Compactified Jacobians and Symmetric Determinantal Hypersurfaces

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Zusammenfassung

German Summary. *For an English introduction see the beginnings of the two parts of this thesis.*

Eine Riemannsche Fläche S ist eine eindimensionale kompakte komplexe Mannigfaltigkeit. Typischerweise wird eine Riemannsche Fläche untersucht, indem man ihre Geradenbündel studiert. Diese sind im wesentlichen zweidimensionale komplexe Mannigfaltigkeiten mit einer surjektiven Abbildung auf S, so daß die Fasern eindimensionale komplexe Vektorräume sind. Alle diese Geradenbündel sind durch die Picard–Varietät $\operatorname{Pic} S$ parametrisiert. Sie ist eine g–dimensionale komplexe Mannigfaltigkeit, wobei g das Geschlecht der Riemannschen Fläche ist. Die Picard–Varietät ist nicht zusammenhängend, daher definiert man die Jacobi–Varietät JS als die Zusammenhangskomponente, die das triviale Geradenbündel, $S \times \mathbb{C} \to S$, enthält. Es gilt $\operatorname{Pic} S \cong \mathbb{Z} \times JS$. Die Jacobi–Varietät ist homeomorph zu einem $2g$–dimensionalen reellen Torus, insbesondere also kompakt.

In der algebraischen Geometrie entsprechen die Riemannschen Flächen den glatten Kurven und die Vektorbündel den lokal freien Garben vom Rang 1. Dort kann man jetzt auch eine singuläre Kurve C betrachten. Die Jacobi–Varietät JC von C ist niemals kompakt, und es stellt sich die Frage nach einer natürlichen Kompaktifizierung $\bar{J}C$ von JC. Die Existenz dieser Kompaktifizierung für Kurven mit ebenen Singularitäten wurde in den Arbeiten von Mayer, Mumford, D'Souza, Altman, Iarrobino, Kleiman und Rego bewiesen [MM, DS, AK, AIK, R]; sie besteht aus den torsionsfreien Garben vom Rang 1. Während man bei den Riemannschen Flächen weiß, daß JS homeomorph zu einem Torus ist, ist die topologische Struktur von $\bar{J}C$ weitgehend ungeklärt.

Die wichtigste topologische Invariante ist die Eulerzahl. Beauville bewies, daß die Eulerzahl von $\bar{J}C$ Null ist, falls das geometrische Geschlecht von C ungleich Null ist, und ferner daß im Falle von Geschlecht Null $\bar{J}C$ homeomorph zu einem direkten Produkt von sogenannten Jacobifaktoren ist, die nur von dem analytischen Typ der auftretenden Singularitäten abhängen [Be1]. Beauville berechnete auch die Eulerzahlen der Jacobifaktoren der Singularitäten $x^p - y^q = 0$ für p, q teilerfremd.

Im ersten Teil dieser Arbeit werden die Jacobifaktoren der ebenen Singularitäten mit den Puiseux–Exponenten (p, q), $(4, 2q, s)$, $(6, 8, s)$ und $(6, 10, s)$ untersucht, wobei $\operatorname{ggT}(p, q) = 1$, $\operatorname{ggT}(qs, 2) = 1$ bzw. $\operatorname{ggT}(s, 2) = 1$ gilt. Für die Eulerzahlen hat man die folgende Tabelle:

Puiseux–Exponenten	Eulerzahl
(p,q)	$\frac{1}{p+q}\binom{p+q}{p}$
$(4,2q,s)$	$\frac{(q+1)(q^2+5q+3)}{12}+\frac{(q+1)^2}{8}s$
$(6,8,s)$	$\frac{229}{2}+\frac{25}{2}s$
$(6,10,s)$	$\frac{511}{2}+\frac{49}{2}s$

Desweiteren wird die Berechnung der Bettizahlen der Jacobifaktoren für die Puiseux–Exponenten (p,q) und $(4,2q,s)$ auf ein kombinatorisches Problem reduziert. Dies führt für die höchsten und niedrigsten Bettizahlen zu expliziten Formeln. Im Falle von $(4,2q,s)$ ergeben sich vermutungsweise Formeln für alle Bettizahlen.

Die Eulerzahlen der kompaktifizierten Jacobi–Varietät haben bisher zwei wichtige Anwendungen gefunden:

Inspiriert durch eine Arbeit von den Physikern Yau und Zaslow bewies Beauville folgendes [YZ, Be1]: Sei $X \subset \mathbb{P}^g$ eine projektive K3–Fläche, die mit einem vollständigen Linearsystem eingebettet wurde. Beim Schneiden von X mit den Hyperebenen des projektiven Raumes entstehen nur endlich viele rationale Kurven; deren Anzahl ist der Koeffizient von q^g in der Potenzreihe $\prod_{n\geq 1}(1-q^n)^{-24}$, dabei muß jede rationale Kurve mit Multiplizität gezählt werden und diese ist gerade die Eulerzahl der kompaktifizierten Jacobi–Varietät der Kurve.

Fantechi, Göttsche und van Straten entdeckten ein weiteres Auftreten dieser Eulerzahlen in der Deformationstheorie der Singularitäten [FGS]: Die Multiplizität des δ–konstanten Stratums im versellen Deformationsraum einer einzweigigen ebenen Singularität ist gerade die Eulerzahl des Jacobifaktors dieser Singularität.

Die ursprüngliche Motivation des Autors für die Untersuchung der kompaktifizierten Jacobi–Varietät singulärer Kurven war die Frage auf wieviele wesentlich verschiedene Weisen die Gleichung einer ebenen Kurve als Determinante einer symmetrischen Matrix mit linearen Einträgen auftreten kann. Zum Beispiel gibt es zwei wesentlich verschiedene Darstellungen der nodalen Kubik $x^3+y^3+xyz=0$, nämlich als Determinante der Matrizen

$$\begin{pmatrix} -y & \frac{1}{2}z & x \\ \frac{1}{2}z & -x & y \\ x & y & 0 \end{pmatrix} \quad \text{und} \quad \begin{pmatrix} -y & 0 & x \\ 0 & -x & y \\ x & y & z \end{pmatrix}.$$

Beauville zeigte, daß es für jede ebene Kurve zumindest eine Darstellung gibt [Be2]. Die Anzahl der Darstellung hängt bei irreduzibelen Kurven von den Eigenschaften der 2–Torsionspunkte der kompaktifizierten Jacobi–Varietät ab.

Die Frage nach der Anzahl dieser Darstellungen ist klassisch. Bereits 1844 zeigte Hesse, daß eine ebene glatte Kubik drei nicht–äquivalente Darstellungen hat [H]. Dixon zeigte 1904, daß die linearen symmetrischen Matrizendarstellungen von glatten ebenen Kurven den ineffektiven Theta–Charakteristiken entsprechen [D]. Dies wurde von Barth für singuläre Kurven verallgemeinert [B].

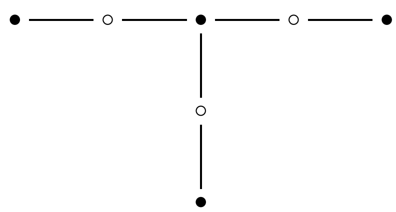
Die offensichtliche Verallgemeinerung des Problems ist die Frage nach den linearen symmetrischen Matrixdarstellungen von Hyperflächen in höher dimensionalen

Räumen. Die allgemeine Theorie dazu findet man bei Catanese [C], Meyer–Brandis [M–B] und Beauville [Be2].

Der zweite Teil dieser Arbeit beschäftigt sich mit den linearen symmetrischen Matrixdarstellungen von Flächen im projektiven 3–Raum. Bereits Salmon wußte, daß jede Fläche, die eine solche Darstellung besitzt, singulär ist und im allgemeinen $\binom{n+1}{3}$ Doppelpunkte besitzt, wobei n der Grad der Fläche oder auch die Größe der Matrix ist [S, p. 495]. Die möglichen Positionen dieser Doppelpunkte wurden von Cayley bestimmt [Ca] und Catanese untersuchte solche Flächen mit nur Doppelpunkten in einem allgemeineren Zusammenhang [C].

Hier interessieren wir uns hauptsächlich für die Frage, welche möglichen Kombinationen von Singularitäten bei Kubiken und Quartiken auftreten können. Die Korrespondenz der rationalen Singularitäten mit den Dynkin–Graphen A_k, D_k, E_6, E_7, E_8 und der elliptischen Singularitäten mit $\tilde{E}_6$ ausnutzend erhalten wir zum Beispiel:

Satz. *Es gibt vier Typen von kubischen linearen symmetrischen Determinantenflächen. Die Kombinationen ihrer Singularitäten sind gegeben durch die Teilgraphen von* $\tilde{E}_6$



die durch das Entfernen von weißen Punkten entstehen. Ferner gibt es linear symmetrische Matrizendarstellungen für alle nicht–normalen kubischen Flächen mit Ausnahme der Vereinigung von einer Quadrik mit einer transversalen Ebene.

Analoge Sätze werden für Quartiken mit isolierten Singularitäten bewiesen, dabei benötigt man jedoch mehrere Ausgangsdiagramme.

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Part I

Compactified Jacobians of Singular Curves

Introduction

Let C be an irreducible and reduced projective curve and $\Sigma \subset C$ its singularities. The generalized Jacobian JC of C consists of the locally free sheaves of rank 1 and degree 0 on C. It is an extention of the Jacobian of the normalization $\tilde{C}$ of C by an affine commutative subgroup of dimension $\delta := \sum_{p\in\Sigma} \delta_{(C,p)}$, thus its dimension equals the arithmetic genus $g_a(C)$ of C. Unfortunately, JC is never compact except when C is smooth, but it is an open subspace of the compactified Jacobian $\bar{J}C$, which consists of all rank one torsion free sheaves $\mathcal{F}$ of degree zero, i.e., $\chi(\mathcal{F}) = 1 - g_a(C)$. The compactified Jacobian is irreducible if and only if all singularities of C are planar [AIK, R]. Only in this case JC is dense in $\bar{J}C$, and $\bar{J}C$ is in fact a compactification of JC. The Euler number $e(\bar{J}C)$ of $\bar{J}C$ is of particular interest because of the following two applications:

Inspired by the work of Yau and Zaslow, Beauville showed that while counting the rational curves in a complete linear system on a K3–surface the Euler number of $\bar{J}C$ is the multiplicity every curve has to be counted with [YZ, B]. Beauville also showed that the Euler number of the compactified Jacobian of a rational curve C equals the Euler number of the compactified Jacobian of the minimal unibranched partial normalization $\check{C}$ of C. Further, for a rational unibranched curve C its compactified Jacobian is homeomorphic to the direct product of compact spaces, the Jacobi factors $J_{(C,p)}$, $p \in \Sigma$, which depend only on the analytic type of the singularities of C. The Jacobi factors can be defined to be the compactified Jacobian of any rational curve with (C,p) as its unique singularity. Hence, it remains to compute the Euler numbers of the Jacobi factors for the unibranched plane singularities.

Fantechi, Götsche, and van Straten proved that the Euler number of the Jacobi factor of a plane singularity (C,p) equals the multiplicity of the δ–constant strata in the base of the semi–universal deformation of the singularity [FGS].

Unfortunately, this surprising result did not help to compute the Euler numbers $e(J_{(C,p)})$. So far the only known Euler numbers of the Jacobi factors are those of the plane singularities with $\mathbb{C}^*$–action, $V(x^p - y^q)$ with $\gcd(p,q) = 1$, whose Euler numbers, $\frac{1}{p+q}\binom{p+q}{p}$, were computed by Beauville. Here, we will use a natural decomposition of the Jacobi factors to compute further examples:

Main Theorem. *The following table assigns to an unibranched plane singularity with characteristic Puiseux exponents which occur in the left column the Euler number of its Jacobian factor:*

Puiseux exponents	Euler number
(p,q) with $\gcd(p,q) = 1$	$\dfrac{1}{p+q}\dbinom{p+q}{p}$
$(4,2q,s)$ with $\gcd(qs,2) = 1$	$\dfrac{(q+1)(q^2+5q+3)}{12} + \dfrac{(q+1)^2}{8}s$
$(6,8,s)$ with $\gcd(s,2) = 1$	$\dfrac{229}{2} + \dfrac{25}{2}s$
$(6,10,s)$ with $\gcd(s,2) = 1$	$\dfrac{511}{2} + \dfrac{49}{2}s$

The reason for the restriction to the above Puiseux exponents is that in these cases a natural decomposition of the Jacobian factor is a cell decomposition into complex cells, $J_X = \bigcup_{i=1}^{e(J_X)} \mathbb{C}^{n_i}$. We show by several examples that this is not the case for the more complicated cases. From the cell decomposition the Betti numbers of the Jacobi factors can be computed by purely combinatorial means. Explicit formulas are harder to derive, we will prove in Section 5 the following:

Theorem. *Let X be a unibranched plane singularity with characteristic Puiseux exponents (p,q) and J_X its $\delta_X = (p-1)(q-1)/2$ dimensional Jacobi factor. Then the odd homology groups of J_X all vanish. The even homology groups are free abelian groups. The ranks of $H_0(J_X), H_2(J_X), \ldots, H_{2(q-\lceil \frac{q}{p} \rceil)}(J_X)$ are the same as the first $q - \lceil \frac{q}{p} \rceil + 1$ coefficients of the power series $\prod_{i=1}^{p-1}(1-t^i)^{-1}$. The ranks of $H_{2\delta_X}(J_X), H_{2\delta_X-2}(J_X), \ldots, H_{2\delta_X-2\lfloor \frac{q}{p} \rfloor}(J_X)$ are the same as the first $\lceil \frac{q}{p} \rceil$ coefficients of the power series $(1-t)^{1-p}$.*

This proves in particular the conjectures of Warmt about the odd homology groups and $H_2(J_X), H_4(J_X)$ [W2, 5.8.4]. An analogous theorem is shown for singularities with characteristic Puiseux exponents $(4, 2q, s)$. In this case one can also describe all Betti numbers conjecturally.

As singularities with the same characteristic Puiseux exponents are topologically equivalent, these theorems provide evidence for the general conjecture that the topology of the compactified Jacobian depends only on the topology of the curve.

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1 Jacobi factors and their cell decomposition

The definition of the Jacobi factor of a singularity X as the compactified Jacobian of any rational curve with X as its unique singularity is unsuitable for us, because the definition is not purely local. We will use Rego's definition, which we will explain in a moment after fixing some notation [R, GP]. We always assume that X is a unibranched plane singularity given by the equation $f \in \mathbb{C}[[x,y]]$. The complete local ring $R = \mathbb{C}[[x,y]]/(f)$ of the singularity has $\tilde{R} = \mathbb{C}[[t]]$ as its normalization. By Puiseux's Theorem there exist $t^n, \varphi \in \mathbb{C}[[t]]$ such that R is embedded as $R \cong \mathbb{C}[[t^n, \varphi]]$ into $\tilde{R} = \mathbb{C}[[t]]$. The conductor of R is $C := \mathrm{Ann}_R(\tilde{R}/R)$. Since X is planar, we have $\delta_X := \delta_R := \dim \tilde{R}/R = \dim R/C$ and $C = (t^{2\delta_R})$ [JP, 5.2.4].

Let M be any torsion free R–module of rank 1. Such a module M can be embedded into $\tilde{R}$. In this situation we define the conductor $C(M)$ of M to be $C(M) := \mathrm{Ann}_R(\tilde{R}/M)$. Because it is an ideal in $\tilde{R}$ as well, we identified it with the natural number $c = c(M)$ such that $C = (t^c) \subset \tilde{R}$. The embedding of M into $\tilde{R}$ can be chosen such that $C \subset M \subset \tilde{R}$ and $\dim \tilde{R}/M = \dim M/C = \delta_R$; we will call such an embedding δ_R–normalized. A δ_R–normalized module M can be considered as a point of the Grassmannian $\mathrm{G}(\tilde{R}/C, \delta_R)$, which consists of the δ_R–dimensional subspaces of $\tilde{R}/C$. The Jacobi factor J_X or J_R of the singularity is the set of points of $\mathrm{G}(\tilde{R}/C, \delta_R)$, which are R–modules. Therefore, $M/C \in \mathrm{G}(\tilde{R}/C, \delta_R)$ lies in J_R if $RM \subseteq M$. This turns out to be a linear condition on $\mathrm{G}(\tilde{R}/C, \delta_R)$ when one considers $\mathrm{G}(\tilde{R}/C, \delta_R)$ to be embedded by the Plücker embedding [GP, 1.4]. Different points of J_R may correspond to isomorphic R–modules. In fact, one has

Theorem 1.1 *The subsets of J_R consisting of isomorphic modules are biregular to affine spaces.*

Proof. Two R–submodules $M_1, M_2 \subset \tilde{R}$ are isomorphic if there is an $x \in Q(\tilde{R}) = \mathbb{C}((t))$ such that $M_2 = xM_1$. If M_1 and M_2 are δ_R–normalized, the order of x must be zero, i.e., $x \in \tilde{R}^*$. Therefore, the subsets of isomorphic modules are the orbits of the action of $(\tilde{R}/C)^*$ on J_R. Since $\mathbb{C}^* \subset (\tilde{R}/C)^*$ acts trivially and the representation of $(\tilde{R}/C)^*/\mathbb{C}^*$ on J_R is unipotent, the orbits are affine spaces by the Theorem of Chevalley–Rosenlicht [CG, 3.14]. □

Unfortunately, there are infinitely many isomorphism classes of torsion free modules of rank 1 if the singularity is not an A_{2k}, E_6, or E_8 singularity by a theorem of Greuel and Knörrer [GK]. To get a finite cell decomposition, we use the natural valuation $v : \tilde{R} = \mathbb{C}[[t]] \to \mathbb{N}$ and decompose J_R according to the images of the modules under the map v. To prove the Main Theorem, we will show that this decomposition is a cell decomposition into affine complex spaces in the cases of the Main Theorem, then we will count the nonempty ones. This will require some work, because the Theorem of Chevalley–Rosenlicht cannot be applied anymore. We start by translating parts of the problem to a combinatorial problem with the help of the valuation v.

