Equivariant holomorphic torsion for a fibre bundle

Inaugural-Dissertation

zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Heinrich-Heine-Universität Düsseldorf

vorgelegt von

Thomas Ueckerdt
aus Berlin

Düsseldorf, Januar 2014

aus dem Institut für Mathematik
der Heinrich-Heine-Universität Düsseldorf

Gedruckt mit der Genehmigung der
Mathematisch-Naturwissenschaftlichen Fakultät der
Heinrich-Heine-Universität Düsseldorf

Referent: Prof. Dr. Köhler
Korreferent: Prof. Dr. Singhof

Tag der mündlichen Prüfung: 08.04.2014

Zusammenfassung

In der vorliegenden Arbeit geht es darum eine verallgemeinerte Produktformel für die äquivariante holomorphe Torsion von bestimmten holomorphen Linienbündeln über speziellen Faserbündeln zu bestimmen.
Desweiteren wird diese Produktformel dazu benutzt die äquivariante holomorphe Torsion für flache Linienbündel über kompakten, gerade-dimensionalen Liegruppen zu bestimmen.
Wir verallgemeinern dabei in beiden Fällen ein bekanntes nicht-äquivariantes Resultat von Stanton in [29].

In bestimmten Fällen zerfällt der Dolbeault-Laplace-Operator für holomorphe Linienbündel über komplexen Faserbündeln in zwei Teile, einen vertikalen Laplace-Operator und einen horizontalen Laplace-Operator.
Diese Aufteilung lässt sich, wie wir in dieser Arbeit zeigen, fortsetzen auf die zugehörige äquivariante spektrale Zeta-funktion. Um genauer zu werden, wir erhalten einen Teil, der nur vom Kern des horizontalen Laplace-Operators abhängt und sich darstellen lässt über die äquivarianten Indexe bestimmter holomorpher Vektorbündel über der Basis des Faserbündels, und einen weiteren Teil, der nur vom Kern des vertikalen Laplace-Operators abhängt und insbesondere weiter zerfällt in die äquivarianten Torsionen von speziellen holomorphen Vektorbündeln über der Basis des Faserbündels.
Für zulässige Wirkungen auf die holomorphen Linienbündel, deren induzierte Wirkung auf die Basis des holomorphen Faserbündels nur isolierte Fixpunkte hat, ergibt sich ein noch einfacherer Ausdruck für die äquivariante holomorphe Torsion.

Im zweiten Teil dieser Arbeit wenden wir diese Zerlegung der Torsion auf ein spezielles Beispiel an.
Wir betrachten das Faserbündel, welches man erhält, wenn man eine kompakte, gerade-dimensionale Liegruppe durch einen maximalen Torus dividiert. Bei den zu untersuchendenden holomorphen Linienbündeln schränken wir uns auf die Klasse der flachen Linienbündel ein.
Die Theorie des ersten Teils, angewendet auf dieses Beispiel, liefert uns einen übersichtlicheren Ausdruck für die äquivariante holomorphe Torsion der Linienbündel.

Abstract

This thesis is dedicated to develop a generalised product formula for the equivariant holomorphic torsion of a holomorphic, Hermitian line bundle over a certain kind of fibre bundle.
Furthermore, we study an example which is given on the one hand, by a holomorphic fibre bundle, consisting of a compact, connected, even-dimensional Lie group modded out by a maximal torus and on the other hand, by flat complex line bundles over this Lie group.
In both parts of this thesis, we generalise a non-equivariant result from Stanton (cf. [29]).

Take a holomorphic line bundle $\mathfrak{L}$ over a holomorphic fibre bundle $E \to M$. There are certain conditions that guaranty a splitting of the Dolbeault-Laplacian on $\mathfrak{L}$ into a horizontal part and a vertical part.
In the first part of this thesis, we show that this splitting sometimes extends to a splitting of the spectral equivariant zeta-function into a part that depends only on the kernel of the horizontal Laplacian, consisting of a sum over various indexes of certain holomorphic vector bundles over M, and a part the depends only on the kernel of the vertical Laplacian. The latter part is given by a sum over equivariant holomorphic torsions of holomorphic vector bundles over M.
For the special case of an admissible action that induces an action on M which has only non-degenerated fixed points, we obtain an even simpler result. This is due to the fact that we can apply the Atiyah-Bott fixed point formula to the sum over the indexes occurring in the first part of the expression for the equivariant holomorphic torsion of $\mathfrak{L}$.

In the second part of this thesis, we study the example of the holomorphic fibre bundle, induced by a compact, even-dimensional Lie group G and a maximal torus T therein. We show that for certain flat line bundles over G the theory of the first part is applicable.
Let $\tilde{g}_0$ be an element of the universal covering group $\tilde{G}$ that covers an element g_0 in G which generates a maximal torus. For the special case of an equivariant action that is essentially given by left multiplication with $\tilde{g}_0$, we obtain an expression for the equivariant holomorphic torsion for the flat line bundle over G that depends only on the roots of G and on the equivariant holomorphic torsions of the line bundle restricted to the maximal torus with $\tilde{g}_0$ induced action.

Acknowledgements

I wish to thank, first and foremost, my supervisor, Professor Kai Köhler. He introduced me to a very interesting subject and supported me whenever questions arose.

Furthermore, I owe my deepest gratitude to my parents, Rainer and Birgit Ueckerdt, who helped me develop a knack for mathematics and who supported my academic career right from the start.
Another thank you goes to my dear colleague Dr. Ferit Deniz. He always had time for me and for the mathimatical questions that occurred to me while working on this thesis or somewhere else.

My deepest gratitude, however, is owed to my fiancee, Anna Abczynski. She supported me wherever she could. In particular, she improved the understandability of this thesis tremendously by pointing out all the small details that were missing in this thesis when I first presented it to her.

Table of contents

1 Introduction

1.1 History and motivation

In the 1930s, Reidemeister [25] and Franz [12] developed an invariant to distinguish certain manifolds that are homotopically equivalent but not homeomorphic. The invariant is called Reidemeister torsion. In particular, they were able to classify the homeomorphism classes of lens spaces.

Later, in the 1970s, Ray and Singer introduced an analytic analog to the Reidemeister torsion, the analytic torsion (cf. [23]). They interpreted the Reidemeister torsion to be the derivative of a ζ-function at $0 \in \mathbb{C}$. The ζ-function is given by the spectrum of a combinatorial Laplacian acting on an elliptic chain complex.
The analytic torsion was defined analogously by taking the de-Rham Laplacian on differential forms and its spectral ζ-function. More general, this analytic torsion can be extended to smooth, flat vector bundles over a compact manifold.

Shortly thereafter, Ray and Singer extended this Ansatz for the analytic torsion further. They applied the same mechanism to Hermitian, holomorphic vector bundles over a compact, complex manifold and defined this way the holomorphic torsion (cf. [24]). The chain complex for the holomorphic torsion consists of the antiholomorphic differential forms with coefficients in the holomorphic, Hermitian vector bundle. The differential of this chain complex is given by the natural $\bar{\partial}$-operator.
In particular, Ray and Singer computed the holomorphic torsion of flat line bundles over a complex torus (in [24]).

In 1978, Stanton derived in [29] that for flat line bundles over certain holomorphic fibre bundles, the computation of the holomorphic torsion of these line bundles simplifies. This is due to a splitting of the ζ-function into two parts. One part mainly depends on the holomorphic torsion of the line bundle restricted to a fibre while the second part is a series over indexes of elliptic operators on vector bundles over the base space of the holomorphic fibre bundle.
In particular, she was able to compute the holomorphic torsion of flat line bundles over compact, even-dimensional Lie groups.

The equivariant holomorphic torsion is a natural equivariant generalisation of the holomorphic torsion. It is of interest for the Arakelov theory. In [22] Köhler and Roessler show that for a fixed point formula in the context of Arakelov theory, analogously to the Lefschetz fixed point formula (cf. [3]), the equivariant holomorphic torsion becomes a main ingredient.

The aim of this thesis is to generalise Stantons result to an equivariant setting, i.e to give a formula for the equivariant holomorphic torsion for fibre bundles.
On the one hand, we show that for suitably good actions the splitting of the ζ-function survives the equivariant approach even for a slightly more general case.
On the other hand, we apply this theory, similar to Stanton, to flat line bundles over

compact, even-dimensional Lie groups. In particular, we give an explicit example for a suitable action on these line bundles and compute its equivariant torsion.

1.2 A brief summary of this thesis

This thesis is divided into two parts.

The first part consists of the Sections 2, 3 as well as Section 4.
In Section 2, we recall common knowledge about complex manifolds, holomorphic vector bundles and equivariant invariants.
Furthermore, we introduce the type of fibre bundle which we investigate later on, the so-called holomorphic fibre bundle. It is a slight generalisation of the definition of a holomorphic fibre bundle given by Stanton in [29].
In Section 3, we define the type of line bundles on which we study the equivariant holomorphic torsion, namely the compatible line bundles. Additionally, we take a closer look at the properties of those bundles with respect to the underlying holomorphic fibre bundle structure and we derive verifiable conditions for the existence of compatible line bundles.
The arguments, we use, for the compatibility are a natural generalisation of the arguments Stanton derives in [29].
Furthermore, we use results from Atiyah and Singer to construct holomorphic vector bundles $\mathcal{W}^{(\lambda;t)}$ over the base space M of a holomorphic fibre bundle $\pi_E : E \to M$ such that the fibre over each point $x \in M$ is given by the λ-Eigenspace of the Laplacian acting on antiholomorphic forms on $E_x := \pi_E^{-1}(x)$ with coefficients in the restricted holomorphic line bundle $\mathfrak{L}\,|_{E_x}$, i.e.

$$\mathcal{W}^{(\lambda;*)} := \bigcup_{x\in M} \operatorname{Ker}\left(\square_{\mathfrak{L}|_{\pi_E^{-1}(x)}} - \lambda\right).$$

In addition to that, we introduce the equivariant action in Section 3. It is an action which is compatible with all those structures, we defined so far. We call it the legitimate action. We complete this section by constructing a morphism, the ψ-morphism, that identifies objects over the total space of the fibre bundle with objects over its base space. In particular, we show that there is a natural action $\gamma^{\mathcal{W}^{(\lambda,t)}}$ on $\mathcal{W}^{(\lambda,t)}$ corresponding to the legitimate action.
We finish the first part of this thesis in Section 4 by computing the equivariant holomorphic ζ-function for legitimate actions on compatible line bundles over holomorphic fibre bundles. The result is accumulated in Theorem 4.1:

Theorem (4.1)**:**
Let $E \to M$ be a holomorphic fibre bundle and let $\mathfrak{L} \to E$ be a compatible, holomorphic, Hermitian line bundle.
Let further on $\vec{\gamma} = \left(\gamma^M, \gamma^E, \gamma^{\mathfrak{L}}\right)$ be a legitimate action on $\mathfrak{L}$.

Then the equivariant ζ-function can be expressed for large $\mathrm{Re}(z)$ *as follows:*

$$Z^{\mathfrak{L}}_{\tilde{\gamma}^{\mathfrak{L}}}(z) = -\sum_{\lambda\neq 0}\lambda^{-z}\sum_t t(-1)^t \mathrm{ind}(\gamma^{\mathcal{W}^{(\lambda;t)}}, \square_{\mathcal{W}^{(\lambda;t)}}) + \sum_t (-1)^t Z^{\mathcal{W}^{(0;t)}}_{\tilde{\gamma}^{\mathcal{W}^{(0;t)}}}(z).$$

Here, $\mathrm{ind}(\gamma^{\mathcal{W}^{(\lambda;t)}}, \square_{\mathcal{W}^{(\lambda;t)}})$ *denotes the equivariant index of the Laplacian acting on* $\mathcal{W}^{(\lambda;t)}$.

In particular, the equivariant ζ-function of $\mathfrak{L} \to E$ is represented one the one hand (in the first term), by differential topological invariants on the base space M belonging to vector bundles $\mathcal{W}^{(\lambda;t)}$ and on the other hand, by the equivariant ζ-functions of the bundles $\mathcal{W}^{(0;t)}$ which are given as the kernel of a Laplacian on the fibres.

The second part of this thesis is contained in Section 5.

Motivated by Stantons non-equivariant result, we give an example in which the equivariant holomorphic torsion can be computed using the theory of the first part. We look at compact, even-dimensional Lie groups and flat line bundles over those.
In Section 5.1, we apply common knowledge about compact Lie groups to show that these Lie groups form a holomorphic fibre bundle over their maximal torus in a natural way.
In Section 5.2, we recall classical topological results about the isomorphism classes of complex line bundles over Lie groups.
In Section 5.3, we recall that a complex line bundle associated to a representation of $\pi_1(G)$ obtains a natural holomorphic structure.
In Section 5.4, we investigate which holomorphic line bundles over Lie groups fulfil the prerequisites of Theorem 4.1. Here, we use essentially differential geometric methods.
Afterwards, in Section 5.5, we apply Theorem 4.1 for a general legitimate action on those line bundles. In particular, we recall some commonly known spectral properties of flat line bundles over the complex torus that imply a very simple structure of the bundles $\mathcal{W}^{(0;t)}$.
We obtain the following theorem:

Theorem (5.1)**:**
Let G be a compact, even-dimensional Lie group and let T be a maximal torus in G.
Let $G \to G/T$ be the corresponding principle fibre bundle equipped with its natural holomorphic fibre bundle structure.
Let further on $\pi_{1,\tilde{G}} : \tilde{G} \to G$ denote the universal cover of G and let $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C} \to G$ be a Hermitian line bundle associated to the principle fibre bundle $\tilde{G} \to G$ through a character χ of $\pi_1(G)$.
Equip $\mathfrak{L} \to G$ with the holomorphic structure $\bar{\partial}_{\mathfrak{L}} = \bar{\partial} + \varepsilon(\pi_G^*(\omega))$ for a $\bar{\partial}$-closed form ω in $\mathfrak{A}^{(0,1)}(G/T)$.
Additionally, let $\vec{\gamma} = \left(\gamma^{\mathfrak{L}}, \gamma^G, \gamma^{G/T}\right)$ be a legitimate action.
Then the equivariant ζ-function is the meromorphic continuation of the following

expression. For large $\mathrm{Re}(z)$ *the* ζ*-function* $Z^{\mathfrak{L}}_{\check{\gamma}^{\mathfrak{L}}}(z)$ *is given by:*

$$\begin{aligned} Z^{\mathfrak{L}}_{\check{\gamma}^{\mathfrak{L}}}(z) = &\sum_{\lambda\neq 0}\lambda^{-z}\sum_t t(-1)^{t+1}\mathrm{ind}(\gamma^{\mathcal{W}(\lambda;t)},\Box_{\mathcal{W}(\lambda;t)}) \\ &+\chi_\gamma(T)\begin{cases} Z^{\mathcal{W}(0;0)}_{\check{\gamma}^{\mathcal{W}(0;0)}}(z) & \text{if } \chi\equiv 1 \\ 0 & \text{if } \chi\not\equiv 1 \end{cases}. \end{aligned}$$

Here, $\chi_\gamma(T)$ *denotes the equivariant Euler characteristic of* T*.*

At last, in Section 5.6, we give an easy example for a legitimate action on these bundles and apply the Theorem 5.1.
The action on $\mathfrak{L} = \tilde{G}\times_\chi \mathbb{C}$ is given by a left multiplication with an element $\tilde{g}_0$ of $\tilde{G}$, i.e.

$$\begin{aligned} \gamma^{\mathfrak{L}} := L_{\tilde{g}_0} : \quad G\times_\chi\mathbb{C} &\longrightarrow G\times_\chi\mathbb{C} \\ [\tilde{g},z]_\chi &\longmapsto [\tilde{g}_0\tilde{g},z]_\chi . \end{aligned}$$

The action $\gamma^{\mathfrak{L}}$ is covering an action $\gamma^G = L_{g_0}$ on G and an action $L^{G/T}_{g_0}$ given as well by left multiplication, this time with $g_0 = \pi_{1,\tilde{G}}(\tilde{g}_0)$.
We obtain the following result:

Theorem (5.2)**:**
In the setting of Theorem 5.1, let $\tilde{g}_0$ *be an element of* $\tilde{G}$ *such that the* $(0,1)$*-form* ω *is left invariant under the pullback with* $L^{G/T}_{g_0}$ *for* $g_0 = \pi_{1,\tilde{G}}(\tilde{g}_0)\in G$*.*
Let $\vec{\gamma}$ *denote the induced legitimate action of* $L_{\tilde{g}_0}$ *on* $\mathfrak{L}$ *given by* $\vec{\gamma} = (L^{G/T}_{g_0}, L_{g_0}, L_{\tilde{g}_0})$*.*
Then the equivariant holomorphic ζ*-function is given for large* $\mathrm{Re}(z)$ *by:*

$$Z^{\mathfrak{L}}_{\check{\gamma}^{\mathfrak{L}}}(z) = -\sum_{\lambda\neq 0}\lambda^{-z}\sum_t t(-1)^t\mathrm{ind}(\gamma^{\mathcal{W}(\lambda;t)},\Box_{\mathcal{W}(\lambda;t)}).$$

In particular, for the special case where $\tilde{g}_0\in\tilde{G}$ has the property that its projection $g_0 := \pi_{1,\tilde{G}}(\tilde{g}_0)\in G$ generates a maximal torus, we obtain a very easy expression for the equivariant holomorphic torsion of $\mathfrak{L}$ if we apply the Atiyah-Bott fixed point formula.

The next result is for rank of G greater than 2.

Corollary (5.49)**:**
In the situation of Theorem 5.2. Let G *be of rank greater than* 2 *and let* g_0 *generate a maximal torus.*
Then the equivariant holomorphic torsion vanishes, i.e.

$$\tau^{\mathfrak{L}}(\check{\gamma}^{\mathfrak{L}}) = 0.$$

For the rank 2 case, the result is a slightly more complicated.

For any $\tilde{g}_0$, we obtain a map:

$$\hat{\Omega}: \quad (G/T)^\gamma \longrightarrow \hat{T}$$

such that for every fixed point $[x]$ in $G/T = \tilde{G}/\hat{T}$ of $L_{g_0}^{G/T} = \gamma^{G/T}$ the action $L_{\tilde{g}_0}$ on the fibre $\pi_{1,\tilde{G}}^{-1} \circ \pi_G^{-1}([x])$ is given by right multiplication with $\hat{\Omega}([x])$, i.e. $\tilde{g}_0 \cdot \tilde{x} = \tilde{x} \cdot \hat{\Omega}([x])$.
This map $\hat{\Omega}$ covers a map Ω:

$$\Omega: \quad (G/T)^\gamma \longrightarrow T$$

such that for every fixed point $[x] \in (G/T)^\gamma$ the action L_{g_0} on the fibre $\pi_G^{-1}([x])$ is given by right multiplication with $\Omega([x])$.
We obtain the subsequent corollary.

Corollary (5.50)**:**
In the situation of Theorem 5.2. Let G be of rank 2 and let g_0 generate a maximal torus. Let furthermore $[x_0]$ denote one arbitrarily chosen fixed point in G/T, i.e. $[x_0] \in (G/T)^\gamma$. The equivariant holomorphic torsion becomes:

$$\tau^{\mathfrak{L}}(\check{\gamma}^{\mathfrak{L}}) = \prod_{\alpha \in R^+} \left(1 - e^{-2\pi i\alpha}\Big(\Omega([x_0])\Big)\right)^{-1} \cdot \sum_{[n] \in W(T)} \tau^{\tilde{\mathfrak{L}}}(\check{\gamma}^{\tilde{\mathfrak{L}}}_{[x_0 \cdot n]}).$$

Here, the product goes over all the positive roots of the Lie group G and $e^{-2\pi i\alpha}$ denotes the global root corresponding to $-\alpha \in R^-$:

$$\begin{array}{rccc} e^{-2\pi i\alpha}: & T & \longrightarrow & U(1) \\ & t = \exp(X) & \longmapsto & e^{-2\pi i\alpha(X)}. \end{array}$$

Furthermore, the sum in the second factor goes over the Weyl group $W(T) = N(T)/T$ of T in G, and it adds up the equivariant holomorphic torsions for the holomorphic line bundle $\mathfrak{L}\mid_T = \tilde{\mathfrak{L}} \to T$ (which is isomorphic to $\hat{T} \times_\chi \mathbb{C}$) and the actions

$$\begin{array}{rccc} \gamma^{\tilde{\mathfrak{L}}}_{[x_0 \cdot n]}: & \hat{T} \times_\chi \mathbb{C} & \longrightarrow & \hat{T} \times_\chi \mathbb{C} \\ & \left[\hat{t}, z\right]_\chi & \longmapsto & \left[\hat{\Omega}([x_0 \cdot n]) \cdot \hat{t}\right]_\chi. \end{array}$$

This way, we obtain an expression for the equivariant torsion of a flat line bundle $\mathfrak{L}$ over the Lie group G that depends only on the element $\tilde{g}_0 \in \tilde{G}$ and on the equivariant holomorphic torsions of the restricted line bundle $\mathfrak{L}\mid_T$ with actions induced by the Weyl group and $\tilde{g}_0$.

2 Preliminaries

The aim of this section is to define the objects we examine throughout this thesis.
In Subsection 2.1, we recall some facts about the moduli space of holomorphic structures for a given complex vector bundle. In particular, we state the result for the special case of a complex line bundle.
In Section 2.2, we recall the definition of the unique holomorphic, Hermitian connection for a holomorphic, Hermitian vector bundle.
In Section 2.3, we define the type of fibre bundles, we want to discuss later on, namely the holomorphic fibre bundles. Furthermore, we examine some of its properties.
Stanton defined in [29] a holomorphic fibre bundle. Our definition is a slight generalisation of hers.
Afterwards, in Section 2.4, we state a definition of smoothness for a vector bundle over a continuous fibre bundle. This is a property, we require later on for the compatibility of the line bundle over a holomorphic fibre bundle. For a vector bundle to be smooth over a fibre bundle is defined originally by Atiyah and Singer in [4]. We adapt their definition and specialise it to our scenario.
Additionally, we introduce some notations for the maps and objects which we use throughout this thesis.
Finally, in Section 2.5, we recall the definition of the equivariant torsion of an action on a holomorphic, Hermitian line bundle over a complex manifold and some equivariant invariants.

2.1 Holomorphic structures of a complex vector bundle

The equivariant holomorphic torsion is an object that belongs to an action on a holomorphic, Hermitian vector bundle over a complex or more general an almost complex manifold.

In this subsection, we give the definition of a holomorphic vector bundle and recall some facts about the space of holomorphic structures of a complex vector bundle over a complex manifold.
We start by giving a definition of a holomorphic vector bundle structure for a complex vector bundle.

Definition 2.1:
Let M be a complex manifold and let further on $\mathcal{Q} \to M$ be a smooth complex vector bundle over M.

- A family $\boldsymbol{\{(U_i, \phi_i)\}_{i\in I}}$ is called a **holomorphic trivialisation** for $\mathcal{Q}$ if the following properties hold.
    - The U_i form an open cover of M, i.e. $M = \bigcup U_i$.

– The ϕ_i are maps

$$\phi_i : \quad \mathcal{Q}\,|_{U_i} \quad \longrightarrow \quad U_i \times \mathbb{C}^m$$

that form local trivialisations of $\mathcal{Q}$ which are compatible with the smooth vector bundle structure of $\mathcal{Q}$.

– The transition maps

$$\phi_i \circ \phi_j^{-1} : \quad (U_i \cap U_j) \times \mathbb{C}^m \quad \longrightarrow \quad (U_i \cap U_j) \times \mathbb{C}^m$$

are biholomorphic.

- Two families of holomorphic trivialisations $\{(U_i, \phi_i)\}_{i\in I}$ and $\{(V_j, \psi_j)\}_{j\in J}$ for $\mathcal{Q}$ are called **equivalent** if their composition is biholomorphic, i.e. if the following maps are biholomorphic

$$\phi_i \circ \psi_j^{-1} : (U_i \cap V_j) \times \mathbb{C}^m \to (U_i \cap V_j) \times \mathbb{C}^m$$

for every pair $(i, j) \in I \times J$ with $U_i \cap V_j \neq \emptyset$.

- We call a tuple $\boldsymbol{(\mathcal{Q}, M, \left[\{(U_i, \phi_i)\}_{i\in I}\right])}$ **holomorphic vector bundle** if $\mathcal{Q} \to M$ is a smooth complex vector bundle and if $\{(U_i, \phi_i)\}_{i\in I}$ represents an equivalence class of holomorphic trivialisations of $\mathcal{Q}$.
We call such an equivalence class of holomorphic trivialisations a **holomorphic structure**.

Remark 2.2:

- For every point $x \in M$ of a complex manifold M the tangent space T_xM has a natural almost complex vector space structure (compare Definition A.1). Therefore, $T_xM \otimes_{\mathbb{R}} \mathbb{C}$ splits (compare appendix A).

$$T_xM \otimes_{\mathbb{R}} \mathbb{C} = T_x^{(0,1)}M \oplus T_x^{(1,0)}M$$

This splitting extends to the complexified tangent bundle (cf. [17])

$$TM \otimes_{\mathbb{R}} \mathbb{C} = T^{(0,1)}M \oplus T^{(1,0)}M$$

and therefore the complexified cotangent bundle splits as well

$$T^*M \otimes_{\mathbb{R}} \mathbb{C} = T^{(0,1),*}M \oplus T^{(1,0),*}M.$$

We denote $\boldsymbol{\mathfrak{A}^{(0,t)}(M)}$ to be the complex vector space of **antiholomorphic forms**. It is given by the smooth sections from M into the complex vector bundle $\Lambda^t T^{(0,1),*}M$.

Let $\mathcal{Q} \to M$ be a complex line bundle over a complex manifold. Let $\mathfrak{A}^{(0,t)}(M, \mathcal{Q})$ denote the space of **antiholomorphic forms with coefficients in $\mathcal{Q}$**, i.e. the space of smooth sections from M into the complex vector bundle $\Lambda^t \left(T^{(0,1),*}M\right) \otimes \mathcal{Q}$.

- On a complex manifold, the exterior differential d on antiholomorphic forms

$$d : \mathfrak{A}^{(0,t)}(M) \longrightarrow \mathfrak{A}^{(0,t+1)}(M) \oplus \mathfrak{A}^{(1,t)}(M)$$

splits $d = \bar{\partial} \oplus \partial$ where the operators $\bar{\partial}$ and ∂ are determined by their target space.

$$\begin{array}{rccc} \bar{\partial} : & \mathfrak{A}^{(0,t)}(M) & \longrightarrow & \mathfrak{A}^{(0,t+1)}(M) \\ \partial : & \mathfrak{A}^{(0,t)}(M) & \longrightarrow & \mathfrak{A}^{(1,t)}(M) \end{array}$$

- For any holomorphic map $f : M \to N$ between complex manifolds, the $\bar{\partial}$-operator commutes with the pullback of antiholomorphic forms, i.e. for any $\alpha \in \mathfrak{A}^{(0,*)}(N)$, we get:

$$\bar{\partial}(f^*\alpha) = f^*(\bar{\partial}\alpha). \tag{1}$$

- Let $(\mathcal{Q}, M, \{(U_i, \phi_i)\}_{i\in I})$ be a holomorphic vector bundle. The holomorphic structure of $\mathcal{Q}$ (compare Definition 2.1) induces a canonical first order operator

$$\bar{\partial}_{\mathcal{Q}} : \mathfrak{A}^{(0,q)}(M, \mathcal{Q}) \to \mathfrak{A}^{(0,q+1)}(M, \mathcal{Q}).$$

The operator $\bar{\partial}_{\mathcal{Q}}$ is locally given by:

$$\begin{tikzcd} \mathfrak{A}^{(0,q)}(U_i, \mathcal{Q}\mid_{U_i}) \arrow[r, "\bar{\partial}_{\mathcal{Q}}"] \arrow[d, "\mathrm{proj}_2\circ\phi_i"'] & \mathfrak{A}^{(0,q+1)}(U_i, \mathcal{Q}\mid_{U_i}) \arrow[d, "\mathrm{proj}_2\circ\phi_i"] \\ \mathfrak{A}^{(0,q)}(U_i, \mathbb{C}^m) \arrow[r, "\bar{\partial}"'] & \mathfrak{A}^{(0,q+1)}(U_i, \mathbb{C}^m). \end{tikzcd} \tag{2}$$

The Diagram (2) may be used to define $\bar{\partial}_{\mathcal{Q}}$ because the resulting operator does not depend on $i \in I$ This is true because the transition maps $\phi_i \circ \phi_j^{-1}$ are holomorphic (compare Equation (1)).
For the same reason, $\bar{\partial}_{\mathcal{Q}}$ does not depend on the family of holomorphic trivialisations that represents the holomorphic structure.

The $\bar{\partial}_{\mathcal{Q}}$-operator has two obvious but important properties.
On the one hand its square vanishes, $\bar{\partial}_{\mathcal{Q}}^2 = 0$, and on the other hand $\bar{\partial}_{\mathcal{Q}}$ has the same symbol as $\bar{\partial} : \mathfrak{A}^{(0,*)}(M, \mathbb{C}^m) \longrightarrow \mathfrak{A}^{(0,*)}(M, \mathbb{C}^m)$, as a differential operator.
Actually, those two properties may be used to define the holomorphic structure on the holomorphic vector bundle. This is shown in [2, Ch. 5, Thm. 5.1].

I will specify this result to the situation at hand.

Corollary 2.3:
*Let $\mathcal{Q} \to M$ be a smooth complex vector bundle over a complex manifold M.
Let further on $\bar{\partial}_{\mathcal{Q}}$ be a first order differential operator acting on $\mathfrak{A}^{(0,*)}(M, \mathcal{Q})$ such that*

$$\bar{\partial}_{\mathcal{Q}} : \mathfrak{A}^{(0,t)}(M, \mathcal{Q}) \longrightarrow \mathfrak{A}^{(0,t+1)}(M, \mathcal{Q}).$$

And, assume $\bar{\partial}_{\mathcal{Q}}^2 = 0$ and suppose $\bar{\partial}_{\mathcal{Q}}$ fulfils the Leibniz Equation (3), i.e. for any smooth form $\alpha \in \mathfrak{A}^{(0,q)}(M)$ and every section $s \in \Gamma(M, \mathcal{Q})$ we get

$$\bar{\partial}_{\mathcal{Q}} (\alpha \otimes s) = \left(\bar{\partial}\alpha\right) \otimes s + (-1)^q \alpha \otimes \left(\bar{\partial}_{\mathcal{Q}} s\right). \tag{3}$$

Then there exists a unique holomorphic structure $\left[\{(U_i, \phi_i)\}_{i \in I}\right]$ on $\mathcal{Q} \to M$ with $\bar{\partial}_{\mathcal{Q}}$ being its corresponding operator (compare Equation (2)).

From now on, we will use Corollary 2.3 without further mentioning it, i.e. we identify holomorphic structures on $\mathcal{Q} \to M$ with their corresponding $\bar{\partial}_{\mathcal{Q}}$ operators and vice verse.

In the special case of $\text{rank}(\mathcal{Q}) = 1$, i.e. $\mathcal{Q}$ is a complex line bundle, we can actually describe the set of holomorphic structures on $\mathcal{Q} \to M$. This is due to the fact that $\text{End}(\mathbb{C})$ is canonically isomorphic to $\mathbb{C}$.

Lemma 2.4:
*Let $\mathfrak{L} \to M$ be a complex line bundle that possesses a holomorphic structure $\bar{\partial}_{\mathfrak{L}}$.
Then the space of holomorphic structures on $\mathfrak{L} \to M$ is an affine space over the vector space of $\bar{\partial}$-closed $(0,1)$-forms on M.
In other words, $\bar{\partial}'_{\mathfrak{L}}$ defines a holomorphic structure for $\mathfrak{L} \to M$ if and only if there is a $\bar{\partial}$-closed differential form $\omega \in \mathfrak{A}^{(0,1)}(M)$ such that*

$$\bar{\partial}'_{\mathfrak{L}} = \bar{\partial}_{\mathfrak{L}} + \varepsilon(\omega)$$

where $\varepsilon(\omega)$ denotes the exterior multiplication with ω from the left hand side.

Proof.
Let $\omega \in \mathfrak{A}^{(0,1)}(M)$ be a $\bar{\partial}$-closed antiholomorphic form.
An easy calculation shows that $\bar{\partial}'_{\mathfrak{L}} := \bar{\partial}_{\mathfrak{L}} + \varepsilon(\omega)$ fulfils the Leibniz Equation (3) and since has $\varepsilon(\omega)$ is a 0th order differential operator, $\bar{\partial}'_{\mathfrak{L}}$ has the same symbol as $\bar{\partial}_{\mathfrak{L}}$.
Furthermore, $\left(\bar{\partial}'_{\mathfrak{L}}\right)^2 = 0$ since $\bar{\partial}\omega = 0$ and therefore, $\bar{\partial}'_{\mathfrak{L}}$ defines a holomorphic structure by Corollary 2.3.
Conversely, suppose $\bar{\partial}'_{\mathfrak{L}}$ defines a holomorphic structure.
The Leibniz Equation (3) now implies:

$$\bar{\partial}'_{\mathfrak{L}} - \bar{\partial}_{\mathfrak{L}} = \varepsilon(\omega) \in \mathfrak{A}^{(0,1)}(M, \mathfrak{L}^* \otimes \mathfrak{L}) \cong \mathfrak{A}^{(0,1)}(M)$$

since $\mathfrak{L}^* \otimes \mathfrak{L}$ is isomorphic to the trivial complex line bundle.
Finally, $\left(\bar{\partial}'_{\mathfrak{L}}\right)^2 = 0$ implies that $\bar{\partial}\omega = 0$, i.e. ω has to be $\bar{\partial}$-closed. □

We now introduce a concept of equivalence for holomorphic vector bundles that are isomorphic as complex vector bundles.

Definition 2.5:

1. Let $\mathfrak{L} \to M$ and $\mathfrak{L}' \to M$ be two holomorphic vector bundles with holomorphic structures $\bar{\partial}_{\mathfrak{L}}$ and $\bar{\partial}_{\mathfrak{L}'}$.
   $\mathfrak{L}$ and $\mathfrak{L}'$ are called **equivalent**, $\boldsymbol{(\mathfrak{L}, \bar{\partial}_{\mathfrak{L}}) \cong (\mathfrak{L}', \bar{\partial}_{\mathfrak{L}'})}$, if there is an isomorphism $g : \mathfrak{L} \to \mathfrak{L}'$ of smooth complex vector bundles which covers the identity map on M such that g commutes with the holomorphic structure, i.e.
   $$\bar{\partial}_{\mathfrak{L}} \circ g = g \circ \bar{\partial}_{\mathfrak{L}'}.$$
2. Two holomorphic structures $\bar{\partial}_{\mathfrak{L}}, \bar{\partial}'_{\mathfrak{L}}$ on one complex vector bundle are **equivalent** if there is an element $g \in C^\infty(M, \mathbb{C}^*)$, such that $\bar{\partial}_{\mathfrak{L}} = \bar{\partial}'_{\mathfrak{L}} + \varepsilon(g^{-1}\bar{\partial}g)$.
   Note that $g^{-1}\bar{\partial}g$ equals $\bar{\partial}(ln(g))$. Therefore, $g^{-1}\bar{\partial}g$ is indeed $\bar{\partial}$-closed.

The next lemma shows that these definitions are strongly related.

Lemma 2.6:
Let $\mathfrak{L} \to M$ be a complex line bundle over a compact, complex manifold. Let further on $\bar{\partial}_{\mathfrak{L}}$ as well as $\bar{\partial}'_{\mathfrak{L}}$ be two holomorphic structures on $\mathfrak{L}$. Then the following two properties are equivalent:

1. $(\mathfrak{L}, \bar{\partial}_{\mathfrak{L}}) \cong (\mathfrak{L}, \bar{\partial}_{\mathfrak{L}'})$, *i.e. the holomorphic line bundle* $(\mathfrak{L}, \bar{\partial}_{\mathfrak{L}})$ *is equivalent to the holomorphic line bundle* $(\mathfrak{L}', \bar{\partial}_{\mathfrak{L}'})$.
2. $\bar{\partial}_{\mathfrak{L}}$ *and* $\bar{\partial}_{\mathfrak{L}'}$ *are equivalent holomorphic structures on* $\mathfrak{L}$.

Proof.
1. $\Rightarrow$ 2. Let $\bar{\partial}'_{\mathfrak{L}} = \bar{\partial}_{\mathfrak{L}} + \varepsilon(\omega)$ and let $g_{\mathfrak{L}} : \mathfrak{L} \to \mathfrak{L}$ be the map defining the equivalence of $(\mathfrak{L}, \bar{\partial}_{\mathfrak{L}})$ and $(\mathfrak{L}, \bar{\partial}_{\mathfrak{L}'})$.
The map $g_{\mathfrak{L}} : \mathfrak{L} \to \mathfrak{L}$ is linear on fibres. Therefore, it can be represented by a function equally named $g : M \to \mathbb{C}^*$.
It follows that for every section $s \in \Gamma(M, \mathfrak{L})$

$$\bar{\partial}_{\mathfrak{L}}\left(g \cdot s\right) = (\bar{\partial}g) \otimes s + g \cdot (\bar{\partial}_{\mathfrak{L}} s).$$

On the other hand, we obtain:

$$\bar{\partial}'_{\mathfrak{L}} s = \bar{\partial}_{\mathfrak{L}} s + \omega \otimes s.$$

Thus, we deduce that $\omega = g^{-1}\bar{\partial} g$ which is what we wanted to show.

1. $\Leftarrow$ 2. Let now $g : M \to \mathbb{C}^*$ be a map such that $\bar{\partial}'_{\mathfrak{L}} = \bar{\partial}_{\mathfrak{L}} + \varepsilon(g^{-1} \cdot \bar{\partial} g)$.
Define the map $g_{\mathfrak{L}} : \mathfrak{L} \to \mathfrak{L}$ to be the multiplication with $\pi^*_{\mathfrak{L}} g$, i.e.

$$\begin{array}{rccc} g_{\mathfrak{L}} : & \mathfrak{L} & \longrightarrow & \mathfrak{L} \\ & l & \longmapsto & g(\pi_{\mathfrak{L}}(l)) \cdot l. \end{array}$$

Obviously, this defines a smooth line bundle isomorphism covering the identity.
What remains to be shown is that $g_{\mathfrak{L}}$ commutes with the holomorphic structure. But this is easily computed. For any $s \in \Gamma(M, \mathfrak{L})$, we obtain:

$$\bar{\partial}_{\mathfrak{L}} \circ g_{\mathfrak{L}}(s) = \bar{\partial}_{\mathfrak{L}}(g \cdot s) = (\bar{\partial} g) \otimes s + g \cdot (\bar{\partial}_{\mathfrak{L}} s) = g\left(\bar{\partial}_{\mathfrak{L}} s + g^{-1}(\bar{\partial} g) \otimes s\right) = g_{\mathfrak{L}}(\bar{\partial}_{\mathfrak{L}'} s)$$

which completes the proof. □

It is well known that the set of all equivalence classes of holomorphic line bundles over a manifold M carries a group structure (cf. [17]).

We finish this subsection by introducing a notation for this set.

Definition 2.7:
Let M be a complex manifold.

1. The **Picard Group**, $\mathbf{Pic}(M)$, of M is the group of equivalence classes of holomorphic line bundles over M where the group multiplication is given by the tensor product.

2. Let $\underline{\mathbb{C}}$ denote the trivial complex line bundle over M.
The **reduced Picard Group**, $\mathbf{Pic^0}(M)$, is the subgroup of $\mathrm{Pic}(M)$ which is given by holomorphic line bundles $\mathfrak{L}$, with the property that $\mathfrak{L}$ are isomorphic to $\underline{\mathbb{C}}$ as smooth complex line bundles.

2.2 Chern connection of a Hermitian, holomorphic vector bundle

In this subsection, we recall the definition of the unique holomorphic, Hermitian connection for any holomorphic, Hermitian vector bundle.

Let $\mathcal{Q} \to M$ be a holomorphic vector bundle with holomorphic structure $\bar{\partial}_{\mathcal{Q}}$ and let further on $h_{\mathcal{Q}}$ be a Hermitian metric on $\mathcal{Q}$.

Definition 2.8:
A **connection** on $\mathcal{Q}$ is an $\mathbb{R}$ linear map

$$\nabla : \Gamma(M, \mathcal{Q}) \to \Gamma(M, T^*M \otimes \mathcal{Q})$$

that fulfils the following Leibniz Equation for every $f \in C^\infty(M)$ and $s \in \Gamma(M, \mathcal{Q})$:

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla s.$$

Any connection ∇ (if complexified) splits into a holomorphic part $\nabla^{(1,0)}$ and an antiholomorphic part $\nabla^{(0,1)}$, i.e. $\nabla = \nabla^{(0,1)} \oplus \nabla^{(1,0)}$ where both summands are given by their target spaces, i.e.

$$\begin{array}{lccl} \nabla^{(1,0)}: & \Gamma(M,\mathcal{Q}) & \longrightarrow & \mathfrak{A}^{(1,0)}(M,\mathcal{Q}) \\ \nabla^{(0,1)}: & \Gamma(M,\mathcal{Q}) & \longrightarrow & \mathfrak{A}^{(0,1)}(M,\mathcal{Q}). \end{array}$$

For any Hermitian, holomorphic vector bundle $(\mathcal{Q}, h_{\mathcal{Q}})$, there is a natural connection on (compare [17] or [19]) which we define now.

Definition 2.9:
The **Chern connection** on $(\mathcal{Q}, h_{\mathcal{Q}})$ is the unique connection $\nabla^{\mathcal{Q}}$ on $\mathcal{Q}$ such that the following properties hold.

- $\nabla^{\mathcal{Q}}$ is a **holomorphic** connection, i.e. its antiholomorphic part $\nabla^{\mathcal{Q},(0,1)}$ equals the holomorphic structure

$$\nabla^{\mathcal{Q},(0,1)} = \bar{\partial}_{\mathcal{Q}}.$$

- $\nabla^{\mathcal{Q}}$ is a **Hermitian connection**, i.e. for any two sections $s, \tilde{s} \in \Gamma(M,\mathcal{Q})$, we get:

$$\bar{\partial}\left(h_{\mathcal{Q}}(s,\tilde{s})\right) = h_{\mathcal{Q}}(\nabla^{\mathcal{Q},(0,1)}s, \tilde{s}) + h_{\mathcal{Q}}(s, \nabla^{\mathcal{Q},(1,0)}\tilde{s}).$$

If we extend the Leibniz Equation to differential forms, the Chern connection induces a natural derivative on $\mathfrak{A}^{(*,\cdot)}(M,\mathcal{Q})$.

Definition 2.10:
There is a natural extension of $\nabla^{\mathcal{Q}}$ to $\mathfrak{A}^{(*,\cdot)}(M,\mathcal{Q}) = \mathfrak{A}^{(*,\cdot)}(M) \otimes \Gamma(M,\mathcal{Q})$, i.e. the complexified differential forms with coefficients in $\mathcal{Q}$.
It is given by

$$\nabla^{\mathcal{Q}}(\alpha \otimes s) := (d\alpha) \otimes s + (-1)^{|\alpha|}\alpha \wedge \left(\nabla^{\mathcal{Q}} s\right) \tag{4}$$

for arbitrary $\alpha \in \mathfrak{A}^{(*,\cdot)}(M)$ and $s \in \Gamma(M,\mathcal{Q})$, $\mathbb{C}$-linear extended to the whole tensor product.

2.3 Holomorphic fibre bundle, definition and properties

This subsection is dedicated to defining and understanding the kind of fibre bundle we want to study later on, namely the holomorphic fibre bundle.
Our definition is a slight generalisation of Stantons definition of a holomorphic fibre bundle in [29]. Most of the properties of Stantons holomorphic fibre bundle extend to

this generalisation. In particular, for a holomorphic line bundle $\mathfrak{L} \to E$, the splitting of the $\bar{\partial}_{\mathfrak{L}}$-operator holds with the same argument (compare Subsection 2.3.1). Furthermore, we adapt her results about how the $\bar{\partial}_{\mathfrak{L}}$-operator commutes with the pullback of antiholomorphic forms on M (compare Subsection 2.3.2).

A holomorphic fibre bundle carries a lot of structures, f.e. a Riemannian metric or a complex structure.
For a holomorphic fibre bundle, these structures need to be comparable for different fibres, therefore the transition functions have to be maps into a structure preserving group. To clarify what we mean by such a structure preserving morphism, we now define $\mathrm{Aut}(F)$ for a complex, Riemannian manifold F.

Definition 2.11:
Let F be a compact, complex Riemannian manifold with complex structure J_F and a compatible (compare Definition A.1) Riemannian metric g_F.

- The **group of biholomorphic maps from** $(\mathbf{F}, J_F)$ **to** $(\mathbf{F}, J_F)$ will be denoted by $\mathbf{Hol}(\boldsymbol{F})$.
- Similarly, we denote the **group of isometries of** $(\mathbf{F}, g_F)$ by $\mathbf{Isom}(\boldsymbol{F})$.
- Now, we define the **automorphism group of** $(\mathbf{F}, J_F, g_F)$ to be the group $\mathbf{Aut}(\boldsymbol{F})$ of all biholomorphic isometries from $(\mathbf{F}, J_F, g_F)$ to $(\mathbf{F}, J_F, g_F)$, i.e.

$$\mathrm{Aut}(F) = \mathrm{Hol}(F) \cap \mathrm{Isom}(F).$$

Remark 2.12:

- A known fact is that we can endow $\mathrm{Hol}(F)$ with a complex Lie group structure (cf. [20]). Its Lie algebra consists of all vector fields having a biholomorphic flow, the so-called holomorphic vector fields $\boldsymbol{\mathfrak{hol}(F)}$. The left invariant almost complex structure on $\mathrm{Hol}(F)$ is given by the almost complex structure on F restricted to the holomorphic vector fields. Similarly, the Lie bracket on $\mathfrak{hol}(F)$ is given by the commutator of the holomorphic vector fields.
  In particular, the integrability of the almost complex structure on F directly implies the integrability of the almost complex structure on $\mathrm{Hol}(F)$ making it a complex structure.
- Similarly, it is known that for any compact Riemannian manifold F $\mathrm{Isom}(F)$ is a compact Lie group (in [20]). Its Lie algebra consists of the Killing vector fields on (F, g_F).
- Unfortunately, it rarely happens for a complex, Riemannian manifold F that $\mathrm{Isom}(F)$ inherits a complex structure from F.

This can be seen as follows: Assume $\mathrm{Isom}(F)$ inherits such a complex structure, then $\mathrm{Aut}(F)$ inherits a complex Lie group structure as well. On the other hand $\mathrm{Aut}(F) \subset \mathrm{Isom}(F)$ is compact.
Every connected, compact, complex Lie group is a complex torus. Consequently, $\mathrm{Aut}(F)$ has to be a finite disjoint union of complex tori.

Now, that we have established a concept of structure preserving automorphisms, it is time to define what a holomorphic fibre bundle should be.

Recall therefore that for a smooth fibre bundle $\pi_E : E \to M$ a **connection** is a horizontal distribution, i.e. a subbundle $T^H E$ which is a direct summand to $T^V E := \mathrm{Ker}(d\pi_E)$ such that $TE = T^H E \oplus T^V E$.

Definition 2.13:
Let F and M be compact, complex manifolds with compatible Riemannian metrics g_F and g_M. Let further on $\pi_E : E \to M$ be a smooth fibre bundle whose fibretype is F. We call the tuple $\boldsymbol{(E, \pi_E, (M, g_M), (F, g_F), T^H E)}$ a **holomorphic fibre bundle** if on the one hand there are local trivialisation $\{(U_k, \phi_k)\}_{k \in I}$ of E such that the maps

$$\phi_{i,x} \circ \phi_{k,x}^{-1} := \phi_i\big|_{E_x} \circ \left(\phi_k\big|_{E_x}\right)^{-1} : \{x\} \times F \xrightarrow{\sim} \{x\} \times F$$

and

$$\begin{array}{rcccl} \widehat{\phi_i \circ \phi_j^{-1}} : & U_i \cap U_j & \longrightarrow & \mathrm{Aut}(F) \subset \mathrm{Hol}(F), & \text{defined by} \\ & u & \longmapsto & \left(f \mapsto \mathrm{proj}_2 \circ \left(\phi_i \circ \phi_j^{-1}\right)(u, f)\right), & \end{array}$$

are holomorphic (*This induces a complex structure on E, compare Lemma 2.16.*) and if on the other hand $T^H E$ is a connection on $E \to M$ with the following properties.

- The complex structure J_E on E preserves the splitting $T^H E \oplus T^V E$.
  Thereby, we mean that the spaces $T^H E$ and $T^V E$ are J_E-invariant, i.e.

$$J_E(T^H E) = T^H E \quad \text{and} \quad J_E(T^V E) = T^V E.$$

- The **connection** $T^H E$ is **of type** $\boldsymbol{(1,1)}$ which means that for

$$T^{H,(0,1)} E := T^{(0,1)} E \cap \left(T^H E \otimes_{\mathbb{R}} \mathbb{C}\right)$$

  the space $\Gamma(E, T^{H,(0,1)} E)$ is closed under commutator brackets, i.e.

$$[\Gamma(E, T^{H,(0,1)} E), \Gamma(T^{H,(0,1)} E)] \subset \Gamma(E, T^{H,(0,1)} E). \tag{5}$$

Remark 2.14:
Our definition of a holomorphic fibre bundle differs from that of Stanton in [29] because she wants her fibre bundle to be associated to a principle fibre bundle whose fibre is a compact Lie group.
For her, that is a natural assumption to make for a holomorphic fibre bundle because she was looking at holomorphic line bundles associated to a representation of the fundamental group of the total space E only.
For us on the other hand, this property belongs to the line bundle over the total space not to the holomorphic fibre bundle itself.
Actually, we generalise this property. The holomorphic line bundles $\mathfrak{L} \to G$ we want to look at are supposed to be smooth vector bundles over the fibre bundle $E \to M$. We define what we mean by this property in Subsection 2.4.

Definition 2.13 looks rather excessive, therefore we now analyse what these properties imply and why we required them, to illuminate their necessity and usefulness.

First of all, as promised above, we show that the holomorphy property for the maps $\phi_{i,x} \circ \phi_{k,x}^{-1}$ and for $\widehat{\phi_i \circ \phi_j^{-1}}$ induces a complex structure on E, making E a complex manifold.
Actually, we prove an equivalence of definitions for the complex structure on the total space E of our holomorphic fibre bundle.

Remark 2.15:
Recall that $F \hookrightarrow E \to M$ is a **complex fibre bundle** if F and M are complex manifolds and the local trivialisations $\{\phi_i, U_i\}_{i\in I}$ induce holomorphic transition maps, i.e.

$$\phi_i \circ \phi_k^{-1} : (U_i \cap U_k) \times F \to (U_i \cap U_k) \times F$$

is holomorphic for every pair $(i, k) \in I \times I$.
In particular, the local trivialisations induce a complex manifold structure on E.

The content of the subsequent lemma is general knowledge. Nonetheless, we state as well as proof it here for a lack of sources to cite from.

Lemma 2.16:

1. *Let $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ be a holomorphic fibre bundle, then E carries the structure of a complex manifold.*
   In particular, $F \hookrightarrow E \to M$ becomes a complex fibre bundle.

2. *Let on the other hand $F \hookrightarrow E \to M$ be a complex fibre bundle.*

Then the maps

$$\phi_{i,x} \circ \phi_{k,x}^{-1} := \phi_i\big|_{E_x} \circ \left(\phi_k\big|_{E_x}\right)^{-1} : \{x\} \times F \xrightarrow{\sim} \{x\} \times F$$

as well as

$$\widehat{\phi_i \circ \phi_j^{-1}} : U_i \cap U_j \longrightarrow \mathrm{Hol}(F)$$

are holomorphic.

Proof.

1. It suffices to show that for a holomorphic fibre bundle the transition map

$$\phi_i \circ \phi_k^{-1} : (U_i \cap U_k) \times F \to (U_i \cap U_k) \times F$$

is holomorphic for any $i, k \in I$ because if this is the case, we can define the complex structure of E locally on $U_i \times F$ and patch it together along those biholomorphic transition maps.
But since $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ is a holomorphic fibre bundle, the following equations:

$$\phi_i \circ \phi_k^{-1}(u, f) = \left(u, \left[\widehat{\phi_i \circ \phi_k^{-1}}(u)\right](f)\right) \tag{6}$$

$$= \left[\phi_{i,u} \circ \phi_{k,u}^{-1}\right](u, f) \tag{7}$$

show that $\phi_i \circ \phi_k^{-1}$ is holomorphic in $u \in U_i \cap U_k$ (Equation (6)) and in $f \in F$ (Equation (7)) and therefore in (u, f).
Hence, $F \hookrightarrow E \to M$ becomes a complex fibre bundle.

2. Suppose now, that $\phi_i \circ \phi_k^{-1}$ are biholomorphic for any pair $(i, k) \in I \times I$.
Consequently, we get that the map

$$\phi_i \circ \phi_k^{-1}\,|_{\{x\}\times F} : \begin{array}{ccc} \{x\} \times F & \longrightarrow & \{x\} \times F \\ f & \longmapsto & \left[\phi_{i,x} \circ \phi_{k,x}^{-1}\right](x, f) \end{array}$$

is holomorphic by restriction for any $x \in U_i \cap U_k$.
It remains to be shown that $u \mapsto \widehat{\phi_i \circ \phi_k^{-1}}(u) \in \mathrm{Hol}(F)$ is holomorphic as well.
Since $\mathrm{Hol}(F)$ is a Lie group (Remark 2.12), its tangent bundle $T\mathrm{Hol}(F)$ is trivial, i.e. bundle isomorphic to $\mathrm{Hol}(F) \times \mathfrak{hol}(F)$.
If we differentiate

$$\widehat{\phi_i \circ \phi_k^{-1}}(x) : \begin{array}{ccc} F & \longmapsto & F \\ f & \longmapsto & \mathrm{proj}_2 \circ \left(\phi_i \circ \phi_k^{-1}\right)(x, f), \end{array}$$

we obtain for $X \in T_x\left(U_i \cap U_k\right)$:

$$\begin{array}{rccc} T_x\widehat{\phi_i \circ \phi_k^{-1}}(X): & F & \longrightarrow & T_{\phi_i\circ\phi_k^{-1}(x)(f)}F \\ & f & \longmapsto & \left(\mathrm{proj}_2 \circ T_{(x,f)}\left(\phi_i \circ \phi_k^{-1}\right)(X,0)\right) \end{array}$$

Now, we make use of the fact that $T_g\mathrm{Hol}(F) = T_eL_g\mathfrak{hol}(F)$ for any $g \in \mathrm{Hol}(F)$ to identify $\left(T_x\widehat{\phi_i \circ \phi_k^{-1}}\right)(X) \in T_{\widehat{\phi_i\circ\phi_k^{-1}}(x)}\mathrm{Hol}(F)$ with a holomorphic vector field $\Phi_{i,k}(x) \in \mathfrak{hol}(F)$.
$\Phi_{i,k}(x) \in \mathfrak{hol}(F)$ is given by

$$f \mapsto \left(T_{\widehat{\phi_i\circ\phi_k^{-1}}(x)(f)}\left(\widehat{\phi_i \circ \phi_k^{-1}}(x)^{-1}\right)\right)\left[\left(\mathrm{proj}_2 \circ T_{(x,f)}\left(\phi_i \circ \phi_k^{-1}\right)\right)(X,0)\right].$$

We see that

$$\mathrm{proj}_2 \circ T_{(x,f)}\left(\phi_i \circ \phi_k^{-1}\right)(J_MX,0) = J_F\left(\mathrm{proj}_2 \circ T_{(x,f)}\left(\phi_i \circ \phi_k^{-1}\right)(X,0)\right)$$

because proj_2 as well as $\phi_i \circ \phi_k^{-1}$ are holomorphic by assumption.
Hence,

$$T_x\widehat{\phi_i \circ \phi_k^{-1}}(J_MX) = J_{Hol(F)}T_x\widehat{\phi_i \circ \phi_k^{-1}}(X)$$

since the complex structure on $\mathrm{Hol}(F)$ is induced by the complex structure on F applied to the holomorphic vector fields.
We deduce that $T_x\widehat{\phi_i \circ \phi_k^{-1}}$ is complex linear and therefore $x \mapsto \widehat{\phi_i \circ \phi_k^{-1}}(x)$ is holomorphic.

□

Remark 2.17:
We can endow E not only with a complex structure, using the complex structures of F and M but with a compatible Riemannian metric $g = g_E$ as well such that:

- the horizontal and the vertical tangent space are perpendicular with regard to g, i.e. $T^HE \perp_g T^VE$,
- the inclusion of the fibre is an isometric immersion, i.e. $\left(\phi_{i,x}^{-1}\right)^* g = g_F$ for all $i \in I$ and all $x \in U_i$,
- $d\pi_E : (T^HE, g\mid_{T^HE\otimes T^HE}) \to (TM, g_M)$ is point wise a linear isometry, i.e. π_E is a Riemannian submersion.

This can be done by using the splitting $TE = T^HE \oplus T^VE$ which is invariant under the complex structure on E.

We now summarise what we know so far.
We have three complex manifolds M, F and E with compatible Riemannian metrics g_M, g_F and g. The projection $\pi_E : E \to M$ is a holomorphic Riemannian submersion where the horizontal tangent space of E is mapped isometrically to the tangent space of M while the inclusions of the fibre F is a holomorphic, Riemannian immersion.

Next in line is to understand why we wanted our connection to be of type $(1,1)$. This is the content of Subsection 2.3.1.
In Subsection 2.3.3, we describe how a holomorphic line bundle $\mathfrak{L}$ over a holomorphic fibre bundle induces holomorphic line bundles over every fibre by reduction, i.e. by pullback under the inclusion.

2.3.1 Splitting properties

Let $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ be a holomorphic fibre bundle.
Let furthermore $\pi_{\mathfrak{L}} : (\mathfrak{L}, h) \to E$ be a Hermitian, holomorphic line bundle, i.e. a holomorphic line bundle $\mathfrak{L}$ with a Hermitian metric h over E. We denote its holomorphic structure by $\bar{\partial}_{\mathfrak{L}}$.

The splitting $TE = T^H E \oplus^{\perp_{g_E}} T^V E$ leads to an orthogonal splitting of the antiholomorphic forms with coefficients in $\mathfrak{L}$. In this subsection, we show that this orthogonal splitting extends to a splitting of the operator $\bar{\partial}_{\mathfrak{L}}$ into a vertical and a horizontal part. This is due to the fact that our connection is of type $(1,1)$ which is why we require this in the first place.

We prove this kind splitting property for the $\bar{\partial}_{\mathfrak{L}}$-operator in slightly more generality. In order to do that, we extend the property to be of type $(1,1)$-property from horizontal distributions to general distributions that are invariant under the complex structure.

Definition 2.18:
Let (E, J_E) be a complex manifold with compatible Riemannian metric g_E.
A J_E-invariant distribution $D \subset TE$ is called of **type $(1,1)$**, if

$$[\Gamma(E, D^{(0,1)}), \Gamma(E, D^{(0,1)})] \subset \Gamma(E, D^{(0,1)}).$$

Remark 2.19:

- Recall that for any distribution $D \subset TE$ on a Riemannian manifold, we get an orthogonal distribution $D^\perp$. In particular, $D = \left(D^\perp\right)^\perp$.
- Let $\mathcal{Q} \longrightarrow E$ be a complex line bundle.
  Every distribution $D \subset TE$ defines a natural subspace $\mathfrak{A}_D^{(0,*)}(E, \mathcal{Q})$ of $\mathfrak{A}^{(0,*)}(E, \mathcal{Q})$ where α lies in $\mathfrak{A}_D^{(0,*)}(E, \mathcal{Q})$ if and only if $\iota_X \alpha = 0$ for every section $X \in \Gamma(E, D^\perp \otimes_{\mathbb{R}} \mathbb{C})$.

In particular, we obtain the following splitting:

$$\mathfrak{A}^{(0,q)}(E,\mathcal{Q}) = \bigoplus_{s+t=q} \mathfrak{A}_D^{(0,s)}(E) \wedge \mathfrak{A}_{D^\perp}^{(0,t)}(E,\mathcal{Q}).$$

If additionally, $\mathcal{Q}$ is a holomorphic vector bundle, then we have a well defined $\bar\partial_{\mathcal{Q}}$ operator which is a first order differential operator $\bar\partial : \mathfrak{A}^{(0,*)}(E,\mathcal{Q}) \to \mathfrak{A}^{(0,*+1)}(E,\mathcal{Q})$. For D and $D^\perp$ of type $(1,1)$ the $\bar\partial_{\mathcal{Q}}$-operator splits into a D and a $D^\perp$ part.

Lemma 2.20:
Let $\pi_{\mathcal{Q}} : (\mathcal{Q},h) \to E$ be a Hermitian, holomorphic vector bundle over a complex Riemannian manifold E with compatible metric.
Furthermore, let $D \subset TE$ be a distribution of type $(1,1)$ such that $D^\perp$ is a distribution of type $(1,1)$ as well.
Then there are two first order differential operators

$$\begin{array}{rlcl} \bar\partial_D : & \mathfrak{A}_D^{(0,s)}(E) \wedge \mathfrak{A}_{D^\perp}^{(0,t)}(E,\mathcal{Q}) & \longrightarrow & \mathfrak{A}_D^{(0,s+1)}(E) \wedge \mathfrak{A}_{D^\perp}^{(0,t)}(E,\mathcal{Q}) \\ \bar\partial_{D^\perp} : & \mathfrak{A}_D^{(0,s)}(E) \wedge \mathfrak{A}_{D^\perp}^{(0,t)}(E,\mathcal{Q}) & \longrightarrow & \mathfrak{A}_D^{(0,s)}(E) \wedge \mathfrak{A}_{D^\perp}^{(0,t+1)}(E,\mathcal{Q}) \end{array}$$

such that

$$\bar\partial_{\mathcal{Q}} = \bar\partial_D + \bar\partial_{D^\perp}.$$

If we denote the orthogonal projection by

$$Q_{p,q} : \mathfrak{A}^{(0,*)}(E,\mathcal{Q}) \longrightarrow \mathfrak{A}_D^{(0,p)}(E) \wedge \mathfrak{A}_{D^\perp}^{(0,q)}(E,\mathcal{Q}),$$

then the operators $\bar\partial_D$ and $\bar\partial_{D^\perp}$ restricted to $\mathfrak{A}_D^{(0,p)}(E) \wedge \mathfrak{A}_{D^\perp}^{(0,q)}(E,\mathcal{Q})$ are given by

$$\bar\partial_D = Q_{p+1,q} \circ \bar\partial_{\mathcal{Q}} \quad \text{and} \quad \bar\partial_{D^\perp} = Q_{p,q+1} \circ \bar\partial_{\mathcal{Q}}.$$

Proof.
Let α be an antiholomorphic form in $\mathfrak{A}_D^{(0,p)}(E) \wedge \mathfrak{A}_{D^\perp}^{(0,q)}(E,\mathcal{Q})$.
We have to show that $\bar\partial_{\mathcal{Q}}\alpha$ lies in

$$\mathfrak{A}_D^{(0,p+1)}(E) \wedge \mathfrak{A}_{D^\perp}^{(0,q)}(E,\mathcal{Q}) \oplus \mathfrak{A}_D^{(0,p)}(E) \wedge \mathfrak{A}_{D^\perp}^{(0,q+1)}(E,\mathcal{Q}).$$

The Leibniz rule (Equation (3)), i.e.

$$\bar\partial_{\mathcal{Q}}(\alpha \wedge \beta) = (\bar\partial_{\mathcal{Q}}\alpha) \wedge \beta + (-1)^{|\alpha|}\alpha \wedge (\bar\partial_{\mathcal{Q}}\beta),$$

enables us to reduce our investigations to the case $\alpha \in \mathfrak{A}^{(0,1)}(E,\mathcal{Q})$.

