a coalgebra structure for which the inclusion is a coalgebra morphism as required.

2.1.9 Semisimple comodules. A right C-comodule M is called *semisimple* in $\mathbf{M}^C$ if every C-monomorphism $N \to M$ is a coretraction, and M is called *simple* if all these monomorphisms are isomorphisms. Proposition 4.13 of [5] shows that if we assume that C is flat as an R-module, any comodule M is simple if and only if M has no nontrivial subcomodules. Furthermore, M is semisimple if and only if every subcomodule of M is a direct summand.

2.1.10 Projective comodules. A comodule P is *projective* in $\mathbf{M}^C$ if for any epimorphism $M \to N$ in $\mathbf{M}^C$, the canonical map $\mathrm{Hom}^C(P, M) \to \mathrm{Hom}^C(P, N)$ is surjective. There is a relationship between the projectivity of a comodule and its projectivity as an R-module (see Lemma 3.22 of [5]): If P is projective in $\mathbf{M}^C$, then P is projective in $\mathbf{M}_R$. If ${}_RC$ is flat, P is projective in $\mathbf{M}^C$ if and only if $\mathrm{Hom}^C(P, -) : \mathbf{M}^C \to \mathbf{M}_R$ is exact.

2.1.11 Hom-tensor relations. Similar to the classical Hom-tensor relations (e.g. [38]), there are Hom-tensor relations in $\mathbf{M}^C$ (see [5], 3.10). For any $M \in \mathbf{M}^C$ the R-linear map

$$\phi : \mathrm{Hom}^C(M, X \otimes_R C) \to \mathrm{Hom}_R(M, X), \;\; f \mapsto (I_X \otimes \varepsilon) \circ f,$$

is bijective, with inverse map $h \mapsto (h \otimes I_X) \circ \rho^M$.

For $X = R$ and $M = C$ the isomorphism ϕ describes the comodule endomorphisms of C. There is an algebra *anti-isomorphim*

$$\phi : \mathrm{End}^C(C) \to C^*, f \mapsto \varepsilon \circ f$$

with inverse map $h \mapsto (h \otimes I_C) \circ \Delta$, and $h \in C^*$ acts on $c \in C$ from the right by

$$c \leftharpoonup h = (h \otimes I_C) \circ \Delta(c) = \sum h(c_{\underline{1}})c_{\underline{2}}.$$

Moreover, there is an algebra *isomorphim*

$$\phi' : \ {}^C\mathrm{End}(C) \to C^*, f \mapsto \varepsilon \circ f$$

with inverse map $h \mapsto (I_C \otimes h) \circ \Delta$, and $h \in C^*$ acts on $c \in C$ from the left by

$$h \rightharpoonup c = (I_C \otimes h) \circ \Delta(c) = \sum c_{\underline{1}} h(c_{\underline{2}}).$$

2.1.12 C-comodules as C^*-modules. Any coalgebra C can be considered as a right C-comodule, and there is a close relationship between comodules of C and modules of C^*.

Any $M \in \mathbf{M}^C$ is a (unital) left C^*-module by

$$\rightharpoonup : C^* \otimes_R M \to M, \quad f \otimes m \mapsto (I_M \otimes f) \circ \varrho^M(m) = \sum m_{\underline{0}} f(m_{\underline{1}})$$

and any morphism $h : M \to N$ in $\mathbf{M}^C$ is a left C^*-module morphism, i.e.,

$$\mathrm{Hom}^C(M,N) \subset {}_{C^*}\mathrm{Hom}(M,N).$$

C is a subgenerator in $\mathbf{M}^C$, that is all C-comodules are subgenerated by C as C-comodules and C^*-modules.

Denote by ${}_{C^*}\mathbf{M}$ the category of left C^*-modules. The above observation shows that there is a faithful functor from $\mathbf{M}^C$ to ${}_{C^*}\mathbf{M}$. $\mathbf{M}^C$ is a full subcategory of ${}_{C^*}\mathbf{M}$ when all the (left) C^*-linear maps between right C-comodules arise from (right) C-comodule morphisms, or when $\mathbf{M}^C$ is isomorphic to $\sigma[{}_{C^*}C]$, that is the full subcategory of left C^*-modules whose objects are ${}_{C^*}C$-subgenerated.

C is said to satisfy the α-condition if the map

$$\alpha_N : N \otimes_R C \to \mathrm{Hom}_R(C^*, N), \quad n \otimes c \mapsto [f \mapsto f(c)n],$$

is injective for every $N \in \mathbf{M}_R$.

2.1.13 Proposition. ([5], 4.2) *The following are equivalent for an R-coalgebra C :*

(a) *C satisfies the α -condition.*
(b) *For any $N \in \mathbf{M}_R$ and $u \in N \otimes_R C$, $(I_N \otimes f)(u) = 0$ for all $f \in C^*$, implies $u = 0$.*
(c) *C is locally projective as an R-module.*

In particular this implies that C is a flat R-module and cogenerated by R.

In the category of comodules, the importance of the α-condition becomes clear from the following observation.

2.1.14 Proposition. ([5], 4.3) *The following are equivalent :*

(a) $\mathbf{M}^C = \sigma[{}_{C^*}C]$.

(b) $\mathbf{M}^C$ *is a full subcategory of* ${}_{C^*}\mathbf{M}$.

(c) *for all* $M, N \in \mathbf{M}^C$, $\mathrm{Hom}^C(M,N) = {}_{C^*}\mathrm{Hom}(M,N)$.

(d) ${}_RC$ *is locally projective.*

2.1.15 (C^*, C^*)**-bimodules.** Since C is a left and right C-comodule, we can study the structure of C as a (C^*, C^*)-bimodule by

$$\begin{aligned} \rightharpoonup &: C^* \otimes_R C \to C, \quad f \otimes c \mapsto f \rightharpoonup c = (I_C \otimes f) \circ \Delta(c) \\ \leftharpoonup &: C \otimes_R C^* \to C, \quad c \otimes g \mapsto c \leftharpoonup g = (g \otimes I_C) \circ \Delta(c), \end{aligned}$$

where $f, g \in C^*$, $c \in C$. The multiplication between two elements in C^* has the following properties :

$$f * g(c) = f(g \rightharpoonup c) = g(c \leftharpoonup f).$$

As a left and right C^*-module, C is faithful. If C is locally projective, then C is a balanced (C^*, C^*)-bimodule, i.e., writing right comodule morphisms on the right side,

$${}_{C^*}\mathrm{End}(C) = \mathrm{End}^C(C) \simeq C^* \simeq {}^C\mathrm{End}(C) = \mathrm{End}_{C^*}(C) \qquad (2.1)$$

In this case a pure R-submodule $D \subset C$ is a subcoalgebra if and only if D is a left and right C^*-submodule.

Let the coalgebra C satisfy the α-condition and let $f : M \to N$ be a comodule morphism. If L is a subcomodule of N, then $(L)f^{-1}$ is a subcomodule of M. If U is a subcomodule of M, then $(U)f$ is a subcomodule of N. By these facts, Lemma 1.1.20 yields :

2.1.16 Lemma. *Let* M, N *be* C*-comodules,* $f \in \mathrm{Hom}^C(M,N)$ *an epimorphism and assume the coalgebra* C *satisfies the* α*-condition.*

(i) *If* $\mathrm{Ker}\, f$ *is a fully invariant subcomodule of* M *and* L *is a fully-invariant subcomodule of* N*, then* $(L)f^{-1}$ *is a fully-invariant subcomodule of* M.

(ii) *If* M *is self-projective and* U *is a fully invariant subcomodule of* M*, then* $(U)f$ *is a fully-invariant subcomodule of* N.

Throughout C will be assumed to satisfied the α-condition.

2.2 Annihilator conditions for comodules and coalgebras

As for modules, we define the following annihilators for comodules. For a C-subcomodule $K \subseteq M$ and an ideal $I \subseteq S := \text{End}^C(M)$, set

$$\begin{aligned} \text{Ann}_S(K) &:= \{f \in S \mid (K)f = 0\} = \pi_K \diamond \text{Hom}^C(M/K, M) \\ \text{Ker } I &:= \bigcap \{\text{Ker } f \mid f \in I\}, \end{aligned}$$

where $\pi_K : M \to M/K$ is the canonical projection comodule homomorphism.

By Lemma 1.1.9 we get

2.2.1 Lemma. *Let M be a C-comodule and $S = \text{End}^C(M)$.*

(i) *For any C-subcomodule $K \subseteq M$,*

$$\text{Ker Ann}_S(K) = \text{Ker } \pi_K \diamond \text{Hom}^C(M/K, M) = K$$

if and only if M is a self-cogenerator comodule.

(ii) *If M is self-injective, then for every finitely generated right ideal $I \subseteq S$,*

$$\text{Hom}^C(M/\text{Ker } I, M) = I.$$

(iii) *If C is self-injective, then for any finitely generated right ideal I of the dual algebra* $\text{End}^C(C) \simeq C^*$,

$$\text{Hom}_R(C/\text{Ker } I, R) \simeq \text{Hom}^C(C/\text{Ker } I, C) = I.$$

Recall dual orthogonal (e.g. 1.1.10) and notice that $A \subset (A^{\perp C^*})^{\perp C}$ always holds. To get the equality, we apply the properties of annihilators ([38], 28.1).

2.2.2 Lemma. *If every factor module of C is R-cogenerated, then for any R-submodule A of C,* $\text{Ker Hom}_R(C/A, R) = A$. *Thus $A = (A^{\perp C^*})^{\perp C}$.*

For any subsets I, J of C^* with $I \subseteq J$, obviously $J^{\perp C} \subseteq I^{\perp C}$. The following lemma uses this fact and Lemma 2.2.2.

2.2.3 Lemma. *Let C be any R-module and C^* its dual module. If every factor module of C is R-cogenerated, then for any R-submodules A, B of C, $A \subseteq B$ if and only if $B^{\perp C^*} \subseteq A^{\perp C^*}$.*

The properties of the annihilator and the kernel are given in the two lemmas below.

2.2.4 Lemma. ([5], 6.2) *Let $A \subset C$ be an R-submodule.*

(i) *If A is a left C^*-submodule of C, then $A^{\perp C^*}$ is a right C^*-submodule.*

(ii) *If A is a (C^*, C^*)-subbimodule of C, then $A^{\perp C^*}$ is an ideal in C^*.*

(iii) *If A is a coideal of C, then $A^{\perp C^*}$ is a subalgebra of C^*.*

2.2.5 Lemma. ([5], 6.3) *Let $I \subset C^*$ be an R-submodule.*

(i) *If I is a right (left) ideal in C^*, then $I^{\perp C}$ is a left (right) C^*-submodule of C.*

(ii) *If I is an ideal in C^*, then $I^{\perp C}$ is a (C^*, C^*)-subbimodule of C.*

Assume R to be a semisimple ring. Then

(i) *If I is a subalgebra, then $I^{\perp C}$ is a coideal.*

(ii) *$A \subset C$ is a coideal if and only if $A^{\perp C^*}$ is a subalgebra.*

We need the following well-known fact from Linear Algebra to see properties of the wedge product to be defined later on.