We have $v(\tilde{R}) = \mathbb{N}$ and the image of R under v is a semi–group Γ. The above properties of the conductor translate into $\#(\mathbb{N} - \Gamma) = \delta_R$ and $\min\{x \in \mathbb{N} | x + \mathbb{N} \subset \Gamma\} = 2\delta_R$. For a module $M \subset \tilde{R}$, we get an associated Γ–semi–module $\Delta := v(M)$, i.e., $\Gamma + \Delta \subseteq \Delta$. If M is δ_R–normalized, then $\#(\mathbb{N} - \Delta) = \delta_R$. We will call a semi–module Δ with this property δ_R–normalized, too. Two Γ–modules are isomorphic if one is the shift of the other by an integer. Corresponding to the definition of the conductor of a module $M \subset \tilde{R}$, we define the conductor $c(\Delta)$ of the semi–module $\Delta \subset \mathbb{N}$ to be the smallest natural number c with $c + \mathbb{N} \subseteq \Delta$.

We call the subset of modules of J_X with associated semi–module Δ simply the Δ–subset of J_X. This decomposition of J_X into Δ–subsets corresponds to the Schubert cell decomposition of the Grassmannian. More precisely, consider the flag in $\tilde{R}/C = \mathbb{C}[[t]]/(t^{2\delta_X})$ given by the ideals (t^i), $i = 1, \ldots, 2\delta_X$, and the Schubert cell decomposition corresponding to it. Then the valuation map $v : \mathbb{C}[[t]] \to \mathbb{N}$ induces a map

$$\begin{array}{ccc} G(\tilde{R}/C, \delta_X) & \longrightarrow & \{S \subset \{0, \ldots, 2\delta_X - 1\} \mid \#S = \delta_X\} \\ \Lambda + C & \longmapsto & v(\Lambda + C) \cap \{0, \ldots, 2\delta_X - 1\}, \end{array}$$

and its fibers are precisely the Schubert cells. Recalling that J_X is the intersection of $G(\tilde{R}/C, \delta_X)$ and a linear subspace L, we see that the Δ–subsets are linear sections of these Schubert cells. We will show in Section 3 that these Δ–subsets are again complex cells in the cases of the Main Theorem. To prove that they form a CW–complex like the Schubert cell decomposition seems incredible tedious mainly because the dimension of the Schubert cells does not drop uniformly during the intersection process. Luckily, this is not needed in order to compute the homology groups with this cell complex. Namely, from the Schubert cell decomposition of $G(\tilde{R}/C, \delta_X)$ we find a descending chain of closed algebraic sets

$$G(\tilde{R}/C, \delta_X) = A_0 \supset A_1 \supset \ldots \supset A_N = \operatorname{span}\{t^{\delta_X}, t^{\delta_X+1}, \ldots, t^{2\delta_X-1}\}$$

such that $A_{i+1} \setminus A_i$ is a complex cell. Intersecting the elements of this chain with the linear space L does not change this —due to the result that the Δ–subsets are complex cells. Using the long exact homology sequence for the pair $A_{i+1} \subseteq A_i$ and the excision theorem, one sees that the Δ–subset decomposition may be treated like a CW–complex for the purposes of the computation of the homology groups. In particular, all odd homology groups are zero, and the even ones are free abelian groups whose rang equals the number of Δ–subsets of the corresponding dimension.

Before attacking the problem of proving that the Δ–subsets are affine in the cases of the Main Theorem, we discuss the Γ–semi–modules. In particular, we need to count them. Later on we need "syzygies" of the generators of a semi–module. However, such a notion seems cumbersome to define. Therefore, we pass over to the graded semi–group algebra $\mathbb{C}[\Gamma] = \operatorname{span}\{t^\gamma | \gamma \in \Gamma\}$ and correspondingly to the graded $\mathbb{C}[\Gamma]$–module $\mathbb{C}[\Delta] = \operatorname{span}\{t^\delta | \delta \in \Delta\}$, where we can use the conventional definition of syzygies. The connection of these objects with an R–module M with $v(M) = \Delta$ is as follows:

Define the initial term of a power series $f = \sum_{i=k}^{\infty} \lambda_i t^i$, $\lambda_k \neq 0$, to be $\operatorname{in}(f) := \lambda_k t^k$ and set $\operatorname{in}(0) := 0$. Then the graded semi–group algebra $\mathbb{C}[\Gamma]$ equals the initial ring $\operatorname{in}(R) := \operatorname{span}\{\operatorname{in}(f) | f \in R\} \subseteq \mathbb{C}[[t]]$. Analogously, for any maximal CM–module M the graded semi–module module $\mathbb{C}[\Delta]$ equals the initial module $\operatorname{in}(M) := \operatorname{span}\{\operatorname{in}(f) | f \in M\} \subseteq \mathbb{C}[[t]]$.

The study of the $\mathbb{C}[\Gamma]$–semi–modules is done in the next section; the proof that the Δ–subsets are complex cells in the following section. Everything concerning the Puiseux exponents $(6, 8, s)$ and $(6, 10, s)$ was moved to Section 4 which is combinatorically more complicated and included only for completeness and the most interested reader. In the final Section 5 the Betti numbers of the Jacobi factors are discussed.

2 The number and the syzygies of the $\mathbb{C}[\Gamma]$–modules

During this section we will always assume that any Γ–semi–module Δ is 0–normalized, i.e., $\min \Delta = 0$, to obtain unique representatives in the isomorphism classes of the semi–modules. In particular, one has $\Gamma \subseteq \Delta$.

For a singularity with only two characteristic Puiseux exponents (p, q), $p < q$, the semi–group Γ is generated by p and q, $\Gamma = \langle p, q \rangle$. To study the Γ–semi–modules, we introduce the notion of a basis for them, modeled after the Apéry–basis for semi–groups (see [JP, A, H] for the semi–group case).

Definition 2.1 *Let $\Gamma = \langle p, q \rangle$. A p–*basis *of a Γ–semi–module Δ is the unique p–tuple $(a_0, \ldots, a_{p-1})$ such that*

$$\Delta = \bigcup_{i=0}^{p-1} (a_i + p\mathbb{N}) \qquad \textit{and} \qquad a_i \equiv iq \mod p.$$

In particular, the $\{a_i\}$ generate Δ as a Γ–semi–module and $c(\Delta) = \max\{a_i\} - p + 1$.

By the definition of the p–basis and $\mathbb{N}q \subset \Gamma \subseteq \Delta$, there exist $\alpha_1, \ldots, \alpha_{p-1} \in \mathbb{N}$ such that

$$a_0 = 0, \quad a_1 = q - \alpha_1 p, \quad a_2 = 2q - \alpha_2 p, \quad \ldots, a_{p-1} = (p-1)q - \alpha_{p-1} p.$$

To simplify the notation, we define $\alpha_0 = 0$. The condition that $\Delta = \bigcup (a_i + p\mathbb{N})$ is a Γ–semi–module is equivalent to $a_i + q \in \Delta$ and — with a cyclic notation of the indices — to $a_i + q \geq a_{i+1}$. The latter is the same as $0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{p-1} \leq q$. Due to the 0–normalization we have $a_i \geq 0$, i.e., $\alpha_i \leq iq/p$.

Proposition 2.2 *For the semi–group $\Gamma = \langle p, q \rangle$, $\gcd(p, q) = 1$, the number of isomorphism classes of Γ–semi–modules is $\frac{1}{p+q}\binom{p+q}{p}$.*

Proof. Beauville proves this result with the help of generating functions [B, 4.3]. Fantechi, Götsche, and van Straten derive this from a local computation in a moduli space for rational curves [FGS, G1]. We give a third, shorter proof using the p–bases. For a moment we normalize our Γ–modules only by $\min(\Delta \cap p\mathbb{N}) = 0$, i.e., $a_0 = 0$. Then by the above arguments all such modules can be obtained by choosing $0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{p-1} \leq q$; hence, there are $\binom{p+q-1}{p-1} = \frac{p}{p+q}\binom{p+q}{p}$ of them. If we shift $\Delta = \bigcup (a_i + p\mathbb{N})$ by $-a_j$, $j = 0, \ldots, p-1$, we obtain an isomorphic semi–module Δ' with $\min(\Delta' \cap p\mathbb{N}) = 0$ and these are also the only shifts of Δ that satisfy the additional condition. Therefore, to get the number of isomorphism classes of Γ–semi–modules, we have to divide the above number by p. □

For the purpose of the next section we need to compute the syzygies of the graded $\mathbb{C}[\Gamma]$–semi–modules. We start with a very general lemma.

Lemma 2.3 *Let Γ be any semi–group $\Gamma \subset \mathbb{N}$ with $\#(\mathbb{N} \setminus \Gamma) < \infty$ and Δ a 0–normalized semi–module. Let $A = (t^{a_1}, \ldots, t^{a_k})$ be a graded generating set of a $\mathbb{C}[\Gamma]$–module $\mathbb{C}[\Delta]$. There is a minimal generating set C of syzygies of A consisting of bivectors, i.e., vectors $v = (0, \ldots, 0, t^{\gamma_i}, 0, \ldots, 0, -t^{\gamma_j}, 0, \ldots, 0) \in \mathbb{C}[\Gamma]^k$ with $A \cdot v = 0$.*

Proof. Clearly, any relation between the generators can be splitted into the sum of graded ones. Next, we show that any graded relation between the generators can be splitted into a linear combination of bivectors that are relations as well. Assume we have a graded vector $w = (w_1 t^{\gamma_1}, \dots, w_n t^{\gamma_n})$ with $A \cdot w = 0$, i.e., $\sum w_i t^{\gamma_i + a_i} = 0$ and $\gamma_i + a_i = const$ for all i with $w_i \neq 0$. Therefore $\sum w_i = 0$. Choose j with $w_j \neq 0$ and set $v_i = (0, \dots, 0, t^{\gamma_i}, 0, \dots, 0, -t^{\gamma_j}, 0, \dots, 0)$ for $i \neq j$ with $w_i \neq 0$ where the nonzero entries are at the positions i and j. Then $\sum w_i v_i = w$ using $\sum w_i = 0$. Finally, we can choose a minimal generating set among all these bivectors using Nakayama's lemma. □

The degree $\deg(v)$ of the above bivector syzygy v is by definition $a_i + \gamma_i$. The bivector syzygy is — up to an unimportant choice of sign — determined by the exponents; hence, we will sometimes use the shorter additive notation

$$a_i + \gamma_i = a_j + \gamma_j.$$

The syzygies of a $\mathbb{C}[\Gamma]$–module for $\Gamma = \langle p, q \rangle$ are nearly obvious.

Proposition 2.4 *Let $\Gamma = \langle p, q\rangle$ and $\Delta = \bigcup_{i=0}^{p-1}(a_i + p\mathbb{N})$ be a Γ–semi–module like above. Then the $\mathbb{C}[\Gamma]$–module $\mathbb{C}[\Delta]$ is generated by $A = (t^{a_0}, t^{a_1}, \dots, t^{a_{p-1}})$ and the syzygies of A are generated minimally by the following p–bivectors:*

$$\begin{array}{ll} v_0 := (t^q, -t^{\alpha_1 p}, 0, \dots, 0), & v_1 := (0, t^q, -t^{(\alpha_2-\alpha_1)p}, 0, \dots, 0), \\ v_2 := (0, 0, t^q, -t^{(\alpha_3-\alpha_2)p}, 0, \dots, 0), & \dots \\ v_{p-2} := (0, \dots, 0, t^q, -t^{(\alpha_{p-1}-\alpha_{p-2})p}), & v_{p-1} := (-t^{(q-\alpha_{p-1})p}, 0, \dots, 0, t^q). \end{array}$$

In particular, the degree of one of these syzygies is greater than $c(\Delta)$.

Proof. Because $\{a_0, \dots, a_{p-1}\}$ generate the Γ–semi–module Δ, A generates the $\mathbb{C}[\Gamma]$–module $\mathbb{C}[\Delta]$. Clearly, all of the above bivectors are syzygies and none of them is a linear combination of the others. Therefore, it remains to show that the above bivectors form a generating set. By Lemma 2.3 all syzygies are generated by bivectors and finding a bivector $(0, \dots, 0, t^{\gamma_i}, 0, \dots, 0, -t^{\gamma_j}, 0, \dots, 0)$ relating t^{a_i} and t^{a_j} is the same as finding γ_i and γ_j with $a_i + \gamma_i = a_j + \gamma_j$. W.l.o.g. assume $i < j$. Because we are looking for a minimal generating set, we may assume that neither $(\gamma_i - q, \gamma_j - q) \in \Gamma^2$ nor $(\gamma_i - p, \gamma_j - p) \in \Gamma^2$, thus either $\gamma_i \in q\mathbb{N}$ and $\gamma_j \in p\mathbb{N}$ or the other way around. Recalling that $a_i \equiv iq \bmod p$ and that q generates the group $\mathbb{Z}/p\mathbb{Z}$, we get that either $\gamma_i = (j-i)q$ and $\gamma_j = (\alpha_j - \alpha_i)p$ or $\gamma_i = (q + \alpha_i - \alpha_j)p$ and $\gamma_j = (p + i - j)q$. Thus we found only the two bivectors

$$\begin{array}{l} (0, \dots, 0, \quad t^{(j-i)q} \quad, 0, \dots, 0, -t^{(\alpha_j-\alpha_i)p}, 0, \dots, 0) \quad \text{and} \\ (0, \dots, 0, -t^{(q+\alpha_i-\alpha_j)p}, 0, \dots, 0, \quad t^{(p+i-j)q}, 0, \dots, 0), \end{array}$$

which are the linear combinations of the elementary bivectors $v_0, \dots, v_{p-1}$. Namely, the first is

$$\sum_{l=i}^{j-1} t^{(j-l-1)q + (\alpha_l - \alpha_i)p} v_l$$

and the second is

$$\sum_{l=j}^{p-1} t^{(p+i-l-1)q + (\alpha_l - \alpha_j)p} v_l + \sum_{l=0}^{i-1} t^{(i-l-1)q + (q+\alpha_l-\alpha_j)p} v_l.$$

This shows that the v_k generate the syzygies. Since $c(\Delta) = \max\{a_k - p + 1\}$, at least one of the degrees of the v_k, $\deg v_k = a_k + q$, is greater than $c(\Delta)$. □

Now we turn to the singularities with the three characteristic Puiseux exponents $(2p, 2q, s)$ with $\gcd(p,q) = 1$, $\gcd(s,2) = 1$, and $2p < 2q < s$. We will give here the general definitions and then restrict ourselves to $(4, 2q, s)$, leaving the $(6, 2q, s)$ case for Section 4. The semi–group Γ is generated by $\gamma_0 := 2p$, $\gamma_1 := 2q$, and $\gamma_2 := (p-1)\gamma_1 + s$ [A, H]. Note that these generators are related by $p\gamma_1 = q\gamma_0$ and $2\gamma_2 = \beta\gamma_1 + \eta\gamma_0$ for suitable $\beta \in \{0, \dots, p-1\}$ and $\eta \in \mathbb{N}$. Any $\gamma \in \Gamma$ can be written uniquely as

$$\gamma = \mu_2\gamma_2 + \mu_1\gamma_1 + \mu_0\gamma_0 \quad \text{with } \mu_2 \in \{0,1\},\ \mu_1 \in \{0,\dots,p-1\},\ \mu_0 \in \mathbb{N}.$$

The same holds for $\gamma \in \mathbb{Z}$ if one allows $\mu_0 \in \mathbb{Z}$. We use this to define a special basis for any Γ–semi–module:

Definition 2.5 *Let $\Gamma = \langle \gamma_0 = 2p,\ \gamma_1 = 2q,\ \gamma_2 = 2(p-1)q + s\rangle$. A $2 \times p$–basis of a Γ–semi–module Δ is the unique $2 \times p$–matrix $\left(\begin{smallmatrix} a_{00} & a_{01} & \cdots & a_{0,p-1} \\ a_{10} & a_{11} & \cdots & a_{1,p-1}\end{smallmatrix}\right)$ such that*

$$\Delta = \bigcup_{\substack{i=0,1 \\ j=0,\dots,p-1}} (a_{ij} + \gamma_0\mathbb{N}) \quad \textit{and} \quad a_{ij} \equiv i\gamma_2 + j\gamma_1 \bmod \gamma_0.$$