Since D and $D^\perp$ are interchangeable, we can assume without loss of generality that α lives in $\mathfrak{A}^{(0,1)}_{D^\perp}(E,\mathcal{Q})$, i.e. $\alpha = \sum_i \alpha_i \otimes s_i$ where $s_i \in \Gamma(E,\mathcal{Q})$ and $\alpha_i \in \mathfrak{A}^{(0,1)}_{D^\perp}(E)$.
Hence, for two $D^{0,1}$ valued vector fields $X, Y \in \Gamma(E, D^{0,1})$:

$$\begin{aligned}\left(\bar\partial_{\mathcal{Q}}\alpha\right)(X,Y) &= \sum_i (\bar\partial\alpha_i)(X,Y)\cdot s_i - \left(\alpha_i \wedge (\bar\partial_{\mathcal{Q}} s_i)\right)(X,Y)\\ &= \sum_i (\bar\partial\alpha_i)(X,Y)\cdot s_i - \underbrace{\alpha_i(X)}_{=0}(\bar\partial_{\mathcal{Q}} s_i)(Y) + \underbrace{\alpha_i(Y)}_{=0}(\bar\partial_{\mathcal{Q}} s_i)(X).\end{aligned}$$

Computing the first term we obtain:

$$\left(\bar\partial\alpha_i\right)(X,Y) = X.\underbrace{\alpha(Y)}_{=0} - Y.\underbrace{\alpha(X)}_{=0} - \alpha(\underbrace{[X,Y]}_{\in\Gamma(E,D^{0,1})}) = 0$$

since the distribution D is of of type $(1,1)$.
We deduce that $\bar\partial_{\mathcal{Q}}\alpha(X,Y) = 0$. □

Remark 2.21:
Let $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ be a holomorphic fibre bundle and $\mathfrak{L} \to E$ be a holomorphic line bundle.
We have a J_E invariant horizontal distribution $D = T^H E$ of type $(1,1)$ and a perpendicular vertical distribution $D^\perp = T^V E$ which is of type $(1,1)$ too since it is the push forward of the complex tangent bundle TF via the holomorphic embedding $i_x : F \hookrightarrow E$.
We deduced that the $\bar\partial_{\mathfrak{L}}$-operator splits into a vertical and a horizontal portion. This is crucial for the whole setting of this work since one of the main aspects, we use later on, is the splitting of the Dolbeault-Laplacian into a vertical and a horizontal part which would fail to hold if the $\bar\partial_{\mathfrak{L}}$-operator wouldn't split.

Definition 2.22:
From now on and throughout this thesis, we will use the following simplified notations:

$$\boldsymbol{\mathfrak{A}^{(0,*)}_H(E)} := \mathfrak{A}^{(0,*)}_{T^H E}(E) \quad \text{and} \quad \boldsymbol{\mathfrak{A}^{(0,*)}_V(E)} := \mathfrak{A}^{(0,*)}_{T^V E}(E)$$

as well as $\boldsymbol{\bar\partial_H} := \bar\partial_{T^H E}$ and $\boldsymbol{\bar\partial_V} := \bar\partial_{T^V E}$.

Remark 2.23:
The antiholomorphic forms split orthogonally

$$\mathfrak{A}^{(0,q)}(E,\mathfrak{L}) = \bigoplus_{s+t=q} \mathfrak{A}^{(0,t)}_H(E) \wedge \mathfrak{A}^{(0,s)}_V(E,\mathfrak{L}),$$

for the L^2-Hermitian metric induced by the Hermitian metrics on $\mathfrak{L}$ and on TE. Furthermore, we saw in Lemma 2.20 and Remark 2.21 that

$$\bar{\partial}_{\mathfrak{L}} : \mathfrak{A}_H^{(0,t)}(E) \wedge \mathfrak{A}_V^{(0,s)}(E,\mathfrak{L}) \longrightarrow \mathfrak{A}_H^{(0,t+1)}(E) \wedge \mathfrak{A}_V^{(0,s)}(E,\mathfrak{L}) \oplus \mathfrak{A}_H^{(0,t)}(E) \wedge \mathfrak{A}_V^{(0,s+1)}(E,\mathfrak{L})$$

splits into a vertical operator $\bar{\partial}_V$ and a horizontal one $\bar{\partial}_H$.

$$\begin{array}{rlcl} \bar{\partial}_V : & \mathfrak{A}_H^{(0,t)}(E) \wedge \mathfrak{A}_V^{(0,s)}(E,\mathfrak{L}) & \longrightarrow & \mathfrak{A}_H^{(0,t)}(E) \wedge \mathfrak{A}_V^{(0,s+1)}(E,\mathfrak{L}) \\ \bar{\partial}_H : & \mathfrak{A}_H^{(0,t)}(E) \wedge \mathfrak{A}_V^{(0,s)}(E,\mathfrak{L}) & \longrightarrow & \mathfrak{A}_H^{(0,t+1)}(E) \wedge \mathfrak{A}_V^{(0,s)}(E,\mathfrak{L}) \end{array}$$

Consequently, the adjoint $\bar{\partial}_{\mathfrak{L}}^*$ splits as well,

$$\bar{\partial}_{\mathfrak{L}}^* : \mathfrak{A}_H^{(0,t)}(E) \wedge \mathfrak{A}_V^{(0,s)}(E,\mathfrak{L}) \longrightarrow \mathfrak{A}_H^{(0,t-1)}(E) \wedge \mathfrak{A}_V^{(0,s)}(E,\mathfrak{L}) \oplus \mathfrak{A}_H^{(0,t)}(E) \wedge \mathfrak{A}_V^{(0,s-1)}(E,\mathfrak{L})$$

into $\boldsymbol{\bar{\partial}_V^*}$ and $\boldsymbol{\bar{\partial}_H^*}$ given by

$$\begin{array}{rlcl} \bar{\partial}_V^* : & \mathfrak{A}_H^{(0,t)}(E) \wedge \mathfrak{A}_V^{(0,s)}(E,\mathfrak{L}) & \longrightarrow & \mathfrak{A}_H^{(0,t)}(E) \wedge \mathfrak{A}_V^{(0,s-1)}(E,\mathfrak{L}) \\ \bar{\partial}_H^* : & \mathfrak{A}_H^{(0,t)}(E) \wedge \mathfrak{A}_V^{(0,s)}(E,\mathfrak{L}) & \longrightarrow & \mathfrak{A}_H^{(0,t-1)}(E) \wedge \mathfrak{A}_V^{(0,s)}(E,\mathfrak{L}). \end{array}$$

2.3.2 Splitting of the holomorphic structure on pullback-forms

Let $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ be a holomorphic fibre bundle and furthermore let $(\mathfrak{L}, h) \to E$ be a Hermitian, holomorphic line bundle.
In this subsection, we investigate how the pullback of an antiholomorphic form on M interferes with the splitting of the $\bar{\partial}_{\mathfrak{L}}$-operator.

The pullback of a $(0,q)$-form on M is obviously a horizontal form. To be more precise the space $\mathfrak{A}_H^{(0,*)}(E)$ of horizontal antiholomorphic forms on E is a $C^\infty(E,\mathbb{C})$ module over the vector space of pullback forms $\pi_E^*(\mathfrak{A}^{(0,*)}(M))$.

$$\mathfrak{A}_H^{(0,*)}(E) = C^\infty(E,\mathbb{C}) \otimes_{\mathbb{C}} \pi_E^* \left(\mathfrak{A}^{(0,*)}(M)\right)$$

It is a finitely generated module since E is compact.

Therefore, it is self-evident to check how the operators $\bar{\partial}_V$ and $\bar{\partial}_H$ act on these pullbacks.

Lemma 2.24:
Let again $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ be a holomorphic fibre bundle and $\mathfrak{L}$ be a holomorphic line bundle over E.
For $\mu \in \mathfrak{A}^{0,q}(M)$ and $\omega \in \mathfrak{A}_V^{(0,p)}(E,\mathfrak{L})$, the following equations hold.

$$\bar{\partial}_V \left(\pi_E^*\mu \wedge \omega\right) = (-1)^q \pi_E^*\mu \wedge \bar{\partial}_V \omega \tag{8}$$
$$\bar{\partial}_H \left(\pi_E^*\mu \wedge \omega\right) = \pi_E^*(\bar{\partial}\mu) \wedge \omega + (-1)^q \pi_E^*\mu \wedge \bar{\partial}_H \omega \tag{9}$$

Proof.
Since $\bar{\partial}_{\mathfrak{L}}$ fulfils the Leibniz rule, so does $\bar{\partial}_V$:

$$\bar{\partial}_V \left(\pi_E^*\mu \wedge \omega\right) = (\bar{\partial}_V \pi_E^*\mu) \wedge \omega + (-1)^q \pi_E^*\mu \wedge \bar{\partial}_V \omega.$$

Hence, we have to check that $\bar{\partial}_V \pi_E^*\mu = 0$ which can be reduced inductively to the cases $q = 0$ and $q = 1$.
Start with $q = 0$ and with a vertical vector field $X \in \Gamma(E, T^{V,(0,1)}E)$, we obtain:

$$(\bar{\partial}_V \pi_E^*\mu)_e(X) = d(\mu \circ \pi_E)_e(X) =: (X.\mu \circ \pi_E)_e = 0$$

since $\mu \circ \pi_E$ is vertically constant.
Now for the $q = 1$ part, take $X \in \Gamma(E, T^{V,(0,1)}E)$ and a horizontal lift $\tilde{Y} \in \Gamma(E, T^{H,(0,1)}E)$ of $Y \in \Gamma(M, T^{(0,1)}M)$:

$$\begin{aligned}(\bar{\partial}_V \pi_E^*\mu)_e(X, \tilde{Y}) &= X.\pi_E^*\mu(\tilde{Y}) - \tilde{Y}.\pi_E^*\mu(X) - \pi_E^*\mu(\underbrace{[X, \tilde{Y}]}_{\in\Gamma(E, T^V E \otimes_{\mathbb{R}} \mathbb{C})}) \\ &= X.\left(\mu(Y) \circ \pi_E\right) - \tilde{Y}.0 - 0 = 0.\end{aligned}$$

Thus, we have shown Equation (8).
Equation (9) follows instantaneously from the Leibniz rule (Equation (3)) and the fact that π_E is a holomorphic map. □

We show next that an equation analogous to Equation (8) holds for the adjoint operator $\bar{\partial}_V^*$ of $\bar{\partial}_V$ as well.

Lemma 2.25:
Let $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ be a holomorphic fibre bundle and $(\mathfrak{L}, h) \to E$ be a holomorphic, Hermitian line bundle over E.
For $\mu \in \mathfrak{A}^{0,q}(M)$ and $\omega \in \mathfrak{A}_V^{(0,p)}(E, \mathfrak{L})$, the following equation holds.

$$\bar{\partial}_V^* \left(\pi_E^*\mu \wedge \omega\right) = (-1)^{|\mu|} \pi_E^*\mu \wedge \bar{\partial}_V^* \omega$$

Proof.
Recall that the metrics g_M and g_E are compatible with the complex structure. Hence, they induce Hermitian metrics on TM and TE. By extension we obtain Hermitian metrics h_M on $\Lambda^\cdot \left(T^{(0,1)M}\right)^*$ and h_E $\Lambda^\cdot \left(T^{(0,1)}E\right)^*$. Denote in abuse of notation by $h = h_E \otimes h$ the Hermitian metric on $\Lambda^\cdot \left(T^{(0,1)}E\right)^* \otimes \mathfrak{L}$.
Let further on $\langle \cdot, \cdot \rangle_{L^2}$ denote the L^2 Hermitian product on differential forms, i.e. on $\mathfrak{A}^{(0,*)}(E, \mathfrak{L})$, induced by h.

We obtain:

$$
\begin{aligned}
\langle \pi_E^*\alpha \wedge \beta, \bar{\partial}_V^* \left(\pi_E^*\mu \wedge \omega\right)\rangle_{L^2} &= \langle \bar{\partial}_V \left(\pi_E^*\alpha \wedge \beta\right), \pi_E^*\mu \wedge \omega\rangle_{L^2} \\
&\overset{(8)}{=} \langle (-1)^{|\alpha|}\pi_E^*\alpha \wedge \bar{\partial}_V\beta, \pi_E^*\mu \wedge \omega\rangle_{L^2} \\
&= \int_E (-1)^{|\alpha|}\pi_E^*\left(h_M(\alpha,\mu)\right) \cdot h(\bar{\partial}_V\beta, \omega)
\end{aligned}
$$

Note that the function $\pi_E^*\left(h_M(\alpha,\mu)\right)$ is constant along each fibre. Thus, its multiplication commutes with the $\bar{\partial}_V$-operator (compare Equation (8)).
Consequently, we get:

$$
\begin{aligned}
\langle \pi_E^*\alpha \wedge \beta, \bar{\partial}_V^* \left(\pi_E^*\mu \wedge \omega\right)\rangle_{L^2} &= \int_E (-1)^{|\alpha|} h\Big(\bar{\partial}_V\Big(\pi_E^*\left(h_M(\alpha,\mu)\right)\cdot\beta\Big), \omega\Big) \\
&= \int_E (-1)^{|\alpha|} h\Big(\pi_E^*\left(h_M(\alpha,\mu)\right)\cdot\beta, \bar{\partial}_V^*\omega\Big) \\
&= \int_E (-1)^{|\alpha|}\pi_E^*\left(h_M(\alpha,\mu)\right) \cdot h(\beta, \bar{\partial}_V^*\omega) \\
&= \langle \pi_E^*\alpha \wedge \beta, (-1)^{|\alpha|}\pi_E^*\mu \wedge \bar{\partial}_V^*\omega\rangle_{L^2}.
\end{aligned}
$$

Now, the assertion follows from the fact that $\pi_E^*\left(h_M(\alpha,\mu)\right) \neq 0$ directly implies that $|\alpha| = |\mu|$. $\square$

The $\bar{\partial}_V$ operator as well as its adjoint $\bar{\partial}_V^*$ act trivially on pullback forms. Therefore, so does their sum.
We define $L_V := \bar{\partial}_V + \bar{\partial}_V^*$ and derive trivially:

$$
L_V\left(\pi_E^*\mu \wedge \omega\right) = (-1)^{|\mu|}\pi_E^*\mu \wedge L_V\omega.
$$

Now, we formally define the vertical as well as the horizontal Laplacian for a holomorphic, Hermitian line bundle over holomorphic fibre bundle.

Definition 2.26:
Let $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ be a holomorphic fibre bundle and $\mathfrak{L} \to E$ be a holomorphic, Hermitian line bundle over E.

- The **vertical Laplacian** $\square_{\boldsymbol{V}}$ is a differential operator acting on $\mathfrak{A}^{(0,*)}(E,\mathfrak{L})$ defined by the following equation.

$$
\square_V := L_V^2 = \bar{\partial}_V\bar{\partial}_V^* + \bar{\partial}_V^*\bar{\partial}_V
$$

- The **horizontal Laplacian** $\square_{\boldsymbol{H}}$ is the corresponding differential operator acting on $\mathfrak{A}^{(0,*)}(E,\mathfrak{L})$ defined analogously to $\square_V$, i.e. we define

$$
\square_H := \bar{\partial}_H\bar{\partial}_H^* + \bar{\partial}_H^*\bar{\partial}_H.
$$

Remark 2.27:
Although we have a splitting $\bar{\partial}_{\mathfrak{L}} = \bar{\partial}_V + \bar{\partial}_H$ and a splitting $\bar{\partial}^*_{\mathfrak{L}} = \bar{\partial}^*_V + \bar{\partial}^*_H$, it is not obvious that this splitting extends to the Laplacian, i.e. in general we have

$$\square_{\mathfrak{L}} \neq \square_V + \square_H.$$

We investigate when such a splitting actually occurs in Section 3.2.

Applying the lemmas above, one property of the vertical Laplacian becomes obvious.

Corollary 2.28:
Let $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ be a holomorphic fibre bundle and $\mathfrak{L} \to E$ be a holomorphic, Hermitian line bundle over E.
Then the following equation holds

$$\square_V(\pi_E^*\mu \wedge \omega) = (\pi_E^*\mu) \wedge (\square_V \omega)$$

for all $\mu \in \mathfrak{A}^{(0,*)}(M)$ and $\omega \in \mathfrak{A}^{(0,*)}(E, \mathfrak{L})$.

2.3.3 Induced holomorphic structure on fibres

Let again $\pi_{\mathfrak{L}} : (\mathfrak{L}, h) \longrightarrow E$ be a Hermitian, holomorphic line bundle over a holomorphic fibre bundle $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$.
Now, E has the structure of a complex manifold (compare Lemma 2.16) and in addition to that, we have local trivialisations $\{(U_k, \phi_k)\}_{k \in I}$ of E such that the embedding of a fibre F, given by

$$\phi_{k,x}^{-1} : \quad \{x\} \times F \cong F \quad \longrightarrow \quad E,$$

is a holomorphic map for any $k \in I$ and any $x \in U_k$.
We can pull the line bundle $\mathfrak{L}$ back under the inclusion $\phi_{k,x}^{-1}$ to obtain a complex line bundle

$$\mathfrak{L}_{k,x} := \left(\phi_{k,x}^{-1}\right)^* \mathfrak{L} \quad \longrightarrow \quad F$$

Recall that the pullback bundle $\psi^*\mathfrak{L}$ of a bundle $\mathfrak{L}$ through a map $\psi : F \to E$ is defined by

$$\psi^*\mathfrak{L} := \{(f, l) \mid \psi(f) = \pi_{\mathfrak{L}}(l)\}$$

where the projection $\pi_{\psi^*\mathfrak{L}}$ is defined by $\pi_{\psi^*\mathfrak{L}}(f, l) := f$.
This fact in mind, we get an induced map

$$\begin{array}{rccc} (\phi_{k,x}^{-1})^* : & \Gamma(E, \mathfrak{L}) & \longrightarrow & \Gamma(F, \mathfrak{L}_{k,x}) \\ & s & \longmapsto & \left\{ f \mapsto \left(f, s(\phi_{k,x}^{-1}(f)) \right) \right\}. \end{array}$$

The induced map $\left(\phi_{k,x}^{-1}\right)^*$ on smooth sections is surjective which can be seen as follows.

The fibre E_x is closed in E.
Now, a known fact for sections into a vector bundle is that a section $s_V \in \Gamma(V, \mathfrak{L})$ defined on a closed subset $i_V : V \hookrightarrow E$, has a smooth continuation $s_E \in \Gamma(E, \mathfrak{L})$ to the rest of E (cf. [5]).

Consequently, we obtain $i_V^* s_E = s_V$ and hence,

$$(\phi_{k,x}^{-1})^* : \quad \Gamma(E, \mathfrak{L}) \quad \longrightarrow \quad \Gamma(F, \mathfrak{L}_{k,x})$$

is surjective.

Of course, the same argument holds for the pullback $(\phi_{k,x}^{-1})^*$, i.e.

$$\begin{array}{rccc} (\phi_{k,x}^{-1})^* : & \mathfrak{A}^{(0,*)}(E, \mathfrak{L}) & \longrightarrow & \mathfrak{A}^{(0,*)}(F, \mathfrak{L}_{k,x}) \\ & \alpha \otimes s & \longmapsto & \left(\left(\phi_{k,x}^{-1}\right)^* \alpha\right) \otimes \left(\phi_{k,x}^{-1}\right)^* s, \end{array}$$

is onto as well.
In abuse of notation, we denote the pullback of complexified differential forms with the same symbol $\left(\phi_{k,x}^{-1}\right)^*$ as the pullback of sections in $\mathfrak{L}$.
It is possible to equip $\mathfrak{L}_{k,x}$ with a holomorphic structure induced by the holomorphic structure on $\mathfrak{L}$.
Recall therefore that a holomorphic structure can be identified with a first order differential operator $\bar{\partial}_{\mathfrak{L}}$ on the antiholomorphic differential forms with coefficients in $\mathfrak{L}$ having the symbol of the $\bar{\partial}$ operator (compare Corollary 2.3). Hence, it suffices to define such an operator $\bar{\partial}_{\mathfrak{L}_{k,x}}$ on $\mathfrak{L}_{k,x}$ in order to equip $\mathfrak{L}_{k,x}$ with a holomorphic structure.
A natural definition would be the following.

$$\bar{\partial}_{\mathfrak{L}_{k,x}} \circ \left(\phi_{k,x}^{-1}\right)^* = \left(\phi_{k,x}^{-1}\right)^* \circ \bar{\partial}_{\mathfrak{L}} \tag{10}$$

A simple calculation shows that this operator is well defined and has the correct behaviour as a differential operator.
Hence, it defines a holomorphic structure on $\mathfrak{L}$.

Actually, we can reduce Equation (10) to

$$\bar{\partial}_{\mathfrak{L}_{k,x}} \circ \left(\phi_{k,x}^{-1}\right)^* = \left(\phi_{k,x}^{-1}\right)^* \circ \left(\bar{\partial}_V + \bar{\partial}_H\right) = \left(\phi_{k,x}^{-1}\right)^* \circ \bar{\partial}_V. \tag{11}$$

Pulling back the Hermitian metric on $\mathfrak{L}$, we obtain a Hermitian metric on $\mathfrak{L}_{k,x}$.
Furthermore, since F is compact and oriented, we get a L^2 inner product on $\mathfrak{A}^{(0,*)}(F, \mathfrak{L}_{k,x})$ and hereby an adjoint operator $\bar{\partial}^*_{\mathfrak{L}_{k,x}}$.
There is a formula for $\bar{\partial}^*_{\mathfrak{L}_{k,x}}$ analogous to Equation (11) which we show next.

Lemma 2.29:
Let $\pi_{\mathfrak{L}} : (\mathfrak{L}, h) \to E$ be a Hermitian, holomorphic line bundle over a holomorphic fibre bundle.
Furthermore, let $\bar{\partial}_{\mathfrak{L}_{k,x}}$ be the holomorphic structure on $\mathfrak{L}_{k,x}$ induced by Equation (11). Then its adjoint operator $\bar{\partial}^*_{\mathfrak{L}_{k,x}}$ is given by:

$$\bar{\partial}^*_{\mathfrak{L}_{k,x}} \circ \left(\phi_{k,x}^{-1}\right)^* = \left(\phi_{k,x}^{-1}\right)^* \circ \bar{\partial}^*_V : \quad \mathfrak{A}_V^{(0,*)}(E, \mathfrak{L}) \quad \longrightarrow \quad \mathfrak{A}^{(0,*)}(F, \mathfrak{L}_{k,x}).$$

Proof.
The main idea of this proof is to make use of the following fact.

If $\mathcal{Q} \to B$ is a holomorphic, Hermitian vector bundle over a complex manifold B, the adjoint $\bar{\partial}_{\mathcal{Q}}$-operator, i.e. $\bar{\partial}^*_{\mathcal{Q}}$, is closely related to $\bar{\partial}_{\mathcal{Q}^*}$ which denotes the $\bar{\partial}$-operator on the dual bundle $\mathcal{Q}^*$.
Their correlation is given by the identity:

$$\bar{\partial}^*_{\mathcal{Q}} = -\bar{*}_{\mathcal{Q}^*} \circ \bar{\partial}_{\mathcal{Q}^*} \circ \bar{*}_{\mathcal{Q}} \tag{12}$$

where $\bar{*}_{\mathcal{Q}} : \mathfrak{A}^{(p,q)}(B, \mathcal{Q}) \to \mathfrak{A}^{(n-p,n-q)}(B, \mathcal{Q}^*)$ denotes the Hodge-Star operator (compare Definition A.4). *This fact is proven for example in [17].*

Now, we return to the situation at hand.
The assertion of Lemma 2.29 can be proven locally, therefore we will omit the $k \in I$ from the notation within this proof, i.e. we denote $\mathfrak{L}_x := \mathfrak{L}_{k,x}$ and $\bar{\partial}_{\mathfrak{L}_x} := \bar{\partial}_{\mathfrak{L}_{k,x}}$.
Additionally, let $\bar{\partial}_{\mathfrak{L}^*_x}$ denote the Dolbeault-operator on $\mathfrak{L}^*_x = \left(\phi_{k,x}^{-1}\right)^* \mathfrak{L}^*$, defined analogously to Equation (11), i.e.

$$\bar{\partial}_{\mathfrak{L}^*_x} \circ \left(\phi_{k,x}^{-1}\right)^* = \left(\phi_{k,x}^{-1}\right)^* \circ \bar{\partial}^*_V.$$

Now, we make use of Lemma A.6 which enables us to exchange the $\left(\phi_{k,x}^{-1}\right)^*$ morphism with the Hodge-Star operator.
In order to do that, observe that the pullback of the volume form dvol_M of the Riemannian manifold M via π_E is a horizontal form of maximal degree.

$$\begin{aligned}\bar{\partial}^*_{\mathfrak{L}_x} \circ \left(\phi_{k,x}^{-1}\right)^* &\overset{(12)}{=} -\bar{*}_{\mathfrak{L}^*_x} \circ \bar{\partial}_{\mathfrak{L}^*_x} \circ \bar{*}_{\mathfrak{L}_x} \circ \left(\phi_{k,x}^{-1}\right)^* \\ &\overset{A.6}{=} -\bar{*}_{\mathfrak{L}^*_x} \circ \bar{\partial}_{\mathfrak{L}^*_x} \circ \left(\phi_{k,x}^{-1}\right)^* \circ \bar{*}_{\mathfrak{L}} \circ \varepsilon\left(\pi_E^* \mathrm{dvol}_M\right)\end{aligned}$$

Now, we apply Equation (11) to obtain the holomorphic structure $\bar{\partial}_{\mathfrak{L}^*_x}$ on $\mathfrak{L}^*_x$ induced by the vertical $\bar{\partial}$-operator $\bar{\partial}_{V,\mathfrak{L}^*}$ defined by the holomorphic structure of $\mathfrak{L}^*$.

Hence, we get:

$$\begin{aligned}\bar{\partial}^*_{\mathfrak{L}_x} \circ \left(\phi^{-1}_{k,x}\right)^* &= - \bar{*}_{\mathfrak{L}^*_x} \circ \left(\phi^{-1}_{k,x}\right)^* \circ \bar{\partial}_{V,\mathfrak{L}^*} \circ \bar{*}_{\mathfrak{L}} \circ \varepsilon\left(\pi^*_E \mathrm{dvol}_M\right) \\ &\overset{A.6}{=} - \left(\phi^{-1}_{k,x}\right)^* \circ \bar{*}_{\mathfrak{L}^*} \circ \varepsilon\left(\pi^*_E \mathrm{dvol}_M\right) \circ \bar{\partial}_{V,\mathfrak{L}^*} \circ \bar{*}_{\mathfrak{L}} \circ \varepsilon\left(\pi^*_E \mathrm{dvol}_M\right)\end{aligned}$$

We showed in Lemma 2.24 that the exterior product of pullbacks of antiholomorphic forms commutes with the operator $\bar{\partial}_V$ up to the sign $(-1)^{\deg}$ where deg denotes the degree of the differential form.
It follows that $\bar{\partial}_{V,\mathfrak{L}^*}$ and $\varepsilon(\mathrm{dvol}_M)$ commute because dvol_M has an even degree.
Consequently, we obtain:

$$\begin{aligned}\bar{\partial}^*_{\mathfrak{L}_x} \circ \left(\phi^{-1}_{k,x}\right)^* &= - \left(\phi^{-1}_{k,x}\right)^* \circ \bar{*}_{\mathfrak{L}^*} \circ \bar{\partial}_{V,\mathfrak{L}^*} \circ \varepsilon\left(\pi^*_E \mathrm{dvol}_M\right) \circ \bar{*}_{\mathfrak{L}} \circ \varepsilon\left(\pi^*_E \mathrm{dvol}_M\right) \\ &\overset{A.6}{=} - \left(\phi^{-1}_{k,x}\right)^* \circ \bar{*}_{\mathfrak{L}^*} \circ \bar{\partial}_{V,\mathfrak{L}^*} \circ \bar{*}_{\mathfrak{L}}\end{aligned}$$

where we applied the second assertion of Lemma A.6 in order to obtain the second equality.
At last, we apply Equation (12) again, bearing in mind that $\left(\phi^{-1}_{k,x}\right)^* \circ \bar{*}_{\mathfrak{L}^*} \circ \bar{\partial}_{H,\mathfrak{L}^*} \circ \bar{*}_{\mathfrak{L}}$ vanishes when restricted to $\mathfrak{A}^{(0,*)}_V(E,\mathfrak{L})$, and receive

$$\bar{\partial}^*_{\mathfrak{L}_x} \circ \left(\phi^{-1}_{k,x}\right)^* = \left(\phi^{-1}_{k,x}\right)^* \circ \bar{\partial}^*_V$$

which finishes the proof. □

Closing this subsection, we summarise that for any Hermitian, holomorphic line bundle $\mathfrak{L}$ over a holomorphic fibre bundle we obtain an induced Hermitian, holomorphic line bundle $\mathfrak{L}_{k,x}$ for every admissible identification of the fibretype F with the fibre $E_x = \pi_E^{-1}(x)$. The holomorphic structure $\bar{\partial}_{\mathfrak{L}_{k,x}}$ as well as its adjoint operator are induced via pullbacks by the vertical $\bar{\partial}$-operator $\bar{\partial}_V$ and its adjoint $\bar{\partial}^*_V$.

2.4 Smooth vector bundle over a fibre bundle

For our purpose we need to look at complex line bundles $\mathfrak{L}$ over a fibre bundle $F \hookrightarrow E \mapsto M$. It will be necessary to understand sections from E into $\mathfrak{L}$ as sections from M into an infinite dimensional vector bundle over M.
Unfortunately, our research hasn't produced a general identification of these section spaces. Therefore, we will have to restrict $\mathfrak{L}$ to the case of a so-called smooth vector bundle over the fibre bundle $F \hookrightarrow E \to M$ which is introduced by Atiyah and Singer in [4].

Remark 2.30:
For $\mathcal{Q}$ to be a smooth vector bundle over a fibre bundle $E \to M$ is a different property than being a smooth vector bundle over the total space E of the fibre bundle.

In order to increase the understandability of this thesis, we will state this definition here, tailored to the case in which we need it.
First, we have to give a definition of the group $\mathrm{Diff}(F, \tilde{\mathcal{Q}})$ for a smooth vector bundle $\tilde{\mathcal{Q}}$ over a compact manifold F as a topological group.

Definition 2.31:
Let $\pi_{\tilde{\mathcal{Q}}} : \tilde{\mathcal{Q}} \longrightarrow F$ be a smooth vector bundle over a compact, connected manifold F.
The group $\mathbf{Diff}(\boldsymbol{F}, \tilde{\boldsymbol{\mathcal{Q}}})$ is given by

$$\mathrm{Diff}(F, \tilde{\mathcal{Q}}) := \left\{ \varphi \in \mathrm{Diff}(\tilde{\mathcal{Q}}, \tilde{\mathcal{Q}}) \;\middle|\; \begin{array}{l} \varphi \text{ covers a diffeo. } \Pi(\varphi) : F \to F \\ \varphi\,|_{\tilde{\mathcal{Q}}_f} : \tilde{\mathcal{Q}}_f \to \tilde{\mathcal{Q}}_{\Pi(\varphi)(f)} \text{ is linear } \forall f \in F \end{array} \right\}$$

where $\mathrm{Diff}(\tilde{\mathcal{Q}}, \tilde{\mathcal{Q}})$ denotes the group of diffeomorphisms from $\tilde{\mathcal{Q}}$ into itself.

In order to define its topology, look at the map $\Pi : \mathrm{Diff}(F, \tilde{\mathcal{Q}}) \to \mathrm{Diff}(F, F)$. Without restriction of generality, suppose (F, g_F) is a Riemannian manifold and $\tilde{\mathcal{Q}}$ has a metric $h_{\tilde{\mathcal{Q}}}$ as well as a metric connection $\nabla^{\tilde{\mathcal{Q}}}$.
Furthermore, we need a classical result from Whitehead [31], stated by Cheeger and Ebin in [11, Thm 5.14], which we repeat adapted to our purpose.

Theorem (Whitehead)**:**
For a compact Riemannian manifold (F, g_F) there is a positive, continuous map

$$r : \quad F \quad \longrightarrow \quad \mathbb{R}^+$$

so-called ***convexity radius****, such that for all $r < r(p)$ the geodesic ball $B_r(p)$ is strongly convex.*

In this context, a subset $X \subset F$ is called **strongly convex** if for any two points $x, y \in \bar{X}$ in the closure of X, there is a unique minimising geodesic $\tau_{x,y} : [0,1] \to F$ connecting x and y and the interior of $\tau_{x,y} :]0,1[\to X$ lies in X.

Corollary 2.32:
There is a positive constant r_0 such that for any $f \in F$ the geodesic ball $B_{r_0}(f)$ is strongly convex.

Now, we apply this corollary to our situation.

Let $\kappa \in \mathrm{Diff}(F,F)$ be a diffeomorphism such that the maximum geodesic distance between f and $\kappa(f)$ is smaller than r_0, i.e. $d_{\text{geod.}}(f,\kappa(f)) < r_0$. Because of Corollary 2.32, we have a unique minimising geodesic γ_f^κ from f to $\kappa(f)$.
Hence, we may identify $\tilde{\mathcal{Q}}_f$ with $\tilde{\mathcal{Q}}_{\kappa(f)}$ by parallel transport along γ_f^κ. Let us denote this map from $\tilde{\mathcal{Q}}$ to $\tilde{\mathcal{Q}}$ covering κ with $\boldsymbol{C(\kappa)}$, i.e. the following diagram commutes.

$$\begin{tikzcd}
\tilde{\mathcal{Q}} \arrow[r, "C(\kappa)"] \arrow[d, "\pi_{\tilde{\mathcal{Q}}}"'] & \tilde{\mathcal{Q}} \arrow[d, "\pi_{\tilde{\mathcal{Q}}}"] \\
F \arrow[r, "\kappa"'] & F
\end{tikzcd}$$

Now, we can define what we mean by a small open neighbourhood of the identity in $\mathrm{Diff}(F,\tilde{\mathcal{Q}})$ which allows us to define the topology of $\mathrm{Diff}(F,\tilde{\mathcal{Q}})$.

Definition 2.33:
Let $\varepsilon > 0$ be a positive constant with $r_0 > \varepsilon$. Let further on $\delta > 0$ be another positive constant.
Define the set $U_{\varepsilon,\delta} \subset \mathrm{Diff}(F,\tilde{\mathcal{Q}})$ given by:

$$U_{\varepsilon,\delta} := \left\{ \varphi \;\middle|\; \begin{array}{ll} a) & max\{d_{\text{geod.}}(\Pi(\varphi)(f), f) \mid f \in F\} < \varepsilon \\ b) & max\{\|\varphi(\tilde{l}) - C(\Pi(\varphi)(\tilde{l}))\|_{h_{\tilde{\mathcal{Q}}}} \mid \tilde{l} \in \tilde{\mathcal{Q}},\ \|\tilde{l}\|_{h_{\tilde{\mathcal{Q}}}} = 1\} < \delta \end{array} \right\}$$

to be open. Here $h_{\tilde{\mathcal{Q}}}$ denotes the Hermitian metric on $\tilde{\mathcal{Q}}$ and $\|\tilde{l}\|_{h_{\tilde{\mathcal{Q}}}}$ is the norm of $\tilde{l} \in \tilde{\mathcal{Q}}$ induced by $h_{\tilde{\mathcal{Q}}}$.
The topology on $\mathrm{Diff}(F,\tilde{\mathcal{Q}})$ is generated (using the group action and inversion, unions and intersections) by $U_{\varepsilon,\delta}$ for arbitrary small ε and δ.

Remark 2.34:

- The topology from Definition 2.33 equals the restriction of the compact-open topology on $\mathrm{Diff}(\tilde{\mathcal{Q}},\tilde{\mathcal{Q}})$ to the subspace $\mathrm{Diff}(F,\tilde{\mathcal{Q}})$.
  This fact can be seen by taking a sequence of maps $\{\varphi_n\}_n \in \mathrm{Diff}(F,\tilde{\mathcal{Q}})$.
  Now, φ_n converges in the topology above if and only if it converges uniformly on every compact subset $K \subset \tilde{\mathcal{Q}}$. But since $\tilde{\mathcal{Q}}$ is a metric space, compact-open topology and topology of compact convergence are one and the same.

- This topology is independent of the choice of g_F, $h_{\tilde{\mathcal{Q}}}$ and $\nabla^{\tilde{\mathcal{Q}}}$ since the compact open topology does not depend on these objects.

- $\mathrm{Diff}(F,\tilde{\mathcal{Q}})$ is Hausdorff because $\tilde{\mathcal{Q}}$ is Hausdorff and the Hausdorff property transports to compact-open-topology.
  It follows that $\mathrm{Diff}(F,\tilde{\mathcal{Q}})$ becomes a topological group.

Before we may define what a smooth vector bundle over a fibre bundle is, we have to repeat one definition. We define what a Lie transformation group of a manifold is. (cf. [21]).

Definition 2.35:
Let M be a manifold and K be a Lie group.
K is called **Lie transformation group of M** if the following two properties hold.

- K is a topological subgroup of $\mathrm{Diff}(M, M)$.
- The map

$$\begin{array}{ccc} K \times M & \longrightarrow & M \\ (k, x) & \longmapsto & k \cdot x \end{array}$$

  is smooth.

Finally, we define a smooth vector bundle over a fibre bundle.

Definition 2.36:
Let $\pi_E : E \longmapsto M$ be a smooth fibre bundle with compact fibre F.
Then $\pi_{\mathcal{Q}} : \mathcal{Q} \longrightarrow E$ is called **smooth vector bundle over the fibre bundle** $E \to M$ if the following properties hold.

- There exists a smooth vector bundle $\tilde{\mathcal{Q}} \to F$ and a Lie group K which is a Lie transformation group K of $\tilde{\mathcal{Q}}$ and a topological subgroup of $\mathrm{Diff}(F, \tilde{\mathcal{Q}})$.
- The bundle $\pi_E \circ \pi_{\mathcal{Q}} : \mathcal{Q} \longrightarrow M$ is a smooth fibre bundle over M with fibre $\tilde{\mathcal{Q}}$ and structure group K, i.e. $\mathcal{Q} \to M$ forms a smooth fibre bundle where the transition functions are smooth maps into K.

$$\begin{tikzcd} \tilde{\mathcal{Q}} \arrow[r, hook] \arrow[d, "\pi_{\tilde{\mathcal{Q}}}"'] & \mathcal{Q} \arrow[d, "\pi_{\mathcal{Q}}"'] \arrow[dd, bend left] \\ F \arrow[r, hook] & E \arrow[d, "\pi_E"'] \\ & M \end{tikzcd}$$

Remark 2.37:
The definition of a smooth vector bundle over a fibre bundle given by Atiyah and Singer in [4] is actually more general, but since we do not need it in that generality, we restricted our definition to the situation at hand.

For a smooth vector bundle $\mathcal{Q} \to E$ over a fibre bundle $E \to M$ with structure group K, we naturally inherit a K-principle fibre bundle $P \to M$.
The bundle P is given by patching together the local transition functions of the bundle $\mathcal{Q} \to M$ which we formally do now.

Let $\{U_i\}_{i \in I}$ be a cover of M such that $\mathcal{Q} \to M$ trivialises over U_i for every $i \in I$ and let further on

$$\widehat{\varphi_i \circ \varphi_j^{-1}} : \quad U_i \cap U_j \quad \longrightarrow K$$

be the smooth transition functions for the bundle $\mathcal{Q} \to M$.
We define the manifold P to be the disjoint union of the $U_i \times K$ modded out by an equivalence relation

$$P := \left(\coprod_{i \in I} U_i \times K \right) \Big/ \sim$$

where the equivalence relation is defined for every $x \in U_i \cap U_j$ to be:

$$U_i \times K \ni (x, k) \sim (x, \widehat{\varphi_i \circ \varphi_j^{-1}}(x) \cdot k) \in U_j \times K.$$

Baum shows in [5] that P defined this way becomes a smooth manifold as well as a K-principle fibre bundle over M with projection:

$$\begin{array}{rccc} \pi_P : & P & \longrightarrow & M \\ & [(x,k)] & \longmapsto & x. \end{array}$$

Remark 2.38:
Let $\tilde{\rho}$ denote the inclusion of K into $\mathrm{Diff}(F, \tilde{\mathcal{Q}})$. By construction of P, we see that $\mathcal{Q}$ is associated to P as fibre bundle over M via $\tilde{\rho}$, i.e.

$$\mathcal{Q} = P \times_{\tilde{\rho}, K} \tilde{\mathcal{Q}} \to M.$$

Furthermore, we can apply the group homomorphism Π (compare Definition 2.31)

$$\Pi : \mathrm{Diff}(F, \tilde{\mathcal{Q}}) \to \mathrm{Diff}(F, F)$$

to obtain a K-action

$$\rho = \Pi \circ \tilde{\rho} : K \to \mathrm{Diff}(F, F).$$

In particular, the fibre bundle E is associated to P as well, i.e.

$$E = P \times_{\rho, K} F.$$

What is not obvious up to this point is that the bundle $\mathcal{Q} = P \times_{\tilde{\rho}} \tilde{\mathcal{Q}} \to E$ is a smooth vector bundle over the manifold $E = P \times_{\rho} F$, in opposition to being a smooth vector bundle over the fibre bundle $E \to M$.
To show this is the content of the next lemma. Furthermore, we introduce some notations for the local trivialisation maps that we use throughout this thesis.

Lemma 2.39:
Let $\pi_{\mathcal{Q}} : \mathcal{Q} \to F$ be a complex vector bundle over a manifold F, let K be a Lie group and let furthermore

$$\tilde{\rho} : K \to \mathrm{Diff}(F, \tilde{\mathcal{Q}})$$

*be a topological group homomorphism making K a Lie transformation group of $\tilde{\mathcal{Q}}$.
Denote by ρ the map*

$$\rho : K \to \mathrm{Diff}(F, F)$$

*given by $\rho = \Pi \circ \tilde{\rho}$ (compare Definition 2.31).
Then for any K-principle fibre bundle $P \to M$ over a manifold M, the bundle*

$$\pi_{\mathcal{Q}} : \quad \mathcal{Q} := P \times_{\tilde{\rho}} \tilde{\mathcal{Q}} \quad \longrightarrow \quad E := P \times_{\rho} F$$

*is a smooth complex vector bundle.
Summarising, we obtain the following commuting diagram.*

$$\begin{tikzcd}
\tilde{\mathcal{Q}} \arrow[r, hook] \arrow[d, "\pi_{\tilde{\mathcal{Q}}}"'] & \mathcal{Q} = P \times_{\tilde{\rho},K} \tilde{\mathcal{Q}} \arrow[d, "\pi_{\mathcal{Q}}"] \arrow[dd, bend left=60] \\
F \arrow[r, hook] & E = P \times_{\rho,K} F \arrow[d, "\pi_E"] \\
 & M
\end{tikzcd} \tag{13}$$

The projection $\pi_{\mathcal{Q}} : \mathcal{Q} \to E$ is given by $\pi_{\mathcal{Q}}([p, \tilde{v}]_{\tilde{\rho}}) := [p, \pi_{\tilde{\mathcal{Q}}}(\tilde{v})]_{\rho}$.

Proof.
Choose local sections $q_i : U_i \to P\,|_{U_i}$ and local trivialisations (φ_i, U_i) of P such that

$$\begin{array}{rccc} \varphi_i : & P\,|_{U_i} & \longrightarrow & U_i \times K \\ & (q_i(x) \cdot k) & \longmapsto & (x, k). \end{array}$$

Denote the transition functions by

$$g_{ij} : \quad U_i \cap U_j \quad \longrightarrow \quad K$$

i.e. $q_i \cdot g_{ij} = q_j$.

Additionally, choose local trivialisations (ψ_j, V_j) for the smooth vector bundle $\tilde{\mathcal{Q}} \to F$.

$$\psi_j : \quad \tilde{\mathcal{Q}}\,|_{V_j} \quad \longrightarrow \quad V_j \times \mathbb{C}^m$$

Now, we define local trivialisations (χ_{ij}, W_{ij}) for $\mathcal{Q} \to E$.
The local base set W_{ij} is given by:

$$W_{ij} := \{[q_i(x), f]_\rho \mid x \in U_i, f \in V_j\} \subset E\,|_{U_i}$$

and take for the local trivialisation map:

$$\begin{array}{rccc} \chi_{ij} : & \mathcal{Q}\,|_{W_{ij}} & \longrightarrow & W_{ij} \times \mathbb{C}^m \\ & [q_i(x), \tilde{v}]_{\tilde{\rho}} & \longmapsto & \left([q_i(x), \pi_{\tilde{\mathcal{Q}}}(\tilde{v})]_\rho, \mathrm{proj}_2(\psi_j(\tilde{v}))\right). \end{array}$$

It obviously fulfils $\pi_{\mathcal{Q}} = \mathrm{proj}_1 \circ \chi_{ij}$.
Its inverse map can be easily deduced to be:

$$\begin{array}{rccc} \chi_{ij}^{-1} : & W_{ij} \times \mathbb{C}^m & \longrightarrow & \mathcal{Q}\,|_{W_{ij}} \\ & ([q_i(x), f]_\rho, \lambda) & \longmapsto & \Big[q_i(x), \psi_j^{-1}(f, \lambda)\Big]_{\tilde{\rho}}. \end{array}$$

We compute the transition map on $W := W_{ij} \cap W_{ab}$:

$$\chi_{ij} \circ \chi_{ab}^{-1} : \quad W \times \mathbb{C}^m \quad \longrightarrow \quad W \times \mathbb{C}^m,$$

and obtain the following expression

$$\begin{aligned} \chi_{ij} \circ \chi_{ab}^{-1}\left([q_a(x), f]_\rho, \lambda\right) &= \chi_{ij}\left(\left[q_a(x), \psi_b^{-1}(f, \lambda)\right]_{\tilde{\rho}}\right) \\ &= \chi_{ij}\left(\left[q_i(x) \cdot g_{ia}(x), \psi_b^{-1}(f, \lambda)\right]_{\tilde{\rho}}\right) \\ &= \chi_{ij}\left(\left[q_i(x), \tilde{\rho}(g_{ia}(x))\left(\psi_b^{-1}(f, \lambda)\right)\right]_{\tilde{\rho}}\right) \\ &= \left(\left[q_i(x), \rho(g_{ia}(x))(f)\right]_\rho, \mathrm{proj}_2 \circ \psi_j \circ \tilde{\rho}(g_{ia}(x)) \circ \psi_b^{-1}(f, \lambda)\right). \end{aligned}$$

This is linear in λ since ψ_i and $\tilde{\rho}(g)$ act $\mathbb{C}$-linearly on the fibres. And the transition function is smooth in (x, f) because K is a Lie transformation group.
Hence the transition functions are smooth in $e = [q_a(x), f]_\rho$. □

2.5 Equivariant torsion and equivariant index

In this subsection, we want to recall the definitions of the equivariant index as well as our main object of interest, namely the equivariant holomorphic torsion.

Let $\pi_{\mathcal{Q}} : \mathcal{Q} \to E$ be a holomorphic, Hermitian vector bundle over a compact, complex, Riemannian manifold (E, g_E) and let $\gamma = (\gamma^{\mathcal{Q}}, \gamma^E)$ be a pair of biholomorphic isometries $\gamma^{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}$ and $\gamma^E : E \to E$ such that

- the following diagram commutes

$$\begin{tikzcd} \mathcal{Q} \arrow[r, "\gamma^{\mathcal{Q}}"] \arrow[d, "\pi_{\mathcal{Q}}"'] & \mathcal{Q} \arrow[d, "\pi_{\mathcal{Q}}"] \\ E \arrow[r, "\gamma^E"'] & E \end{tikzcd}$$

- and the map $\gamma^{\mathcal{Q}} |_{\mathcal{Q}_x := \pi_{\mathcal{Q}}^{-1}(x)} : \mathcal{Q}_x \to \mathcal{Q}_{\gamma^E(x)}$ is a complex linear isometry.

Definition 2.40:
There is a natural, γ-induced action $\check{\boldsymbol{\gamma}}^{\mathcal{Q}}$ on antiholomorphic forms with coefficients in $\mathcal{Q}$. It is given for $\alpha \in \mathfrak{A}^{(0,*)}(E)$ and $s \in \Gamma(E, \mathcal{Q})$, by:

$$\check{\gamma}^{\mathcal{Q}} (\alpha \otimes s)_e := \left\{ \left(\left(\gamma^E\right)^{-1} \right)^* \alpha \right\}_e \otimes \gamma^{\mathcal{Q}} \left(s(\left(\gamma^E\right)^{-1}(e)) \right),$$

and extended linearly onto $\mathfrak{A}^{(0,*)}(E) \otimes \Gamma(E, \mathcal{Q})$.

Remark 2.41:
The map $\gamma^{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}$ is biholomorphic. Therefore, $\bar{\partial}_{\mathcal{Q}}$ commutes with $\check{\gamma}^{\mathcal{Q}}$. On the other hand, $\check{\gamma}^{\mathcal{Q}}$ is an isometry on $\mathfrak{A}^{(0,*)}(E, \mathcal{Q})$. Hence, it commutes with $\bar{\partial}^*_{\mathcal{Q}}$ as well.
Consequently, we obtain that the Laplacian $\square_{\mathcal{Q}} = \bar{\partial}_{\mathcal{Q}} \bar{\partial}^*_{\mathcal{Q}} + \bar{\partial}^*_{\mathcal{Q}} \bar{\partial}_{\mathcal{Q}}$ commutes with the action $\check{\gamma}^{\mathcal{Q}}$ on $\mathfrak{A}^{(0,*)}(E, \mathcal{Q})$, too.
It follows for any λ in the spectrum $\sigma(\square_{\mathcal{Q}}) \subset \mathbb{R}^+_0$ of $\square_{\mathcal{Q}}$ that $\check{\gamma}^{\mathcal{Q}}$ acts on the Eigenspace

$$\mathrm{Eig}_\lambda(\square^{(0,q)}_{\mathcal{Q}}) := \left\{ \alpha \in \mathfrak{A}^{(0,q)}(E, \mathcal{Q}) \mid \square_{\mathcal{Q}} \alpha = \lambda \cdot \alpha \right\}$$

by restriction.

The fact that $\check{\gamma}^{\mathcal{Q}}$ acts on the Eigenspace $\mathrm{Eig}_\lambda(\square^{(0,q)}_{\mathcal{Q}})$ for any λ is now used to define two invariants, on the one hand the equivariant index which depends on the 0-Eigenspace of $\square_{\mathcal{Q}}$ and on the other hand the equivariant torsion which depends on the Eigenspaces corresponding to the non-zero Eigenvalues.

Definition 2.42:
The **equivariant index $\mathbf{ind}(\boldsymbol{\gamma}^{\mathcal{Q}}, \square_{\mathcal{Q}})$** of a Hermitian, holomorphic vector bundle $\mathcal{Q}$ over M is given by

$$\mathrm{ind}(\gamma^{\mathcal{Q}}, \square_{\mathcal{Q}}) := \sum_{q \geq 0} (-1)^q \mathrm{Tr} \left(\check{\gamma}^{\mathcal{Q}} |_{\mathrm{Ker}(\square^{(0,q)}_{\mathcal{Q}})} \right)$$

In order to define the equivariant holomorphic torsion, we have define the equivariant ζ-function at first.

Definition 2.43:
The **equivariant ζ-function $Z^{\mathcal{Q}}_{\check{\gamma}^{\mathcal{Q}}}$** is now formally defined by:

$$Z^{\mathcal{Q}}_{\check{\gamma}^{\mathcal{Q}}}(s) := \sum_{q\geq 0}(-1)^{q+1}q \sum_{\lambda\in\sigma(\Box_{\mathcal{Q}})\setminus\{0\}} \lambda^{-s}\cdot \mathrm{Tr}\left(\check{\gamma}^{\mathcal{Q}}|_{\mathrm{Eig}_\lambda(\Box^{(0,q)}_{\mathcal{Q}})}\right)$$

for $s\in\mathbb{C}$ with sufficiently large real part $\mathrm{Re}(s) >> 0$ where this series converges absolutely.

There is always a real constant c such that $Z^{\mathcal{Q}}_{\check{\gamma}^{\mathcal{Q}}}(s)$ converges absolutely if the $\mathrm{Re}(s) > c$. Furthermore, $Z^{\mathcal{Q}}_{\check{\gamma}^{\mathcal{Q}}}$ can be continued meromorphically to the complex plane and this continuation has no pole at $0\in\mathbb{C}$. (*This fact is proven for the non-equivariant case, i.e. for $\gamma^{\mathcal{Q}} = \mathrm{id}_{\mathcal{Q}}$ in [28]. The equivariant case is proven analogously.*)

The fact that the equivariant ζ-function is holomorphic at $0\in\mathbb{C}$ is now used to define the equivariant torsion (cf. [22]).

Definition 2.44:
The **equivariant holomorphic torsion** $\tau^{\mathcal{Q}}(\check{\gamma}^{\mathcal{Q}})$ is defined by:

$$\tau^{\mathcal{Q}}(\check{\gamma}^{\mathcal{Q}}) := (Z^{\mathcal{Q}}_{\check{\gamma}^{\mathcal{Q}}})'(0).$$

Remark 2.45:
Obviously, two holomorphic, Hermitian vector bundles structures $\bar{\partial}_0$ and $\bar{\partial}_1$ on $\mathcal{Q}$ that are equivalent in the sense of Definition 2.5 induce the same equivariant torsion if the equivalence is an isometry as well, i.e. if $\bar{\partial}_1 = \bar{\partial}_0 + \varepsilon(g^{-1}\bar{\partial}g)$ for $g: E\to U(1)$.

3 Compatible line bundles and legitimate equivariance

In this section, we introduce the setting which we want to work in.

In order to obtain statements about the holomorphic torsion of line bundles over a holomorphic fibre bundle, it is necessary to restrict to so-called compatible line bundles. In Subsection 3.1, we define what a compatible line bundle is and what this restriction implies from arbitrary holomorphic line bundles implies. The class of compatible line bundles is a generalisation of the line bundles Stanton looks at in [29].
Furthermore, we recall some results from Atiyah and Singer (cf. [4]) to obtain vector bundles

$$\mathcal{W}^{(\lambda;*)} = \bigcup_{x \in M} \operatorname{Ker}\left(\Box_{\mathfrak{L}|_{E_x}} - \lambda\right).$$

A very important property that needs to hold for a holomorphic line bundle, in order to make it compatible, is the splitting of the Laplacian into a vertical and a horizontal part. In Subsection 3.2, we investigate when such a splitting occurs, i.e. what conditions lead to such a splitting. The proof of these vanishing conditions is an extension of Stantons results about flat holomorphic line bundles.
In the following two Subsections 3.3 and 3.4, we identify the λ-Eigenspace of the vertical Laplacian $\Box_V$ on E with antiholomorphic forms on the base M of our holomorphic fibre bundle with coefficients in the holomorphic, Hermitian vector bundle $\mathcal{W}^{(\lambda,*)}$ via a morphism ψ. This enables us later on to translate the problem of calculating the equivariant holomorphic torsion of $\mathfrak{L}$ from E to computing differential-topological invariants on M and vice verse.
Subsection 3.5 introduces the equivariant setting. Since we don't want to loose the orthogonal splitting of $\Box_{\mathfrak{L}}$ into vertical and horizontal parts, we need to restrict the equivariant setting to actions $\vec{\gamma}$ that are legitimate. We define what this means exactly in Subsection 3.5.
At last, we show that we can translate the legitimate $\vec{\gamma}$ action on $\mathfrak{L} \to E$ via the morphism ψ to an action γ on the holomorphic vector bundles $\mathcal{W}^{(\lambda;t)}$ over M. This is the subject of Subsection 3.6.

3.1 Setting

This section is about defining what a compatible line bundle is and furthermore about giving a short survey of some properties of a compatible line bundle.

We start right away with the definition of a compatible line bundle over a holomorphic fibre bundle.

Definition 3.1:
Let $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ be a holomorphic fibre bundle as given by Definition 2.13, and let furthermore $\mathfrak{L}$ be a holomorphic, Hermitian line bundle over E.
$\mathfrak{L} \to E$ is called **compatible line bundle** for the holomorphic fibre bundle if the following properties hold.

1. $\mathfrak{L}$ is a smooth vector bundle over the fibre bundle $E \to M$ in the sense of Definition 2.36.
   The smooth vector bundle structure $\mathfrak{L} \to E$ is the one induced by Lemma 2.39. Furthermore, the fibre type of the smooth fibre bundle $\mathfrak{L} \to M$ is a smooth, holomorphic, Hermitian line bundle $\tilde{\mathfrak{L}} \to F$ such that every element $k \in K$ becomes a morphism $\tilde{\rho}(k) : \tilde{\mathfrak{L}} \to \tilde{\mathfrak{L}}$ that is a fibrewise Hermitian, linear map and that respects the holomorphic structure of $\tilde{\mathfrak{L}} \to F$.

2. The Laplacian $\Box_{\mathfrak{L}} = \bar{\partial}_{\mathfrak{L}} \bar{\partial}_{\mathfrak{L}}^* + \bar{\partial}_{\mathfrak{L}}^* \bar{\partial}_{\mathfrak{L}}$ splits into a vertical

   $$\Box_V = \bar{\partial}_V \bar{\partial}_V^* + \bar{\partial}_V^* \bar{\partial}_V$$

   as well as a horizontal part:

   $$\Box_H = \bar{\partial}_H \bar{\partial}_H^* + \bar{\partial}_H^* \bar{\partial}_H,$$

   i.e.

   $$\Box_{\mathfrak{L}} = \Box_V + \Box_H.$$

   In Subsection 3.2, we examine when such a splitting occurs.

3. The holomorphic structure of the line bundle $\mathfrak{L}$ restricted to the fibres E_x is fixed in the following way.
   There is a family of local trivialisations $\{(\phi_i, U_i)\}_{i \in I}$ of E, such that the induced holomorphic structure (compare Section 2.3.3) on $\left(\phi_{i,x}^{-1}\right)^* \mathfrak{L} \to F$, given by

   $$\bar{\partial}_{\left(\phi_{i,x}^{-1}\right)^* \mathfrak{L}} := \left(\phi_{i,x}^{-1}\right)^* \circ \bar{\partial}_V$$

   is the same holomorphic structure that $\tilde{\mathfrak{L}}$ naturally induces, i.e. such that for every $x \in U_i$, we obtain

   $$\left(\phi_{i,x}^{-1}\right)^* \mathfrak{L} \cong \tilde{\mathfrak{L}}$$

   as holomorphic, Hermitian line bundles (in the sense of Definition 2.5).

Remark 3.2:
Observe that forgetting the holomorphic structure, the isomorphism class of the smooth vector bundle $\left(\phi_{i,x}^{-1}\right)^* \mathfrak{L}$ does not depend on x if we choose a path connected, i.e. connected, local trivialisation base set U_i. (cf. [1])

In the subsequent remark, we explain why we need the first property of the definition of a compatible line bundle.
Stanton in [29] is able to circumvent the need for the line bundle $\mathfrak{L}$ to be smooth (in the sense of Definition 2.36) over the holomorphic fibre bundle because she already assumed the holomorphic fibre bundle to be associated to a compact principle fibre bundle in the first place. For the line bundles she looks at, this is sufficient to imply their smoothness over the fibre bundle.
In our, more general, case we use some properties for smooth vector bundles over a fibre bundle given in [4]. We now summarise the important facts that Atiyah and Singer showed in [4].

Remark 3.3:
Let $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ be a holomorphic fibre bundle and $\mathfrak{L} \to E$ be a compatible line bundle.
We need a survey of some further properties.

- For a compatible line bundle $\mathfrak{L} \to E$ the structure group of the bundle $\mathfrak{L} \to M$ becomes a Lie group K.
  Furthermore, there is a K-principle fibre bundle $P \to M$ such that the bundles $\mathfrak{L} \to M$ and $E \to M$ are associated (compare Remark 2.38).
  Explicitly stated, we obtain $\mathfrak{L} = P \times_{\tilde{\rho}} \tilde{\mathfrak{L}} \to M$ and $E = P \times_\rho F \to M$ for group homomorphisms

  $$\begin{array}{rccc} \tilde{\rho}: & K & \longrightarrow & \mathrm{Diff}(F, \tilde{\mathfrak{L}}) \\ \rho: & K & \longrightarrow & \mathrm{Diff}(F, F). \end{array}$$

  Additionally, $\tilde{\rho}$ and ρ induce a representation

  $$\check{\rho}: \quad K \quad \longrightarrow \quad \mathrm{Aut}(\mathfrak{A}^{(0,*)}(F, \tilde{\mathfrak{L}}))$$

  given for any differential form $\alpha \in \mathfrak{A}^{(p,q)}(F, \tilde{\mathfrak{L}})$ by

  $$\check{\rho}(k)\,(\alpha) := \tilde{\rho}(k)\left(\rho\left(k^{-1}\right)^*(\alpha)\right).$$

- Atiyah and Singer show in [4] that for $\mathfrak{L}$ being a smooth vector bundle (in the sense of Definition 2.36) there is a Fréchet bundle

  $$\mathbb{V}^{(0,*)} := \bigcup_{x \in M} \mathbb{V}_x^{(0,*)} \quad \longrightarrow \quad M$$

associated to P. Its fibre $\mathbb{V}_x^{(0,*)}$ over the point $x \in M$ is the set of vertical forms restricted to a fibre E_x of E, i.e.

$$\mathbb{V}_x^{(0,*)} = \mathfrak{A}_V^{(0,*)}(E_x, \mathfrak{L}\mid_{E_x}).$$

Hence, $\mathbb{V}_x^{(0,*)} \cong \mathfrak{A}^{(0,*)}(F, \tilde{\mathfrak{L}})$ as Fréchet space and consequently the fibre bundle $\mathbb{V}^{(0,*)}$ has the fibretype $\mathfrak{A}^{(0,*)}(F, \tilde{\mathfrak{L}})$.

$$\begin{tikzcd} \mathfrak{A}^{(0,*)}(F, \tilde{\mathfrak{L}}) \arrow[r, hook] & \mathbb{V}^{(0,*)} \arrow[d] \\ & M \end{tikzcd}$$

Summarising, we obtain:

$$\mathbb{V}^{(0,*)} = P \times_{\check{\rho},K} \mathfrak{A}^{(0,*)}(F, \tilde{\mathfrak{L}}).$$

- Because of Property 1 of Definition 3.1, the holomorphic structure $\bar{\partial}_{\tilde{\mathfrak{L}}}$ on $\tilde{\mathfrak{L}} \to F$ is invariant under $\check{\rho}(k)$ for any $k \in K$.
Additionally, $\rho(k)$ acts as a Hermitian isometry covering an isometry $\rho(k)$ of F for every $k \in K$. Therefore, $\check{\rho}(k)$ commutes with the $\bar{\partial}^*_{\tilde{\mathfrak{L}}}$-operator as well.
It follows that $\bar{\partial}_{\tilde{\mathfrak{L}}} + \bar{\partial}^*_{\tilde{\mathfrak{L}}}$ induces an elliptic operator on $\mathfrak{A}^{(0,*)}(F, \tilde{\mathfrak{L}})$ which is invariant under the K-action $\check{\rho}$. Thus, $\bar{\partial}_{\tilde{\mathfrak{L}}} + \bar{\partial}^*_{\tilde{\mathfrak{L}}}$ defines a "constant", in particular continuous, section in $\Gamma\left(M, \mathrm{End}\left(\mathbb{V}^{(0,*)}\right)\right)$. Analogously, so does $\Box_{\tilde{\mathfrak{L}}} = \left(\bar{\partial}_{\tilde{\mathfrak{L}}} + \bar{\partial}^*_{\tilde{\mathfrak{L}}}\right)^2$.
Consequently, $\Box_{\tilde{\mathfrak{L}}}$ defines a continuous family of elliptic operators. (cf. [4])

- Again, following [4], we get for each Eigenvalue λ of $\Box_{\tilde{\mathfrak{L}}}$ a complex vector bundle $\mathcal{W}^{(\lambda;*)} \to M$ of finite and constant rank over M.
It is given by

$$\pi_{\mathcal{W}^{(\lambda;*)}} : \quad \mathcal{W}^{(\lambda;*)} = \mathrm{Ker}(\Box_{\tilde{\mathfrak{L}}} - \lambda) \quad \longrightarrow \quad M.$$

The bundle $\mathcal{W}^{(\lambda;*)}$ is associated to P as well because it is a restriction of $\mathbb{V}^{(0,*)}$ to the kernel of $\Box_{\tilde{\mathfrak{L}}}$, i.e.

$$\mathcal{W}^{(\lambda;*)} = P \times_{\check{\rho},K} \mathrm{Ker}\left(\Box_{\tilde{\mathfrak{L}}} - \lambda\right).$$

- Observe that Atiyah and Singer prove that $\mathcal{W}^{(\lambda;*)}$ is a continuous vector bundle, and not explicitly a smooth vector bundle.
On the other hand, any continuous vector bundle over a smooth manifold has a unique smooth vector bundle structure compatible with its continuous vector bundle structure (cf. [16, Ch. 4, Thm 3.5.]).
It follows that $\mathcal{W}^{(\lambda;t)}$ can be regarded as a smooth vector bundle.
Later on, in Section 3.4, we show that $\mathcal{W}^{(\lambda;t)}$ becomes a Hermitian, holomorphic vector bundle.