2.2.6 Lemma. *Let M, N be vector spaces over a field k and $u \in M \otimes_k N$. Denote $M^* = \mathrm{Hom}_k(M, k)$ and $N^* = \mathrm{Hom}_k(N, k)$. The following assertions are equivalent :*

(a) $M^* \otimes_k N^*(u) = 0$.

(b) $(M \otimes_k N)^*(u) = 0$.

(c) $u = 0$.

For subsets $A \subseteq C$ and $I \subseteq C^*$, putting

$$\begin{aligned} \mathrm{Ann}_{C^*}(A) &:= \{f \in C^* \mid f \rightharpoonup A = 0\} \\ \mathrm{Ann}_C(I) &:= \{c \in C \mid I \rightharpoonup c = 0\}, \end{aligned}$$

we have the following relationship :

2.2.7 Lemma. *Let C be a coalgebra and $A \subset C$.*

(i) *If A is an R-submodule of C, then $\mathrm{Ann}_{C^*}(A) \subset A^{\perp C^*}$.*

(ii) *If A is a left and right subcomodule (a left and right C^*-submodule) of C, then $A^{\perp C^*} = \mathrm{Ann}_{C^*}(A)$.*

Proof. (i) Let $g \in \mathrm{Ann}_{C^*}(A)$. Then $g \rightharpoonup a = 0$ for any $a \in A$ and $(I_C \otimes g) \circ \Delta(a) = 0$. Applying ε we obtain

$$\begin{aligned} 0 &= \varepsilon((I_C \otimes g) \circ \Delta(a)) = \sum \varepsilon(a_{\underline{1}} g(a_{\underline{2}})) \\ &= \sum \varepsilon(a_{\underline{1}}) g(a_{\underline{2}}) = g \sum (\varepsilon(a_{\underline{1}}) a_{\underline{2}}) = g(a). \end{aligned}$$

Thus $\mathrm{Ann}_{C^*}(A) \subset A^{\perp C^*}$.

(ii) It is sufficient to prove that $A^{\perp C^*} \subset \mathrm{Ann}_{C^*}(A)$. Take any $f \in A^{\perp C^*}$, that is $f(a) = 0$ for any $a \in A$. Then

$$f \rightharpoonup a = \sum (I_C \otimes_R f) \circ \Delta(a) = \sum a_{\underline{1}} f(a_{\underline{2}}) = 0,$$

since $\Delta(a) \in C \otimes_R A$. Thus $A^{\perp C^*} \subset \mathrm{Ann}_{C^*}(A)$. □

Notice that for any subcoalgebra A of C holds $A^{\perp C^*} = \mathrm{Ann}_{C^*}(A)$, since by 2.1.7, A is a left and right subcomodule of C.

2.2.8 Lemma. *Let C be a coalgebra and $J \subset C^*$.*

(i) *If J is an R-submodule of C^*, then $\mathrm{Ann}_C(J) \subset J^{\perp C}$.*

(ii) *If J is a two-sided ideal of C^*, then $J^{\perp C} = \mathrm{Ann}_C(J)$.*

Proof. (i) Let $c \in \mathrm{Ann}_C(J)$. Then $h \rightharpoonup a = 0$ for any $h \in J$ and $(I_C \otimes h) \circ \Delta(c) = 0$. Applying ε we obtain

$$\begin{aligned} 0 &= \varepsilon((I_C \otimes h) \circ \Delta(c)) = \sum \varepsilon(c_{\underline{1}} h(c_{\underline{2}})) \\ &= \sum \varepsilon(c_{\underline{1}}) h(c_{\underline{2}}) = h \sum (\varepsilon(c_{\underline{1}}) c_{\underline{2}}) = h(c). \end{aligned}$$

Thus $\mathrm{Ann}_C(J) \subset J^{\perp C}$.

(ii) Notice that by Lemma 2.2.5 part (ii), $J^{\perp C}$ is a (C^*, C^*)-subbimodule of C. It is sufficient to prove that $J^{\perp C} \subset \mathrm{Ann}_C(J)$. Take any $c \in J^{\perp C}$, that is $l(c) = 0$ for any $l \in J$. Then

$$l \rightharpoonup c = \sum (I_C \otimes_R l) \circ \Delta(c) = \sum c_{\underline{1}} l(c_{\underline{2}}) = 0,$$

since $\Delta(c) \in C \otimes_R J^{\perp C}$. Thus $J^{\perp C} \subset \mathrm{Ann}_C(J)$. □

2.2.9 Lemma. *For any proper (C^*, C^*)-subbimodule A of C, if C/A is R-cogenerated, then $\mathrm{Ann}_{C^*}(A) \neq 0$ and equivalently C satisfies condition $(**)$ as a C^*-module. Thus if R is a cogenerator in ${}_R\mathbf{M}$, then $\mathrm{Ann}_{C^*}(A) \neq 0$.*

Proof. Since C/A is R-cogenerated, there is some $\tilde{f} : C/A \to R$ such that $0 \neq f := \pi_A \diamond \tilde{f} : C \to R$. Thus $\mathrm{Ann}_{C^*}(A) \neq 0$. □

2.3 Prime comodules

The study of prime and strongly prime comodules has been investigated by Rodrigues [29] mainly for coalgebras over a field k. The definition of prime comodules over an R-coalgebra C is the following.

2.3.1 Definition. A right C-comodule M is called *prime* if M is prime as a left C^*-module.

From 1.2.2 we have characterizations of prime comodules.

2.3.2 Prime comodules. *For a right C-comodule M, the following are equivalent :*

(a) *M is a prime comodule.*

(b) $\mathrm{Ann}_{C^*}(K) = \mathrm{Ann}_{C^*}(M)$ *for any subcomodule K of M.*

(c) $C^*/\mathrm{Ann}_{C^*}(M)$ *is cogenerated by K for any subcomodule K of M.*

(d) $C^*/\mathrm{Ann}_{C^*}(M)$ *is cogenerated by K for any fully invariant subcomodule K of M.*

If these conditions hold, then :

(i) *$\overline{C^*}$ is a prime algebra which is finitely generated as an R-module.*

(ii) *$R/\mathrm{Ann}_R(\overline{C^*})$ is a prime ring.*

If M is a faithful C^*-module, then the conditions (a)-(d) are also equivalent to the injective hull $E(M)$ of M in ${}_{C^*}\mathbf{M}$ being a prime C^*-module.

Proof. Transfer from Lemma 1.2.10.

(i) Consider a cyclic $\overline{C^*}$-submodule of M, say $U := \overline{C^*} \rightharpoonup m$, for some $m \in M$. According to the Finiteness Theorem 2 ([5],4.12), U is a finitely generated R-module with generators say $\{u_1, \ldots, u_k\}$. Define a mapping

$$\varphi : \overline{C^*} \to U^k, \quad f \mapsto (f \rightharpoonup u_1, \ldots, f \rightharpoonup u_1) = f \rightharpoonup (u_1, \ldots, u_k) \in U^k.$$

If $f \rightharpoonup (u_1, \ldots, u_k) = 0$, then $f \rightharpoonup u_i = 0$ and $f \rightharpoonup (r_1u_1 + \ldots + r_ku_k) = 0$ for any $r_i \in R$, $i = 1, 2, \ldots, k$. Thus $f \rightharpoonup U = 0$, i.e., $f \in \mathrm{Ann}_{\overline{C^*}}(U) = 0$, since M is prime. It means that the map φ is a monomorphism and $\overline{C^*}$ is a submodule of a finitely generated R-module and moreover, $\overline{C^*}$ is a comodule. Thus $\overline{C^*}$ is a finitely generated C^*-module and, by the Finiteness Theorem 2, is a finitely generated R-module.

(ii) By Lemma 1.1.2. □

The following is the adopted version of Proposition 1.2.3.

2.3.3 Proposition. *Let M be a comodule which satisfies condition $(*fi)$ as C^*-module, $S = \mathrm{End}^C(M)$ and $\overline{C^*} = C^*/\mathrm{Ann}_{C^*}(M)$.*

(i) *M is prime if and only if $\overline{C^*}$ is a prime ring.*

(ii) *If M is prime then M_S is prime (and S is a prime ring).*

2.3.4 Proposition. *Let M be a comodule with $\mathrm{Soc}(M) \neq 0$. If M is prime, then $\overline{C^*}$ is a simple artinian algebra and finitely generated as an R-module. Thus M is a homogeneous semisimple comodule.*

Proof. Consider a simple $\overline{C^*}$-submodule of M, say $V := \overline{C^*} \rightharpoonup m$, for some $m \in M$. Then by 2.3.2 part (i), $\overline{C^*}$ is a direct summand of the homogeneous semisimple C^*-module V^k, i.e., $\overline{C^*}$ is a simple artinian algebra.

Notice that if M is a faithful C^*-module, then C^* is a simple artinian algebra and finitely generated as an R-module. □

Proposition 3.2 of [11] is a corollary of our Proposition 2.3.4.

2.3.5 Corollary. *If R is a perfect ring and M is a prime comodule over the R-coalgebra C, then $C^*/\mathrm{Ann}_{C^*}(M)$ is a simple artinian algebra.*

Proof. If R is a perfect ring, then M satisfies the descending chain condition for finitely generated R-submodules. The Finiteness Theorem then implies the descending chain condition on finitely generated C-subcomodules (see [5], 4.16). Thus $\mathrm{Soc}(M) \neq 0$, then apply Proposition 2.3.4. □

2.3.6 Prime coalgebras. *For a coalgebra C, the following are equivalent :*

(a) *C is prime as a right C-comodule.*

(b) *$\mathrm{Ann}_{C^*}(A) = 0$ for any non-zero right subcomodule A of C.*

(c) *C^* is cogenerated by A for any non-zero right subcomodule A of C.*

(d) *C^* is cogenerated by A for any non-zero (C^*, C^*)-subbimodule A of C.*

(e) *The injective hull $\widehat{C}$ of C in ${}_{C^*}\mathbf{M}$ is a prime C^*-module.*

If these conditions hold, then :

(i) *C is a finitely generated R-module and $\mathbf{M}^C = {}_{C^*}\mathbf{M}$.*

(ii) *Every C-cogenerated comodule is prime.*

(iii) *Every projective comodule P is prime.*

(iv) *C^* is a prime algebra and finitely generated as R-module.*

Proof. Considering C as a right C-comodule we get the equivalences and (iv) from 2.3.2.

(i) By 2.3.2, $C^* \in \mathbf{M}^C$ and hence C is finitely generated as R-module. Now apply 4.7 of [5].

(ii)-(iii) Transfer the situation of Proposition 1.2.7 into $\mathbf{M}^C$. □

The following is a special case of Proposition 2.3.3.

2.3.7 Proposition. *Let C be a coalgebra and for any non-zero (C^*, C^*)-subbimodule A of C, $\mathrm{Ann}_{C^*}(C/A) \neq 0$.*

(i) *C is prime as right C-comodule if and only if C^* is prime.*

(ii) *If C is prime as right C-comodule, then C is prime as left C-comodule (and C^* is prime).*

2.3.8 Proposition. *Let C be a coalgebra and prime as right C-comodule. If $\mathrm{Soc}(C) \neq 0$, then*

(i) *C^* is a simple algebra and finitely generated as R-module.*

(ii) *If C is cocommutative, then C^* is a field.*

Proof. (i) By Proposition 2.3.4. (ii) By Corollary 1.2.12. □

If C is prime as right C-comodule, then for any fully invariant subbicomodule A of C, $\mathrm{Ann}_{C^*}(A) = A^{\perp C^*} = 0$ (see Lemma 2.2.7).