Here, the a_{0J} are even and the a_{1J} are odd numbers. Again, there exist $\alpha_{ij} \in \mathbb{N}$ $(\alpha_{00} := 0)$ with

$$a_{ij} = i\gamma_2 + j\gamma_1 - \alpha_{ij}\gamma_0 \qquad \text{for } i \in \{0,1\},\ j \in \{0,\dots,p-1\}.$$

The fact that Δ is a Γ–semi–module is equivalent to $a_{ij} + \gamma_1, a_{ij} + \gamma_2 \in \Delta$. With a cyclic notation of the indices, the first is equivalent to $a_{ij} + \gamma_1 \geq a_{i,j+1}$ and the second to $a_{0j} + \gamma_2 \geq a_{1j}$ and $a_{1j} + \gamma_2 \geq a_{0,j+\beta}$, using the above relation $2\gamma_2 = \beta\gamma_1 + \eta\gamma_0$. Expressed in terms of the α_{ij}, this means

$$\begin{array}{ccccccccccc}
0 & \leq & \alpha_{01} & \leq\cdots\leq & \alpha_{0,p-1-\beta} & \leq & \alpha_{0,p-\beta} & \leq\cdots\leq & \alpha_{0,p-1} & \leq & q \\
\wedge\!| & & \wedge\!| & & \wedge\!| & & \wedge\!| & & \wedge\!| & & \\
\alpha_{10} & \leq & \alpha_{11} & \leq\cdots\leq & \alpha_{1,p-1-\beta} & \leq & \alpha_{1,p-\beta} & \leq\cdots\leq & \alpha_{1,p-1} & \leq & q+\alpha_{10} \\
\wedge\!| & & \wedge\!| & & \wedge\!| & & \wedge\!| & & \wedge\!| & & \\
\eta+\alpha_{0\beta} & & \eta+\alpha_{0,\beta+1} & & \eta+\alpha_{0,p-1} & & \eta+q+\alpha_{00} & & \eta+q+\alpha_{0,\beta-1} & &
\end{array}$$

The 0–normalization of Δ is equivalent to $a_{ij} \geq 0$ or to $\alpha_{00} = 0$, $\alpha_{0j} < jq/p$ and $\alpha_{1j} < (\gamma_2 + j\gamma_1)/\gamma_0$. In particular, we have $\alpha_{0,p-1} < q - q/p$ sharpening $\alpha_{0,p-1} \leq q$. For $j \leq p - 1 - \beta$ we get

$$\alpha_{1j} < \frac{2\gamma_2 - \gamma_2 + j\gamma_1}{\gamma_0} = \frac{\eta\gamma_0 + (\beta + j)\gamma_1 - \gamma_2}{\gamma_0} \leq \eta + \frac{(p-1)\gamma_1 - \gamma_2}{\gamma_0} < \eta$$

and for any $j \leq p - 1$ we obtain similarly

$$\alpha_{1j} < \frac{\eta\gamma_0 + (\beta + j)\gamma_1 - \gamma_2}{\gamma_0} \leq \eta + \frac{(p-1)\gamma_1}{\gamma_0} + \frac{(p-1)\gamma_1 - \gamma_2}{\gamma_0} < \eta + q.$$

Therefore, the 0–normalization of the Γ–semi–module already implies the last row of the inequalities between the α_{ij} in the above diagram. Now, we are ready to compute the number of Γ–semi–modules for $p = 2$.

Proposition 2.6 *The number of* 0*–normalized* Γ*–semi–modules for the semi–group* $\Gamma = \langle 4, 2q, 2q + s\rangle$ *with* $\gcd(qs, 2) = 1$ *is*

$$\frac{(q+1)(2q^2 + 4q + 3)}{12} + s\frac{(q+1)^2}{8}.$$

Proof. We have to count the triples $\alpha = (\alpha_{01}, \alpha_{10}, \alpha_{11})$ with the restrictions

$$\begin{array}{ccccc} 0 & \le & \alpha_{01} & \le & q/2 \\ \wedge\!| & & \wedge\!| & & \\ \alpha_{10} & \le & \alpha_{11} & \le & q+\alpha_{10}\,; \\ \wedge\!| & & \wedge\!| & & \\ \frac{2q+s}{4} & & \frac{4q+s}{4} & & \end{array}$$

thus we have to count the elements of

$$A := \left\{\alpha \in [0, \tfrac{q}{2}] \times [0, \tfrac{2q+s}{4}] \times \mathbb{N} \mid \max\{\alpha_{01}, \alpha_{10}\} \le \alpha_{11} \le \min\{q+\alpha_{10}, q+\tfrac{s}{4}\}\right\}.$$

We set

$$A_0 = \left\{\alpha \in [0, \tfrac{q}{2}] \times [0, \tfrac{2q+s}{4}] \times \mathbb{N} \mid \alpha_{10} \le \alpha_{11} \le q+\alpha_{10}\right\}$$

$$A_1 = \left\{\alpha \in [0, \tfrac{q}{2}] \times [0, \tfrac{2q+s}{4}] \times \mathbb{N} \mid \alpha_{10} \le \alpha_{11} < \alpha_{01}\right\}$$

$$A_2 = \left\{\alpha \in [0, \tfrac{q}{2}] \times [0, \tfrac{2q+s}{4}] \times \mathbb{N} \mid q+\tfrac{s}{4} < \alpha_{11} \le q+\alpha_{10}\right\}.$$

Due to $\alpha_{01} \le q/2 \le q+\alpha_{10}$ and $\alpha_{10} \le q/2+s/4$, we have $A_1, A_2 \subseteq A_0$. Obviously, also $A = A_0 \setminus (A_1 \cup A_2)$ and $A_1 \cap A_2 = \emptyset$ holds. Therefore, the number of of elements of A is $\#A_0 - \#A_1 - \#A_2$ or

$$\left\lceil\frac{q}{2}\right\rceil \cdot \left\lceil\frac{2q+s}{4}\right\rceil \cdot (q+1) - \sum_{\alpha_{01}=1}^{\lfloor \frac{q}{2}\rfloor} \sum_{\alpha_{10}=0}^{\alpha_{01}} (\alpha_{01}-\alpha_{10}) - \left\lceil\frac{q}{2}\right\rceil \sum_{\alpha_{10}=\lceil \frac{s}{4}\rceil}^{\lfloor \frac{2q+s}{4}\rfloor} \left(q+\alpha_{10} - \left\lfloor q+\frac{s}{4}\right\rfloor\right).$$

Using the substitution $q = 2\bar{q}+1$ and $s = 4\bar{s}+1$ resp. $s = 4\bar{s}+3$ for the intermediate steps, the above sum can be evaluated easily to obtain the number in the statement of the proposition. □

Later we will show that some of the Γ–semi–modules cannot occur as the 0–normalization of an associated semi–module of a maximal CM–module over the local ring of the singularity. We call the ones that occur admissible. Their combinatorial definition is as follows:

Definition 2.7 *Let $\Gamma = \langle \gamma_0 = 2p,\ \gamma_1 = 2q,\ \gamma_2 = (p-1)\gamma_1 + s\rangle$ with $p = 2$ or $(p,q) \in \{(3,4),(3,5)\}$. A 0–normalized Γ–semi–module Δ is* admissible *iff*

$$\Delta \cap \{a_{0j} + s \mid j = 0, \ldots, p-1\} \neq \emptyset.$$

Proposition 2.8 *Let $\Gamma = \langle 4, 2q, 2q+s\rangle$ be the above semi–group for $p = 2$. The number of admissible Γ–semi–modules is*

$$\frac{(q+1)(q^2+5q+3)}{12} + \frac{(q+1)^2}{8}s.$$

Proof. We are going to count the nonadmissible semi–modules, i.e., the 0–normalized semi–modules with $s, a_{01}+s \notin \Delta$. We have

$$\begin{aligned} s &= \gamma_2 - \gamma_1 = \gamma_2 + \gamma_1 - q\gamma_0 \equiv a_{11} \bmod \gamma_0 \\ a_{01} + s &= \qquad\qquad \gamma_2 - \alpha_{01}\gamma_0 \quad\ \equiv a_{10} \bmod \gamma_0. \end{aligned}$$

Hence, $s, a_{01}+s \notin \Delta$ is equivalent to $s < a_{11}$ and $a_{01}+s < a_{10}$, i.e., $\alpha_{11} < q$ and $\alpha_{10} < \alpha_{01}$. Together with the conditions for Δ being a 0–normalized semi–module,

$$\alpha_{01} < \tfrac{q}{2}, \quad \alpha_{10} < \tfrac{2q+s}{4}, \quad \max\{\alpha_{01}, \alpha_{10}\} \le \alpha_{11} \le \min\{q+\alpha_{10}, q+\tfrac{s}{4}\},$$

the nonadmissible semi–modules correspond to the triples $(\alpha_{01}, \alpha_{10}, \alpha_{11}) \in \mathbb{N}^3$ with

$$\alpha_{01} < \tfrac{q}{2}, \quad \alpha_{10} < \alpha_{01}, \quad \alpha_{01} \leq \alpha_{11} < q.$$

Clearly, the numbers of these is

$$\sum_{\alpha_{01}=0}^{\lfloor \frac{q}{2} \rfloor} \alpha_{01}(q - \alpha_{01}) = \frac{1}{2}\binom{q+1}{3}.$$

We obtain the number of admissible semi–modules as the difference of the number of all semi–modules (Proposition 2.6) and this term. □

We already know from Lemma 2.3 that the syzygies of a graded generating set of a $\mathbb{C}[\Gamma]$–module $\mathbb{C}[\Delta]$ are generated by bivectors. We are going to select a small subset of these bisectors, which generate the syzygies of degree less than the conductor $c(\Delta)$. Later on, only these syzygies will be of interest to us.

Proposition 2.9 *Let* $\Gamma = \langle \gamma_0 = 4,\ \gamma_1 = 2q,\ \gamma_2 = \gamma_1 + s \rangle$, *and* $\Delta = \langle 0, a_{01}; a_{10}, a_{11} \rangle$ *like above. The* $\mathbb{C}[\Gamma]$*–module* $\mathbb{C}[\Delta]$ *is generated by* $A = (1, t^{a_{01}}, t^{a_{10}}, t^{a_{11}})$, *and the syzygies of* A *of degree less than* $c(\Delta)$ *are generated by*

$$(t^{\gamma_1}, -t^{\alpha_{01}\gamma_0}, 0, 0), \quad (-t^{(q-\alpha_{01})\gamma_0}, t^{\gamma_1}, 0, 0), \quad \textit{and} \ \ (t^{\gamma_2}, 0, -t^{\alpha_{10}\gamma_0}, 0).$$

Proof. By Lemma 2.3 there is a generating set of syzygies consisting of bivectors. Any bivector syzygy of degree d may be written additively as

$$a_{ij} + (\xi_2\gamma_2 + \xi_1\gamma_1 + \xi_0\gamma_0) = a_{kl} + (\zeta_2\gamma_2 + \zeta_1\gamma_1 + \zeta_0\gamma_0) = d$$

with $\xi_1, \xi_2, \zeta_1, \zeta_2 \in \{0, 1\}$ and $\xi_0, \zeta_0 \in \mathbb{N}$. We may assume that the bivector syzygy is not the multiple of another; hence, for all $r \in \{0, 1, 2\}$ one of the ξ_r, ζ_r is zero.

Recall that $c(\Delta) = \max\{a_{ij}\} - \gamma_0 + 1 \leq c(\Gamma)$. We have $a_{00} = 0$, $a_{01} \leq \gamma_1$, $a_{10} + \gamma_1 \geq a_{11}$, and $a_{11} + \gamma_1 \geq a_{10}$, thus $a_{1j} + \gamma_2 > a_{1j} + \gamma_1 > c(\Delta)$. Consequently, of all possible bivector syzygies only the following four may be of degree less than $c(\Delta)$:

$$\begin{aligned} a_{00} + \gamma_1 &= a_{01} + \alpha_{01}\gamma_0 & \quad a_{01} + \gamma_1 &= a_{00} + (q - \alpha_{01})\gamma_0 \\ a_{00} + \gamma_2 &= a_{10} + \alpha_{10}\gamma_0 & \quad a_{01} + \gamma_2 &= a_{11} + (\alpha_{11} - \alpha_{01})\gamma_0. \end{aligned}$$

However, the degree of the last relation is also greater than $c(\Delta)$ due to $a_{00}, a_{10} \leq \gamma_2$. □

3 The cell decomposition

We return to our singularity $(X,0)$ with local ring R. Its Jacobi factor $J_R \subset G(\tilde{R}/C,\delta_R)$ consists of the torsion free modules of rank 1 and was decomposed according to their associated semi–modules. We will show that these subsets are biregular to an affine space $\mathbb{C}^N$. Given a Γ–semi–module Δ we are going to explicitly construct all R–modules $M \subseteq \tilde{R} = \mathbb{C}[[t]]$ with associated semi–module Δ. There is a small technical problem: The associated semi–module Δ_M of a module M is δ_R–normalized, while we were using 0–normalized semi–modules in the last section for notational convenience. However, if $d := \min \Delta_M$ then $-d+\Delta_M$ is 0–normalized and the module $t^{-d}M \subseteq \tilde{R}$ has this semi–module as associated semi–module. With the help of this obvious bijection we may continue to assume that the occurring Γ—semi–modules are 0–normalized.

We start our proof with several remarks about the elements of $M \subset \mathbb{C}[[t]]$. Any element $x = \sum_{k\in\mathbb{N}} \lambda_k t^k \in M$ can be normalized as follows: To get the coefficient of the initial term $\lambda_{v(x)}t^{v(x)}$ equal to one, we multiply by $1/\lambda_{v(x)}$. Then one removes the terms $\lambda_\delta t^\delta$, $\delta \in \Delta$, $\delta > v(x)$, in increasing order by subtracting suitable multiples of elements $y \in M$ with $v(y) = \delta$; thus as the normal form of x we obtain a polynomial of type

$$t^{v(x)} + \sum_{k\in]v(x),\infty[\setminus\Delta} \lambda_k t^k.$$

There is only one normalized x of a fixed order $v(x)$, because the difference of two such lies in M and has no powers of t which can occur as an initial term and must therefore vanish.

The same ideas lead to a reduction algorithm for an element $x \in \mathbb{C}[[t]]$ with respect to a set $\{m_0,\dots,m_n\} \subset \mathbb{C}[[t]]$: Let Δ be the Γ–semi–group generated by $\{v(m_0),\dots,v(m_n)\}$. Set $x_0 = x \in \mathbb{C}[[t]]$. Starting with $i=0$ we do for increasing $i \in \mathbb{N}$ the following: If $i \notin \Delta$, set $s_i = 0$ and $x_{i+1} = x_i$. If $i \in \Delta$, then locate the t^i–term, $\tilde{\lambda}_i t^i$, in x_i, find $s_i \in R$, $j_i \in \{0,\dots,n\}$ with $\tilde{\lambda}_i t^i = s_i m_{j_i}$ and set $x_{i+1} = x_i - s_i m_{j_i}$. The x_i converge to an

$$x_\infty = \sum_{k\notin\Delta} \mu_k t^k.$$

Unfortunately, x_∞ depends in general on the choices made. However, this does not make the reduction process useless. Its main application is the following: If the $m_0,\dots,m_n$ generate the R–module M and $v(M) = \Delta$, then $x \in M$ iff $x_\infty = 0$. Namely, on the one hand if $x_\infty = 0$ then the algorithm yields $x = \sum s_i m_{j_i} \in M$, on the other hand if $x_\infty \neq 0$ then $v(x_\infty) \notin \Delta$ and this implies $x_\infty \notin M$ and $x = x_\infty + \sum s_i m_{j_i} \notin M$.

Often one starts with a module M and then picks normalized generators $m_0,\dots,m_n$ such that $\Delta := v(M)$ is generated as a Γ–semi–module by $v(m_0),\dots,v(m_n)$. We will call such a set a Δ–generating set of M. We write the generators as

$$\begin{aligned} m_0 &= \quad 1+ \sum_{k\in]0,\infty[\setminus\Delta} \lambda_k^0 t^k &&= \quad 1+ \sum_{k\notin\Delta} \lambda_k^0\, t^k \\ m_1 &= t^{a_1}+ \sum_{k\in]a_1,\infty[\setminus\Delta} \lambda_{k-a_1}^1 t^k &&= t^{a_1}+ \sum_{a_1+k\notin\Delta} \lambda_k^1\, t^{a_1+k} \\ &\vdots \\ m_n &= t^{a_n}+ \sum_{k\in]a_n,\infty[\setminus\Delta} \lambda_{k-a_n}^n t^k &&= t^{a_n}+ \sum_{a_n+k\notin\Delta} \lambda_k^n\, t^{a_n+k}. \end{aligned}$$

The special choice of the lower indices of the λ will be crucial later on.