In Subsection 3.3, we show that for every Eigenvalue λ of $\Box_V$ the λ-Eigenforms of $\Box_V$ can be identified with sections from M into $\mathcal{W}^{(\lambda;*)}$.
This is essential when we use the bundles $\mathcal{W}^{(\lambda;*)}$ to express the equivariant holomorphic torsion of $\mathfrak{L} \to E$ in Section 4.

However, before we delve deeper into this problem, we try at first to understand the second property of Definition 3.1.
Therefore, we look for verifiable conditions for the Laplace splitting property. This is the subject of the following subsection.

3.2 Laplace splitting property

So far, we have explained what the first property of the definition of a compatible line bundle implies.

This subsection is about the second property of Definition 3.1, the Laplace splitting property. It generalises the analogous statements of Stanton in [29]. The proofs of those statements are quite similar.

First, we simplify the problem of verifying this property. Afterwards in Subsection 3.2.1, we give a specific, verifiable and sufficient condition for the occurrence of such a splitting in the case where F is a compact Kähler manifold. This is especially interesting for the second part of this thesis where we look at the holomorphic torsion of holomorphic line bundles over Lie groups. There, the fibre of the considered holomorphic fibre bundle is a complex torus with bi-invariant Kähler metric.
In Subsection 3.2.2, we state as well as prove a useful consequence of the Laplace splitting property.

Now, let $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ be a holomorphic fibre bundle and let $\mathfrak{L} \to E$ be a holomorphic line bundle over E.
Lemma 2.20 shows that the $\bar{\partial}_{\mathfrak{L}}$ operator, defining the holomorphic structure of $\mathfrak{L}$, splits into a vertical part $\bar{\partial}_V$ as well as a horizontal part $\bar{\partial}_H$. It suggests itself to check if a similar property holds for the Laplacian $\Box_{\mathfrak{L}} = \bar{\partial}_{\mathfrak{L}}\bar{\partial}_{\mathfrak{L}}^* + \bar{\partial}_{\mathfrak{L}}^*\bar{\partial}_{\mathfrak{L}}$.
We show in this section that there are certain conditions that guarantee such a splitting.

First of all, we introduce some notations for the operators needed to describe this splitting property.

Definition 3.4:
All of those operators are differential operators on $\mathfrak{A}^{(0,*)}(E, \mathfrak{L})$:

- $L_V := \bar{\partial}_V + \bar{\partial}_V^*$
- $L_H := \bar{\partial}_H + \bar{\partial}_H^*$
- $\Box_V := (L_V)^2 = \bar{\partial}_V^*\bar{\partial}_V + \bar{\partial}_V\bar{\partial}_V^*$
- $\Box_H := (L_H)^2 = \bar{\partial}_H^*\bar{\partial}_H + \bar{\partial}_H\bar{\partial}_H^*$

- $L := \Box_{\mathfrak{L}} - \Box_H - \Box_V = L_V L_H + L_H L_V$

Observe that the splitting of the Laplacian is equivalent to the vanishing of the operator

$$L = \Box_{\mathfrak{L}} - \Box_H - \Box_V$$

on antiholomorphic forms on E with coefficients in $\mathfrak{L}$.

Now, we show that it suffices to verify that L vanishes on vertical antiholomorphic forms $\mathfrak{A}_V^{(0,*)}(E, \mathfrak{L})$ because if it vanishes there, it vanishes everywhere.

Lemma 3.5:
$L \equiv 0$ on $\mathfrak{A}^{(0,*)}(E, \mathfrak{L})$ if and only if L vanishes on $\mathfrak{A}_V^{(0,q)}(E, \mathfrak{L})$ for all q.

Proof.
One direction is obvious. I follows directly from $\mathfrak{A}_V^{(0,q)}(E, \mathfrak{L}) \subset \mathfrak{A}^{(0,*)}(E, \mathfrak{L})$.

We proof the other direction in the following way.
At first, we show that L is the sum of an operator A and its adjoint. Consequently, L vanishes if and only if A does. Afterwards, we show that A vanishes on all antiholomorphic forms if and only if it vanishes on vertical forms only. At last, we summarise that A vanishes on vertical forms if and only if L does.

Now, taking a closer look at L, we notice:

$$L = \left(\bar{\partial}_H L_V + L_V \bar{\partial}_H\right) + \left(\bar{\partial}_H L_V + L_V \bar{\partial}_H\right)^* = A + A^*,$$

where $A := \bar{\partial}_H L_V + L_V \bar{\partial}_H$.
L vanishes if and only if A vanishes on $\mathfrak{A}^{(0,*)}(E, \mathfrak{L})$ because of the following argument.

Observe that the operators A and A^* restricted to the space $\mathfrak{A}_H^{(0,s)}(E) \wedge \mathfrak{A}_V^{(0,*)}(E, \mathfrak{L})$ map as follows.

$$\begin{array}{rlcl} A: & \mathfrak{A}_H^{(0,s)}(E) \wedge \mathfrak{A}_V^{(0,*)}(E, \mathfrak{L}) & \longrightarrow & \mathfrak{A}_H^{(0,s+1)}(E) \wedge \mathfrak{A}_V^{(0,*)}(E, \mathfrak{L}) \\ A^*: & \mathfrak{A}_H^{(0,s)}(E) \wedge \mathfrak{A}_V^{(0,*)}(E, \mathfrak{L}) & \longrightarrow & \mathfrak{A}_H^{(0,s-1)}(E) \wedge \mathfrak{A}_V^{(0,*)}(E, \mathfrak{L}). \end{array}$$

Now, the target spaces on the right hand side are linearly independent. Hence, so are the images of A and A^*.
Thus, $(A + A^*)\mu$ vanishes for any antiholomorphic form $\mu \in \mathfrak{A}_H^{(0,s)}(E) \wedge \mathfrak{A}_V^{(0,*)}(E, \mathfrak{L})$ if and only if $A\mu$ as well as $A^*\mu$ vanish.
On the other hand, $A = 0$ leads to $A^* = 0$.
Consequently, we obtain $A + A^* = 0$. Therefore in order to proof the assertion, we just have to prove it for A.

We now show that A vanishes on $\mathfrak{A}^{(0,*)}(E, \mathfrak{L})$ if and only if $A = 0$ on $\mathfrak{A}_V^{(0,*)}(E, \mathfrak{L})$.

Recall that every antiholomorphic form $\eta \in \mathfrak{A}^{(0,*)}(E,\mathfrak{L})$ is a finite sum of antiholomorphic forms of the form $(\pi_E^*\mu) \wedge \omega$ with $\mu \in \mathfrak{A}^{(0,*)}(M)$ and $\omega \in \mathfrak{A}_V^{(0,*)}(E,\mathfrak{L})$.
Hence, we can restrict our considerations to the case where η equals $(\pi_E^*\mu) \wedge \omega$.
Let μ be in $\mathfrak{A}^{(0,*)}(M)$ and ω in $\mathfrak{A}_V^{(0,*)}(E,\mathfrak{L})$. We described how $\bar{\partial}_V, \bar{\partial}_H$ (Lemma 2.24) and $\bar{\partial}_V^*$ (Lemma 2.25) commute with the pullback of differential forms on M.
We apply A on $\pi_E^*\mu \wedge \omega$

$$A((\pi_E^*\mu) \wedge \omega) = \left(\bar{\partial}_H L_V + L_V \bar{\partial}_H\right) ((\pi_E^*\mu) \wedge \omega).$$

For $\bar{\partial}_H$ and for L_V the Leibniz equation holds.
Therefore, we obtain:

$$\begin{aligned}
A((\pi_E^*\mu) \wedge \omega) =& (-1)^{|\mu|} \left(\bar{\partial}_H(\pi_E^*\mu)\right) \wedge (L_V\omega) + (\pi_E^*\mu) \wedge \left(\bar{\partial}_H(L_V\omega)\right) \\
&+ L_V \left[(\bar{\partial}_H(\pi_E^*\mu)) \wedge \omega + (-1)^{|\mu|}(\pi_E^*\mu) \wedge (\bar{\partial}_H\omega)\right] \\
=& \cancel{(-1)^{|\mu|} \left(\bar{\partial}_H(\pi_E^*\mu)\right) \wedge (L_V\omega)} + (\pi_E^*\mu) \wedge \left(\bar{\partial}_H(L_V\omega)\right) \\
&+ \left(L_V\bar{\partial}_H(\pi_E^*\mu)\right) \wedge \omega + \cancel{(-1)^{|\mu|+1} \left(\bar{\partial}_H(\pi_E^*\mu)\right) \wedge (L_V\omega)} \\
&+ (\pi_E^*\mu) \wedge (L_V(\bar{\partial}_H\omega)) \\
=& (\pi_E^*\mu) \wedge \left(\bar{\partial}_H L_V + L_V\bar{\partial}_H\right)\omega + \left(L_V(\bar{\partial}_H(\pi_E^*\mu))\right) \wedge \omega.
\end{aligned}$$

Now, the assumption for A follows directly from:

$$L_V\bar{\partial}_H(\pi_E^*\mu) = L_V(\pi_E^*\bar{\partial}\mu) = 0.$$

We summarise: L vanishes on $\mathfrak{A}^{(0,*)}(E,\mathfrak{L})$ if and only if A vanishes on $\mathfrak{A}^{(0,*)}(E,\mathfrak{L})$, which happens if and only if $A = 0$ on $\mathfrak{A}_V^{(0,*)}(E,\mathfrak{L})$.
On the other hand L vanishes on $\mathfrak{A}_V^{(0,*)}(E,\mathfrak{L})$ if and only if A does, which directly implies $L = 0$ on all $\mathfrak{A}^{(0,*)}(E,\mathfrak{L})$. □

For a general holomorphic fibre bundle, this is as far as we get in understanding the Laplace splitting property.
Fortunately, we can derive a much more explicit condition under which the L-operator vanishes if the fibretype F of our holomorphic fibre bundle is a Kähler manifold.
This is the content of the subsequent subsection.

3.2.1 Holomorphic fibre bundles of Kähler fibretype

Before we can state conditions for the vanishing of L, we have to do some preparatory, somewhat technical work.
Let $(E, \pi_E, (M, g_M), (F, g_F), T^HE)$ be a holomorphic fibre bundle whose fibretype F is a Kähler manifold.
Let us denote the Kähler form on F by $\omega_F \in \mathfrak{A}^{(1,1)}(F)$.

For every $x \in M$ every admissible embedding of the fibre, i.e. an embedding coming from restrictions of the holomorphic trivialisations $\phi_{k,x}^{-1} : F \hookrightarrow E$ is holomorphic. Hence, we can define a holomorphic structure $\bar{\partial}_{\mathfrak{L}_{k,x}}$ on the pullback bundle $\mathfrak{L}_{k,x} := (\phi_{k,x}^{-1})^* \mathfrak{L}$ over F as we saw in Equation (10) in Section 2.3.3.
Furthermore, the Kähler form ω_F induces an endomorphism of $\mathfrak{A}^{(\cdot,*)}(E, \mathfrak{L}_{k,x})$, by exterior multiplication, i.e.:

$$\begin{array}{rccc} \varepsilon(\omega_F): & \mathfrak{A}^{(\cdot,*)}(E, \mathfrak{L}_{k,x}) & \longrightarrow & \mathfrak{A}^{(\cdot+1,*+1)}(E, \mathfrak{L}_{k,x}) \\ & \alpha & \longmapsto & \omega_F \wedge \alpha. \end{array}$$

Definition 3.6:
The **Lefschetz operator Λ_F**

$$\Lambda_F : \mathfrak{A}^{(\cdot,*)}(E, \mathfrak{L}_{k,x}) \longrightarrow \mathfrak{A}^{(\cdot-1,*-1)}(E, \mathfrak{L}_{k,x}),$$

is defined to be the adjoint operator of $\varepsilon(\omega_F)$ for the L^2-inner product $(\cdot,\cdot)$ induced by the pullback $(\phi_{k,x}^{-1})^*$.

Remark 3.7:

- For a holomorphic fibre bundle, the transition maps $\phi_{k,x} \circ \phi_{l,x}^{-1} : F \to F$ are by definition biholomorphic isometries (compare Definition 2.13).
  Hence, the Kähler form ω_F is invariant under the pullback via $\phi_{k,x} \circ \phi_{l,x}^{-1}$, i.e.

  $$\left(\phi_{k,x} \circ \phi_{l,x}^{-1}\right)^* \omega_F = \omega_F.$$

  It follows that ω_F induces a differential form $\omega_V \in \mathfrak{A}_V^{(1,1)}(E)$ such that

  $$(\phi_{k,x}^{-1})^* \omega_V = \omega_F.$$

- Consequently, the Lefschetz operator has an analogous operator Λ_V,

  $$\Lambda_V : \mathfrak{A}_H^{(s,t)}(E) \wedge \mathfrak{A}_V^{(p,q)}(E, \mathfrak{L}) \longrightarrow \mathfrak{A}_H^{(s,t)}(E) \wedge \mathfrak{A}_V^{(p-1,q-1)}(E, \mathfrak{L}),$$

  which is the adjoint of $\varepsilon(\omega_V)$.

- Look at the following subspaces of $T^V E \otimes_{\mathbb{R}} \mathbb{C}$:

  $$\begin{array}{rcll} T^{V,(1,0)}E & := & \left(T^V E\right)^{(1,0)} & \text{and} \\ T^{V,(0,1)}E & := & \left(T^V E\right)^{(0,1)}. & \end{array}$$

  A short computation shows that for a local orthonormal frame $\{E_l\}_{l \in J}$ of $T^{V,(1,0)}E$ with corresponding local orthonormal frame $\left\{\bar{E}_l\right\}_{l \in J}$ of $T^{V,(0,1)}E$, Λ_V is given by:

  $$\Lambda_V = -i \sum_{l \in J} \iota_{\bar{E}_l} \iota_{E_l}. \tag{14}$$

  Here, ι_X denotes the contraction with X via the Hermitian form.

For F being a Kähler manifold and $\mathfrak{L}_{k,x}$ being a Hermitian, holomorphic vector bundle, there are the so-called Kähler identities (cf. [19]). They simplify the computation of the adjoint $\bar{\partial}^*_{\mathfrak{L}_{k,x}}$ to the operator $\bar{\partial}_{\mathfrak{L}_{k,x}}$. Explicitly stated in Equation (15) below.

$$\bar{\partial}^*_{\mathfrak{L}_{k,x}} = -i[\Lambda, \nabla^{\mathfrak{L}_{k,x},(1,0)}]. \tag{15}$$

Here, $\nabla^{\mathfrak{L}_{k,x}} = \nabla^{\mathfrak{L}_{k,x},(0,1)} \oplus \nabla^{\mathfrak{L}_{k,x},(1,0)}$ denotes the unique holomorphic, Hermitian connection on $(\mathfrak{L}_{k,x}, (\phi^{-1}_{k,x})^* h)$ (compare Definition 2.9).
Now, together with Lemma 2.29 the Kähler identities can be used to get an explicit expression for $\bar{\partial}^*_V$.

Lemma 3.8:
Let $\nabla^{\mathfrak{L}} = \nabla^{\mathfrak{L},(1,0)} \oplus \nabla^{\mathfrak{L},(0,1)}$ denote the unique holomorphic, Hermitian connection on the Hermitian, holomorphic line bundle $(\mathfrak{L}, h) \to E$.
Let furthermore $\nabla^{V,(1,0)}$ denote the restriction of $\nabla^{\mathfrak{L},(1,0)}$ to the vertical part, i.e. the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{A}_H^{(p,q)}(E) \wedge \mathfrak{A}_V^{(s,t)}(E,\mathfrak{L}) \arrow[dd, "\nabla^{V,(1,0)}"'] \arrow[dr, "\nabla^{\mathfrak{L},(1,0)}"] & \\
 & \mathfrak{A}^{(p+s+1,q+t)}(E,\mathfrak{L}) \arrow[dl, "\mathrm{proj}"] \\
\mathfrak{A}_H^{(p,q)}(E) \wedge \mathfrak{A}_V^{(s+1,t)}(E,\mathfrak{L}). &
\end{tikzcd}$$

Then, with notations from above, the following identity holds:

$$\bar{\partial}^*_V = -i[\Lambda_V, \nabla^{V,(1,0)}] : \quad \mathfrak{A}^{(0,*)}(E,\mathfrak{L}) \quad \longrightarrow \quad \mathfrak{A}^{(0,*-1)}(E,\mathfrak{L}). \tag{16}$$

Proof.
The proof is divided into two steps. First, we show that it suffices to show Equation (16) for the pullback of vertical forms only and second we proof Equation (16) for pullbacks of vertical forms.
1) Let η be a form in $\mathfrak{A}_H^{(0,q)}(E) \wedge \mathfrak{A}_V^{(0,p)}(E,\mathfrak{L})$. Since M is compact, we can identify η with the finite sum

$$\eta = \sum_l (\pi_E^* \alpha_l) \wedge \beta_l$$

where $\alpha_l \in \mathfrak{A}^{(0,q)}(M)$ and $\beta_l \in \mathfrak{A}_V^{(0,p)}(E,\mathfrak{L})$.
We apply Lemma 2.25 to the right hand side of Equation (16) and obtain:

$$\bar{\partial}^*_V \eta = \sum_l (-1)^q (\pi_E^* \alpha_l) \wedge \bar{\partial}^*_V \beta_l.$$

If we look at the the left hand side of Equation (16), we observe that, for any differential forms $\alpha \in \mathfrak{A}^{(s_1,s_2)}(M)$ and $\beta \in \mathfrak{A}^{(t_1,t_2)}(E,\mathfrak{L})$, the following two identities hold:

$$\Lambda_V\left(\pi_E^*\alpha \wedge \beta\right) \overset{(14)}{=} \underbrace{(\Lambda_V \pi_E^*\alpha)}_{=0} \wedge \beta + \underbrace{(-1)^{2\cdot|\alpha|}}_{=1} \pi_E^*\alpha \wedge \Lambda_V \beta,$$

$$\nabla^{V,(1,0)}\left(\pi_E^*\alpha \wedge \beta\right) = \underbrace{((\partial_V)\pi_E^*\alpha)}_{=0} \wedge \beta + (-1)^{s_1+s_2}\pi_E^*\alpha \wedge \nabla^{V,(1,0)}\beta$$

where ∂_V is given analogously to $\nabla^{V,(1,0)}$, i.e. by the following commuting diagram

$$\begin{tikzcd}
\mathfrak{A}_H^{(s_1,s_2)}(E) \wedge \mathfrak{A}_V^{(t_1,t_2)}(E) \arrow[dd, "\partial_V"'] \arrow[dr, "d"] & \\
& \mathfrak{A}^{(s_1+t_1+1,s_2+t_2)}(E) \oplus \mathfrak{A}^{(s_1+t_1,s_2+t_2+1)}(E) \arrow[dl, "\text{proj}"] \\
\mathfrak{A}_H^{(s_1,s_2)}(E) \wedge \mathfrak{A}_V^{(t_1+1,t_2)}(E). &
\end{tikzcd}$$

We conclude that we may restrict our considerations to vertical forms since the right hand side of Equation (16) applied to η simplifies to:

$$[\Lambda_V, \nabla^{V,(1,0)}]\eta = \sum_l (-1)^q (\pi_E^*\alpha_l) \wedge \left([\Lambda_V, \nabla^{V,(1,0)}]\beta_l\right),$$

hence, without restrictions to generality η is in $\mathfrak{A}_V^{(0,*)}(E,\mathfrak{L})$.

2) Two vertical forms $\eta, \eta' \in \mathfrak{A}_V^{(0,*)}(E,\mathfrak{L})$ coincide if and only if $(\phi_{k,x}^{-1})^*\eta = (\phi_{k,x}^{-1})^*\eta'$ for all $x \in M$ and $k \in I$ such that $x \in U_k \subset M$.
This implies that we now have to check

$$(\phi_{k,x}^{-1})^* \circ \bar{\partial}_V^* = (\phi_{k,x}^{-1})^* \circ \left(-i\left[\Lambda_V, \nabla^{V,(1,0)}\right]\right).$$

On the one hand, Lemma 2.29 implies that $(\phi_{k,x}^{-1})^* \circ \bar{\partial}_V^* = \bar{\partial}_{\mathfrak{L}_{k,x}}^* \circ (\phi_{k,x}^{-1})^*$. On the other hand, we know that $\bar{\partial}_{\mathfrak{L}_{k,x}}^*$ can be calculated via the Kähler identities as in Equation (15), i.e. $\bar{\partial}_{\mathfrak{L}_{k,x}}^* = -i[\Lambda_F, \nabla^{\mathfrak{L}_{k,x},(1,0)}]$.
To put it in a nutshell, we have to check

$$(\phi_{k,x}^{-1})^* \circ \left([\Lambda_V, \nabla^{V,(1,0)}]\right)\eta = \left[\Lambda_F, \nabla^{\mathfrak{L}_{k,x},(1,0)}\right] \circ (\phi_{k,x}^{-1})^*\eta \tag{17}$$

for any vertical form $\eta \in \mathfrak{A}_V^{(0,q)}(E,\mathfrak{L})$ and every $x \in M$.
Without loss of generality, let η be given by $\alpha \otimes s$ with $\alpha \in \mathfrak{A}_V^{(0,q)}(E)$ and $s \in \Gamma(E,\mathfrak{L})$.

The left hand side of Equation (17) now becomes:

$$\begin{aligned}(\phi_{k,x}^{-1})^* \circ \left(\left[\Lambda_V, \nabla^{V,(1,0)}\right]\right) \eta =&(\phi_{k,x}^{-1})^* \circ \Lambda_V \circ \nabla^{V,(1,0)}(\alpha \otimes s)\\ =&(\phi_{k,x}^{-1})^* \circ \Lambda_V \left(\partial_V \alpha \otimes s + (-1)^q \alpha \wedge \nabla^{V,(1,0)} s\right)\end{aligned}$$

The embedding $\phi_{k,x}^{-1} : F \to E_x$ is a holomorphic, isometric immersion.
Consequently, we obtain:

$$(\phi_{k,x}^{-1})^* \circ \left(\Lambda_V \circ \nabla^{V,(1,0)}\right) \eta = \Lambda_F \circ (\phi_{k,x}^{-1})^* \left((\partial_V \alpha) \otimes s + (-1)^q \alpha \wedge \nabla^{V,(1,0)} s\right). \tag{18}$$

For the first term, observe that $(\phi_{k,x}^{-1})^*$ commutes with the ∂-operator on $\mathfrak{A}^{(0,*)}(E)$ because of the subsequent computation and because it commutes with d and $\bar{\partial}$.

$$(\phi_{k,x}^{-1})^* \circ \partial_V = (\phi_{k,x}^{-1})^* \circ \partial = (\phi_{k,x}^{-1})^* \circ (d - \bar{\partial}) = (d - \bar{\partial}) \circ (\phi_{k,x}^{-1})^* = \partial \circ (\phi_{k,x}^{-1})^* \tag{19}$$

For the second term, we study the pullback properties of the $\nabla^{V,(1,0)}$-operator on sections. Let therefore s, t be two sections in $\Gamma(E, \mathfrak{L})$. The Hermitian and holomorphic property of the Chern connection $\nabla^{\mathfrak{L}}$ now directly implies the subsequent computation.

$$\begin{aligned}\bar{\partial}\left((\phi_{k,x}^{-1})^* h^{\mathfrak{L}}(s,t)\right) =&(\phi_{k,x}^{-1})^* \left(\bar{\partial}\left(h^{\mathfrak{L}}(s,t)\right)\right)\\ =&(\phi_{k,x}^{-1})^* \left(h^{\mathfrak{L}}\left(\nabla^{V,(1,0)} s, t\right) + h^{\mathfrak{L}}\left(s, \bar{\partial}_V t\right)\right)\\ =&h^{\mathfrak{L}_{k,x}}\left((\phi_{k,x}^{-1})^* \left(\nabla^{V,(1,0)} s\right), (\phi_{k,x}^{-1})^* t\right)\\ &+ h^{\mathfrak{L}_{k,x}}\left((\phi_{k,x}^{-1})^* s, \underbrace{(\phi_{k,x}^{-1})^* (\bar{\partial}_V t)}_{=\bar{\partial}\left((\phi_{k,x}^{-1})^* t\right)}\right)\end{aligned}$$

Conversely, the Chern connection $\nabla^{\mathfrak{L}_{k,x}}$ on $\mathfrak{L}_{k,x}$ implies:

$$\begin{aligned}\bar{\partial}\left((\phi_{k,x}^{-1})^* h^{\mathfrak{L}}(s,t)\right) =&\bar{\partial}\left(h^{\mathfrak{L}_{k,x}}\left((\phi_{k,x}^{-1})^* s, (\phi_{k,x}^{-1})^* t\right)\right)\\ =&h^{\mathfrak{L}_{k,x}}\left(\nabla^{\mathfrak{L}_{k,x},(1,0)} (\phi_{k,x}^{-1})^* s, (\phi_{k,x}^{-1})^* t\right)\\ &+ h^{\mathfrak{L}_{k,x}}\left((\phi_{k,x}^{-1})^* s, \bar{\partial}(\phi_{k,x}^{-1})^* t\right)\end{aligned}$$

Thus, we obtain the following equality

$$h^{\mathfrak{L}_{k,x}}\left((\phi_{k,x}^{-1})^* \left(\nabla^{V,(1,0)} s\right), (\phi_{k,x}^{-1})^* t\right) = h^{\mathfrak{L}_{k,x}}\left(\nabla^{\mathfrak{L}_{k,x},(1,0)} (\phi_{k,x}^{-1})^* s, (\phi_{k,x}^{-1})^* t\right)$$

for arbitrary sections s and t into $\mathfrak{L}$.
Consequently, we obtain:

$$(\phi_{k,x}^{-1})^* \circ \nabla^{V,(1,0)}(s) = \nabla^{\mathfrak{L}_{k,x},(1,0)} \circ (\phi_{k,x}^{-1})^*(s) \tag{20}$$

At last, we insert Equations (19) and (20) into Equation (18) and obtain:

$$\begin{aligned}(\phi_{k,x}^{-1})^* \circ \left([\Lambda_V, \nabla^{V,(1,0)}]\right)\eta =& \Lambda_F \circ \Big(\partial\left(\circ(\phi_{k,x}^{-1})^*\alpha\right) \otimes (\phi_{k,x}^{-1})^* s \\ &+ (-1)^q (\phi_{k,x}^{-1})^*\alpha \wedge \nabla^{\mathfrak{L}_{k,x},(1,0)} \circ (\phi_{k,x}^{-1})^* s\Big) \\ =& \Lambda_F \circ \nabla^{\mathfrak{L}_{k,x},(1,0)} \circ (\phi_{k,x}^{-1})^* (\alpha \otimes s)\end{aligned}$$

and therefore,

$$(\phi_{k,x}^{-1})^* \circ \left([\Lambda_V, \nabla^{V,(1,0)}]\right)\eta = \left[\Lambda_F, \nabla^{\mathfrak{L}_{k,x},(1,0)}\right] \circ (\phi_{k,x}^{-1})^* (\alpha \otimes s)$$

which finishes the proof. □

We have to take one last step, the subsequent lemma, before we can state as well as proof our vanishing condition for the Kähler case.
Recall therefore that we defined the $\bar\partial_H$-operator to act on $\mathfrak{A}^{(0,*)}(E,\mathfrak{L})$ in Definition 2.22. This definition may be extended to the whole space $\mathfrak{A}^*_{\mathbb{C}}(E,\mathfrak{L})$ via the following diagram:

$$\begin{tikzcd} \mathfrak{A}_H^{(s_1,s_2)}(E) \wedge \mathfrak{A}_V^{(t_1,t_2)}(E,\mathfrak{L}) \arrow[dd, "\bar\partial_H"'] \arrow[dr, "\bar\partial"] & \\ & \mathfrak{A}^{(s_1+t_1,s_2+t_2+1)}(E,\mathfrak{L}) \arrow[dl, "\mathrm{proj}"] \\ \mathfrak{A}_H^{(s_1,s_2+1)}(E) \wedge \mathfrak{A}_V^{(t_1,t_2)}(E,\mathfrak{L}). & \end{tikzcd}$$

Of course, we may not assume that $\bar\partial = \bar\partial_H + \bar\partial_V$ on the whole space $\mathfrak{A}^*_{\mathbb{C}}(E,\mathfrak{L})$ anymore.

Lemma 3.9:
*Let Y be in $\Gamma(E, T^V E \otimes_{\mathbb{R}} \mathbb{C})$ and let X be a vector field in $\Gamma(E, T^{H,(0,1)}E)$.
Furthermore, let $\omega \in \mathfrak{A}_V^{(p,q)}(E,\mathfrak{L})$ be a vertical differential form.
Then the following equation holds:*

$$\left(\iota_X \bar\partial_H\right)(\iota_Y \omega) = -\iota_{[Y,X]}\omega + \iota_Y\left[\left(\iota_X\bar\partial_H\right)\omega\right]. \tag{21}$$

Proof.
Let $\omega \in \Gamma(E, \Lambda^q \left(T^{V,\mathbb{C}}E\right)^* \otimes \mathfrak{L}) = \mathfrak{A}^q_{\mathbb{C}}(E,\mathfrak{L})$ and $Z_1, \ldots, Z_{q-1} \in \Gamma(E, T^{V,\mathbb{C}}E)$.
Since both sides of Equation (21) are $\mathbb{C}$ linear, we can without restrictions to generality assume $\omega = \alpha \otimes s$ with $\alpha \in \mathfrak{A}^q_{\mathbb{C}}(E)$ and $s \in \Gamma(E,\mathfrak{L})$.
Within this proof, we use the following abbreviation.

For the index set $J = \{1, \dots, q-1\}$ and a subset $I = \{i_1, \dots, i_r \mid i_s < i_{s+1}\} \subset J$ we denote the tuple $(Z_{i_1}, \dots, Z_{i_r})$ by Z_I.
We now compute:

$$\begin{aligned} B := \left(\iota_Y \left(\left(\iota_X \bar{\partial}_H\right) \omega\right)\right)(Z_J) =& (\bar{\partial}_H \omega)(X, Y, Z_1, \dots, Z_{q-1}) \\ =& (\bar{\partial}_H(\alpha \otimes s))(X, Y, Z_J). \end{aligned}$$

Applying the Leibniz Equation (4) we obtain

$$B = (\bar{\partial}_H \alpha)(X, Y, Z_J) \cdot s + (-1)^q (\alpha \wedge \bar{\partial}_H s)(X, Y, Z_J)$$

We denote the complexified derivative of a C^∞-function $f \in C^\infty(E)$ in the direction of a complexified vector field $V \in \Gamma(E, T^{\mathbb{C}}E)$ by $X.f$, i.e. $df(X) = X.f$.
Then using the definition of the exterior differential d, we obtain:

$$\begin{aligned} B =& \left[X.\alpha(Y, Z_J) - \alpha([X,Y], Z_J) - \sum_{k=1}^{q-1} (-1)^k \alpha\left([X, Z_k], Y, Z_{J\setminus\{k\}}\right) \right] \cdot s \\ &+ \alpha(Y, Z_J) \cdot (\bar{\partial}_H s)(X) \\ =& \left[X.\left\{(\iota_Y \alpha)(Z_J)\right\} - (\iota_{[X,Y]}\alpha)(Z_J) + \sum_{k=1}^{q-1} (-1)^k (\iota_Y \alpha)([X, Z_k], Z_{J\setminus\{k\}}) \right] s \\ &+ (\iota_Y \alpha)(Z_J) \cdot (\bar{\partial}_H s)(X) \\ =& \underbrace{\left(\bar{\partial}_H(\iota_Y \alpha)\right)(X, Z_J) \cdot s + (-1)^{q-1} (\iota_Y \alpha \wedge \bar{\partial}_H s)(X, Z_J)}_{=\left(\bar{\partial}_H(\iota_Y \alpha \otimes s)\right)(X, Z_J)} - (\iota_{[X,Y]}\alpha)(Z_J) s \\ =& \iota_X \bar{\partial}_H(\iota_Y \omega)(Z_J) - (\iota_{[X,Y]}\omega)(Z_J), \end{aligned}$$

which completes the proof. □

Now, we can finally prove a vanishing-condition for L.

Recall that L maps from the vector space of antiholomorphic, vertical $(0,q)$-forms, i.e. $\mathfrak{A}_V^{(0,q)}(E, \mathfrak{L})$, to the vector space $\mathfrak{A}_H^{(0,1)}(E) \wedge \mathfrak{A}_V^{(0,q-1)}(E, \mathfrak{L})$.
Hence, $L\omega = 0$ for any $\omega \in \mathfrak{A}_V^{(0,q)}(E, \mathfrak{L})$ if and only if $\iota_X(L\omega)$ vanishes for every horizontal vector field $X \in \Gamma(E, T^H E)$.

For the subsequent proposition, let Q denote the orthogonal projection

$$Q : \mathfrak{A}^{(1,1)}(E) \longrightarrow \mathfrak{A}_H^{(0,1)}(E) \wedge \mathfrak{A}_V^{(1,0)}(E). \tag{22}$$

Proposition 3.10:
*Let $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ be a holomorphic fibre bundle and let $\mathfrak{L} \to E$ be a Hermitian, holomorphic line bundle over E.
Furthermore, let $\{E_l\}_{l \in J}$ be a local orthonormal frame in $T^{V,(1,0)}E$ and let $\{\bar{E}_l\}_{l \in J}$ denote its complex conjugated orthonormal frame in $T^{V,(0,1)}E$.*

For any vertical form $\omega \in \mathfrak{A}_V^{(0,q)}(E,\mathfrak{L})$ and any vector field $X \in \Gamma(M,TM)$ with horizontal lift $\tilde{X} \in \Gamma(E,T^H E)$, we obtain:

$$\iota_{\tilde{X}}(L\omega) = -i \sum_k \left\{ \iota_{[\tilde{X},\bar{E}_k]} \nabla^{\mathfrak{L},(1,0)}_{E_k} + \iota_{\bar{E}_k} \nabla^{\mathfrak{L},(1,0)}_{[\tilde{X},E_k]} \right\} \omega - i\Lambda_V \iota_{\tilde{X}} \left[Q\left(\left(\nabla^{\mathfrak{L}}\right)^2 \right) \wedge \omega \right]$$

where Q denotes the projection from Equation (22).
Note that $[E_i,\tilde{X}]$ and $[\bar{E}_i,\tilde{X}]$ are vertical because the commutator of a horizontal lift with a vertical vector field is always vertical.

Remark 3.11:
Proposition 3.10 particularly implies that for a flat vector bundle $\mathfrak{L}$, i.e. $\left(\nabla^{\mathfrak{L}}\right)^2 = 0$, it suffices to show that $[\bar{E}_k,\tilde{X}] = 0$ and $[E_k,\tilde{X}] = 0$ for L to vanish.

Proof of Proposition 3.10.
We start by applying Lemma 3.8, i.e. Equation (16), to the left hand side.
We substitute $\bar{\partial}_V^*$ and obtain:

$$\begin{aligned} (\iota_{\tilde{X}} L)\omega =& \iota_{\tilde{X}}(\bar{\partial}_H \bar{\partial}_V^* + \bar{\partial}_V^* \bar{\partial}_H)\omega \\ \overset{(16)}{=}& -i\iota_{\tilde{X}} \left\{ \bar{\partial}_H \Lambda_V \nabla^{V,(1,0)} + \Lambda_V \nabla^{V,(1,0)} \bar{\partial}_H \right\} \omega. \end{aligned} \tag{23}$$

Here, we make use of the fact that $\Lambda_V \bar{\partial}_H \omega = 0$ (as well as $\Lambda_V \omega = 0$) because $\bar{\partial}_H \omega$ (or ω respectively) is an antiholomorphic form, i.e. $\bar{\partial}_H \omega \in \mathfrak{A}^{(0,q+1)}(E,\mathfrak{L})$ (and $\omega \in \mathfrak{A}^{(0,1)}(E,\mathfrak{L})$).
The next step is to transform the first term of the right hand side of this equation.
Therefore, we swap the $\iota_{\tilde{X}} \bar{\partial}_H$-operator with the Λ_V-operator, using Lemma 3.9, i.e. Equation (21) repeatedly.

$$\begin{aligned} \iota_{\tilde{X}} \bar{\partial}_H \Lambda_V \nabla^{V,(1,0)} \omega &= \sum_k \iota_{\tilde{X}} \bar{\partial}_H \iota_{\bar{E}_k} \iota_{E_k} \nabla^{V,(1,0)} \omega \\ &\overset{(21)}{=} \sum_k \left\{ -\iota_{[\bar{E}_k,\tilde{X}]} + \iota_{\bar{E}_k} \iota_{\tilde{X}} \bar{\partial}_H \right\} \iota_{E_k} \nabla^{V,(1,0)} \omega \\ &\overset{(21)}{=} \sum_k \left\{ -\iota_{[\bar{E}_k,\tilde{X}]} \iota_{E_k} - \iota_{\bar{E}_k} \iota_{[E_k,\tilde{X}]} + \iota_{\bar{E}_k} \iota_{E_k} \iota_{\tilde{X}} \bar{\partial}_H \right\} \nabla^{V,(1,0)} \omega \end{aligned}$$

Now, swapping the contractions in the first two terms, we obtain:

$$\begin{aligned} \iota_{\tilde{X}} \bar{\partial}_H \Lambda_V \nabla^{V,(1,0)} \omega = &\sum_k \left\{ \iota_{[\tilde{X},\bar{E}_k]} \iota_{E_k} + \iota_{\bar{E}_k} \iota_{[\tilde{X},E_k]} \right\} \nabla^{V,(1,0)} \omega \\ &+ \Lambda_V \iota_{\tilde{X}} \bar{\partial}_H \nabla^{V,(1,0)} \omega. \end{aligned}$$

Inserting this into Equation (23) and using the identity $\Lambda_V \iota_{\tilde{X}} = \iota_{\tilde{X}} \Lambda_V$, we finally get:

$$\begin{aligned}(\iota_{\tilde{X}} L)\omega = & - i \sum_k \left\{ \iota_{[\tilde{X}, \bar{E}_k]} \iota_{E_k} + \iota_{\bar{E}_k} \iota_{[\tilde{X}, E_k]} \right\} \nabla^{V,(1,0)} \omega \\ & - i \Lambda_V \iota_{\tilde{X}} \underbrace{\left\{ \bar{\partial}_H \nabla^{V,(1,0)} + \nabla^{V,(1,0)} \bar{\partial}_H \right\}}_{=\varepsilon\left(Q(\nabla^{\mathfrak{L}})^2\right)} \omega\end{aligned}$$

which completes the proof. □

Summarising, Proposition 3.10 gives us an explicit formula for L, and therefore, we are able to check if for a given holomorphic, Hermitian line bundle $\mathfrak{L}$ over a holomorphic fibre bundle, the Laplacian splits.

3.2.2 A consequence of the Laplace splitting property

We now derive that the splitting of the Laplacian leads to the commutation of both of its parts, i.e. $\square_V \square_H = \square_H \square_V$.

Lemma 3.12:
Let $\mathfrak{L} \to E$ be a holomorphic, Hermitian line bundle over a holomorphic fibre bundle such that the Laplacian $\square_{\mathfrak{L}}$ on $\mathfrak{L}$ splits, i.e. $\square_{\mathfrak{L}} = \square_V + \square_H$
Then the following identities hold:

$$\bar{\partial}_V \bar{\partial}_H + \bar{\partial}_H \bar{\partial}_V = 0 = \bar{\partial}_V^* \bar{\partial}_H^* + \bar{\partial}_H^* \bar{\partial}_V^* \tag{24}$$
$$\bar{\partial}_V^* \bar{\partial}_H + \bar{\partial}_H \bar{\partial}_V^* = 0 = \bar{\partial}_V \bar{\partial}_H^* + \bar{\partial}_H^* \bar{\partial}_V. \tag{25}$$

In particular, we obtain:

$$\square_H \bar{\partial}_V^{(*)} = \bar{\partial}_V^{(*)} \square_H \quad \text{as well as} \quad \square_V \bar{\partial}_H^{(*)} = \bar{\partial}_H^{(*)} \square_V$$

and the Laplacians commute, i.e. $\square_V \square_H = \square_H \square_V$.

Proof.
We already know that $\bar{\partial}_{\mathfrak{L}}$ splits into a vertical and a horizontal part and so does its adjoint $\bar{\partial}_{\mathfrak{L}}^*$.
Note that $\bar{\partial}_{\mathfrak{L}}^2 = 0 = \left(\bar{\partial}_{\mathfrak{L}}^*\right)^2$, and note furthermore that

$$\mathfrak{A}^{(0,q)}(E, \mathfrak{L}) = \bigoplus_{s+t=q} \mathfrak{A}_H^{(0,s)}(E) \wedge \mathfrak{A}_V^{(0,t)}(E, \mathfrak{L})$$

is an orthogonal direct sum, especially its summands are linearly independent.

We deduce that

$$\bar\partial_V^2 = \bar\partial_H^2 = \bar\partial_V\bar\partial_H + \bar\partial_H\bar\partial_V = 0$$
$$\left(\bar\partial_V^*\right)^2 = \left(\bar\partial_H^*\right)^2 = \bar\partial_V^*\bar\partial_H^* + \bar\partial_H^*\bar\partial_V^* = 0$$

which directly implies Equation (24).
Now, for Equation (25), we observe that the Laplace splitting property:

$$\Box_{\mathfrak{L}} = \bar\partial_{\mathfrak{L}}\bar\partial_{\mathfrak{L}}^* + \bar\partial_{\mathfrak{L}}^*\bar\partial_{\mathfrak{L}} = \Box_V + \Box_H$$

is equivalent to

$$\bar\partial_V\bar\partial_H^* + \bar\partial_H^*\bar\partial_V = 0 \text{ and}$$
$$\bar\partial_V^*\bar\partial_H + \bar\partial_H\bar\partial_V^* = 0.$$

We conclude that Equation (25) holds which completes the proof. □

Corollary 3.13:
There is an orthonormal Hilbert base of $\mathfrak{A}^{(0,*)}(E, \mathfrak{L})$ consisting of Eigenforms for both $\Box_V$ and $\Box_H$.

3.3 Bijection of certain section spaces

This subsection is dedicated to the proof of Proposition 3.14, below.
It is about a vector space isomorphism between the λ-Eigensections of the vertical Laplace operator $\Box_V$ and the antiholomorphic forms on M with coefficients in the associated bundle $\mathcal{W}^{(\lambda;*)}$.
Recall that we described the bundle $\mathcal{W}^{(\lambda;*)}$ in Remark 3.3.
We denote the operator $\Box_{\boldsymbol{V}}^{[\boldsymbol{s},\boldsymbol{t}]}$ to be the vertical Laplacian acting on the space

$$\mathfrak{A}_H^{(0,s)}(E) \wedge \mathfrak{A}_V^{(0,t)}(E, \mathfrak{L}).$$

Proposition 3.14:
Let $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ be a holomorphic fibre bundle and let $\mathfrak{L} \to E$ be a compatible holomorphic line bundle of fibre type $\tilde{\mathfrak{L}} \to F$.
Furthermore, let K denote a Lie group, $\pi_P : P \to M$ a K principle fibre bundle and

$$\rho: \; K \to \; \mathrm{Aut}(F) \quad \textit{as well as} \quad \tilde\rho: \; K \; \to \; \mathrm{Diff}(F, \tilde{\mathfrak{L}})$$

group homomorphisms such that

$$E = P \times_\rho F \quad \textit{and} \quad \mathfrak{L} = P \times_{\tilde\rho} \tilde{\mathfrak{L}}.$$

Let λ be an Eigenvalue of $\Box_V$ and $\mathcal{W}^{(\lambda;*)}$ be the vector bundle $P \times_{\check\rho} \mathrm{Ker}(\Box_{\tilde{\mathfrak{L}}} - \lambda)$ described in Remark 3.3.

Furthermore, let the space of λ-Eigenforms of $\square_V$ be denoted by

$$\mathrm{Eig}_\lambda\left(\square_V^{[s,t]}\right) := \mathrm{Ker}(\square_V - \lambda) \cap \mathfrak{A}_H^{(0,s)}(E) \wedge \mathfrak{A}_V^{(0,t)}(E, \mathfrak{L}).$$

Then there is an isomorphism

$$\psi : \ \Gamma(M, \mathcal{W}^{(\lambda;*)}) \ \xrightarrow{\sim} \ \mathrm{Eig}_\lambda\left(\square_V^{[0,*]}\right) \subset \mathfrak{A}_V^{(0,*)}(E, \mathfrak{L}),$$

This isomorphism can be extended to an isomorphism

$$\psi : \ \mathfrak{A}^{(0,s)}(M, \mathcal{W}^{(\lambda;t)}) \ \xrightarrow{\sim} \ \mathrm{Eig}_\lambda\left(\square_V^{[s,t]}\right) \subset \mathfrak{A}_H^{(0,s)}(E) \wedge \mathfrak{A}_V^{(0,t)}(E, \mathfrak{L}).$$

Definition 3.15:
We call the isomorphism $\psi : \mathfrak{A}^{(0,s)}\left(M, \mathcal{W}^{(\lambda;t)}\right) \xrightarrow{\sim} \mathrm{Eig}_\lambda\left(\square_V^{[s,t]}\right)$ from the proposition above, i.e. Proposition 3.14, **ψ-morphism**.

We divide the proof of Proposition 3.14 into smaller pieces.
At first as a direct consequence of Lemma 3.16 (in Corollary 3.17), we obtain an isomorphism between the vertical antiholomorphic forms, $\mathfrak{A}_V^{(0,*)}(E, \mathfrak{L})$, and a subspace of $C^\infty\left(P \times F, \Lambda^*\left(T^{(0,1)}F\right)^* \otimes \tilde{\mathfrak{L}}\right)$.
Later on in Corollary 3.20 following from Lemma 3.18, we see that the latter space can be identified with $C^\infty(P, \mathfrak{A}^{(0,*)}(F, \tilde{\mathfrak{L}}))^{\check{\rho}}$.
In Lemma 3.22, we check the compatibility of these identifications above with the vertical Laplace operator.
This finally leads to an isomorphism between

$$\mathrm{Eig}_\lambda(\square_V^{0,t}) = \mathrm{Ker}\left(\square_V - \lambda\right) \cap \left(\mathfrak{A}_V^{(0,t)}(E, \mathfrak{L})\right)$$

and

$$C^\infty\left(P, \mathrm{Eig}_\lambda\left(\square_{\tilde{\mathfrak{L}}}^{(0,t)}\right)\right)^{\check{\rho}}.$$

The latter one is by standard arguments for associated bundles in one to one correspondence to the space of sections $\Gamma\left(M, P \times_{\check{\rho}} \mathrm{Eig}_\lambda\left(\square_{\tilde{\mathfrak{L}}}^{(0,t)}\right)\right)$.

Lemma 3.16:
Let K be a Lie group and let $P \to M$ be a K principle fibre bundle over a compact manifold M.
Furthermore, let E be fibre bundle over M, associated to P via a group homomorphism $\rho : K \to \mathrm{Diff}(F, F)$, whose fibre type F is a compact manifold, i.e.

$$E = P \times_\rho F.$$

Additionally, let $\tilde{\mathcal{Q}} \to F$ be a smooth vector bundle and $\tilde{\rho} : K \to \mathrm{Diff}(F, \tilde{\mathcal{Q}})$ be a continuous group homomorphism covering ρ, making K a Lie transformation group of $\tilde{\mathcal{Q}}$. Denote the induced vector bundle over E by:

$$\mathcal{Q} := P \times_{\tilde{\rho}} \tilde{\mathcal{Q}}.$$

At last, let $C^\infty(P \times F, \tilde{\mathcal{Q}})^{i,ii}$ denote the set of smooth maps

$$\tilde{s} : \quad P \times F \quad \longrightarrow \quad \tilde{\mathcal{Q}}$$

with the properties:

$$\begin{array}{lcr} i) & \tilde{s}(p, f) \in \tilde{\mathcal{Q}}_f & \textit{and} \\ ii) & \tilde{s}\Big(p \cdot k^{-1}, \rho\,(k)\,(f)\Big) = \tilde{\rho}\,(k)\,\Big(\tilde{s}(p, f)\Big). & \end{array}$$

Then there is an isomorphism $\tilde{}$ of vector spaces

$$\begin{array}{rccc} \tilde{} : & \Gamma(E, \mathcal{Q}) & \longrightarrow & C^\infty(P \times F, \tilde{\mathcal{Q}})^{i,ii} \\ & s & \longmapsto & \tilde{s}. \end{array}$$

Proof.
Let $s \in \Gamma(E, \mathcal{Q})$ be a smooth section and let $e = [p, k]_\rho$ be in E. Then $s(e)$ lies in $\mathcal{Q}_e$. Therefore, it has the form $s(e) = [p, \tilde{s}(p, f)]_{\tilde{\rho}}$.
The desired bijection is now given by:

$$\begin{array}{rccc} \tilde{} : & \Gamma(E, \mathcal{Q}) & \longrightarrow & C^\infty(P \times F, \tilde{\mathcal{Q}})^{i,ii} \\ & s & \longmapsto & \tilde{s} \end{array}$$

Note that for $\tilde{}$ to be well defined, $\tilde{s}$ has to fulfil the second property *ii*).
Furthermore, s is a section from E to $\mathcal{Q}$, in particular, $\pi_{\mathcal{Q}} \circ s = \mathrm{id}_E$. This is corresponds on the $\tilde{s}$ side to property *i*).
Conversely, any map $\tilde{s} \in C^\infty(P \times F, \tilde{\mathcal{Q}})^{i,ii}$ induces a map $s : E \to \mathcal{Q}$ that fulfils $\pi_{\mathcal{Q}} \circ s = \mathrm{id}_E$.
Summarising, we obtain

$$\begin{array}{ccl} \pi_{\mathcal{Q}} \circ s & \longleftrightarrow & i)\ \pi_{\tilde{\mathcal{Q}}} \circ \tilde{s} = \mathrm{proj}_2 \\ s([p, f]_\rho) = s(\big[p \cdot k^{-1}, \rho(k)(f)\big]_\rho) & \longleftrightarrow & ii)\ \tilde{s}(p \cdot k^{-1}, \rho(k)(f)) = \tilde{\rho}(k)\,(\tilde{s}(p, f))\,. \end{array}$$

What remains to be shown is that the smoothness of s induces the smoothness of $\tilde{s}$ and vice verse.
In order to do that, we will borrow some notations from Lemma 2.39. In particular, we denote the local trivialisations the same way.

We deduce:

$$\begin{array}{rlrlll} s : E \longrightarrow \mathcal{Q} \text{ is } C^\infty & \Leftrightarrow & \chi_{ij} \circ s_{|W_{ij}} : & W_{ij} & \longrightarrow & W_{ij} \times \mathbb{C}^m & \text{is } C^\infty \ \forall i,j \\ & \Leftrightarrow & \mathrm{proj}_2 \circ \chi_{ij} \circ s_{|W_{ij}} : & W_{ij} & \longrightarrow & \mathbb{C}^m & \text{is } C^\infty \ \forall i,j \end{array}$$

On the other hand W_{ij} is diffeomorphic to $U_i \times V_j$ through $q_i \times \mathrm{id}_{V_j}$ and

$$\mathrm{proj}_2 \circ \chi_{ij} \circ s_{|W_{ij}} \circ \left[q_i \times \mathrm{id}_{V_j}\right]_\rho = \mathrm{proj}_2 \circ \psi_j \circ \tilde{s} \circ (q_i \times \mathrm{id}_{V_j}).$$

Therefore,

$$\begin{array}{lrllll} s \text{ is } C^\infty & \Leftrightarrow \ \mathrm{proj}_2 \circ \psi_j \circ \tilde{s} \circ \left(q_i \times \mathrm{id}_{V_j}\right) : & U_i \times V_j & \to & \mathbb{C}^m & \text{is } C^\infty \ \forall i,j \\ & \Leftrightarrow \ \tilde{s} \circ \left(\varphi_i^{-1}(\cdot, e) \times \mathrm{id}_F\right) : & U_i \times F & \to & \tilde{\mathcal{Q}} & \text{is } C^\infty \ \forall i \end{array}$$

This fact implies that if $\tilde{s} : P \times F \to \tilde{\mathcal{Q}}$ is smooth, then so is its corresponding map s. For the opposite direction, suppose now that s is a smooth map. The considerations above showed that s smooth directly implies that the map $\tilde{s} \circ \left(\varphi_i^{-1}(\cdot, e) \times \mathrm{id}_F\right)$ is smooth as well.
In order to show that $\tilde{s}$ itself is C^∞, we have to check that, locally for each U_i,

$$\tilde{s} \circ \left(\varphi_i^{-1} \times \mathrm{id}_F\right) : \quad U_i \times K \times F \quad \longrightarrow \quad \tilde{\mathcal{Q}},$$

i.e. the map $(x, k, f) \longmapsto \tilde{s}(q_i(x) \cdot k, f)$, is smooth.
But this is true because of property $ii)$ which implies

$$\tilde{s}(q_i(x) \cdot k, f) = \tilde{\rho}\left(k^{-1}\right) \circ \tilde{s}\left(\varphi_i^{-1}(x, e), \rho(k^{-1})(f)\right)$$

and this is smooth in (x, k, f) as composition of smooth maps. □

For our compatible line bundle $\mathfrak{L} = P \times_{\tilde{\rho}} \tilde{\mathfrak{L}} \to E$ over a holomorphic fibre bundle, we can apply Lemma 3.16 not only on sections $\Gamma(E, \mathfrak{L})$ but on the vertical antiholomorphic forms $\mathfrak{A}_V^{(0,q)}(E, \mathfrak{L})$ with coefficients in $\mathfrak{L}$ as well since for

$$\rho^* : \quad K \longrightarrow \quad \mathrm{Diff}(F, \Lambda^q T^{(0,1)} F),$$

given by $\rho^*(k) := \rho(k^{-1})^*$, we obtain:

$$\Lambda^q \left(T^{V,(0,1)} E\right)^* = P \times_{\rho^*} \Lambda^q \left(T^{(0,1)} F\right)^*.$$

And consequently we get for $\rho^* \otimes \tilde{\rho}$:

$$\Lambda^q \left(T^{V,(0,1)} E\right)^* \otimes \mathfrak{L} = P \times_{\rho^* \otimes \tilde{\rho}} \Lambda^q \left(T^{(0,1)} F\right)^* \otimes \tilde{\mathfrak{L}}.$$

Corollary 3.17:
For a holomorphic fibre bundle $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ and a compatible line bundle $\mathfrak{L}$, i.e.

$$\mathfrak{L} = P \times_{\tilde{\rho}} \tilde{\mathfrak{L}} \quad \longrightarrow \quad E = P \times_\rho F,$$

we get an isomorphism of vector spaces

$$\begin{array}{rccc} \tilde{} : & \mathfrak{A}_V^{(0,q)}(E, \mathfrak{L}) & \longrightarrow & C^\infty(P \times F, \Lambda^q \left(T^{(0,1)}F\right)^* \otimes \tilde{\mathfrak{L}})^{i,ii} \\ & s & \longmapsto & \tilde{s} \end{array}$$

The next step is to find a bijection of $C^\infty(P \times F, \Lambda^q \left(T^{(0,1)}F\right)^* \otimes \tilde{\mathfrak{L}})^{i,ii}$ on the one hand and a subspace of $C^\infty(P, \mathfrak{A}^{(0,q)}(F, \tilde{\mathfrak{L}}))$ on the other hand.
Therefore, let

$$C^\infty \left(P \times F, \Lambda^q \left(T^{(0,1)}F\right)^* \otimes \tilde{\mathfrak{L}}\right)^i$$

be the subspace of smooth maps $\tilde{s}$ from $P \times F$ to $\Lambda^q \left(T^{(0,1)}F\right)^* \otimes \tilde{\mathfrak{L}}$ that fulfil:

$$i) \;\; \tilde{s}(p, f) \in \left(\Lambda^q \left(T^{(0,1)}F\right)^* \otimes \tilde{\mathfrak{L}}\right)_f.$$

Obviously, we have the following inclusion:

$$C^\infty \left(P \times F, \Lambda^q \left(T^{(0,1)}F\right)^* \otimes \tilde{\mathfrak{L}}\right)^{i,ii} \subset C^\infty \left(P \times F, \Lambda^q \left(T^{(0,1)}F\right)^* \otimes \tilde{\mathfrak{L}}\right)^i.$$

Now, we show that $C^\infty(P \times F, \Lambda^q \left(T^{(0,1)}F\right)^* \otimes \tilde{\mathfrak{L}})^i$ and $C^\infty(P, \mathfrak{A}^{(0,q)}(F, \tilde{\mathfrak{L}}))$ are isomorphic.

The map describing the desired isomorphism is actually a very basic one.

If we forget any structure, let A, B, C be sets and let $\mathrm{Map}(A, B)$ denote the maps from A to B.
The space $\mathrm{Map}(A \times B, C)$ is isomorphic to the space $\mathrm{Map}(A, \mathrm{Map}(B, C))$ in a natural way.
The identification is given by

$$\widehat{} : \;\; \mathrm{Map}(A \times B, C) \quad \longrightarrow \quad \mathrm{Map}(A, \mathrm{Map}(B, C)),$$

where $\hat{f}(a) : b \mapsto f(a, b)$ and inversely $f(a, b) := \left(\hat{f}(a)\right)(b)$.

Unfortunately, if we return to the category of smooth manifolds, the translation of properties like continuity or smoothness via $\widehat{}$ are much more intricate.
The subsequent lemma shows that in the case of $C^\infty(P \times F, \tilde{\mathcal{Q}})^i$ everything behaves just fine.

Beforehand, we state the following theorem stated by Trèves in [30, Ch.40, Thm. 40.1]. We present it, tailored to the situation at hand.

Theorem (Trèves Thm 40.1)**:**
Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be open sets and let $C = \mathbb{R}^k$. Then the map

$$\widehat{\ } : \mathrm{Map}(A \times B, C) \longrightarrow \mathrm{Map}(A, \mathrm{Map}(B, C)),$$

introduced above, induces an isomorphism of topological vector spaces between $C^\infty(A \times B, \mathbb{R})$ and $C^\infty(A, C^\infty(B, \mathbb{R}))$ by restriction.
Here, the topology on $C^\infty(X, \mathbb{R})$ is given by the Fréchet topology.

We give a brief summary of the Fréchet space structure on $\Gamma(F, \mathcal{Q})$ in Appendix B.

Lemma 3.18:
Let B, F be smooth manifolds and let $\pi_{\tilde{\mathcal{Q}}} : \tilde{\mathcal{Q}} \to F$ be a smooth vector bundle.
Let furthermore $C^\infty(B \times F, \tilde{\mathcal{Q}})^i$ denote the following topological vector space:

$$C^\infty(B \times F, \tilde{\mathcal{Q}})^i := \left\{ g : B \times F \to \tilde{\mathcal{Q}} \mid \pi_{\tilde{\mathcal{Q}}} \circ g = \mathrm{proj}_2 \right\}.$$

Then there is an isomorphism $\widehat{\ }$ of the topological vector spaces $C^\infty(B \times F, \tilde{\mathcal{Q}})^i$ and $C^\infty(B, \Gamma(F, \tilde{\mathcal{Q}}))$.

$$\widehat{\ } : \quad C^\infty(B \times F, \tilde{\mathcal{Q}})^i \quad \xrightarrow{1:1} \quad C^\infty(B, \Gamma(F, \tilde{\mathcal{Q}}))$$

Proof.
The bundle $\tilde{\mathcal{Q}} \to F$ is locally trivial. We denote the local trivialisation maps for $\tilde{\mathcal{Q}} \to F$ by (ψ_j, V_j), i.e..

$$\psi_j : \quad \tilde{\mathcal{Q}}\mid_{V_j} \quad \longrightarrow \quad V_j \times \mathbb{C}^m$$

Now, for $g \in C^\infty(B \times F, \tilde{\mathcal{Q}})^i$ the map

$$\hat{g} : \quad B \quad \longrightarrow \quad \Gamma(F, \tilde{\mathcal{Q}})$$

is smooth if and only if the maps $\hat{g}_j$ defined by

$$\begin{aligned} \hat{g}_j : \quad B &\longrightarrow \quad \Gamma(V_j, \tilde{\mathcal{Q}}\mid_{V_j}) \\ b &\longmapsto \quad \hat{g}(b)\mid_{V_j} \end{aligned}$$

are smooth for all $j \in J$ where $\hat{g}_j(b)$ denotes the restriction of $g(b)$ to the local trivialisation base set $V_j \subset F$ of $\tilde{\mathcal{Q}}$ (compare Corollary B.5).

In particular, $\hat{g}$ is smooth if and only if the maps

$$\begin{array}{rccc} \psi_j \circ \hat{g}_j : & B & \longrightarrow & C^\infty(V_j, V_j \times \mathbb{C}^m) \\ & b & \longmapsto & \psi_j \circ \hat{g}_j(b) \end{array}$$

are smooth for any $j \in J$.
Now, for every $b \in B$ and every $j \in J$, the map

$$\psi_j \circ \hat{g}_j(b) : \quad V_j \quad \longrightarrow \quad V_j \times \mathbb{C}^m$$

has got the form $\psi_j \circ \hat{g}_j(b) = \mathrm{id}_{V_j} \times \hat{h}_j(b)$ for some $\hat{h}_j(b) \in C^\infty(V_j, \mathbb{C}^m)$.
Looking at the Fréchet structure, we see that $\hat{g} : B \to \Gamma(F, \tilde{\mathcal{Q}})$ is smooth if and only if

$$\hat{h}_j : \quad B \quad \longrightarrow \quad C^\infty(V_j, \mathbb{C}^m)$$

is smooth for every $j \in J$ (compare Corollary B.6).
Reading this statement through charts, we can apply Trèves Theorem 40.1, stated above.
Hence, $\hat{g}$ is smooth if and only if

$$\begin{array}{rccc} h_j : & B \times V_j & \longrightarrow & \mathbb{C}^m \\ & (b, f) & \longmapsto & \left(\hat{h}_j(b)\right)(f) = \mathrm{proj}_2 \circ \psi_j \circ (\hat{g}(b))\,(f) \end{array}$$

is smooth for any $j \in J$.
On the other hand h_j is obviously smooth if and only if

$$\begin{array}{rccc} \mathrm{proj}_2 \times h_j : & B \times V_j & \longrightarrow & V_j \times \mathbb{C}^m \\ & (b, f) & \longmapsto & \Big(f, h_j(b, f)\Big) \end{array}$$

is smooth.
Now, by applying ψ_j^{-1}, we see that this is equivalent to

$$\psi_j^{-1} \circ (\mathrm{proj}_2 \times h_j) : \quad B \times V_j \quad \longrightarrow \quad \tilde{\mathcal{Q}}$$

being smooth for any $j \in J$, since ψ is a diffeomorphism.
But, if we make use of Property $i)$, we obtain

$$\psi_j^{-1} \circ (\mathrm{proj}_2 \times h_j)\,(b, f) = \psi_j^{-1}\left(f, \mathrm{proj}_2 \circ \psi_j \circ \underbrace{(\hat{g}(b))\,(f)}_{=g(b,f)}\right) \stackrel{i)}{=} g(p, f).$$

Consequently, we obtain:

$$\begin{array}{rccc} \psi_j^{-1} \circ (\mathrm{proj}_2 \times h_j) : & B \times V_j & \longrightarrow & \tilde{\mathcal{Q}} \\ & (b, f) & \longmapsto & g(b, f), \end{array}$$

and therefore $\hat{g}$ is smooth if and only if g is smooth which finishes the proof. $\square$

We obtain a direct consequence of the Lemma 3.18 by choosing $B = P$.