2.3.9 Example. Let R be a commutative ring. Consider the free R-module $T := R^n$, where $n \in \mathbb{N}$, T is a finitely generated and projective R-module. $C := T^* \otimes_R T$ is a coalgebra (see [9]) and moreover, as R-module,

$$C = (R^n)^* \otimes_R R^n \simeq \mathrm{End}_R(R^n_R),$$

that is the matrix coalgebra of all $n \times n$ matrices over R, which we denote as $\mathbb{M}^n(R)$. Let $\{e_{ij}\}_{1\leq i,j\leq n}$ be the canonical basis of $\mathbb{M}^n(R)$. Then the coproduct and counit of C are

$$\begin{aligned} \Delta : \mathbb{M}^n(R) &\rightarrow \mathbb{M}^n(R) \otimes_R \mathbb{M}^n(R), \; e_{ij} \mapsto \sum_{k=1}^{n} e_{ik} \otimes_R e_{kj}. \\ \varepsilon : \mathbb{M}^n(R) &\rightarrow R, \;\; e_{ij} \mapsto \delta_{ij}. \end{aligned}$$

For the dual algebra of C, there are anti-algebra morphisms :

$$C^* \simeq \mathrm{End}_R((R^n)^*_R) \simeq \mathrm{End}_R((R^*)^n_R) \simeq \mathbb{M}_n(\mathrm{End}_R(R^*_R)) \simeq \mathbb{M}_n(R),$$

the matrix ring of all $n \times n$ matrices over R. Thus if R is prime then C^* is a prime algebra.

The fully invariant subcomodules of $C = \mathbb{M}^n(R)$ are (C^*, C^*)-subbimodules, that is the two-sided ideals of $\mathbb{M}_n(R)$, and hence are of the form $\mathbb{M}_n(I)$, where I is an ideal of R.

Since $\mathrm{Ann}_{\mathbb{M}_n(R)}(\mathbb{M}_n(R)/\mathbb{M}_n(I)) \neq 0$, C satisfies $(*fi)$, and thus C is prime as C^*-module (by Proposition 1.2.3 (ii)).

2.4 Coprime comodules

2.4.1 Definition. A right C-comodule M is called *coprime* if M is coprime as a C^*-module.

Here 1.3.2 and Lemma 1.3.3 read as follows.

2.4.2 Coprime comodules. *Let M be a right C-comodule with $S = \mathrm{End}^C(M)$.*

(i) *The following assertions are equivalent :*
   - (a) *M is a coprime comodule.*
   - (b) $\mathrm{Ann}_{C^*}(M/K) = \mathrm{Ann}_{C^*}(M)$ *for any proper subcomodule K of M.*
   - (c) $C^*/\mathrm{Ann}_{C^*}(M)$ *is cogenerated by M/K for any proper subcomodule K of M.*
   - (d) $C^*/\mathrm{Ann}_{C^*}(M)$ *is cogenerated by M/K for any proper fully invariant subcomodule K of M.*

(ii) *If M is coprime, then $C^*/\mathrm{Ann}_{C^*}(M)$ is prime.*

(iii) *If $C^*/\mathrm{Ann}_{C^*}(M)$ is prime and for any proper (fully invariant) subcomodule K of M holds $\mathrm{Ann}_{C^*}(K) \neq \mathrm{Ann}_{C^*}(M)$, then M is coprime.*

By Proposition 1.3.8 and Lemma 1.3.9, factor comodules and the direct sums of copies of a coprime comodule are again coprime.

2.4.3 Proposition. *Let M be a right C-comodule*

(i) *If M is coprime and K is a proper subcomodule of M, then M/K is coprime.*

(ii) $M^{(\Lambda)}$ *is coprime if and only if* M *is coprime.*

(iii) *If* M *is coprime and* $\mathrm{Rad}(M) \neq M$, *then* $\overline{C^*} := C^*/\mathrm{Ann}_{C^*}(M)$ *is a simple algebra and finitely generated as* R*-module.*

Proof. (iii) By Proposition 1.3.10, $\overline{C^*} := C^*/\mathrm{Ann}_{C^*}(M)$ is a primitive algebra. Then the proof is similar to 2.3.4. □

Applying 2.4.2 to a coalgebra C yields the first part of the next observation.

2.4.4 Coprime coalgebras. *Let* C *be a coalgebra.*

(1) *The following are equivalent :*

(a) C *is coprime as a right* C*-comodule.*

(b) $\mathrm{Ann}_{C^*}(C/A) = 0$ *for any proper right subcomodule* A *of* C.

(c) C^* *is cogenerated by* C/A *for any proper right subcomodule* A *of* C.

(d) C^* *is cogenerated by* C/A *for any proper* (C^*, C^*) *subbimodule* A *of* C.

(2) *If the conditions (a)-(d) hold, then :*

(i) C^* *is prime.*

(ii) *If* C *is cocommutative, then* C^* *is an integral domain.*

(iii) *For any proper fully-invariant* (C^*, C^*)*-subbimodule* A *of* C, C/A *is coprime as* C^**-module.*

(iv) $\mathrm{Rad}(C) \neq C$ *implies* C^* *is a simple algebra and finitely generated as* R*-module.*

(v) *If* C *is cocommutative with* $\mathrm{Rad}(C) \neq C$, *then* C^* *is a field.*

(3) *If* C^* *is prime and for any proper* (C^*, C^*)*-subbimodule* A *of* C *holds* $\mathrm{Ann}_{C^*}(A) \neq 0$, *then* C *is coprime as a right* C*-comodule.*

Proof. 2(i) and 3 follow from 2.4.2.
(ii) C^* is prime and commutative.
(iii) and (iv) follow Proposition 2.4.3.
(v) is a consequence of (iv). □

By Proposition 1.3.11 we have

2.4.5 Proposition. *Let* $p : P \to C$ *be a small epimorphism in* $\mathbf{M}^C$. *If* C *is coprime as comodule, then* P *is a coprime comodule.*

A coalgebra C with C^* prime may not be coprime if it does not satisfy the condition $(**)$, i.e., for any proper (C^*, C^*)-subbimodule A of C holds $\mathrm{Ann}_{C^*}(A) \neq 0$. We can see this in the following example.

2.4.6 Example. Divided power coalgebra. Let H be a free R-module with basis $\{c_m \mid m \in \mathbb{N}\}$. Define the comultiplication

$$\Delta : H \to H \otimes_R H, \quad c_m \mapsto \sum_{i=0,m} c_i \otimes c_{m-i}$$

and the counit by $\varepsilon(c_m) = \delta_{0,m}$. H is a coalgebra and the dual algebra H^* has multiplication for $f, g \in H^*$,

$$(f * g)(c_m) := \sum_{i=0,m} f(c_i)g(c_{m-i})$$

and unit $u : R \to H^*$ where $u(\alpha)(c_m) = \alpha\delta_{0,m}$ for any $\alpha \in R$, $m \in \mathbb{N}$.

There is an isomorphism (see Example 1.3.8 of [7])

$$\Phi : H^* \to R[[X]], \quad f \mapsto \sum_{m \geq 0} f(c_m)X^m.$$

Notice that the formal power series ring $R[[X]]$ is prime provided R is a prime ring.

As a special case one may take $H = R[X]$ with comultiplication

$$\Delta : R[X] \to R[X] \otimes_R R[X], \quad X^m \mapsto \sum_{i=0,m} X^i \otimes X^{m-i},$$

and the counit is $\varepsilon(X^m) = \delta_{0,m}$.

If R is a field, then $R[[X]]$ is a prime ring, and $R[X]$ is a coprime comodule by 2.4.4 part (3).

For $R = \mathbb{Z}$, the primeness of $\mathbb{Z}[[X]]$ does not imply the coprimeness of $\mathbb{Z}[X]$, since for the subcomodule $n\mathbb{Z}[X]$, for $0 \neq n \in \mathbb{N}$,

$$\mathrm{Ann}_{\mathbb{Z}[X]^*}(\mathbb{Z}[X]/n\mathbb{Z}[X]) \neq 0.$$

2.5 Endo-prime comodules

2.5.1 Definition. A right C-comodule M is called *endo-prime* if for any non-zero fully invariant subcomodule K of M, $\mathrm{Ann}_S(K) = 0$, where $S = \mathrm{End}^C(M)$.

An endo-prime comodule M can be considered as a prime right module over its endomorphism ring $\mathrm{End}^C(M)$. Now 1.4.2 yields :

2.5.2 Endo-prime comodules. *Let M be a right C-comodule with $S = \mathrm{End}^C(M)$.*

(i) *M is endo-prime if and only if M_S is prime.*

(ii) *If M is endo-prime, then S is a prime ring.*

(iii) *If the comodule M is fi-retractable and S is a prime ring, then M is endo-prime.*

(iv) *If N is a simple comodule, then $N^{(\Lambda)}$ is endo-prime for any set Λ.*

2.5.3 Proposition. *Let the comodule M satisfy $(*fi)$ as C^*-module. If M is a prime comodule then it is an endo-prime comodule.*

Proof. By Proposition 2.3.3. □

The coalgebra C is endo-prime provided it is prime over the $\mathrm{End}^C(C) \simeq C^*$ acting from the right. Thus 2.5.2 yields :

2.5.4 Endo-prime coalgebras. *Let C be a coalgebra.*

(i) *C is prime as a left C-comodule if and only if C is endo-prime.*

(ii) *If C is endo-prime, then C^* is prime.*

(iii) *If C is fi-retractable as a right C^*-module and C^* is a prime ring, then C is endo-prime.*

2.5.5 Example. Recall Example 2.3.9 and take $\mathbb{Z}$ as the ground ring, $C = \mathbb{M}^n(\mathbb{Z})$, $n \in \mathbb{N}$, i.e., a matrix coalgebra of all $n \times n$ matrices over $\mathbb{Z}$. The dual algebra of C is $C^* \simeq \mathbb{M}_n(\mathbb{Z})$, hence it is prime.

For any ideal $\mathbb{M}_n(I)$ of $\mathbb{M}_n(\mathbb{Z})$, where I is an ideal of $\mathbb{Z}$, holds

$$\mathrm{Ann}_{\mathbb{M}_n(\mathbb{Z})}(\mathbb{M}_n(\mathbb{Z})/\mathbb{M}_n(I)) \neq 0.$$

Thus C is prime as C^*-module and by Corollary 2.5.3, C is endo-prime. □

2.6 Endo-coprime comodules

2.6.1 Definition. A comodule M is called *endo-coprime* if for any proper fully invariant subcomodule K of M, $\mathrm{Ann}_S(M/K) = 0$, where $S = \mathrm{End}^C(M)$.

2.6.2 Endo-coprime comodules. *Let M be a comodule and $S = \mathrm{End}^C(M)$.*

(i) *M is endo-coprime if and only if M_S is coprime.*

(ii) *If M is endo-coprime, then S is prime.*

(iii) *If M is fi-coretractable and S is prime, then M is endo-coprime.*

Proof. By 1.5.2.

Notice that any simple C-comodule M is endo-coprime.