Now we start the other way around. Given Δ with generators $\{a_0 = 0, \dots, a_n\}$, let $m_0, \dots, m_n$ be like above and M the module generated by them. Clearly, $\Delta \subseteq v(M)$. We want to see which conditions the λ must satisfy such that $v(M) = \Delta$; we consider them now as variables. To count the number of the λ, we introduce the *gap counting function* g_Δ by $g_\Delta(k) = \#([k,\infty[\setminus\Delta)$ for $k \in \mathbb{N}$, i.e., g_Δ counts the gaps of Δ greater than or equal to k. With this notation, there are $\sum_{j=0}^n g_\Delta(a_j)$ of the λ variables.

We want to consider syzygies between the m_j as well as between their initial terms. For a graded vector $r = (r_j) \in \bigoplus_j R(-a_j)$ we define the *initial vector* as follows: Let $\delta := \min\{v(r_j)+a_j\}$. Then $\mathrm{in}(r) = (s_j)$ with $s_j = \mathrm{in}(r_j)$ if $v(r_j)+a_j = \delta$ and 0 otherwise. The important consequence is that if r is a syzygy of the generators (m_j) of M then $\mathrm{in}(r)$ is a syzygy of $(\mathrm{in}(m_j)) = (t^{a_j})$.

Our leading idea for the following is

Lemma 3.1 *With the above notation let M be the R–module generated by $\{m_0, \dots, m_n\}$. Further, let $V \subset \bigoplus_j R(-a_j)$ such that the initial vectors $\{\mathrm{in}(r) | r \in V\}$ of V generate the syzygies of the generating set $A = (t^{a_j})$ of $\mathbb{C}[\Delta]$. Then $v(M) = \Delta$ if and only if for each $r = (r_j) \in V$ the following holds:*

Let $\delta := \min\{v(r_j m_j)\}$. Then the initial terms of $\sum r_j m_j$ cancel, i.e., $v(\sum r_j m_j) > \delta$, and there exist $s_j \in R$ with $v(s_j m_j) > \delta$ and $\sum r_j m_j = \sum s_j m_j$.

We will call $\sum s_j m_j$ a *higher order expression* of $\sum r_j m_j$. Note that a higher order expression can be obtained trivially if $\delta \geq c(\Delta)$ because $t^{c(\Delta)+k} \in M$ for $k \in \mathbb{N}$. A higher order expression for a term $T = \sum r_j m_j$ may be found by reducing it with the above algorithm. If T reduces to zero, then the algorithm produces an expression with $T = \sum s_i m_{j_i}$ with $v(s_i m_{j_i}) > \delta$ which we can reorder to get the higher order expression. If T reduces to $T_\infty \neq 0$, then $\mathrm{in}(T_\infty) \in v(M) \setminus \Delta$ showing $v(M) \neq \Delta$.

Proof. Clearly, we always have $v(M) \supseteq \Delta$. We have equality iff there is no element $x = \sum r_j m_j \in M$ with $v(x) \notin \Delta$. We claim that this is the case if and only if the following holds:

$(*)$ For any $\delta \in \Delta$ and $r = (r_j) \in \bigoplus_j R(-a_j)$ with $v(r_j) + a_j = \delta$ or $r_j = 0$ such that the initial terms of $\sum r_j m_j$ cancel, i.e., $v(\sum r_j m_j) > \delta$, $\sum r_j m_j$ can be expressed as $\sum s_j m_j$ with $v(s_j m_j) > \delta$.

To prove this claim, assume that $(*)$ holds, and there is an $x = \sum r_j m_j \in M$ with $v(x) \notin \Delta$. Further, we assume that the linear combination is chosen such that $\delta := \min\{v(r_j m_j)\} \in \Delta$ is maximal. Since $\delta \in \Delta$ and $v(x) \notin \Delta$, the initial terms of $\sum r_j m_j$ cancel and by $(*)$ there exist s_j with $x = \sum r_j m_j = \sum s_j m_j$ and $v(s_j m_j) > \delta$. But this contradicts the maximality assumption on $\min\{v(r_j m_j)\}$.

The other way around, assume we have r_j such that $(*)$ fails. Set $x_0 := \sum r_j m_j$ and reduce it with the above algorithm to x_∞. x_0 cannot be reduced to zero because $\sum r_j m_j = x_0 = \sum_{i=0}^{\infty} s_i m_{j_i}$ would show that $(*)$ holds for r_j. Therefore, $x_\infty \neq 0$ and $v(x_\infty) \notin \Delta$ shows $v(M) = \Delta$.

Cancellation of the initial terms in $(*)$ means precisely that $\mathrm{in}(r)$ is a syzygy of the generating set $A = (\mathrm{in}(m_j)) = (t^{a_j})$ of $\mathbb{C}[\Delta]$. Note that if $(*)$ holds for r then it holds for all r' with $\mathrm{in}(r') = \mathrm{in}(r)$, because $\sum r'_j m_j = \sum r_j m_j + \sum (r'_j - r_j) m_j = \sum (s_j + r_j - r'_j) m_j$ and $v(s_j), v(r_j - r'_j) > v(r'_j)$. Therefore, it is enough to check $(*)$ for a set of vectors which generate the syzygies of $\mathbb{C}[\Delta]$. □

We now study the different cases of characteristic Puiseux exponents separately. We start again with a unibranched plane singularity that has the characteristic Puiseux exponents (p, q). Then by Puiseux's Theorem the local ring R of the singularity is isomorphic to a ring $\mathbb{C}[[t^p, \varphi]] \subset \mathbb{C}[[t]] \cong \tilde{R}$ where $\varphi = t^q +$ higher order terms. Further $\Gamma = \langle p, q \rangle$. Let Δ be any 0–normalized Γ–semi–module. As a generating set for Δ we choose the unique p–basis $(a_0, \ldots, a_{p-1})$ and define m_j as above. By Proposition 2.4 and Lemma 3.1 the module M has $v(M) = \Delta$ iff higher order expressions for the following p terms can be found:

$$\begin{aligned} T^j &:= \varphi m_j - t^{(\alpha_{j+1} - \alpha_j)p} m_{j+1} =: \sum_{k=1}^{\infty} c_k^j t^{a_j + q + k} \qquad j = 0, \ldots, p-2 \\ T^{p-1} &:= \varphi m_{p-1} - t^{(q - \alpha_{p-1})p} m_0 \ \ =: \sum_{k=1}^{\infty} c_k^{p-1} t^{a_{p-1} + q + k}. \end{aligned}$$

We study the coefficients c_k^j more closely. For $k \in \mathbb{N}$ define the *gap function* $\tilde{g}_\Delta$ by $\tilde{g}_\Delta(k) = 1$ if $k \notin \Delta$ and 0 otherwise. Then with cyclic index notation

$$c_k^j = \tilde{g}_\Delta(a_j + k)\lambda_k^j - \tilde{g}_\Delta(a_{j+1} + k)\lambda_k^{j+1} + \text{polynomial in } \lambda_l^j \text{ with } l < k.$$

To find the higher order expressions for the T^j, we reduce the T^j by the above algorithm. We denote the resulting terms by $\tilde{T}^j$. These terms must vanish, otherwise $v(M) \neq \Delta$. The terms $\tilde{T}^j$ have only powers of t whose exponents do not lie in Δ. Let us study the coefficients of these t–powers more closely. There are two important observations: The first is that during this process a coefficient c_k^j is only modified by the addition of polynomials in the λ_l^i with $l < k$, except it is made to vanish. The second concerns the occurrence of the λ_k^j and λ_k^{j+1} in the final coefficients $\tilde{c}_k^j$. If $\tilde{g}_\Delta(a_j + k) = 0$ or $\tilde{g}_\Delta(a_{j+1} + k) = 0$, then $a_j + k \in \Delta$ or $a_{j+1} + k \in \Delta$ and further $a_j + k + q = a_{j+1} + k + (\alpha_{j+1} - \alpha_j)p \in \Delta$ for $j < p - 1$ resp. $a_{p-1} + k + q = a_0 + k + (q + \alpha_0 - \alpha_{p-1})p \in \Delta$ for $j = p - 1$, showing that the t–power $t^{a_j + q + k}$ will be eliminated in the process. Therefore, all c_k^j which do not have a $\lambda_k^j - \lambda_k^{j+1}$ term vanish during this process. Thus in the end the remaining coefficients $\tilde{c}_k^j$ with $a_j + q + k \notin \Delta$ are of the form

$$\tilde{c}_k^j = \lambda_k^j - \lambda_k^{j+1} + \text{polynomial in } \lambda_l^* \text{ with } l < k,$$

and there are $\sum_{j=0}^{p-1} g_\Delta(a_j + q)$ of these. The vanishing of these coefficients is equivalent to M being an R–module with associate semi–module Δ. For fixed k we may view $\tilde{c}_k^j = 0$ as an inhomogeneous linear equation system in the variables λ_k^j. Because of Proposition 2.4 there is a $J \in \{0, \ldots, p-1\}$ with $c(T_J) \geq c(\Delta)$ and consequently $\tilde{T}_j = 0$, therefore there are at most $p - 1$ nonzero equations and the linear system is in row echelon form. Hence, we can easily obtain a dependency of some of the λ_k^* on the other λ_k^* and the λ_l^* with $l < k$. Finally, we substitute successively the solutions for the λ with lower index less than k into the solutions for the λ with lower index k; thereby obtaining an explicit form of the equations $\tilde{c}_k^j = 0$, expressing some λ–variables as polynomial functions of the other.

Summarizing, we have shown that all possible coefficients for the m_i such that $v(M) = \Delta$ can be obtained as the graph of a polynomial function in $\sum g_\Delta(a_j) - \sum g_\Delta(a_j + q)$ variables. Setting $d = \delta_R - g_\Delta(0)$, we note that different values for the remaining free λ–variables lead to different modules $t^d M$ and modulo C to different points of $G(\tilde{R}/C, \delta_R)$, because of the normalized form of the m_j. Thus we have proved

Theorem 3.2 *Let R be the local ring of a unibranched plane singularity with characteristic Puiseux exponents (p, q) and Δ be a δ_R–normalized $\langle p, q\rangle$–semi–module, whose 0–normalization Δ_0 has the p–basis $(a_0, \ldots, a_{p-1})$. Then the subset of modules of J_R with associated semi–module Δ is biregular to an affine space $\mathbb{C}^N$ with*

$$N = \sum_{j=0}^{p-1} \left(g_{\Delta_0}(a_j) - g_{\Delta_0}(a_j + q) \right),$$

where for a $k \in \mathbb{N}$ the number $g_{\Delta_0}(k) := \#([k, \infty[\setminus \Delta_0)$ is the number of gaps in Δ_0 equal to or greater than k.

Since the number of $\langle p, q\rangle$–semi–modules is $\frac{1}{p+q}\binom{p+q}{p}$ by Proposition 2.2, the Jacobi factor J_R has a cell decomposition into the same number of complex cells. In particular, its Euler number is also $\frac{1}{p+q}\binom{p+q}{p}$, proving the main theorem in this case.

Now we treat the case of a singularity with characteristic Puiseux exponents $(4, 2q, s)$ using the notation of the preceeding section. The local ring R of such a singularity is isomorphic to $\mathbb{C}[[t^4, \varphi]] \subset \mathbb{C}[[t]]$, where $\varphi = t^{2q} + t^s+$ higher order terms [Z, p. 784]. Let $\psi \in R$ be the normalized element with $v(\psi) = \gamma_2 = 2q + s$. A 0–normalized Γ–semi–module Δ has a 2×2–basis $(a_{00} = 0, a_{01}; a_{10}, a_{11})$, thus we have the ansatz

$$m_{00} = 1 + \sum_{k \in]0,\infty[\setminus \Delta} \lambda_k^{00} t^k \qquad m_{01} = t^{a_{01}} + \sum_{k \in]a_{01},\infty[\setminus \Delta} \lambda_{k-a_{01}}^{01} t^k$$

$$m_{10} = t^{a_{10}} + \sum_{k \in]a_{10},\infty[\setminus \Delta} \lambda_{k-a_{10}}^{10} t^k \qquad m_{11} = t^{a_{11}} + \sum_{k \in]a_{11},\infty[\setminus \Delta} \lambda_{k-a_{11}}^{11} t^k$$

for the generators of an R–module M with associated semi–module Δ. By Proposition 2.9 and Lemma 3.1 we have $v(M) = \Delta$ iff we can find higher order expressions for the three terms

$$T^1 := \varphi m_{00} - t^{4\alpha_{01}} m_{01} =: \sum_{k=1}^{\infty} c_k^1 t^{\gamma_1 + k}$$

$$T^2 := t^{4(q-\alpha_{01})} m_{00} - \varphi m_{01} =: \sum_{k=1}^{\infty} c_k^2 t^{a_{01} + \gamma_1 + k}$$

$$T^3 := \psi m_{00} - t^{4\alpha_{10}} m_{10} =: \sum_{k=1}^{\infty} c_k^3 t^{\gamma_2 + k}.$$

We follow the same strategy as before: reduce T^1, T^2, T^3 with respect to $\{m_{ij}\}$ and solve the equations given by the remaining coefficients. However, the resulting equations are not so easy to solve, and we have to take more care in the reduction process of T_1 and T_2, which we think of being processed at the same time with increasing index of the λ variables. First, we note that

$$\begin{aligned} c^1_k = c^2_k &= \tilde{g}_\Delta(k)\lambda^{00}_k - \tilde{g}_\Delta(a_{01}+k)\lambda^{01}_k \quad \text{for } k = 1,\dots,s-\gamma_1-1 \\ c^1_{s-\gamma_1} &= \tilde{g}_\Delta(s-\gamma_1)\lambda^{00}_{s-\gamma_1} - \tilde{g}_\Delta(a_{01}+s-\gamma_1)\lambda^{01}_{s-\gamma_1} + 1 \\ c^2_{s-\gamma_1} &= \tilde{g}_\Delta(s-\gamma_1)\lambda^{00}_{s-\gamma_1} - \tilde{g}_\Delta(a_{01}+s-\gamma_1)\lambda^{01}_{s-\gamma_1} - 1 \end{aligned}$$

by the special form of φ. We want to organize the reduction process in such a way that in the intermediate stages the coefficients $\tilde{c}^1_k, \tilde{c}^2_k$ satisfy $\tilde{c}^1_k = \tilde{c}^2_k$ for $k = 1,\dots,s-\gamma_1-1$ and $\tilde{c}^1_{s-\gamma_1} - \tilde{c}^2_{s-\gamma_1} = 2$ as long as possible. Because $v(T^1), v(T^2) \geq \gamma_1$ and $\gamma_1 + 2\mathbb{N} \subset \Gamma$, the even powers of t, $\tilde{c}^1_{2k}t^{\gamma_1+2k}$ in T^1 and $\tilde{c}^2_{2k}t^{a_{01}+\gamma_1+2k}$ in T^2, can be eliminated by subtracting elements of the form $\tilde{c}^1_{2k}t^{4l}\varphi^i m_{00}$, $i \in \{0,1\}$. Whereas one has to use different pairs of (l,i) for T^1 and T^2, the coefficient $\tilde{c}^1_{2k}$ is equal to $\tilde{c}^2_{2k}$ for $0 < 2k < s-\gamma_1$. Therefore, the coefficients of the resulting terms differ only in and after the $(s-\gamma_1)$–th t–power term; in particular, the differences $\tilde{c}^1_j - \tilde{c}^2_j$ for $j = 1,\dots,s-\gamma_1$ are the same before and after the subtraction. The lower odd powers of t in T^1 and T^2 we do not eliminate at all while (a) the degree is less than s resp. $a_{01}+s$ and (b) there has not been an odd degree γ_1+2k+1 resp. $a_{01}+\gamma_1+2k+1$ where both degrees lie in Δ. After that we eliminate as many powers of t as possible in the usual way. By Lemma 3.1 $v(M) = \Delta$ holds iff the remaining coefficients $\tilde{c}^1_k, \tilde{c}^2_k, \tilde{c}^3_k$ of the reduced terms $\tilde{T}^1, \tilde{T}^2, \tilde{T}^3$ vanish. Our special treatment of the lower odd powers of t does not influence this, because for each of them $\tilde{c}^1_{2k+1}t^{\gamma_1+2k+1}$ resp. $\tilde{c}^2_{2k+1}t^{a_{01}+\gamma_1+2k+1}$ which we might have removed by subtraction there was a nonremovable $\tilde{c}^2_{2k+1}t^{a_{01}+\gamma_1+2k+1}$ resp. $\tilde{c}^1_{2k+1}t^{\gamma_1+2k+1}$ term, which forces $\tilde{c}^2_{2k+1}$ resp. $\tilde{c}^1_{2k+1}$ to vanish and with it the other one due to $\tilde{c}^1_{2k+1} = \tilde{c}^2_{2k+1}$. The advantage of this process is that we keep the difference $\tilde{c}^1_k - \tilde{c}^2_k$ fixed as long as possible. Let us exploit this.

We show that the 0–normalized associated semi–module Δ of an R–module has to be admissible, i.e., $\{s, a_{01}+s\} \cap \Delta \neq \emptyset$. Assume that this is not the case. Since $\{s, a_{01}+s\} \equiv \{1,3\} \bmod 4$, there is no odd number equal or less than s in Δ and condition (b) is satisfied up to $(s, a_{01}+s)$; hence $\tilde{c}^1_{s-\gamma_1} - \tilde{c}^2_{s-\gamma_1} = 2$ even at the end of the reduction process. Therefore, not both coefficients $\tilde{c}^1_{s-\gamma_1}, \tilde{c}^2_{s-\gamma_1}$ can vanish at the same time, and we cannot find higher order expressions for T^1 and T^2 simultaneously.