Corollary 3.19:
In the setting from Lemma 3.16, there is an isomorphism $\hat{}$ *between the topological vector spaces* $C^\infty(P \times F, \tilde{\mathcal{Q}})^i$ *and* $C^\infty(P, \Gamma(F, \tilde{\mathcal{Q}}))$.

$$\hat{}: \quad C^\infty(P \times F, \tilde{\mathcal{Q}})^i \xrightarrow{1:1} C^\infty(P, \Gamma(F, \tilde{\mathcal{Q}}))$$

This corollary on the other hand has as a direct consequence the subsequent corollary.

Corollary 3.20:
Let $\check{\rho}$ *be the induced action of* K *on* $\mathfrak{A}^{(0,*)}(F, \tilde{\mathcal{Q}})$ *which is given for any* $k \in K$ *and every* $\alpha \in \mathfrak{A}^{(0,*)}(F, \tilde{\mathcal{Q}})$ *by:*

$$\check{\rho}(k)(\alpha) := \tilde{\rho}(k)\left(\rho\left(k^{-1}\right)^*(\alpha)\right).$$

Denote by $C^\infty(P, \Gamma(F, \tilde{\mathcal{Q}}))^{\check{\rho}}$ *the vector space of* K *equivariant maps from* P *into the section space* $\Gamma(F, \tilde{\mathcal{Q}})$.
Then we get an isomorphism between $C^\infty(P \times F, \tilde{\mathcal{Q}})^{i,ii}$ *and* $C^\infty(P, \Gamma(F, \tilde{\mathcal{Q}}))^{\check{\rho}}$,

$$\hat{}: \quad C^\infty(P \times F, \tilde{\mathcal{Q}})^{i,ii} \xrightarrow{1:1} C^\infty(P, \Gamma(F, \tilde{\mathcal{Q}}))^{\check{\rho}},$$

by restriction.

Proof. Let g be in $C^\infty(P \times F, \tilde{\mathcal{Q}})^{i,ii}$. We have to verify that $\hat{g}$ is $\check{\rho}$-equivariant.
Therefore, take $p \in P$, $k \in K$ and $f \in f$ and compute:

$$\begin{aligned}
\hat{g}(p \cdot k)(f) &\overset{!}{=} g(p \cdot k, f) = g\left(p \cdot k, \rho(k^{-1}) \circ \rho(k)(f)\right) \\
&\overset{ii)}{=} \tilde{\rho}(k^{-1})\big(g(p, \rho(k)(f))\big) \overset{!}{=} \tilde{\rho}(k^{-1})\left(\hat{g}(p)(\rho(k)(f))\right) \\
&= \Big(\tilde{\rho}(k^{-1}) \circ \rho(k)^* \hat{g}(p)\Big)(f) = \Big(\check{\rho}(k^{-1})\left(\hat{g}(p)\right)\Big)(f).
\end{aligned}$$

Here, we marked the equalities coming from the isomorphism $\hat{}$ with an $'!'$.
It follows that $\hat{g}$ lies in $C^\infty(P, \Gamma(F, \tilde{\mathcal{Q}}))^{\check{\rho}}$.

For the other direction, let $\hat{g}$ be in $C^\infty(P, \Gamma(F, \tilde{\mathcal{Q}}))^{\check{\rho}}$.
We now have to show that g fulfils the property $ii)$.
We compute for $p \in P$, $k \in K$ and $f \in f$:

$$\begin{aligned}
g\left(p \cdot k, \rho(k^{-1})(f)\right) &\overset{!}{=} \Big(\hat{g}(p \cdot k)\Big)(\rho(k^{-1})(f)) = \Big(\check{\rho}(k^{-1})(\hat{g}(p))\Big)(\rho(k^{-1})(f)) \\
&= \tilde{\rho}(k^{-1})\left(\hat{g}(p)(f)\right) \overset{!}{=} \tilde{\rho}(k^{-1})(g(p, f)),
\end{aligned}$$

which is what we wanted to show. $\square$

Corollary 3.21:
For a holomorphic fibre bundle $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ *and a compatible line bundle* $\mathfrak{L} \to E$*. We have an isomorphism*

$$\widehat{\ } \circ \widetilde{\ } = \widehat{\widetilde{\ }} : \ \mathfrak{A}_V^{(0,*)}(E, \mathfrak{L}) \ \longrightarrow \ C^\infty(P, \mathfrak{A}^{(0,*)}(F, \tilde{\mathfrak{L}}))^{\check{\rho}}.$$

One question has been unanswered so far.
What happens under the correspondence $\widehat{\widetilde{\ }}$ above with the λ-Eigenspaces of $\Box_V$?
To answer this question is the next step.

In the definition of a compatible line bundle we fixed our holomorphic structure on $\mathfrak{L}$ in such a way that the vertical Laplacian $\Box_V$ corresponds to the Laplacian $\Box_{\tilde{\mathfrak{L}}}$ on $\tilde{\mathfrak{L}} \to F$. This property has the indispensable consequence that the morphism $\widehat{\widetilde{\ }}$ from above, identifying $\mathfrak{A}_V^{(0,*)}(E, \mathfrak{L})$ with $C^\infty(P, \mathfrak{A}^{(0,*)}(F, \tilde{\mathfrak{L}})^{\check{\rho}})$ restricts to a morphism:

$$\mathfrak{A}_V^{(0,*)}(E, \mathfrak{L}) \supset \mathrm{Eig}_\lambda(\Box_V^{0,*}) \leftrightarrow C^\infty(P, \mathrm{Eig}_\lambda(\Box_F))^{\check{\rho}} \subset C^\infty(P, \mathfrak{A}^{(0,*)}(F, \tilde{\mathfrak{L}}))^{\check{\rho}}$$

This is the content of the following lemma.

Lemma 3.22:
Let α *be a form in* $\mathfrak{A}_V^{(0,q)}(E, \mathfrak{L})$ *and let* $\hat{\tilde{\alpha}} \in C^\infty(P, \mathfrak{A}^{(0,q)}(F, \tilde{\mathfrak{L}}))^{\check{\rho}}$ *be its image under* $\widehat{\widetilde{\ }}$*. Then we get for every* $p \in P$*:*

$$\left(\widehat{\widetilde{\Box_V \alpha}}\right)(p) = \Box_{\tilde{\mathfrak{L}}}\left(\hat{\tilde{\alpha}}(p)\right).$$

Proof.
For the local trivialisation maps, we stick to the notations we have evolved so far, nonetheless we will repeat them here to make this proof easier to understand.
We have an open cover $\{U_i\}_{i \in I}$ of M and local sections $q_i : U_i \to P$ which induce local trivialisations:

- for $P \to M$,

$$\begin{array}{rccc} \varphi_i : & P\,|_{U_i} & \longrightarrow & U_i \times K \\ & (q_i(x) \cdot k) & \longmapsto & (x, k), \end{array}$$

- for $E \to M$,

$$\begin{array}{rccc} \phi_i : & E\,|_{U_i} & \longrightarrow & U_i \times F \\ & [q_i(x), f]_\rho & \longmapsto & (x, f), \end{array}$$

- and for $\mathfrak{L} = P \times_{\tilde{\rho}} \tilde{\mathfrak{L}}$ seen as an associated fibre bundle over M, we introduce local trivialisations

$$\begin{array}{rccc} \tilde{\phi}_i : & \mathfrak{L}\mid_{\pi_E^{-1}(U_i)} & \longrightarrow & U_i \times \tilde{\mathfrak{L}} \\ & \left[q_i(x), \tilde{l}\right]_{\tilde{\rho}} & \longmapsto & (x, \tilde{l}). \end{array}$$

They generalise to local trivialisations

$$\begin{array}{rccc} \tilde{\phi}_i : & \Lambda^* \left(T^{V,(0,1)}E\right)^* \otimes \mathfrak{L}\mid_{\pi_E^{-1}(U_i)} & \longrightarrow & U_i \times \Lambda^* \left(T^{(0,1)}F\right)^* \otimes \tilde{\mathfrak{L}} \\ & \omega \otimes \tilde{l} & \longmapsto & \left(x, \left(\phi_{i,x}^{-1}\right)^* \omega \otimes \tilde{\phi}_{i,x}(\tilde{l})\right) \end{array}$$

where $\phi_{i,x}$ denotes the induced map

$$\phi_{i,x} : \ \pi_E^{-1}(x) \ \longrightarrow \ F$$

and $\tilde{\phi}_{i,x}$ likewise.

Keeping these maps in mind, we can start the actual proof.
For an $\alpha \in \mathfrak{A}_V^{(0,q)}(E, \mathfrak{L})$, we obtain by construction that $\hat{\tilde{\alpha}}$ evaluated at $q_i(x)$ is given by

$$\left(\hat{\tilde{\alpha}}(q_i(x))\right)(f) = \tilde{\phi}_{i,x} \circ \alpha(q_i(x), f) = \tilde{\phi}_{i,x} \circ \alpha \circ \phi_{i,x}^{-1}(f).$$

Let $\mathfrak{L}_{i,x}$ denote the pullback bundle $\left(\phi_{i,x}^{-1}\right)^* \mathfrak{L}$.
A small computation shows that $\left(\hat{\tilde{\alpha}}(q_i(x))\right)$ decomposes into a pullback

$$\left(\phi_{i,x}^{-1}\right)^* : \ \mathfrak{A}_V^{(0,*)}(E, \mathfrak{L}) \ \longrightarrow \ \mathfrak{A}^{(0,*)}\left(F, \left(\phi_{i,x}^{-1}\right)^* \mathfrak{L}\right)$$

and the natural vector bundle homomorphism identifying $\tilde{\mathfrak{L}}$ and $\mathfrak{L}_{i,x}$ as an equivalence of Hermitian, holomorphic line bundles over F

$$\begin{array}{rccc} \Phi_{i,x} : & \mathfrak{L}_{i,x} & \longrightarrow & \tilde{\mathfrak{L}} \\ & \left(f, \left[q_i(x), \tilde{l}\right]_{\tilde{\rho}}\right) & \longmapsto & \tilde{l} \end{array}$$

in the following way:

$$\left(\hat{\tilde{\alpha}}(q_i(x))\right)(f) = \left(\Phi_{i,x} \circ \left(\phi_{i,x}^{-1}\right)^* \alpha\right)(f).$$

Now, since $\mathfrak{L} \to E$ is a compatible line bundle, Property 3 of Definition 3.1 implies

$$\begin{aligned} \left(\widehat{\widetilde{\bar{\partial}_V \alpha}}(q_i(x))\right)(f) &= \left(\Phi_{i,x} \circ \left(\phi_{i,x}^{-1}\right)^* \bar{\partial}_V \alpha\right)(f) \\ &= \left\{\bar{\partial}_{\tilde{\mathfrak{L}}} \left(\Phi_{i,x} \circ \left(\phi_{i,x}^{-1}\right)^* \alpha\right)\right\}(f) = \left(\bar{\partial}_{\tilde{\mathfrak{L}}} \hat{\tilde{\alpha}}\right)(f). \end{aligned}$$

An analogous result holds for $\bar{\partial}_V^*$ because of Lemma 2.29.
Therefore the assertion is proven. □

Finally, we are able complete the proof of Proposition 3.14.
Therefore, recall the following fact. Let $\pi_P : P \longrightarrow M$ be a K-principle fibre bundle and let $\mathcal{Q} = P \times_\chi V$ be an associated vector bundle for a representation $\chi : K \longrightarrow \mathrm{Gl}(V)$. Then the space of sections from M into $\mathcal{Q}$ is isomorphic to the K-χ-equivariant C^∞-maps from P to V (cf. [5]).
This isomorphism is explicitly given by:

$$\begin{array}{ccc} C^\infty(P,V)^\chi & \longrightarrow & \Gamma(M,\mathcal{Q}) \\ f & \longmapsto & \left\{x \mapsto [p_x, f(p_x)]_\chi\right\} \end{array} \tag{26}$$

where p_x lies in the fibre over $x \in M$, i.e. $\pi_P(p_x) = x$.

Proof of Proposition 3.14.
If we assume the existence of the first isomorphism of Proposition 3.14, i.e.

$$\psi : \quad \Gamma(M, \mathcal{W}^{(\lambda;*)}) \quad \xrightarrow{\sim} \quad \mathrm{Eig}_\lambda\left(\Box_V^{[0,*]}\right) \subset \mathfrak{A}_V^{(0,*)}(E,\mathfrak{L}),$$

it can be extended to the space $\mathfrak{A}^{(0,s)}(M, \mathcal{W}^{(\lambda;t)}) \cong \mathfrak{A}^{(0,s)}(M) \otimes \Gamma(M, \mathcal{W}^{(\lambda;t)})$ as follows.

Take $\alpha \in \mathfrak{A}^{(0,s)}(M)$ and $s \in \Gamma(M, \mathcal{W}^{(\lambda;t)})$ now the ψ-morphism extends via

$$\psi(\alpha \otimes s) := (\pi_E^*\alpha) \wedge \psi(s). \tag{27}$$

This extension respects the Eigenspace structure because for any $\alpha \in \mathfrak{A}^{(0,s)}(M)$ and any $\beta \in \mathfrak{A}_H^{(0,t_1)}(E) \wedge \mathfrak{A}_V^{(0,t_2)}(E,\mathfrak{L})$, we obtain:

$$\Box_V^{[s+t_1,t_2]}\Big((\pi_E^*\alpha) \wedge \beta\Big) = (\pi_E^*\alpha) \wedge \Big(\Box_V^{[t_1,t_2]}\beta\Big)$$

which has been shown in Corollary 2.28.

Thus, we have to construct the first isomorphism only, i.e.

$$\psi : \quad \Gamma(M, \mathcal{W}^{(\lambda;*)}) \quad \longrightarrow \quad \mathrm{Eig}_\lambda(\Box_V^{[0,*]}) \subset \mathfrak{A}_V^{(0,*)}(E,\mathfrak{L})$$

Let α be in $\mathfrak{A}_V^{(0,*)}(E,\mathfrak{L})$.
We apply the Corollaries 3.17 and 3.21 and obtain a unique map $\hat{\tilde{\alpha}} \in C^\infty\left(P, \mathfrak{A}^{(0,t)}(F,\tilde{\mathfrak{L}})\right)^{\check{\rho}}$ corresponding to α.
Lemma 3.22 now implies that α is a λ-Eigenform of $\Box_V$ if and only if $\hat{\tilde{\alpha}}(p)$ is a λ-Eigenform of $\Box_{\tilde{\mathfrak{L}}}$ for every $p \in P$.
Consequently, the map $\widehat{\tilde{\ }}$ restricts to

$$\widehat{\tilde{\ }} : \quad \mathrm{Eig}_\lambda(\Box_V^{[0,*]}) \quad \longrightarrow \quad C^\infty(P, \mathrm{Eig}_\lambda(\Box_{\tilde{\mathfrak{L}}}))^{\check{\rho}}.$$

However, the bundle $\mathcal{W}^{(\lambda;*)}$ is associated to P, i.e. $\mathcal{W}^{(\lambda;*)} = P \times_{\check{\rho}} \mathrm{Eig}_\lambda(\Box_{\tilde{\mathfrak{L}}})$ (compare Remark 3.3).

Now, as we mentioned above, sections into an associated vector bundle can be identified with K-equivariant smooth maps from P to the fibretype (compare Equation (26)).
In particular,

$$\Gamma(M, \mathcal{W}^{(\lambda;*)}) = \Gamma(M, P \times_{\check{\rho}} \mathrm{Eig}_\lambda(\square_{\tilde{\mathfrak{L}}})) \overset{1:1}{\longleftrightarrow} C^\infty(P, \mathrm{Eig}_\lambda(\square_{\tilde{\mathfrak{L}}}))^{\check{\rho}}$$

which completes the proof. $\square$

Now, that we have constructed the ψ-morphism, we can use it to express the equivariant ζ-function of $\mathfrak{L}$ in terms and objects that depend on M as well as on the bundles $\mathcal{W}^{(\lambda;*)}$.
In order to do that, we have to carry every information we have of $\mathfrak{L}$, like its Hermitian metric or its holomorphic structure, forward along ψ^{-1}.
This is the content of the subsequent subsection.

3.4 Induced holomorphic, Hermitian structure on the Eigenspace vector bundles

Throughout this subsection let $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ be a holomorphic fibre bundle and let $\mathfrak{L} \to E$ be a compatible line bundle over E.

The ψ-morphism, constructed in the preceding subsection, allows us to equip the complex vector bundle $\mathcal{W}^{(\lambda;*)}$ with a Hermitian and a holomorphic structure for each λ.
To do this explicitly is the content of this subsection.

Lemma 3.23:
For every λ and every t the bundle $\mathcal{W}^{(\lambda;t)}$ is a holomorphic vector bundle. Its holomorphic structure $\bar\partial_{\mathcal{W}^{(\lambda;t)}} := \bigoplus_s \bar\partial^s_{\mathcal{W}^{(\lambda;t)}}$ is given by the following diagram.

$$\begin{tikzcd}
\mathfrak{A}^{(0,s)}(M, \mathcal{W}^{(\lambda;t)}) \arrow[r, "\psi"] \arrow[d, "\bar\partial^s_{\mathcal{W}^{(\lambda;t)}}"'] & \mathrm{Eig}_\lambda(\square_V^{[s,t]}) \arrow[d, "\bar\partial_H"] \\
\mathfrak{A}^{(0,s+1)}(M, \mathcal{W}^{(\lambda;t)}) & \mathrm{Eig}_\lambda(\square_V^{[s+1,t]}) \arrow[l, "\psi^{-1}"]
\end{tikzcd}$$

In the similar way, Bismut uses in [8] that the horizontal Dolbeault-operator induces a holomorphic structure on his push forward bundles.

Proof.
By Corollary 2.3, we only have to check that $\bar\partial_{\mathcal{W}^{(\lambda;t)}}$ fulfils $\bar\partial^2_{\mathcal{W}^{(\lambda;t)}} = 0$ as well as the Leibniz equation (Equation (3)).
The first property is easy to see since $\bar\partial_H^2 = 0$. Consequently, we obtain

$$\bar\partial^2_{\mathcal{W}^{(\lambda;t)}} = \psi^{-1} \circ \bar\partial_H^2 \circ \psi = 0.$$

For the Leibniz equation, let α be in $\mathfrak{A}^{(0,q)}(M)$ and take $s \in \Gamma(M, \mathcal{W}^{(\lambda;t)})$. Using Equation (27), i.e. the extension of ψ to antiholomorphic forms, we compute:

$$\begin{aligned}\bar{\partial}_{\mathcal{W}^{(\lambda;t)}}(\alpha \otimes s) =& \psi^{-1} \circ \bar{\partial}_H \circ \psi(\alpha \otimes s) \overset{(27)}{=} \psi^{-1} \circ \bar{\partial}_H (\pi_E^* \alpha \otimes \psi(s)) \\ =& \psi^{-1} \circ \left(\pi_E^*(\bar{\partial}\alpha) \otimes \psi(s) + (-1)^q (\pi_E^* \alpha) \wedge \bar{\partial}_H \psi(s)\right) \\ =& \left(\bar{\partial}\alpha\right) \otimes s + (-1)^q \alpha \wedge \left(\bar{\partial}_{\mathcal{W}^{(\lambda;t)}} s\right).\end{aligned}$$

□

Similarly to the definition of a holomorphic structure on $\mathcal{W}^{(\lambda;t)}$, we can equip the space $\mathfrak{A}^{(0,s)}(M, \mathcal{W}^{(\lambda;t)})$ with a Hermitian metric on $\mathcal{W}^{(\lambda;t)}$ which makes

$$\psi: \quad \mathfrak{A}^{(0,s)}(M, \mathcal{W}^{(\lambda;t)}) \quad \longrightarrow \quad \mathrm{Eig}_\lambda(\Box_V^{[s,t]})$$

a linear isometry for the induced the L^2-metric on both sides.

Let $h^{\tilde{\mathfrak{L}}}$ denote the Hermitian metric on $\tilde{\mathfrak{L}}$ as well as its extension to the complex vector bundle $\Lambda^t \left(T^{(0,1)}F\right)^* \otimes \tilde{\mathfrak{L}}$.

We construct a Hermitian metric $h^{\mathcal{W}^{(\lambda;t)}}$ on $\mathcal{W}^{(\lambda;t)} = P \times_{\check{\rho}} \mathrm{Eig}_\lambda(\Box_{\tilde{\mathfrak{L}}}^{(0,t)})$ as follows.

For any point $x \in M$ and every $p \in \pi_E^{-1}(x) \subset E$ let $[p,v]_{\check{\rho}}$ and $[p,w]_{\check{\rho}}$ be elements in $\mathcal{W}^{(\lambda;t)}$.

In particular, v and w are antiholomorphic forms in $\mathfrak{A}^{(0,t)}(F, \tilde{\mathfrak{L}})$. We define:

$$h_x^{\mathcal{W}^{(\lambda;t)}}\left([p,v]_{\check{\rho}}, [p,w]_{\check{\rho}}\right) := \int_F h^{\tilde{\mathfrak{L}}}(v,w) \mathrm{dvol}_F.$$

Lemma 3.24:
The ψ-morphism

$$\psi: \quad \mathfrak{A}^{(0,s)}(M, \mathcal{W}^{(\lambda;t)}) \quad \longrightarrow \quad \mathrm{Eig}_\lambda(\Box_V^{[s,t]})$$

becomes a linear isometry of Hermitian vector spaces.
The Hermitian metrics on both sides are the L^2 metrics induced by $h^{\mathfrak{L}}$ on the bundle $\Lambda^t \left(T^{V,(0,1)}E\right)^* \otimes \mathfrak{L}$ on the left hand side and by $h^{\mathcal{W}^{(\lambda;t)}}$ on the right hand side.

Proof.
This follows directly from the fact that $\tilde{\mathfrak{L}}$ is isomorphic to $\left(\phi_{i,x}^{-1}\right)^* \mathfrak{L}$ as a Hermitian complex line bundles over F and from the Fubini Theorem for fibre bundles. □

Now, that we made ψ a Hermitian isometry and $\mathcal{W}^{(\lambda;t)}$ a holomorphic vector bundle, we will introduce some notations.

We denote the operator operator corresponding to $\bar{\partial}_H + \bar{\partial}_H^*$, by

$$D_{\mathcal{W}^{(\lambda;t)}} = \bigoplus_s D^s_{\mathcal{W}^{(\lambda;t)}}: \quad \mathfrak{A}^{(0,*)}(M, \mathcal{W}^{(\lambda;t)}) \quad \longrightarrow \quad \mathfrak{A}^{(0,*)}(M, \mathcal{W}^{(\lambda;t)}). \tag{28}$$

Consequently $D_{\mathcal{W}^{(\lambda;t)}}$ is given explicitly as the left hand side of this commuting diagram:

$$\begin{tikzcd}
\mathfrak{A}^{(0,s)}(M,\mathcal{W}^{(\lambda;t)}) \arrow[r, "\psi"] \arrow[d, "D^s_{(\lambda,t)}"'] & \mathrm{Eig}_\lambda(\Box_V^{[s,t]}) \arrow[d, "\bar{\partial}_H+\bar{\partial}_H^*"] \\
\mathfrak{A}^{(0,s\pm1)}(M,\mathcal{W}^{(\lambda;t)}) & \mathrm{Eig}_\lambda(\Box_V^{[s\pm1,t]}). \arrow[l, "\psi^{-1}"]
\end{tikzcd}$$

Remark 3.25:
Observe that because ψ is an isometry and $\bar{\partial}_H$ corresponds to $\bar{\partial}_{\mathcal{W}^{(\lambda;t)}}$ (compare Lemma 3.23), it follows that

$$D_{\mathcal{W}^{(\lambda;t)}} = \bar{\partial}_{\mathcal{W}^{(\lambda;t)}} + \bar{\partial}^*_{\mathcal{W}^{(\lambda;t)}}$$

becomes an elliptic operator and its square

$$\Box_{\mathcal{W}^{(\lambda;t)}} = \bigoplus_s \Box^s_{\mathcal{W}^{(\lambda;t)}} = D^2_{\mathcal{W}^{(\lambda;t)}}.$$

is a generalised Laplace operator, i.e. a second order differential operator whose main symbol is given by the metric (cf. [6]).

Up to this point, we have described what the compatibility of a complex line bundle over a holomorphic fibre bundle implies.
The aim of this thesis however is to give a nice expression for the equivariant torsion of such a line bundle. Therefore, we have to define what kind of actions on $\mathfrak{L} \to E$ we want to admit.
We do this in the subsequent subsection.

3.5 Equivariant setting

This subsection is dedicated to introducing the setting for the equivariance, i.e. introducing the kind of actions that we use later on, the so-called legitimate action.

We start by giving a definition of a legitimate action.

Definition 3.26:
Let $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ be a holomorphic fibre bundle and let $\mathfrak{L} \to E$ be a compatible holomorphic line bundle over E.
A triple $\vec{\gamma} = (\gamma^M, \gamma^E, \gamma^{\mathfrak{L}})$ of diffeomorphisms is called a **legitimate action** if the following four properties hold.

1. The maps $\gamma^M : M \to M$ and $\gamma^E : E \to E$ are biholomorphic isometries.

2. the map $\gamma^{\mathfrak{L}} : \mathfrak{L} \to \mathfrak{L}$ is a covers γ^E which is itself covers the map γ^M, i.e. the following diagram commutes,

$$\begin{tikzcd} \mathfrak{L} \arrow[r, "\pi_{\mathfrak{L}}"] \arrow[d, "\gamma^{\mathfrak{L}}"'] & E \arrow[r, "\pi_E"] \arrow[d, "\gamma^E"] & M \arrow[d, "\gamma^M"] \\ \mathfrak{L} \arrow[r, "\pi_{\mathfrak{L}}"'] & E \arrow[r, "\pi_E"'] & M \end{tikzcd}$$

3. The map $\gamma^{\mathfrak{L}}$ sends fibres of $\mathfrak{L} \to E$ linearly and isometrically to fibres of $\mathfrak{L} \to E$, i.e. for every $e \in E$ and $\mathfrak{L}_e = \pi_{\mathfrak{L}}^{-1}(e) \subset \mathfrak{L}$ the map

$$\gamma^{\mathfrak{L}}\,|_{\mathfrak{L}_e} : \quad \mathfrak{L}_e \quad \longrightarrow \quad \mathfrak{L}_{\gamma^E(e)}$$

is a linear isometry.

4. The induced map $\check{\gamma}^{\mathfrak{L}} : \mathfrak{A}^{(0,*)}(E, \mathfrak{L}) \to \mathfrak{A}^{(0,*)}(E, \mathfrak{L})$ (cf. Definition 2.40 or below in Remark 3.27) commutes with $\bar{\partial}_{\mathfrak{L}}$.

Remark 3.27:

- Recall that the induced action $\check{\gamma}^{\mathfrak{L}}$ (compare Definition 2.40) on antiholomorphic forms $\mathfrak{A}^{(0,q)}(E, \mathfrak{L})$ is given, for $\alpha \in \mathfrak{A}^{(0,q)}(E)$ and $s \in \Gamma(E, \mathfrak{L})$, by

$$\left\{\check{\gamma}^{\mathfrak{L}}(\alpha \otimes s)\right\}_e := \left\{\left(\left(\gamma^E\right)^{-1}\right)^* \alpha\right\}_e \otimes \gamma^{\mathfrak{L}} \left(s(\left(\gamma^E\right)^{-1}(e))\right)$$

and extended linearly to $\mathfrak{A}^{(0,q)}(E, \mathfrak{L})$.

- Note that $\gamma^{\mathfrak{L}}$ and γ^E are isometries and that $\check{\gamma}^{\mathfrak{L}}$ commutes with $\bar{\partial}_{\mathfrak{L}}$. Therefore, $\check{\gamma}^{\mathfrak{L}}$ commutes with the vertical and horizontal Dolbeault-operator individually, i.e.

$$[\check{\gamma}^{\mathfrak{L}}, \bar{\partial}_V] = 0 = [\check{\gamma}^{\mathfrak{L}}, \bar{\partial}_H]. \tag{29}$$

- For a legitimate action $\vec{\gamma}$, the map $\check{\gamma}^{\mathfrak{L}}$ becomes an isometry commuting with $\bar{\partial}_V$ and $\bar{\partial}_H$. Hence, it commutes with $\bar{\partial}^*_{\mathfrak{L}}$ as well as with $\bar{\partial}^*_H$ and $\bar{\partial}^*_V$.
It follows that it leaves Eigenspaces of $\Box_H$ and $\Box_V$ invariant.

In Lemma 3.12 we have already seen that for a compatible line bundle $\mathfrak{L}$ the horizontal and the vertical Laplacian commute, i.e. $[\Box_V, \Box_H] = 0$. This implies that both operators have a common orthonormal Hilbert base of Eigenforms.
Now, because $\Box_{\mathfrak{L}} = \Box_H + \Box_V$ this orthogonal Hilbert base consists of Eigenforms for $\Box_{\mathfrak{L}}$ as well.

Let us use the following nomenclature:

- From now on, let $\Box^{[s,t]}$ denote the restriction of the Laplacian $\Box_{\mathfrak{L}}$ to the subspace $\mathfrak{A}_H^{(0,s)}(E) \wedge \mathfrak{A}_V^{(0,t)}(E, \mathfrak{L})$.
  Note that this notation does not conflict with our previous notations where $\Box_{\mathfrak{L}}^{(0,q)}$ denotes the Laplacian $\Box_{\mathfrak{L}}$ restricted to the space $\mathfrak{A}^{(0,q)}(E, \mathfrak{L})$.
  In particular, we have the following identity:
  $$\Box_{\mathfrak{L}}^{(0,q)} = \bigoplus_{s+t=q} \Box^{[s,t]}.$$
- Analogously, we denote $\Box_V$ (or $\Box_H$) restricted to $\mathfrak{A}_H^{(0,s)}(E) \wedge \mathfrak{A}_V^{(0,t)}(E, \mathfrak{L})$ by $\Box_V^{[s,t]}$ (or $\Box_H^{[s,t]}$).
- Furthermore, let $L(\lambda, \mu, s, t)$ be an abbreviation for the space given by the intersection of the λ-Eigenspace of $\Box_V^{[s,t]}$ with the μ-Eigenspace of $\Box_H^{[s,t]}$, i.e.
  $$L(\lambda, \mu, s, t) := \mathrm{Eig}_\lambda(\Box_V^{[s,t]}) \cap \mathrm{Eig}_\mu(\Box_H^{[s,t]}). \tag{30}$$
  Obviously, $L(\lambda, \mu, s, t)$ is a subspace of the Eigenspace of $\Box^{[s,t]}$ with Eigenvalue $\lambda + \mu$ and the following identity holds:
  $$\mathrm{Eig}_\lambda(\Box^{[s,t]}) = \bigoplus_{\mu+\nu=\lambda} L(\mu, \nu, s, t). \tag{31}$$

We now have fully developed our setting.

In the next subsection, we describe how the the legitimate action $\vec{\gamma}$ translates via the ψ-morphism to an action γ on $\mathcal{W}^{(\lambda;t)}$.

3.6 The psi-morphism and legitimate action

This subsection is dedicated to transferring the necessary properties of the legitimate action $\vec{\gamma}$ on on $L(\lambda, \mu, s, t)$ to the space $\mathfrak{A}^{(0,s)}\left(M, \mathcal{W}^{(\lambda;t)}\right)$ using the ψ-morphism.

The map ψ is a linear isometry.
Therefore, we can construct an equivalent of the $\vec{\gamma} = (\gamma^M, \gamma^E, \gamma^{\mathfrak{L}})$ induced action $\check{\gamma}^{\mathfrak{L}}$ on $L(\lambda, \mu, s, t)$ for the space $\mathrm{Eig}_\mu(\Box_{\mathcal{W}^{(\lambda;t)}}^{(0,s)})$.
We denote this action on $\mathfrak{A}^{(0,s)}(M, \mathcal{W}^{(\lambda;t)})$ by γ.
Slightly more general, i.e. extended to $\mathfrak{A}^{(0,s)}(M, \mathcal{W}^{(\lambda;t)})$, γ is expressed by the following commutative diagram, i.e.:

$$\begin{tikzcd}
\mathfrak{A}^{(0,s)}(M, \mathcal{W}^{(\lambda;t)}) \arrow[r, "\psi"] \arrow[d, "\gamma"'] & \mathrm{Eig}_\lambda(\Box_V^{[s,t]}) \arrow[d, "\check{\gamma}^{\mathfrak{L}}"'] \\
\mathfrak{A}^{(0,s)}(M, \mathcal{W}^{(\lambda;t)}) & \mathrm{Eig}_\lambda(\Box_V^{[s,t]}). \arrow[l, "\psi^{-1}"]
\end{tikzcd}$$

What is not obvious, right now, is that the action γ actually splits into an element $\gamma^{\mathcal{W}^{(\lambda;t)}}$ in $\mathrm{Diff}(M, \mathcal{W}^{(\lambda;t)})$ and the pullback via $(\gamma^M)^{-1}$, i.e. in our usual nomenclature we write $\gamma = \check{\gamma}^{\mathcal{W}^{(\lambda;t)}}$ (compare Definition 2.40).

To show this property, which is stated explicitly in the following proposition, is the task of this subsection.

Proposition 3.28:
Let $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ be a holomorphic fibre bundle, $\mathfrak{L} \to E$ be a compatible holomorphic line bundle over E and let $\vec{\gamma} = (\gamma^M, \gamma^E, \gamma^{\mathfrak{L}})$ be a legitimate action.
Let further on γ denote the induced action on $\mathfrak{A}^{(0,*)}(M, \mathcal{W}^{(\lambda;t)})$ given by

$$\gamma = \psi^{-1} \circ \check{\gamma}^{\mathfrak{L}} \circ \psi.$$

There is a bundle morphism

$$\gamma^{\mathcal{W}^{(\lambda;t)}} : \ \mathcal{W}^{(\lambda;t)} \longrightarrow \mathcal{W}^{(\lambda;t)}$$

covering the action γ^M and acting linearly and isometrically on fibres such that γ and $\check{\gamma}^{\mathcal{W}^{(\lambda;t)}}$ coincide, i.e. for any differential form $\alpha \in \mathfrak{A}^{(0,s)}(M, \mathcal{W}^{(\lambda;t)})$ the following identity holds:

$$\gamma \cdot \alpha = \gamma^{\mathcal{W}^{(\lambda;t)}} \circ \left(\left(\left(\gamma^M\right)^{-1} \right)^* \alpha \right) = \check{\gamma}^{\mathcal{W}^{(\lambda;t)}} \alpha.$$

Remark 3.29:
Before we proof this proposition, we would like to give an argument why this seems plausible. This argument is summarised in the following diagram:

$$\begin{tikzcd}
\mathcal{W}_x^{(\lambda;t)} \arrow[r, "\psi"] \arrow[d, "\gamma^{\mathcal{W}^{(\lambda;t)}}"'] & \mathfrak{A}^{(0,t)}(E_x, \mathfrak{L}\,|_{E_x}) \arrow[r, "\text{cont.}"] & \mathfrak{A}_V^{(0,t)}(E, \mathfrak{L}) \arrow[d, "\check{\gamma}^{\mathfrak{L}}"] \\
\mathcal{W}_{\gamma^M(x)}^{(\lambda;t)} & \mathfrak{A}^{(0,t)}\left(E_{\gamma^M(x)}, \mathfrak{L}\,|_{E_{\gamma^M(x)}}\right) \arrow[l, "\psi^{-1}"'] & \mathfrak{A}_V^{(0,t)}(E, \mathfrak{L}) \arrow[l, "i^*_{E_{\gamma^M(x)}}"']
\end{tikzcd}$$

The map we seek should be visualised in this way. First, use the morphism ψ on an element of $\mathcal{W}_x^{(\lambda;t)}$. This of course may not be done directly since ψ acts on sections into $\mathcal{W}^{(\lambda;t)}$ and not on elements. However, suppose ψ can be restricted in this way, then we continue the corresponding $(0,t)$-form on E_x to the whole space E, apply $\check{\gamma}^{\mathfrak{L}}$ and restrict this $(0,t)$-form to the image of E_x under γ^E, i.e. to $E_{\gamma^M(x)}$. Then we use ψ^{-1} and get an element of $\mathcal{W}_{\gamma^M(x)}^{(\lambda;t)}$.

The actual proof of this identity will be motivated by this diagram, but since this diagram has some technical difficulties, we want to circumvent, like the restriction of ψ

to elements of $\mathcal{W}^{(\lambda;t)}$, the independence of the resulting map from the continuation of the $(0,t)$-form to E and the unclarity of the smoothness of the resulting map $\gamma^{\mathcal{W}^{(\lambda;t)}}$, we approach our problem in a slightly different way.

The proof of Proposition 3.28 is split into several lemmas.
First, in Lemma 3.30, we reduce the question to sections from M into $\mathcal{W}^{(0;*)}$.
Afterwards, we make an Ansatz for the bundle morphism $\gamma^{\mathcal{W}^{(0;*)}}$ in Equation (36) and show in Lemma 3.38 that this Ansatz is valid.
Directly thereafter, in Lemma 3.39, we proof that the map $\gamma^{\mathcal{W}^{(0;*)}}$ is smooth map.
At last, we proof that the action γ is indeed given by $\check{\gamma}^{\mathcal{W}^{(0;*)}}$.

Now, for the proof of Proposition 3.28, we first show that we can restrict to the case of sections into $\mathcal{W}^{(\lambda;t)}$.

Lemma 3.30:
Let α be in $\mathfrak{A}^{(0,*)}(M)$ and s be an element of $\Gamma(M,\mathcal{W}^{(\lambda;t)})$. Let furthermore $\gamma=\psi^{-1}\circ\check{\gamma}^{\mathfrak{L}}\circ\psi$ denote the action on $\mathfrak{A}^{(0,*)}(M,\mathcal{W}^{(\lambda;t)})$ induced by a legitimate action $\vec{\gamma}=(\gamma^M,\gamma^E,\gamma^{\mathfrak{L}})$.
Then the γ-action on $\alpha\otimes s$ splits as follows:

$$\gamma(\alpha\otimes s)=\left(\left[\left(\gamma^M\right)^{-1}\right]^*\alpha\right)\otimes\gamma(s).$$

Proof.
This splitting follows directly from $\psi(\alpha\otimes s)=(\pi_E^*\alpha)\wedge\psi(s)$ (compare Equation (27)) through the subsequent computation.

$$\gamma(\alpha\otimes s)=\psi^{-1}\circ\check{\gamma}^{\mathfrak{L}}\circ\psi(\alpha\otimes s)=\psi^{-1}\circ\check{\gamma}^{\mathfrak{L}}\Big[(\pi_E^*\alpha)\wedge\psi(s)\Big]$$

Recall that $\check{\gamma}^{\mathfrak{L}}$ acts on ordinary differential forms, i.e. forms without coefficients in $\mathfrak{L}$, as a pullback via $(\gamma^E)^{-1}$. Hence, we obtain:

$$\gamma(\alpha\otimes s)=\psi^{-1}\left(\left\{\left[\left(\gamma^E\right)^{-1}\right]^*(\pi_E^*\alpha)\right\}\wedge\check{\gamma}^{\mathfrak{L}}(\psi(s))\right).$$

Now, $\vec{\gamma}$ is legitimate. In particular, γ^E covers γ^M. It follows that

$$\begin{aligned}\gamma(\alpha\otimes s)=&\psi^{-1}\left(\left\{\pi_E^*\left(\left[\left(\gamma^M\right)^{-1}\right]^*\alpha\right)\right\}\wedge\check{\gamma}^{\mathfrak{L}}(\psi(s))\right)\\=&\left(\left[\left(\gamma^M\right)^{-1}\right]^*\alpha\right)\otimes\underbrace{\psi^{-1}\left(\check{\gamma}^{\mathfrak{L}}(\psi(s))\right)}_{=\gamma(s)}\end{aligned}$$

which proves the assertion. □

The construction of the ψ-morphism makes explicit use of the fact that we can identify sections in $\Gamma(M, \mathcal{W}^{(\lambda;*)})$ with maps in $C^\infty(P, \mathrm{Eig}_\lambda(\Box_{\tilde{\mathfrak{L}}}))^{\check{\rho}}$ and that λ-Eigenforms of $\Box_V^{[0,*]}$, can be identified with the latter space as well.
Thus, it suggests itself to study the induced action of $\check{\gamma}^{\mathfrak{L}}$ on $C^\infty(P, \mathfrak{A}^{(0,t)}(F, \tilde{\mathfrak{L}}))^{\check{\rho}}$.

Remark 3.31:
For a legitimate action $\vec{\gamma} = (\gamma^M, \gamma^E, \gamma^{\mathfrak{L}})$, there is a bijective map $\gamma^P : P \to P$, not necessarily smooth or continuous, covering the action of γ^M such that

$$\gamma^P(p \cdot k) = \gamma^P(p) \cdot k. \tag{32}$$

This map γ^P is neither unique nor naturally excelled.

Lemma 3.32:
Let $\vec{\gamma} = (\gamma^M, \gamma^E, \gamma^{\mathfrak{L}})$ be a legitimate action and let $\gamma^P : P \to P$ be a map covering γ^M which is K-equivariant, i.e. it fulfils Equation (32).
Then there are maps $\gamma^{\tilde{\mathfrak{L}}} : P \to \mathrm{Diff}(F, \tilde{\mathfrak{L}})$ and $\gamma^F : P \to \mathrm{Aut}(F)$ depending on γ^P such that the following equations hold.

$$\begin{array}{llccc} 1) & \gamma^{\mathfrak{L}}: & P \times_{\tilde{\rho}} \tilde{\mathfrak{L}} & \longrightarrow & P \times_{\tilde{\rho}} \tilde{\mathfrak{L}} \\ & & \left[p, \tilde{l}\right]_{\tilde{\rho}} & \longmapsto & \left[\gamma^P(p), \gamma^{\tilde{\mathfrak{L}}}(p)(\tilde{l})\right]_{\tilde{\rho}} \\ 2) & \gamma^E: & P \times_\rho F & \longrightarrow & P \times_\rho F \\ & & [p, f]_\rho & \longmapsto & \left[\gamma^P(p), \left(\gamma^F(p)\right)(f)\right]_\rho \end{array}$$

They are correlated via the following equation

$$\gamma^F(p) \circ \pi_{\tilde{\mathfrak{L}}} = \pi_{\tilde{\mathfrak{L}}} \circ \gamma^{\tilde{\mathfrak{L}}}(p).$$

Remark 3.33:
Recall that E (and likewise $\mathfrak{L}$) are quotients of $P \times F$ (or likewise $P \times \tilde{\mathfrak{L}}$).
Therefore, we have a natural quotient map $P \times F \to E$ (or $P \times \tilde{\mathfrak{L}} \to \mathfrak{L}$).

The following diagram commutes:

$$\begin{tikzcd}
P \times \tilde{\mathfrak{L}} \arrow[rrr, "\gamma^P \times \gamma^{\tilde{\mathfrak{L}}}"] \arrow[dr] \arrow[ddd, "\mathrm{id}_P \times \pi_{\tilde{\mathfrak{L}}}"'] & & & P \times \tilde{\mathfrak{L}} \arrow[dl] \arrow[ddd, "\mathrm{id}_P \times \pi_{\tilde{\mathfrak{L}}}"] \\
& \mathfrak{L} \arrow[r, "\gamma^{\mathfrak{L}}"] \arrow[d, "\pi_{\mathfrak{L}}"'] & \mathfrak{L} \arrow[d, "\pi_{\mathfrak{L}}"] & \\
& E \arrow[r, "\gamma^E"'] & E & \\
P \times F \arrow[ur] \arrow[rrr, "\gamma^P \times \gamma^F"'] & & & P \times F \arrow[ul]
\end{tikzcd}$$

where the unlabelled arrows represent the natural quotient maps.

Proof of Lemma 3.32.
Let $\tilde{l}$ be an element in $\tilde{\mathfrak{L}}$, $p \in P$ and let $x = \pi_P(p)$ be the projection of p to M.
The action $\vec{\gamma} = (\gamma^M, \gamma^E, \gamma^{\mathfrak{L}})$ is legitimate. In particular, $\gamma^{\mathfrak{L}}$ covers a γ^M,

$$\begin{tikzcd} \mathfrak{L} \arrow[r, "\gamma^{\mathfrak{L}}"] \arrow[d] & \mathfrak{L} \arrow[d] \\ M \arrow[r, "\gamma^M"'] & M \end{tikzcd} \qquad \begin{tikzcd} \left[p, \tilde{l}\right]_{\tilde{\rho}} \arrow[r, mapsto, "\gamma^{\mathfrak{L}}"] \arrow[d, mapsto] & \gamma^{\mathfrak{L}}\left(\left[p, \tilde{l}\right]_{\tilde{\rho}}\right) \arrow[d, mapsto] \\ x \arrow[r, mapsto, "\gamma^M"'] & \gamma^M(x). \end{tikzcd}$$

Hence, for an element $\left[p, \tilde{l}\right]_{\tilde{\rho}} \in \mathfrak{L}_x := \mathfrak{L}\mid_{\pi_E^{-1}(x)}$, its image $\gamma^{\mathfrak{L}}\left(\left[p, \tilde{l}\right]_{\tilde{\rho}}\right)$ under $\gamma^{\mathfrak{L}}$ has to lie in the fibre $\mathfrak{L}_{\gamma^M(x)}$.
On the other hand, the map

$$\begin{array}{rccc} [q]_{\tilde{\rho}}: & \tilde{\mathfrak{L}} & \longrightarrow & \mathfrak{L}_{\gamma^M(x)} \\ & \tilde{l} & \longmapsto & \left[q, \tilde{l}\right]_{\tilde{\rho}} \end{array}$$

defines a smooth bijection for any $q \in \pi_P^{-1}(\gamma^M(x))$.
It follows that the map $\gamma^{\tilde{\mathfrak{L}}}$ can be constructed as follows:

$$\gamma^{\tilde{\mathfrak{L}}}(p) := \left[\gamma^P(p)\right]_{\tilde{\rho}}^{-1} \circ \gamma^{\mathfrak{L}} \circ [p]_{\tilde{\rho}}\,.$$

Now, an analogous construction gives us γ^F and their relation follows directly from the covering $\gamma^{\mathfrak{L}}$ over γ^E, i.e. we obtain

$$\gamma^F(p) := \left[\gamma^P(p)\right]_{\rho}^{-1} \circ \gamma^E \circ [p]_{\rho}\,.$$

□

Remark 3.34:

- The K-equivariance of γ^P, compare Equation (32), implies a similar equivariance for $\gamma^{\tilde{\mathfrak{L}}}$ as well as for γ^F.

$$\gamma^{\tilde{\mathfrak{L}}}(p \cdot k) = \tilde{\rho}(k^{-1}) \circ \gamma^{\tilde{\rho}}(p) \circ \tilde{\rho}(k) \tag{33}$$
$$\gamma^F(p \cdot k) = \rho(k^{-1}) \circ \gamma^F(p) \circ \rho(k) \tag{34}$$

- The inverse map of γ^E (or respectively of $\gamma^{\mathfrak{L}}$) can be expressed, using γ^P and γ^F (or respectively $\gamma^{\tilde{\mathfrak{L}}}$) as well.
  For any $f \in F$ and $p \in P$, we obtain:

$$\left(\gamma^E\right)^{-1}\left([p, f]_{\rho}\right) = \left[(\gamma^P)^{-1}(p), \left(\gamma^F\left((\gamma^P)^{-1}(p)\right)\right)^{-1}(f)\right]_{\rho}. \tag{35}$$

  The equation for $\left(\gamma^{\mathfrak{L}}\right)^{-1}$ is given analogously.

For the subsequent Lemma, recall that the action $\check{\gamma}^{\mathfrak{L}}$ is given by $\gamma^{\mathfrak{L}} \circ \left[\left(\gamma^E\right)^{-1}\right]^*$.

Lemma 3.35:
Let $\vec{\gamma} = (\gamma^M, \gamma^E, \gamma^{\mathfrak{L}})$ be a legitimate action and let further on γ be the action on $\mathfrak{A}^{(0,*)}(M, \mathcal{W}^{(0;*)})$ induced by $\check{\gamma}^{\mathfrak{L}}$.
For $\alpha \in \mathfrak{A}_V^{(0,t)}(E, \mathfrak{L})$ let $\hat{\tilde{\alpha}}$ be its corresponding element in $C^\infty(P, \mathfrak{A}^{(0,t)}(F, \tilde{\mathfrak{L}}))^{\check{\rho}}$.
Then the γ-induced action on $\hat{\tilde{\alpha}}$ is given implicitly via

$$\check{\gamma}^{\mathfrak{L}}\left(\left[p, \hat{\tilde{\alpha}}(p)\right]_{\check{\rho}}\right) = \left[p, \gamma^{\tilde{\mathfrak{L}}}(q) \circ \left\{\left(\gamma^F(q)\right)^{-1}\right\}^* \left(\hat{\tilde{\alpha}}(q)\right)\right]_{\check{\rho}} =: \left[p, \widetilde{\gamma^{\tilde{\mathfrak{L}}}(q)}\left(\hat{\tilde{\alpha}}(q)\right)\right]_{\check{\rho}}$$

where we used $\left(\gamma^P\right)^{-1}(p) = q$ for reasons of simplicity.

Proof.
We will proof this lemma for sections α in $\Gamma(E, \mathfrak{L})$ only. The generalisation to vertical differential forms is more tedious but not more complicated.
Let again $q = (\gamma^P)^{-1}(p)$, we obtain:

$$\begin{aligned}\left(\check{\gamma}^{\mathfrak{L}}\alpha\right)_{[p,f]_\rho} &= \gamma^{\mathfrak{L}}\left(\alpha\left(((\gamma^E)^{-1})([p,f]_\rho)\right)\right) \overset{(35)}{=} \gamma^{\mathfrak{L}}\left(\alpha\left(\left[q, \left(\gamma^F(q)\right)^{-1}(f)\right]_\rho\right)\right) \\ &= \gamma^{\mathfrak{L}}\left(\left[q, \hat{\tilde{\alpha}}(q)\left(\left(\gamma^F(q)\right)^{-1}(f)\right)\right]_{\tilde{\rho}}\right)\end{aligned}$$

Now the claim of the lemma follows directly from the definition of $\gamma^{\tilde{\mathfrak{L}}}$. □

Remark 3.36:
In the previous lemma, we used the commutative diagram

$$\begin{tikzcd} \tilde{\mathfrak{L}} \arrow[r, "\gamma^{\tilde{\mathfrak{L}}}(p)"] \arrow[d, "\pi_{\tilde{\mathfrak{L}}}"'] & \tilde{\mathfrak{L}} \arrow[d, "\pi_{\tilde{\mathfrak{L}}}"] \\ F \arrow[r, "\gamma^F(p)"'] & F \end{tikzcd}$$

to obtain an action

$$\widetilde{\gamma^{\tilde{\mathfrak{L}}}(q)} : \quad \mathfrak{A}^{(0,t)}(F, \tilde{\mathfrak{L}}) \quad \longrightarrow \quad \mathfrak{A}^{(0,t)}(F, \tilde{\mathfrak{L}})$$

for every $q \in P$ corresponding to $\check{\gamma}^{\mathfrak{L}}$ on $\mathfrak{A}_V^{(0,*)}(E, \mathfrak{L})$, in the sense of

$$\check{\gamma}^{\mathfrak{L}}\left[p, \hat{\tilde{\alpha}}(p)\right]_{\check{\rho}} = \left[p, \widetilde{\gamma^{\tilde{\mathfrak{L}}}(q)}\left(\hat{\tilde{\alpha}}(q)\right)\right]_{\check{\rho}}.$$

Here again, p equals $\gamma^P(q)$.

We now posses all necessary tools to make an Ansatz for $\gamma^{\mathcal{W}^{(\lambda;t)}}$.

Definition 3.37:
Let $w = [p, \beta]_{\check{\rho}}$ be an element of $\mathcal{W}^{(\lambda;*)} = P \times_{\check{\rho}} \mathrm{Eig}_\lambda(\Box_{\tilde{\mathfrak{L}}})$.
We define:

$$\gamma^{\mathcal{W}^{(\lambda;*)}}\left([p,\beta]_{\check{\rho}}\right) := \left[\gamma^P(p), \widetilde{\gamma^{\tilde{\mathfrak{L}}}(p)}(\beta)\right]_{\check{\rho}}. \tag{36}$$

So far our Ansatz seems to have a strong dependency on the choice of γ^P which is quite undesirable. Therefore, we need to show that it does not depend on the choice of γ^P at all.
This is the purpose of the following lemma. Furthermore, we show that the Ansatz above does not depend on the representing element of $[p, \beta]_{\check{\rho}}$.

Lemma 3.38:
The Ansatz of Equation (36) is well defined and does not depend on the choice of γ^P.

Proof.
Recall that the maps

$$\tilde{\rho}: \quad K \quad \longrightarrow \quad \mathrm{Diff}(F, \tilde{\mathfrak{L}}) \qquad \text{and} \qquad \rho: \quad K \quad \longrightarrow \quad \mathrm{Diff}(F, F)$$

induce the map

$$\check{\rho}: \quad K \quad \longrightarrow \quad \mathrm{Aut}(\mathfrak{A}^{(0,*)}(F, \tilde{\mathfrak{L}})),$$

explicitly expressed by $\check{\rho}(k) = \tilde{\rho}(k) \circ \rho\left(k^{-1}\right)^*$ (compare Remark 3.3).
That Equation (36) is well defined, follows directly from the equivariance of $\widetilde{\gamma^{\tilde{\mathfrak{L}}}}$ which can be derived by the equivariances of $\gamma^{\tilde{\mathfrak{L}}}$ and γ^F given in Equations (33) and (34).
Therefore, we derive the equivariance of $\widetilde{\gamma^{\tilde{\mathfrak{L}}}}$ and obtain:

$$\begin{aligned}
\widetilde{\gamma^{\tilde{\mathfrak{L}}}(p \cdot k)} &= \gamma^{\tilde{\mathfrak{L}}}(p \cdot k) \circ \left(\gamma^F(p \cdot k)^{-1}\right)^* \\
&\overset{(34),(33)}{=} \tilde{\rho}(k^{-1}) \circ \gamma^{\tilde{\mathfrak{L}}}(p) \circ \tilde{\rho}(k) \circ \left(\rho(k^{-1}) \circ \gamma^F(p)^{-1} \circ \rho(k)\right)^* \\
&= \underbrace{\left[\tilde{\rho}(k^{-1}) \circ \rho(k)^*\right]}_{=\check{\rho}(k^{-1})} \circ \left[\gamma^{\tilde{\mathfrak{L}}}(p) \circ \left(\gamma^F(p)^{-1}\right)^*\right] \circ \underbrace{\left[\tilde{\rho}(k) \circ \rho(k^{-1})^*\right]}_{=\check{\rho}(k)} \\
&= \check{\rho}(k^{-1}) \circ \widetilde{\gamma^{\tilde{\mathfrak{L}}}(p)} \circ \check{\rho}(k).
\end{aligned}$$

Consequently, for $\beta \in \mathrm{Eig}_\lambda(\Box_{\tilde{\mathfrak{L}}})$ the following identity holds:

$$\begin{aligned}
\gamma^{\mathcal{W}^{(\lambda;*)}}\left(\left[p\cdot k, \check{\rho}(k^{-1})(\beta)\right]_{\check{\rho}}\right) \overset{(36)}{=} & \left[\gamma^P(p)\cdot k, \widetilde{\gamma^{\tilde{\mathfrak{L}}}(p\cdot k)} \circ \check{\rho}(k^{-1})(\beta)\right]_{\check{\rho}} \\
= & \left[\gamma^P(p), \check{\rho}(k)\circ \widetilde{\gamma^{\tilde{\mathfrak{L}}}(p\cdot k)} \circ \check{\rho}(k^{-1})(\beta)\right]_{\check{\rho}} \\
= & \left[\gamma^P(p), \widetilde{\gamma^{\tilde{\mathfrak{L}}}(p)}(\beta)\right]_{\check{\rho}} \\
= & \gamma^{\mathcal{W}^{(\lambda;t)}}\left([p,\beta]_{\check{\rho}}\right).
\end{aligned}$$

We conclude that the Ansatz for $\gamma^{\mathcal{W}^{(\lambda;*)}}$ does not depend on the choice of the representing element for the equivalence class $[p,\beta]_{\check{\rho}} = \left[p\cdot k, \check{\rho}(k^{-1})(\beta)\right]_{\check{\rho}}$.

What remains to be shown is that our Ansatz does not depend on γ^P.
Let therefore η^P be another bijective and K-equivariant map from P to P covering the map γ^M and let furthermore $\eta^{\tilde{\mathfrak{L}}}$ as well as η^F be the other corresponding maps (compare Lemma 3.32).
Now, γ^P covers γ^M, as does η^P. Therefore, we obtain for every $p \in P$:

$$\pi_P(\eta^P(p)) = \pi_P(\gamma^P(p)) = \gamma^M(\pi_P(p)).$$

It follows that there exists a map $g^P : P \to K$ such that:

$$\eta^P(p) = \gamma^P(p)\cdot g^P(p).$$

Observe that the K-equivariance of γ^P and η^P implies that g^P is actually a pullback of a map a map $g : M \to K$, i.e.

$$\eta^P(p) = \gamma^P(p)\cdot g(\pi_P(p)). \tag{37}$$

We use Equation (37) to express the relations between η^F and γ^F as well as between $\eta^{\tilde{\mathfrak{L}}}$ and $\gamma^{\tilde{\mathfrak{L}}}$.
We obtain:

$$\begin{aligned}
\eta^{\tilde{\mathfrak{L}}}(p) = & \tilde{\rho}\left(g(\pi_P(p))^{-1}\right)\circ \gamma^{\tilde{\mathfrak{L}}}(p) \\
\eta^F(p) = & \rho\left(g(\pi_P(p))^{-1}\right)\circ \gamma^F(p).
\end{aligned}$$

This implies, for $x = \pi_P(p)$:

$$\widetilde{\eta^{\tilde{\mathfrak{L}}}(p)} = \eta^{\tilde{\mathfrak{L}}}(p)\circ \left[\left((\eta^F)(p)\right)^{-1}\right]^* = \check{\rho}(g(x)^{-1})\circ \widetilde{\gamma^{\tilde{\mathfrak{L}}}(p)}$$

Let again q be $(\eta^P)^{-1}(p) = (\gamma^P)^{-1}(p)\cdot g(\pi_P(q))$.

We obtain for a β in $\mathrm{Eig}_\lambda(\Box_{\tilde{\mathfrak{L}}})$:

$$\begin{aligned}\left[\eta^P(p), \widetilde{\eta^{\tilde{\mathfrak{L}}}(p)}(\beta)\right]_{\check{\rho}} &= \left[\gamma^P(p)\cdot g(x), \check{\rho}(g(x)^{-1})\circ \widetilde{\gamma^{\tilde{\mathfrak{L}}}(p)}(\beta)\right]_{\check{\rho}} \\ &= \left[\gamma^P(p), \widetilde{\gamma^{\tilde{\mathfrak{L}}}(p)}(\beta)\right]_{\check{\rho}}.\end{aligned}$$

Consequently, $\gamma^{\mathcal{W}^{(\lambda;*)}}$ does not depend on the choice of the map γ^P which finishes the proof. $\Box$

Up to this point we showed that the map $\gamma^{\mathcal{W}^{(\lambda;t)}}$, given by Equation (36), is a well defined map and that it is independent of the choice of γ^P.
Before we can finally proof Proposition 3.28, we have to show that $\gamma^{\mathcal{W}^{(\lambda;*)}}$ is smooth.

Lemma 3.39:
The map $\gamma^{\mathcal{W}^{(\lambda;t)}} : \mathcal{W}^{(\lambda;t)} \to \mathcal{W}^{(\lambda;t)}$, defined in Equation (36), is smooth.

Proof.
Again, we proof this Lemma for sections α in $\Gamma(E,\mathfrak{L})\cap\mathrm{Eig}_\lambda(\Box_V)$ only. The generalisation to vertical differential forms is again more tedious but not more complicated.
That $\gamma^{\mathcal{W}^{(\lambda;0)}}$ is smooth can be checked locally by taking local trivialisations of $\mathcal{W}^{(\lambda;0)}$ which are induced by smooth local sections $q_i : U_i \longrightarrow P$ with $i \in I$.
Let,

$$\begin{array}{rccc} [q_i]_{\check{\rho}} : & U_i \times \mathrm{Eig}_\lambda(\Box^{(0,0)}_{\tilde{\mathfrak{L}}}) & \longrightarrow & \mathcal{W}^{(\lambda;0)}_x \\ & (x,\beta) & \longmapsto & [q_i(x),\beta]_{\check{\rho}} \end{array}$$

denote the induced local trivialisation maps of $\mathcal{W}^{(\lambda;t)}$.
Now, fix an $i \in I$ and assume without loss of generality that $\gamma^M(U_i) = U_j$.
The idea is to choose $\gamma^P : P \to P$ to be the map sending $q_i(x)$ to $q_j(\gamma^M(x))$ continued K-equivariantly to $\pi_P^{-1}(U_i)$ and arbitrarily outside of $\pi_P^{-1}(U_i)$ (within the constrictions we demanded above, i.e. K-equivariant and covering γ^M).
Therefore, let now γ^P be given on $\pi_P^{-1}(U_i)$ by:

$$\begin{array}{rccc} \gamma^P\mid_{\pi_P^{-1}(U_i)} : & P\mid_{U_i} & \longrightarrow & P\mid_{U_j} \\ & q_i(x)\cdot k & \longmapsto & q_j(\gamma^M(x))\cdot k \end{array} \tag{38}$$

This is valid, because $\gamma^{\mathcal{W}^{(0;*)}}$ does not depend on the choice of γ^P which we showed in Lemma 3.38.

We now show that the map $\gamma^{\mathcal{W}^{(\lambda;0)}}$ read through those local trivialisations is smooth, i.e. the top map of the following diagram:

$$\begin{tikzcd}
U_i \times \mathrm{Eig}_\lambda(\Box^{(0,0)}_{\tilde{\mathfrak{L}}}) \arrow[r, "\delta_{ij}^{\mathcal{W}^{(\lambda;0)}}"] \arrow[d, "{[q_i]_{\check{\rho}}}"'] & U_j \times \mathrm{Eig}_\lambda(\Box^{(0,0)}_{\tilde{\mathfrak{L}}}) \arrow[d, "{[q_j]_{\check{\rho}}}"] \\
\mathcal{W}^{(\lambda;0)}\,|_{U_i} \arrow[r, "\gamma^{\mathcal{W}^{(\lambda;0)}}"'] \arrow[d] & \mathcal{W}^{(\lambda;0)}\,|_{U_j} \arrow[d] \\
U_i \arrow[r, "\gamma^M"'] & U_j
\end{tikzcd}$$

is smooth.
We compute for $x \in U_i$ and $\beta \in \mathrm{Eig}_\lambda(\Box^{(0,0)}_{\tilde{\mathfrak{L}}})$:

$$\begin{aligned}
\delta_{ij}^{\mathcal{W}^{(\lambda;0)}}(x,\beta) &= [q_j]^{-1}_{\check{\rho}} \circ \gamma^{\mathcal{W}^{(\lambda;0)}} \circ [q_i]_{\check{\rho}}\left(x,\beta\right) = [q_j]^{-1}_{\check{\rho}} \circ \gamma^{\mathcal{W}^{(\lambda;0)}}\Big([q_i(x),\beta]_{\check{\rho}}\Big) \\
&= [q_j]^{-1}_{\check{\rho}}\left(\left[\gamma^P(q_i(x)), \gamma^{\tilde{\mathfrak{L}}}\widetilde{(q_i(x))}(\beta)\right]_{\check{\rho}}\right) \\
&\overset{(38)}{=} [q_j]^{-1}_{\check{\rho}}\left(\left[q_j(\gamma^M(x)), \gamma^{\tilde{\mathfrak{L}}}\widetilde{(q_i(x))}(\beta)\right]_{\check{\rho}}\right) \\
&= \left(\gamma^M(x), \gamma^{\tilde{\mathfrak{L}}}\widetilde{(q_i(x))}(\beta)\right).
\end{aligned}$$

Consequently, it remains to be shown that the map

$$\begin{array}{rccc}
\gamma^{\tilde{\mathfrak{L}}}\widetilde{(q_i(\cdot))}: & U_i \times \mathrm{Eig}_\lambda(\Box^{(0,0)}_{\tilde{\mathfrak{L}}}) & \longrightarrow & \mathrm{Eig}_\lambda(\Box^{(0,0)}_{\tilde{\mathfrak{L}}}) \\
& (x,\beta) & \longmapsto & \gamma^{\tilde{\mathfrak{L}}}\widetilde{(q_i(x))}\beta
\end{array} \tag{39}$$

is smooth and since $\mathrm{Eig}_\lambda(\Box^{(0,0)}_{\tilde{\mathfrak{L}}})$ is finite dimensional and Equation (39) is linear in the β, it is smooth in β.
We now determine the smoothness properties of

$$\begin{array}{lcll}
x & \longmapsto & \gamma^{\tilde{\mathfrak{L}}}(q_i(x)) & \text{as well as} \\
x & \longmapsto & \gamma^F(q_i(x)), &
\end{array}$$

in order to show that $x \mapsto \gamma^{\tilde{\mathfrak{L}}}\widetilde{(q_i(x))}\beta$ is smooth for any β.