The coalgebra C is endo-coprime as comodule provided it is coprime over the $\mathrm{End}^C(C) \simeq C^*$ acting from the right. We apply 2.6.2 to characterize

2.6.3 Endo-coprime coalgebras. *Let C be a coalgebra.*

(i) *C is coprime as a left C-comodule if and only if C is endo-coprime.*

(ii) *If C is endo-coprime, then C^* is prime.*

(iii) *If C is fi-coretractable and C^* is prime, then C is endo-coprime.*

2.6.4 Proposition. *Let R be a cogenerator in ${}_R\mathbf{M}$ and C an R-coalgebra. Then the following are equivalent :*

(a) *C is coprime as a left C-comodule.*

(b) *C is endo-coprime as a left C-comodule.*

(c) *C is coprime as a right C-comodule.*

(d) *C is endo-coprime as a right C-comodule.*

(e) *C^* is a prime algebra.*

Proof. (a) $\Longleftrightarrow$ (d) By Proposition 2.6.3 part (i).

(b) $\Longleftrightarrow$ (c) is symmetric to (a) $\Longleftrightarrow$ (d).

(c) $\Longleftrightarrow$ (e) Recall Lemma 2.2.9 and 2.4.4 part (3).

(a) $\Longleftrightarrow$ (e) is symmetric to (c) $\Longleftrightarrow$ (e). □

2.7 Fully prime comodules

2.7.1 Definition. A comodule M is called *fully prime* if for any non-zero fully invariant subcomodule K of M, M is K-cogenerated.

A product of fully invariant subcomodules of M is defined by

$$K *_M L := K\mathrm{Hom}^C(M, L).$$

2.7.2 Fully prime comodules. *The following are equivalent for a comodule M :*

(a) *M is a fully prime comodule.*

(b) $\mathrm{Rej}(M, K) = 0$ *for any non-zero fully invariant subcomodule $K \subset M$.*

(c) $K *_M L \neq 0$ *for any non-zero fully invariant subcomodules $K, L \subset M$.*

(d) $\mathrm{Rej}(-, M) = \mathrm{Rej}(-, K)$ *for any non-zero fully invariant subcomodule K of M, i.e., any M-cogenerated comodule is also K-cogenerated.*

If these conditions hold and $\mathrm{Soc}(M) \neq 0$*, then*

(i) *M is semisimple.*

(ii) $\overline{C^*} := C^*/\mathrm{Ann}_{C^*}(M)$ *is a simple artinian algebra and finitely generated as R-module.*

Proof. From 1.6.3 we get the equivalences. M is fully prime implies it is prime. Now apply Proposition 2.3.4 to get (i) and (ii). □

2.7.3 Fully prime subcomodules. A fully invariant subcomodule N of M is called *fully prime* in M if for any fully invariant subcomodules K, L of M, the relation $K *_M L \subseteq N$ implies $K \subseteq N$ or $L \subseteq N$.

By the characterization of fully prime comodule above, the comodule M is fully prime if zero is a fully prime subcomodule .

With some additional condition, we can characterize a prime comodule as a comodule which is cogenerated by any of its subcomodules (compare with Proposition 1.6.7).

2.7.4 Proposition. *For a comodule M with $(*fi)$ as a C^*-module, the following are equivalent :*

(a) *M is a prime comodule and fi-retractable as a C^*-module.*

(b) *M is a fully prime comodule.*

Applying Proposition 1.6.5 and Lemma 2.1.16 yields

2.7.5 Proposition. *Let N be a proper fully-invariant subcomodule of M.*

(i) *If N is fully prime in M, then M/N is a fully prime comodule.*

(ii) *If M is self-projective and M/N is fully prime, then N is fully prime in M.*

For $M = C$ the assertions in 2.7.2 yields

2.7.6 Fully prime coalgebras. *The following are equivalent for a coalgebra C :*

(a) *C is fully prime as a right C-comodule.*

(b) $\mathrm{Rej}(C, A) = 0$ *for any non-zero (C^*, C^*)-subbimodule A of C.*

(c) $A *_C B \neq 0$ *for any non-zero (C^*, C^*)-subbimodules A, B of C.*

(d) $\mathrm{Rej}(-, C) = \mathrm{Rej}(-, A)$ *for any non-zero (C^*, C^*)-subbimodule A of C, i.e., any C-cogenerated coalgebra is also A-cogenerated.*

If these conditions hold and $\mathrm{Soc}(C) \neq 0$*, then*

(i) *C is a semisimple C^*-module.*

(ii) *C^* is a simple artinian and finitely generated as R-module.*

We apply Proposition 2.7.4 to obtain

2.7.7 Proposition. *Let C be a coalgebra. If for any (C^*, C^*)-subbimodule A of C holds* $\mathrm{Ann}_{C^*}(C/A) \neq 0$*, then the following are equivalent :*

(a) *C is a prime coalgebra and fi-retractable as C^*-module.*

(b) *C is a fully prime coalgebra.*

Applying Proposition 2.7.5 yields

2.7.8 Proposition. *Let A be a proper (C^*, C^*)-subbimodule of C.*

(i) *If A is a fully prime subcomodule in C, then C/A is a fully prime right C-comodule.*

(ii) *If C is self-projective in $\mathbf{M}^C$ and C/A is a fully prime right C-comodule, then A is fully prime in C.*

2.8 Fully coprime comodules

2.8.1 Definition. A comodule M is called *fully coprime* if for any proper fully invariant subcomodule K of M, M is M/K-generated.

For any fully invariant subcomodules $K, L \subset M$, we have the internal coproduct (see coproduct 1.1)

$$\begin{aligned} K :_M L &:= \bigcap\{(L)f^{-1} \mid f \in \mathrm{End}^C(M), K \subseteq \mathrm{Ker} f\} \\ &= \mathrm{Ker}\ \pi_K \diamond \mathrm{Hom}^C(M/K, M) \diamond \pi_L. \end{aligned}$$

We characterize fully coprime comodules :

2.8.2 Fully coprime comodules. *Let M be a C-comodule and $S = \mathrm{End}^C(M)$. The following are equivalent :*

(a) *M is a fully coprime comodule.*
(b) *If $K :_M L = M$, then $K = M$ or $L = M$, for any fully invariant subcomodules K, L of M.*
(c) *$K :_M L \neq M$ for any proper fully invariant subcomodules K, L of M.*
(d) *$\mathrm{Tr}(M/K, -) = \mathrm{Tr}(M, -)$ for any proper fully invariant subcomodules K of M, i.e. any M-generated comodule is also M/K-generated.*

If these conditions hold, then :

(i) *M is coprime, fi-coretractable and indecomposable as (C^*, S)-bimodule.*
(ii) *If $\mathrm{Rad}(M) \neq M$, then $\overline{C^*} := C^*/\mathrm{Ann}_{C^*}(M)$ is a simple algebra and finitely generated as R-module. Moreover, M is homogeneous semisimple.*

Proof. By 1.7.3 we get the equivalences.

(i) By Proposition 1.7.8 and Lemma 1.7.5.

(ii) By Proposition 1.7.15 part (iii), $\overline{C^*} := C^*/\mathrm{Ann}_{C^*}(M)$ is a primitive algebra. Then the proof is similar to 2.3.4. □

2.8.3 Fully coprime subcomodule. Let M be a comodule and $N \subset M$ be a fully invariant subcomodule. We say that N is *fully coprime* in M if for any fully invariant subcomodules $K, L \subset M$, $N \subseteq K :_M L$ implies $N \subset K$ or $N \subset L$.

By 2.8.2, M is a fully coprime comodule if and only if M is fully coprime in M.

2.8.4 Proposition. *Let M be a self-cogenerator right comodule and $S = \mathrm{End}^C(M)$.*

(i) *If S is prime, then M is fully coprime.*

(ii) *If M is self-injective and fully coprime, then M is endo-coprime and hence S is prime.*

(iii) *If M is endo-coprime then M is fully coprime.*

Proof. By Lemma 1.7.10 and Lemma 1.7.11. □

As a consequence of Proposition 2.8.4, notice that if M is a self-injective self-cogenerator, then fully coprimeness and endo-coprimeness of M coincide, and it is equivalent to S being a prime ring.

Putting $M = C$ we obtain from 2.8.2 :

2.8.5 Fully coprime coalgebras. *The following are equivalent for a coalgebra C :*

(a) *C is fully coprime as a right C-comodule.*

(b) *If $A :_C B = C$, then $A = C$ or $B = C$, for any (C^*, C^*)-subbimodules A, B of C.*

(c) *$A :_C B \neq C$ for any proper (C^*, C^*)-subbimodules A, B of C.*

(d) *$\mathrm{Tr}(C/A, -) = \mathrm{Tr}(C, -)$ for any proper (C^*, C^*)-subbimodule A of C, i.e. any C-generated coalgebra is also C/A-generated.*

If these conditions hold, then :

(i) *C is indecomposable as (C^*, C^*)-bimodule.*

(ii) *If $\mathrm{Rad}(C) \neq C$, then C^* is a simple algebra and finitely generated as R-module. Moreover, C is homogeneous semisimple as a comodule.*

(iii) *For any projective comodule P in $\mathbf{M}^C$, $\mathrm{Rad}(P) = 0$.*

Proof. (iii) See Proposition 1.7.15 part (ii). □

2.8.6 Proposition. *Let C be a fully coprime coalgebra, P a C-generated comodule and $p : P \to C$ a small epimorphism in $\mathbf{M}^C$.*

(i) *If P is self-projective, then P is fully coprime.*

(ii) *If P is projective in $\mathbf{M}^C$, then C is projective in $\mathbf{M}^C$, i.e. $C \simeq P$.*

Proof. By Proposition 1.7.16. □

2.9 Strongly prime and strongly coprime comodules

2.9.1 Definition. A comodule M is called *strongly prime* if it is strongly prime as C^*-module.

This property extends to the self-injective hull and we have

2.9.2 Strongly prime comodules. *For a comodule M denote its injective hull in $\mathbf{M}^C$ as $\widehat{M}$. The following are equivalent :*

(a) *M is a strongly prime comodule.*

(b) *M is subgenerated by each of its non-zero subcomodules.*

(c) *$\widehat{M}$ is generated by each of its nonzero (fully-invariant) subcomodules.*

(d) *$\widehat{M}$ has no non-trivial fully invariant subcomodules.*

If these conditions hold and $\mathrm{Soc}M \neq 0$*, then*

(i) *M is homogeneous semisimple.*

(ii) *$\overline{C^*} := C^*/\mathrm{Ann}_{C^*}(M)$ is a simple algebra and finitely generated as R-module.*

Proof. Apply 1.8.2 to get the equivalences.

(i) and (ii) follow Proposition 1.8.5 and the finitely generation is obtained similarly to the proof of Proposition 2.3.4. □

As an immediate corollary of 2.9.2 we obtain Theorem 3.3 and Corollary 3.5 of [11].