Now, we will show that when Δ is admissible the final equations $\tilde{c}^i_k = 0$ are solvable. We claim that either $\tilde{c}^i_k$ is already zero or

$$\tilde{c}^1_k = \lambda^{00}_k - \lambda^{01}_k + \dots, \quad \tilde{c}^2_k = \lambda^{00}_k - \lambda^{01}_k + \dots, \quad \tilde{c}^3_k = \lambda^{00}_k - \lambda^{10}_k + \dots,$$

where the dots stand for polynomials in the λ with lower index less than k. This follows as before. We discuss as an example the coefficient $\tilde{c}^1_k$. Looking at the definition of $\tilde{c}^1_k$, we see immediately that

$$\tilde{c}^1_k = \tilde{g}_\Delta(k)\lambda^{00}_k - \tilde{g}_\Delta(a_{01}+k)\lambda^{01}_k + \dots.$$

Now $\tilde{g}_\Delta(k) = 0$ or $\tilde{g}_\Delta(a_{01}+k) = 0$ implies $k \in \Delta$ or $a_{01}+k \in \Delta$ thus $\gamma_1 + k = a_{01} + k + 4\alpha_{01} \in \Delta$ and the term $\tilde{c}^1_k t^{\gamma_1+k}$ will be eliminated.

Again, we solve the equations $\tilde{c}^i_k = 0$ first for fixed index k and then successively substitute the solutions for the index less than k into the solutions for index k. The difficulty is that $\tilde{c}^1_k$ and $\tilde{c}^2_k$ have the same term $\lambda^{00}_k - \lambda^{01}_k$, and it is therefore impossible to solve these equations for λ_k^{00} and λ_k^{01} when $\tilde{c}_k^1$ and $\tilde{c}_k^2$ are nonzero and not the same. We have to treat two cases separately.

Let us assume that the smallest odd number $n \in \Delta \cap [\gamma_1, \infty[$ is less or equal to s. We visualize which coefficients can be eliminated by the following diagram:

	0	1	2	3	$\cdots$	$n-\gamma_1$	$+1$	$+2$	$+3$	$+4$	$\cdots$
T^1	0	$=/\circ$	$=/\times$	$=/\circ$	$\cdots$	$\times$	$\times$		$\times$	$\times$	$\cdots$
T^2	0	$=/\circ$	$=/\times$	$=/\circ$	$\cdots$		$\times$	$\times$	$\times$		$\cdots$

The k–th column stands for the coefficients of t^{γ_1+k} and $t^{a_{01}+\gamma_1+k}$ in T_1 and T_2; "$=$" stands for equal coefficients in the these terms, "$\times$" for "coefficient can be eliminated" and "$\circ$" for "coefficient cannot be eliminated". After the $(n-\gamma_1)$–th coefficient at least one of the coefficients with the same odd index k can be eliminated, because the corresponding t–powers have the degrees $(k, a_{01}+k)$, hence they are $(1,3)$ or $(3,1)$ modulo 4 and both are greater than n, thus one of them lies in $n + 4\mathbb{N} \subset \Delta$.

Therefore, up to the index $n-\gamma_1$, $\tilde{c}_k^1 = \tilde{c}_k^2$, and after that at least one of the $\tilde{c}_k^1$, $\tilde{c}_k^2$ vanishes; thus there is at most one equation for each odd index and solving it is trivial. In addition, a higher order expression for T^3 is trivially obtained because the order of each of its t–powers is greater than the conductor $c(\Delta) = \max\{a_{ij}\} - 3 \leq \max\{n, n+\gamma_1, a_{01}\} \leq \gamma_2$. Therefore, we found explicit polynomial equations for the $g_\Delta(\gamma_1)$ nontrivial equations $\tilde{c}_k^1 = 0$ and the additional $g_\Delta(a_{01}+n)$ nontrivial equations $\tilde{c}_k^2 = 0$ with $k \geq n-\gamma_1$. This shows that the Δ–subset of J_X is a complex cell of dimension

$$\sum g_\Delta(a_{ij}) - g_\Delta(\gamma_1) - g_\Delta(a_{01}+n).$$

The second case we have to consider is when the smallest odd number of Δ is greater than s. Because Δ is admissible, we have $a_{01}+s \in \Delta$, in a diagram

	0	1	2	3	$\cdots$	$s-\gamma_1$	$+1$	$+2$	$+3$	$+4$	$\cdots$
T^1	0	$=/\circ$	$=/\times$	$=/\circ$	$\cdots$	$\circ$	$\times$		$\times$		$\cdots$
T^2	0	$=/\circ$	$=/\times$	$=/\circ$	$\cdots$	$\times$	$\times$		$\times$	$\times$	$\cdots$

During the reduction process for the first $s-\gamma_1-1$ coefficients we use only multiples of m_{00}. To eliminate the $t^{a_{01}+s}$–term of T^2, we have to use a multiple of m_{10} due to $a_{01}+s \equiv a_{10} \bmod 4$. Because of $c^2_{s-\gamma_1} = -1 + \ldots$, we add $(1+\ldots)t^{a_{01}+s-a_{10}}m_{10}$ to T^2, in particular we add λ^{10}_{2+4l} to the coefficient $\tilde{c}^2_{s-\gamma_1+2+4l}$ for $a_{01}+s+2+4l \notin \Delta$. Tracking the variables λ_k^{00}, λ_k^{01}, λ_k^{10} with the greatest index in these coefficients we find

$$\tilde{c}^1_{s-\gamma_1+2+4l} = \lambda^{00}_{s-\gamma_1+2+4l} - \lambda^{01}_{s-\gamma_1+2+4l} + \ldots$$

$$\tilde{c}^2_{s-\gamma_1+2+4l} = \lambda^{00}_{s-\gamma_1+2+4l} - \lambda^{01}_{s-\gamma_1+2+4l} + \lambda^{10}_{2+4l} + \ldots$$

at this point, and this will not change later during the reduction process except when $\tilde{c}^1_{s-\gamma_1+2+4l}$ is made to vanish. Hence, $\tilde{c}^1_{s-\gamma_1+2+4l} = \tilde{c}^2_{s-\gamma_1+2+4l} = 0$ can be solved for λ^{10}_{2+4l} and $\lambda^{00}_{s-\gamma_1+2+4l}$. Due to $a_{01}+s+4l \in \Delta$, we have $\tilde{c}^2_{s-\gamma_1+4l} \equiv 0$ and solving $\tilde{c}^1_{s-\gamma_1+4l} = \tilde{c}^2_{s-\gamma_1+4l} = 0$ is trivial. Plugging these solutions back into T^1 and T^2, we see that with our reduction process we have found higher order expressions for T^1 and T^2 of the form

$$\varphi m_{00} - t^{4\alpha_{01}} m_{01} = f_{00}m_{00} + f_{01}m_{01} + f_{10}m_{10} + f_{11}m_{11}$$

$$t^{4(q-\alpha_{01})}m_{00} - \varphi m_{01} = g_{00}m_{00} + g_{01}m_{01} - t^{4(\alpha_{10}-\alpha_{01})}m_{10} + g_{10}m_{10} + g_{11}m_{11}$$

with $v(f_{00}m_{00}), v(f_{01}m_{01}) > \gamma_1,\quad v(f_{10}m_{10}), v(f_{11}m_{11}) > s$

and $v(g_{00}m_{00}), v(g_{01}m_{01}) > a_{01}+\gamma_1,\ v(g_{10}m_{10}), v(g_{11}m_{11}) > a_{01}+s.$

The amazing fact is that from these two equations we can find a higher order expression for T^3 without imposing further restrictions on the λ–variables. We multiply the first equation by φ and the second by $t^{4\alpha_{01}}$, then all products on the left hand sides are of order $2\gamma_1$. We subtract the second from the first equation and move the terms from the left hand side to the right hand side to obtain

$$0 = h_{00}m_{00} + h_{01}m_{01} - t^{4\alpha_{10}}m_{10} + h_{10}m_{10} + h_{11}m_{11} \qquad (+)$$

$$\text{with } v(h_{00}m_{00}), v(h_{01}m_{01}) > 2\gamma_1,\ v(h_{10}m_{10}), v(h_{11}m_{11}) > s + \gamma_1 = \gamma_2.$$

Assume that $v(h_{00}m_{00}), v(h_{01}m_{01}) < \gamma_2$, then cancellation of the initial terms takes place in $h_{00}m_{00} + h_{01}m_{01}$, i.e., $(\mathrm{in}(h_{00}), \mathrm{in}(h_{01}))$ is a syzygy of $(\mathrm{in}(m_{00}), \mathrm{in}(m_{01}))$. As the syzygies of $(\mathrm{in}(m_{00}), \mathrm{in}(m_{01}))$ are generated by $(\mathrm{in}(\varphi), -t^{4\alpha_{01}})$ and $(t^{4(q-\alpha_{01})}, -\mathrm{in}(\varphi))$, we can find $r_1, r_2 \in R$ with

$$(\mathrm{in}(h_{00}), \mathrm{in}(h_{01})) = \mathrm{in}(r_1)(\mathrm{in}(\varphi), -t^{4\alpha_{01}}) + \mathrm{in}(r_2)(t^{4(q-\alpha_{01})}, -\mathrm{in}(\varphi));$$

thus in

$$T := r_1T^1 + r_2T^2 = (r_1\varphi + r_2t^{4(q-\alpha_{01})})m_{00} + (-r_1t^{4\alpha_{01}} - r_2\varphi)m_{01}$$

the coefficients of m_{00} and m_{01} have also the same initial terms, $(\mathrm{in}(h_{00}), \mathrm{in}(h_{01}))$. From the higher order expressions of T^1 and T^2 we obtain one for T, $\sum s_{ij}m_{ij}$. We subtract $T - \sum s_{ij}m_{ij} = 0$ from (+) to get rid of the initial terms of $h_{00}m_{00}$ and $h_{01}m_{01}$ in (+) without changing any of the extra conditions. Continuing this way we arrive at the stage where we may assume that $v(h_{00}m_{00}), v(h_{01}m_{01}) \geq \gamma_2$.

As γ_2 is the smallest odd number in Γ and $v(m_{01}) > 0$ is even, we conclude that $v(h_{01}m_{01}) = v(h_{01}) + v(m_{01}) > \gamma_2$. Therefore, the cancellation of the initial terms in (+) takes places between $h_{00}m_{00}$ and $t^{4\alpha_{10}}m_{10}$ with $v(h_{00}) = \gamma_2$, providing a higher order expression for the term $h_{00}m_{00} + t^{4\alpha_{10}}m_{10}$, which is essentially T^3 and may be used instead of it. Namely, because the syzygy $(\mathrm{in}(h_{00}), -t^{4\alpha_{10}})$ between $\mathrm{in}(m_{00})$ and $\mathrm{in}(m_{10})$ together with the above two syzygies between $\mathrm{in}(m_{00})$ and $\mathrm{in}(m_{01})$ generate all the syzygies of $\mathbb{C}[\Delta]$ below the degree of the conductor $c(\Delta)$, the conditions of the Lemma 3.1 are satisfied. As we solved the $g_\Delta(\gamma_1)$ nontrivial equations $\tilde{c}_k^1 = 0$ and the additional $g_\Delta(a_{01} + s)$ nontrivial equations $\tilde{c}_k^2 = 0$ by polynomial functions we have shown that the Δ–subset of the Jacobi factor J_R is a complex cell of dimension

$$\sum g_\Delta(a_{ij}) - g_\Delta(\gamma_1) - g_\Delta(a_{01} + s).$$

Summarizing we proved

Theorem 3.3 *Let R be the local ring of a unibranched plane singularity with characteristic Puiseux exponents $(4, 2q, s)$ and Δ be a δ_R–normalized $\langle 4, \gamma_1 = 2q, \gamma_2 = 2q + s\rangle$–semi–module, whose 0–normalization Δ_0 has a 2×2–basis $(a_{00} = 0, a_{01}; a_{10}, a_{11})$. Then the subset of modules of J_R with associated semi–module Δ is nonempty if Δ_0 is admissible, i.e., $\{s, a_{01} + s\} \cap \Delta_0 \neq \emptyset$. In this case it is biregular to an affine space $\mathbb{C}^N$ with*

$$N = \sum g_{\Delta_0}(a_{ij}) - g_{\Delta_0}(\gamma_1) - g_{\Delta_0}(a_{01} + n)$$

where n is the smallest odd number in $(\Delta_0 \cup \{s\}) \cap [\gamma_1, \infty[$.

As a consequence the number of admissible semi–modules (Proposition 2.8) is the Euler number of J_R, as stated in the main theorem.

In the next section we are going to prove an analogous theorem for the characteristic Puiseux exponents $(6, 8, s)$ and $(6, 10, s)$. It seems that there are no further Puiseux exponents where such a theorem holds. The rows of the following table consist of a ring R and its associated semi–group Γ together with the 0–normalization of a Γ–semi–module Δ such that the Δ–subset of J_R is not affine, but $\mathbb{C}^N \times \mathbb{C}^*$, a union of two affine spaces, or worse.

R	Γ	Δ_0
$\mathbb{C}[t^6, t^{14} + t^{15}]$	$\langle 6, 14, 43\rangle$	$\langle 0, 8, 16, 23, 31, 39\rangle$
$\mathbb{C}[t^6, t^{14} + t^{17}]$	$\langle 6, 14, 45\rangle$	$\langle 0, 8, 16, 23, 31, 39\rangle$
$\mathbb{C}[t^6, t^9 + t^{10}]$	$\langle 6, 9, 19\rangle$	$\langle 0, 3, 7, 10, 17, 20\rangle$
$\mathbb{C}[t^6, t^9 + t^{13}]$	$\langle 6, 9, 22\rangle$	$\langle 0, 3, 7, 10, 17, 20\rangle$
$\mathbb{C}[t^9, t^{12} + t^{14}]$	$\langle 9, 12, 38\rangle$	$\langle 0, 3, 13, 28, 32, 35\rangle$
$\mathbb{C}[t^{10}, t^{14} + t^{17}]$	$\langle 10, 14, 73\rangle$	$\langle 0, 4, 16, 31, 37\rangle$
$\mathbb{C}[t^8, t^{12} + t^{14} + t^{15}]$	$\langle 8, 12, 26, 53\rangle$	$\langle 0, 4, 13, 17, 19, 22\rangle$

4 The Puiseux exponents $(6, 8, s)$ and $(6, 10, s)$

In this section we deal with the cases when the characteristic Puiseux exponents of the singularity are $(6, 8, s)$ and $(6, 10, s)$. The above examples suggest that these two cases are the last ones where the natural cell decomposition of the Jacobi factor is affine. The basic ideas for the proof of this are the same as in the $(4, 2q, s)$ case, but the arguments have to be sharpened. In particular, the combinatorics of the Γ–semi–modules is more complicated. As most of this section is very technical, we recommend it only for the most interested reader.

We follow the proof for the $(4, 2q, s)$ case. Recall that Γ is generated by $\gamma_0 = 6$, $\gamma_1 = 2q$, and $\gamma_2 = 2\gamma_1 + s$ and that a 0–normalized Γ–semi–module has a 2×3–basis, see Definition 2.5. We compute the number of Γ–semi–modules.