By prerequisite, we know that $(\gamma^E)^{-1}$ as well as $\gamma^{\mathfrak{L}}$ are smooth. Thus, so are the maps $\delta_{ij}^{\tilde{\mathfrak{L}}} : U_i \times \tilde{\mathfrak{L}} \to \tilde{\mathfrak{L}}$ and $\delta_{ij}^F : U_i \times F \to F$ defined as follows.

$$\begin{tikzcd}
U_j \times F \arrow[r, "(\gamma^M)^{-1} \times \delta_{ij}^F"] \arrow[d] & U_i \times F \arrow[d] \\
E\,|_{U_j} \arrow[r, "(\gamma^E)^{-1}"'] & E\,|_{U_i}
\end{tikzcd}
\qquad
\begin{tikzcd}
U_i \times \tilde{\mathfrak{L}} \arrow[r, "\gamma^M \times \delta_{ij}^{\tilde{\mathfrak{L}}}"] \arrow[d] & U_j \times \tilde{\mathfrak{L}} \arrow[d] \\
\mathfrak{L}\,|_{U_i} \arrow[r, "\gamma^{\mathfrak{L}}"'] & \mathfrak{L}\,|_{U_j}
\end{tikzcd}$$

We evaluate these maps explicitly, using the equations of Lemma 3.32 defining γ^F and $\gamma^{\tilde{\mathfrak{L}}}$:

$$\begin{aligned}
\gamma^M \times \delta_{ij}^{\tilde{\mathfrak{L}}}(x, \tilde{l}) &= [q_j]_{\tilde{\rho}}^{-1} \circ \gamma^{\mathfrak{L}} \circ [q_i]_{\tilde{\rho}} \left(x, \tilde{l}\right) = [q_j]_{\tilde{\rho}}^{-1} \circ \gamma^{\mathfrak{L}} \circ \left[q_i(x), \tilde{l}\right]_{\tilde{\rho}} \\
&= [q_j]_{\tilde{\rho}}^{-1} \left(\left[\gamma^P(q_i(x)), \gamma^{\tilde{\mathfrak{L}}}(q_i(x)) \left(\tilde{l}\right)\right]_{\tilde{\rho}} \right) \\
&\overset{(38)}{=} [q_j]_{\tilde{\rho}}^{-1} \left(\left[q_j(\gamma^M(x)), \gamma^{\tilde{\mathfrak{L}}}(q_i(x)) \left(\tilde{l}\right)\right]_{\tilde{\rho}} \right) \\
&= \left(\gamma^M(x), \gamma^{\tilde{\mathfrak{L}}}(q_i(x)) \left(\tilde{l}\right)\right).
\end{aligned}$$

Hence, we obtain that the map

$$\begin{array}{rccc}
\delta_{ij}^{\tilde{\mathfrak{L}}} : & U_i \times \tilde{\mathfrak{L}} & \longrightarrow & \tilde{\mathfrak{L}} \\
 & (x, \tilde{l}) & \longmapsto & \gamma^{\tilde{\mathfrak{L}}}(q_i(x))(\tilde{l})
\end{array}$$

is smooth. And analogously, the map

$$\begin{array}{rccc}
\delta_{ij}^F : & U_i \times F & \longrightarrow & F \\
 & (\gamma^M(x), f) & \longmapsto & \left[\gamma^F(q_i(x))\right]^{-1}(f),
\end{array}$$

is smooth, too.

We conclude that for any $\beta \in \Gamma(F, \tilde{\mathfrak{L}})$, the map $\delta_{ij}^{\beta} := \delta_{ij}^{\tilde{\mathfrak{L}}} \circ \left(\mathrm{id}_{U_i} \times (\beta \circ \delta_{ij}^F)\right)$, explicitly given by

$$\begin{array}{rccc}
\delta_{ij}^{\beta} : & U_i \times F & \longrightarrow & \tilde{\mathfrak{L}} \\
 & (x, f) & \longmapsto & \underbrace{\delta_{ij}^{\tilde{\mathfrak{L}}}\left(x, \beta \circ (\delta_{ij}^F)\left(\gamma^M(x), f\right)\right)}_{\left[\widetilde{\gamma^{\tilde{\mathfrak{L}}}(q_i(x))}(\beta)\right](f)},
\end{array}$$

is a composition of smooth maps. Hence, it is smooth itself.

Additionally, we observe that

$$\begin{aligned}
\left(\pi_{\tilde{\mathfrak{L}}} \circ \delta_{ij}^{\beta}\right)(x, f) =& \pi_{\tilde{\mathfrak{L}}} \left(\left[\widetilde{\gamma^{\tilde{\mathfrak{L}}}(q_i(x))}(\beta)\right](f) \right) \\
=& \pi_{\tilde{\mathfrak{L}}} \left(\left[\gamma^{\tilde{\mathfrak{L}}}(q_i(x)) \circ \left\{\left(\gamma^F\left(q_i(x)\right)\right)^{-1}\right\}^* \beta\right](f) \right) \\
=& f.
\end{aligned}$$

Consequently, δ_{ij}^{β} lies in

$$C^\infty(U_i \times F, \tilde{\mathfrak{L}})^i := \{\kappa \in C^\infty(U_i \times F, \tilde{\mathfrak{L}}) \mid \pi_{\tilde{\mathfrak{L}}} \circ \kappa = \mathrm{proj}_2\}.$$

On the other hand, we saw in Lemma 3.18 that the vector spaces $C^\infty(U_i \times F, \tilde{\mathfrak{L}})^i$ and $C^\infty(U_i, \Gamma(F, \tilde{\mathfrak{L}}))$ are canonically isomorphic.
This leads to

$$\begin{array}{rccc} \widehat{\delta_{ij}^{\beta}}(\cdot,\beta): & U_i & \to & \Gamma(F,\tilde{\mathfrak{L}}) \\ & x & \mapsto & \widetilde{\gamma^{\tilde{\mathfrak{L}}}(q_i(x))}(\beta) \end{array}$$

being smooth for every β as well.
In particular, we obtain by restriction that for every $\beta \in \mathrm{Eig}_\lambda(\square_{\tilde{\mathfrak{L}}}^{(0,0)})$ the map

$$\begin{array}{rccc} \widehat{\delta_{ij}^{\beta}}(\cdot,\beta): & U_i & \to & \mathrm{Eig}_\lambda(\square_{\tilde{\mathfrak{L}}}^{(0,0)}) \\ & x & \mapsto & \widetilde{\gamma^{\tilde{\mathfrak{L}}}(q_i(x))}(\beta) \end{array}$$

is smooth.
Thus, the map

$$\begin{array}{rccc} \widehat{\delta_{ij}}(\cdot): & U_i & \to & \mathrm{Gl}\left(\mathrm{Eig}_\lambda(\square_{\tilde{\mathfrak{L}}}^{(0,0)})\right) \\ & x & \mapsto & \widetilde{\gamma^{\tilde{\mathfrak{L}}}(q_i(x))} \end{array}$$

is smooth which can be seen by choosing a base for $\mathrm{Eig}_\lambda(\square_{\tilde{\mathfrak{L}}}^{(0,0)})$.
We finally obtain that the map from Equation (39),

$$\begin{array}{rccc} \widetilde{\gamma^{\tilde{\mathfrak{L}}}(q_i(\cdot))}: & U_i \times \mathrm{Eig}_\lambda(\square_{\tilde{\mathfrak{L}}}^{(0,0)}) & \longrightarrow & \mathrm{Eig}_\lambda(\square_{\tilde{\mathfrak{L}}}^{(0,0)}) \\ & (x,\beta) & \longmapsto & \widetilde{\gamma^{\tilde{\mathfrak{L}}}(q_i(x))}\beta, \end{array}$$

is a product of two smooth maps and therefore smooth itself which finishes the proof. $\square$

Summarising the lemmas above, we were able to construct a smooth map $\gamma^{\mathcal{W}^{(\lambda;t)}}$ covering the map γ^M.

$$\begin{tikzcd} \mathcal{W}^{(\lambda;t)} \arrow[r, "\gamma^{\mathcal{W}^{(\lambda;t)}}"] \arrow[d] & \mathcal{W}^{(\lambda;t)} \arrow[d] \\ M \arrow[r, "\gamma^M"'] & M \end{tikzcd}$$

Now we can finally prove the Proposition 3.28.

Proof of Proposition 3.28.
The map $\gamma^{\mathcal{W}^{(\lambda;t)}}$ is obviously $\mathbb{C}$-linear on fibres.
Therefore, what is left to check is that the γ-action on $\mathfrak{A}^{(0,*)}(M, \mathcal{W}^{(\lambda;t)})$ decomposes into a pullback with $(\gamma^M)^{-1}$ and this map $\gamma^{\mathcal{W}^{(\lambda;t)}}$ constructed above.

Lemma 3.30 shows that if we proof this assertion for sections $\Gamma(M, \mathcal{W}^{(\lambda;t)})$, it holds for antiholomorphic forms $\mathfrak{A}^{(0,*)}(M, \mathcal{W}^{(\lambda;t)})$ as well.
Furthermore, because of Lemma 3.35, we may compare the induced actions on the C^∞ functions.
Let w be a section in $\Gamma(M, \mathcal{W}^{(\lambda;t)})$ and let $\breve{w} \in C^\infty(P, \mathrm{Eig}_\lambda(\square^{(0,t)}_{\tilde{\mathfrak{L}}}))^{\breve{\rho}}$ correspond to W, i.e. for any $x \in M$ and any $p \in \pi_P^{-1}(x)$ we obtain:

$$w(x) = [p, \breve{w}(p)]_{\breve{\rho}} \,.$$

Now, Lemma 3.35 states that we get for $q = (\gamma^P)^{-1}(p)$:

$$(\gamma \cdot w)_x = \left[p, \widetilde{\gamma^{\tilde{\mathfrak{L}}}(q)} \left(\breve{w}\,(q)\right)\right]_{\breve{\rho}} \,.$$

On the other hand, we get by the definition of $\gamma^{\mathcal{W}^{(\lambda;t)}}$:

$$\gamma^{\mathcal{W}^{(\lambda;t)}} \left(\left((\gamma^M)^{-1}\right)^* w\right)_x = \gamma^{\mathcal{W}^{(\lambda;t)}} \left([q, \breve{w}\,(q)]_{\breve{\rho}}\right) = \left[p, \widetilde{\gamma^{\tilde{\mathfrak{L}}}\,(q)}\,(\breve{w}(q))\right]_{\breve{\rho}} \,.$$

Both expressions are equal which finally proofs the assertion. $\square$

We now have evolved the theory of legitimate actions on compatible line bundles over holomorphic fibre bundles as far as we need it.

What comes next is to apply this theory to the problem of computing the equivariant holomorphic torsion for those compatible line bundles. This will be the content of the next section.

4 The equivariant torsion for fibre bundles

Up to this point, we have introduced as well as examined our setting. We have stated specific conditions for a fibre bundle (*to be holomorphic*), a line bundle (*to be compatible*) and an action (*to be legitimate*) for which we want to study the equivariant ζ-function.

Now, it is time to move on towards our goal which has been to derive a suitably nice formula for the equivariant holomorphic torsion of a compatible holomorphic line bundle $\mathfrak{L}$ over a holomorphic fibre bundle $E \to M$.

At first, in Section 4.1, we show that the equivariant ζ-function splits into two parts, each one depending on the nonzero spectrum of one of the operators $\square_V$ and $\square_H$ only. This is the equivariant generalisation of analogous results from Stanton in [29].
In Section 4.1, we present Theorem 4.1 our most general result. It generalises the analogous non-equivariant result of Stanton in a natural way.
It gives us the equivariant holomorphic ζ-function $Z^{\mathfrak{L}}_{\gamma}$ of $\mathfrak{L}$ expressed through objects living on the base M of the holomorphic fibre bundle $E \to M$.
In the last subsection, i.e. Section 4.3, we specialise this general result to the case of γ^M having only non-degenerate fixed points in M. This is achieved by applying the Atiyah-Bott's fixed point formula.

Let throughout this section $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ be a holomorphic fibre bundle, $\mathfrak{L} \to E$ be a compatible line bundle over E and let $\gamma = (\gamma^M, \gamma^E, \gamma^{\mathfrak{L}})$ be a legitimate action. Furthermore, we keep the notations we developed so far.

4.1 Splitting of the zeta-function

This subsection is dedicated to uncover a splitting of our equivariant ζ-function into a horizontal and a vertical part. This splitting is due to the fact that the Eigenspaces of $\square_{\mathfrak{L}}$ contribute to the ζ-function only if they are a 0-Eigenspace for either $\square_H$ or $\square_V$.
We summarise what we educe in this subsection in Proposition 4.3.

We start at the definition of the equivariant ζ-function (compare Definition 2.43).
Let $\sigma(\square_{\mathfrak{L}})$ denote the spectrum of $\square_{\mathfrak{L}}$.
Now, for $z \in \mathbb{C}$ with sufficiently large $\mathrm{Re}(z)$, the ζ-function is given by:

$$Z^{\mathfrak{L}}_{\tilde{\gamma}^{\mathfrak{L}}}(z) = \sum_{q \geq 0} (-1)^{q+1} q \sum_{0 \neq \nu \in \sigma(\square_{\mathfrak{L}})} \nu^{-z} \cdot \mathrm{Tr}\left((\gamma^{\mathfrak{L}})^* |_{\mathrm{Eig}_\nu(\square^{(0,q)}_{\mathfrak{L}})} \right).$$

For a compatible line bundle $\mathfrak{L}$ the Eigenspace $\mathrm{Eig}_\nu(\square^{(0,q)}_{\mathfrak{L}})$ decomposes into a direct sum of Eigenspaces $L(\lambda, \mu, s, t)$, given by

$$L(\lambda, \mu, s, t) = \mathrm{Eig}_\lambda(\square^{[s,t]}_V) \cap \mathrm{Eig}_\mu(\square^{[s,t]}_H),$$

with $\lambda + \mu = \nu$ and $s + t = q$ (compare Equation (30) and Equation (31)).

The action $\vec{\gamma}$ is legitimate, therefore, following Remark 3.27, $\check{\gamma}^{\mathfrak{L}}$ commutes with both $\Box_V$ and $\Box_H$. Hence, $L(\lambda, \mu, s, t)$ is invariant under $\check{\gamma}^{\mathfrak{L}}$.
We obtain the following equation.

$$\begin{aligned} Z^{\mathfrak{L}}_{\check{\gamma}^{\mathfrak{L}}}(z) =& \sum_{t,s}(-1)^{s+t+1}(s+t) \sum_{\lambda+\mu\neq 0} (\lambda+\mu)^{-z} \mathrm{Tr}\left[\check{\gamma}^{\mathfrak{L}}|_{L(\lambda,\mu,s,t)}\right] \\ =& - \sum_{\lambda+\mu\neq 0} (\lambda+\mu)^{-z} \left(\sum_{t,s} t(-1)^{s+t} + \sum_{t,s} s(-1)^{s+t}\right) \mathrm{Tr}\left[\check{\gamma}^{\mathfrak{L}}|_{L(\lambda,\mu,s,t)}\right] \end{aligned}$$

Recall that $\Box_H$ as well as $\Box_V$ are given as a square of a self-adjoint operator. Therefore, their spectrum is non-negative.
Consequently, the sum over $\lambda+\mu\neq 0$ splits into a sum $\lambda \neq 0$ with arbitrary μ and a sum $\lambda = 0$ with $\mu \neq 0$.

$$\begin{aligned} Z^{\mathfrak{L}}_{\check{\gamma}^{\mathfrak{L}}}(z) =& - \sum_{\lambda\neq 0,\mu} (\lambda+\mu)^{-z} \left(\sum_{t,s} t(-1)^{s+t} + \sum_{t,s} s(-1)^{s+t}\right) \mathrm{Tr}\left[\check{\gamma}^{\mathfrak{L}}|_{L(\lambda,\mu,s,t)}\right] \\ & - \sum_{\mu\neq 0} \mu^{-z} \left(\sum_{t,s} t(-1)^{s+t} + \sum_{t,s} s(-1)^{s+t}\right) \mathrm{Tr}\left[\check{\gamma}^{\mathfrak{L}}|_{L(0,\mu,s,t)}\right] \end{aligned} \tag{40}$$

This might look slightly more complicated than the original expression of the equivariant ζ-function, but the following lemma shows that some parts of this decomposed ζ-function simply vanish.
Its effect on the sum above is depicted below in Corollary 4.2.
This lemma is the equivariant generalisation of a property Stanton showed in [29].

Lemma 4.1:
In the situation above, the following identities hold.

a) The Eigenspace for nonzero Eigenvalues of either $\Box_H$ or $\Box_V$ can be split as follows.

$$L(\lambda\neq 0,\mu,s,t) = \bar{\partial}_V L(\lambda,\mu,s,t-1) \oplus \bar{\partial}_V^* L(\lambda,\mu,s,t+1) \tag{41}$$
$$L(\lambda,\mu\neq 0,s,t) = \bar{\partial}_H L(\lambda,\mu,s-1,t) \oplus \bar{\partial}_H^* L(\lambda,\mu,s+1,t) \tag{42}$$

b) The trace of the $(\check{\gamma}^{\mathfrak{L}})$-action splits as well.

$$\begin{aligned} \mathrm{Tr}(\check{\gamma}^{\mathfrak{L}}|_{L(\lambda\neq 0,\mu,s,t)}) =& \mathrm{Tr}(\check{\gamma}^{\mathfrak{L}}|_{\bar{\partial}_V L(\lambda,\mu,s,t-1)}) + \mathrm{Tr}(\check{\gamma}^{\mathfrak{L}}|_{\bar{\partial}_V^* L(\lambda,\mu,s,t+1)}) \\ \mathrm{Tr}(\check{\gamma}^{\mathfrak{L}}|_{L(\lambda,\mu\neq 0,s,t)}) =& \mathrm{Tr}(\check{\gamma}^{\mathfrak{L}}|_{\bar{\partial}_H L(\lambda,\mu,s-1,t)}) + \mathrm{Tr}(\check{\gamma}^{\mathfrak{L}}|_{\bar{\partial}_H^* L(\lambda,\mu,s+1,t)}) \end{aligned}$$

c) The following two traces are equal.

$$\begin{aligned} \mathrm{Tr}(\check{\gamma}^{\mathfrak{L}}|_{\bar{\partial}_V L(\lambda\neq 0,\mu,s,t-1)}) =& \mathrm{Tr}(\check{\gamma}^{\mathfrak{L}}|_{\bar{\partial}_V^* L(\lambda,\mu,s,t)}) \\ \mathrm{Tr}(\check{\gamma}^{\mathfrak{L}}|_{\bar{\partial}_H L(\lambda,\mu\neq 0,s-1,t)}) =& \mathrm{Tr}(\check{\gamma}^{\mathfrak{L}}|_{\bar{\partial}_H^* L(\lambda,\mu,s,t)}) \end{aligned}$$

Proof.

a) For the proof of Equation (41), let $\alpha \in L(\lambda, \mu, s, t-1)$ be an Eigenform for some $\lambda \neq 0$ and for an arbitrary μ.
Both, the horizontal Laplacian, $\Box_H$, as well as the vertical Laplacian, $\Box_V$, commute with $\bar{\partial}_V^{(*)}$ (compare Lemma 3.12).

$$\begin{aligned}
&\Box_V \bar{\partial}_V \alpha = \bar{\partial}_V \Box_V \alpha = \lambda \bar{\partial}_V \alpha \\
&\Box_H \bar{\partial}_V \alpha = (\Box_{\mathfrak{L}} - \Box_H) \bar{\partial}_V \alpha = \bar{\partial}_V (\Box_{\mathfrak{L}} - \Box_H) \alpha = \mu \bar{\partial}_V \alpha.
\end{aligned}$$

Hence, $\bar{\partial}_V \alpha$ is an element of $L(\lambda, \mu, s, t)$.
The form α has been chosen arbitrary in $L(\lambda, \mu, s, t-1)$. Thus, we conclude that

$$\bar{\partial}_V L(\lambda, \mu, s, t-1) \subset L(\lambda, \mu, s, t).$$

Analogously, we observe that

$$\bar{\partial}_V^* L(\lambda, \mu, s, t+1) \subset L(\lambda, \mu, s, t).$$

The intersection between $\bar{\partial}_V L(\lambda, \mu, s, t-1)$ and $\bar{\partial}_V^* L(\lambda, \mu, s, t+1)$ as subspaces of $L(\lambda, \mu, s, t)$ is $\{0\}$. This is due to the fact that, for $\beta = \bar{\partial}_V \alpha = \bar{\partial}_V^* \omega$ in $L(\lambda, \mu, s, t)$, we obtain

$$\lambda \beta = \Box_V \beta = \bar{\partial}_V \underbrace{(\bar{\partial}_V^*)^2}_{=0} \omega + \bar{\partial}_V^* \underbrace{(\bar{\partial}_V)^2}_{=0} \alpha = 0.$$

On the other hand, we assumed $\lambda \neq 0$, hence, β vanishes.
Now, Equation (41) holds if and only if the map

$$\begin{array}{rccc}
\varphi: & L(\lambda, \mu, s, t-1) \oplus L(\lambda, \mu, s, t+1) & \longrightarrow & L(\lambda, \mu, s, t) \\
& (\alpha \oplus \omega) & \longmapsto & \bar{\partial}_V \alpha + \bar{\partial}_V^* \omega
\end{array}$$

is surjective.
This can be shown by finding a right inverse φ^{-1} such that $\varphi \circ \varphi^{-1} = \mathrm{id}\big|_{L(\lambda,\mu,s,t)}$.
We claim that this inverse map is given by $\varphi^{-1} := \frac{1}{\lambda}(\bar{\partial}_V^* + \bar{\partial}_V)$.
We verify this, by evaluating it for an arbitrary β in $L(\lambda, \mu, s, t)$.

$$\varphi \circ \varphi^{-1}(\beta) = \varphi\left(\frac{1}{\lambda}\left(\bar{\partial}_V^* \beta \oplus \bar{\partial}_V \beta\right)\right) = \frac{1}{\lambda}\left(\underbrace{\bar{\partial}_V \bar{\partial}_V^* + \bar{\partial}_V^* \bar{\partial}_V}_{=\Box_V}\right) \beta = \beta$$

This proves Equation (41).
Analogously, Equation (42) can be shown.

b) We want to show that the splitting of $L(\lambda, \mu, s, t)$ from Equations (41) and (42) is compatible with the $\check{\gamma}^{\mathfrak{L}}$ action.

We already know that $\check{\gamma}^{\mathfrak{L}}|_{L(\lambda,\mu,s,t)} \in \mathrm{Aut}\,(L(\lambda,\mu,s,t))$, i.e. it maps $L(\lambda,\mu,s,t)$ onto itself.
Since $[\check{\gamma}^{\mathfrak{L}}, \bar{\partial}_V^{(*)}] = 0$, the image of $\check{\gamma}^{\mathfrak{L}}$ restricted to the space $\bar{\partial}_V^{(*)} L(\lambda,\mu,s,t)$ is contained in $\bar{\partial}_V^{(*)} L(\lambda,\mu,s,t)$ which shows the first equation of b).
Analogous considerations, using $[\check{\gamma}^{\mathfrak{L}}, \bar{\partial}_H^{(*)}] = 0$, can be made for the second equation.

c) At last, the first identity of c) can be seen as follows.
For a non-vanishing λ, the vector spaces $\bar{\partial}_V L(\lambda,\mu,s,t-1)$ and $\bar{\partial}_V^* L(\lambda,\mu,s,t)$ are isomorphic.
The isomorphism is explicitly given by:

$$\begin{array}{rccc} \tilde{\varphi}: & \bar{\partial}_V L(\lambda,\mu,s,t-1) & \longrightarrow & \bar{\partial}_V^* L(\lambda,\mu,s,t) \\ & \alpha & \longmapsto & \frac{1}{\sqrt{\lambda}} \bar{\partial}_V^* \alpha. \end{array}$$

Its inverse map is:

$$\begin{array}{rccc} \tilde{\varphi}^{-1}: & \bar{\partial}_V^* L(\lambda,\mu,s,t) & \longrightarrow & \bar{\partial}_V L(\lambda,\mu,s,t-1) \\ & \alpha & \longmapsto & \frac{1}{\sqrt{\lambda}} \bar{\partial}_V \alpha. \end{array}$$

Now, the first identity of c) holds because $\check{\gamma}^{\mathfrak{L}}$ commutes with $\tilde{\varphi}$, i.e.

$$[\tilde{\varphi}, \check{\gamma}^{\mathfrak{L}}] = 0.$$

Again, the second equation follows analogously.

$\square$

We now summarise what Lemma 4.1 implies for the sums in our expression of the ζ-function.

Corollary 4.2:
For $\lambda \neq 0$, the following equation holds for every μ:

$$\sum_t (-1)^t \mathrm{Tr}(\check{\gamma}^{\mathfrak{L}}|_{L(\lambda,\mu,s,t)}) = 0.$$

Analogously, we obtain for $\mu \neq 0$ and arbitrary λ the subsequent identity:

$$\sum_{s=0}^{m} (-1)^s \mathrm{Tr}(\check{\gamma}^{\mathfrak{L}}|_{L(\lambda,\mu,s,t)}) = 0.$$

Proof.
The first equation follows directly from

$$\begin{aligned}\mathrm{Tr}(\check{\gamma}^{\mathfrak{L}}|_{L(\lambda,\mu,s,t)}) &\overset{b)}{=} \mathrm{Tr}(\check{\gamma}^{\mathfrak{L}}|_{\bar{\partial}_V L(\lambda,\mu,s,t-1)}) + \mathrm{Tr}(\check{\gamma}^{\mathfrak{L}}|_{\bar{\partial}_V^* L(\lambda,\mu,s,t+1)}) \\ &\overset{c)}{=} \mathrm{Tr}(\check{\gamma}^{\mathfrak{L}}|_{\bar{\partial}_V L(\lambda,\mu,s,t-1)}) + \mathrm{Tr}(\check{\gamma}^{\mathfrak{L}}|_{\bar{\partial}_V L(\lambda,\mu,s,t)})\end{aligned}$$

and the obvious fact that for any finite sequence $\{a_n\}_{n\in\{-1,\dots,m\}}$ with $a_{-1} = a_m = 0$ the alternating series over $a_n + a_{n+1}$ vanishes, i.e.

$$\sum_{n=0}^{m}(-1)^n\,(a_{n-1} + a_n) = 0.$$

Almost the same proof holds for the second equation. □

We now apply Corollary 4.2 to our expression for the ζ-function (compare Equation (40)). This simplifies the expression for the equivariant ζ-function significantly.

Proposition 4.3:
The equivariant ζ-function may be reduced to computing traces of $\check{\gamma}^{\mathfrak{L}}$ on the kernels of $\Box_H$ and $\Box_V$ in the following way.

$$\begin{aligned}Z^{\mathfrak{L}}_{\check{\gamma}^{\mathfrak{L}}}(z) = &-\sum_{\lambda\neq 0}\lambda^{-z}\sum_t t(-1)^t\sum_{s=0}^{m}(-1)^s\mathrm{Tr}\left[\check{\gamma}^{\mathfrak{L}}\Big|_{\mathrm{Ker}\left(\Box_H\big|_{\mathrm{Eig}_\lambda\left(\Box_V^{[s,t]}\right)}\right)}\right] \\ &-\sum_{\mu\neq 0}\mu^{-z}\sum_{s=0}^{m}s(-1)^s\sum_t(-1)^t\mathrm{Tr}\left[\check{\gamma}^{\mathfrak{L}}\Big|_{\mathrm{Ker}\left(\Box_V\big|_{\mathrm{Eig}_\mu\left(\Box_H^{[s,t]}\right)}\right)}\right]\end{aligned}$$

Proof.
We start with the expression for the ζ-function which we developed as in Equation (40).

Now, we apply Corollary 4.2 repeatedly.

$$\begin{aligned}
Z^{\mathfrak{L}}_{\check{\gamma}^{\mathfrak{L}}}(z) = &- \sum_{\lambda\neq 0,\mu} (\lambda+\mu)^{-z} \left(\sum_{t,s} t(-1)^{s+t} + \cancel{\sum_{t,s} s(-1)^{s+t}} \right) \operatorname{Tr}\left[\check{\gamma}^{\mathfrak{L}}|_{L(\lambda,\mu,s,t)} \right] \\
&- \sum_{\mu\neq 0} \mu^{-z} \left(\cancel{\sum_{t,s} t(-1)^{s+t}} + \sum_{t,s} s(-1)^{s+t} \right) \operatorname{Tr}\left[\check{\gamma}^{\mathfrak{L}}|_{L(0,\mu,s,t)} \right] \\
= &- \sum_{\lambda\neq 0,\mu} (\lambda+\mu)^{-z} \sum_{t} t(-1)^{t} \sum_{s=0}^{m} (-1)^{s} \operatorname{Tr}\left[\check{\gamma}^{\mathfrak{L}}|_{L(\lambda,\mu,s,t)} \right] \\
&- \sum_{\mu\neq 0} \mu^{-z} \sum_{s=0}^{m} s(-1)^{s} \sum_{t} (-1)^{t} \operatorname{Tr}\left[\check{\gamma}^{\mathfrak{L}}|_{L(0,\mu,s,t)} \right]
\end{aligned}$$

Again, we use the Corollary 4.2 to reduce our first summand.
This time, we can eliminate all the sums where $\mu \neq 0$.

$$\begin{aligned}
Z^{\mathfrak{L}}_{\check{\gamma}^{\mathfrak{L}}}(z) = &- \sum_{\lambda\neq 0} \lambda^{-z} \sum_{t} t(-1)^{t} \sum_{s=0}^{m} (-1)^{s} \operatorname{Tr}\left[\check{\gamma}^{\mathfrak{L}}|_{L(\lambda,0,s,t)} \right] \\
&- \sum_{\mu\neq 0} \mu^{-z} \sum_{s=0}^{m} s(-1)^{s} \sum_{t} (-1)^{t} \operatorname{Tr}\left[\check{\gamma}^{\mathfrak{L}}|_{L(0,\mu,s,t)} \right]
\end{aligned}$$

This finishes the proof. $\square$

4.2 Applying the psi-correspondence

So far, we have used that the $\check{\gamma}^{\mathfrak{L}}$ operator commutes with $\bar{\partial}_V^{(*)}$ as well as $\bar{\partial}_H^{(*)}$ to simplify our expression of the equivariant ζ-function. Now, it depends only on the action of $\check{\gamma}^{\mathfrak{L}}$ on the kernels of $\square_H$ and $\square_V$.
In this subsection, we apply the ψ-morphism, described in Section 3.3 and summarised in Proposition 3.14, to our situation.

The ψ-morphism identifies λ-Eigenforms of $\square_V$ with antiholomorphic differential forms on the base space M with coefficients in a holomorphic vector bundle $\mathcal{W}^{(\lambda;*)}$.
The action $\check{\gamma}^{\mathfrak{L}}$ on $\mathrm{Eig}_\lambda(\square_V)$ corresponds to an action γ on $\mathfrak{A}^{(0,*)}(M, \mathcal{W}^{(\lambda;*)})$ under this isomorphism.
We showed that γ is a composition of a vector bundle morphism $\gamma^{\mathcal{W}^{(\lambda;*)}}$ and a pullback along $(\gamma^M)^{-1}$ (compare Proposition 3.28).

In Section 3.4, we, furthermore, defined an operator $D_{\mathcal{W}^{(\lambda;t)}}$ acting on $\mathfrak{A}^{(0,*)}(M, \mathcal{W}^{(\lambda;t)})$ (Equation (28)). It is the operator corresponding to $\bar{\partial}_H + \bar{\partial}_H^*$ via ψ.
Its square, $\square_{\mathcal{W}^{(\lambda;t)}}$, is the Dolbeault-Laplacian acting on $\mathfrak{A}^{(0,*)}(M, \mathcal{W}^{(\lambda;t)})$.

It is convenient to translate our expression for $Z^{\mathfrak{L}}_{\gamma}$ under the ψ-correspondence since sometimes there is better knowledge about the existence or the structures of holomorphic vector bundles over M.

We start at the expression for the equivariant ζ-function developed in Proposition 4.3. Now, we apply ψ as follows:

$$\begin{aligned} Z^{\mathfrak{L}}_{\check{\gamma}^{\mathfrak{L}}}(z) = &- \sum_{\lambda\neq 0} \lambda^{-z} \sum_t t(-1)^t \underbrace{\sum_{s=0}^m (-1)^s \mathrm{Tr}\left[\gamma\big|_{\mathrm{Ker}\left(\square^{(0,s)}_{\mathcal{W}^{(\lambda;t)}}\right)}\right]}_{=\mathrm{ind}\left(\gamma^{\mathcal{W}^{(\lambda;t)}},\square_{\mathcal{W}^{(\lambda;t)}}\right)} \\ &- \sum_{\mu\neq 0} \mu^{-z} \sum_{s=0}^m s(-1)^s \sum_t (-1)^t \mathrm{Tr}\left[\check{\gamma}^{\mathfrak{L}}\Big|_{\mathrm{Ker}\left(\square_V\big|_{\mathrm{Eig}_\mu\left(\square_H^{[s,t]}\right)}\right)}\right]. \end{aligned}$$

The first term becomes a sum over equivariant indexes

$$\begin{aligned} Z^{\mathfrak{L}}_{\check{\gamma}^{\mathfrak{L}}}(z) = &- \sum_{\lambda\neq 0} \lambda^{-z} \sum_t t(-1)^t \mathrm{ind}(\gamma^{\mathcal{W}^{(\lambda;t)}},\square_{\mathcal{W}^{(\lambda;t)}}) \\ &+ \sum_t (-1)^t \underbrace{\sum_{s=0}^m s(-1)^{s+1} \sum_{\mu\neq 0} \mu^{-z} \mathrm{Tr}\left[\gamma\big|_{\mathrm{Eig}_\mu\left(\square^{(0,s)}_{\mathcal{W}^{(0;t)}}\right)}\right]}_{=Z^{\mathcal{W}^{(0;t)}}_{\check{\gamma}^{\mathcal{W}^{(0;t)}}}(z)} \end{aligned} \tag{43}$$

while the second term can be expressed through ζ-functions of $\mathcal{W}^{(0;t)}$.
Here, $\mathrm{ind}(\gamma^{\mathcal{W}^{(\lambda;t)}},\square_{\mathcal{W}^{(\lambda;t)}})$ denotes the equivariant index given by Definition 2.42.

Equation (43) is our final result for the general case, therefore we summarise it in a theorem.

Theorem 4.1:
Let $(E,\pi_E,(M,g_M),(F,g_F),T^HE)$ be a holomorphic fibre bundle and let $\mathfrak{L}\to E$ be a compatible, holomorphic, Hermitian line bundle.
Let further on γ be a legitimate action on $\mathfrak{L}$.
Then the equivariant ζ-function can be expressed as follows:

$$Z^{\mathfrak{L}}_{\check{\gamma}^{\mathfrak{L}}}(z) = - \sum_{\lambda\neq 0} \lambda^{-z} \sum_t t(-1)^t \mathrm{ind}(\gamma^{\mathcal{W}^{(\lambda;t)}},\square_{\mathcal{W}^{(\lambda;t)}}) + \sum_t (-1)^t Z^{\mathcal{W}^{(0;t)}}_{\check{\gamma}^{\mathcal{W}^{(0;t)}}}(z).$$

The non-equivariant ζ-function is a special case of the equivariant ζ-function for the trivial legitimate action $\gamma^{\mathfrak{L}} = \mathrm{id}_{\mathfrak{L}}$.

Corollary 4.4:
For a compatible holomorphic, Hermitian line bundle $\mathfrak{L}\to E$ over a holomorphic fibre bundle $(E,\pi_E,(M,g_M),(F,g_F),T^HE)$, the ζ-function has the following form:

$$Z^{\mathfrak{L}}(z) = - \sum_{\lambda\neq 0} \lambda^{-z} \sum_t t(-1)^t \mathrm{ind}(\square_{\mathcal{W}^{(\lambda;t)}}) + \sum_t (-1)^t Z^{\mathcal{W}^{(0;t)}}(z) \tag{44}$$

Remark 4.5:
Originally, Stanton shows the assertion of Corollary 4.4 in [29].
In particular, she applies it on some holomorphic line bundles over a compact even-dimensional Lie group.
She computes the non-equivariant holomorphic torsion of those line bundles over a compact even-dimensional Lie group and shows that the holomorphic torsion of one of these line bundles equals the holomorphic torsion of the restriction of this line bundle to a maximal torus. Furthermore, the holomorphic torsion of these restricted line bundles over the torus are known (cf. [24]).
The main arguments for this Lie group result are on the one hand that the second summand in (44) vanishes and on the other hand that the Atiyah-Singer Theorem applied to $\mathrm{ind}(\square_{\mathcal{W}(\lambda;t)})$ simplifies the expression for first summand significantly.
Stantons result for Lie groups is our main motivation in trying to apply Theorem 4.1 to Lie groups.
We approach this example in Section 5.

Theorem 4.1 can be used directly to describe the equivariant torsion $\tau^{\mathfrak{L}}(\check{\gamma}^{L})$ (compare Definition 2.44).

Corollary 4.6:
Let $\mathfrak{L}$ be a compatible holomorphic, Hermitian line bundle over a holomorphic fibre bundle $(E, \pi_E, (M, g_M), (F, g_F), T^H E)$ and let $\vec{\gamma}$ be a legitimate action.
Denote by Θ the meromorphic continuation of the map

$$z \longmapsto -\sum_{\lambda\neq 0} \lambda^{-z} \sum_{t} t(-1)^t \mathrm{ind}(\gamma^{\mathcal{W}(\lambda;t)}, \square_{\mathcal{W}(\lambda;t)})$$

to the complex plane $\mathbb{C}$.
Now, the equivariant torsion of $\mathfrak{L}$ corresponding to the action $\vec{\gamma}$ is given by:

$$\tau^{\mathfrak{L}}(\check{\gamma}^{\mathfrak{L}}) = \Theta'(0) + \sum_{t} \left(\tau^{\mathcal{W}(0;2t)}(\check{\gamma}^{\mathcal{W}(0;2t)}) - \tau^{\mathcal{W}(0;2t+1)}(\check{\gamma}^{\mathcal{W}(0;2t+1)})\right).$$

4.3 Special case: isolated non-degenerated fixed points

For a special case of a legitimate action $\vec{\gamma}$, there is a further simplification of the expression of the ζ-function given in Theorem 4.1.

In order to state it, we have to recall the definition of an isolated and non-degenerated fixed point of an isometry.

Definition 4.7:
Let M be a Riemannian manifold and let γ^M be an isometry of M.

- A fixed point $x \in M$ of γ^M is called **isolated** if there is an $\varepsilon > 0$ such that there is no other fixed point of γ^M in an ε-neighbourhood of x.
- A fixed point $x \in M$ of γ^M is called **non-degenerated** if there is no vector X in T_xM such that $\left(T_x\gamma^M\right)(X) = X$.

In particular, every non-degenerated fixed point is isolated.

Now, in the setting from Theorem 4.1, we look at the special case where the $\vec{\gamma}$-action on the lowest level, i.e. $\gamma^M : M \to M$, has only non-degenerated fixed points.
Here, we can use the Atiyah-Bott fixed point formula (compare [6, Ch. 6, Thm 6.6]) to calculate the term $\operatorname{ind}(\gamma^{\mathcal{W}(\lambda;t)}, \Box_{\mathcal{W}(\lambda;t)})$.

In order to make it easier for the reader follow, we will not recite the whole theorem but a corollary (compare [6, Ch. 6, Cor 6.8]) tailored to the situation at hand.
We explain the notations used in this corollary in the subsequent remark.

Theorem (Corollary following from Atiyah-Bott)**:**
"If M is a compact complex manifold with holomorphic vector bundle $\mathcal{W} \to M$, and γ is a holomorphic transformation of $\mathcal{W} \to M$, then γ acts on the $\bar{\partial}$-cohomology spaces $H^{0,i}(M, \mathcal{W})$. If the action of γ on M has only isolated non-degenerate fixed points, then

$$\sum_i (-1)^i \operatorname{Tr}(\gamma, H^{0,i}(M, \mathcal{W})) = \sum_{x_0 \in M^\gamma} \frac{\operatorname{Tr}(\gamma^{\mathcal{W}}_{x_0})}{\det_{T^{1,0}_{x_0}M}(1 - \gamma_{x_0}^{-1})}."$$

Remark 4.8:
We now clarify the notations above.

- The cohomology $H^{0,i}(M, \mathcal{W})$ is defined to be the kernel of $\Box^{(0,i)}_{\mathcal{W}}$ and the action of γ on $H^{0,i}(M, \mathcal{W})$ is the action $\check{\gamma}^{\mathfrak{L}}$ (compare Definition 2.40) on $\mathfrak{A}^{(0,*)}(M, \mathcal{W})$ restricted to the kernel of $\Box^{(0,i)}_{\mathcal{W}}$.
- The symbol $\boldsymbol{M^\gamma}$ denotes the fixed point set of γ^M.
- For any fixed point $x_0 \in M^\gamma$ the map $\boldsymbol{\gamma_{x_0}}$ denotes the restriction of $T_{x_0}\gamma^M$ to the space $T^{\mathbb{C}}_{x_0}M$. Similarly, $\boldsymbol{\gamma^{\mathcal{W}}_{x_0}}$ is the restriction of $\gamma^{\mathcal{W}}$ to the fibre $\mathcal{W}_{x_0}$.
- In the expression above, the determinant of $1 - \gamma_{x_0}^{-1}$ is taken on the restriction of $1 - \gamma_{x_0}^{-1}$ to the invariant subspace $T^{(1,0)}_{x_0}M$.

These identifications in mind, we observe that the left hand side equals $\operatorname{ind}(\gamma^{\mathcal{W}}, \Box_{\mathcal{W}})$ (compare Definition 2.42).

We apply the Atiyah-Bott theorem and obtain the following corollary.

Corollary 4.9:
In the setting from Theorem 4.1, suppose that $\gamma^M : M \to M$ is a biholomorphic isometry which has only isolated, non-degenerated fixed points.
Then the formula for the equivariant holomorphic Zeta function can be simplified to:

$$Z^{\mathfrak{L}}_{\check{\gamma}^{\mathfrak{L}}}(z) = \sum_{x_0 \in M^\gamma} \left[\det_{T^{1,0}_{x_0} M}(1 - \gamma^{-1}_{x_0})\right]^{-1} \cdot Z^{\mathfrak{L}_{x_0}}_{\check{\gamma}^{\mathfrak{L}_{x_0}}}(z) + \sum_t (-1)^t Z^{\mathcal{W}^{(0;t)}}_{\gamma}(z) \tag{45}$$

where we use the notations from above.
In particular, the equivariant torsion is now given by:

$$\begin{aligned} \tau^{\mathfrak{L}}(\check{\gamma}^{\mathfrak{L}}) = & \sum_{x_0 \in M^\gamma} \left[\det_{T^{1,0}_{x_0} M}(1 - \gamma^{-1}_{x_0})\right]^{-1} \cdot \tau^{\mathfrak{L}_{x_0}}(\check{\gamma}^{\mathfrak{L}_{x_0}}) \\ & + \sum_t \left(\tau^{\mathcal{W}^{(0;2t)}}(\check{\gamma}^{\mathcal{W}^{(0;2t)}}) - \tau^{\mathcal{W}^{(0;2t+1)}}(\check{\gamma}^{\mathcal{W}^{(0;2t+1)}})\right). \end{aligned}$$

This finishes the first part of this thesis.

The last section is dedicated to apply the theory, we evolved so far, to a specific example. We study the equivariant holomorphic torsion of a flat line bundle over a compact Lie group.

5 Equivariant torsion for Lie groups

In this section, we apply Theorem 4.1 to an explicit example.
Every example has to fulfil a lot of prerequisites. There has to be a holomorphic fibre bundle in the sense of Definition 2.13, a compatible holomorphic, Hermitian line bundle (compare Definition 3.1) and a legitimate group action, described in Definition 3.26. Therefore, it requires a lot of preparing to apply Theorem 4.1 to an example.

However, there is one class of examples that seems to be the most manageable for this kind of investigation, the case where the total space of the holomorphic fibre bundle is given by a compact, even-dimensional Lie group. We chose the holomorphic fibre bundle to be the Lie group over the homogeneous space which is given by factorising a maximal torus out of the Lie group.

This section is dedicated to analyse this example.

At first, in Section 5.1, we apply known results for compact Lie groups to show that every compact, even-dimensional Lie group induces a natural holomorphic fibre bundle structure in the sense of Definition 2.13.

In Section 5.2, we recall classical results about the set of flatable smooth complex line bundles over an even-dimensional, compact Lie group G. In particular, we recall that every even-dimensional, compact semi-simple Lie group admits only flatable complex line bundles.
Afterwards, in Subsection 5.3, we recall a commonly known result about holomorphic structures on a complex line bundle which is associated to a principle fibre bundle with discrete fibre.

Later on, in Section 5.4, we examine which holomorphic and Hermitian structures we can endow on our line bundles in order to make them compatible with the holomorphic fibre bundle structure of $G \to G/T$.
In Sub-Subsection 5.4.1, we show that $\tilde{G} \times_\chi \mathbb{C}$ becomes a smooth vector bundle over the fibre bundle $G \to G/T$, i.e. $\mathfrak{L} = \tilde{G} \times_{\tilde{\rho}} \tilde{\mathfrak{L}} \to G/T$.
Further on, in Sub-Subsections 5.4.2 as well as 5.4.3, we show that the natural holomorphic structure and Hermitian metric on $\tilde{\mathfrak{L}}$ are respected by the action $\tilde{\rho}$.
Additionally, we derive the implications for the possible holomorphic structures on $\mathfrak{L}$ in Sub-Subsection 5.4.4.
At last, in Sub-Subsection 5.4.5, we generalise the result for the Laplace splitting property, shown by Stanton, to the more general holomorphic structures on $\mathfrak{L}$.

In Section 5.5, we restate Theorem 4.1 tailored to the situation at hand, i.e. for the set of holomorphic line bundles over G with flatable, smooth complex line bundle structure.
Here, we apply known facts about the kernel of the Laplacian $\Box_{\tilde{\mathfrak{L}}}$ for flat line bundles over the torus.

Finally, in Section 5.6, we look at an easy example for a legitimate action. We take $\gamma^{\mathfrak{L}}$ to be $L_{\tilde{g}_0}$, i.e. the left transition with an element $\tilde{g}_0 \in \tilde{G}$.
Here, we apply classical facts about the left action of G on G/T to obtain a very simple

result for the equivariant ζ-function for this special case of the legitimate action.
Its result for general $\tilde{g}_0 \in \tilde{G}$ is summarised in Theorem 5.2.
At last, for the special case where $\pi_{1,\tilde{G}}(\tilde{g}_0) \in G$ generates a maximal torus, we deduce a very convenient result for the equivariant holomorphic torsion.
It is stated in Corollary 5.49 as well as in Corollary 5.50.

5.1 General setting

The aim of this subsection is to specify the holomorphic fibre bundle of our example. Therefore, we apply common knowledge about compact Lie groups in order to obtain the necessary structures.

Let $E := G$ be a compact, real even-dimensional Lie group with bi-invariant metric g_G. Such a metric always exists since G is compact. In particular, we still have a degree of freedom left because we still may chose $(g_G)_h$ at one point of $h \in G$.
Furthermore, let $F := T \subset G$ be a maximal torus. We denote the Lie algebra of G with $\mathfrak{g}$ and the Lie algebra of T with $\mathfrak{t}$.
We obtain a smooth principle fibre bundle $T \hookrightarrow G \to G/T$ over the homogeneous space G/T.

At first, we have to show that this fibre bundle naturally induces a holomorphic fibre bundle structure in the sense of Definition 2.13.

Remark 5.1:
By prerequisite, G is compact. Therefore, G/T becomes a reductive homogeneous space, i.e. there is an $\mathrm{Ad}\,|_T$ invariant complement $\mathfrak{m}$ of $\mathfrak{t}$, s.t. $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$. This can be seen by the subsequent argument.

Chose $\mathfrak{m}$ to be the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$ for the bi-invariant metric g_G.
Now, g_G is bi-invariant and therefore $\mathfrak{m}$ is $\mathrm{Ad}\,|_T$-invariant.

As a direct consequence, we obtain the following facts.

- For a reductive homogeneous space, the theory of principle fibre bundles and associated bundles (cf. [5]) now gives us the tangent space of the base space, i.e. $T(G/T)$, as an associated vector bundle to the principle fibre bundle $G \to G/T$,

  $$T(G/T) \cong G \times_{\mathrm{Ad},T} \mathfrak{m}.$$

  In particular, every $\mathrm{Ad}\,|_T$-invariant structure on $\mathfrak{m}$ directly induces a corresponding structure on $T(G/T)$.

- For example, we obtain a metric $g_{G/T}$ induced by the bi-invariant metric g_G.

- Furthermore, we obtain a smooth horizontal distribution $T^H G \subset TG$ by left translation of $\mathfrak{m}$, as follows.

Denote, for $g \in G$, the left transition with g with

$$\begin{array}{rccc} L_g: & G & \longrightarrow & G \\ & h & \longmapsto & g \cdot h. \end{array}$$

We define the distribution $T^H_g G := T_e L_g(\mathfrak{m})$ for every g in G.
Observe now that $L_g : G \longrightarrow G$ covers a diffeomorphism $L_g^{G/T}$, i.e. the following diagram commutes.

$$\begin{tikzcd} G \arrow[r, "L_g"] \arrow[d, "\pi_G"'] & G \arrow[d, "\pi_G"] \\ G/T \arrow[r, "L_g^{G/T}"'] & G/T \end{tikzcd}$$

Furthermore, note that $T_e\pi_G : \mathfrak{m} \longrightarrow T_{[e]}G/T$ is an isomorphism.
Consequently, $T^H G$ becomes a horizontal distribution.

So far, we have a fibre bundle $T \hookrightarrow G \to G/T$, a connection $T^H G$ of the fibre bundle $G \to G/T$ and a Riemannian, bi-invariant metric g_G on G. The latter one is fixed up to a choice of a scalar product on $\mathfrak{g}$.
Now, we construct a complex structure for the manifold G.

A known fact for a compact Lie group G is, that its Lie algebra $\mathfrak{g}$ is a product of an Abelian Lie algebra $\mathfrak{h}$ and a semi-simple Lie algebra $\mathfrak{g}_s$ (cf. [18, Ch. 4 Cor. 4.25.]). Furthermore, the the semi-simple part $\mathfrak{g}_s$ possesses a maximal Abelian sub-Lie algebra $\mathfrak{h}_s$ such that $\mathfrak{t} = \mathfrak{h} \oplus \mathfrak{h}_s$.

Using representation theory (cf. [13]) we obtain the following splitting of the complexified semi-simple Lie algebra.

$$\mathfrak{g}_s \otimes_{\mathbb{R}} \mathbb{C} = (\mathfrak{h}_s \otimes_{\mathbb{R}} \mathbb{C}) \oplus \bigoplus_{\alpha \in R^+} (\mathfrak{g}_{s,\alpha} \oplus \mathfrak{g}_{s,-\alpha}) \, .$$

Here, R^+ denotes an arbitrary, but fixed, set of positive roots, g_α denotes the root space for the root α and $h_s \otimes_{\mathbb{R}} \mathbb{C}$ denotes the complexified Cartan algebra.

Using this decomposition, we define the almost complex structure $J_{\mathfrak{g}}$ on the Lie algebra $\mathfrak{g}$ via:

$$\begin{aligned} (\mathfrak{g})^{1,0} &= (\mathfrak{h} \oplus \mathfrak{h}_s)^{1,0} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_{s,\alpha} \\ (\mathfrak{g})^{0,1} &= (\mathfrak{h} \oplus \mathfrak{h}_s)^{0,1} \oplus \bigoplus_{\alpha \in R^-} \mathfrak{g}_{s,\alpha} \end{aligned} \tag{46}$$

where we choose an arbitrary almost complex structure on $\mathfrak{h} \oplus \mathfrak{h}_s = \mathfrak{t}$.

Let J_G be the left translated almost complex structure on G induced by Equation (46), i.e. for any $\tilde{X} \in T_g G$ given by $\tilde{X} = T_e L_g(X)$ we define

$$J_G(\tilde{X}) := T_e L_g(J_{\mathfrak{g}}(X)).$$

Samelson shows in [27] that J_G is integrable, i.e. G becomes a complex manifold. Furthermore, he proves that J_G naturally induces a complex structure $J_{G/T}$ on $M = G/T$.
To be more explicit, Samelson showed that G, T and G/T are complex manifolds with complex structures J_G, $J_T := J_G \mid_{\mathfrak{t}}$ and $J_{G/T}$ induced by $J_G \mid_{\mathfrak{m}}$.
Note that $J_{G/T}$ is a well defined map due to the fact that the root spaces $\mathfrak{g}_{s,\alpha}$ are invariant under $\mathrm{Ad} \mid_T$. It follows that the projection π_G is a holomorphic map.

The metric g_G as well as the complex structure J_G may be chosen compatible on $\mathfrak{g} \subset T_e G$ (in the sense of Definition A.1, i.e. J_G is an isometry of $(\mathfrak{g}, g_G)$).
Now, because they are left invariant, they stay compatible on all of G.
This compatibility extends to (T, J_T, g_T) where g_T is given by g_G through restriction. Furthermore, $J_{G/T}$ and $g_{G/T}$ are compatible because $g_{G/T}$ as well as $J_{G/T}$ are given by restriction of g_G or J_G to the subspace $\mathfrak{m} \subset T_e G$.

We now check that $(G, \pi_G, (G/T, g_{G/T}), (T, g_T), T^H G)$ fulfils the prerequisites of a holomorphic fibre bundle (Definition 2.13).

- The map $\pi_G : G \longrightarrow G/T$ defines a smooth (principle) fibre bundle whose fibretype is the maximal torus T. ✓
- Furthermore, (T, g_T) and $(G/T, g_{G/T})$ are complex manifolds with compatible Riemannian metrics. ✓
- The set G has a complex manifold structure and π_G is a holomorphic map whose differential has constant maximal rank.
  Now, the implicit function theorem for holomorphic functions implies that there are local holomorphic sections

  $$q_k : U_k \longrightarrow G \cap \pi_G^{-1}(U_k).$$

  Consequently, the local trivialisations

  $$\begin{array}{rccc} \phi_k^{-1}: & U_k \times T & \longrightarrow & \pi_G^{-1}(U_k) \\ & (x,t) & \longmapsto & q_k(x) \cdot t \end{array}$$

  are holomorphic maps.
  Therefore, so are the transition functions

  $$\phi_k \circ \phi_l^{-1} : (U_k \cap U_l) \times T \longrightarrow (U_k \cap U_l) \times T.$$

  Now, we apply Lemma 2.16 and obtain that the maps

  $$\phi_{i,x} \circ \phi_{k,x}^{-1} : \quad F \quad \longrightarrow \quad F$$

as well as

$$\widehat{\phi_i \circ \phi_j^{-1}} : \quad U_i \cap U_j \quad \to \quad Hol(F)$$

are holomorphic. ✓

- Additionally, we note that $\phi_{i,x} \circ \phi_{k,x}^{-1}$ is an isometry for each i and each k.
- We have a direct sum decomposition $TG = T^V G \oplus T^H G$, i.e. we have a connection $\mathfrak{m} = T^H G$ on $G \to G/T$. ✓
- The connection $T^H G$ is J_G invariant because $\mathfrak{m}$ is $J_G \mid_{T_e G}$ invariant by construction and because the J_G is given by left transition of $J_G \mid_{T_e G}$. ✓
- The connection $\mathfrak{m}$ is of type $(1,1)$ because

$$[\mathfrak{g}_{s,\alpha}, \mathfrak{g}_{s,\beta}] \subset \mathfrak{g}_{s,\alpha+\beta} \tag{47}$$

  and because $\alpha + \beta \in R^-$ if α and β are negative roots (cf. [13]).

Finally, our chosen metric g_G fulfils the properties of Remark 2.17 which is shown in the subsequent consideration.

- By construction, the horizontal space and the vertical space are orthogonal with respect to g_G. ✓
- Furthermore, g_G is left invariant. Hence, every inclusion of the fibre $t \mapsto g \cdot t$ is an isometric immersion. ✓
- The projection π_G is a Riemannian submersion which follows directly from the definition of $g_{G/T}$. ✓

We summarise the information we collected so far in the following corollary.

Corollary 5.2:
The tuple $\big(G, \pi_G, \big(G/T, g_{G/T}\big), (T, g_T), T^H G\big)$*, with notations from above, is a holomorphic fibre bundle in the sense of Definition 2.13.*

From now on, throughout this section $\big(G, \pi_G, \big(G/T, g_{G/T}\big), (T, g_T), T^H G\big)$ denotes the holomorphic fibre bundle defined in the preceding subsection.

We have shown that $G \to G/T$ becomes a holomorphic fibre bundle in a natural way.

In the next subsection, we recall some well known facts about the set of smooth complex line bundles over Lie groups.

5.2 Line bundles over even-dimensional Lie groups

Now, that we have seen that $\left(G, \pi_G, \left(G/T, g_{G/T}\right), (T, g_T), T^H G\right)$ becomes a holomorphic fibre bundle, we study the set of holomorphic line bundles over G. This is the content of this subsection.

The results of this subsection are commonly known. Nonetheless, we give a small survey of the facts for the convenience of the reader.

Remark 5.3:
This subsection is not necessary for the understanding of the rest of this thesis. It merely states what we lose when we restrict to the case where the line bundles $\mathfrak{L}$ over G are associated to the universal covering principle fibre bundle $\pi_{1,\tilde{G}} : \tilde{G} \to G$ via a representation $\chi : \pi_1(G) \to \mathbb{C}^*$.

A natural question about holomorphic line bundles over compact even dimensional Lie groups is the following: How many isomorphism classes of holomorphic line bundles exist? In order to answer this question at least partially, we describe at first the isomorphism classes of complex line bundles, i.e. we ignore their holomorphic structures at first.
It is a known fact that the isomorphism classes of complex line bundles over G form an Abelian group and that the first Chern class describes an isomorphism between this group and $H^2(G, \mathbb{Z})$ (compare [14]). Therefore, in order to answer our question, we have to study the group $H^2(G, \mathbb{Z})$.
For our example, we want to study flat line bundles over a compact Lie group. Therefore, it suggests itself to investigate how big the restriction from arbitrary line bundles to flat line bundles really is. This is the aim of the subsequent subsection.

5.2.1 Flat line bundles over compact Lie groups

We denote the universal covering space of the Lie group G with the symbol $\tilde{G}$.
The space $\tilde{G}$ has a natural Lie group structure such that the projection

$$\pi_{1,\tilde{G}} : \quad \tilde{G} \quad \longrightarrow \quad G$$

becomes a Lie group homomorphism.
In particular, the map $\pi_{1,\tilde{G}}$ describes a $\pi_1(G)$-principle fibre bundle over G.

$$\begin{tikzcd} \pi_1(G) \arrow[r, hook] & \tilde{G} \arrow[d, "\pi_{1,\tilde{G}}"] \\ & G \end{tikzcd}$$

Furthermore, $\pi_1(G) = \pi_{1,\tilde{G}}^{-1}(e)$ is embedded as a subgroup in $\tilde{G}$.

The following facts are well known, nonetheless, we summarise them here for the convenience of the reader. Additionally, we give a sketch of the proof.

Lemma 5.4:
Let $\mathfrak{L}$ be a complex line bundle over a connected Lie group G, then the following properties are equivalent.

1. *The line bundle $\mathfrak{L}$ is flat, i.e. admits a flat connection.*
2. *The first real Chern class of $\mathfrak{L}$ vanishes.*
3. *The bundle $\mathfrak{L}$ is an associated vector bundle via a representation χ of the fundamental group of G, i.e. $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$.*

Remark 5.5:
The fact that every flat vector bundle is associated to $\tilde{G}$ via a representation of the fundamental group is a special case of the so-called **Riemann-Hilbert correspondence**.

Proof of Lemma 5.4.

- 1. and 2. are equivalent because of the following argument.
  Let $\mathfrak{L}$ be flat and let $\nabla^{\mathfrak{L}}$ be its flat connection, i.e $\left(\nabla^{\mathfrak{L}}\right)^2 = 0$.
  We apply Chern-Weil theory (cf. [5]) and obtain:
  $$c_1(\mathfrak{L}) \otimes_{\mathbb{Z}} \mathbb{R} = \left[-\frac{1}{2\pi i}\left(\nabla^{\mathfrak{L}}\right)^2\right] = 0.$$
  On the other hand, the same equation shows that if $c_1(\mathfrak{L}) \otimes_{\mathbb{Z}} \mathbb{R} = 0$, then there exists a connection $\nabla^{\mathfrak{L}}$ such that $\left(\nabla^{\mathfrak{L}}\right)^2 = 0$.
- That 1. follows from 3. can be seen as follows.
  Let $\mathfrak{L}$ be associated to a representation of $\pi_1(G)$, i.e. $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$.
  We can construct a covariant derivative ∇^A on $\mathfrak{L}$ through a connection one form A on $\tilde{G}$ (cf. [5]). On the other hand, $\tilde{G} \to G$ is a discrete covering. In particular, the vertical space at each point is zero.
  Thus, there is but one connection one form, namely $A = 0$.
  Now, $A = 0$ implies $(\nabla^A)^2 = 0$.
- What remains to be shown is that 3. follows from $\mathfrak{L}$ being flat.
  Therefore, suppose that $\nabla^{\mathfrak{L}}$ is a flat connection on $\mathfrak{L}$.
  Now, although we don't want to outline the whole theory, we apply some properties from holonomy theory (cf. [5]).
  Every vector bundle is associated to a connected K-principle fibre bundle, now called P, i.e. $\mathfrak{L} = P \times_\rho \mathbb{C}$.
  By prerequisite, $\nabla^{\mathfrak{L}}$ is flat. Now, the theorem of **Ambrose-Singer** (cf. [5, Ch. 4. Satz 4.5.]) states that the holonomy group $\mathrm{Hol}(\nabla^{\mathfrak{L}})$ of $\nabla^{\mathfrak{L}}$ is discrete.

This, on the other hand, implies (cf. again [5, Ch. 4. Satz 4.4.]) that P can be reduced to a connected principle fibre bundle Q with discrete structure group such that $\mathfrak{L} = Q \times_{\hat{\chi}} \mathbb{C}$. Furthermore, the connection $\nabla^{\mathfrak{L}}$ comes from a connection one form on Q.

$$\begin{tikzcd}
\mathrm{Hol}(\nabla^{\mathfrak{L}}) \arrow[r, hook] \arrow[d, "i_{\mathrm{Hol}}"'] & Q \arrow[d, "i_Q"] \\
K \arrow[r, hook] & P \arrow[d, "\pi_P"] \\
 & G
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \times \mathbb{C} \arrow[rr, "i_Q \times \mathrm{id}_{\mathbb{C}}"] \arrow[dr] & & P \times \mathbb{C} \arrow[dl] \\
 & Q \times_{\hat{\chi}} \mathbb{C} \cong \mathfrak{L} \cong P \times_\rho \mathbb{C} \arrow[d] & \\
 & G &
\end{tikzcd}$$

But, the fibre $Q \to G$ is discrete and Q is connected. Therefore, Q becomes a connected covering of G and thus, there is a covering $\pi : \tilde{G} \to Q$.
It follows that $\mathfrak{L}$ is associated to $\tilde{G}$, i.e. $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$ where χ denotes the representation of $\pi_1(G)$ induced by $\hat{\chi}$.