2.9.3 Corollary. *Let C be an R-coalgebra, R a perfect ring. Then the following are equivalent:*

(a) *M is a prime comodule.*

(b) *M is a strongly prime comodule.*

(c) *M is generated by each non-zero subcomodule of M.*

(d) *M has no non-trivial fully invariant subcomodules.*

(e) *M is a homogeneous semisimple right C-comodule.*

(f) *$C^*/\mathrm{Ann}_{C^*}(M)$ is a simple artinian ring and finitely generated R-module.*

Proof. R is a perfect ring implies $\mathrm{Soc}(M) \neq 0$. Now we apply Proposition 2.9.2. □

For $M = C$, 2.9.2 yields

2.9.4 Strongly prime coalgebras. *For a coalgebra C with C-injective hull $\widehat{C}$, the following are equivalent :*

(a) *C is strongly prime as a right C-comodule.*

(b) *$\widehat{C}$ is generated by each of its nonzero subcomodules.*

(c) *$\widehat{C}$ has no non-trivial (C^*, C^*)-subbimodule.*

If these conditions hold and $\mathrm{Soc}(C) \neq 0$, then

(i) *C is homogeneous semisimple.*

(ii) *C^* is simple and finitely generated as an R-module.*

(iii) *Every comodule is strongly prime.*

(iv) *For any finitely generated projective comodule P, $\mathrm{End}^C(P) \simeq \mathrm{End}_{C^*}(P)$ is strongly prime.*

Proof. (iii) and (iv) follow Proposition 1.8.4. □

2.9.5 Corollary. *Let C be an R-coalgebra, R a perfect ring. Then the following assertions are equivalent:*

(a) *C is prime as a right C-comodule.*

(b) *C is strongly prime as a right C-comodule.*

(c) *C is homogeneous semisimple.*

(d) *C^* is a simple ring and a finitely generated R-module.*

Proof. By Corollary 2.9.3. □

2.9.6 Definition. A comodule M is called *strongly coprime* if it is strongly coprime as C^*-module.

2.9.7 Strongly coprime comodules. *Let M be a strongly coprime comodule.*

(1) *For any proper fully-invariant subcomodule K of M, M/K is strongly coprime.*

(2) *If* $\mathrm{Rad}(M) \neq M$*, then :*

(i) *M is homogeneous semisimple.*

(ii) *$\overline{C^*} := C^*/\mathrm{Ann}_{C^*}(M)$ is a simple algebra and finitely generated as R-module.*

Proof. (1) By Proposition 1.8.7.

(2) By Proposition 1.8.10, $\overline{C^*} := C^*/\mathrm{Ann}_{C^*}(M)$ is a primitive algebra. Then similar to the proof of 2.3.4 we get $\overline{C^*}$ is a simple algebra. □

The coproduct of copies of a strongly coprime comodule is again strongly coprime (Lemma 1.8.8). Recall that M is *duprime* if for any fully invariant subcomodule K of M holds $M \in \sigma[K]$ or $M \in \sigma[M/K]$. By definition it is obvious that any strongly coprime comodule is duprime. The convers will be true if that comodule is self-injective.

2.9.8 Strongly coprime coalgebras. *Let C be strongly coprime as a right C-comodule.*

(1) *If* $\mathrm{Rad}(C) \neq C$*, then :*

(i) *C is homogeneous semisimple.*

(ii) *C^* is a simple algebra and finitely generated as R-module.*

(2) *Let $p : P \to C$ be a small epimorphism in $\mathbf{M}^C$. Then :*

(i) *If P is self-projective as C^*-module, then P is strongly coprime.*

(ii) *If P is projective in $\mathbf{M}^C$, then $C \simeq P$.*

Proof. (1) By 2.9.7. (2) By Proposition 1.8.11. □

2.9.9 Proposition. *Let C be a self-injective C^*-module. Then the following assertions are equivalent :*

(a) *C is fully coprime as a right C-comodule.*

(b) *C is strongly coprime as a right C-comodule.*

(c) *C is duprime as a right C-comodule.*

If R is a cogenerator in ${}_R\mathbf{M}$, then these conditions are equivalent to C^* being a prime algebra.

Proof. By Proposition 1.8.12 we get the equivalences. Moreover, C is fully coprime as a right C-comodule implies it is coprime as a right C-comodule.

If R is a cogenerator in ${}_R\mathbf{M}$, then C satisfies $(**)$-condition as C^*-module (by Lemma 2.2.9) and is a self-cogenerator.

We apply Proposition 2.6.4 to get that C is coprime as a right C-comodule if and only if C^* is a prime algebra.

Moreover, by Corollary 1.7.12, C^* is prime if and only if C is fully coprime as a C-comodule, as required. □

2.10 Wedge Product of Subcomodules

2.10.1 Wedge product of subcomodules. For any two fully invariant subcomodules $K, L \subset M$, put

$$\begin{aligned} K \wedge^M L &:= \text{Ker } \pi_K \diamond \text{Hom}^C(M/K, M) \diamond \pi_L \diamond \text{Hom}^C(M/L, M) \quad (2.2) \\ &= \text{Ker } (\text{Ann}_S(K) \diamond \text{Ann}_S(L)). \quad (2.3) \end{aligned}$$

Notice that it is always true that $K :_M L \subseteq K \wedge^M L$, where "=" holds provided M is a self-cogenerator (see Proposition 1.7.9).

For any two (C^*, C^*)-subbimodules $A, B \subset C$ we have

$$\begin{aligned} A \wedge^C B &:= \text{Ker } \pi_A \diamond \text{Hom}^C(C/A, C) \diamond \pi_B \diamond \text{Hom}^C(C/B, C) \quad (2.4) \\ &= \text{Ker } \pi_A \diamond \text{Hom}_{C^*}(C/A, C) \diamond \pi_B \diamond \text{Hom}_{C^*}(C/B, C) \quad (2.5) \end{aligned}$$

and as a direct consequence :

2.10.2 Lemma. *Let A and B be (C^*, C^*)-subbimodules of the coalgebra C. Then $A :_C B \subseteq A \wedge^C B$, where "=" holds provided C is a self-cogenerator as C-comodule.*

Recall a property from 2.1.15 : for any $f, g \in C^*$ and $c \in C$,

$$f * g(c) = f(g \rightharpoonup c) = g(c \leftharpoonup f).$$

Thus for any proper subbicomodules A, B of C, $(C/A)^* * (C/B)^*$ is a two-sided ideal in C^*. By Lemma 2.2.5 part (ii), $((C/A)^* * (C/B)^*)^{\perp C}$ is a (C^*, C^*)-subbimodule of C.

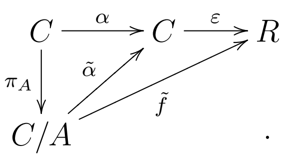
From 3.12 of [5] we know that for any $\beta, \gamma \in \text{End}^C(C)$ and $c \in C$ holds

$$(c)(\beta \diamond \varepsilon) * (\gamma \diamond \varepsilon) = (c)(\beta \diamond \gamma \diamond \varepsilon).$$

Take any proper subbicomodule A of C and any $\tilde{f} \in (C/A)^*$. There is $\alpha := \Delta \diamond (\pi_A \diamond \tilde{f} \otimes I_C) \in \mathrm{End}^C(C)$, such that

$$(\Delta \diamond (\pi_A \diamond \tilde{f} \otimes I_C)) \diamond \varepsilon = \pi_A \diamond \tilde{f} \in C^*,$$

and $(A)\alpha = (A)\Delta \diamond (\pi_A \diamond \tilde{f} \otimes I_C) = 0$. Moreover for these α, there is $\tilde{\alpha} : C/A \to C$ such that $\alpha = \pi_A \diamond \tilde{\alpha}$, i.e., there is a commutative diagram



Consider now

$$\begin{aligned}(A^{\perp C^*} * B^{\perp C^*})^{\perp C} &= \bigcap \{\mathrm{Ker}\ f * g \mid f \in A^{\perp C^*}, g \in B^{\perp C^*}\} \\ &= \bigcap \{\mathrm{Ker}\ \Delta \diamond (f \otimes g) \mid f \in A^{\perp C^*}, g \in B^{\perp C^*}\}.\end{aligned}$$

By definition, $A^{\perp C^*} * B^{\perp C^*})^{\perp C} = ((C/A)^* * (C/B)^*)^{\perp C}$

2.10.3 Lemma. *Let C be a coalgebra. For any (proper) subbicomodules $A, B \subset C$,*

$$A \ \wedge^C \ B = (A^{\perp C^*} * B^{\perp C^*})^{\perp C}.$$

Proof. Consider the coalgebra C as a left C^*-module and comodule morphisms as C^*-module homomorphisms. We want to show that

$$\mathrm{Ker}\ \pi_A \diamond \mathrm{Hom}^C(C/A, C) \diamond \pi_B \diamond \mathrm{Hom}^C(C/B, C) \subseteq ((C/A)^* * (C/B)^*)^{\perp C}.$$

Let

$$u \in \mathrm{Ker}\ \pi_A \diamond \mathrm{Hom}^C(C/A, C) \diamond \pi_B \diamond \mathrm{Hom}_C(C/B, C)$$

and take any $\tilde{f} \in (C/A)^*$, $\tilde{g} \in (C/B)^*$. Then using the notation above, there are $\alpha := \Delta \diamond (\pi_A \diamond \tilde{f} \otimes I_C)$ and $\beta := \Delta \diamond (\pi_B \diamond \tilde{g} \otimes I_C)$ in $\mathrm{End}^C(C)$ such that

$$\begin{aligned}(u)(\pi_A \diamond \tilde{f}) * (\pi_B \diamond \tilde{g}) &= (u)(\alpha \diamond \varepsilon) * (\beta \diamond \varepsilon) \\ &= (u)(\pi_A \diamond \tilde{\alpha} \diamond \varepsilon) * (\pi_B \diamond \tilde{\beta} \diamond \varepsilon) \\ &= (u)(\pi_A \diamond \tilde{\alpha} \diamond \pi_B \diamond \tilde{\beta}) \diamond \varepsilon = 0.\end{aligned}$$

Conversely, we want to show that

$((C/A)^* * (C/B)^*)^{\perp C} \subseteq \text{Ker } \pi_A \diamond \text{Hom}^C(C/A, C) \diamond \pi_B \diamond \text{Hom}^C(C/B, C)$.

$((C/A)^* * (C/B)^*)^{\perp C}$ is a (C^*, C^*)-subbimodule of C.

Let $v \in ((C/A)^* * (C/B)^*)^{\perp C}$. Then $v \in \text{Ann}_C((C/A)^* * (C/B)^*)$ (by Lemma 2.2.8) and we have

$$\begin{aligned} 0 &= (v)\Delta \diamond (I_C \otimes \pi_A \diamond \tilde{f} * \pi_B \diamond \tilde{g}) \\ &= (v)\Delta \diamond (I_C \otimes \alpha \diamond \varepsilon * \beta \diamond \varepsilon) \\ &= (v)\Delta \diamond (I_C \otimes \pi_A \diamond \tilde{\alpha} \diamond \varepsilon * \pi_B \diamond \tilde{\beta} \diamond \varepsilon) \\ &= (v)\Delta \diamond (I_C \otimes \pi_A \diamond \tilde{\alpha} \diamond \pi_B \diamond \tilde{\beta} \diamond \varepsilon) \\ &= (v)\Delta \diamond (I_C \otimes \pi_A \diamond \tilde{\alpha} \diamond \pi_B \diamond \tilde{\beta}) \diamond (I_C \otimes \varepsilon) \\ &= (v)(\pi_A \diamond \tilde{\alpha} \diamond \pi_B \diamond \tilde{\beta}) \diamond \Delta \diamond (I_C \otimes \varepsilon) \\ &= (v)(\pi_A \diamond \tilde{\alpha} \diamond \pi_B \diamond \tilde{\beta}), \end{aligned}$$

as needed. □

Let A and B be (C^*, C^*)-subbimodules of C and consider $\text{Ker } \Delta \diamond (\pi_A \otimes \pi_B)$.