Proposition 4.1 *The number of* 0*–normalized* Γ*–semi–modules for the semi–group* $\Gamma = \langle 6, 2q, 4q + s\rangle$ *with* $\gcd(q, 3) = \gcd(s, 2) = 1$ *is*

$$\frac{(q+1)(q+2)(7q^3+24q^2+29q+15)}{180} + s\frac{(q+1)^2(q+2)^2}{72}.$$

Proof. This time we have to count the number of 5–tuples $\alpha = (\alpha_{01}, \alpha_{02}, \alpha_{10}, \alpha_{11}, \alpha_{12})$ which satisfy

$$\begin{array}{ccccccc} & & q/3 & & 2q/3 & & \\ & & \vee & & \vee & & \\ 0 & \leq & \alpha_{01} & \leq & \alpha_{02} & & \\ \wedge\!| & & \wedge\!| & & \wedge\!| & & \\ \alpha_{10} & \leq & \alpha_{11} & \leq & \alpha_{12} & \leq & q+\alpha_{10}. \\ \wedge & & \wedge & & \wedge & & \\ \frac{4q+s}{6} & & \frac{6q+s}{6} & & \frac{8q+s}{6} & & \end{array}$$

Let A be the set of these. We may view this set as $A = \bar{A} \setminus (A_3 \cup A_4)$ with

$$\bar{A} = \Big\{\alpha \in \mathbb{N}^5 \mid \alpha_{01} < \tfrac{q}{3}, \alpha_{01} \leq \alpha_{02} < \tfrac{2q}{3}; \alpha_{1j} < \tfrac{(4+2j)q+s}{6}, \\ \alpha_{10} \leq \alpha_{11} \leq \alpha_{12} \leq q + \alpha_{10}\Big\}$$

$$\begin{aligned} A_3 &= \bar{A} \cap \{\alpha \in \mathbb{N}^5 \mid \alpha_{02} > \alpha_{12}\} \\ &= \{\alpha \in \mathbb{N}^5 \mid \alpha_{01} < \tfrac{q}{3}, \alpha_{01} \leq \alpha_{02} < \tfrac{2q}{3}; \alpha_{10} \leq \alpha_{11} \leq \alpha_{12} < \alpha_{02}\} \\ A_4 &= \bar{A} \cap \{\alpha \in \mathbb{N}^5 \mid \alpha_{02} \leq \alpha_{12}, \alpha_{01} > \alpha_{11}\} \\ &= \{\alpha \in \mathbb{N}^5 \mid \alpha_{01} < \tfrac{q}{3}, \alpha_{01} \leq \alpha_{02} < \tfrac{2q}{3}; \alpha_{10} \leq \alpha_{11} < \alpha_{01}, \alpha_{02} \leq \alpha_{12} \leq q + \alpha_{10}\}. \end{aligned}$$

We split $\bar{A}$ again as $\bar{A} = A_0 \setminus (A_1 \cup A_2)$ with

$$\begin{aligned} A_0 &= \{\alpha \in \mathbb{N}^5 \mid \alpha_{01} < \tfrac{q}{3}, \alpha_{01} \leq \alpha_{02} < \tfrac{2q}{3}; \alpha_{10} < \tfrac{4q+s}{6}, \alpha_{10} \leq \alpha_{11} \leq \alpha_{12} \leq q + \alpha_{10}\} \\ A_1 &= A_0 \cap \{\alpha \in \mathbb{N}^5 \mid \alpha_{11} > \tfrac{6q+s}{6}\} \\ &= \{\alpha \in \mathbb{N}^5 \mid \alpha_{01} < \tfrac{q}{3}, \alpha_{01} \leq \alpha_{02} < \tfrac{2q}{3}; \tfrac{s}{6} < \alpha_{10} < \tfrac{4q+s}{6}, \\ &\qquad \tfrac{6q+s}{6} < \alpha_{11} \leq \alpha_{12} \leq q + \alpha_{10}\} \\ A_2 &= \bar{A} \cap \{\alpha \in \mathbb{N}^5 \mid \alpha_{11} < \tfrac{6q+s}{6}, \alpha_{21} > \tfrac{8q+s}{6}\} \\ &= \{\alpha \in \mathbb{N}^5 \mid \alpha_{01} < \tfrac{q}{3}, \alpha_{01} \leq \alpha_{02} < \tfrac{2q}{3}; \tfrac{2q+s}{6} < \alpha_{10} < \tfrac{4q+s}{6}, \alpha_{10} \leq \alpha_{11} < \tfrac{6q+s}{6}, \\ &\qquad \tfrac{8q+s}{6} < \alpha_{12} \leq q + \alpha_{10}\}. \end{aligned}$$

By definition the $A_1,\dots,A_4 \subset A_0$ are pairwise disjoint, thus $\#A = \#A_0 - \sum_{i=1}^4 \#A_i$. The sets A_i are written down in such a way that when one reads the inequalities from the left to the right there are only restrictions on the newly appearing variables; hence, they can be counted, for example

$$\#A_1 = \sum_{\alpha_{01}=0}^{\lfloor \frac{q}{3} \rfloor} \sum_{\alpha_{02}=\alpha_{01}}^{\lfloor \frac{2q}{3} \rfloor} \sum_{\alpha_{10}=\lceil \frac{s}{6} \rceil}^{\lfloor \frac{4q+s}{6} \rfloor} \sum_{\alpha_{11}=\lceil \frac{6q+s}{6} \rceil}^{q+\alpha_{10}} (q+1+\alpha_{10}-\alpha_{11}).$$

It is possible to evaluate these sums and obtain for $\#A$ the number in the statement. □

Next we count the Γ–semi–modules of which we show later that they are not the 0–normalization of an associated semi–module of a torsion free module over the local ring of the singularity.

Proposition 4.2 *Let $\Gamma = \langle 6, 2q, 4q+s \rangle$ be the above semi–group with $q \in \{4,5\}$. The number of admissible Γ–semi–modules is*

$$\frac{(q+1)(q+2)(7q^3+24q^2+29q+15)}{180} - \frac{2(4q+7)}{15}\binom{q+2}{4} + s\frac{(q+1)^2(q+2)^2}{72}.$$

Proof. A proof of this Proposition can be obtained by mixing the ideas of the proofs of Proposition 2.8 and Proposition 4.1. □

Evaluating this formula for $q = 4, 5$, one obtains the numbers given in the statement of the Main Theorem.

It remains to compute the syzygies of the canonical generators of the $\mathbb{C}[\Gamma]$–module $\mathbb{C}[\Delta]$. We continue to use the notation of Section 2

Proposition 4.3 *Let $\Gamma = \langle \gamma_0 = 2p,\ \gamma_1 = 2q,\ \gamma_2 = 2(p-1)q+s \rangle$, choose $\beta \in \{0,\dots,p-1\}$ and $\eta \in \mathbb{N}$ such that $2\gamma_2 = \beta\gamma_1 + \eta\gamma_0$. Further, let Δ be a 0–normalized Γ–semi–module with 2×3–basis (a_{ij}). Then the $\mathbb{C}[\Gamma]$–module $\mathbb{C}[\Delta]$ is generated by $A = (t^{a_{ij}})$, and the syzygies of this generating set are generated by the following additively written bivector syzygies:*

- $a_{ij} + \gamma_1 = a_{i,j+1} + *\gamma_0$
- $a_{0j} + \gamma_2 = a_{1,j-\mu_j} + \mu_j\gamma_1 + *\gamma_0$ *where μ_j is chosen maximal under the condition $a_{0j} + \gamma_2 \geq a_{1,j-\mu_j} + \mu_j\gamma_1$.*
- $a_{1j} + \gamma_2 = a_{0,j+\beta-\nu_j} + \nu_j\gamma_1 + *\gamma_0$ *where ν_j is chosen maximal under the condition $a_{1j} + \gamma_2 \geq a_{0,j+\beta-\nu_j} + \nu_j\gamma_1$.*

Here, we use cyclic index notation and $*$ stands for an easily computed unique natural number.

Proof. By Lemma 2.3 there is a generating set of syzygies consisting of bivectors. Any bivector syzygy of degree d may be written additively as

$$a_{ij} + \xi_2\gamma_2 + \xi_1\gamma_1 + \xi_0\gamma_0 = a_{lk} + \zeta_2\gamma_2 + \zeta_1\gamma_1 + \zeta_0\gamma_0 = d$$

with $\xi_2, \zeta_2 \in \{0,1\}$, $\xi_1, \zeta_1 \in \{0,\dots,p-1\}$, and $\xi_0, \zeta_0 \in \mathbb{N}$. We may assume that the bivector syzygy is not the multiple of another; hence, for all $r \in \{0,1,2\}$ one of the $\xi_r, \zeta_r = 0$ is zero. Considering the above relations modulo γ_0 and using $2\gamma_2 = \beta\gamma_1 + \eta\gamma_0$ one sees

$$i + \xi_2 \equiv l + \zeta_2 \mod 2 \quad \text{and} \quad j + \xi_1 + \beta\delta^2_{i+\xi_2} \equiv k + \zeta_1 + \beta\delta^2_{l+\zeta_2} \mod p;$$

here δ_n^2 is the Kronecker δ–symbol, i.e., $\delta_n^2 = 0$ except for $n = 2$ where $\delta_2^2 = 1$. In particular, a minimal syzygy between the $\{a_{ij}\}$ for fixed i does not involve a γ_2. Therefore, any such bivector syzygy is of the type $a_{ij} + k\gamma_1 = a_{i,j+k} + *\gamma_0$, and thus a combination of the $a_{il} + \gamma_1 = a_{i,l+1} + *\gamma_0$ for $l = j, \dots, j+k-1$. Next, we note that a syzygy $a_{0j} + \gamma_2 + k\gamma_1 = a_{1,j+k} + *\gamma_0$ is a combination of $a_{0j} + \gamma_2 = a_{1,j} + *\gamma_0$ and $a_{1j} + k\gamma_1 = a_{1,j+k} + *\gamma_0$, and similarly for $a_{1j} + \gamma_2 + k\gamma_1 = a_{0,j+\beta+k} + *\gamma_0$. Also, if there exists a relation of the type $a_{0j} + \gamma_2 + *\gamma_0 = a_{1,j-k} + k\gamma_1$, it can be obtained from $a_{0j} + \gamma_2 = a_{1,j} + *_1\gamma_0$ and $a_{1,j-k} + k\gamma_1 = a_{1,j} + *_2\gamma_0$ using the assumed relation to see that $*_2 \geq *_1$. Again, an analogous statement holds for $a_{1j} + \gamma_2 + *\gamma_0 = a_{0,j+\beta-k} + k\gamma_1$.

Thus it remains to show that all relations of the type $a_{ij} + \gamma_2 = \dots$ can be obtained from the ones in the statement of the theorem. Since the ones with the most γ_1's on the right hand side for each fixed a_{0j} or a_{1j} on the left hand side are the ones in the statement, the other can be obtained from them by replacing $a_{ij} + k\gamma_1$ on the right hand side by the corresponding $a_{i,j+k} + *\gamma_0$. $\square$

Our main interest are the syzygies whose degree is less than the conductor of the module. Let us isolate these for $p = 3$.

Corollary 4.4 *Let $\Gamma = \langle \gamma_0 = 6,\ \gamma_1 = 2q,\ \gamma_2 = 4q + s\rangle$, and $\Delta = \bigcup (a_{ij} + \gamma_0\mathbb{N})$ like above. The $\mathbb{C}[\Gamma]$–module $\mathbb{C}[\Delta]$ is generated by $A = (1, t^{a_{01}}, t^{a_{02}}, t^{a_{10}}, t^{a_{11}}, t^{a_{12}})$, and the syzygies of A of degree less than $c(\Delta)$ are generated by the following additively written bivector syzygies:*

— *If $\max\{a_{ij}\} = a_{0J}$ for a suitable $J \in \{1, 2\}$ use the following relations:*

- $a_{02} + \gamma_1 = a_{00} + *\gamma_0$ *if* $J = 1$ *or* $a_{00} + \gamma_1 = a_{01} + *\gamma_0$ *if* $J = 2$.
- $a_{1j} + \gamma_1 = a_{1,j+1} + *\gamma_0$ *for* $j \in \{0, 1, 2\}$

— *If $\max\{a_{ij}\} = a_{1J}$ for a suitable $J \in \{0, 1, 2\}$, choose $K, L \in \{0, 1, 2\} \setminus \{I\}$ with $K + 1 \equiv L \bmod 3$ and use the following relations:*

- $a_{0j} + \gamma_1 = a_{0,j+1} + *\gamma_0$ *for* $j \in \{0, 1, 2\}$
- $a_{1K} + \gamma_1 = a_{1L} + *\gamma_0$
- $a_{0K} + \gamma_2 = a_{1K} + *\gamma_0$
- $a_{0L} + \gamma_2 = a_{1K} + \gamma_1 + *\gamma_0$ *if* $a_{0L} + \gamma_2 \geq a_{1K} + \gamma_1$
  $a_{0L} + \gamma_2 = a_{1L} + *\gamma_0$ *else.*

Proof. Obviously, any relation involving the maximal a_{ij} has degree greater than $c(\Delta)$. Therefore, for the first case it is enough to remark that because of $\max\{a_{ij}\} = a_{0J} \leq 2\gamma_1 < \gamma_2$ any relation involving a γ_2 has also degree greater than $c(\Delta)$. For the more general second case we need only to argue for $a_{1j} + \gamma_2 > c(\Delta)$. By the definition of the a_{ij} we have $a_{0j} \leq 2\gamma_1$, $a_{1,j+1} \leq a_{1j} + \gamma_1$, and $a_{1,j+2} \leq a_{1j} + 2\gamma_1$; hence, $c(\Delta) < \max\{a_{lk}\} \leq a_{1j} + 2\gamma_1 < a_{1j} + \gamma_2$. $\square$

Finally, it remains to prove that the Δ–subsets of the Jacobi factor J_R are affine. As shown by examples at the end of the last section this is probably only possible for the characteristic Puiseux exponents $(6, 8, s)$ and $(6, 10, s)$.

Theorem 4.5 *Let R be the local ring of a unibranched plane singularity with characteristic Puiseux exponents $(6, 8, s)$ or $(6, 10, s)$ and Γ its associated semi-group. Let Δ be a δ_R–normalized Γ–semi–module. Then the Δ–subset of J_R is biregular to an affine space $\mathbb{C}^N$ and nonempty iff the 0–normalization of Δ is admissible.*

The proof proceeds as before. Again, the local ring of a singularity is isomorphic to $\mathbb{C}[[t^6,\varphi]] \in \mathbb{C}[[t]]$, where $\varphi = t^{\gamma_1}+t^s+\ldots$, because by a coordinate transformation any t–power whose exponent lies in $\Gamma \setminus \{\gamma_1\}$ or in $((\gamma_1-\gamma_0)+\Gamma)\setminus\{\gamma_1\}$ can be eliminated [Z, p. 784]. These two characteristic Puiseux exponents series are the only ones — apart from the ones already discussed — where there are no t–powers in φ between the t–powers to the second and third Puiseux exponent. We denote the normalized element of R of order $\gamma_2 = 2\gamma_1+s$ by ψ.

Let (a_{ij}) be the 2×3–basis of the 0–normalization of Δ. We will work during this proof only with the 0–normalization and may therefore denote it by Δ as well. For an R–module M with associated semi–module Δ we have the ansatz

$$m_{ij} = t^{a_{ij}} + \sum_{k\in]a_{ij},\infty[\setminus\Delta} \lambda^{ij}_{k-a_{ij}} t^k$$

for its six generators. These generators must satisfy the condition of Lemma 3.1 for the syzygies of Corollary 4.4. The most interesting syzygies are the ones between 1, $t^{a_{01}}$, and $t^{a_{02}}$. They lead to the terms

$$\begin{aligned} T^1 &:= \varphi m_{00} - t^{6\alpha_{01}} m_{01} &&=: \sum_{k=1}^{\infty} c^1_k t^{\gamma_1+k}\\ T^2 &:= \varphi m_{01} - t^{6(\alpha_{02}-\alpha_{01})} m_{02} &&=: \sum_{k=1}^{\infty} c^2_k t^{a_{01}+\gamma_1+k}\\ T^3 &:= \varphi m_{02} - t^{6(q-\alpha_{02})} m_{00} &&=: \sum_{k=1}^{\infty} c^3_k t^{a_{02}+\gamma_1+k}\end{aligned}$$

for which we have to find higher order expressions. We proceed as before: reduce T^1, T^2, T^3 with respect to $\{m_{ij}\}$ in some modified way, solve the equations given by the remaining coefficients for a fixed index k and successively substitute these solutions into each other.

As before we find that

$$\begin{aligned} &c^j_k = \tilde g_\Delta(a_{0,j-1}+k)\lambda^{0,j-1}_k - \tilde g_\Delta(a_{0j}+k)\lambda^{0j}_k \quad \text{for } k=1,\ldots,s-\gamma_1-1 \quad \text{and}\\ &c^j_{s-\gamma_1} = \tilde g_\Delta(a_{0,j-1}+s-\gamma_1)\lambda^{0,j-1}_{s-\gamma_1} - \tilde g_\Delta(a_{0j}+s-\gamma_1)\lambda^{0j}_{s-\gamma_1} + 1.\end{aligned}$$

We note that the sum $c^1_k+c^2_k+c^3_k$ is zero for $k=1,\ldots,s-\gamma_1-1$ and $c^1_{s-\gamma_1} + c^2_{s-\gamma_1} + c^3_{s-\gamma_1} = 3$. These are the invariants that we want to keep as long as possible during our modified reduction process. First, we consider the elimination of the even t–powers. Since all even numbers greater than or equal to $2\gamma_1-4$ are contained in Γ, we can subtract — for fixed even k — appropriate multiples of m_{00} from all T^j to eliminate the terms $\tilde c^j_k t^{a_{0j}+\gamma_1+k}$ when $\gamma_1+k \geq 2\gamma_1-4$. This does not change the sum conditions, because modulo the ideals (t^{s+1}), $(t^{a_{01}+s+1})$, resp. $(t^{a_{02}+s+1})$ we subtract $\tilde c^1_k t^{\gamma_1+k} m_{00}$ from T^1, $\tilde c^2_k t^{a_{01}+\gamma_1+k} m_{00}$ from T^2, and $\tilde c^3_k t^{a_{02}+\gamma_1+k} m_{00} = -(\tilde c^1_k+\tilde c^2_k) t^{a_{02}+\gamma_1+k} m_{00}$ from T^3. For $\gamma_1 = 8$ this leaves only the terms $c^j_2 t^{a_{0j}+\gamma_1+2}$ to discuss. If $a_{01}=2$, then $a_{00}+\gamma_1+2, a_{02}+\gamma_1+2 \in a_{01}+\Gamma$ and $a_{01}+\gamma_1+2, a_{02}+\gamma_1+2 \in a_{00}+\Gamma$; thus, we can subtract $\tilde c^1_2 \varphi m_{01}$ from T^1, $\tilde c^2_2 t^{12} m_{00}$ from T^2, and add $\tilde c^1_2 t^{6(4-\alpha_{02})} m_{01} + \tilde c^2_2 \varphi t^{6(3-\alpha_{02})} m_{00}$ to T^3. Due to $\tilde c^3_2 = -\tilde c^1_2 - \tilde c^2_2$, this eliminates all the $\tilde c^j_2$ coefficients and leaves the sum condition intact. If $a_{01}=8$ and $a_{02}\in\{4,10\}$, then $a_{00}+\gamma_1+2, a_{01}+\gamma_1+2 \in a_{02}+\Gamma$ and $a_{01}+\gamma_1+2, a_{02}+\gamma_1+2 \in a_{00}+\Gamma$ and an analogous subtraction and addition works. The case of $a_{01}=8$ and $a_{02}=16$ is trivial because here m_{01} and m_{02} must be the normalization of φm_{00} resp. $\varphi^2 m_{00}$. For $\gamma_1 = 10$ the same ideas work, because we can always find two indices j_1, j_2 such that

$$\#\left(\{a_{0j} + \gamma_1 + 2k \mid j = 0,1,2\} \cap (a_{0j_\varrho} + \Gamma)\right) \geq 2 \quad \text{for} \quad \varrho = 1,2 \quad \text{and}$$
$$\{a_{0j} + \gamma_1 + 2k \mid j = 0,1,2\} \subset (a_{0j_1} + \Gamma) \cup (a_{0j_2} + \Gamma).$$

An analogous result does not hold for $\gamma_1 > 10$.