□

We conclude that the first integer Chern class $c_1(\mathfrak{L})$ of a flat line $\mathfrak{L}$ has to lie in the torsion ideal of the cohomology ring with integer coefficients.

Corollary 5.6:
A Lie group G admits no non-flat complex line bundles if and only if the second cohomology group with coefficients in $\mathbb{R}$ vanishes, i.e. $H^2(G, \mathbb{R}) = 0$.

Remark 5.7:
An arbitrary Lie group has, in general, a non-vanishing second cohomology group $H^2(G, \mathbb{R})$.
Take, for instance, the torus $T = \mathbb{C}^n/\Lambda$ (Λ being a lattice in $\mathbb{C}^n$ of maximal rank).
We obtain, by [7, Ch. 1.3. Lemma 1.3.1], that

$$H^2(T, \mathbb{Z}) = \Lambda^2 \mathrm{Hom}(\pi_1(T), \mathbb{Z}) \cong \mathbb{Z}^{\binom{2n}{2}}.$$

In particular, $H^2(T, \mathbb{Z})$ has no torsion at all, and consequently, $H^2(T, \mathbb{R}) \cong \mathbb{R}^{\binom{2n}{2}}$ does not vanish.
Even more, if $H^2(T, \mathbb{Z})$ has no torsion, every flat line bundle $\mathfrak{L} \to T$ has to be trivial as a smooth complex line bundle.

It is a known fact that every Abelian, compact, connected Lie group is isomorphic to a torus. Therefore, we look at maximal non-Abelian Lie groups, the semi-simple Lie groups. This is the content of the subsequent subsection.

5.2.2 The second cohomology group of compact Lie groups

The main objective of this subsection is to show that every complex line bundle over a compact, semi-simple Lie group admits a flat connection, i.e. is flat.

First, we state some facts about compact, connected Lie groups in general. Later on, we restrict to the semi-simple case.

The cohomology of a Lie group G with values in $\mathbb{Z}$ is strongly related to its homotopy groups.
This helps us significantly because there is a general result about the second homotopy group of a Lie group stated in [9] which states that for a connected Lie group G the second homotopy vanishes, i.e. $\pi_2(G) = 0$.

This fact has a nice well known consequence.

> **Corollary 5.8:**
> *Every complex line bundle $\hat{\mathfrak{L}}$ over $\tilde{G}$ is trivial.*
> *In particular, we obtain that for every line bundle $\mathfrak{L} \to G$, the bundle $\tilde{\pi}^*\mathfrak{L} \to \tilde{G}$ is isomorphic to the trivial line bundle.*

Proof.
The assertion is, more or less, a direct consequence of the theorem of Hurewicz (cf. [15, Ch. 4, Thm. 4.37]).
In our case, we have for any base point $\tilde{g} \in \tilde{G}$ that the first two homotopy groups of $\tilde{G}$ vanish, i.e.

$$\pi_1(\tilde{G}, \tilde{g}) \cong \pi_2(\tilde{G}, \tilde{g}) \cong 0.$$

The theorem of Hurewicz now implies the existence of an isomorphism of groups

$$\pi_1(\tilde{G}, \tilde{g}) \cong H_1(\tilde{G}, \mathbb{Z}) \cong 0 \cong \pi_2(\tilde{G}, \tilde{g}) \cong H_2(\tilde{G}, \mathbb{Z}).$$

In order to make any predication on the cohomology of $\tilde{G}$, we now apply the universal coefficient theorem for cohomology (cf. [15, Ch. 3, Thm. 3.2]) which states that there is an exact sequence

$$0 \to \mathrm{Ext}(H_{n-1}(\tilde{G}, \mathbb{Z}), \mathbb{Z}) \to H^n(\tilde{G}, \mathbb{Z}) \to \mathrm{Hom}\Big(H_n(\tilde{G}, \mathbb{Z}), \mathbb{Z}\Big) \to 0$$

for any $n \in \mathbb{N}$.
Hence, for the special case $n = 2$, we obtain $H^2(\tilde{G}, \mathbb{Z}) \cong 0$ since $H_1(\tilde{G}, \mathbb{Z})$ as well as $H_2(\tilde{G}, \mathbb{Z})$ vanish.
We conclude that the second cohomology of $\tilde{G}$ vanishes.
Furthermore, the first Chern class is a bijection between isomorphism classes of complex line bundles and the second cohomology group. Consequently, we observe that $\tilde{G}$ admits only one isomorphism-class of complex line bundles, namely the trivial one. □

This is as far as we get, assuming G to be an arbitrary compact, connected Lie group. We already stated that not every complex line bundle over the complex torus is flat or trivial (compare Remark 5.7).
Thus, we have to restrict our consideration at this point to compact, semi-simple Lie groups.

A well known fact for a compact, semi-simple Lie group G is that its second real cohomology vanishes, i.e. $H^2(G, \mathbb{R})$ (cf. [26]).

Therefore, the following corollary summarises what statements we obtain for line bundles over semi-simple, compact, connected Lie groups.

Corollary 5.9:
Let G be a compact, connected and semi-simple Lie group.
Then every line bundle $\mathfrak{L}$ admits a flat connection. Furthermore, there is a representation $\chi : \pi_1(G) \to U(1)$ such that $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$.
We can restrict to the case of unitary representations χ here, due to the fact that $\pi_1(G)$ is finite, i.e. compact.

5.3 Holomorphic structures on associated line bundles

Up to now, we tried to understand the set of smooth complex line bundles over compact, connected Lie groups. It is now time to focus on the holomorphic structure if there is one.

Let from now on G be an even-dimensional, compact Lie group. And let $\mathfrak{L}$ be the associated line bundle $\mathfrak{L} := \tilde{G} \times_\chi \mathbb{C}$ for a representation $\chi : \pi_1(G) \to \mathbb{C}^*$.

First of all, we show in the following lemma that $\mathfrak{L}$ admits a natural holomorphic structure.

Lemma 5.10:
In the setting from above, i.e. $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$, there is a natural holomorphic structure on $\mathfrak{L}$ induced by χ.

Proof.
The theory of associated vector bundles (cf. [5]) gives us a natural isomorphism between differential forms on the base G with coefficients in the associated vector bundle on the one hand and on the other hand horizontal differential forms on the total space of the principle fibre bundle with values in the fibretype of the vector bundle that are χ equivariant.
We denote this isomorphism by κ_χ, i.e.

$$\kappa_\chi : \quad \mathfrak{A}^*(G, \tilde{G} \times_\chi \mathbb{C}) \quad \longrightarrow \quad \mathfrak{A}^*(\tilde{G}, \mathbb{C})^{\chi,\mathrm{hor}}.$$

The morphism κ_χ is given as follows.

Let $\alpha \in \mathfrak{A}^*(G)$ and let s be a section $s \in \Gamma(G, \mathfrak{L})$, then

$$\kappa_\chi(\alpha \otimes s) := \left(\pi_{1,\tilde{G}}^* \alpha\right) \otimes \kappa_\chi(s).$$

Furthermore, κ_χ evaluated on the section s is given implicitly for any $g \in G$ and every $\tilde{g} \in \pi_{1,\tilde{G}}^{-1}(g)$ by:

$$s(g) = [\tilde{g}, (\kappa_\chi(s))(\tilde{g})]_\chi.$$

Fortunately, the principle fibre bundle is a discrete covering. Hence, every differential form is horizontal.
In addition, the covering $\tilde{G} \to G$ is holomorphic by construction, i.e. respects the complex structure.
Thus, we obtain

$$\kappa_\chi : \ \mathfrak{A}^{(0,*)}(G, \tilde{G} \times_\chi \mathbb{C}) \ \longrightarrow \ \mathfrak{A}^{(0,*)}(\tilde{G})^\chi.$$

On the space on the right hand side, we have a natural $\bar{\partial}$-operator,

$$\bar{\partial} : \ \mathfrak{A}^{(0,*)}(\tilde{G})^\chi \ \longrightarrow \ \mathfrak{A}^{(0,*)}(\tilde{G})^\chi.$$

It maps χ-equivariant forms to χ-equivariant forms because, on the one hand, the deck transformations are holomorphic while, on the other hand, $\chi(\sigma) \in \mathbb{C}^*$ becomes a linear map for every $\sigma \in \pi_1(G)$.
Now, the natural holomorphic structure on $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$ is defined via the following diagram.

$$\begin{tikzcd}
\Gamma(G, \tilde{G} \times_\chi \mathbb{C}) \arrow[r, "\bar{\partial}_{\mathfrak{L}}"] \arrow[d, "\kappa_\chi"'] & \mathfrak{A}^{(0,1)}(G, \tilde{G} \times_\chi \mathbb{C}) \\
C^\infty(\tilde{G})^\chi \arrow[r, "\bar{\partial}"'] & \mathfrak{A}^{(0,1)}(\tilde{G})^\chi \arrow[u, "\kappa_\chi^{-1}"']
\end{tikzcd}$$

Obviously, $\bar{\partial}_{\mathfrak{L}}^2 = 0$, and additionally, an easy computation shows that $\bar{\partial}_{\mathfrak{L}}$ indeed fulfils the Leibniz equation.
Consequently, it defines a holomorphic structure on $\mathfrak{L}$. □

This holomorphic structure, defined in Lemma 5.10, is almost never the only one.
We have seen in Lemma 2.4 that for holomorphic line bundles $\mathfrak{L} \to G$, the space of holomorphic structures is an affine space over the vector space $\mathrm{Ker}\left(\bar{\partial}\mid_{\mathfrak{A}^{(0,1)}(G)}\right)$.

Now, that we have explained what line bundles we initially want to look at, namely those that are associated to a representation of $\pi_1(G)$, it is time to grind this setting to get compatible line bundles over the holomorphic fibre bundle
$(G, \pi_G, (G/T, g_{G/T}), (T, g_T), T^H G)$.

5.4 Compatible line bundles

This section is dedicated to understand the conditions that the line bundle $\mathfrak{L}$ given by $\tilde{G} \times_\chi \mathbb{C}$ has to fulfil in order to be compatible in the sense of Definition 3.1.

Not every holomorphic line bundle $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$ is compatible with our holomorphic fibre bundle $(G, \pi_G, (G/T, g_{G/T}), (T, g_T), T^H G)$.

In order to check which constraints we have to enforce in order to achieve the compatibility, we investigate each of the three defining properties of Definition 3.1 individually.

This subsection is split into six sub-subsections.
At first in Section 5.4.1, we investigate the smooth line bundle property.
In particular, we state a line bundle $\tilde{\mathfrak{L}} \to T$ for a given line bundle $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$ such that $\mathfrak{L} \to G/T$ becomes a smooth fibre bundle whose fibre type is the line bundle $\tilde{\mathfrak{L}} \to T$.
Furthermore, we show that $\mathfrak{L}$ becomes a smooth line bundle over the fibre bundle $G \to G/T$ in the sense of Definition 2.36. Its Lie transformation group $\hat{T}$ is a discrete covering of T.
We already stated that every flat line bundle on the torus is trivial. In Section 5.4.2, we recall how the representations $\chi : \pi_1(G) \to U(1)$ induce different holomorphic structures on $\tilde{\mathfrak{L}} = T \times \mathbb{C}$.
Additionally, we show that $\hat{T}$ respects this holomorphic structure.
In Section 5.4.3, we equip $\tilde{\mathfrak{L}}$ as well as $\mathfrak{L}$ with a Hermitian metric and we show that $\hat{T}$ acts Hermitian.
Directly thereafter, in Section 5.4.4, we deduce which holomorphic structures on $\mathfrak{L}$ we may admit such that the holomorphic structure on $\tilde{\mathfrak{L}}$ is induced by the holomorphic structure on $\mathfrak{L}$.
The last remaining property of Definition 3.1, i.e. the Laplace splitting property, is investigated in Section 5.4.5.
Finally, in Section 5.4.6, we summarise the previous sub-subsections by giving sufficient conditions for $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$ to be compatible.

5.4.1 Possible line bundles over the maximal torus

This sub-subsection is dedicated to show that $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$ is smooth in the sense of Definition 2.36. In particular, we state a group homomorphism $\tilde{\rho}$ such that $\mathfrak{L}$ is associated to the principle fibre bundle $\tilde{G} \to G/T$ via $\tilde{\rho}$.

If we want $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$ to be a compatible line bundle, we have to ask what its fibre type $\tilde{\mathfrak{L}} \to T$ as a bundle over G/T should be.
The natural choice of the bundle $\tilde{\mathfrak{L}} \to T$ would be the restriction $\tilde{\mathfrak{L}} := \mathfrak{L} \mid_T = \pi_{\mathfrak{L}}^{-1}(T)$.
This is by construction a smooth complex line bundle over T.

Now, $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$ is an associated vector bundle to the principle fibre bundle $\tilde{G} \to G$.
Therefore, it sounds plausible that $\tilde{L}$ is associated via a representation of $\pi_1(G)$ as well.

We show that this is indeed true.

But before we do this, we recall some facts from the theory of covering spaces of compact Lie groups.
This is essentially a repetition of what Stanton does in [29].

- Observe that G/T is a simply connected, connected space for a connected Lie group G with maximal torus T.
- For the fibre bundle $T \overset{i}{\hookrightarrow} G \overset{\pi_G}{\to} G/T$ we get a long exact sequence of homotopy groups (compare [15, Ch. 4, Thm. 4.41])
$$0 = \pi_1(G/T) \leftarrow \pi_1(G) \overset{i_*}{\leftarrow} \pi_1(T) \overset{\delta}{\leftarrow} \pi_2(G/T) \leftarrow 0 = \pi_2(G).$$
It follows that δ is injective, i.e. we get a subgroup
$$\mathrm{im}(\delta) = \delta(\pi_2(G/T)) \subset \pi_1(T).$$
Hence, we get a covering $\hat{T} \to T$ corresponding to this subgroup $\mathrm{im}(\delta)$.
- In [29, Ch. 8], Stanton shows explicitly that there is a $\hat{T}$-principle fibre bundle $\hat{T} \hookrightarrow \tilde{G} \to G/T$ such that we obtain the homogeneous space $G/T = \tilde{G}/\hat{T}$.
Its deck transformation group is $\pi_1(T)/\mathrm{im}(\delta) \cong \pi_1(G)$ since $\pi_1(T) \cong \mathbb{Z}^k$ is Abelian and because the sequence above is exact.
In particular, the following diagram commutes.
$$\begin{tikzcd}
\hat{T} \arrow[r, hook, "\hat{i}"] \arrow[d, "\pi_{1,\hat{T}}"'] & \tilde{G} \arrow[d, "\pi_{1,\tilde{G}}"'] \arrow[dr, "\pi_{\tilde{G}/\hat{T}}"] & \\
 & & G/T \\
T \arrow[r, hook, "i"'] & G \arrow[ur, "\pi_{G/T}"'] &
\end{tikzcd}$$

We use these facts to prove the following lemma.

Lemma 5.11:
Let χ be a character of $\pi_1(G)$, i.e. a representation of $\pi_1(G)$, such that $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$.
Additionally, let i denote the inclusion $i : T \hookrightarrow G$.
Then the bundle $i^*\mathfrak{L} \to T$ is associated to the principle fibre bundle
$$\begin{tikzcd}
\pi_1(G) \arrow[r, hook] & \hat{T} \arrow[d, "\pi_{1,\hat{T}}"] \\
 & T
\end{tikzcd}$$
via the same representation χ, i.e.
$$\tilde{\mathfrak{L}} := i^*\mathfrak{L} \cong \hat{T} \times_\chi \mathbb{C}.$$

Proof.
By prerequisite, the bundle $\mathfrak{L}$ is associated to $\tilde{G} \to G$, i.e. $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$.
Hence, every element in $\tilde{\mathfrak{L}}$ has the form $(t, [\hat{t}, z]_\chi)$ such that

$$\pi_{1,\hat{T}}(\hat{t}) = \pi_{1,\tilde{G}}(\hat{t}) = t.$$

We write down the isomorphism explicitly:

$$\begin{array}{ccc} \tilde{\mathfrak{L}} & \longrightarrow & \hat{T} \times_\chi \mathbb{C} \\ (t, [\hat{t}, z]_\chi) & \longmapsto & [\hat{t}, z]_\chi \,. \end{array}$$

This map is well defined, smooth, bijective and $\mathbb{C}$-linear on fibres. □

Remark 5.12:
One might ask, why we made it so explicit when the following simple argument would have sufficed.

The vector bundle $\mathfrak{L}$ is flat and so is its restriction $\tilde{\mathfrak{L}}$.
This implies, by Remark 5.7, that $\tilde{\mathfrak{L}}$ is isomorphic to the trivial line bundle as a smooth, complex vector bundle.

Although, this is essentially true, we don't want to look at the complex vector bundle structure of $\tilde{\mathfrak{L}} \to T$ only, and there is a natural holomorphic structure on $\tilde{\mathfrak{L}} \to T$ induced by the representation χ. (cf. Lemma 5.10)

Now, that we have a candidate for the bundle $\tilde{\mathfrak{L}} \to T$, we still have to show that $\mathfrak{L} \to G$ is a smooth fibre bundle in the sense of Definition 2.36, and that its structure group is indeed a Lie transformation group.
This is the content of the subsequent proposition.

Proposition 5.13:
Let $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$ be a complex line bundle over G associated to a representation χ of $\pi_1(G)$.
Then $\mathfrak{L} = \tilde{G} \times_{\tilde{\rho},\hat{T}} \tilde{\mathfrak{L}}$ becomes a smooth fibre bundle over G/T associated to the principle fibre bundle $\hat{T} \hookrightarrow \tilde{G} \to G/T$.
The action $\tilde{\rho}$ is given by

$$\begin{array}{cccc} \tilde{\rho}: & \hat{T} & \longrightarrow & \mathrm{Diff}(\tilde{\mathfrak{L}}, \tilde{\mathfrak{L}}) \\ & \hat{s} & \longmapsto & \left\{ [\hat{t}, z]_\chi \mapsto [\hat{t}\hat{s}, z]_\chi \right\}. \end{array}$$

In particular, we obtain $G = \tilde{G} \times_\rho T \to G/T$ for an action ρ given by:

$$\begin{array}{cccc} \rho: & \hat{T} & \longrightarrow & \mathrm{Diff}(T, T) \\ & \hat{s} & \longmapsto & \left\{ t \mapsto \pi_{1,\hat{T}}(\hat{s}) \cdot t \right\}. \end{array}$$

Proof.
At first, we show that $\tilde{\rho}$ is a well defined group action.
Note that $\pi_{1,\hat{T}} : \hat{T} \to T$ is a Lie group homomorphism, hence, $\pi_1(G) = \mathrm{Ker}(\pi_{1,\hat{T}})$ is a subgroup of $\hat{T}$.
This sub-group-property is all we need to show $\tilde{\rho}$ is well defined.
For $\sigma \in \pi_1(G)$, we get:

$$\tilde{\rho}(\hat{s})(\left[\hat{t} \cdot \sigma, \chi(\sigma^{-1})(z)\right]_{\chi}) = \left[\sigma \cdot \hat{t}\hat{s}, \chi(\sigma^{-1})(z)\right]_{\chi} = \left[\hat{t}\hat{s}, z\right]_{\chi} \stackrel{!}{=} \tilde{\rho}(\hat{s})(\left[\hat{t}, z\right]_{\chi}).$$

Now, that we have made sense of the bundle $\tilde{G} \times_{\tilde{\rho},\hat{T}} \tilde{\mathfrak{L}}$, we may try to write down an isomorphism between $\mathfrak{L} \to G/T$ and $\tilde{G} \times_{\tilde{\rho},\hat{T}} \tilde{\mathfrak{L}} \to G/T$.
An Ansatz for this morphism is:

$$\begin{array}{rccc} \nu : & \tilde{G} \times_{\tilde{\rho},\hat{T}} \tilde{\mathfrak{L}} & \to & \mathfrak{L} \\ & \left[\tilde{g}, \left[\hat{t}, z\right]_{\chi}\right]_{\tilde{\rho}} & \mapsto & \left[\tilde{g} \cdot \hat{t}, z\right]_{\chi}. \end{array} \tag{48}$$

This map induces the identity map on the base space G/T since

$$\pi_{\tilde{G}/\hat{T}}(\tilde{g}) = \pi_{\tilde{G}/\hat{T}}(\tilde{g} \cdot \hat{t}) \in G/T.$$

Thus, the following diagram commutes:

$$\begin{tikzcd} \tilde{G} \times_{\tilde{\rho},\hat{T}} \tilde{\mathfrak{L}} \arrow[rr, "\nu"] \arrow[dr] & & \mathfrak{L} \arrow[dl] \\ & G/T & \end{tikzcd}.$$

Furthermore, ν is obviously surjective and smooth.
Hence, what remains to be shown is that ν is injective.
Suppose therefore that we have elements $\tilde{g}_k \in \tilde{G}$, $\hat{t}_k \in \hat{T}$ and $z_k \in \mathbb{C}$ such that

$$\left[\tilde{g}_0\hat{t}_0, z_0\right]_{\chi} = \left[\tilde{g}_1\hat{t}_1, z_1\right]_{\chi}.$$

We obtain:

$$\pi_{1,\tilde{G}}(\tilde{g}_0\hat{t}_0) = \pi_{\mathfrak{L}}\left(\left[\tilde{g}_0\hat{t}_0, z_0\right]_{\chi}\right) \stackrel{!}{=} \pi_{\mathfrak{L}}\left(\left[\tilde{g}_1\hat{t}_1, z_1\right]_{\chi}\right) = \pi_{1,\tilde{G}}(\tilde{g}_1\hat{t}_1).$$

It follows that there is an element σ in $\pi_1(G)$ such that

$$\tilde{g}_0\hat{t}_0 \cdot \sigma = \tilde{g}_1\hat{t}_1 \quad \text{or equivalently} \quad \tilde{g}_1 = \tilde{g}_0\hat{t}_0\sigma\hat{t}_1^{-1}.$$

Consequently, we obtain $\left[\tilde{g}_1\hat{t}_1, z_1\right]_{\chi} = \left[\tilde{g}_0\hat{t}_0, \chi(\sigma)(z_1)\right]_{\chi}$ which directly implies that z_0 equals $\chi(\sigma)(z_1)$.
We conclude:

$$\left[\tilde{g}_0, \left[\hat{t}_0, z_0\right]_{\chi}\right]_{\tilde{\rho}} = \left[\tilde{g}_0, \left[\hat{t}_0\sigma, z_1\right]_{\chi}\right]_{\tilde{\rho}} = \left[\tilde{g}_0\hat{t}_1^{-1}\hat{t}_0\sigma, \left[\hat{t}_1, z_1\right]_{\chi}\right]_{\tilde{\rho}} = \left[\tilde{g}_1, \left[\hat{t}_1, z_1\right]_{\chi}\right]_{\tilde{\rho}}$$

which proofs the injectivity of ν. □

Remark 5.14:
The embedding $\hat{T}$ into $\mathrm{Diff}(T, \tilde{\mathfrak{L}})$ is continuous because for arbitrary $\hat{t}$ and sufficiently small $\varepsilon > 0$, the set $B_\varepsilon(\hat{t}) := \left\{\hat{s} \mid d_{\text{geod.}}(\hat{s}, \hat{t}) < \varepsilon\right\}$ maps into $U_{\varepsilon,\delta}$ (compare with Definition 2.33) for arbitrary small δ since

$$\begin{aligned} &a) \quad d_{\text{geod.}}\left(\Pi(\hat{s}\hat{t}^{-1})(x), x\right) = d_{\text{geod.}}\left(\hat{s}, \hat{t}\right) < \varepsilon && \forall x \in T \\ &b) \quad \tilde{\rho}(\hat{s}\hat{t}^{-1})\left(\left[\hat{x}, z\right]_\chi\right) = \left[\hat{x}\hat{s}\hat{t}^{-1}, z\right]_\chi = C\left(\Pi(\hat{s}\hat{t}^{-1})\left(\left[\hat{x}, z\right]_\chi\right)\right). \end{aligned}$$

The notations Π and C are transferred from Definition 2.33 as well.
On the other hand, the map

$$\begin{array}{ccc} \hat{T} \times \tilde{\mathfrak{L}}_\chi & \longrightarrow \tilde{\mathfrak{L}}_\chi & \\ (\hat{s}, \left[\hat{t}, z\right]_\chi) & \longmapsto & \left[\hat{s}\hat{t}, z\right]_\chi \end{array}$$

is smooth which makes $\hat{T}$ a Lie transformation group of $\tilde{\mathfrak{L}}_\chi$.

Corollary 5.15:
Let χ be a character of $\pi_1(G)$.
Then $\mathfrak{L} := \tilde{G} \times_\chi \mathbb{C}$ becomes a smooth, complex line bundle over the fibre bundle $G \to G/T$ in the sense of Definition 2.36.
Its fibre type is the smooth, complex line bundle $\tilde{\mathfrak{L}} = \hat{T} \times_\chi \mathbb{C} \to T$.

The aim of this subsection, i.e. Section 5.4, is to show that $\mathfrak{L}$ becomes compatible. Therefore, in order to fulfil the first property of Definition 3.1, we still have to show that $\tilde{\rho}$ respects the holomorphic structure and acts Hermitian.
But, so far, we did not introduce a holomorphic and Hermitian structure on $\tilde{\mathfrak{L}}$.
The following two sub-subsections are dedicated to describe these two structures on $\tilde{\mathfrak{L}}$ and to show that $\tilde{\rho}$ indeed acts on $\tilde{\mathfrak{L}}$ as wanted, i.e. Hermitian and respecting the holomorphic structure.

5.4.2 Admissible holomorphic structures on the line bundle over the torus

This sub-subsection is dedicated to describe the set of holomorphic structures on the complex line bundle $\tilde{\mathfrak{L}} = \hat{T} \times_\chi \mathbb{C}$ and to show that $\tilde{\rho}$ respects these holomorphic structures.

Recall that we saw in Section 5.3 that every the representation $\chi : \pi_1(G) \to \mathbb{C}^*$ induces a holomorphic structure on $\tilde{\mathfrak{L}} = \hat{T} \times_\chi \mathbb{C}$. On the other hand, every flat complex line bundle over the torus is trivial (compare Remark 5.7).
Thus, we study the different holomorphic structures induced by those representations χ on the trivial line bundle.

Although, it is not necessary, for the rest of this thesis, to understand the different holomorphic structures on the trivial line bundle $\tilde{\mathfrak{L}}_1 := T \times \mathbb{C}$ induced by different characters χ, we investigate this for the convenience of the reader.

The reader, who wants to skip this investigation, may resume reading at Lemma 5.21 on page 116.

The general theory of holomorphic line bundles over complex tori is described and proven in [7]. We are interested in the group $\mathrm{Pic}^0(T)$ only, i.e. the holomorphic line bundles that are isomorphic to the trivial line bundle as smooth complex vector bundles. Therefore, we will not review the whole theory here.

However, there is one fact stated in [7] that we need here. It is a statement included in the **Appell-Humbert Theorem**.

Theorem (Appell-Humbert)**:**
Let Λ be a lattice of maximal rank in $\mathbb{C}^n$ and let further on T be the quotient space $\mathbb{C}^n/\Lambda$, then the map

$$\mathrm{Hom}(\Lambda, U(1)) \quad \longrightarrow \quad \mathrm{Pic}^0(T)$$

which sends every unitary character $\chi : \Lambda = \pi_1(T) \longrightarrow U(1)$ to its associated, holomorphic vector bundle $\mathbb{C}^n \times_\chi \mathbb{C}$ is an isomorphism of groups.

Remark 5.16:
The theorem of Appell-Humbert implies that we obtain all possible equivalence classes of holomorphic structures on $T \times \mathbb{C}$ that way.
In particular, it suffices to chose unitary representations χ, although $\pi_1(T)$ is not finite.

Now, in order to compute the different holomorphic structures on $\tilde{\mathfrak{L}}_1$, we write down an explicit isomorphism of vector bundles between $\tilde{\mathfrak{L}}_1$ and $\tilde{\mathfrak{L}}_\chi := \mathbb{C}^n \times_\chi \mathbb{C}$.

Lemma 5.17:
Let $\chi : \Lambda \longrightarrow U(1)$ be a character and let T be a torus given by $T = \mathbb{C}^n/\Lambda$ for a lattice $\Lambda \subset \mathbb{C}^n$ of maximal rank.
Furthermore, let ω_χ be in $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{R})$ such that $\chi(\lambda) = e^{i\omega_\chi(\lambda)}$ for every $\lambda \in \Lambda$.

1. *We get an isomorphism of smooth vector bundles induced by ω_χ:*

$$\begin{tikzcd} \tilde{\mathfrak{L}}_1 \arrow[rr, "\psi_{\omega_\chi}"] \arrow[dr] & & \tilde{\mathfrak{L}}_\chi \arrow[dl] \\ & T & \end{tikzcd} \qquad \text{via} \qquad \begin{tikzcd} \left([\hat{t}]\, , z\right) \arrow[rr, mapsto, "\psi_{\omega_\chi}"] \arrow[dr, mapsto] & & \left[\hat{t}, e^{-i\omega_\chi(\hat{t})} \cdot z\right]_\chi \arrow[dl, mapsto] \\ & [\hat{t}] & \end{tikzcd}$$

depending on ω_χ.

2. *Every other choice* $\tilde{\omega}_\chi \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{R})$ *with* $\chi(\lambda) = e^{i\tilde{\omega}_\chi(\lambda)}$ *fulfils:*
$$(\omega_\chi - \tilde{\omega}_\chi)(\Lambda) \subset 2\pi\mathbb{Z}. \tag{49}$$
3. *The isomorphism* ψ_{ω_χ} *differs from* $\psi_{\tilde{\omega}_\chi}$ *by a map* $\mu : T \to U(1)$.

Proof.

1. At first, we check that ψ_{ω_χ} is well defined:
$$\begin{tikzcd}
(\left[\hat{t}\right], z) \arrow[r, "\psi_{\omega_\chi}"] \arrow[d, equal] & \left[\hat{t}, e^{-i\omega_\chi(\hat{t})} \cdot z\right]_\chi \arrow[d, equal] \\
(\left[\hat{t}+\lambda\right], z) \arrow[r, "\psi_{\omega_\chi}"'] & \left[\hat{t}+\lambda, e^{-i\omega_\chi(\hat{t}+\lambda)} \cdot z\right]_\chi . \checkmark
\end{tikzcd}$$
The right vertical equality holds because
$$\left[\hat{t}+\lambda, e^{-i\omega_\chi(\hat{t}+\lambda)} z\right]_\chi = \left[\hat{t}, \chi(\lambda) e^{-i\omega_\chi(\hat{t})} \underbrace{e^{-i\omega_\chi(\lambda)}}_{=\chi(\lambda)^{-1}} z\right]_\chi = \left[\hat{t}, e^{-i\omega_\chi(\hat{t})} z\right]_\chi .$$
ψ_{ω_χ} is obviously smooth, bijective and $\mathbb{C}$-linear on fibres, hence, it is an isomorphism of smooth complex line bundles.

2. The map χ is defined on Λ.
We deduce that we obtain for every $\lambda \in \Lambda$:
$$\chi(\lambda) = e^{i\omega_\chi(\lambda)} = e^{i\tilde{\omega}_\chi(\lambda)}.$$
Equivalently, we observe that $e^{i(\omega_\chi(\lambda) - \tilde{\omega}_\chi(\lambda))} = 1$.
Now, Equation (49) follows trivially.

3. Let $\delta\omega$ denote the difference $\omega_\chi - \tilde{\omega}_\chi \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{R})$.
It follows, by 2., that
$$\delta\omega(\Lambda) \subset 2\pi\mathbb{Z}.$$
We compute for $\hat{t} \in \mathbb{C}^n$ and $z \in \mathbb{C}$:
$$\begin{aligned}
\psi_{\tilde{\omega}_\chi}(\left[\hat{t}\right], z) &= \left[\hat{t}, e^{-i(\omega_\chi(t) - \delta\omega(\hat{t}))} \cdot z\right]_\chi = \underbrace{e^{-i\delta\omega(\hat{t})}}_{=:\tilde{\mu}(\hat{t})} \left[\hat{t}, e^{-i\omega_\chi(\hat{t})} \cdot z\right]_\chi \\
&= \tilde{\mu}(\hat{t}) \cdot \psi_{\omega_\chi}(\left[\hat{t}\right], z)
\end{aligned}$$
and $\mu(\left[\hat{t}\right]) := \tilde{\mu}(\hat{t})$ is a well defined map on T because $\delta\omega(\Lambda) \subset 2\pi\mathbb{Z}$.

$\square$

The next step is to compare the different holomorphic structures on the trivial line bundle $\tilde{\mathfrak{L}}_1 = T \times \mathbb{C}$ induced by the characters χ.

We saw in Lemma 5.10 that the natural holomorphic structure on $\tilde{\mathfrak{L}}_\chi = \hat{T} \times_\chi \mathbb{C}$ is given via the following diagram

$$\begin{tikzcd}
\Gamma(T, \hat{T} \times_\chi \mathbb{C}) \arrow[r, "\bar{\partial}_{\tilde{\mathfrak{L}}_\chi}"] \arrow[d, "\hat{\kappa}_\chi"'] & \mathfrak{A}^{(0,1)}(T, \hat{T} \times_\chi \mathbb{C}) \\
C^\infty(\hat{T}, \mathbb{C})^\chi \arrow[r, "\bar{\partial}"'] & \mathfrak{A}^{(0,1)}(\hat{T})^\chi \arrow[u, "\hat{\kappa}_\chi^{-1}"']
\end{tikzcd}$$

for the isomorphism

$$\hat{\kappa}_\chi : \quad \mathfrak{A}^{(0,*)}(T, \hat{T} \times_\chi \mathbb{C}) \quad \longrightarrow \quad \mathfrak{A}^{(0,*)}(\hat{T})^\chi$$

where we added a hat to the symbol of the map κ_χ in order to discern it from the map

$$\kappa_\chi : \quad \mathfrak{A}^{(0,*)}(G, \tilde{G} \times_\chi \mathbb{C}) \quad \longrightarrow \quad \mathfrak{A}^{(0,*)}(\tilde{G})^\chi.$$

We now apply the bundle isomorphism ψ_{ω_χ} in order describe the holomorphic structure on $T \times \mathbb{C}$ induced by $\tilde{\mathfrak{L}}_\chi$.
Note therefore that $\omega_\chi \in \mathrm{Hom}_\mathbb{R}(\mathbb{C}^n, \mathbb{R}) = \mathrm{Hom}_\mathbb{R}(T_eT, \mathbb{R})$ can be identified canonically with a one form in $\mathfrak{A}^1(T)$ because the co-tangent bundle of the torus is trivial.

Lemma 5.18:
The holomorphic structure $\bar{\partial}_{\omega_\chi}$ on $T \times \mathbb{C}$ induced by the isomorphism of complex line bundles

$$\psi_{\omega_\chi} : \quad T \times \mathbb{C} = \tilde{\mathfrak{L}}_1 \quad \longrightarrow \quad \tilde{\mathfrak{L}}_\chi,$$

defined in Lemma 5.17, is given by

$$\bar{\partial}_{\omega_\chi} = \bar{\partial} - i\varepsilon \left(\mathrm{proj}^{(0,1)}(\omega_\chi) \right)$$

where $\mathrm{proj}^{(0,1)}$ denotes the projection onto the antiholomorphic subbundle of the complexified cotangent bundle $(T^*T) \otimes_\mathbb{R} \mathbb{C}$.
In particular, the following diagram commutes for $\hat{\kappa}_1 = \hat{\kappa}_{\chi_1 \equiv 1}$:

$$\begin{tikzcd}
\Gamma(T, \tilde{\mathfrak{L}}_\chi) \arrow[r, "\psi_{\omega_\chi}^{-1}"] \arrow[d, "\bar{\partial}_{\tilde{\mathfrak{L}}_\chi}"'] & \Gamma(T, T \times \mathbb{C}) \arrow[r, "\hat{\kappa}_1"] \arrow[d] & C^\infty(\mathbb{C}^n, \mathbb{C})^{\chi_1} \arrow[d, "\bar{\partial}_{\omega_\chi}"] \\
\mathfrak{A}^{(0,1)}(T, \tilde{\mathfrak{L}}_\chi) \arrow[r, "\psi_{\omega_\chi}^{-1}"'] & \mathfrak{A}^{(0,1)}(T, T \times \mathbb{C}) \arrow[r, "\hat{\kappa}_1"'] & \mathfrak{A}^{(0,1)}(\mathbb{C}^n, \mathbb{C})^{\chi_1}
\end{tikzcd}$$

Now, since $\omega_\chi \in \mathrm{Hom}_\mathbb{R}(\mathbb{C}^n, \mathbb{R}) \subset \mathfrak{A}^1(T)$ is real valued, we obtain
$\bar{\partial}_{\omega_\chi} = \bar{\partial} + \frac{1}{2}\varepsilon(\omega_\chi - iJ\omega_\chi)$.

Proof.
The morphism ψ_{ω_χ} is covered by a morphism

$$\begin{array}{rccc} \tilde{\psi}_{\omega_\chi}: & \tilde{T}\times\mathbb{C} & \longrightarrow & \tilde{T}\times\mathbb{C} \\ & (\tilde{t},z) & \longmapsto & (\tilde{t},e^{-i\omega_\chi(\tilde{t})}\cdot z), \end{array}$$

thus, the following diagram commutes:

$$\begin{tikzcd} \tilde{T}\times\mathbb{C} \arrow[r, "\tilde{\psi}_{\omega_\chi}"] \arrow[d] & \tilde{T}\times\mathbb{C} \arrow[d] \\ \tilde{\mathfrak{L}}_1 \arrow[r, "\psi_{\omega_\chi}"'] & \tilde{\mathfrak{L}}_\chi. \end{tikzcd}$$

In particular, $\tilde{\psi}_{\omega_\chi}$ gives rise to a map

$$\check{\psi}_{\omega_\chi}: \quad \mathfrak{A}^{(0,*)}(\tilde{T})^{\chi_1\equiv 1} \longrightarrow \mathfrak{A}^{(0,*)}(\tilde{T})^{\chi}$$

which maps periodic antiholomorphic forms to χ-equivariant antiholomorphic forms both living on the universal covering $\tilde{T}=\mathbb{C}^n$ of T.
It is given by multiplication with the function

$$\begin{array}{rccc} e^{-i\omega_\chi}: & \tilde{T} & \longrightarrow & U(1) \\ & \tilde{t} & \longmapsto & e^{-i\omega_\chi(\tilde{t})}. \end{array}$$

Now, look at the following commuting diagram for $\tilde{T}=\mathbb{C}^n$ and $\chi_1\equiv 1$.

$$\begin{tikzcd} C^\infty(\tilde{T},\mathbb{C})^\chi \arrow[ddd, "\bar{\partial}"'] & & & C^\infty(\tilde{T},\mathbb{C})^{\chi_1} \arrow[lll, "\check{\psi}_{\omega_\chi}"'] \arrow[ddd, "\bar{\partial}_{\omega_\chi}"] \\ & \Gamma(T,\tilde{\mathfrak{L}}_\chi) \arrow[ul, "\hat{\kappa}_\chi"] \arrow[r, "\psi_{\omega_\chi}^{-1}"] \arrow[d, "\bar{\partial}_{\tilde{\mathfrak{L}}_\chi}"'] & \Gamma(T,\tilde{\mathfrak{L}}_1) \arrow[ur, "\hat{\kappa}_1"] \arrow[d, "\bar{\partial}_{\omega_\chi}"] & \\ & \mathfrak{A}^{(0,1)}(T,\tilde{\mathfrak{L}}_\chi) \arrow[r, "\psi_{\omega_\chi}^{-1}"'] \arrow[dl, "\hat{\kappa}_\chi"] & \mathfrak{A}^{(0,1)}(T,\tilde{\mathfrak{L}}_1) \arrow[dr, "\hat{\kappa}_1"] & \\ \mathfrak{A}^{(0,1)}(\tilde{T},\mathbb{C})^\chi \arrow[rrr, "\check{\psi}_{\omega_\chi}^{-1}"'] & & & \mathfrak{A}^{(0,1)}(\tilde{T},\mathbb{C})^{\chi_1} \end{tikzcd}$$

In order to compute $\bar{\partial}_{\omega_\chi}$, it is sufficient to walk around the outer rectangle of this diagram.
Let f be a periodic function on $\tilde{T}$, i.e. $f\in C^\infty(\tilde{T},\mathbb{C})^{\chi_1}$.

We obtain:

$$\begin{aligned}\bar\partial_{\omega_\chi} f =& \check\psi_{\omega_\chi}^{-1} \circ \bar\partial \circ \check\psi_{\omega_\chi} f \\ =& \check\psi_{\omega_\chi}^{-1} \circ \bar\partial \left(e^{-i\omega_\chi} \cdot f\right) \\ =& \check\psi_{\omega_\chi}^{-1} \left(-i\mathrm{proj}^{(0,1)}(\omega_\chi) \cdot e^{-i\omega_\chi} \cdot f + e^{-i\omega_\chi} \cdot \bar\partial f\right) \\ =& \bar\partial f - i\mathrm{proj}^{(0,1)}(\omega_\chi) \cdot f\end{aligned}$$

which completes the proof. □

Although, it is already clear, by the Appell-Humbert theorem, that the equivalence class of holomorphic structures on $\tilde{\mathfrak{L}}_1$ induced by ψ_{ω_χ} does not depend on the choice of ω_χ, we show this once more in the subsequent corollary.

Corollary 5.19:
A different $\tilde\omega_\chi \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{R})$ for the same representation χ, gives us a holomorphic structure $\bar\partial_{\tilde\omega_\chi}$ on $T \times \mathbb{C}$ equivalent to $\bar\partial_{\omega_\chi}$.

Proof.
Lemma 5.17) states that two isomorphisms ψ_{ω_χ} and $\psi_{\tilde\omega_\chi}$ differ by a map $\mu : T \to U(1)$ which is given for $\tilde t \in \tilde T$ through $\mu(\left[\tilde t\right]) = e^{i\delta\omega(\tilde t)}$ where $\delta\omega$ denotes the difference $\tilde\omega_\chi - \omega_\chi$. Consequently, we obtain:

$$\bar\partial_{\tilde\omega_\chi} - \bar\partial_{\omega_\chi} = -i\varepsilon\left(\mathrm{proj}^{(0,1)}(\delta\omega)\right) = \varepsilon(\bar\mu\bar\partial\mu).$$

Thus, both holomorphic structures on $\tilde{\mathfrak{L}}_1 = T \times \mathbb{C}$ are equivalent by Definition 2.5. □

Remark 5.20:
For a complex torus T, the group $\mathrm{Pic}^0(T)$ is very well understood. Using the Appell-Humbert theorem, we see that it is isomorphic to $\mathrm{Hom}(\Lambda, U(1))$.
In addition to that, we have the following exact sequence of groups.

$$0 \to \mathrm{Hom}(\Lambda, \mathbb{Z}) \xrightarrow{2\pi\cdot} \mathrm{Hom}(\Lambda, \mathbb{R}) \xrightarrow{\exp(i\cdot)} \mathrm{Hom}(\Lambda, U(1)) \to 1$$

We obtain $\mathrm{Hom}(\Lambda, U(1)) \cong \mathrm{Hom}(\Lambda, \mathbb{R})/\mathrm{Hom}(\Lambda, \mathbb{Z}) \cong \mathbb{R}^{2n}/\Lambda^\vee$ where $\Lambda^\vee$ denotes the dual lattice.
Consequently, $\mathrm{Pic}^0(T)$ has the structure of a torus.

As a last statement concerning the holomorphic structure $\bar\partial_{\tilde{\mathfrak{L}}_\chi}$, we show that $\tilde\rho$ is compatible with this holomorphic structure.

Lemma 5.21:
Let $\tilde{\rho} : \hat{T} \longrightarrow \mathrm{Diff}(T, \tilde{\mathfrak{L}}_\chi)$ be the action described in Proposition 5.13.
Let furthermore $\check{\rho}$ denote the $\tilde{\rho}$-induced action on $\mathfrak{A}^{(0,*)}(T, \tilde{\mathfrak{L}}_\chi)$. (compare Remark 3.3).
Then the following identity holds for every $\hat{s} \in \hat{T}$:

$$\check{\rho}(\hat{s}) \circ \bar{\partial}_{\tilde{\mathfrak{L}}_\chi} = \bar{\partial}_{\tilde{\mathfrak{L}}_\chi} \circ \check{\rho}(\hat{s}),$$

i.e. the action $\tilde{\rho}$ respects the holomorphic structure $\bar{\partial}_{\tilde{\mathfrak{L}}_\chi}$.

Proof.
Without loss of generality, assume that $\hat{T} = \tilde{T}$. We have to show that the following diagram commutes.

$$\begin{tikzcd}
\mathfrak{A}^{(0,q)}(T, \tilde{\mathfrak{L}}_\chi) \arrow[r, "\check{\rho}(\hat{s})"] \arrow[d, "\bar{\partial}_{\tilde{\mathfrak{L}}_\chi}"'] & \mathfrak{A}^{(0,q)}(T, \tilde{\mathfrak{L}}_\chi) \arrow[d, "\bar{\partial}_{\omega_\chi}"] \\
\mathfrak{A}^{(0,1+1)}(T, \tilde{\mathfrak{L}}_\chi) \arrow[r, "\check{\rho}(\hat{s})"'] & \mathfrak{A}^{(0,1+1)}(T, \tilde{\mathfrak{L}}_\chi)
\end{tikzcd}$$

Now, let $\tilde{s} \in \tilde{T}$ lie in $\pi^{-1}_{1,\tilde{T}}(s)$ and let furthermore f be in $\Gamma(T, \tilde{\mathfrak{L}}_\chi)$.
We obtain for $t \in T$ and $\tilde{t} \in \pi^{-1}_{1,\tilde{T}}(t)$:

$$\begin{aligned}
\Big(\check{\rho}(\tilde{s})\Big)(f)(t) =& \tilde{\rho}(\tilde{s}) \left(f(s^{-1} \cdot t)\right) = \tilde{\rho}(\tilde{s}) \left(\left[\tilde{t} \cdot \tilde{s}^{-1}, \left(\kappa_\chi(f)\right)(\tilde{t} \cdot \tilde{s}^{-1})\right]_\chi\right) \\
=& \left[\tilde{t}, \left(\kappa_\chi(f)\right)(\tilde{t} \cdot \tilde{s}^{-1})\right]_\chi = \left[\tilde{t}, \left((L_{\tilde{s}^{-1}})^* \kappa_\chi(f)\right)(\tilde{t})\right]_\chi
\end{aligned}$$

where $L_{\tilde{s}}$ denotes the left translation on $\tilde{T}$ by the element $\tilde{s}$.
Consequently, the following identity holds:

$$\check{\rho}(\tilde{s})(f) = \kappa_\chi^{-1} \circ (L_{\tilde{s}^{-1}})^* \circ \kappa_\chi(f). \tag{50}$$

This equality extends from sections $\Gamma(T, \tilde{\mathfrak{L}}_\chi)$ to antiholomorphic forms $\mathfrak{A}^{(0,*)}(T, \tilde{\mathfrak{L}}_\chi)$.
Observe now that the following diagram commutes.

$$\begin{tikzcd}
\mathfrak{A}^{(0,q)}(\tilde{T}, \mathbb{C})^\chi \arrow[rrr, "(L_{\tilde{s}^{-1}})^*"] \arrow[ddd, "\bar{\partial}"'] & & & \mathfrak{A}^{(0,q)}(\tilde{T}, \mathbb{C})^\chi \arrow[ddd, "\bar{\partial}_{\omega_\chi}"] \\
& \mathfrak{A}^{(0,q)}(T, \tilde{\mathfrak{L}}_\chi) \arrow[ul, "\hat{\kappa}_\chi"'] \arrow[r, "\check{\rho}(\hat{s})"] \arrow[d, "\bar{\partial}_{\tilde{\mathfrak{L}}_\chi}"'] & \mathfrak{A}^{(0,q)}(T, \tilde{\mathfrak{L}}_\chi) \arrow[ur, "\hat{\kappa}_\chi"] \arrow[d, "\bar{\partial}_{\omega_\chi}"] & \\
& \mathfrak{A}^{(0,1+1)}(T, \tilde{\mathfrak{L}}_\chi) \arrow[dl, "\hat{\kappa}_\chi"] \arrow[r, "\check{\rho}(\hat{s})"'] & \mathfrak{A}^{(0,q+1)}(T, \tilde{\mathfrak{L}}_\chi) \arrow[dr, "\hat{\kappa}_\chi"'] & \\
\mathfrak{A}^{(0,1+1)}(\tilde{T}, \mathbb{C})^\chi \arrow[rrr, "(L_{\tilde{s}^{-1}})^*"'] & & & \mathfrak{A}^{(0,1+1)}(\tilde{T}, \mathbb{C})^\chi
\end{tikzcd}$$

The left and right trapezoid commute because of the definition of the natural holomorphic structure $\bar{\partial}_{\tilde{\mathfrak{L}}_\chi}$ and the upper and lower trapezoid commute because of Equation (50).
At last, the outer rectangle commutes because $L_{\tilde{s}} : \tilde{T} \longrightarrow \tilde{T}$ is obviously biholomorphic.
Therefore, the inner rectangle commutes and the assertion is proven. $\square$

Summarising, we showed that $\tilde{\mathfrak{L}}_\chi$, although being isomorphic to the trivial complex line bundle $\tilde{\mathfrak{L}}_1 = T \times \mathbb{C}$, does have an excelled, in general non-trivial, equivalence class of holomorphic structures. Furthermore, this class of holomorphic structures is respected by the action $\tilde{\rho}$ of $\hat{T}$.

Now, in order to show the first property of Definition 3.1, we still have to equip $\tilde{\mathfrak{L}}_\chi$ with a Hermitian metric such that $\tilde{\rho}$ acts Hermitian on $\tilde{\mathfrak{L}}_\chi$. This is the content of the subsequent sub-subsection.

5.4.3 Invariant Hermitian metric

In the previous subsection, we investigated the natural holomorphic structure on the line bundle $\tilde{\mathfrak{L}}_\chi = \mathbb{C}^n \times_\chi \mathbb{C}$ and showed that it is respected by the action $\tilde{\rho}$.

In this sub-subsection, we define a natural Hermitian metric $h^{\tilde{\mathfrak{L}}_\chi}$ on $\tilde{\mathfrak{L}}_\chi$ which is $\tilde{\rho}$ invariant.

We already restricted our representation to be unitary, i.e. $\chi : \pi_1(G) \to U(1)$.
Thus, we inherit an induced Hermitian metric by

$$h_t^{\tilde{\mathfrak{L}}_\chi}(\left[\hat{t}, z_0\right]_\chi, \left[\hat{t}, z_1\right]_\chi) := z_0 \cdot \bar{z}_1. \tag{51}$$

This metric is well defined and as the following equation shows it is $\tilde{\rho}$-invariant.

$$\begin{aligned} h_t^{\tilde{\mathfrak{L}}_\chi}\left(\left[\hat{t}, z_0\right]_\chi, \left[\hat{t}, z_1\right]_\chi \right) =& z_0 \cdot \bar{z}_1 = h_{t\cdot s}^{\tilde{\mathfrak{L}}_\chi}\left(\left[\hat{t}\hat{s}, z_0\right]_\chi, \left[\hat{t}\hat{s}, z_1\right]_\chi \right) \\ =& h_{t\cdot s}^{\tilde{\mathfrak{L}}_\chi}\left(\tilde{\rho}(\hat{s}) \left[\hat{t}, z_0\right]_\chi, \tilde{\rho}(\hat{s}) \left[\hat{t}, z_1\right]_\chi \right) \\ =& \left(\tilde{\rho}(\hat{s})^* h^{\tilde{\mathfrak{L}}_\chi}\right)_t \left(\left[\hat{t}, z_0\right]_\chi, \left[\hat{t}, z_1\right]_\chi \right) \end{aligned}$$

This $\tilde{\rho}$-invariance is necessary as well as sufficient to induce a Hermitian metric $h^{\mathfrak{L}_\chi}$ on $\mathfrak{L}_\chi = \tilde{G} \times_\chi \mathbb{C}$.
To show this is the content of the next lemma.

Lemma 5.22:
Let ν be the isomorphism from Equation (48), i.e.:

$$\begin{array}{rccc} \nu : & \tilde{G} \times_{\tilde{\rho},\hat{T}} \tilde{\mathfrak{L}}_\chi & \to & \mathfrak{L}_\chi = \tilde{G} \times_\chi \mathbb{C} \to G \\ & \left[\tilde{g}, \left[\hat{t}, z\right]_\chi\right]_{\tilde{\rho}} & \mapsto & \left[\tilde{g} \cdot \hat{t}, z\right]_\chi . \end{array}$$

We obtain an induced Hermitian metric $h^{\mathfrak{L}_\chi}$ on $\mathfrak{L}_\chi$, given by:

$$h^{\mathfrak{L}_\chi}\left([\tilde{g}, z_0]_\chi, [\tilde{g}, z_1]_\chi\right) = z_0 \cdot \bar{z}_1,$$

analogously to Equation (51).

Proof.
For $g = \pi_{1,\tilde{G}}(\tilde{g})$ the ν-induced Hermitian metric on $\mathfrak{L}_\chi$ is given by

$$\begin{aligned} h_g^{\mathfrak{L}_\chi}([\tilde{g}, z_0]_\chi, [\tilde{g}, z_1]_\chi) =& h_g^{\mathfrak{L}_\chi}\left(\nu\left(\left[\tilde{g}, [\hat{e}, z_0]_\chi\right]_{\tilde{\rho}}\right), \nu\left(\left[\tilde{g}, [\hat{e}, z_1]_\chi\right]_{\tilde{\rho}}\right)\right) \\ :=& h_e^{\tilde{\mathfrak{L}}_\chi}\left([\hat{e}, z_0]_\chi, [\hat{e}, z_1]_\chi\right) = z_0 \cdot \bar{z}_1 \end{aligned}$$

The $\tilde{\rho}$-invariance of $h^{\tilde{\mathfrak{L}}_\chi}$ now implies that this expression does not depend on the representing elements $\left[\tilde{g}, [\hat{e}, z_k]_\chi\right]_{\tilde{\rho}}$. □

Up to this point, we have shown the following facts.
The bundle $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$ is a smooth complex line bundle over the manifold G that is a smooth vector bundle over the fibre bundle $G \to G/T$. In particular, its fibre type as a bundle over G/T is $\tilde{\mathfrak{L}}_\chi := \hat{T} \times \chi\mathbb{C}$ which itself is a Hermitian, holomorphic line bundle.
Furthermore, $\mathfrak{L}_\chi$ is equipped with a Hermitian metric $h^{\mathfrak{L}_\chi}$ such that the induced Hermitian metric $h^{\tilde{\mathfrak{L}}_\chi}$ on $\tilde{\mathfrak{L}}_\chi$ is $\tilde{\rho}$ invariant.

Even further, we know that $\mathfrak{L}_\chi$ has a naturally excelled holomorphic structure $\bar{\partial}_{\mathfrak{L}_\chi}$.

What we show in the subsequent sub-subsection is, that the natural holomorphic structure $\bar{\partial}_{\mathfrak{L}_\chi}$ on $\mathfrak{L}_\chi$ induces the natural holomorphic structure $\bar{\partial}_{\tilde{\mathfrak{L}}_\chi}$ on $\tilde{\mathfrak{L}}_\chi$. More general, we describe a set of holomorphic structures $\bar{\partial}_{\mathfrak{L}}$ on $\mathfrak{L}_\chi$ such that the induced holomorphic structure on $\tilde{\mathfrak{L}}_\chi$ is the naturally excelled holomorphic structure $\bar{\partial}_{\tilde{\mathfrak{L}}_\chi}$.

5.4.4 Implications for the holomorphic structure on the line bundle over the Lie group

In this sub-subsection, we investigate the third property of Definition 3.1, i.e. we give a specific condition on the holomorphic structures $\bar{\partial}_{\mathfrak{L}}$ on $\mathfrak{L} = \mathfrak{L}_\chi$ such that the pullback under the inclusion $\phi_{k,x}^{-1} : T \hookrightarrow G$ of the fibre induces the naturally excelled holomorphic structure $\bar{\partial}_{\tilde{\mathfrak{L}}_\chi}$ on $\tilde{\mathfrak{L}}_\chi$.

At this point, we have seen that for a unitary representation χ of $\pi_1(G)$ every associated line bundle $\mathfrak{L} = \mathfrak{L}_\chi := \tilde{G} \times_\chi \mathbb{C}$ has a natural Hermitian metric $h^{\mathfrak{L}_\chi}$.
Furthermore, $\mathfrak{L}_\chi$ becomes a smooth vector bundle over the fibre bundle $G \to G/T$ in the sense of Definition 2.36. Its fibretype as a bundle $\mathfrak{L} \to G/T$ is $\tilde{\mathfrak{L}}_\chi = \mathbb{C}^n \times_\chi \mathbb{C}$ which is a

Hermitian, holomorphic line bundle over T, with Hermitian metric $h^{\tilde{\mathfrak{L}}_\chi}$ and holomorphic structure $\bar{\partial}_{\tilde{\mathfrak{L}}_\chi}$.
The structure group of the fibre bundle $\mathfrak{L} \to G/T$ is $\hat{T}$ and its action is denoted by $\tilde{\rho}$.

The next step is to check the implications of the third defining property for compatible holomorphic line bundles from Definition 3.1 on the holomorphic structure of $\mathfrak{L} = \mathfrak{L}_\chi$.
The third property required us to enforce a "constant" vertical holomorphic structure on $\mathfrak{L}$ which is understood through local trivialisations.

We realised (compare Lemma 5.10) that $\mathfrak{L}_\chi = \tilde{G} \times_\chi \mathbb{C}$ as well as $\tilde{\mathfrak{L}}_\chi$ have an excelled equivalence class of holomorphic structures induced by χ. We denote them by $\bar{\partial}_{\mathfrak{L}_\chi}$ as well as $\bar{\partial}_{\tilde{\mathfrak{L}}_\chi}$.

Remark 5.23:
Let $\{U_i\}_{i\in I}$ be an open cover for G/T and let

$$q_i : \quad U_i \quad \longrightarrow \quad \pi_G^{-1}(U_i) \subset G$$

be local holomorphic sections into the principle fibre bundle $G \to G/T$.
Furthermore, let ϕ_i denote the local trivialisations of $G \to G/T$ induced by q_i, i.e.

$$\begin{array}{rrcl} \phi_i : & \pi_G^{-1}(U_i) & \longrightarrow & U_i \times T \\ & q_i(x) \cdot t & \longmapsto & (x,t). \end{array}$$

In particular, we obtain for every $i \in I$ and any $x \in U_i \subset G/T$ a natural embedding of the fibre T into the total space G:

$$\begin{array}{rrcl} \phi_{i,x}^{-1} : & T & \longrightarrow & G_x \\ & t & \longmapsto & q_i(x) \cdot t. \end{array}$$

Let, additionally,

$$\tilde{q}_i : \quad U_i \quad \longrightarrow \quad \tilde{G}$$

be a lift of $q_i : U_i \longrightarrow G$, i.e. $\pi_{1,\tilde{G}} \circ \tilde{q}_i = q_i$.
We obtain the following isomorphism of vector bundles:

$$\begin{array}{rrcl} \Phi_{i,x} : & \tilde{\mathfrak{L}}_\chi & \longrightarrow & \left(\phi_{i,x}^{-1}\right)^* \mathfrak{L}_\chi \\ & \left[\hat{t}, z\right]_\chi & \longmapsto & \left(\pi_{1,\tilde{G}}(\hat{t}), \left[\tilde{q}_i(x) \cdot \hat{t}, z\right]_\chi\right). \end{array}$$

between $\tilde{\mathfrak{L}}_\chi$ and the pullback bundle $\left(\phi_{i,x}^{-1}\right)^* \mathfrak{L}_\chi$ via $\phi_{i,x}$.

Lemma 5.24:
The map $\Phi_{i,x}$ is an equivalence between $\left(\tilde{\mathfrak{L}}_\chi, \bar{\partial}_{\tilde{\mathfrak{L}}_\chi}\right)$ and $\left(\left(\phi_{i,x}^{-1}\right)^* \mathfrak{L}_\chi, \left(\phi_{i,x}^{-1}\right)^* \bar{\partial}_{\mathfrak{L}_\chi}\right)$ as holomorphic line bundles in the sense of Definition 2.5.