Over any commutative ring, it always holds

$$\text{Ker } \Delta \diamond (\pi_A \otimes \pi_B) \subseteq (A^{\perp C^*} * B^{\perp C^*})^{\perp C}.$$

To see this, take any $c \in \text{Ker } \Delta \diamond (\pi_A \otimes \pi_B)$. By definition,

$$(c)(f * g) = (c)\Delta \diamond (f \otimes g) = (c)\Delta \diamond (\pi_A \otimes \pi_B) \diamond (\tilde{f} \otimes \tilde{g}) = 0,$$

thus $c \in (A^{\perp C^*} * B^{\perp C^*})^{\perp C}$.

The convers is true for coalgebras over fields. We give a short proof of this well-kown fact.

2.10.4 Lemma. *Let A, B be subcoalgebras of C. If C is a coalgebra over a field k, then*

$$\text{Ker } \Delta \diamond (\pi_A \otimes \pi_B) = (A^{\perp C^*} * B^{\perp C^*})^{\perp C}. \tag{2.6}$$

Proof. Let $d \in (A^{\perp C^*} * B^{\perp C^*})^{\perp C}$. For all $f \in A^{\perp C^*}$ and $g \in B^{\perp C^*}$, writing $f = \pi_A \diamond \tilde{f}$ and $g = \pi_B \diamond \tilde{g}$, where $\tilde{f} \in (C/A)^*$ and $\tilde{g} \in (C/B)^*$ such that

$$0 = (d)(f * g) = (d)\Delta \diamond (f \otimes g) = (d)\Delta \diamond (\pi_A \otimes \pi_B) \diamond (\tilde{f} \otimes \tilde{g}).$$

By Lemma 2.2.6 we conclude that $(d)\Delta \diamond (\pi_A \otimes \pi_B) = 0$. This implies $d \in \text{Ker } \Delta \diamond (\pi_A \otimes \pi_B)$. □

The wedge product of subcoalgebras of a coalgebra over a field has been investigated by Sweedler in [31].

2.11 Review of coprime coalgebras over field

Coprime coalgebras over a field k were defined by the wedge product (see for example Jara, Merino, Ruiz [17] and Nekooei-Torkzadeh [26]). We call it *wedge coprime* to distinguish it from other notions of coprime.

2.11.1 Definition. A subcoalgebra D of a coalgebra C over a field k is called *wedge coprime in* C if for any subcoalgebras A, B of C, $D \subseteq A \wedge^C B$ implies $D \subseteq A$ or $D \subseteq B$. The coalgebra C is called *wedge coprime* if it is wedge coprime in C.

Recall Lemma 2.10.2 and Lemma 2.10.3 for self-cogenerator coalgebras C. Take any (C^*, C^*)-subbimodules A and B of C, then

$$A :_C B = A \;\wedge^C\; B = (A^{\perp C^*} * B^{\perp C^*})^{\perp C}.$$

If C is over a field k, any subbimodule is a subcoalgebra and Ker $\Delta \diamond (\pi_A \otimes \pi_B) = A \;\wedge^C\; B$ (see Lemma 2.10.4). Hence the wedge coprime coalgebras defined in [17] and [26] are special cases of the fully coprime coalgebras in our work.

2.11.2 Lemma. *If C is a coalgebra over a field, then the following assertions are equivalent :*

(a) *C is coprime as a right (left) C-comodule.*

(b) *C is fully coprime as a right (left) C-comodule.*

(c) *C is endo-coprime as a left (right) C-comodule.*

(d) *C^* is a prime ring.*

Proof. Over a field, C is a self-injective self-cogenerator. Now apply Corollary 1.7.12 and Proposition 2.6.3. □

We have the following identities :

2.11.3 Proposition. *If C is a coalgebra over a field and I, J are two-sided ideals that are finitely generated as right ideals in C^*, then*

$$I^{\perp C} \wedge^C J^{\perp C} = (I * J)^{\perp C}, \quad (I^{\perp C} \wedge^C J^{\perp C})^{\perp C^*} = ((I * J)^{\perp C})^{\perp C^*} = I * J.$$

Proof. Applying 2.2.1 part (iii) to coalgebras over a field yields $I = (I^{\perp C})^{\perp C^*}$. For any finitely generated ideals I, J of C^* and by Lemma 2.10.4 we have

$$I^{\perp C} \wedge^C J^{\perp C} = ((I^{\perp C})^{\perp C^*} * (J^{\perp C})^{\perp C^*})^{\perp C} = (I * J)^{\perp C}. \quad \square$$

For any finitely generated right ideal I of C^*, $(I^{\perp C})^{\perp C^*} = I$. Moreover we obtain a relationship between prime ideals in C^* and its dual orthogonal subcomodules in C.

2.11.4 Proposition. *Let C be a coalgebra over a field k. Let D be a subcoalgebra of C. If for any subcoalgebras A, B of C, $D \subseteq A \wedge^C B$ implies $D \subseteq A$ or $D \subseteq B$, then $D^{\perp C^*}$ is a prime ideal in C^*.*

Proof. Let $I, J \subseteq C^*$ be finitely generated ideals with $I * J \subseteq D^{\perp C^*}$, then as $((D^{\perp C^*})^{\perp C})^{\perp C^*} = D^{\perp C^*}$ we have

$$\begin{aligned} (D^{\perp C^*})^{\perp C} &\subseteq (I * J)^{\perp C} \\ ((I * J)^{\perp C})^{\perp C^*} &\subseteq ((D^{\perp C^*})^{\perp C})^{\perp C^*} = D^{\perp C^*} \end{aligned}$$

By Proposition 2.11.3, $(I^{\perp C} \wedge^C J^{\perp C})^{\perp C^*} = ((I * J)^{\perp C})^{\perp C^*} \subseteq D^{\perp C^*}$. Therefore $D \subseteq I^{\perp C} \wedge^C J^{\perp C}$ and either $D \subseteq I^{\perp C}$, which implies $I \subseteq (I^{\perp C})^{\perp C^*} \subseteq D^{\perp C^*}$ or $D \subseteq J^{\perp C}$ which implies $J \subseteq (J^{\perp C})^{\perp C^*} \subseteq D^{\perp C^*}$. □

Recall Proposition 1.2 of [26].

2.11.5 Proposition. *Let C be a coalgebra over a field k. A subcoalgebra D of C is wedge coprime in C if and only if $D^{\perp C^*}$ is a prime ideal of C^*.*

Notice that for an arbitrary prime ideal in C^*, its dual orthogonal need not be a wedge coprime subcoalgebra in C. However, it is true for the zero ideal and we obtain

2.11.6 Corollary. *Let C be a coalgebra over a field k and C^* its dual algebra. C is a wedge coprime coalgebra if and only if C^* is a prime algebra.*

Proof. Use Proposition 2.11.5 and take $D = C$. □

Xu, Lu, Zhu [41] investigated the properties of a coalgebra by describing properties of its dual algebra. Theorem 3 of [41] is included in

2.11.7 Coprime coalgebras over fields. *Let C be a coalgebra over a field k. Then the following are equivalent :*

(a) *C is a wedge coprime coalgebra.*

(b) *C is coprime as a left (right) C-comodule.*

(c) *C is fully coprime as a right C-comodule.*

(d) *C is a strongly coprime as a right C-comodule.*

(e) *C^* is a prime algebra.*

(f) *For any proper (C^*, C^*)-subbimodule A of C,* $\mathrm{Hom}^C(C, A) = 0$.

(g) *For any* $0 \neq f \in C^*$, $C = C \leftharpoonup (f * C^*)$.

(h) *For any ideal* $0 \neq I \subset C^*$, $C = C \leftharpoonup I$.

(i) *For any* $0 \neq f \in C^*$, $C = (C^* * f) \rightharpoonup C$.

(j) *For any ideal* $0 \neq I \subset C^*$, $C = I \rightharpoonup C$.

Proof. The coalgebra C over a field k is a self-cogenerator and hence fi-coretractable. By the isomorphism $\mathrm{End}^C(C) \simeq C^*$ we have $(C)\mathrm{End}^C(C) = C \leftharpoonup C^*$. Now apply Proposition 1.1.14 to get the equivalences (e) $\Longleftrightarrow$ (f) $\Longleftrightarrow$ (g) $\Longleftrightarrow$ (h).

(a) $\Longleftrightarrow$ (c) Over a field, the wedge product and $:_C$ coincide.

(b) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (e) C is a self-injective self-cogenerator, hence we can apply Corollary 2.11.2.

(c) $\Longleftrightarrow$ (d) By Proposition 2.9.9.

(e) $\Longleftrightarrow$ (i) is symmetric to (e) $\Longleftrightarrow$ (g). It also follows by Theorem 3 of [41].

(e) $\Longleftrightarrow$ (j) is symmetric to (e) $\Longleftrightarrow$ (h). $\square$

Recall that our definition of coprimeness was related to fully invariant subcomodules $A \subset C$. Now we observe some properties of (not necessarily fully invariant) subcomodules of the coalgebra C as follow.

2.11.8 Proposition. *Let C be a coalgebra over a field k. The following are equivalent :*

(a) *C^* has no zero-divisor.*

(b) *For any proper left C^*-submodule A of C,* $\mathrm{Hom}^C(C, A) = 0$.

(c) *For any* $0 \neq f \in C^*$, $C \leftharpoonup f = C$.

Proof. The coalgebra C over a field k is a self-cogenerator and hence coretractable. By the isomorphism $\mathrm{End}^C(C) \simeq C^*$ we have $(C)\mathrm{End}^C(C) = C \leftharpoonup C^*$. Now apply Proposition 1.1.13.

The implication (a) $\Longleftrightarrow$ (c) also follows by the Corollary in [41]. $\square$

2.11.9 Example. ([26], Example 1.1) Let C be a vector space over k with basis $\{c_i\}_{i=0}^{\infty}$. If $\Delta(c_i) = c_i \otimes c_i$ and $\varepsilon(c_i) = 1$, $i = 1, 2, \ldots$, then C is a k-coalgebra. $A =< c_i >$ is a subcoalgebra of C generated by a single element and A is a simple subcoalgebra. Thus it is a wedge coprime subcoalgebra.

2.11.10 Example. ([26], Example 1.2) Let C be a free module with basis $\{c_i\}_{i=0}^{\infty}$. $\Delta(c_i) = \sum_{j=0}^{i} c_j \otimes c_{i-j}$ and $\varepsilon(c_i) = \delta_{i,0}$, $i = 1, 2, \ldots$, then (C, Δ, ε) is a coalgebra. The subcoalgebra $< c_0 >$ is a simple subcoalgebra then it is coprime. Since the subcoalgebra generated by $\{c_0, c_1, \ldots, c_i\}$, $i = 1, 2, \ldots$ is not coprime and the subcoalgebra generated by c_i is equal to the subcoalgebra generated by $\{c_0, c_1, \ldots, c_i\}$, the only wedge coprime subcoalgebras are $< c_0 >$ and C.