Now, we consider the elimination of the odd t–powers. If we can eliminate only one of the terms $\tilde{c}_k^2 t^{a_{01}+\gamma_1+k}$ of T^2 or $\tilde{c}_k^3 t^{a_{02}+\gamma_1+k}$ of T^3 for an odd index $k < s - \gamma_1$ and also not $\tilde{c}_k^1 t^{\gamma_1+k}$ of T^1, we do not eliminate at all. Because we later force the remaining two coefficients to be zero, the third will be zero as well due to the sum condition. Therefore, we will still find a higher order expression for T^1, T^2, T^3 by this modified reduction process. As soon as we find an odd index n with $\gamma_1 + n \in \Delta$ or $a_{01} + \gamma_1 + n \in \Delta$ and $a_{02} + \gamma_1 + n \in \Delta$, we eliminate all possible t–powers. We claim that these conditions imply that at least one of each of the following triples of the odd exponents $(a_{0j} + \gamma_1 + n + 2k)_{j=0,1,2}$ lies in Δ — with the exception of the trivial case of $a_{01} = \gamma_1$ and $a_{02} = 2\gamma_1$. If $\gamma_1 + n \in \Delta$ this is obvious, as $\{a_{0j} + \gamma_1 + n + 2k \mid j = 0,1,2\} \equiv \{1,3,5\} \bmod 6$ and $a_{0j} + \gamma_1 + n + 2k > \gamma_1 + n$ and hence $\{a_{0j} + \gamma_1 + n + 2k \mid j = 0,1,2\} \cap (\gamma_1 + n + 6\mathbb{N})$ is nonempty. If $a_{01} + \gamma_1 + n \in \Delta$ and $a_{02} + \gamma_1 + n \in \Delta$, this statement has to be checked case by case. We do this with the help of the following diagrams that indicate which odd terms can be eliminated; the second column stands for t–powers with the exponents $(a_{0j} + \gamma_1 + n)_{j=0,1,2}$, the third for the exponents $(a_{0j} + \gamma_1 + n + 2)_{j=0,1,2}$ and the last one for the exponents $(a_{0j} + \gamma_1 + n + 4)_{j=0,1,2}$. Since $\Delta + 6\mathbb{N} \subset \Delta$, it is enough to consider only the next two odd numbers. For $\gamma_1 = 8$ we get

	$+0$	$+2$	$+4$
$a_{00} = 0$		$\times$	
$a_{01} = 2$	$\times$		
$a_{02} = ?$	$\times$		$\times$

	$+0$	$+2$	$+4$
$a_{00} = 0$			$\times$
$a_{01} = 8$	$\times$	$\times$	
$a_{02} = 4$	$\times$		

	$+0$	$+2$	$+4$
$a_{00} = 0$			
$a_{01} = 8$	$\times$	$\times$	
$a_{02} = 10$	$\times$		$\times$

For $\gamma_1 = 10$ we get

	$+0$	$+2$	$+4$
$a_{00} = 0$			$\times$
$a_{01} = 4$	$\times$		
$a_{02} = ?$	$\times$	$\times$	

	$+0$	$+2$	$+4$
$a_{00} = 0$		$\times$	
$a_{01} = 10$	$\times$	$\times$	$\times$
$a_{02} = 2$	$\times$		

	$+0$	$+2$	$+4$
$a_{00} = 0$			
$a_{01} = 10$	$\times$		$\times$
$a_{02} = 8$	$\times$	$\times$	

	$+0$	$+2$	$+4$
$a_{00} = 0$			
$a_{01} = 10$	$\times$		$\times$
$a_{02} = 14$	$\times$	$\times$	

We are ready to prove that for any R–modules M its 0–normalized associated semi–module Δ is admissible, i.e., it has a nonempty intersection with $\{s, a_{01} + s, a_{02} + s\}$. If it were empty, then by the above discussion we have to apply only operations during the reduction that do not change the sum condition; hence, the requirement that after the normalization process the coefficients $\tilde{c}^1_{s-\gamma_1}, \tilde{c}^2_{s-\gamma_1}, \tilde{c}^3_{s-\gamma_1}$ have to vanish contradicts the fact that their sum is three.

We need to show that in the remaining cases the equations can be solved by expressing some of the λ–variables as polynomials of the other. The coefficients c_k^1, c_k^2, c_k^3 are of the form

$$c_k^j = \tilde{g}_\Delta(a_{0,j-1} + k)\lambda_k^{0,j-1} - \tilde{g}_\Delta(a_{0j} + k)\lambda_k^{0j} + \dots$$

where the lower dots stand for polynomials in the λ with indices less than k. During the reduction process some of the c_1^k, c_2^k, c_3^k are made to vanish, in particular those where the gap function $\tilde{g}_\Delta$ assumes the value zero by the usual arguments. In the end we are left with either zero coefficients or coefficients $\tilde{c}_k^j$ that look like

$$\tilde{c}_k^1 = \lambda_k^{00} - \lambda_k^{01} + \dots \quad \tilde{c}_k^2 = \lambda_k^{01} - \lambda_k^{02} + \dots \quad \tilde{c}_k^3 = \lambda_k^{02} - \lambda_k^{00} + \dots .$$

For fixed k we can obviously solve the equations $\tilde{c}_k^j = 0$ for $\lambda_k^{00}, \lambda_k^{01}, \lambda_k^{02}$ if their sum is zero, which is the case for $k < \min\{n, s - \gamma_1\}$, or if at least one of them is zero, which is always the case for $k \geq n$. Thus it remains to discuss the coefficients with indices in the range $]s - \gamma_1, n[$. This range is nonempty only if Δ contains no odd number less or equal to s and either $a_{01} + s \in \Delta$ or $a_{02} + s \in \Delta$, but not both. Let us start with $a_{01} + s \in \Delta$. Assume that we reduced all T^j for the t–powers with exponents less then $a_{0j} + s$. Because $a_{01} + s \equiv \gamma_1 + s \equiv \gamma_2 + 2\gamma_1 \equiv a_{12} \bmod 6$ we can subtract $\tilde{c}^2_{s-\gamma_1} t^{a_{01}+s-a_{12}} m_{12}$ from the term T^2 to eliminate the $\tilde{c}^2_{s-\gamma_1} t^{a_{01}+s}$ term. As the constant term of the original $c^2_{s-\gamma_1}$ is -1, the constant term of $\tilde{c}^2_{s-\gamma_1}$ is -1, too. Thus we are adding λ_k^{12} to $\tilde{c}^2_{s-\gamma_1+k}$ for all k with $a_{01} + s + k \notin \Delta$. Tracking again the variables $\lambda_k^{0j}, \lambda_k^{12}$ with the greatest index, we find that at this moment in the process we have for the coefficients $\tilde{c}^j_{s-\gamma_1+k}$ with $a_{0j} + s + k \notin \Delta$ — the others are made to vanish later on anyway —

$$\tilde{c}^1_{s-\gamma_1+k} = \lambda^{00}_{s-\gamma_1+k} - \lambda^{01}_{s-\gamma_1+k} + \dots \quad \tilde{c}^2_{s-\gamma_1+k} = \lambda^{01}_{s-\gamma_1+k} - \lambda^{02}_{s-\gamma_1+k} + \lambda_k^{12} + \dots$$
$$\tilde{c}^3_{s-\gamma_1+k} = \lambda^{02}_{s-\gamma_1+k} - \lambda^{00}_{s-\gamma_1+k} + \dots$$

and this will not change later in the process. Now there is no difficulty in solving these equations for $\lambda^{00}_{s-\gamma_1+k}$, $\lambda^{02}_{s-\gamma_1+k}$, and λ_k^{12}.

The case of $a_{02} + s \in \Delta$ is similar, one uses a multiple of m_{10} for the term T^3. In the whole we have shown so far:

The existence of higher order expressions for the terms T^1, T^2, T^3 can be expressed as a polynomial dependence of some of the λ–variables on the other λ–variables.

Now we have to find higher order expressions for the terms derived from the remaining syzygies of the canonical generating set of $\mathbb{C}[\Delta]$. The case where $\max\{a_{ij}\} = a_{0J}$ — see Corollary 4.4 – is nearly trivial. In fact, as there is only one interesting cancellation of initial terms between the m_{00}, m_{01}, m_{02} getting the condition of Lemma 3.1 to hold for it is trivial and the above discussion is not needed here. The three cyclic cancellations of initial terms between the m_{10}, m_{11}, m_{12} derived from the syzygies between $(t^{a_{10}}, t^{a_{11}}, t^{a_{12}}) \in \mathbb{C}[\Delta]$ are

$$\begin{aligned} T^4 &:= \varphi m_{10} - t^{6(\alpha_{11}-\alpha_{10})} m_{11} &&=: \sum_{k=1}^{\infty} c_k^4 t^{a_{10}+\gamma_1+k} \\ T^5 &:= \varphi m_{11} - t^{6(\alpha_{12}-\alpha_{11})} m_{12} &&=: \sum_{k=1}^{\infty} c_k^5 t^{a_{11}+\gamma_1+k} \\ T^6 &:= \varphi m_{12} - t^{6(q+\alpha_{10}-\alpha_{12})} m_{10} &&=: \sum_{k=1}^{\infty} c_k^6 t^{a_{12}+\gamma_1+k}. \end{aligned}$$

These terms are easily expressed as higher order expressions. Namely, the coefficients have again the typical form

$$c_k^4 = \lambda_k^{10} - \lambda_k^{11} + \dots \quad c_k^5 = \lambda_k^{11} - \lambda_k^{12} + \dots \quad c_k^6 = \lambda_k^{12} - \lambda_k^{10} + \dots ;$$

here we suppress the gap function in front of the λ, because the coefficients where it is relevant will be made to vanish later on. We will show that for fixed k at least one of the coefficients vanishes during the reduction process. Let $J \in \{0,1,2\}$ be such that $\min_j\{a_{1j}\} = a_{1J}$. Then we see that $\{a_{1j} + \gamma_1 + k \mid j = 0,1,2\} \cap 6\mathbb{N} \neq \emptyset$ for even k and $\{a_{1j} + \gamma_1 + k \mid j = 0,1,2\} \cap (a_{1J} + 6\mathbb{N}) \neq \emptyset$ for odd k by considering the numbers modulo 6; thus at least one of the t–powers $t^{a_{1j}+\gamma_1+k}$, $j = 0,1,2$, can be eliminated. In the end, at most two of three equations $\tilde{c}_k^4 = \tilde{c}_k^5 = \tilde{c}_k^6 = 0$ are nontrivial and solving them for one or two of the λ_k^{1j} is easy.

We turn to the case of the syzygies of $\mathbb{C}[\Delta]$ described in Corollary 4.4, where $\max\{a_{ij}\} = a_{1J}$ and $K, L \in \{0,1,2\}$ with $K + 1 \equiv L$ and $K + 2 \equiv J$ modulo 3. We have to find higher order expressions for the terms

$$
\begin{aligned}
T^4 &:= \varphi m_{1K} - t^{6*} m_{1L} &&=: \sum_{k=1}^{\infty} c_k^4 t^{a_{1K}+\gamma_1+k} \\
T^5 &:= \psi m_{0K} - t^{6(\alpha_{1K}-\alpha_{0K})} m_{1K} &&=: \sum_{k=1}^{\infty} c_k^5 t^{a_{0K}+\gamma_2+k} \\
T^6 &:= \psi m_{0L} - \varphi t^{6*} m_{1K} &&=: \sum_{k=1}^{\infty} c_k^6 t^{a_{0L}+\gamma_2+k} \\
T^{6\prime} &:= \psi m_{0L} - t^{6(\alpha_{1L}-\alpha_{0L})} m_{1L} &&=: \sum_{k=1}^{\infty} c_k^{6\prime} t^{a_{0L}+\gamma_2+k}
\end{aligned}
$$

where one uses T^6 if $a_{0L} + \gamma_2 \geq a_{1K} + \gamma_1$ and $T^{6\prime}$ otherwise. The coefficients are

$$
\begin{aligned}
c_k^4 &= \lambda_k^{1K} - \lambda_k^{1L} + \dots \qquad & c_k^5 &= \lambda_k^{0K} - \lambda_k^{1K} + \dots \\
c_k^6 &= \lambda_k^{0L} - \lambda_k^{1K} + \dots \qquad & c_k^{6\prime} &= \lambda_k^{0L} - \lambda_k^{1L} + \dots,
\end{aligned}
$$

where we suppressed the gap function again.

Now if Δ contains an odd number $n \leq s + 6$ — for example s itself — then $n + \gamma_1, n + 2\gamma_1 \in \Delta$ and $\{n, n + \gamma_1, n + 2\gamma_1\} \equiv \{1,3,5\}$ mod 6, thus $c(\Delta) \leq n + 2\gamma_1 - 6 + 1 \leq \gamma_2 + 1$. Consequently $c(\Delta) \leq \gamma_2$, because $\gamma_2 \in \Gamma \subset \Delta$. Therefore, the only t–powers in the terms $T^4, T^5, T^6, T^{6\prime}$ whose exponents may be less than $c(\Delta)$ occur in the term T^4. Solving the coefficients of its reduction is trivial, even if we already used up either λ_k^{1K} or λ_k^{1L} before.

Another exceptional case in the treatment of T^1, T^2, T^3 was when there is an even number k with $a_{01} + s - k, a_{02} + s - k \in \Delta$. Choose $I \in \{0,1,2\}$ such that $a_{01} + s - k \equiv a_{1I}$ mod 6, then $a_{1I} \leq a_{01} + s - k \leq \gamma_2 - \gamma_1 - 2$, $a_{1,I+1} \leq \gamma_2 - 2$, $a_{1,I+2} \leq \gamma_2 + \gamma_1 - 2$ and we get $c(\Delta) \leq \gamma_2 + \gamma_1 - 2 - 6 + 1 \leq \gamma_2 + 3$ due to $\gamma_1 \leq 10$. Now the terms T^5, T^6 resp. $T^{6\prime}$ have order greater then $a_{0K} + \gamma_2$ and $a_{0L} + \gamma_2$. One of the a_{0K}, a_{0L} is at least 2, thus the order of the corresponding term is equal to or greater than the conductor $c(\Delta)$ and a higher order expression can be found trivially. Therefore, we need to consider only one of the terms $T^5, T^6/T^{6\prime}$ besides T^4. They contain the so far unused variables $\lambda_k^{1K}, \lambda_k^{1L}$ and finding higher order expressions for them is easy.

The final exceptional case we had during the search for higher order expressions for T^1, T^2, T^3 was when $a_{01} = \gamma_1$ and $a_{02} = 2\gamma_1$. If $\max\{a_{1i}\} \neq a_{12}$, then $a_{0K} + \gamma_2$ or $a_{0L} + \gamma_2$ equals $2\gamma_1 + \gamma_2 < c(\Delta)$ and we argue as before. If $\max\{a_{1i}\} = a_{12}$, then the term T^5 is essentially T^6, because m_{01} is the normalization of φm_{00}. Therefore, we are left again with only two terms, T^4, T^5, to consider, which we can solve easily.