Proof.
Denote by $\tilde{\phi}_{i,x}^{-1}$ the inclusion of $\hat{T}$ into $\tilde{G}$ induced by $\tilde{q}_i(x)$:

$$\begin{array}{rccc} \tilde{\phi}_{i,x}^{-1}: & \hat{T} & \longrightarrow & \tilde{G} \cap \pi_{1,\tilde{G}}^{-1}(G_x) \\ & \hat{t} & \longmapsto & \tilde{q}_i(x) \cdot \hat{t}. \end{array}$$

An easy computation shows that the pullback with $\tilde{\phi}_{i,x}^{-1}$ sends χ-equivariant antiholomorphic forms on $\tilde{G}$ to χ-equivariant antiholomorphic forms on $\hat{T}$,

$$\left(\tilde{\phi}_{i,x}^{-1}\right)^* : \quad \mathfrak{A}^{(0,*)}(\tilde{G})^\chi \quad \longmapsto \quad \mathfrak{A}^{(0,*)}(\hat{T})^\chi. \tag{52}$$

Furthermore, $\tilde{\phi}_{i,x}^{-1}$ is a holomorphic immersion.
Hence, the pullback commutes with the $\bar{\partial}$-operator, i.e. the following diagram commutes:

$$\begin{tikzcd} \mathfrak{A}^{(0,*)}(\tilde{G}) \arrow[r, "\bar{\partial}"] \arrow[d, "(\tilde{\phi}_{i,x}^{-1})^*"'] & \mathfrak{A}^{(0,*+1)}(\tilde{G}) \arrow[d, "(\tilde{\phi}_{i,x}^{-1})^*"] \\ \mathfrak{A}^{(0,*)}(\hat{T}) \arrow[r, "\bar{\partial}"'] & \mathfrak{A}^{(0,*+1)}(\hat{T}) \end{tikzcd} \tag{53}$$

Now, in order to proof the assertion we have to check that the following diagram commutes

$$\begin{tikzcd} \Gamma\left(T, \left(\phi_{i,x}^{-1}\right)^* \mathfrak{L}_\chi\right) \arrow[r, "(\phi_{i,x}^{-1})^*\bar{\partial}_{\mathfrak{L}_\chi}"] \arrow[d, "\Phi_{i,x}^{-1}"'] & \mathfrak{A}^{(0,1)}\left(G, \left(\phi_{i,x}^{-1}\right)^* \mathfrak{L}_\chi\right) \arrow[d, "\Phi_{i,x}^{-1}"] \\ \Gamma(T, \tilde{\mathfrak{L}}_\chi) \arrow[r, "\bar{\partial}_{\tilde{\mathfrak{L}}_\chi}"'] & \mathfrak{A}^{(0,1)}(G, \tilde{\mathfrak{L}}_\chi) \end{tikzcd}$$

Observe, that $\phi_{i,x}^{-1} : T \longrightarrow G$ is injective and furthermore that $\phi_{i,x}^{-1}(T)$ is closed in G.
Hence, $\left(\phi_{i,x}^{-1}\right)^*$ is surjective.
Consequently, it suffices to show that the outer rectangle of the subsequent diagram commutes in order to show the assertion.

$$\begin{tikzcd} \Gamma(G, \mathfrak{L}_\chi) \arrow[r, "\bar{\partial}_{\mathfrak{L}_\chi}"] \arrow[d, "(\phi_{i,x}^{-1})^*"'] & \mathfrak{A}^{(0,1)}(G, \mathfrak{L}_\chi) \arrow[d, "(\phi_{i,x}^{-1})^*"] \\ \Gamma\left(T, \left(\phi_{i,x}^{-1}\right)^* \mathfrak{L}_\chi\right) \arrow[r, dotted, "(\phi_{i,x}^{-1})^*\bar{\partial}_{\mathfrak{L}_\chi}"] \arrow[d, "\Phi_{i,x}^{-1}"'] & \mathfrak{A}^{(0,1)}\left(G, \left(\phi_{i,x}^{-1}\right)^* \mathfrak{L}_\chi\right) \arrow[d, "\Phi_{i,x}^{-1}"] \\ \Gamma(T, \tilde{\mathfrak{L}}_\chi) \arrow[r, "\bar{\partial}_{\tilde{\mathfrak{L}}_\chi}"'] & \mathfrak{A}^{(0,1)}(G, \tilde{\mathfrak{L}}_\chi) \end{tikzcd}$$

We embed the diagram above into the following diagram

$$\begin{tikzcd}
C^\infty(\tilde{G},\mathbb{C})^\chi \arrow[rrr, "\bar{\partial}"] \arrow[dr, "\kappa_\chi^{-1}"] \arrow[dddd, "(\tilde{\phi}_{i,x}^{-1})^*"'] & & & \mathfrak{A}^{(0,1)}(\tilde{G},\mathbb{C})^\chi \arrow[dddd, "(\tilde{\phi}_{i,x}^{-1})^*"] \\
& \Gamma(G,\mathfrak{L}_\chi) \arrow[r, "\bar{\partial}_{\mathfrak{L}_\chi}"] \arrow[d, "(\phi_{i,x}^{-1})^*"'] & \mathfrak{A}^{(0,1)}(G,\mathfrak{L}_\chi) \arrow[ur, "\kappa_\chi"] \arrow[d, "(\phi_{i,x}^{-1})^*"] & \\
& \Gamma\left(T,\left(\phi_{i,x}^{-1}\right)^*\mathfrak{L}_\chi\right) \arrow[r, dotted, "(\phi_{i,x}^{-1})^*\bar{\partial}_{\mathfrak{L}_\chi}"] \arrow[d, "\Phi_{i,x}^{-1}"'] & \mathfrak{A}^{(0,1)}\left(G,\left(\phi_{i,x}^{-1}\right)^*\mathfrak{L}_\chi\right) \arrow[d, "\Phi_{i,x}^{-1}"] & \\
& \Gamma(T,\tilde{\mathfrak{L}}_\chi) \arrow[r, "\bar{\partial}_{\tilde{\mathfrak{L}}_\chi}"'] & \mathfrak{A}^{(0,1)}(G,\tilde{\mathfrak{L}}_\chi) \arrow[dr, "\hat{\kappa}_\chi"] & \\
C^\infty(\hat{T},\mathbb{C})^\chi \arrow[rrr, "\bar{\partial}"'] \arrow[ur, "\hat{\kappa}_\chi^{-1}"'] & & & \mathfrak{A}^{(0,1)}(\hat{T},\mathbb{C})^\chi
\end{tikzcd}$$

Now, following the statement above (Equation (52) and Diagram (53)) the outer rectangle commutes.
Furthermore, by the definition of the excelled holomorphic structure on $\mathfrak{L}_\chi$ as well as on $\tilde{\mathfrak{L}}_\chi$, compare Lemma 5.10, the upper an the lower trapezoid commute as well.
Additionally, by the definition of the induced holomorphic structure on pullback bundles, the following diagram commutes as well.

$$\begin{tikzcd}
\Gamma(G,\mathfrak{L}_\chi) \arrow[r, "\bar{\partial}_{\mathfrak{L}_\chi}"] \arrow[d, "(\phi_{i,x}^{-1})^*"'] & \mathfrak{A}^{(0,1)}(G,\mathfrak{L}_\chi) \arrow[d, "(\phi_{i,x}^{-1})^*"] \\
\Gamma\left(T,\left(\phi_{i,x}^{-1}\right)^*\mathfrak{L}_\chi\right) \arrow[r, "(\phi_{i,x}^{-1})^*\bar{\partial}_{\mathfrak{L}_\chi}"'] & \mathfrak{A}^{(0,1)}\left(G,\left(\phi_{i,x}^{-1}\right)^*\mathfrak{L}_\chi\right)
\end{tikzcd}$$

Thus, it is left to show that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{A}^{(0,*)}(\tilde{G})^\chi \arrow[rr, "(\tilde{\phi}_{i,x}^{-1})^*"] \arrow[d, "\kappa_\chi^{-1}"'] & & \mathfrak{A}^{(0,*)}(\hat{T})^\chi \\
\mathfrak{A}^{(0,*)}(G,\mathfrak{L}_\chi) \arrow[r, "(\phi_{i,x}^{-1})^*"'] & \mathfrak{A}^{(0,*)}\left(T,\left(\phi_{i,x}^{-1}\right)^*\mathfrak{L}_\chi\right) \arrow[r, "\Phi_{i,x}^{-1}"'] & \mathfrak{A}^{(0,*)}\left(T,\tilde{\mathfrak{L}}_\chi\right). \arrow[u, "\hat{\kappa}_\chi"']
\end{tikzcd} \tag{54}$$

The previous diagram obviously commutes if and only if it commutes for sections or C^∞ maps respectively.
Now, take $f \in C^\infty(\tilde{G})^\chi$ and compute the resulting maps in the previous diagram.
On the upper lane, we obtain

$$\left(\left(\tilde{\phi}_{i,x}^{-1}\right)^* f\right)(\hat{t}) = f\left(\tilde{q}_i(x)\cdot\hat{t}\right).$$

The other way round, we obtain for $\pi_{1,\hat{T}}(\hat{t}) = t \in T$

$$\begin{aligned}
\left(\kappa_\chi^{-1}(f)\right)(g) &= [\tilde{g}, f(\tilde{g})]_\chi \\
\left(\left(\phi_{i,x}^{-1}\right)^* \circ \kappa_\chi^{-1}(f)\right)(t) &= \left(t, \left[\tilde{q}_i(x) \cdot \hat{t}, f(\tilde{q}_i(x) \cdot \hat{t})\right]_\chi\right) \\
\left(\Phi_{i,x}^{-1} \circ \left(\phi_{i,x}^{-1}\right)^* \circ \kappa_\chi^{-1}(f)\right)(t) &= \left[\hat{t}, f(\tilde{q}_i(x) \cdot \hat{t})\right]_\chi
\end{aligned}$$

which finally leads to

$$\left(\hat{\kappa}_\chi \circ \Phi_{i,x}^{-1} \circ \left(\phi_{i,x}^{-1}\right)^* \circ \kappa_\chi^{-1}(f)\right)(\hat{t}) = f(\tilde{q}_i(x) \cdot \hat{t}).$$

We conclude that Diagram (54) commutes which finishes the proof. □

Corollary 5.25:
Let $\bar{\partial}_{\mathfrak{L}_\chi}$ denote the naturally excelled holomorphic structure on $\mathfrak{L}_\chi$ (compare Lemma 5.10).
Then any other holomorphic structure $\bar{\partial}'_{\mathfrak{L}_\chi} = \bar{\partial}_{\mathfrak{L}_\chi} + \varepsilon(\omega)$ on $\mathfrak{L}_\chi$ induces the canonical holomorphic structure $\bar{\partial}_{\tilde{\mathfrak{L}}_\chi}$ on $\tilde{\mathfrak{L}}_\chi$ if and only if ω is a closed horizontal $(0,1)$-form.
In particular, $\bar{\partial}'_{\mathfrak{L}_\chi}$ induces $\bar{\partial}_{\tilde{\mathfrak{L}}_\chi}$, then ω lies in $\mathfrak{A}_H^{(0,1)}(G)$.

Summarising, we have shown that if we equip $\mathfrak{L} = \mathfrak{L}_\chi$ with the Hermitian metric $h^{\mathfrak{L}} = h^{\mathfrak{L}_\chi}$ and with a holomorphic structure $\bar{\partial}_{\mathfrak{L}} = \bar{\partial}_{\tilde{\mathfrak{L}}_\chi} + \varepsilon(\omega_H)$ for a $\bar{\partial}$-closed horizontal $(0,1)$-form ω, then $\mathfrak{L}$ becomes a smooth vector bundle over the fibre bundle $G \to G/T$ and $\mathfrak{L}$ fulfils the properties 1. and 3. of the definition of a compatible line bundle (Definition 3.1) over the holomorphic fibre bundle $(G, \pi_G, (G/T, g_{G/T}), (T, g_T), T^H G)$ described in Subsection 5.1.

Therefore, what remains to be shown is when $(\mathfrak{L}, h^{\mathfrak{L}}, \bar{\partial}_{\mathfrak{L}})$ fulfils the Laplace splitting property.

5.4.5 Investigation of the Laplace splitting property

In this subsection, we discuss the last remaining, i.e. the second, property in the definition of a compatible line bundle. In other words, we check when the Laplacian $\square_{\mathfrak{L}_\chi}$ splits into a vertical and a horizontal part.

Let, as above, $(\mathfrak{L} = \mathfrak{L}_\chi = \tilde{G} \times_\chi \mathbb{C}, h^{\mathfrak{L}_\chi}, \bar{\partial}_{\mathfrak{L}})$ be the smooth Hermitian line bundle whose holomorphic structure $\bar{\partial}_{\mathfrak{L}}$ is given as $\bar{\partial}_{\mathfrak{L}_\chi} + \varepsilon(\omega_H)$. Again, $\bar{\partial}_{\mathfrak{L}_\chi}$ denotes the natural holomorphic structure (compare Lemma 5.10) and let ω_H be a $\bar{\partial}$-closed horizontal $(0,1)$-form.

In this subsection, we give a specific condition for our holomorphic structure that leads to the splitting of the Laplacian $\square_{\mathfrak{L}}$.

In Chapter 3.2, we have given sufficient conditions for the splitting of the Laplacian $\square_{\mathfrak{L}}$.

In our example, the fibretype T of the holomorphic fibre bundle is a complex torus with a bi-invariant metric. In particular, T is a Kähler manifold. Hence, we can apply the results of Subsection 3.2.1 for the Kähler fibretype.

Lemma 5.26:
Let $\mathfrak{L} := \mathfrak{L}_\chi = \tilde{G} \times_\chi \mathbb{C}$ be a Hermitian, holomorphic line bundle.
Its holomorphic structure $\bar{\partial}_\mathfrak{L}$ is given by $\bar{\partial}_\mathfrak{L} = \bar{\partial}_{\mathfrak{L}_\chi} + \varepsilon(\omega_H)$ for a $\bar{\partial}$-closed horizontal antiholomorphic one-form ω_H on G.
Furthermore, its Hermitian metric $h^{\mathfrak{L}_\chi}$ is induced by the character $\chi : \pi_1(G) \longrightarrow U(1)$ (compare Section 5.4.3).
Then the Laplace operator $\square_\mathfrak{L}$ splits if for every horizontal lift $\tilde{X}$ of an antiholomorphic vector field X on M, i.e. $X \in \Gamma(M, T^{(0,1)}M)$, and every vertical vector field Y in $\Gamma(G, T^{V,(1,0)}G)$ the following equation holds:

$$\left\{\bar{\partial}(\overline{\omega_H}) - \partial(\omega_H)\right\}(\tilde{X}, Y) = Y.(\omega_H(\tilde{X}))$$

In particular, $\square_\mathfrak{L}$ splits if $\omega_H = \pi_G^*(\omega)$ for a $\bar{\partial}$-closed $(0,1)$-form ω on M.

Proof.
Let $E_i \in \mathfrak{t}^{(1,0)}$ together with $\bar{E}_j \in \mathfrak{t}^{(0,1)}$ denote an orthonormal base of $\mathfrak{t} \otimes_\mathbb{R} \mathbb{C}$ and let $\tilde{X}$ denote the horizontal lift of a vector field X on M to G.
Furthermore, let $\Lambda_V = -i \sum_{l \in J} \iota_{\bar{E}_l} \iota_{E_l}$ denote the vertical Lefschetz operator and let $\nabla^\mathfrak{L}$ be the Chern connection for $\left(\mathfrak{L}, \bar{\partial}_\mathfrak{L}, h^{\mathfrak{L}_\chi}\right)$ (compare Definition 2.9).
By Proposition 3.10, it suffices to check that the following term vanishes for all complexified vector fields X on G/T and all antiholomorphic forms $\eta \in \mathfrak{A}_V^{(0,*)}(G, \mathfrak{L})$.

$$\sum_k \left\{\iota_{[\tilde{X}, \bar{E}_k]} \nabla^{\mathfrak{L},(1,0)}_{E_k} + \iota_{\bar{E}_k} \nabla^{\mathfrak{L},(1,0)}_{[\tilde{X}, E_k]}\right\} \eta + \Lambda_V \iota_{\tilde{X}} \mathrm{proj}_{\mathfrak{A}_H^{(0,1)} \wedge \mathfrak{A}_V^{(1,0)}(G)} \left((\nabla^\mathfrak{L})^2\right) \wedge \eta \tag{55}$$

Note that $T \hookrightarrow G \to G/T$ is a principle fibre bundle. Thus, the commutator of a fundamental vector field and a horizontal lift is zero because the connection, i.e. the horizontal tangent subspace, is right-invariant (cf. [5]).
Consequently, the first term of Equation (55) vanishes.
This reduces our workload significantly because now $\square_\mathfrak{L}$ splits if the following term vanishes.

$$\mathrm{proj}_{\mathfrak{A}_H^{(0,1)} \wedge \mathfrak{A}_V^{(1,0)}(G)} \left((\nabla^\mathfrak{L})^2\right).$$

Hence, we compute the curvature $(1,1)$-form $(\nabla^\mathfrak{L})^2$ induced by the holomorphic structure $\bar{\partial}_\mathfrak{L} = \bar{\partial}_{\mathfrak{L}_\chi} + \varepsilon(\omega_H)$.
Therefore, let $\tilde{\nabla}^{\mathfrak{L},(0,1)} \oplus \tilde{\nabla}^{\mathfrak{L},(1,0)}$ denote the Chern connection corresponding to the natural flat holomorphic structure $\bar{\partial}_{\mathfrak{L}_\chi}$, i.e. $\tilde{\nabla}^{\mathfrak{L},(0,1)} = \bar{\partial}_{\mathfrak{L}_\chi}$.

We obtain the following identities:

$$\begin{aligned} \nabla^{\mathfrak{L},(0,1)} &= \tilde{\nabla}^{\mathfrak{L},(0,1)} + \varepsilon(\omega_H) \qquad \text{and} \\ \nabla^{\mathfrak{L},(1,0)} &= \tilde{\nabla}^{\mathfrak{L},(1,0)} - \varepsilon(\overline{\omega_H}). \end{aligned}$$

We conclude

$$(\nabla^{\mathfrak{L}})^2 = \varepsilon \left\{ \bar{\partial}(\overline{\omega_H}) - \partial(\omega_H) \right\}$$

Recall that ω_H is a $(0,1)$-forms and therefore $(\nabla^{\mathfrak{L}})^2$ is indeed a $(1,1)$-form.
Now, let $\tilde{X}$ be again a horizontal lift of a vector field $X \in \Gamma(M, T^{(0,1)}M)$ and let furthermore $Y \in \Gamma(G, T^{V,(1,0)}G)$ be a vertical vector field.
We obtain:

$$\begin{aligned} \left\{ \bar{\partial}(\overline{\omega_H}) - \partial(\omega_H) \right\} (\tilde{X}, Y) =& \tilde{X}.\underbrace{(\overline{\omega_H}(Y) - \omega_H(Y))}_{=0} - \underbrace{(\overline{\omega_H} - \omega_H)(\overbrace{[\tilde{X}, Y]}^{\text{vertical}})}_{=0} \\ &- Y.\left(\overline{\omega_H}(\tilde{X}) - \omega_H(\tilde{X}) \right). \end{aligned}$$

We used that the commutator of a horizontal lift and a vertical vector field are vertical. Furthermore, note that $\overline{\omega_H}$ is a $(1,0)$-form. Therefore, $\overline{\omega_H}(Y) = 0$ and we finally deduce:

$$\left\{ \bar{\partial}(\overline{\omega_H}) - \partial(\omega_H) \right\} (\tilde{X}, Y) = Y.(\omega_H(\tilde{X}))$$

which finishes the proof. □

The preceding lemma shows that for every line bundle $\mathfrak{L}_\chi = \tilde{G} \times_\chi \mathbb{C}$ with induced metric $h^{\mathfrak{L}_\chi}$ there are holomorphic structures such that $\mathfrak{L}_\chi$ fulfils the Laplace splitting property.

5.4.6 Summary

Now, we have found sufficiently many conditions such that the line bundle $\mathfrak{L}$ becomes compatible.
We summarise them in the next proposition.

Proposition 5.27:
Let $\chi : \pi_1(G) \to U(1)$ be a unitary representation of $\pi_1(G)$ and let $\mathfrak{L} = \mathfrak{L}_\chi$ be the complex line bundle over G which is associated to the universal covering space $\tilde{G}$ of G, i.e. $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$.
On the one hand, let $\mathfrak{L}$ be equipped with a Hermitian metric given by the unitary representation χ (compare Section 5.4.3).
On the other hand, endow $\mathfrak{L}$ with the holomorphic structure $\bar{\partial}_{\mathfrak{L}} := \bar{\partial}_{\mathfrak{L}_\chi} + \varepsilon(\pi_G^*\omega)$ where $\bar{\partial}_{\mathfrak{L}_\chi}$ denotes the natural flat holomorphic structure on $\mathfrak{L}_\chi$ and where $\omega \in \mathfrak{A}^{(0,1)}(G/T)$ is a $\bar{\partial}$-closed form on G/T.
Then the holomorphic line bundle $(\mathfrak{L}, \bar{\partial}_{\mathfrak{L}})$ becomes a compatible holomorphic line bundle for the holomorphic fibre bundle $(G, \pi_G, (G/T, g_{G/T}), (T, g_T), T^H G)$.

Proof. By construction $\mathfrak{L}$ becomes a smooth complex line bundle over the total space $\tilde{G}$.

1. We show in Corollary 5.15 that $\mathfrak{L}$ is a smooth vector bundle over the fibre bundle $G \to G/T$ with fibre type $\tilde{\mathfrak{L}}_\chi \to T$.
   It is associated to the principle fibre bundle $\tilde{G} \to G/T$ via a group action $\tilde{\rho}$, i.e. the following diagram commutes.

$$\begin{tikzcd}
 & \tilde{G} \times \mathbb{C} \arrow[dl, "\mathrm{proj}_1"'] \arrow[dr, "\sim"] & & \tilde{G} \times \tilde{\mathfrak{L}} \arrow[d, "\sim"] \\
\tilde{G} \arrow[dr, "\pi_{1,\tilde{G}}"'] & & \mathfrak{L} = \tilde{G} \times_\chi \mathbb{C} \arrow[dl, "\pi_{\mathfrak{L}}"] \arrow[r, leftrightarrow, "\nu"] & \tilde{G} \times_{\tilde{\rho}} \tilde{\mathfrak{L}} \arrow[dd] \\
 & G \arrow[d, "\pi_G"'] & & \\
 & G/T \arrow[rr, equal, "\mathrm{id}_{G/T}"'] & & G/T
\end{tikzcd}$$

   In particular, $\tilde{\mathfrak{L}}_\chi \to T$ becomes a holomorphic line bundle with holomorphic structure $\bar{\partial}_{\tilde{\mathfrak{L}}_\chi}$ induced by χ given in Lemma 5.10.
   Additionally, following Section 5.4.3, $\mathfrak{L}$ as well as $\tilde{\mathfrak{L}}_\chi$ are equipped with a Hermitian metric which is $\tilde{\rho}$-invariant.
   Now, $\tilde{\rho}$ acts compatible with the holomorphic structure because of Lemma 5.21 and it acts isometric as well (compare again Section 5.4.3).

2. By Lemma 5.26, the Laplacian $\Box_{\mathfrak{L}}$ splits.

3. At last, Lemma 5.24 implies that the induced holomorphic structure on $\tilde{\mathfrak{L}}_\chi$ coincides with $\bar{\partial}_{\tilde{\mathfrak{L}}_\chi}$.

$\square$

From now on, we will talk only about complex line bundles $\mathfrak{L}$ over G that satisfy the prerequisites of the proposition above.

5.5 Results for a general legitimate action

In the previous subsection, we have given a class of holomorphic line bundles over the holomorphic fibre bundle $(G, \pi_G, (G/T, g_{G/T}), (T, g_T), T^H G)$ that are compatible. Thus, we may apply Theorem 4.1 on those for any legitimate group action, i.e. for any tuple $\vec{\gamma} = (\gamma^{G/T}, \gamma^G, \gamma^{\mathfrak{L}})$ satisfying the properties of Definition 3.26.

In this subsection, more precisely in Sub-Subsection 5.5.1, we recall the commonly known fact that the cohomology group of $\tilde{\mathfrak{L}}_\chi$, i.e. $H^{(0,q)}(T, \tilde{\mathfrak{L}}_\chi)$, has a very simple structure. For the convenience of the reader, this is proven here, too.

This property has nice consequences for the bundle $\mathcal{W}^{(0;t)}$ which simplifies the splitting formula of the equivariant torsion for any legitimate group action $\vec{\gamma}$.
We apply this in Subsection 5.5.2.

5.5.1 The Eigenspace vector bundle for the zero-Eigenvalue

The bundle $\mathcal{W}^{(0;t)} = \tilde{G} \times_{\tilde{\rho}} \operatorname{Ker}\left(\Box^{(0,t)}_{\tilde{\mathfrak{L}}_\chi}\right)$ has fibre type $H^{(0,q)}(T, \tilde{\mathfrak{L}})$ and is therefore strongly dependent on the exact structure of $H^{(0,q)}(T, \tilde{\mathfrak{L}})$.
For $\tilde{\mathfrak{L}} = \tilde{T} \times_\chi \mathbb{C}$, with induced holomorphic structure, we now show that there are two possible cases.
On the one hand, if $\chi \equiv 1$, then $H^{(0,q)}(T) \mathrel{\widehat{=}} \Lambda^q \left(\mathfrak{t}^{(0,1)}\right)^*$ and on the other hand, if $\chi \not\equiv 1$, we obtain $H^{(0,q)}(T, \tilde{\mathfrak{L}}) = 0$.

Lemma 5.28:
Let T be a torus given by $T = \mathbb{C}^n/\Lambda$ for a lattice Λ of maximal rank in $\mathbb{C}^n$ and let $\chi : \Lambda \to U(1)$ be a unitary representation of Λ. Let further on $\tilde{\mathfrak{L}}_\chi$ be the associated holomorphic line bundle $\tilde{\mathfrak{L}}_\chi = \mathbb{C}^n \times_\chi \mathbb{C}$ with holomorphic, Hermitian structure as above (compare Lemma 5.10 and Subsection 5.4.3).

1. *If $\chi \equiv 1$, then the Dolbeault cohomology $H^{(0,q)}(T, \tilde{\mathfrak{L}}_\chi)$ equals $\Lambda^q \left(\mathfrak{t}^{(0,1)}\right)^*$.*
2. *If $\chi \not\equiv 1$, then the Dolbeault cohomology $H^{(0,q)}(T, \tilde{\mathfrak{L}}_\chi)$ vanishes.*

Remark 5.29:
The content of this lemma is commonly known. Nonetheless, we prove it here for a lack of sources to cite from.

Proof of Lemma 5.28.
At first, recall that the tangent bundle of the torus is trivial. Hence, we obtain the following identification:

$$\mathfrak{A}^{(0,q)}(T, \tilde{\mathfrak{L}}_\chi) \cong \Lambda^q \left(\mathfrak{t}^{(0,1)}\right)^* \otimes_{\mathbb{C}} \Gamma(T, \tilde{\mathfrak{L}}_\chi).$$

This splitting is compatible with the Laplace-Eigenspace decomposition because the Laplacian commutes with the exterior product with $\alpha \in \Lambda^q \left(\mathfrak{t}^{(0,1)}\right)^*$.
Thus, we obtain

$$H^{(0,q)}(T, \tilde{\mathfrak{L}}_\chi) \cong \Lambda^t \left(\mathfrak{t}^{(0,1)}\right)^* \otimes H^{(0,0)}(T, \tilde{\mathfrak{L}}_\chi),$$

and therefore, we may restrict to the case where $q = 0$.
We stated in Lemma 5.10 that the sections $\Gamma(T, \tilde{\mathfrak{L}}_\chi)$ can be naturally identified with

χ-equivariant complex C^∞ functions on $\mathbb{C}^n$ and that the holomorphic structure on $\tilde{\mathfrak{L}}_\chi$ is induced via this identification.
Furthermore, the Hermitian structure is transferred via this isomorphism

$$\hat{\kappa}_\chi : \quad \Gamma(T, \tilde{T} \times_\chi \mathbb{C}) \quad \longrightarrow \quad C^\infty(\tilde{T}, \mathbb{C})^\chi,$$

as well.
Thus, we obtain for a section $s \in \Gamma(T, \tilde{\mathfrak{L}}_\chi)$:

$$\hat{\kappa}_\chi \left(\Box_{\tilde{\mathfrak{L}}_\chi} s \right) = \frac{1}{2} \Delta \left(\hat{\kappa}_\chi(s) \right)$$

where Δ denotes the Laplace-Beltrami operator on $\mathbb{C}^n$.
It follows that

$$H^{(0,0)}(T, \tilde{\mathfrak{L}}_\chi) \cong \mathrm{Ker}(\Delta) \cap C^\infty(\mathbb{C}^n, \mathbb{C})^\chi.$$

Consequently, $H^{(0,0)}(T, \tilde{\mathfrak{L}}_\chi)$ can be identified with the χ-equivariant, harmonic, complex valued functions on $\mathbb{C}^n$ in a natural way.
A known fact for harmonic functions is the so-called **maximum principle**. It states that if $f : \mathbb{C}^n \to \mathbb{R}$ is harmonic and $K \subset \mathbb{C}$ is a compact set, then $f \mid_K$ attains its maximum as well as its minimum at ∂K the boundary of K.
This principle together with the fact that $f : \mathbb{C}^n \to \mathbb{C}$ is harmonic if and only if its real and imaginary parts are harmonic leads to $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are monotonous along any affine line in $\mathbb{C}^n$.

1. If $\chi \equiv 1$, then

   $$\begin{array}{rcll} \mathrm{Re}(f(x)) & = & \mathrm{Re}(f(x+\lambda)) & \text{and likewise} \\ \mathrm{Im}(f(x)) & = & \mathrm{Im}(f(x+\lambda)). & \end{array}$$

   Hence, for any $x \in \mathbb{C}^n$ and any $\lambda \in \Lambda$ and any $t \in \mathbb{R}$, we obtain:

   $$\begin{array}{rcll} \mathrm{Re}(f(x+t\cdot\lambda)) & = & \mathrm{Re}(f(x)) & \text{and analogously} \\ \mathrm{Im}(f(x+t\cdot\lambda)) & = & \mathrm{Im}(f(x)), & \end{array}$$

   because $\mathrm{Re}(f)$ as well as $\mathrm{Im}(f)$ are monotonous on this line.
   But, Λ is of maximal rank, i.e. $\mathbb{C}^n = \mathrm{span}_{\mathbb{R}}(\Lambda)$, and therefore, f has to be constant in order to be harmonic. Conversely, any constant function is obviously harmonic.
   We conclude that $H^{(0,0)}(T, \tilde{\mathfrak{L}}_\chi) = \mathbb{C}$, i.e. it equals the vector space of constant complex functions on $\mathbb{C}^n$.

2. If $\chi \not\equiv 1$, fix one $\lambda \in \Lambda$ such that $U(1) \ni \chi(\lambda) \neq 1$.
   Without restriction of generality we can assume that the homomorphism

   $$\begin{array}{rrcl} \theta : & \mathbb{Z} & \longrightarrow & U(1) \\ & z & \longmapsto & \chi(z \cdot \lambda) \end{array}$$

   is injective because of the following consideration.

Suppose there is a nontrivial kernel $\mathrm{Ker}(\theta) = z_0\mathbb{Z}$. Then $\chi(z_0\lambda) = 1$, and consequently,

$$f(x+\lambda) = f(x + z_0\lambda).$$

Now, following the argument of $\chi \equiv 1$, we see that any harmonic function f has to be constant along $t \mapsto x + t\cdot\lambda$ for any $x \in \mathbb{C}^n$.
It follows $f \equiv 0$ because

$$f(x) = f(x+\lambda) = \chi(\lambda)^{-1} f(x)$$

for any $x \in \mathbb{C}^n$ and $\chi(\lambda) \neq 1$.

Therefore, let $\theta : \mathbb{Z} \to U(1)$ be injective. Equivalently, we obtain that the set $\theta(\mathbb{Z})$ is dense in $U(1)$, i.e. $\overline{\theta(\mathbb{Z})} = U(1)$. More precisely, we get $\overline{\theta(\mathbb{N})} = U(1)$ as well as $\overline{\theta(-\mathbb{N})} = U(1)$.
It follows that there is a sequence $\{a_k \mid a_k \in \mathbb{Z}\}_{k\in\mathbb{N}}$ such that

$$-1 = \lim_{k\to\infty} \theta(a_k) = \left(\lim_{k\to\infty} \theta(-a_k)\right)^{-1} = -1$$

which implies that for any $x \in \mathbb{C}^n$ and any harmonic function $f : \mathbb{C}^n \to \mathbb{C}$

$$\lim_{k\to\infty} f(x - a_k\lambda) = -f(x) = \lim_{k\to\infty} f(x + a_k\lambda).$$

But $\mathrm{Re}(f)$ as well as $\mathrm{Im}(f)$ are monotonous.
We conclude that $\mathrm{Re}(f)$ as well as $\mathrm{Im}(f)$ have to be constant along the affine line

$$\begin{array}{rccc} l_x : & \mathbb{R} & \longrightarrow & \mathbb{C}^n \\ & t & \longmapsto & x + t\cdot\lambda \end{array}$$

for every x.
Therefore, $f \equiv 0$ since

$$f(x) = \chi(\lambda) f(x+\lambda)$$

and $\chi(\lambda) \neq 1$.
As a direct consequence, we obtain $H^{(0,0)}(T, \tilde{\mathfrak{L}}_\chi) \mathrel{\widehat{=}} H^{(0,q)}(T, \tilde{\mathfrak{L}}_\chi) = 0$ for every non-trivial representation χ.

□

The imminent implication for $\mathcal{W}^{(0;t)}$ of the preceding lemma is stated in the subsequent corollary.

Corollary 5.30:
Let $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$ be a holomorphic line bundle, as described above, then the cohomology bundle $\mathcal{W}^{(0;q)}$ is trivial.
Furthermore, we obtain:

$$\mathcal{W}^{(0;q)} = \begin{cases} G/T \times \Lambda^q \left(\mathfrak{t}^{(0,1)}\right)^* & \text{if } \chi \equiv 1 \\ G/T \times \{0\} & \text{if } \chi \not\equiv 1. \end{cases}$$

Proof.
Looking at the local trivialisations of $T \to G \to G/T$, we see that the transition functions are given by left multiplication with elements in T.
But this multiplication leaves $\mathfrak{t}$ invariant. The assertion follows. □

The holomorphic structures on $\mathcal{W}^{(0;t)} = G/T \times \Lambda^t \left(\mathfrak{t}^{(0,1)}\right)^*$ differ from the natural holomorphic structure on a trivial vector bundle, i.e. induced by the complex structure on G/T.
More precisely, they differ by the $\bar{\partial}$-closed $(0,1)$-form $\omega \in \mathfrak{A}^{(0,1)}(G/T)$ that defines the difference between the natural holomorphic structure $\bar{\partial}_{\mathfrak{L}_\chi}$ on $\mathfrak{L} = \mathfrak{L}_\chi$ and the actual holomorphic structure $\bar{\partial}_{\mathfrak{L}} = \bar{\partial}_{\mathfrak{L}_\chi} + \varepsilon(\pi_G^* \omega)$. (compare Proposition 5.4.6)

We are now able to specify Theorem 4.1 to this situation.

5.5.2 Equivariant torsion formulae

Now, that we have done the preliminary work it is time for some results.

For the first result, recall that we defined the equivariant index $\mathrm{ind}(\gamma^{\mathcal{Q}}, \square_{\mathcal{Q}})$ as well as for any holomorphic, Hermitian vector bundle $\mathcal{Q}$ and any suitable action γ (compare Definition 2.42).
Note further on that the equivariant index for the trivial line bundle over a compact Kähler manifold F equals the equivariant Euler-characteristic $\chi_\gamma(F)$.

Theorem 5.1:

- *Let $(G, \pi_G, (G/T, g_{G/T}), (T, g_T), T^H G)$ be the holomorphic fibre bundle described in Subsection 5.1.*

- *Let χ be a unitary character of $\pi_1(G)$ and let $\mathfrak{L} = \mathfrak{L}_\chi := \tilde{G} \times_\chi \mathbb{C} \to G$ be the complex line bundle associated to the $\pi_1(G)$-principle fibre bundle $\hat{T} \hookrightarrow \tilde{G} \to G$. Choose on $\mathfrak{L}$ the Hermitian metric inherited by the unitary representation χ, as described in Subsection 5.4.3 and equip further on $\mathfrak{L}_\chi \to G$ with the holomorphic structure $\bar{\partial}_{\mathfrak{L}} = \bar{\partial}_{\mathfrak{L}_\chi} + \varepsilon(\pi_G^*(\omega))$ where $\bar{\partial}_{\mathfrak{L}_\chi}$ denotes the naturally excelled holomorphic structure on $\mathfrak{L}_\chi$ (compare Lemma 5.10) and where $\omega \in \mathfrak{A}^{(0,1)}(G/T)$ is $\bar{\partial}$-closed.*

- *In addition, let* $\vec{\gamma} = \left(\gamma^{\mathfrak{L}}, \gamma^{G}, \gamma^{G/T}\right)$ *be a legitimate action.*

Then for sufficiently large $\mathrm{Re}(z)$ *the equivariant* ζ*-function equals:*

$$Z^{\mathfrak{L}}_{\tilde{\gamma}^{\mathfrak{L}}}(z) = -\sum_{\lambda\neq 0} \lambda^{-z} \sum_{t} t(-1)^t \mathrm{ind}(\gamma^{\mathcal{W}^{(\lambda;t)}}, \Box_{\mathcal{W}^{(\lambda;t)}})$$
$$+ \begin{cases} \chi_\gamma(T) Z^{\mathcal{W}^{(0;0)}}_{\gamma}(z) & \text{if } \chi \equiv 1 \\ 0 & \text{if } \chi \not\equiv 1 \end{cases}.$$

Remark 5.31:
In the theorem above, $\mathcal{W}^{(\lambda;t)}$ denotes again the holomorphic, Hermitian vector bundle on G/T, given by

$$\mathcal{W}^{(\lambda;t)} = \tilde{G} \times_{\check{\rho}} \mathrm{Ker}\left(\Box^{(0,t)}_{\tilde{\mathfrak{L}}_\chi} - \lambda\right),$$

and $\gamma^{\mathcal{W}^{(\lambda;t)}}$ denotes again the action on $\mathcal{W}^{(\lambda;t)}$ covering $\gamma^{G/T}$ on G/T induced by $\vec{\gamma}$.

In order to proof this theorem, we will need the following lemma first.

Lemma 5.32:
In the setting of Theorem 5.1 with $\chi \equiv 1$*, the splitting*

$$\mathrm{Ker}(\Box^{[s,t]}_V) = \pi^*_G(\mathfrak{A}^{(0,s)}(G/T)) \wedge H^{(0,t)}(T)$$

is compatible with the Eigenspace distribution of $\Box_H$*, i.e.*

$$\mathrm{Eig}_\mu\left(\Box^{[s,t]}_H\right) \cap \mathrm{Ker}\left(\Box^{[s,t]}_V\right) = \left\{\mathrm{Eig}_\mu\left(\Box^{[s,0]}_H\right) \cap \mathrm{Ker}\left(\Box^{[s,0]}_V\right)\right\} \wedge \mathrm{Ker}\left(\Box^{[0,t]}_V\right).$$

Remark 5.33:
Recall that

$$\Box_H = \left(\nabla^{H,(0,1)}\right)^* \left(\nabla^{H,(0,1)}\right) + \left(\nabla^{H,(0,1)}\right)\left(\nabla^{H,(0,1)}\right)^*$$

where $\bar{\partial}_{\mathfrak{L}} = \nabla^{H,(0,1)} \oplus \nabla^{V,(0,1)}$ and for the vertical Laplacian accordingly.
For the subsequent proof $\bar{\partial}_{\mathfrak{L}_\chi} = \bar{\partial}_H + \bar{\partial}_V$ in opposition to the identification

$$\bar{\partial}_H = \left(\nabla^{H,(0,1)}\right)$$

anywhere else throughout this thesis.

In particular, we obtain:

$$\begin{aligned} \nabla^{V,(0,1)} &= \bar{\partial}_V \\ \nabla^{H,(0,1)} &= \bar{\partial}_H + \varepsilon(\pi_G^*\omega) \end{aligned} \tag{56}$$

Proof of Lemma 5.32.
We already know (compare Corollary 2.28) that

$$\mathrm{Ker}\left(\Box_V^{[s,t]}\right) = \left(\pi_G^*\mathfrak{A}^{(0,s)}(G/T)\right) \wedge \mathrm{Ker}\left(\Box_V^{[0,t]}\right).$$

Furthermore, it is obvious that

$$\mathrm{Ker}(\Box_V^{[0,t]}) = \Lambda^t\left(\mathfrak{t}^{(0,1)}\right)^*.$$

The proof is divided in two steps.
First, we show that we obtain for every $\eta \in \mathfrak{A}_H^{(0,s)}(G)$ and every $\alpha \in \Lambda^t\left(\mathfrak{t}^{(0,1)}\right)^*$:

$$\Box_H\left(\eta \wedge \alpha\right) = (\Box_H \eta) \wedge \alpha. \tag{57}$$

Thereafter, we show the claim of the lemma.
1) In order to show Equation (57), we show at first that the analogous equation holds for $\nabla^{H,(0,1)}$, i.e.

$$\nabla^{H,(0,1)}\left(\eta \wedge \alpha\right) = \left(\nabla^{H,(0,1)}\eta\right) \wedge \alpha.$$

In order to do that, we compute:

$$\underbrace{\nabla^{H,(0,1)}}_{=\bar{\partial}_H + \varepsilon(\pi_G^*\omega)}\left(\eta \wedge \alpha\right) = \left(\nabla^{H,(0,1)}\eta\right) \wedge \alpha + (-1)^s \eta \wedge \left(\bar{\partial}_H \alpha\right).$$

Now, observe that $\bar{\partial}_H\alpha$ vanishes because of the following argument.

Let X be in $T^{(0,1)}(G/T)$ with horizontal lift $\tilde{X}$, let Y be a vertical vector field in $\mathfrak{t}^{(0,1)}$ and let $\beta \in \left(\mathfrak{t}^{(0,1)}\right)^*$ be a vertical $(0,1)$-form.
We obtain:

$$(\bar{\partial}_H\beta)(\tilde{X}, Y) = \tilde{X}.\beta(Y) - \beta([\tilde{X}, Y]) = 0. \tag{58}$$

Here, the first term vanishes, because $\beta(Y)$ is constant and the second term is zero because the commutator of a fundamental vector field with a horizontal lift vanishes. Hence, we get for a form $\alpha \in \Lambda^t\left(\mathfrak{t}^{(0,1)}\right)^*$ of potentially higher degree that $\bar{\partial}_H\alpha = 0$ as a direct consequence from the antiderivativity of $\bar{\partial}_H$ as well as Equation (58).

Now, in order to show Equation (57), it suffices to show that

$$\left(\nabla^{H,(0,1)}\right)^*\left(\eta\wedge\alpha\right)=\left(\left(\nabla^{H,(0,1)}\right)^*\eta\right)\wedge\alpha.$$

The vertical distribution $T^VG\to G$ is trivial

$$T^VG=G\times\mathfrak{t}$$

for $\mathfrak{t}\subset\mathfrak{g}$.
In particular, the antiholomorphic forms in G split as follows:

$$\begin{aligned}\mathfrak{A}^{(0,r)}(G)&=\bigoplus_{s+t=r}\mathfrak{A}_H^{(0,s)}(G)\wedge H^{(0,t)}(T)=\bigoplus_{s+t=r}\mathfrak{A}_H^{(0,s)}(G)\wedge\operatorname{Ker}(\Box_V^{[0,t]})\\&=\bigoplus_{s+t=r}\mathfrak{A}_H^{(0,s)}(G)\wedge\Lambda^t(t^{(0,1)})^*.\end{aligned}$$

Furthermore, note that the adjoint of the operator $\left(\nabla^{H,(0,1)}\right)$ is given by

$$\left(\nabla^{H,(0,1)}\right)^*=\bar\partial_H^*+\iota^h_{\pi_G^*(\omega)}$$

which follows directly from Equation (56).
Now, take $\beta_H\in\mathfrak{A}_H^{(0,s-1)}(G)$ and $\beta_V\in\Lambda^t\left(\mathfrak{t}^{(0,1)}\right)^*$.
By the definition of the adjoint operator, the following equation holds:

$$\begin{aligned}\int_G h\left(\beta_H\wedge\beta_V,\bar\partial_H^*\left(\eta\wedge\alpha\right)\right)&=\int_G h\left(\bar\partial_H\left(\beta_H\wedge\beta_V\right),\eta\wedge\alpha\right)\\&=\int_G h\left(\beta_V,\alpha\right)\cdot h\left(\bar\partial_H\beta_H,\eta\right).\end{aligned}$$

But, $h\left(\beta_V,\alpha\right)$ is constant and therefore,

$$\int_G h\left(\beta_H\wedge\beta_V,\bar\partial_H^*\left(\eta\wedge\alpha\right)\right)=\int_G h\left(\beta_H,\bar\partial_H^*\eta\right)\cdot h\left(\beta_V,\alpha\right).$$

Consequently, we obtain:

$$\bar\partial_H^*\left(\eta\wedge\alpha\right)=\left(\bar\partial_H^*\eta\right)\wedge\alpha.$$

On the other hand, we note that

$$\iota^h_{\pi_G^*(\omega)}\left(\eta\wedge\alpha\right)=\left(\iota^h_{\pi_G^*(\omega)}\eta\right)\wedge\alpha+(-1)^s\eta\wedge\underbrace{\left(\iota^h_{\pi_G^*(\omega)}\alpha\right)}_{=0}.$$

We conclude

$$\left(\nabla^{H,(0,1)}\right)^*\left(\eta\wedge\alpha\right)=\left(\left(\nabla^{H,(0,1)}\right)^*\eta\right)\wedge\alpha$$

which directly implies Equation (57).

2) We are ready to proof the claim of this lemma.
Let δ be in $\mathrm{Eig}_\mu\left(\Box_H^{[s,t]}\right) \cap \mathrm{Ker}\left(\Box_V^{[s,t]}\right)$.
Then $\delta = \sum_k \eta_k \wedge \alpha_k$ with α_k linearly independent in $\Lambda^t\left(\mathfrak{t}^{(0,1)}\right)^*$ and $\eta_k \in \mathfrak{A}_H^{(0,s)}(G)$.
It follows from the first step of this proof, i.e. Equation (57), that

$$\sum_k (\mu \cdot \eta_k) \wedge \alpha_k = \mu \cdot \delta = \Box_H^{[s,t]} \delta \overset{(57)}{=} \sum_k (\Box_H^{[s,0]} \eta_k) \wedge \alpha_k.$$

The linear independence of the α_k now implies that $\Box_H \eta_k = \mu \cdot \eta_k$, i.e.

$$\delta \in \mathrm{Eig}_\mu\left(\Box_H^{[s,0]}\right) \wedge \mathrm{Ker}\left(\Box_V^{[0,t]}\right).$$

Additionally, we see that

$$0 = \Box_V^{[s,t]} \delta = \sum_k \left(\left(\Box_V^{[s,0]} \eta_k\right) \wedge \alpha_k + \eta_k \wedge \underbrace{\left(\Box_V^{[0,t]} \alpha_k\right)}_{=0} \right).$$

We conclude that η_k lies in $\mathrm{Ker}\left(\Box_V^{[s,0]}\right)$ and consequently

$$\delta \in \left\{ \mathrm{Eig}_\mu\left(\Box_H^{[s,0]}\right) \cap \mathrm{Ker}\left(\Box_V^{[s,0]}\right) \right\} \wedge \mathrm{Ker}\left(\Box_V^{[0,t]}\right).$$

For the other direction, let now δ be in $\left\{ \mathrm{Eig}_\mu\left(\Box_H^{[s,0]}\right) \cap \mathrm{Ker}\left(\Box_V^{[s,0]}\right) \right\} \wedge \mathrm{Ker}\left(\Box_V^{[0,t]}\right)$.
Then δ is again of the form $\delta = \sum_k \eta_k \wedge \alpha_k$ with α_k linearly independent in $\Lambda^t\left(\mathfrak{t}^{(0,1)}\right)^*$ and $\eta_k \in \mathfrak{A}_H^{(0,s)}(G)$.
In particular, we obtain:

$$\Box_H^{[s,t]} \delta = \Box_H^{[s,t]} \left(\sum_k \eta_k \wedge \alpha_k \right) \overset{(57)}{=} \sum_k (\Box_H^{[s,0]} \eta_k) \wedge \alpha_k = \sum_k \mu \cdot \eta_k \wedge \alpha_k = \mu \cdot \delta$$

which implies $\delta \in \mathrm{Eig}_\mu\left(\Box_H^{[s,t]}\right)$.
Similarly, we compute

$$\Box_V^{[s,t]} \delta = \sum_k \left(\underbrace{\left(\Box_V^{[s,0]} \eta_k\right)}_{=0} \wedge \alpha_k + \eta_k \wedge \underbrace{\left(\Box_V^{[0,t]} \alpha_k\right)}_{=0} \right) = 0.$$

and therefore $\delta \in \mathrm{Ker}\left(\Box_V^{[s,t]}\right)$. $\Box$

Corollary 5.34:
For $\chi \equiv 0$, the following equality holds:

$$\mathrm{Eig}_\mu\left(\square_H^{[s,0]}\right) \cap \mathrm{Ker}\left(\square_V^{[s,0]}\right) = \pi_G^*\left(\mathrm{Eig}_\mu(\square^{(0,s)}_{\mathcal{W}^{(0;0)}})\right)$$

Proof. Let δ be in $\mathrm{Eig}_\mu\left(\square_H^{[s,0]}\right) \cap \mathrm{Ker}\left(\square_V^{[s,0]}\right)$.
We apply the ψ-morphism (compare Definition 3.15) and obtain

$$\psi^{-1}(\delta) \in \mathrm{Eig}_\mu\left(\square^{(0,s)}_{\mathcal{W}^{(0;0)}}\right) \subset \mathfrak{A}^{(0,s)}(G/T, \mathcal{W}^{(0;0)}) \cong \mathfrak{A}^{(0,s)}(G/T, \mathbb{C}) = \mathfrak{A}^{(0,s)}(G/T).$$

In particular, we obtain

$$\delta = \psi \circ \psi^{-1}(\delta) = \pi_G^*\left(\psi^{-1}(\delta)\right) \in \pi_G^*\left(\mathrm{Eig}_\mu\left(\square^{(0,s)}_{\mathcal{W}^{(0;0)}}\right)\right).$$

On the other hand, let $\delta = \pi_G^*\eta$ for an η living in $\mathrm{Eig}_\mu(\square^{(0,s)}_{\mathcal{W}^{(0;t)}})$.
Then the ψ-morphism implies:

$$\delta = \psi(\eta) \overset{!}{\in} \mathrm{Eig}_\mu\left(\square_H^{[s,0]}\right) \cap \mathrm{Ker}\left(\square_V^{[s,0]}\right)$$

because of the definition of $\mathcal{W}^{(0;0)}$ and because of the definition of $\square^{(0,s)}_{\mathcal{W}^{(0;0)}}$. $\square$

Having proven this corollary, we can go straight to the proof of Theorem 5.1.

Proof of Theorem 5.1.
We want to show the following identity:

$$Z^{\mathfrak{L}}_{\tilde{\gamma}^{\mathfrak{L}}}(z) = -\sum_{\lambda \neq 0} \lambda^{-z} \sum_t t(-1)^t \mathrm{ind}(\gamma^{\mathcal{W}^{(\lambda;t)}}, \square_{\mathcal{W}^{(\lambda;t)}}) + \begin{cases} \chi_\gamma(T) Z^{\mathcal{W}^{(0;0)}}_{\gamma^{\mathcal{W}^{(0;0)}}}(z) & \text{if } \chi \equiv 1 \\ 0 & \text{if } \chi \not\equiv 1 \end{cases}.$$

We are in the setting of Theorem 4.1 which we showed in Proposition 5.27.
Consequently, we obtain:

$$Z^{\mathfrak{L}}_{\tilde{\gamma}^{\mathfrak{L}}}(z) = -\sum_{\lambda \neq 0} \lambda^{-z} \sum_t t(-1)^t \mathrm{ind}(\gamma^{\mathcal{W}^{(\lambda;t)}}, \square_{\mathcal{W}^{(\lambda;t)}}) + \sum_t (-1)^t Z^{\mathcal{W}^{(0;t)}}_{\tilde{\gamma}^{\mathcal{W}^{(0;t)}}}(z) \tag{59}$$

Now for $\chi \not\equiv 1$, Corollary 5.30 states that the bundle $\mathcal{W}^{(0;t)}$ is the trivial bundle $G/T \times \{0\}$.
Consequently, the second term in Equation (59) vanishes.
For $\chi \equiv 0$, $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$ becomes the trivial complex line bundle over G, i.e. $\mathfrak{L} = G \times \mathbb{C}$ (compare again Corollary 5.30).

Look at the second term of Equation (59).
We apply the ψ-morphism backwards (compare Equation (43)) and obtain:

$$\begin{aligned}\sum_t (-1)^t Z^{\mathcal{W}^{(0;t)}}_{\check{\gamma}^{\mathcal{W}^{(0;t)}}}(z) &= -\sum_t (-1)^t \sum_{\mu\neq 0} \mu^{-z} \mathrm{Tr}\left[\gamma\mid_{\mathrm{Eig}_\mu(\square^{(0,s)}_{\mathcal{W}^{(0;t)}})}\right] \\ &= -\sum_{\mu\neq 0}\mu^{-z}\sum_s s(-1)^s \sum_t (-1)^t \mathrm{Tr}\left[\check{\gamma}^{\mathfrak{L}}\big|_{\mathrm{Ker}\left(\square^{[s,t]}_V\right)\cap \mathrm{Eig}_\mu\left(\square^{[s,t]}_H\right)}\right]\end{aligned}$$

Furthermore, we already saw, in Lemma 5.32 and Corollary (5.34), that

$$\mathrm{Eig}_\mu\left(\square^{[s,t]}_H\right) \cap \mathrm{Ker}\left(\square^{[s,t]}_V\right) = \pi^*_G\left(\mathrm{Eig}_\mu(\square^{(0,s)}_{\mathcal{W}^{(0;0)}})\right) \wedge \mathrm{Ker}\left(\square^{[0,t]}_V\right).$$

In particular, $\check{\gamma}^{\mathfrak{L}}$ acts on both factors of the right hand side separately since it commutes with $\square_V$ as well as with $\square_H$.
Thus, we obtain,

$$\begin{aligned}&\sum_t (-1)^t Z^{\mathcal{W}^{(0;t)}}(z) \\ &= -\sum_{\mu\neq 0}\mu^{-z}\sum_s s(-1)^s\sum_t(-1)^t \mathrm{Tr}\left[\check{\gamma}^{\mathfrak{L}}\big|_{H^{(0,t)}(T)\otimes \pi^*_G\left(\mathrm{Eig}_\mu(\square^{(0,s)}_{\mathcal{W}^{(0;0)}})\right)}\right] \\ &\qquad = \underbrace{-\sum_{\mu\neq 0}\mu^{-z}\sum_{s=0}^{m} s(-1)^s \mathrm{Tr}\left[\mathrm{Eig}_\mu(\square^{(0,s)}_{\mathcal{W}^{(0;0)}})\right]}_{=Z^{\mathcal{W}^{(0;0)}}_\gamma(z)} \cdot \underbrace{\sum_t (-1)^t \mathrm{Tr}\left[\check{\gamma}^{\mathfrak{L}}\big|_{H^{(0,t)}(T)}\right]}_{=\chi_\gamma(T)}\end{aligned}$$

which finishes this proof. $\square$

Remark 5.35:
Recall that for $\chi \equiv 1$ the holomorphic structure on $\mathfrak{L}_\chi = \tilde{G}\times_\chi \mathbb{C}$ is given by $\bar{\partial}_{\mathfrak{L}_\chi} + \varepsilon(\pi^*_G\omega)$ and obviously $\bar{\partial}_{\mathfrak{L}_\chi} = \bar{\partial}$.
Now, the term $Z^{\mathcal{W}^{(0;0)}}_\gamma(z)$ is equal to $Z^{G/T\times\mathbb{C}}_\gamma(z)$ if we do not forget that the holomorphic structure on $\mathcal{W}^{(0;0)} \cong G/T\times\mathbb{C}$ has to be $\bar{\partial}+\varepsilon(\omega)$ instead of the trivial holomorphic structure $\bar{\partial}$.

Naturally, there is a specialised version of Theorem 5.1 for the case of non-degenerated (in particular isolated) fixed points as well. (compare Corollary 4.9)

Corollary 5.36:
Suppose that $\gamma^{G/T}: G/T \to G/T$ has only isolated, non-degenerated fixed points.

Then the formula for the equivariant holomorphic ζ-function becomes:

$$Z^{\mathfrak{L}}_{\check{\gamma}^{\mathfrak{L}}}(z) = \sum_{x_0 \in (G/T)^\gamma} \left(\det_{T^{1,0}_{x_0} G/T} \left(1-\gamma_{x_0}^{-1}\right)^{-1}\right) \cdot Z^{\mathfrak{L}_{x_0}}_{\check{\gamma}^{\mathfrak{L}_{x_0}}}(z) + \begin{cases} \chi_\gamma(T) Z^{\mathcal{W}^{(0;0)}}_\gamma(z) & \text{if } \chi \equiv 1 \\ 0 & \text{if } \chi \not\equiv 1 \end{cases}.$$

Here, analogously to Corollary 4.9,

- $(G/T)^\gamma$ *denotes the fixed point set of* $\gamma^{G/T}$,
- $\mathfrak{L}_{x_0} = \mathfrak{L}\,|_{\pi_G^{-1}(x_0)}$,
- γ_{x_0} *is the linear map* $T_{x_0}\gamma^{G/T}$ *for each* $x_0 \in (G/T)^\gamma$
- *and* $\check{\gamma}^{\mathfrak{L}_{x_0}}$ *is the action on* $\mathfrak{A}^{(0,*)}(\pi_G^{-1}(x_0), \mathfrak{L}_{x_0})$ *induced by* $\check{\gamma}^{\mathfrak{L}}$.

Remark 5.37:
This is as far as we get using a general legitimate group action. What comes next is to give a relatively simple example for a legitimate group action.

5.6 An example of legitimate group actions

Now, it is time to give an example for a legitimate action in order to apply the formulae of the previous subsection.

Therefore, the task at hand is to find a triple $\vec{\gamma} = (\gamma^{G/T}, \gamma^G, \gamma^{\mathfrak{L}})$ of diffeomorphisms that satisfy the defining properties 1., 2., 3. and 4. of Definition 3.26, i.e of a legitimate action.

The easiest approach is to look again at the definition of the bundle $\mathfrak{L} = \mathfrak{L}_\chi$.
We already required that the line bundle $\mathfrak{L}$ over G is associated to the universal covering bundle $\tilde{G} \to G$ via a unitary representation χ of $\pi_1(G)$.
It follows that we have a natural left action of $\tilde{G}$ on $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$. It is given for any element $\tilde{g}_0 \in \tilde{G}$ by

$$\begin{array}{cccc} L_{\tilde{g}_0}: & \mathfrak{L}_\chi & \longrightarrow & \mathfrak{L}_\chi \\ & [\tilde{g}, z]_\chi & \longmapsto & [\tilde{g}_0 \cdot \tilde{g}, z]_\chi \end{array}$$

For every $\tilde{g}_0 \in \tilde{G}$ this map $L_{\tilde{g}_0}$ is covering maps on G and on G/T, given by:

$$\begin{array}{cccc} L_{g_0}: & G & \longrightarrow & G \\ & g & \longmapsto & g_0 \cdot g \\ & & \text{and} & \\ L^{G/T}_{g_0} & G/T & \longmapsto & G/T \\ & [g] & \longmapsto & [g_0 \cdot g] \end{array}$$

for $g_0 = \pi_{1,\tilde{G}}(\tilde{g}_0)$.

We show at first, in the subsequent lemma, that $\vec{\gamma} = (\gamma^{G/T}, \gamma^G, \gamma^{\mathfrak{L}})$ for $\gamma^{\mathfrak{L}} = L_{\tilde{g}_0}$ is legitimate for the natural holomorphic structure $\bar{\partial}_{\mathfrak{L}_\chi}$ on $\mathfrak{L}_\chi$.

Lemma 5.38:
Le $\mathfrak{L} = \mathfrak{L}_\chi$ be the compatible line bundle over G equipped with the natural holomorphic structure $\bar{\partial}_{\mathfrak{L}_\chi}$ given in Lemma 5.10.
Then the action $\vec{\gamma} = \left(\gamma^{\mathfrak{L}}, \gamma^G, \gamma^{G/T}\right)$ given by $\gamma^{\mathfrak{L}} = L_{\tilde{g}_0}$ is legitimate for an arbitrarily fixed element $\tilde{g}_0$ of $\tilde{G}$.

Proof.
We prove this assertion chronologically, i.e. following the ordering of Definition 3.26.

1. Note that $\gamma^{G/T} = L^{G/T}_{g_0}$ as well as $\gamma^G = L_{g_0}$ are holomorphic isometries since the complex structure on G and on G/T is left invariant by construction as is the Hermitian metric.✓

2. A simple calculation shows that the following diagram indeed commutes.✓

$$\begin{tikzcd}
\mathfrak{L} \arrow[r, "\pi_{\mathfrak{L}}"] \arrow[d, "\gamma^{\mathfrak{L}}"'] & G \arrow[r, "\pi_G"] \arrow[d, "\gamma^G"] & G/T \arrow[d, "\gamma^{G/T}"] \\
\mathfrak{L} \arrow[r, "\pi_{\mathfrak{L}}"'] & G \arrow[r, "\pi_G"'] & G/T
\end{tikzcd}$$

3. Furthermore, $\gamma^{\mathfrak{L}}$ sends fibres in $\mathfrak{L} \to G$ linearly and isometric on fibres because

$$h_{g_0 g}\left([\tilde{g}_0\tilde{g}, z_0]_\chi, [\tilde{g}_0\tilde{g}, z_1]_\chi\right) = z_0 \cdot \bar{z}_1 = h_g\left([\tilde{g}, z_0]_\chi, [\tilde{g}, z_1]_\chi\right),$$

   by construction of the Hermitian metric on $\mathfrak{L}$.✓

4. The last property of Definition 3.26 is the only one that actually depends on the holomorphic structure of $\mathfrak{L}$.
   We want to show that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{A}^{(0,*)}(G, \mathfrak{L}) \arrow[r, "\bar{\partial}_{\mathfrak{L}_\chi}"] \arrow[d, "\tilde{\gamma}^{\mathfrak{L}}"'] & \mathfrak{A}^{(0,*+1)}(G, \mathfrak{L}) \arrow[d, "\tilde{\gamma}^{\mathfrak{L}}"] \\
\mathfrak{A}^{(0,*)}(G, \mathfrak{L}) \arrow[r, "\bar{\partial}_{\mathfrak{L}_\chi}"'] & \mathfrak{A}^{(0,*+1)}(G, \mathfrak{L}).
\end{tikzcd}$$

   Now, the holomorphic structure $\bar{\partial}_{\mathfrak{L}_\chi}$ is defined via the one to one correspondence

$$\kappa_\chi : \quad \mathfrak{A}^{(0,*)}(G, \tilde{G} \times_\chi \mathbb{C}) \quad \longrightarrow \quad \mathfrak{A}^{(0,*)}(\tilde{G})^\chi.$$

Explicitly stated, the natural holomorphic structure on $\mathfrak{L}_\chi$ is given by the following commuting diagram:

$$\begin{tikzcd}
\mathfrak{A}^{(0,*)}(G,\mathfrak{L}) \arrow[r, "\bar{\partial}_{\mathfrak{L}_\chi}"] \arrow[d, "\kappa_\chi"'] & \mathfrak{A}^{(0,*+1)}(G,\mathfrak{L}) \arrow[d, "\kappa_\chi"] \\
\mathfrak{A}^{(0,*)}(\tilde{G})^\chi \arrow[r, "\bar{\partial}"'] & \mathfrak{A}^{(0,*+1)}(\tilde{G})^\chi .
\end{tikzcd}$$

Now, let α be in $\mathfrak{A}^{(0,q)}(G)$ which corresponds to $\tilde{\alpha}$ in $\mathfrak{A}^{(0,q)}(\tilde{G})^{\chi_1\equiv 1}$ and let l be in $\Gamma(G,\mathfrak{L})$ corresponding to $\tilde{l}\in C^\infty(\tilde{G},\mathbb{C})^\chi$.
We obtain:

$$\begin{aligned}
\check{\gamma}^{\mathfrak{L}}(\alpha\otimes l)_g =& \gamma^{\mathfrak{L}}\left(\left[\tilde{g}_0^{-1}\tilde{g}, \left(L^*_{\tilde{g}_0^{-1}}\left(\tilde{\alpha}\otimes\tilde{l}\right)\right)_{\tilde{g}}\right]_\chi\right) \\
=& \left[\tilde{g}, \left(L^*_{\tilde{g}_0^{-1}}\left(\tilde{\alpha}\otimes\tilde{l}\right)\right)_{\tilde{g}}\right]_\chi .
\end{aligned}$$

Consequently, the subsequent diagram commutes.

$$\begin{tikzcd}
\mathfrak{A}^{(0,*)}(G,\mathfrak{L}) \arrow[r, "\check{\gamma}^{\mathfrak{L}}"] \arrow[d, "\kappa_\chi"'] & \mathfrak{A}^{(0,*)}(G,\mathfrak{L}) \arrow[d, "\kappa_\chi"] \\
\mathfrak{A}^{(0,*)}(\tilde{G})^\chi \arrow[r, "L^*_{\tilde{g}_0^{-1}}"'] & \mathfrak{A}^{(0,*)}(\tilde{G})^\chi
\end{tikzcd}$$

Now, look at the following cube.

$$\begin{tikzcd}
& \mathfrak{A}^{(0,q)}(G,\mathfrak{L}) \arrow[rr, "\bar{\partial}_{\mathfrak{L}_\chi}"] \arrow[dl, "\kappa_\chi"'] \arrow[dd, dotted, "\check{\gamma}^{\mathfrak{L}}"' near start] & & \mathfrak{A}^{(0,q+1)}(G,\mathfrak{L}) \arrow[dl, "\kappa_\chi"'] \arrow[dd, "\check{\gamma}^{\mathfrak{L}}"] \\
\mathfrak{A}^{(0,q)}(\tilde{G},\mathbb{C})^\chi \arrow[rr, "\bar{\partial}" near end] \arrow[dd, "L^*_{\tilde{g}_0^{-1}}"'] & & \mathfrak{A}^{(0,q+1)}(\tilde{G},\mathbb{C})^\chi \arrow[dd, "L^*_{\tilde{g}_0^{-1}}" near start] & \\
& \mathfrak{A}^{(0,q)}(G,\mathfrak{L}) \arrow[rr, dotted, "\bar{\partial}_{\mathfrak{L}_\chi}"' near start] \arrow[dl, dotted, "\kappa_\chi"] & & \mathfrak{A}^{(0,q+1)}(G,\mathfrak{L}) \arrow[dl, "\kappa_\chi"] \\
\mathfrak{A}^{(0,q)}(\tilde{G},\mathbb{C})^\chi \arrow[rr, "\bar{\partial}"'] & & \mathfrak{A}^{(0,q+1)}(\tilde{G},\mathbb{C})^\chi &
\end{tikzcd}$$

The surfacing rectangles on the left and on the right side as well as on the upper and the lower side commute. Further on, the surfacing diagram in front commutes because the left multiplication with $\tilde{g}_0^{-1}$ is holomorphic.
Consequently, it follows that the whole cube commutes and in particular the diagram in the back which we wanted to show.✓

$\square$

Corollary 5.39:
For the holomorphic structure of $\mathfrak{L}$ given by $\bar\partial_{\mathfrak{L}} = \bar\partial_{\mathfrak{L}_\chi} + \varepsilon(\pi_G^*(\omega))$, the action $\gamma^{\mathfrak{L}} = L_{\tilde g_0}$ induces a legitimate action $\vec\gamma$ if and only if

$$\left(L^{G/T}_{g_0^{-1}}\right)^* \omega = \omega.$$

5.6.1 Investigations on isolated fixed points

In this subsection, we discuss when the legitimate action $\gamma^{\mathfrak{L}} = L_{\tilde g_0}$ induces an $\gamma^{G/T} = L^{G/T}_{g_0}$ that has only isolated and non-degenerate fixed points.
We deduce that this is the general case.