Examples of wedge coprime path coalgebras over a field are studied by Jara et.al. ([17], Example 3.2).

2.12 Colocalization in $\mathbf{M}^C$

Studies of localization and colocalization of coalgebras over a field have been done by some authors, for example Năstăsescu and Torrecillas ([24] and [25]), Gómez-Torrecillas, Năstăsescu, and Torrecillas [12], Jara, Merino, Navarro, and Ruiz [18].

The main objective of this section is to apply colocalization in the category $\mathbf{M}^C$. Throughout this section we assume again C to satisfy the α-condition, that is, we identify $\mathbf{M}^C \simeq \sigma[{}_{C^*}C]$.

2.12.1 Rational Functor. The subfunctor of the identity

$$\mathrm{Rat}^C : {}_{C^*}\mathbf{M} \to \mathbf{M}^C, M \mapsto \mathrm{Rat}^C(M),$$

where $\mathrm{Rat}^C(M) = \mathcal{T}^C(M) = \sum\{\mathrm{Im}\, f \mid f \in \mathrm{Hom}_{C^*}(U, M), U \in \mathbf{M}^C\}$, is called the *rational functor.* $\mathrm{Rat}^C(M) = M$ for $M \in {}_{C^*}\mathbf{M}$ if and only if $M \in \mathbf{M}^C$. $\mathbf{M}^C = {}_{C^*}\mathbf{M}$ if and only if ${}_RC$ is finitely generated and projective (see [5], 4.7).

A coalgebra C is called *right semiperfect* if every simple right comodule has a projective hull in $\mathbf{M}^C$.

Notice that the exactness of the functor Rat^C is closely related to the existence of enough projective in $\mathbf{M}^C$. For coalgebras over QF-ring the two properties are in fact equivalent (see 9.6 of [5]).

2.12.2 Proposition. *Let R be a QF-ring and C an R-coalgebra. C is a right semiperfect coalgebra if and only if the functor* Rat^C *is exact.*

As observed in Lemma 1.9.1, pseudo-projective comodules imply special properties of the trace functor. In our setting this reads as follows :

2.12.3 Lemma. *For $P \in \mathbf{M}^C$ the following are equivalent :*

(a) *P is pseudo-projective in $\mathbf{M}^C$.*

(b) *The trace functor $\mathrm{Tr}(P,-) : \mathbf{M}^C \to \mathbf{M}^C$ preserves epimorphisms.*

(c) *$\mathrm{Tr}(P/\mathrm{Tr}(P,N)) = 0$ for all $N \in \mathbf{M}^C$ and the class*

$$\{X \in \mathbf{M}^C \mid \mathrm{Tr}(P,X) = 0\}$$

is closed under factor modules.

By Lemma 1.9.2 we obtain

2.12.4 Lemma. *For a preradical τ for $\mathbf{M}^C$, the following are equivalent :*

(a) *τ is cohereditary.*

(b) *There is a pseudo-projective module $P \in \mathbf{M}^C$ such that $\mathcal{T}_\tau = \mathrm{Gen}\ (P)$.*

By applying Lemma 1.9.4, pseudo-projective comodules P in ${}_{C^*}\mathbf{M}$ can be characterized by their trace ideal $\mathrm{Tr}(P,C^*) \subseteq C^*$.

2.12.5 Lemma. *For a comodule P with trace ideal $I = \mathrm{Tr}(P,C^*)$, the following are equivalent :*

(a) *P is pseudo-projective in ${}_{C^*}\mathbf{M}$.*

(b) *For every C^*-module L, $\mathrm{Tr}(P,L) = IL$.*

(c) *$P = IP$.*

If these conditions hold, then $I^2 = I$ and $\mathrm{Gen}(P) = \mathrm{Gen}(I)$.

Transferring Lemma 1.9.5 yields

2.12.6 Lemma. *Assume that* Rat^C *is exact, then*

(i) *$\mathbf{M}^C$ is closed under small epimorphisms in ${}_{C^*}\mathbf{M}$.*

(ii) *If a comodule P is projective in $\mathbf{M}^C$, then P is projective in ${}_{C^*}\mathbf{M}$.*

(iii) *If a comodule P is pseudo-projective in $\mathbf{M}^C$, then P is pseudo-projective in ${}_{C^*}\mathbf{M}$.*

By Proposition 1.9.8 applied to $\mathbf{M}^C$, the existence of colocalization of a comodule in the category of C-comodules is guaranteed in the following case.

2.12.7 Proposition. *Suppose there is a projective generator comodule in $\mathbf{M}^C$. Then for an idempotent preradical τ for $\mathbf{M}^C$, the following assertions are equivalent :*

(a) *Every comodule in $\mathbf{M}^C$ has a τ-colocalization, i.e., there is a τ-colocalization functor from $\mathbf{M}^C \to \mathbf{M}^C$.*

(b) *τ is cohereditary, i.e. $\mathcal{F}_\tau$ is cohereditary.*

2.12.8 Proposition. *Let $f : P \to M$ be a pseudo-projective cover in $\mathbf{M}^C$ and $I = \mathrm{Tr}(P, C^*)$. If the rational functor Rat^C is exact, then I is an idempotent ideal of C^* and $IP = P$.*

Proof. Assume Rat^C is exact. From Lemma 2.12.6, P is also pseudo-projective in ${}_{C^*}\mathbf{M}$. Thus we apply Lemma 2.12.5 to get that I is idempotent. It is clear from Lemma 1.9.4 that $IP = P$. □

As a special case, if the coalgebra C is a finitely generated projective R-module, i.e., $\mathbf{M}^C = {}_{C^*}\mathbf{M}$, then the assertions in Proposition 2.12.8 are fulfilled (see 2.12.1). We have the following property from Lemma 1.9.9.

2.12.9 Lemma. *Let P be a right C-comodule, $S = \mathrm{End}_{C^*}(P) \simeq \mathrm{End}^C(P)$ and $I = \mathrm{Tr}(P, C^*)$. For any $M \in \mathbf{M}^C$ let*

$$\psi_M : P \otimes_S \mathrm{Hom}_C(P, M) \to M, \quad p \otimes f \mapsto (p)f.$$

Then $I \rightharpoonup \mathrm{Ker}\ \psi_M = 0$ and $I \rightharpoonup \mathrm{Coker}\ \psi_M = 0$.

τ-colocalization of any comodule M is given by Proposition 1.9.10 :

2.12.10 Proposition. *Let τ be a cohereditary radical for $\mathbf{M}^C$, assume that $I = \tau(C^*)$ is an idempotent ideal, and put*

$$\mathcal{T}_\tau = \{M \in \mathbf{M}^C \mid I \rightharpoonup M = M\}, \quad \mathcal{F}_\tau = \{M \in \mathbf{M}^C \mid I \rightharpoonup M = 0\}.$$

Let $P \in \mathcal{T}_\tau$ and $\mathcal{F}_\tau$-projective with $\mathrm{Tr}(P, C^*) = I$ and put $S = \mathrm{End}^C(P)$. For any $M \in \mathbf{M}^C$,

$$\psi_M : P \otimes_S \mathrm{Hom}^C(P, M) \to M, \; p \otimes f \mapsto (p)f,$$

is a τ-colocalization of M.

If the rational functor Rat^C is exact and P is pseudo-projective in $\mathbf{M}^C$, then by Proposition 2.12.8, the condition in Proposition 2.12.10 are satisfied for $\tau = \mathrm{Tr}(P, -)$.

2.12.11 Remark. For coalgebras C over a field k, colocalization in the category $\mathbf{M}^C$ induced by a coidempotent subcoalgebra $B \subset C$ (i.e. $B \wedge^C B = B$) is studied by Năstăsescu-Torrecillas [25]. Their techniques and results depend heavily on the fact that they are working over a base field.

2.12.12 The class of M-corational comodules. As for R-modules, the class of M-corational comodules in $\mathbf{M}^C$ is defined by

$$\tilde{\mathcal{C}r}_M = \{X \in \mathbf{M}^C \mid \mathrm{Hom}^C(M, X/Y) = 0, \text{ for all } Y \subset X\}$$

and the corresponding torsion class is

$$\tilde{\mathcal{C}r}^{\circ}_M = \{N \in \sigma[M] \mid \mathrm{Hom}^C(N, X) = 0, \text{ for all } X \in \mathcal{C}r_M\}.$$

Moreover, if there exists a pseudo-projective cover $f : P \to M$ in $\mathbf{M}^C$, by Proposition 1.9.13 we can characterize the class $\tilde{\mathcal{C}r}_M$ as

$$\tilde{\mathcal{C}r}_M = \{X \in \mathbf{M}^C \mid \mathrm{Hom}^C(P, X) = 0\}$$

and $P/\mathrm{Tr}(P, \mathrm{Ker}\ f) \to M$ is a corational cover. Furthermore, by assuming $I = \mathrm{Tr}(P, C^*)$ to be idempotent, we have the cohereditary torsion theory

$$\tilde{\mathcal{C}r}^{\circ}_M = \{N \in \mathbf{M}^C \mid I \rightharpoonup N = N\}, \quad \tilde{\mathcal{C}r}_M = \{N \in \mathbf{M}^C \mid I \rightharpoonup N = 0\}.$$

For the cohereditary torsion theory $(\tilde{\mathcal{C}r}^{\circ}_M, \tilde{\mathcal{C}r}_M)$, we colocalize the comodule M in the following way (see Proposition 1.9.14).

2.12.13 Proposition. *Let $f : P \to M$ be a projective hull in $\mathbf{M}^C$. Then $\mathrm{Tr}(P, -)$ induces the M-corational torsion theory, $P/\mathrm{Tr}(P, \mathrm{Ker}\ f)$ is a $\tilde{\mathcal{C}r}_M$-projective module, and $P/\mathrm{Tr}(P, \mathrm{Ker}\ f) \to M$ is the $Tr(P, -)$-colocalization of M.*

By Proposition 1.9.15 and Corollary 1.9.16 we have

2.12.14 Proposition. *Let M be a self-projective comodule and $S = \mathrm{End}^C(M)$.*

(i) *If M is an endo-coprime comodule, then it is copolyform.*

(ii) *If M is fi-coretractable and S is prime, then M is copolyform.*

(iii) *If M is a self-injective self-cogenerator and fully coprime, then M is copolyform.*

(iv) *If M is a self-injective self-cogenerator and strongly coprime, then M is copolyform.*

2.12.15 Proposition. *Let C be an R-coalgebra with $\mathrm{Rad}(C) \neq C$ and P a projective comodule in $\mathbf{M}^C$. If C is fully or strongly coprime as C^*-module, then $\mathrm{Rad}(P) = 0$ and P is copolyform.*

Proof. By Proposition 1.9.17. □

2.12.16 Projective hull of C. *Let C be an R-coalgebra and $p : P \to C$ a projective hull in $\mathbf{M}^C$.*

(i) *If C is coprime as a right C-comodule, then $\overline{C^*} := C^*/\mathrm{Ann}_{C^*}(P)$ is a simple algebra and finitely generated as R-module.*

(ii) *If C is endo-coprime and pseudo-projective, then P is copolyform.*

(iii) *If C is strongly coprime as a right C-comodule, then $C \simeq P$.*

(iv) *If P is C-generated and C is fully coprime as a right C-comodule, then C^* is a simple algebra which is finitely generated as an R-module.*

Proof. (i) Since C is coprime, P is also coprime. By projectivity, $\mathrm{Rad}(P) \neq P$. By Proposition 2.4.3 part (iii), C^* is a simple algebra and finitely generated as R-module.