The content of this sub-subsection is common knowledge.

We start with the following lemma. It states the existence of at least one fixed point for the action $L_{\tilde g_0}$ for one $\tilde g_0 \in \tilde G$.

Lemma 5.40:
Let $\tilde g_0 \in \tilde G$ be arbitrarily fixed, and let $g_0 \in G$ denote its projection under $\pi_{1,\tilde G}$.
Then the map

$$\begin{array}{rccc} L^{G/T}_{g_0}: & G/T & \longrightarrow & G/T \\ & [g] & \longmapsto & [g_0 \cdot g] \end{array}$$

has at least one fixed point.

Proof.
Note that $[x_0] \in G/T$ is a fixed point of $L^{G/T}_{g_0}$ if and only if $g_0 x_0 = x_0 t_0$ for an element $t_0 \in T$. Equivalently, we obtain $g_0 = x_0 t_0 x_0^{-1}$.
For a compact connected Lie group, every element g lies in a maximal torus T' (cf. [10]).
In particular so does g_0.
Now, every maximal torus T' is conjugated to T (again cf. [10]), i.e. there is an $x \in G$ such that $T' = xTx^{-1}$ and consequently $g_0 = x_0 t_0 x_0^{-1}$ for elements $x_0 \in G$ and $t_0 \in T$.
We conclude that every action $L^{G/T}_{g_0}$ has a fixed point. $\square$

The next lemma is dedicated to the properties of the case, where $g_0 \in G$ is generating a maximal torus.
Recall therefore that g_0 generates a maximal torus T' in G if the closure of $g_0^{\mathbb{Z}}$ is a torus of the same dimension as T.

Before we state as well as prove the lemma, we recall the definition of the Weyl group for a torus T' in a compact Lie group G.

Definition 5.41:
Let G be a compact Lie group and $T' \subset G$ a toric subgroup.
The **Weyl group $W(T')$** is given by the following quotient:

$$W(T') = N(T')/T'$$

where $N(T') \subset G$ denotes the normaliser of T' in G, i.e. the maximal subgroup of G such that T' is a normal subgroup of $N(T')$.
More explicitly $N(T')$ is given by:

$$N(T') := \left\{ g \in G \mid gT'g^{-1} = T' \right\}.$$

Remark 5.42:
For a maximal torus T', the order of the Weyl group $W(T')$ is finite. (cf. [10])

Lemma 5.43:
Let g_0 be a generating element of a maximal torus $T' \subset G$. Then the following properties hold.

1. *The map*

$$L_{g_0}^{G/T} : \quad G/T \quad \longrightarrow \quad G/T$$

   has finitely many fixed points and the number of fixed points $\#\,(G/T)^{\gamma}$ equals the order of the Weyl group $W(T)$ of T in G.

2. *Every fixed point of G/T is non-degenerated.*

Proof.

1. Let $[x_0]$ be a fixed point of $L_{g_0}^{G/T}$ which exists by Lemma 5.40.
   We obtain

$$x_0 t_0 x_0^{-1} = g_0.$$

   Now, g_0 generates a maximal torus T' if and only if t_0 generates T.
   The Weyl group $W(T)$ acts transitively and free on the fixed points $(G/T)^{\gamma}$ in the following way.
   Let nT be an element of $W(T) = N(T)/T$, then the $W(T)$-action is given by:

$$\begin{aligned} \delta : \quad W(T) \times (G/T)^{\gamma} &\longrightarrow (G/T)^{\gamma} \\ (nT, xT) &\longmapsto xnT. \end{aligned} \tag{60}$$

This map is well defined since on the one hand a different choice of the representative for the elements of $W(T)$ and G/T leads to

$$x(n \cdot t) \in xnT \quad \text{and} \quad (xt)n = xn \cdot \underbrace{n^{-1}tn}_{\in T} \in xnT,$$

while on the other hand a fixed point xT maps to a fixed point xnT because

$$g_0 \in xTx^{-1} = xnTn^{-1}x^{-1}.$$

Note that δ obviously describes a group action of $W(T)$ on $(G/T)^\gamma$.
Now, in order to show that the number of fixed points equals the order of the Weyl group, we have to show that the $W(T)$-action δ is simply transitive.
We define for the chosen fixed point x_0T above the map $\delta_{[x_0]}$ to be:

$$\begin{array}{rccc} \delta_{[x_0]}: & W(T) & \longrightarrow & (G/T)^\gamma \\ & nT & \longmapsto & \delta(nT, x_0T) = x_0nT. \end{array}$$

The action δ is simply transitive if and only if $\delta_{[x_0]}$ defines a bijection.
Now, $\delta_{[x_0]}(nT) = x_0T$ if and only if $nT = T$, i.e. $n \in T$, which is equivalent to nT being the neutral element in $W(T)$. Hence, $\delta_{[x_0]}$ is injective.
For any other fixed point x_1T, we conclude

$$g_0 = x_1t_1x_1^{-1}$$

for $t_1 \in T$. Thus, we obtain

$$t_0 = x_0^{-1}x_1t_1x_1^{-1}x_0.$$

Now, because g_0 is generating T', the t_k are generating T.
Therefore, it follows that:

$$T = \overline{(t_0)^{\mathbb{Z}}} = \overline{(x_0^{-1}x_1t_1x_1^{-1}x_0)^{\mathbb{Z}}} = x_0^{-1}x_1\overline{(t_1)^{\mathbb{Z}}}x_1^{-1}x_0 = x_0^{-1}x_1Tx_1^{-1}x_0.$$

We conclude that $x_0^{-1}x_1T$ lies in $W(T)$ or equivalently that $x_1T = x_0nT$ for an $nT \in W(T)$. This leads to $x_1T = \delta_{[x_0]}(nT)$.
And consequently, $\delta_{[x_0]}$ is surjective which finishes the proof of 1..

2. To show that each fixed point is non-degenerate, we make use of the fact that G/T is reductive as a homogeneous space, i.e. that our Lie algebra $\mathfrak{g}$ splits (compare Section 5.1) into the vertical space $\mathfrak{h} \oplus \mathfrak{h}_s$ and an $\mathrm{Ad}(T)$ invariant complement

$$\mathfrak{m} = \mathfrak{g} \cap \bigoplus_{\alpha \in R} \mathfrak{g}_{s,\alpha}.$$

The $\mathrm{Ad}(T)$ invariance of $\mathfrak{m}$ follows from that of $\mathfrak{g}_{s,\alpha}$.

Using again common knowledge about principle fibre bundles and reductive homogeneous spaces (cf. [5]), we obtain the following isomorphism of vector bundles

$$\begin{tikzcd} G \times_{Ad} \mathfrak{m} \arrow[rr, "\cong"] \arrow[dr] & & T(G/T) \arrow[dl] \\ & G/T. & \end{tikzcd}$$

It is given for $\mathfrak{m} \subset T_eG$ by

$$\begin{array}{rccc} d\pi_G \circ dL_g : & G \times_{\mathrm{Ad}} \mathfrak{m} & \longrightarrow & T(G/T) \\ & [g, X_{\mathfrak{m}}]_{\mathrm{Ad}} & \longmapsto & (d\pi_G)_g \circ (dL_g)_e\,(X_{\mathfrak{m}}). \end{array}$$

Now, let $[x]$ be a fixed point of $\gamma^{G/T}$, i.e. $g_0 x = x t_0$ and take an element $X \in T_{[x]}G/T$ which is represented by an element $[x, X_{\mathfrak{m}}]_{\mathrm{Ad}} \in G \times_{\mathrm{Ad}} \mathfrak{m}$.
The subsequent computation shows how $\left(dL_{g_0}^{G/T}\right)_{[x]}$ acts on X.

$$\begin{aligned} \left(dL_{g_0}^{G/T}\right)_{[x]} X = & \left(dL_{g_0}^{G/T}\right)_{[x]} \circ (d\pi_G)_x \circ (dL_x)_e\,(X_{\mathfrak{m}}) \\ =& (d\pi_G)_{x\cdot t_0} \circ \underbrace{(dL_{g_0})_x \circ (dL_x)_e}_{=\left(dL_{x\cdot t_0}\right)_e}(X_{\mathfrak{m}}) \\ =& (d\pi_G)_{x\cdot t_0} \circ (dL_x)_{t_0} \circ (dR_{t_0})_e \circ \mathrm{Ad}(t_0)(X_{\mathfrak{m}}) \\ =& (d\pi_G)_x \circ (dL_x)_e \circ \mathrm{Ad}(t_0)(X_{\mathfrak{m}}) \end{aligned}$$

Therefore, $\left(dL_{g_0}^{G/T}\right)_{[x_0]} X = X$ holds if and only if $\mathrm{Ad}(t_0)(X_{\mathfrak{m}}) = X_{\mathfrak{m}}$.
But, since g_0 is generating a maximal torus, so is t_0.
This implies that

$$\mathrm{Ad}(t_0)(X_{\mathfrak{m}}) = X_{\mathfrak{m}}$$

if and only if $X_{\mathfrak{m}}$ is $\mathrm{Ad}(T)$ invariant.
Seeing this in relation to the fact that $\mathfrak{t} = \mathfrak{h} \oplus \mathfrak{h}_s$ is a maximal Abelian subalgebra, we conclude:

$$X_{\mathfrak{m}} \in \mathfrak{m} \cap \mathfrak{t} = \{0\}.$$

Hence, we obtain $X_{\mathfrak{m}} = 0$.
We see that the only vector $X \in T_{[x_0]}G/T$ left invariant by $L_{g_0}^{G/T}$ is the 0-vector and therefore, $[x_0]$ is a non-degenerate fixed point.

□

Remark 5.44:
The set of elements in a torus T that generate this torus is a dense set. Actually, the set of elements in T that do not generate the torus are countable. In particular, they have Lebesgue measure 0.
This fact transfers to any compact Lie group, i.e. the set of elements in G that do not generate a maximal torus are countable and in particular, they have Lebesgue measure 0. Furthermore, the set of elements that generate a maximal torus is dense in G.

From now on for the rest of this sub-subsection, suppose that g_0 is generating a maximal torus.

The subsequent lemma shows how $\gamma^G = L_{g_0}$ acts on fibres $G_{[x_0]} = \pi_G^{-1}([x_0])$ over a fixed point $[x_0] \in (G/T)^\gamma$.

Lemma 5.45:
There is a map $\Omega : (G/T)^\gamma \to T$ with the following properties.

1. *For every $[x] \in (G/T)^\gamma$ and every $g \in \pi_G^{-1}([x])$, the map L_{g_0} restricted to the fibre $G_{[x]}$ is given by*

$$\begin{array}{rccc} L_{g_0}|_{G_{[x]}}: & G_{[x]} & \longrightarrow & G_{[x]} \\ & g & \longmapsto & g \cdot \Omega([x]). \end{array}$$

2. *The map Ω and the action δ of the Weyl group (compare Equation (60)) are correlated.*
For any $nT \in W(T)$ and any fixed point $[x] \in (G/T)^\gamma$, we obtain:

$$\Omega(\delta(nT, [x])) = n^{-1}\Omega([x])n. \tag{61}$$

Remark 5.46:
If there exists a map $\Omega : (G/T)^\gamma \to T$ fulfilling the first property of Lemma 5.45, then this map is obviously unique.

Proof of Lemma 5.45.

1. We construct Ω explicitly.
We already know that if $x \in (G/T)^\gamma$, we get an $t_0 \in T$ such that

$$L_{g_0}(x) = g_0 \cdot x = x \cdot t_0.$$

We define $\Omega([x])$ to be t_0, i.e. implicitly

$$g_0 \cdot x = x \cdot \Omega(\pi_G(x)).$$

This is well defined since for another representative $x \cdot s$ of $[x]$, we obtain

$$g_0 \cdot (x \cdot s) = (g_0 \cdot x) \cdot s = (x \cdot \Omega(\pi_G(x))) \cdot s = (x \cdot s) \cdot \Omega(\pi_G(x))$$

where the last equality holds because T is Abelian.

2. For the second assertion recall that for a fixed point $[x] \in (G/T)^\gamma$ and an element nT of the Weyl group, we obtain

$$\delta(nT, [x]) = [xn] \in (G/T)^\gamma.$$

Consequently, we get:

$$\begin{aligned} g_0 \cdot x &= x \cdot \Omega([x]) && \text{as well as} \\ g_0 \cdot x \cdot n &= x \cdot n \cdot \Omega([x \cdot n]). \end{aligned}$$

Separating g_0, we obtain

$$g_0 = x \cdot \Omega([x]) \cdot x = x \cdot n \cdot \Omega([x \cdot n]) \cdot n^{-1} \cdot x^{-1}$$

which finally leads to

$$\Omega([x]) = n \cdot \Omega([x \cdot n]) \cdot n^{-1}.$$

We note that the preceding expression does not depend on the representing element n of $nT \in W(T)$.

□

Corollary 5.47:
There is a unique lift $\hat{\Omega} : (G/T)^\gamma \longrightarrow \hat{T}$ of Ω such that

1. *For every $\tilde{g} \in \pi_{\tilde{G}}^{-1}([x])$, we get $L_{\tilde{g}_0}(\tilde{g}) = \tilde{g} \cdot \hat{\Omega}([x])$.*
2. *$\pi_{1,\tilde{G}} \circ \hat{\Omega} = \Omega$.*

5.6.2 Equivariant holomorphic torsion

In this sub-subsection, we present our results for the equivariant holomorphic torsion for the legitimate action described above, i.e. $\vec{\gamma} = \left(\gamma^{G/T}, \gamma^G, \gamma^{\mathfrak{L}}\right)$ with $\gamma^{\mathfrak{L}} = L_{\tilde{g}_0}$ for an arbitrary but fixed element $\tilde{g}_0 \in \tilde{G}$.

We summarise the result for an unconditioned choice of $\tilde{g}_0$ in $\tilde{G}$ in the following theorem.

Theorem 5.2:
Let G be a compact, connected, real even-dimensional Lie group and let $T \subset G$ be a maximal torus of G. And equip G with a holomorphic fibre bundle structure, as described in Subsection 5.1.
Let furthermore $\chi : \pi_1(G) \to U(1)$ be a character of $\pi_1(G)$ and let $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C}$ be the Hermitian line bundle over G associated via this character.
Furthermore, equip $\mathfrak{L}$ with a holomorphic structure $\bar{\partial}_{\mathfrak{L}}$, given by $\bar{\partial}_{\mathfrak{L}} = \bar{\partial}_{\mathfrak{L}_\chi} + \varepsilon(\pi_G^*(\omega))$, where $\bar{\partial}_{\mathfrak{L}_\chi}$ denotes the natural holomorphic structure on $\mathfrak{L}_\chi = \tilde{G} \times_\chi \mathbb{C}$, we described in Lemma5.10 and where ω is a $\bar{\partial}$-closed form in $\mathfrak{A}^{(0,1)}(G/T)$.
At last, let $\tilde{g}_0$ be an element of $\tilde{G}$ such that for $g_0 = \pi_{1,\tilde{G}}(\tilde{g}_0)$ the $(0,1)$-form ω is left invariant under the pullback with $L_{g_0}^{G/T}$.
Let $\vec{\gamma} = (\gamma^{G/T}, \gamma^G, \gamma^{\mathfrak{L}}) = (L_{g_0}^{G/T}, L_{g_0}, L_{\tilde{g}_0})$ denote the induced legitimate action of $\tilde{g}_0$ on $\mathfrak{L}$, given by

$$\gamma^{\mathfrak{L}} := L_{\tilde{g}_0} : \begin{array}{ccc} \mathfrak{L} & \longrightarrow & \mathfrak{L} \\ [\tilde{g}, z]_\chi & \longmapsto & [\tilde{g}_0\tilde{g}, z]_\chi \end{array}.$$

Then we obtain the following expression for the equivariant holomorphic ζ-function for sufficiently large $\mathrm{Re}(z)$*:*

$$Z_{\tilde{\gamma}^{\mathfrak{L}}}^{\mathfrak{L}}(z) = -\sum_{\lambda \neq 0} \lambda^{-z} \sum_t t(-1)^t \mathrm{ind}(\gamma^{\mathcal{W}^{(\lambda;t)}}, \Box_{\mathcal{W}^{(\lambda;t)}}).$$

Proof.
We start using Theorem 5.1 to get the following expression for the equivariant ζ-function.

$$\begin{aligned} Z_{\tilde{\gamma}^{\mathfrak{L}}}^{\mathfrak{L}}(z) = &-\sum_{\lambda \neq 0} \lambda^{-z} \sum_t t(-1)^t \mathrm{ind}(\gamma^{\mathcal{W}^{(\lambda;t)}}, \Box_{\mathcal{W}^{(\lambda;t)}}) \\ &+ \begin{cases} \chi_\gamma(T) Z_\gamma^{\mathcal{W}^{(0;0)}}(z) & \text{if } \chi \equiv 1 \\ 0 & \text{if } \chi \not\equiv 1 \end{cases} \end{aligned}$$

The second term is 0 because $\chi_\gamma(T)$ is 0 for $\gamma^{\mathfrak{L}} = L_{\tilde{g}_0}$ which we show now.
Recall that $H^{0,q}(T, T \times \mathbb{C}) = \Lambda^q \left(\mathfrak{t}^{(0,1)}\right)^*$ and that the $\vec{\gamma}$ action on the cohomology of $T \times \mathbb{C}$ is given by $\gamma = L_{g_0}^* : \Lambda^q \left(\mathfrak{t}^{(0,1)}\right)^* \to \Lambda^q \left(\mathfrak{t}^{(0,1)}\right)^*$.
But since the Lie algebra consists of left invariant vector fields the $\vec{\gamma}$ action on $H^{0,q}(T, T \times \mathbb{C})$ is trivial.
We conclude:

$$\chi_\gamma(T) = \chi(T) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = (1-1)^n = 0.$$

$\Box$

Naturally, we get a corollary out of Theorem 5.2 that holds for isolated, non-degenerated fixed points of $\gamma^{G/T}$.

Corollary 5.48:
In the setting from Theorem 5.2.
If $g_0 = \pi_{1,\tilde{G}}(g_0)$ generates a maximal torus in G.
Then the expression for the equivariant ζ-function simplifies to:

$$Z^{\mathfrak{L}}_{\check{\gamma}^{\mathfrak{L}}}(z) = \left[\det_{T^{1,0}_{[x_0]}G/T}(1-\gamma^{-1}_{[x_0]})\right]^{-1} \cdot \sum_{[n]\in W(T)} Z^{\tilde{\mathfrak{L}}}_{\check{\gamma}^{\tilde{\mathfrak{L}}_\chi}_{[x_0\cdot n]}}(z).$$

Here, $[x_0] \in (G/T)^\gamma$ denotes an arbitrary chosen fixed point of $L^{G/T}_{g_0}$ that always exists (compare Lemma 5.40). Further on, $W(T)$ denotes the Weyl group of T in G.
The equivariant ζ-function on the right hand side is that of the holomorphic line bundle

$$\pi_{\tilde{\mathfrak{L}}_\chi} : \quad \tilde{\mathfrak{L}}_\chi := \hat{T} \times_\chi \mathbb{C} \quad \longrightarrow \quad T$$

with a group action

$$\begin{array}{rccc} \gamma^{\tilde{\mathfrak{L}}}_{[x_0\cdot n]} := L^{\tilde{\mathfrak{L}}}_{\hat{\Omega}([x_0\cdot n])} : & \tilde{\mathfrak{L}}_\chi & \longrightarrow & \tilde{\mathfrak{L}}_\chi \\ & \left[\hat{t}, z\right]_\chi & \longmapsto & \left[\hat{\Omega}([x_0\cdot n])\cdot \hat{t}, z\right]_\chi \end{array}$$

where $\hat{\Omega}$ is constructed as above (compare Corollary 5.47).

Proof.
Starting at the formula for the equivariant ζ-function of Theorem 5.2 and compare this with the formula from Corollary 5.36, we obtain:

$$Z^{\mathfrak{L}}_{\check{\gamma}^{\mathfrak{L}}}(z) = \sum_{[x]\in (G/T)^\gamma} \left(\det_{T^{1,0}_{[x]}G/T}\left(1-\gamma^{-1}_{[x]}\right)^{-1}\right) \cdot Z^{\mathfrak{L}_{[x]}}_{\check{\gamma}^{\mathfrak{L}_{[x]}}}(z)$$

Recall that every fixed point $[x]$ differs from $[x_0]$ by an element of $W(T)$, i.e. there is an $[n] \in W(T)$ such that

$$[x] = \delta(nT, [x_0]) = [x_0]\cdot [n]$$

(compare Equation (60)).
In the proof of Lemma 5.43 we saw, that $T_{[x]}(G/T)$ is isomorphic to $\mathfrak{m}$ as complex vector space and furthermore, that the map $\gamma^{-1}_{[x]}$ corresponds to the map $\mathrm{Ad}(\Omega([x]))$ via that isomorphism.

It follows, using the Ω-δ-equivariance, i.e. Equation (61), that we obtain:

$$\begin{aligned}
\det\nolimits_{T^{1,0}_{[x_0\cdot n]}G/T}\left(1-\gamma^{-1}_{[x_0\cdot n]}\right) &= \det\nolimits_{\mathfrak{m}^{(1,0)}}\left(1-\mathrm{Ad}(\Omega([x_0\cdot n])^{-1})\right)\\
&\overset{(61)}{=} \det\nolimits_{\mathfrak{m}^{(1,0)}}\left(\mathrm{Ad}\left([n]^{-1}\right)\circ\left(1-\mathrm{Ad}(\Omega([x_0])^{-1})\right)\circ\mathrm{Ad}\left([n]\right)\right)\\
&= \det\nolimits_{\mathfrak{m}^{(1,0)}}\left(1-\mathrm{Ad}(\Omega([x_0])^{-1})\right)\\
&= \det\nolimits_{T^{1,0}_{[x_0]}G/T}\left(1-\gamma^{-1}_{[x_0]}\right).
\end{aligned}$$

Thus, it does not depend on the choice of the fixed point $[x_0]$.
At last, we simplify the expression $Z^{\mathfrak{L}|_{\pi_G^{-1}([x])}}_{\tilde{\gamma}^{\mathfrak{L}}_{[x]}}(z)$ by translating it for $[x]=[x_0\cdot n]$ to an expression on $\tilde{\mathfrak{L}}_\chi\to T$.
Therefore, choose $\tilde{x}_0\in\pi^{-1}_{\tilde{G}}([x_0])$ and $\tilde{n}\in\pi_{1,\tilde{G}}(n)$.
Now, look at the map

$$\begin{array}{rccc}
[\tilde{x}_0\cdot\tilde{n}]_{\tilde{\rho}}: & \tilde{\mathfrak{L}} & \longrightarrow & \mathfrak{L}\,|_{\pi_G^{-1}([x_0\cdot n])}\\
& [\hat{s},z]_\chi & \longmapsto & [\tilde{x}_0\cdot\tilde{n}\cdot\hat{s},z]_\chi\,.
\end{array}\tag{62}$$

With the notations for the local trivialisations from Remark 5.23, take an $i\in I$ such that $[x_0\cdot n]\in U_i$.
It follows that $\tilde{x}_0\cdot\tilde{n}=\tilde{q}_i([x_0\cdot n])\cdot\hat{t}$ for one $\hat{t}\in\hat{T}$.
The following diagram commutes

$$\begin{tikzcd}
\tilde{\mathfrak{L}}_\chi \arrow[rrr, "{[\tilde{x}_0\cdot\tilde{n}]_{\tilde{\rho}}}"] \arrow[dr, "\tilde{\rho}(\hat{t})"'] & & & \mathfrak{L}\,|_{\pi_G^{-1}([x_0\cdot n])}\\
& \tilde{\mathfrak{L}}_\chi \arrow[r, "\Phi_{i,[x_0\cdot n]}"'] & \left(\phi^{-1}_{i,[x_0\cdot n]}\right)^*\mathfrak{L}_\chi \arrow[ur, "\left(\phi_{i,[x_0\cdot n]}\right)^*"'] &
\end{tikzcd}$$

because for $t\cdot s=\pi_{1,\hat{T}}(\hat{t}\cdot\hat{s})$ we obtain

$$\begin{aligned}
\left(\left(\phi_{i,[x_0\cdot n]}\right)^*\circ\Phi_{i,[x_0\cdot n]}\circ\tilde{\rho}(\hat{t})\right)\left([\hat{s},z]_\chi\right) &= \left(\left(\phi_{i,[x_0\cdot n]}\right)^*\circ\Phi_{i,[x_0\cdot n]}\right)\left(\left[\hat{t}\hat{s},z\right]_\chi\right)\\
&= \left(\phi_{i,[x_0\cdot n]}\right)^*\left(t\cdot s,\left[\tilde{q}_i([x_0\cdot n])\cdot\hat{t}\cdot\hat{s},z\right]_\chi\right)\\
&= \Bigg[\underbrace{\tilde{q}_i([x_0\cdot n])\cdot\hat{t}}_{=\tilde{x}_0\cdot\tilde{n}}\cdot\hat{s},z\Bigg]_\chi\\
&= [\tilde{x}_0\cdot\tilde{n}]_{\tilde{\rho}}\left([\hat{s},z]_\chi\right).
\end{aligned}$$

In particular, Equation (62) defines a biholomorphic isometry because $\tilde{\rho}(\hat{t})$ is one and so is $(\phi_{i,x})^*\circ\Phi_{i,x}$ (compare Lemma 5.24).

The action $\gamma^{\mathfrak{L}}$ restricted to $\mathfrak{L}\mid_{\pi_G^{-1}([x_0\cdot n])}$ is translated via $[\tilde{x}_0\cdot\tilde{n}]_{\tilde{\rho}}$ onto $\tilde{\mathfrak{L}}$ to the action $L^{\tilde{\mathfrak{L}}}_{\hat{\Omega}([x_0\cdot n])}$, explicitly given by

$$L^{\tilde{\mathfrak{L}}}_{\hat{\Omega}([x_0\cdot n])}\left([\hat{s},z]_\chi\right) := \left[\hat{\Omega}([x_0\cdot n])\cdot\hat{s},z\right]_\chi .$$

Consequently, we obtain:

$$Z^{\mathfrak{L}|_{\pi_G^{-1}([x_0\cdot n])}}_{\check{\gamma}^{\mathfrak{L}}[x_0\cdot n]}(z) = Z^{\tilde{\mathfrak{L}}}_{\check{\gamma}^{\tilde{\mathfrak{L}}}_{[x_0\cdot n]}}(z)$$

which finishes the proof. $\square$

Corollary 5.49:
In the situation of Corollary 5.48, if G has rank (which is defined as the real dimension of the maximal torus T) greater than 2 than the equivariant holomorphic ζ-function simplifies to

$$Z^{\mathfrak{L}}_{\check{\gamma}^{\mathfrak{L}}}(z) = \begin{cases} \chi_\gamma(T)Z^{\mathcal{W}^{(0;0)}}_{\gamma}(z) & \text{if } \chi\equiv 1\\ 0 & \text{if } \chi\not\equiv 1\end{cases}$$

because here the equivariant holomorphic torsion of a holomorphic line bundle vanishes. In particular, the equivariant holomorphic torsion vanishes, i.e.

$$\tau^{\mathfrak{L}}(\check{\gamma}^{\mathfrak{L}}) = 0.$$

Proof.
Let k denote in this proof the rank of G.
We just have to show that $Z^{\tilde{\mathfrak{L}}}_{\check{\gamma}^{\tilde{\mathfrak{L}}_\chi}_{[x_0\cdot n]}}(z)$ vanishes.
Starting at the definition of the equivariant ζ-function (compare Def. 2.43), we obtain for large $\mathrm{Re}(z)$:

$$Z^{\tilde{\mathfrak{L}}}_{\check{\gamma}^{\tilde{\mathfrak{L}}_\chi}_{[x_0\cdot n]}}(z) = \sum_{\lambda\in\sigma(\Box_{\mathfrak{L}^\chi_{x_0}})\setminus\{0\}} \lambda^{-z}\cdot\sum_{q\geq 0}^{k/2}(-1)^{q+1}q\cdot\mathrm{Tr}\left(\check{\gamma}^{\tilde{\mathfrak{L}}_\chi}_{[x_0\cdot n]}|_{\mathrm{Eig}_\lambda(\Box^{(0,q)}_{\mathfrak{L}^\chi_{x_0}})}\right).$$

Note that the Eigenspace decomposes as follows

$$\mathrm{Eig}_\lambda(\Box^{(0,q)}_{\mathfrak{L}_{x_0}}) = \Lambda^q\left(\mathfrak{t}^{(0,1)}\right)^*\otimes\mathrm{Eig}_\lambda(\Box^{(0,0)}_{\mathfrak{L}_{x_0}})$$

and note furthermore that $\check{\gamma}^{\tilde{\mathfrak{L}}_\chi}_{[x_0\cdot n]}$ covers a left transition on T that acts trivially on the cohomology, i.e. on $\left(\mathfrak{t}^{(0,1)}\right)^*$.

Consequently, we obtain:

$$
\begin{aligned}
Z^{\tilde{\mathfrak{L}}}_{\check{\gamma}^{\tilde{\mathfrak{L}}_X}_{[x_0\cdot n]}}(z) =& \sum_{\lambda\in\sigma(\square_{\mathfrak{L}^X_{x_0}})\setminus\{0\}} \lambda^{-z}\cdot \operatorname{Tr}\left(\check{\gamma}^{\tilde{\mathfrak{L}}_X}_{[x_0\cdot n]}\big|_{\operatorname{Eig}_\lambda(\square^{(0,0)}_{\mathfrak{L}^X_{x_0}})}\right)\cdot \sum_{q\geq 0}^{k/2}(-1)^{q+1}q\binom{n}{q}\\
=& \sum_{\lambda\in\sigma(\square_{\mathfrak{L}^X_{x_0}})\setminus\{0\}} \lambda^{-z}\cdot \operatorname{Tr}\left(\check{\gamma}^{\tilde{\mathfrak{L}}_X}_{[x_0\cdot n]}\big|_{\operatorname{Eig}_\lambda(\square^{(0,0)}_{\mathfrak{L}^X_{x_0}})}\right)\cdot \sum_{q\geq 0}^{k/2-1} k\cdot(-1)^{q}q\binom{n-1}{q}\\
=& \sum_{\lambda\in\sigma(\square_{\mathfrak{L}^X_{x_0}})\setminus\{0\}} \lambda^{-z}\cdot \operatorname{Tr}\left(\check{\gamma}^{\tilde{\mathfrak{L}}_X}_{[x_0\cdot n]}\big|_{\operatorname{Eig}_\lambda(\square^{(0,0)}_{\mathfrak{L}^X_{x_0}})}\right)\cdot k\cdot(1-1)^{k/2-1}\\
=&0.
\end{aligned}
$$

□

The determinant can be expressed using weights of the adjoint representation.

Corollary 5.50:
In the situation of Corollary 5.48 where the rank of G equals 2, we compute the equivariant torsion via:

$$
\tau^{\mathfrak{L}}(\check{\gamma}^{\mathfrak{L}}) = \prod_{\alpha\in R^+}\left(1-e^{-2\pi i\alpha}\Big(\Omega([x_0])\Big)\right)^{-1}\cdot \sum_{[n]\in W(T)} \tau^{\tilde{\mathfrak{L}}}(\check{\gamma}^{\tilde{\mathfrak{L}}}_{[x_0\cdot n]})
$$

where the product goes over all the positive roots of the Lie group G and where $e^{-2\pi i\alpha}$ denotes the global root corresponding to $-\alpha\in R^-$:

$$
\begin{array}{rccc}
e^{-2\pi i\alpha}: & T & \longrightarrow & U(1)\\
& t=\exp(X) & \longmapsto & e^{-2\pi i\alpha(X)}.
\end{array}
$$

Proof. Corollary 5.48 directly implies the following formula for the equivariant holomorphic torsion of $\mathfrak{L}$:

$$
\tau^{\mathfrak{L}}(\check{\gamma}^{\mathfrak{L}}) = \det\nolimits_{T^{1,0}_{[x_0]}G/T}\left(1-\gamma^{-1}_{[x_0]}\right)^{-1}\cdot \sum_{[n]\in W(T)} \tau^{\tilde{\mathfrak{L}}}(\check{\gamma}^{\tilde{\mathfrak{L}}}_{[x_0\cdot n]}).
$$

Now, observe that $\gamma_{[x_0]} = T_{[x_0]}L^{G/T}_{g_0}$ and furthermore, that

$$
T^{(1,0)}_{[x_0]}G/T = (T_{x_0}\pi_G)\circ(T_eL_{x_0})\left(\bigoplus_{\alpha\in R^+}\mathfrak{g}_\alpha\right).
$$

Using $g_0 \cdot x_0 = x_0 \cdot \Omega([x_0])$, we now compute:

$$\begin{aligned} T_{[x_0]} L_{g_0}^{G/T} \circ (T_{x_0}\pi_G) \circ (T_e L_{x_0})\,(X) &= \left(T_{x_0 \cdot \Omega([x_0])}\pi_G\right) \circ T_{x_0} L_{g_0} \circ (T_e L_{x_0})\,(X) \\ &= \left(T_{x_0 \cdot \Omega([x_0])}\pi_G\right) \circ T_{x_0} L_{g_0} \circ (T_e L_{x_0})\,(X) \\ &= (T_{x_0}\pi_G) \circ (T_e L_{x_0}) \circ \mathrm{Ad}\,(\Omega([x_0]))\,(X) \end{aligned}$$

Consequently, we obtain:

$$\det_{T^{1,0}_{[x_0]} G/T}\left(1 - \gamma_{[x_0]}^{-1}\right) = \det_{\bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha}\left(1 - \mathrm{Ad}\,(\Omega([x_0]))^{-1}\right)$$

We apply the root space composition and finally obtain:

$$\det_{T^{1,0}_{[x_0]} G/T}\left(1 - \gamma_{[x_0]}^{-1}\right) = \prod_{\alpha \in R^+} \left(1 - e^{2\pi i \alpha}(\Omega([x_0])^{-1})\right)$$

which finishes the proof. $\square$

This finally reduces the computation of the equivariant holomorphic torsion for the holomorphic, Hermitian line bundle $\mathfrak{L} = \tilde{G} \times_\chi \mathbb{C} \to G$ with natural Hermitian metric and a holomorphic structure $\bar{\partial}_{\mathfrak{L}} = \bar{\partial}_{\mathfrak{L}_\chi} + \varepsilon(\pi_G^*(\omega))$ for the action $L_{\tilde{g}_0} : \mathfrak{L} \to \mathfrak{L}$ to a computation of the equivariant torsion of the Hermitian, holomorphic line bundle $\tilde{\mathfrak{L}}_\chi \to T$ with holomorphic structure $\bar{\partial}_{\tilde{\mathfrak{L}}_\chi} : \tilde{\mathfrak{L}} \to \tilde{\mathfrak{L}}$ for different actions $\gamma_{[x]}^{\tilde{\mathfrak{L}}}$.
In particular, the result does not depend on the $(0,1)$-form ω.

A Appendix: Linear algebra

In this part of the appendix, we introduce the linear algebra setting that we need. In particular, give a definition of an almost complex vector space and recall some useful definitions and properties of Hermitian vector spaces and operators thereon. Furthermore, we state two identities for the Hodge-Star-operator that we need throughout this thesis.

This part of the appendix is more or less common knowledge. Nonetheless, some conventions have to be made. Furthermore, in order to simplify the process of understanding for the reader, we choose to repeat some of the necessary definitions and results.

Throughout this thesis we will constantly use the fact, that the fibre of the (co-)tangent space over a point in a complex manifold is a complex vector space and that some operators like the Hodge-Star operator can be described by linear algebra purely. Therefore, it sometimes suffices to understand these objects on such a low level.

The first object of interest will be an almost complex vector space.

Definition A.1:

- The tuple (V, J_V) where V is a real even dimensional vector space and J_V is an automorphism of V such that $J_V^2 = -\mathrm{id}_V$ is called **almost complex vector space**.
  The map J_V is called **almost complex structure** on V.
- The triple (V, J_V, g_V) is called **Hermitian, almost complex vector space** if (V, J_V) is an almost complex vector space and g_V is a Euclidean metric which is **compatible** with the almost complex structure J_V, i.e. J_V is an isometry of (V, g_V).

Remark A.2:

- Every almost complex vector space (V, J_V) has a natural orientation given by a base $e_1, J_V(e_1), \ldots, e_n, J_V(e_n)$.
- We may define an almost complex structure $J_V^\flat$ on the dual space V^*, via

$$\left(J_V^\flat(\alpha)\right)(v) := \alpha(J_V(v))$$

  for any $\alpha \in V^*$ and $v \in V$.
- In particular, for a Hermitian, almost complex vector space, if $e_1, \ldots, e_n$ is orthonormal with respect to g_V and if furthermore $J_V(e_k)$ does not lie in the linear

hull of $e_1, \dots, e_n$ for any $k = 1 \dots n$, then $e_1, J_V(e_1) \dots, e_n, J_V(e_n)$ forms an orthonormal base of V.
In particular, this base defines an orientation on V that does not depend on the choice of the $e_1, \dots, e_n$.
Further on, its dual base given by $e^1, J_V^\flat(e^1), \dots, e^n, J_V^\flat(e_n)$ is orthonormal as well.

- For a Hermitian, almost complex vector space, we obtain the **canonical volume form**:

$$\mathrm{dvol}_V := e^1 \wedge J_V(e^1) \wedge \dots \wedge e^n \wedge J_V(e^n) \in \Lambda^{2n} V^*.$$

Let from now on for the time being (V, J_V, g_V) be a Hermitian, almost complex vector space.

Remark A.3:

- The complexified vector space $V_\mathbb{C} := V \otimes_\mathbb{R} \mathbb{C}$ splits into its **holomorphic** part,

$$V^{(1,0)} := \mathrm{Ker}(J_V - i),$$

and its **antiholomorphic** part,

$$V^{(0,1)} := \mathrm{Ker}(J_V + i),$$

i.e. we obtain:

$$V_\mathbb{C} = V^{(1,0)} \oplus V^{(0,1)}.$$

Likewise, its dual space $V_\mathbb{C}^* := V^* \otimes_\mathbb{R} \mathbb{C}$ splits, too:

$$V_\mathbb{C}^* = V^{*(1,0)} \oplus V^{*(0,1)}.$$

This splitting may be extended to the entire exterior algebra of $V_\mathbb{C}^*$:

$$\Lambda^q V_\mathbb{C}^* = \bigoplus_{r+s=q} \Lambda^r V^{*(1,0)} \wedge \Lambda^s V^{*(1,0)} =: \bigoplus_{r+s=q} \Lambda^{(r,s)} V^*$$

- On $V_\mathbb{C}$, we have a Hermitian product h_V induced by g_V given through

$$h_V(v \otimes_\mathbb{R} z, \hat{v} \otimes_\mathbb{R} y) := g_V(v, \hat{v}) \cdot z\bar{y}.$$

We denote its analog on $V_\mathbb{C}^*$ by $h_V^\flat$ and in abuse of notation, $h_V^\flat$ will also be used for the extended Hermitian form on the exterior algebra $\Lambda^\cdot V_\mathbb{C}^*$.

The operator we define right now helps us to compute the dual of a certain operator, namely the $\bar{\partial}$-operator, on complexified differential forms with coefficients in a Hermitian vector bundle.

Definition A.4:
Let (W, h_W) be a finite dimensional Hermitian vector space.
There is a natural operator

$$\bar{*}_{V\otimes W}: \quad \left(\Lambda^{(p,q)}V^*\right)\otimes W \quad \longrightarrow \quad \left(\Lambda^{(n-p,n-q)}V^*\right)\otimes W^*,$$

the **Hodge-Star operator**.
It is given implicitly through:

$$\left(h_V^\flat \otimes h_W\right)(\alpha\otimes w, \beta\otimes w')\mathrm{dvol}_V^{g_V} = \alpha \wedge \left(\iota_w \bar{*}_{V\otimes W}(\beta\otimes w')\right)$$

extended linearly onto the whole tensor product.
For $w \in W$, the operator ι_w denotes the mapping $W_{\mathbb{C}}^*$ to $\mathbb{C}$ putting w into the first component, i.e. $(\iota_w\kappa)(w_1,\ldots,w_{q-1}) := \kappa(w, w_1, \ldots, w_{q-1})$.

Remark A.5:

- Recall that

$$\left(h_V^\flat \otimes h_W\right)(\alpha\otimes w, \beta\otimes w') = h_V^\flat(\alpha,\beta)\cdot h_W(w,w')$$

  and therefore

$$\bar{*}_{V\otimes W}(\beta\otimes w') = (\bar{*}_V\beta)\otimes h_W(\cdot, w') \tag{63}$$

  for an operator

$$\bar{*}_V : \Lambda^{(p,q)}V^* \to \Lambda^{(n-p,n-q)}V^*.$$

- $\bar{*}_{V\otimes W}$ and $\bar{*}_V$ are $\mathbb{C}$-antilinear maps.

Now, that we have defined, what the Hodge-Star operator is, we will proof a small Lemma, that we use for the splitting of the $\bar{\partial}^*$ operator into a horizontal and a vertical part. Actually it will be used directly, when we introduce the induced holomorphic structure of a line bundle, if we restrict it to a complex submanifold, in our case this submanifold is the fibre of a holomorphic fibre bundle.

Lemma A.6:
Let (V, J_V, g_V), (U, J_U, g_U) be two Hermitian, almost complex vector spaces and let $\varphi : U \hookrightarrow V$ be an isometric embedding that is compatible with the almost complex structures, i.e. $\varphi(J_U(u)) = J_V(\varphi(u))$.
Then for any Hermitian vector space (W, h_W) the following two statements hold.

$$\begin{aligned} \bar{*}_{U\otimes W} \circ \varphi^* &= \varphi^* \circ \bar{*}_{V\otimes W} \circ \varepsilon(\mathrm{dvol}_{\varphi(U)^\perp}) \\ \bar{*}_{V\otimes W} \mid_{\Lambda^\cdot \varphi(U)^*} &= \varepsilon\left(\mathrm{dvol}_{\varphi(U)^\perp}\right) \circ \bar{*}_{V\otimes W} \circ \varepsilon\left(\mathrm{dvol}_{\varphi(U)^\perp}\right) \mid_{\Lambda^\cdot \varphi(U)^*} \end{aligned}$$

Here, $\varepsilon(\alpha)$ denotes the wedge-product with α from the left hand side and $\varphi(U)^\perp$ is the orthogonal complement of $\varphi(U)$ in V regarding g_V.

Proof.
Because of Equation (63), the first statement is true if and only if it is true for $\bar{*}_U$ and $\bar{*}_V$ without the W part.
Observe now that $V = \varphi(U) \oplus^\perp \varphi(U)^\perp$ which leads to a splitting

$$\Lambda^\cdot V^* = \Lambda^\cdot \left(\varphi(U)^*\right) \wedge \Lambda^\cdot \left(\left(\varphi(U)^\perp\right)^*\right),$$

using the musical isomorphism $v \mapsto g_V(\cdot, v)$ from V to V^*.
In particular, we obtain $\mathrm{dvol}_V = \mathrm{dvol}_{\varphi(U)} \wedge \mathrm{dvol}_{\varphi(U)^\perp}$.
Now let ω be in $\Lambda^p\left(\varphi(U)^* \otimes_{\mathbb{R}} \mathbb{C}\right) \wedge \Lambda^q\left(\left(\varphi(U)^\perp\right)^* \otimes_{\mathbb{R}} \mathbb{C}\right)$ with non-zero q.
We observe that, on the left hand side, the pullback $\varphi^*\omega = 0$ and, on the right hand side, $\varepsilon(\mathrm{dvol}_{\varphi(U)^\perp})\omega = 0$.
Therefore, the statement holds for these forms.
So let from now on q equal zero, i.e. $\omega \in \Lambda^p\left(\varphi(U)^* \otimes_{\mathbb{R}} \mathbb{C}\right)$.
Now, look at the defining equation for the Hodge-Star-operator $\bar{*}_V$ on V.
The expression:

$$h^\flat_V(\beta, \omega \wedge \mathrm{dvol}_{\varphi(U)^\perp})\mathrm{dvol}_V = \beta \wedge \bar{*}_V\left(\omega \wedge \mathrm{dvol}_{\varphi(U)^\perp}\right)$$

is zero if β is not of the type $\beta = \beta' \wedge \mathrm{dvol}_{\varphi(U)^\perp}$ for a β' in $\Lambda^\cdot\left(\varphi(U)^* \otimes_{\mathbb{R}} \mathbb{C}\right)$.
On the other hand, we get for $\beta = \beta' \wedge \mathrm{dvol}_{\varphi(U)^\perp}$:

$$h^\flat_V(\beta, \omega \wedge \mathrm{dvol}_{\varphi(U)^\perp})\mathrm{dvol}_V = \beta' \wedge \bar{*}_V\left(\omega \wedge \mathrm{dvol}_{\varphi(U)^\perp}\right) \wedge \mathrm{dvol}_{\varphi(U)^\perp}. \tag{64}$$

Further on, computing the left hand side, we obtain:

$$\begin{aligned} h^\flat_V(\beta, \omega \wedge \mathrm{dvol}_{\varphi(U)^\perp})\mathrm{dvol}_V =& h^\flat_V(\beta', \omega)\mathrm{dvol}_V \\ =& h^\flat_U(\varphi^*(\beta'), \varphi^*(\omega))\mathrm{dvol}_{\varphi(U)} \wedge \mathrm{dvol}_{\varphi(U)^\perp}. \end{aligned} \tag{65}$$

Now, we compare Equation (64) and Equation (65) and derive:

$$\begin{aligned}\varphi^*\left(\beta' \wedge \bar{*}_V\left(\omega \wedge \mathrm{dvol}_{\varphi(U)^\perp}\right)\right) =& h_U^\flat(\varphi^*(\beta'), \varphi^*(\omega))\varphi^*\mathrm{dvol}_{\varphi(U)} \\ =& \varphi^*(\beta')) \wedge \bar{*}_U\left(\varphi^*\omega\right)\end{aligned}$$

which, holding for any β', completes the proof of the first statement.

The second statement can be easily seen, using an orthonormal frame. □

B Fréchet space of sections in a vector bundle

The content of this section is to define and understand the Fréchet structure of the vector space of smooth sections from a manifold into a vector bundle.

First of all, we define what a Fréchet space is (compare [30]).

Definition B.1:
A **Fréchet space** is a topological vector space $\mathbb{F}$, with the following properties:

- $\mathbb{F}$ is metrizable.
- $\mathbb{F}$ is complete.
- $\mathbb{F}$ is locally convex, i.e. there is a basis $\{B_k\}_{k\in K}$ of the topology of $\mathbb{F}$, such that each base set B_k is a convex subset of $\mathbb{F}$.

For maps into a Fréchet space $\mathbb{F}$, there is a concept of differentiability.

Definition B.2:
Let $\mathbb{F}$ be a Fréchet space, $\Omega \subset \mathbb{R}^n$ be an open subset and $h : \Omega \longrightarrow \mathbb{F}$ be a map.

- h is **partially differentiable in $\boldsymbol{x} \in \boldsymbol{\Omega}$** if for each $l = 1, \dots, n$ there is an element $\frac{\partial h}{\partial x_l}(x) \in \mathbb{F}$ such that
$$\lim_{0 \neq t \to 0} t^{-1}\left(h(x + t \cdot e_l) - h(x)\right) = \frac{\partial h}{\partial x_l}(x).$$
- h is **differentiable in $\boldsymbol{x} \in \boldsymbol{\Omega}$** if it is partially differentiable in x and if the following equation holds:
$$\lim_{y \to x}\left(h(y) - h(x) - \sum_{l=1}^{n}(y_l - x_l) \cdot \frac{\partial h}{\partial x_l}(x)\right) = 0.$$
- h is **(partially) differentiable** on Ω if it is (partially) differentiable in x for every x in Ω.

Let M be a compact Riemannian manifold and $\mathcal{Q} \longrightarrow M$ be a smooth vector bundle with Euclidean metric and a metric connection ∇.

Lemma B.3:
Let $V \subset M$ be an open subset.
Then the space of smooth sections from V into $\mathcal{Q}$ denoted by $\Gamma(V, \mathcal{Q}\,|_V)$ becomes a Fréchet space.
In particular, the space $\Gamma(M, \mathcal{Q})$ is a Fréchet space.

Proof. First, we have to define a topology on $\Gamma(V, \mathcal{Q}|_V)$.
Take therefore the following set of semi-norms on $\Gamma(V, \mathcal{Q}|_V)$. For each compact subset $K \subset V$ and each $l \in \mathbb{N}$, define for $s \in \Gamma(V, \mathcal{Q}|_V)$:

$$\|s\|_{l,K} := \sup_{t \leq l} \left(\sup_{x \in K} \|\nabla^t s\|_x \right),$$

where ∇^t denotes the map

$$\nabla^t : \quad \Gamma(V, \mathcal{Q}|_V) \quad \longrightarrow \quad \Gamma\left(V, \left(\bigotimes_{i=1}^t T^*M\right) \otimes \mathcal{Q}|_V\right),$$

induced be the metric connection ∇ as well as by the Levi-Civita connection on TM.
The norm on $\Gamma(V, \bigotimes_{i=1}^t T^*M \otimes \mathcal{Q}|_V)|_x$ is given by the metric on M as well as the Euclidean metric on $\mathcal{Q}$.
That $\Gamma(V, \mathcal{Q}|_V)$ is a real topological vector space is obvious.
The local convexity can be seen by taking ε balls in $\Gamma(V, \mathcal{Q}|_V)$.
That $\Gamma(V, \mathcal{Q}|_V)$ is metrizable follows from the fact that every open set $U \subset V$ can be approximated by a sequence of compact sets $\{K_i\}_{i\in\mathbb{N}}$ such that $K_i \subset \mathrm{int}(K_{i+1}) \subset U$ for every $i \in \mathbb{N}$ and the fact that

$$\|s\|_{l,K_i} \leq \|s\|_{k,K_{i+1}}$$

for every $i \in \mathbb{N}$ (compare [30, Ch. 10.3] for the analogous result for $C^\infty(V, \mathbb{R}^m)$).
The completeness follows now from the theorem of uniform convergence (compare again [30, Ch. 10.3] for the $C^\infty(V, \mathbb{R}^m)$ case). □

Now, $\mathcal{Q} \to M$ is a vector bundle. Hence, we may chose a finite covering $\{V_j\}_{j\in J}$ of M such that $\mathcal{Q} \longrightarrow M$ becomes locally trivial, i.e. such that there are smooth maps

$$\psi_j : \quad \mathcal{Q}|_{V_j} \quad \longrightarrow \quad V_j \times \mathbb{R}^m.$$

Lemma B.4:
Let now $\Omega \subset \mathbb{R}^n$ be an open subset.
A map

$$\varphi : \quad \Omega \quad \longrightarrow \quad \Gamma(M, \mathcal{Q})$$

is differentiable if and only if the induced maps

$$\varphi|_{V_j} : \quad \Omega \quad \longrightarrow \quad \Gamma(V_j, \mathcal{Q}|_{V_j})$$

are differentiable for all $j \in J$.

Proof. "$\Rightarrow$": Let $\varphi : \Omega \longrightarrow \Gamma(M, \mathcal{Q})$ be differentiable.

Consequently, there are $\frac{\partial\varphi}{\partial x_k}(x) \in \Gamma(M, \mathcal{Q})$ for every $x \in \Omega$ such that

$$\lim_{y\to x} \left\| \varphi(y) - \varphi(x) - \sum_k (y_k - x_k) \frac{\partial\varphi}{\partial x_k}(x) \right\|_{l,K} = 0$$

Defining $\frac{\partial\varphi|_{V_j}}{\partial x_k}(x)$ to be $\frac{\partial\varphi}{\partial x_k}(x) \mid_{V_j}$, we obtain for every $x \in \Omega$, every compact subset $K \subset V_j$ and every $l \in \mathbb{N}$:

$$\begin{aligned}\lim_{t\to 0} &\left\| \varphi \mid_{V_j} (x + t \cdot e_k) - \varphi \mid_{V_j} (x) - \frac{\partial\varphi \mid_{V_j}}{\partial x_k}(x) \right\|_{l,K} \\ &= \lim_{t\to 0} \left\| \varphi(x + t \cdot e_k) - \varphi(x) - \frac{\partial\varphi}{\partial x_k}(x) \right\|_{l,K} = 0.\end{aligned}$$

Thus $\varphi \mid_{V_j}$ is partially differentiable.

Almost the same observation for the continuity of $\frac{\partial\varphi|_{V_j}}{\partial x_k}(x)$.

$$\begin{aligned}\lim_{y\to x} &\left\| \varphi \mid_{V_j} (y) - \varphi \mid_{V_j} (x) - \sum_k (y_k - x_k) \frac{\partial\varphi \mid_{V_j}}{\partial x_k}(x) \right\|_{l,K} \\ &= \lim_{y\to x} \left\| \varphi(y) - \varphi(x) - \sum_k (y_k - x_k) \frac{\partial\varphi}{\partial x_k}(x) \right\|_{l,K} = 0.\end{aligned}$$

Hence, $\varphi \mid_{V_j}$ is differentiable.

"$\Leftarrow$": Suppose now, that for every $j \in J$ the map

$$\varphi \mid_{V_j}\colon \quad \Omega \quad \longrightarrow \quad \Gamma(V_j, \mathcal{Q} \mid_{V_j})$$

is differentiable.
We now use the sheave property of the vector space of sections, i.e. we use that if two sections $s_i \in \Gamma(V_i, \mathcal{Q} \mid_{V_i})$ and $s_j \in \Gamma(V_i, \mathcal{Q} \mid_{V_i})$ coincide on $V_i \cap V_j$, then there is a section $s_{ij} \in \Gamma(V_i \cup V_j, \mathcal{Q} \mid_{V_i \cup V_j})$ such that $s_i = s_{ij} \mid_{V_i}$ and $s_j = s_{ij} \mid_{V_j}$.
Observe that we have not made use of the fact that M is compact in the "$\Rightarrow$"-direction of this proof.
We conclude that by "$\Rightarrow$" the following equation holds.

$$\frac{\partial\varphi \mid_{V_i}}{\partial x_k}(x) \mid_{V_i \cap V_j} = \frac{\partial\varphi \mid_{V_j}}{\partial x_k}(x) \mid_{V_i \cap V_j}$$

Consequently, we obtain a section $\frac{\partial\varphi}{\partial x_k}(x)$ for every $x \in \Omega$ and for each $k = 1 \ldots n$ such that

$$\frac{\partial\varphi}{\partial x_k}(x) \mid_{V_j} = \frac{\partial\varphi \mid_{V_i}}{\partial x_k}(x).$$

Now, we show the two limit formulae for the differentiability of φ:

$$\lim_{t\to 0}\left\|\varphi(x+t\cdot e_k)-\varphi(x)-\frac{\partial\varphi}{\partial x_k}(x)\right\|_{l,K}$$
$$\leq \lim_{t\to 0}\sum_j\left\|\varphi\,|_{V_j}\,(x+t\cdot e_k)-\varphi\,|_{V_j}\,(x)-\frac{\partial\varphi\,|_{V_j}}{\partial x_k}(x)\right\|_{l,K\cap V_j}=0$$

as well as

$$\lim_{y\to x}\left\|\varphi(y)-\varphi(x)-\sum_k(y_k-x_k)\frac{\partial\varphi}{\partial x_k}(x)\right\|_{l,K}$$
$$\leq \lim_{y\to x}\left\|\varphi\,|_{V_j}\,(y)-\varphi\,|_{V_j}\,(x)-\sum_k(y_k-x_k)\frac{\partial\varphi\,|_{V_j}}{\partial x_k}(x)\right\|_{l,K\cap V_j}=0$$

Consequently, φ is differentiable. □

This property extends to higher differentiabilities by induction.

Corollary B.5:
Let $\Omega\subset\mathbb{R}^n$ be an open subset.
A map

$$\varphi:\quad \Omega \quad\longrightarrow\quad \Gamma(M,\mathcal{Q})$$

is C^∞-smooth if and only if the induced maps

$$\varphi\,|_{V_j}:\quad \Omega \quad\longrightarrow\quad \Gamma(V_j,\mathcal{Q}\,|_{V_j})$$

are C^∞-smooth for all $j\in J$.

The vector bundle $\mathcal{Q}\to M$ becomes trivial over $V_j\subset M$, hence, we obtain an isomorphism of vector spaces:

$$\mathrm{proj}_2\circ\psi_j:\Gamma(V_j,\mathcal{Q}\,|_{V_j})\quad\longrightarrow\quad C^\infty(V_j,\mathbb{R}^m)$$

Without restrictions to generality V_j is a chart, i.e. there is a diffeomorphism

$$\phi_j:V_j\quad\longrightarrow\quad U_j$$

with $U_i\subset\mathbb{R}^p$ open.
We identify now V_j with U_j to make it less difficult.

The space $C^\infty(V_j,\mathbb{R}^m)$ becomes a Fréchet space if we apply the standard topology, i.e. the Fréchet space structure, given by the following semi-norms.

Let $K \subset V_j$ be a compact subset and $l \in \mathbb{N}$, define the semi-norm $\|\cdot\|_{l,K}$ for $f \in C^\infty(V_j, \mathbb{R}^m)$ to be:

$$\|f\|_{l,K} := \sup_{|\vec{t}| \leq l} \left(\sup_{x \in K} \left\| \frac{\partial^{\vec{t}} f}{\partial x}(x) \right\| \right),$$

where for a tuple $\vec{t} = (t_0, \ldots, t_p) \subset \mathbb{N}^n$ the degree of $\vec{t}$ is given by $|\vec{t}| := t_1 + \ldots + t_n$ and the operator $\frac{\partial^{\vec{t}}}{\partial x}$ is given by

$$\frac{\partial^{\vec{t}} f}{\partial x} := \left(\prod_{i=1}^{p} \frac{\partial^{t_i}}{\partial x_i^{t_i}} \right) f.$$

Now, take a $s \in \Gamma(V_j, \mathcal{Q}|_{V_j})$ and take a $K \subset V_j$, then there is are positive constants C_1, C_2 such that:

$$C_1 \|s\|_{l,K} \leq \|\mathrm{proj}_2 \circ \psi_j(s)\|_{l,K} \leq C_2 \|s\|_{l,K}.$$

This is due to the fact that the Euclidean metrics on $\bigotimes T^*M \otimes V_j$ differ from the standard metric on $\bigotimes (\mathbb{R}^p)^* \otimes \mathbb{R}^m$ by smooth functions. These functions have on K a maximum as well as a minimum.

Corollary B.6:
Let $\Omega \subset \mathbb{R}^n$ be an open subset.
A map

$$\varphi : \quad \Omega \quad \longrightarrow \quad \Gamma(M, \mathcal{Q})$$

is C^∞-smooth if and only if the induced maps

$$\mathrm{proj}_2 \circ \psi_j \circ \varphi|_{V_j} : \quad \Omega \quad \longrightarrow \quad C^\infty(V_j, \mathbb{R}^m)$$

are C^∞-smooth for all $j \in J$.

References

[1] M. F. Atiyah. *K-theory.* Lecture notes by D. W. Anderson. W. A. Benjamin, Inc., New York-Amsterdam, 1967.

[2] M. F. Atiyah, N. J. Hitchin, and I. M. Singer. Self-duality in four-dimensional Riemannian geometry. *Proc. Roy. Soc. London Ser. A*, 362(1711):425–461, 1978.

[3] M. F. Atiyah and G. B. Segal. The index of elliptic operators. II. *Ann. of Math. (2)*, 87:531–545, 1968.

[4] M. F. Atiyah and I. M. Singer. The index of elliptic operators. IV. *Ann. of Math. (2)*, 93:119–138, 1971.

[5] H. Baum. *Eichfeldtheorie.* Springer London, Limited, 2010.

[6] N. Berline, E. Getzler, and M. Vergne. *Heat kernels and Dirac operators*, volume 298 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[7] C. Birkenhake and H. Lange. *Complex Abelian Varieties*, volume 302 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2004.

[8] J.-M. Bismut. The Atiyah-Singer index theorem for families of Dirac Operators: Two heat equation proofs. *Invent. math*, 83:91–151, 1986.

[9] A. Borel. Topology of Lie groups and characteristic classes. *Bull. Amer. Math. Soc.*, 61:397–432, 1955.

[10] T. Bröcker and T. tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. Translated from the German manuscript, Corrected reprint of the 1985 translation.

[11] J. Cheeger and D. G. Ebin. *Comparison theorems in Riemannian geometry.* North-Holland Publishing Co., Amsterdam, 1975. North-Holland Mathematical Library, Vol. 9.

[12] W. Franz. Über die Torsion einer Überdeckung. *J. Reine Angew. Math. 173*, 1935.

[13] W. Fulton and J. Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[14] P. Griffiths and J. Harris. *Principles of algebraic geometry.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.

[15] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[16] M. W. Hirsch. *Differential topology.* Springer-Verlag, New York, 1976. Graduate Texts in Mathematics, No. 33.

[17] D. Huybrechts. *Complex geometry.* Universitext. Springer-Verlag, Berlin, 2005. An introduction.

[18] A. W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics.* Birkhäuser Boston Inc., Boston, MA, 1996.

[19] S. Kobayashi. *Differential geometry of complex vector bundles*, volume 15 of *Publications of the Mathematical Society of Japan.* Princeton University Press, Princeton, NJ, 1987. Kanô Memorial Lectures, 5.

[20] S. Kobayashi. *Transformation groups in differential geometry.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1972 edition.

[21] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol I.* Interscience Publishers, a division of John Wiley & Sons, New York-Lond on, 1963.

[22] K. Köhler and D. Roessler. A fixed point formula of Lefschetz type in Arakelov geometry. I. Statement and proof. *Invent. Math.*, 145(2):333–396, 2001.

[23] D. B. Ray and I. M. Singer. R-torsion and the Laplacian on Riemannian manifolds. *Advances in Math.*, 7:145–210, 1971.

[24] D. B. Ray and I. M. Singer. Analytic torsion for complex manifolds. *Ann. of Math. (2)*, 98:154–177, 1973.

[25] K. Reidemeister. Homotopieringe und Linsenräume. *Abh. Math. Sem. Univ. Hamburg*, 11(1):102–109, 1935.

[26] H. Samelson. Topology of Lie groups. *Bull. Amer. Math. Soc.*, 58:2–37, 1952.

[27] H. Samelson. A class of complex-analytic manifolds. *Portugaliae Math.*, 12:129–132, 1953.

[28] C. Soulé. *Lectures on Arakelov geometry*, volume 33 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1992. With the collaboration of D. Abramovich, J.-F. Burnol and J. Kramer.

[29] N. K. Stanton. Holomorphic R-torsion for fiber bundles. *Amer. J. Math.*, 100(3):547–578, 1978.

[30] F. Trèves. *Topological vector spaces, distributions and kernels.* Dover Publications Inc., Mineola, NY, 2006. Unabridged republication of the 1967 original.

[31] J. H. Whitehead. Convex regions in the geometry of paths. *Oxford Journals*, os-3, 1932.

Erklärung

Die hier vorgelegte Dissertation habe ich eigenständig und ohne unerlaubte Hilfe angefertigt.
Die Dissertation wurde in der vorgelegten oder in ähnlicher Form noch bei keiner anderen Institution eingereicht.
Ich habe bisher keine erfolglosen Promotionsversuche unternommen.

Thomas Ueckerdt Düsseldorf, 21. Januar 2014