(ii) By Proposition 1.9.18 part (i).

(iii) By 2.9.8 part 3(ii).

(iv) By Proposition 2.8.6 part (ii). □

2.13 Colocalization of coalgebras

Replacing the comodule M in Proposition 2.12.13 by the coalgebra C yields the $\mathrm{Tr}(P, -)$-colocalization for C.

2.13.1 Proposition. *Let $f : P \to C$ be a projective hull in $\mathbf{M}^C$. Then* $\mathrm{Tr}(P,-)$ *induces the C-corational torsion theory,* $P/\mathrm{Tr}(P, \mathrm{Ker}\ f)$ *is a* $\tilde{Cr}_M$-*projective comodule, and* $P/\mathrm{Tr}(P, \mathrm{Ker}\ f) \to C$ *is the* $Tr(P,-)$-*colocalization of* C.

Moreover, by Lemma 2.12.10 we get

2.13.2 Lemma. *Let τ be a cohereditary radical for $\mathbf{M}^C$, assume that $I = \tau(C^*)$ is an idempotent ideal, and put*

$$\mathcal{T}_\tau = \{M \in \mathbf{M}^C \mid I \rightharpoonup M = M\}, \quad \mathcal{F}_\tau = \{M \in \mathbf{M}^C \mid I \rightharpoonup M = 0\}.$$

Let $P \in \mathcal{T}_\tau$ and $\mathcal{F}_\tau$-projective with $\mathrm{Tr}(P, C^*) = I$ *and put* $S = \mathrm{End}^C(P)$. *For the coalgebra C,*

$$\psi_C : P \otimes_S \mathrm{Hom}^C(P, C) \to C, \ p \otimes f \mapsto (p)f,$$

is a τ-colocalization of C.

One may ask if $P \otimes_S \mathrm{Hom}^C(P, C)$ allows for a coalgebra structure and the map ψ_C in Lemma 2.13.2 can be understood as a morphism of coalgebras. By the fact that $\mathrm{Hom}^C(P, C) \simeq \mathrm{Hom}_R(P, R) = P^*$ (see Lemma 2.1.11) we may focus on the question if $P \otimes_S P^*$ allows for a coalgebra structure.

It is well known that this is the case if P is a finitely generated and projective R-module and this structure was extended to direct sums of modules of this type.

For the investigation of direct sums of modules the following technical observation is helpful (see [40], Section 6). For a direct sum of modules $P = \bigoplus_\Lambda P_\lambda$, denote by $\varepsilon_\lambda : P_\lambda \to P$ and $\pi_\lambda : P \to P_\lambda$ the canonical injections and projections. Recall that the identity of P can be written as the formal sum $\sum_\Lambda \pi_\lambda \varepsilon_\lambda$.

For any direct sum $P = \bigoplus_\Lambda P_\lambda$ of finitely generated C-comodules P_λ and any C-comodule N (e.g. Section 51 of [38]),

$$\widehat{\mathrm{Hom}}^{\mathrm{C}}(P, N) = \{f \in \mathrm{Hom}^C(P, N) \mid (P_\lambda)f = 0 \text{ for almost all } \lambda \in \Lambda\},$$

and $T := \widehat{\mathrm{End}}^{\mathrm{C}}(P)$. Then $\widehat{\mathrm{Hom}}^{\mathrm{C}}(P, N)$ is a left T-module and

$$\widehat{\mathrm{Hom}}^{\mathrm{C}}(P, N) \simeq \bigoplus_\Lambda \mathrm{Hom}^C(P_\lambda, N) \simeq T \otimes_S \mathrm{Hom}^C(P, N), \tag{2.7}$$

where $S = \mathrm{End}^C(P)$ (see [38], 51.2).

2.13.3 Lemma. ([40], Lemma 6.1) *With the notation above,*

$$P \otimes_T (\bigoplus_\Lambda \mathrm{Hom}^C(P_\lambda, N)) \simeq P \otimes_T (T \otimes_S \mathrm{Hom}^C(P, N)) \simeq P \otimes_S \mathrm{Hom}^C(P, N).$$

With this isomorphism the comodule structure of modules that are finitely generated and projective as R-modules can be extended to direct sums of modules of this type.

2.13.4 $P \otimes_S P^*$ **as left comodule.** Consider a family $\{P_\lambda\}_\Lambda$ of comodules $P_\lambda \in \mathbf{M}^C$ such that each P_λ is finitely generated and projective as R-module. Then $P = \bigoplus_\Lambda P_\lambda$ is in $\mathbf{M}^C$. Since all P_λ^* are left C-comodules (see 3.11 of [5]), their direct sum $\bigoplus_\Lambda P_\lambda^*$ is a left C-comodule. For $S = \mathrm{End}^C(P)$, Lemma 2.13.3 yields the isomorphism $P \otimes_S P^* \simeq P \otimes_T (\bigoplus_\Lambda P_\lambda^*)$ which makes $P \otimes_S P^*$ to a left C-comodule.

The construction of coalgebras for finitely generated projective R-modules can also be extended to direct sums of modules of this type.

2.13.5 Coalgebra structure on direct sums. Consider a family $\{P_\lambda\}_\Lambda$ of right C-comodules that are finitely generated and projective as R-modules with dual basis $p_{\lambda_i} \in P_\lambda$, $\pi_{\lambda_i} \in P_\lambda^*$, $\lambda_i \in I_\lambda$, $\lambda \in \Lambda$.

By Lemma 2.13.3 and using the notation above there is an isomorphism

$$P \otimes_S P^* \simeq P \otimes_T \widehat{\mathrm{Hom}}_{\mathrm{R}}(P, R).$$

Let $p \otimes_T f \in P \otimes_T \widehat{\mathrm{Hom}}_{\mathrm{R}}(P, R)$.

We define a coproduct and a counit for $P \otimes_T \widehat{\mathrm{Hom}}_{\mathrm{R}}(P, R)$ by

$$\begin{aligned} \Delta(p \otimes_T f) &= \sum_{\lambda_i} p \otimes_T \pi_{\lambda_i} \otimes_R p_{\lambda_i} \otimes_T f, \\ \varepsilon(p \otimes_T f) &= (p)f. \end{aligned}$$

Notice that the sum in the expression for Δ is finite.

By properties of the dual basis,

$$\begin{aligned} (\varepsilon \otimes I) \circ \Delta(p \otimes_T f) &= \sum_{\lambda_i} (p)\pi_{\lambda_i} p_{\lambda_i} \otimes f = p \otimes_T f, \\ (I \otimes \varepsilon) \circ \Delta(p \otimes_T f) &= \sum_{\lambda_i} p \otimes_T \pi_{\lambda_i} \otimes_R (p_{\lambda_i})f \\ &= p \otimes_T \sum_{\lambda_i} \pi_{\lambda_i}(p_{\lambda_i})f = p \otimes_T f. \end{aligned}$$

The coassociativity follows from

$$\begin{aligned}(I \otimes \Delta) \circ \Delta(p \otimes_T f) &= \sum_{\lambda_i, \mu_j} p \otimes_T \pi_{\lambda_i} \otimes_R p_{\lambda_i} \otimes_T \pi_{\mu_j} \otimes_R p_{\mu_j} \otimes_T f \\ &= (\Delta \otimes I) \circ \Delta(p \otimes_T f).\end{aligned}$$

Thus $P \otimes_S P^*$ is a coalgebra.

The coalgebra structure on $P \otimes_S P^*$ as given here was considered in [10] (where it is called *infinite comatrix coalgebra*) and [40].

2.13.6 Theorem. *Let $\{P_\lambda\}$ be a family of finitely generated C-comodules which are projective in ${}_{C^*}\mathbf{M}$ and $P = \bigoplus_\Lambda P_\lambda$, $S = \mathrm{End}^C(P)$. Then the colocalization of C with respect to $\mathrm{Tr}(P, -)$ is*

$$\psi : P \otimes_S P^* \to C, \quad p \otimes f \mapsto (p)f.$$

$P \otimes_S P^*$ has a coalgebra structure and ψ is a coalgebra morphism.

Proof. By Lemma 2.13.2 and 2.13.5, $P \otimes_S P^*$ has a coalgebra structure. It is pointed out in the proof of 6.4 of [40], ψ is a coalgebra morphism. □

Recall that a module $N \in \sigma[M]$ is called *semihereditary* in $\sigma[M]$ if every finitely generated submodule of N is projective in $\sigma[M]$. A ring R is called *left semihereditary* if ${}_R R$ is semihereditary in ${}_R\mathbf{M}$.

In the following case there is a generating set of comodules with the properties required in 2.13.5.

2.13.7 Lemma. *Consider the following condition on a ring R and a coalgebra C.*

(i) *R is left semihereditary and C is projective as an R-module.*

(ii) *R is von Neumann regular and C is projective as R-module.*

(iii) *R is a semisimple ring (e.g., a field).*

(iv) *$\mathbf{M}^C$ has a set of finitely generated projective generators.*

Then $\mathbf{M}^C$ has a generating set of comodules which are finitely generated and projective as R-modules.

Proof. (i) The finitely generated subcomodules of $C^{(\mathbb{N})}$ form a generating set; they are finitely generated R-submodules of the projective right R-module $C^{(\mathbb{N})}$ (see 39.13 of [38]).

(ii),(iii) These conditions imply the conditions in (i) (see 37.3 of [38]).

(iv) If $M \in \mathbf{M}^C$ is finitely generated and projective as comodule, then it is also finitely generated and projective as R-module. □

For a ring A, the Lambek torsion theory is induced by the injective hull $E(A)$ of A and the correspondence torsion class is

$$\mathcal{T}_\tau = \{N \in {}_A\mathrm{M} \mid \mathrm{Hom}(N, E(A)) = 0\}.$$

Localization of A yields the maximal ring of quotients. For commutative prime rings A this yields the quotient field. This theory can be transferred to module categories of type $\sigma[M]$ and the resulting quotient of prime modules are of interest.

Thus the question arises if dually the existence of a projective hull of M in $\sigma[M]$ leads to a colocalization of strongly coprime modules. In particular, does the existence of a projective hull $P \to C$ of the strongly coprime coalgebra C as comodule yield a colocalization? This is not the case for the following reason : If C is a strongly coprime comodule, then P is also strongly coprime. By the projectivity of P, $\mathrm{Rad}(P) \neq P$ and by Proposition 1.8.10, $\mathrm{Rad}(P) = 0$. Thus $C \simeq P$.

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Die hier vorgelegte Dissertation habe ich eigenständig und ohne unerlaubte Hilfe angefertigt. Die Dissertation wurde in der vorgelegten oder in ähnlicher Form noch bei keiner anderen Institution eingereicht. Ich habe bisher keine erfolglosen Promotionsversuche unternommen.

Düsseldorf, den Juni 2006

Indah Emilia Wijayanti