Now we turn to the cases of $a_{01}+s\in\Delta$ or $a_{02}+s\in\Delta$, but $s\notin\Delta$, where we had to make use of the λ–variables in m_{12} resp. m_{10} during the search for the higher order expression for T^1,T^2,T^3. In fact, we found the following

$$\begin{aligned}
\varphi m_{00}-t^{6\alpha_{01}}m_{01} &= \sum_{j=1}^{3} f_{0j}m_{0j}+\sum_{j=1}^{3} f_{1j}m_{1j}\\
\varphi m_{01}-t^{6(\alpha_{02}-\alpha_{01})}m_{02} &= \varepsilon t^{6(\alpha_{12}-\alpha_{01}-q)}m_{12}+\sum_{j=1}^{3} g_{0j}m_{0j}+\sum_{j=1}^{3} g_{1j}m_{1j}\\
\varphi m_{02}-t^{6(q-\alpha_{02})}m_{00} &= \eta t^{6(\alpha_{10}-\alpha_{02})}m_{10}+\sum_{j=1}^{3} h_{0j}m_{0j}+\sum_{j=1}^{3} h_{1j}m_{1j}
\end{aligned}$$

with

$$\begin{array}{lll}
v(f_{0j}m_{0j})>\gamma_1 & v(g_{0j}m_{0j})>a_{01}+\gamma_1 & v(h_{0j}m_{0j})>a_{02}+\gamma_1\\
v(f_{1j}m_{1j})>s & v(g_{1j}m_{1j})>a_{01}+s & v(h_{1j}m_{1j})>a_{02}+s.
\end{array}$$

Here, $\varepsilon=1$ if $a_{01}+s\in\Delta$ and $\eta=1$ if $a_{02}+s\in\Delta$, otherwise they are 0.

Assume now that $a_{01}+s\in\Delta$. As $a_{01}+s\equiv a_{12}\bmod 6$, we find $a_{12}\le a_{01}+s\le\gamma_1+s=\gamma_2-\gamma_1$. Thus we have either $c(\Delta)\le\gamma_2$, which can be treated like above, or $\max\{a_{1j}\}=a_{11}$. In the latter case we have to consider the following syzygies of $\mathbb{C}[\Delta]$

$$\begin{aligned}
a_{12}+\gamma_1 &= a_{10}+6(q+\alpha_{10}-\alpha_{12})\\
a_{02}+\gamma_2 &= a_{12}+6(\alpha_{12}-\alpha_{02})\\
a_{00}+\gamma_2 &= a_{12}+\gamma_1+6(\alpha_{12}-q)
\end{aligned}$$

and the corresponding T^4,T^5,T^6 terms. The T^4 term is the only one involving the variables λ_k^{10} and a higher order expression can be found by an appropriate choice of these. Higher order expressions for T^5, T^6 can be obtained from the equations (+). Namely, multiply the first equation by φ^2, the second by $\varphi t^{6\alpha_{01}}$, the third by $t^{6\alpha_{02}}$ and add them to obtain after moving the left hand side to the right hand side:

$$0=\varphi t^{6(\alpha_{12}-q)}m_{12}+\eta t^{6\alpha_{10}}m_{10}+\sum_{j=1}^{3}u_{0j}m_{0j}+\sum_{j=1}^{3}u_{1j}m_{1j}$$

with $v(u_{1j}m_{1j})>\gamma_2$. As in the case with the Puiseux exponents $(4,2q,s)$ we replace the multiples of T^1,T^2,T^3 by their higher order expressions to achieve that $v(u_{0j}m_{0j})\ge\gamma_2$. In fact, as the first odd number in Γ is γ_2 and $v(m_{01}),v(m_{02})\ge 2$ are even, we find $v(u_{01}m_{01}),v(u_{02}m_{02})>\gamma_2$. Therefore, we got a higher order expression for the cancellation of the initial terms in

$$\varphi t^{6(\alpha_{12}-q)}m_{12}+\eta t^{6\alpha_{10}}m_{10}-(1+\eta)\psi m_{00}.$$

Replacing $\eta t^{6(\alpha_{12}-q)}T^4$ by its higher order expression, which was found earlier, we find the higher order expression for T^6 or a term that can take the place of T^6 in Lemma 3.1.

To obtain the higher order expression for T^5, we multiply the above equations with different elements, namely $\varphi t^{6(q-\alpha_{02})}$, $t^{6(q-\alpha_{02}+\alpha_{01})}$, and φ^2 before adding them and obtain this time

$$0=t^{6(\alpha_{12}-\alpha_{02})}m_{12}+\eta\varphi^2t^{6(\alpha_{10}-\alpha_{02})}m_{10}+\sum_{j=1}^{3}w_{0j}m_{0j}+\sum_{j=1}^{3}w_{1j}m_{1j}$$

with $v(w_{1j}m_{1j}) > a_{02}+\gamma_2$. Again, we use the higher order expressions for T^1, T^2, T^3 to get $v(w_{0j}m_{0j}) \geq a_{02}+\gamma_2$. With further use of these we can achieve that $v(w_{00}m_{00}), v(w_{01}m_{01}) > a_{02}+\gamma_2$, thus the only terms of the least order $a_{02}+\gamma_2$ are the first two terms and $w_{02}m_{02}$. Now, if $\eta = 0$ then we may view the above equation as a higher order expression for T^5. If $\eta = 1$ then the order of T^5 is greater than $a_{10}+2\gamma_1+6(\alpha_{10}-\alpha_{02}) > a_{10}+\gamma_1 \geq a_{11} \geq c(\Delta)$ and a higher order expression is obtained trivially.

The remaining regular case is when $a_{02}+s \in \Delta$. As $a_{02}+s \equiv a_{10} \equiv \gamma_2 \bmod 6$ this is only a weak restriction on a_{10}. Let us assume that $s, a_{01}+s \notin \Delta$, otherwise we are in one of the above cases. In addition, we assume $c(\Delta) > \gamma_2$, i.e., $c(\Delta) \geq \gamma_2+2$, because otherwise the same arguments as in the special cases apply. We claim that $\max\{a_{ij}\} = a_{12}$. If we had $\max\{a_{ij}\} = a_{11}$, then $\gamma_2 + 2 \leq c(\Delta) = a_{11} - 5 = \gamma_2 + \gamma_1 - 6\alpha_{11} - 5$; hence $\alpha_{11} = 0$ and $a_{11} = \gamma_2 + \gamma_1$. This implies $a_{01} = \gamma_1$ and $a_{10} = \gamma_2$ and from $a_{02} + s \geq a_{10}$ we get $a_{02} = 2\gamma_1$, but this was a special case discussed above.

Because of $\max\{a_{ij}\} = a_{12}$ the syzygies of $\mathbb{C}[\Delta]$ of degree below $c(\Delta)$ are generated by:

$$\begin{aligned}
&a_{10}+\gamma_1 = a_{11}+6(\alpha_{11}-\alpha_{10}) && \\
&a_{00}+\gamma_2 = a_{10}+6\alpha_{10} && \\
&a_{01}+\gamma_2 = a_{10}+\gamma_1+6(\alpha_{10}-\alpha_{01}) && \text{if} \quad a_{01}+\gamma_2 \geq a_{10}+\gamma_1 \\
&a_{01}+\gamma_2 = a_{11}+6(\alpha_{11}-\alpha_{01}) && \text{else.}
\end{aligned}$$

A higher order expression for the term T^5 corresponding to the second syzygy can be derived from (+) (with $\varepsilon = 0$ and $\eta = 1$) by multiplying the three equations with φ^2, $\varphi t^{6\alpha_{01}}$, $t^{6\alpha_{02}}$ respectively and adding them to obtain

$$0 = t^{6\alpha_{10}}m_{10} + \sum_{j=1}^{3} u_{0j}m_{0j} + \sum_{j=1}^{3} u_{1j}m_{1j}$$

with $v(u_{1j}m_{1j}) > \gamma_2$. The usual argument leads to a higher order expression for T^5.

If $a_{01}+\gamma_2 \geq a_{10}+\gamma_1$, then we can also derive a higher order expression for T^6 from (+) by multiplying the equations by $t^{6(q-\alpha_{01})}$, φ^2, $\varphi t^{6(\alpha_{02}-\alpha_{01})}$ adding them and proceeding as before. A higher order expression for the term T^4 can now be found by reducing it and solving the remaining coefficients for the variables λ_k^{11}, which occur only in T^4.

At last when $a_{01}+\gamma_2 < a_{10}+\gamma_1$, we use the variables λ_k^{11} to get a higher order expression for $T^{6\prime}$. We claim that a higher order expression for T^4 can be found trivially because its order is greater than $c(\Delta)$. From $a_{01}+\gamma_2 < a_{10}+\gamma_1$ we conclude $a_{01}+\gamma_2 \leq a_{10}+\gamma_1-6$ and $a_{12} \leq a_{01}+\gamma_2+\gamma_1 \leq a_{10}+2\gamma_1-6$, thus we have $c(\Delta) \leq a_{10}+2\gamma_1-2\cdot 6+1 < a_{10}+\gamma_1$.

5 Betti numbers

For any plane singularity X the Jacobi factor J_X is δ_X–dimensional. More precisely, the subset $\mathrm{Pic}^0(X)$ of free modules of J_X is biregular to $\mathbb{C}^{\delta_X}$ and J_X is its closure. Rego proved that the number of components of $J_x \setminus \mathrm{Pic}^0(X)$ equals the multiplicity of the singularity X minus one [R]. Such results and more follow from purely combinatorial reasoning for singularities which possess an affine cell decomposition. We start our discussion with some notations:

Definition 5.1 *For the semi–group $\Gamma = \langle p, q\rangle \subset \mathbb{N}$, $\gcd(p,q) = 1$, we denote the 0–normalized semi–modules by $\mathrm{Mod}(\Gamma)$. The dimension of a Γ–semi–module Δ with p–basis $(a_0 = 0, a_1, \ldots, a_{p-1})$ is defined as*

$$\dim\Delta := \sum_{j=0}^{p-1} \left(g_\Delta(a_j) - g_\Delta(a_j + q)\right) = \sum_{j=0}^{p-1} \#\left([a_j, a_j + q[\setminus\Delta\right).$$

Analogously, for the semi–group $\Gamma = \langle 4, 2q, 2q+s\rangle \subset \mathbb{N}$, $\gcd(2, qs) = 1$, we denote the admissible *0–normalized semi–modules by $\mathrm{Mod}(\Gamma)$. The dimension of an* admissible *Γ–semi–module Δ with 2×2–basis $(a_{00} = 0, a_{01}; a_{10}, a_{11})$ is defined as*

$$\dim\Delta := \sum_{i,j=0}^{1} g_\Delta(a_{ij}) - g_\Delta(\gamma_1) - g_\Delta(a_{01} + n),$$

where $n := \min\left(\{s\} \cup (\Delta \cap [\gamma_1, \infty[\, \cap (1 + 2\mathbb{N}))\right)$.

The codimension of Δ is $\operatorname{codim}\Delta := \delta_\Gamma - \dim\Delta$, where $\delta_\Gamma := \dim\Gamma$. Thereby, the semi–modules are splitted into the disjoint subsets

$$\mathrm{Mod}_d(\Gamma) := \{\Delta \in \mathrm{Mod}(\Gamma) \mid \dim\Delta = d\}$$

or dually

$$\mathrm{Mod}^d(\Gamma) := \{\Delta \in \mathrm{Mod}(\Gamma) \mid \operatorname{codim}\Delta = d\}.$$

Either geometrically from the next theorem or combinatorically from the proofs of the following Theorems, we will see that the values of the functions dim and codim lie in the range $[0, \delta_\Gamma]$ and $\mathrm{Mod}^0(\Gamma) = \{\Gamma\}$ as well as $\mathrm{Mod}_0(\Gamma) = \{\mathbb{N}\}$.

As an immediate consequence of the affine cell decomposition of the Jacobi factors and the remarks in Section 1 we have

Theorem 5.2 *Let X be a unibranched plane singularity with characteristic Puiseux exponents (p,q) or $(4, 2q, s)$. Let Γ be its associated semi–group and and J_X its Jacobi factor. Then the odd (co–)homology groups of J_X are zero, and the even (co–)homology group are free abelian groups with Betti numbers*

$$h_{2d}(J_X) = \#Mod_d(\Gamma) \qquad \text{and} \qquad h^{2d}(J_X) = \#Mod^d(\Gamma).$$

It is easy to write a computer program that computes all Γ–semi–modules together with their dimension. We discuss the results for the singularities with Puiseux exponents (p,q) first. For the singularities with Puiseux exponents $(2,q)$ and $(3,q)$ one obtains the following list, which has an obvious construction rule.

(p,q)	δ_X	$e(X)$	h^0	h^2	h^4	h^6	h^8									
(2,3)	1	2	1	1												
(2,5)	2	3	1	1	1											
(2,7)	3	4	1	1	1	1										
(2,9)	4	5	1	1	1	1	1									
(2,11)	5	6	1	1	1	1	1	1								
(3,2)	1	2	1	1												
(3,4)	3	5	1	2	1	1										
(3,5)	4	7	1	2	2	1	1									
(3,7)	6	12	1	2	3	2	2	1	1							
(3,8)	7	15	1	2	3	3	2	2	1	1						
(3,10)	9	22	1	2	3	4	3	3	2	2	1	1				
(3,11)	10	26	1	2	3	4	4	3	3	2	2	1	1			
(3,13)	12	35	1	2	3	4	5	4	4	3	3	2	2	1	1	
(3,14)	13	40	1	2	3	4	5	5	4	4	3	3	2	2	1	1

The Betti numbers for the singularities A_{2k}, E_6, E_8, i.e., for the singularities with the characteristic Puiseux exponents $(2,q)$, $(3,4)$, and $(3,5)$, have been computed by Cook [C] and Warmt [W1]. For $p \geq 4$ an explicit formula for the Betti numbers seems difficult to find. A long list of examples is included so that the reader may try himself.

(p,q)	δ_X	$e(X)$	$h^0\, h^2\, h^4\, h^6\, h^8\, \ldots$
(4,3)	3	5	1 2 1 1
(4,5)	6	14	1 3 3 3 2 1 1
(4,7)	9	30	1 3 5 5 5 4 3 2 1 1
(4,9)	12	55	1 3 6 7 8 7 7 5 4 3 2 1 1
(4,11)	15	91	1 3 6 9 10 11 10 10 8 7 5 4 3 2 1 1
(4,13)	18	140	1 3 6 10 12 14 14 14 13 12 10 8 7 5 4 3 2 1 1
(4,15)	21	204	1 3 6 10 14 16 18 18 18 17 16 14 12 10 8 7 5 4 3 2 1 1
(4,17)	24	285	1 3 6 10 15 18 21 22 23 22 22 20 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,19)	27	385	1 3 6 10 15 20 23 26 27 28 27 27 25 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,21)	30	506	1 3 6 10 15 21 25 29 31 33 33 33 32 31 29 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,23)	33	650	1 3 6 10 15 21 27 31 35 37 39 39 39 38 37 35 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,25)	36	819	1 3 6 10 15 21 28 33 38 41 44 45 46 45 45 43 42 39 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,27)	39	1015	1 3 6 10 15 21 28 35 40 45 48 51 52 53 52 52 50 49 46 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,29)	42	1240	1 3 6 10 15 21 28 36 42 48 52 56 58 60 60 60 59 58 56 54 51 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,31)	45	1496	1 3 6 10 15 21 28 36 44 50 56 60 64 66 68 68 68 67 66 64 62 59 56 52 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(5,2)	2	3	1 1 1
(5,3)	4	7	1 2 2 1 1
(5,4)	6	14	1 3 3 3 2 1 1
(5,6)	10	42	1 4 6 7 7 5 5 3 2 1 1
(5,7)	12	66	1 4 7 9 10 9 8 6 5 3 2 1 1
(5,8)	14	99	1 4 8 11 13 13 12 10 9 6 5 3 2 1 1
(5,9)	16	143	1 4 9 13 16 17 17 15 13 11 9 6 5 3 2 1 1
(5,11)	20	273	1 4 10 16 22 25 28 27 26 23 21 17 15 11 9 6 5 3 2 1 1
(5,12)	22	364	1 4 10 17 24 29 33 34 34 31 29 25 22 18 15 11 9 6 5 3 2 1 1
(5,13)	24	476	1 4 10 18 26 33 38 41 42 40 38 34 31 26 23 18 15 11 9 6 5 3 2 1 1
(5,14)	26	612	1 4 10 19 28 37 43 48 50 50 48 44 41 36 32 27 23 18 15 11 9 6 5 3 2 1 1
(5,16)	30	969	1 4 10 20 31 43 53 61 67 70 71 68 66 60 56 50 45 38 34 27 23 18 15 11 9 6 5 3 2 1 1
(5,17)	32	1197	1 4 10 20 32 45 57 67 75 80 83 82 80 75 71 64 59 52 46 39 34 27 23 18 15 11 9 6 5 3 2 1 1
(5,18)	34	1463	1 4 10 20 33 47 61 73 83 90 95 96 95 91 87 80 75 67 61 53 47 39 34 27 23 18 15 11 9 6 5 3 2 1 1
(5,19)	36	1771	1 4 10 20 34 49 65 79 91 100 107 110 111 108 104 98 92 84 78 69 62 54 47 39 34 27 23 18 15 11 9 6 5 3 2 1 1
(5,21)	40	2530	1 4 10 20 35 52 71 89 106 119 131 138 144 144 143 138 133 124 118 108 100 90 82 71 64 54 47 39 34 27 23 18 15 11 9 6 5 3 2 1 1
(5,22)	42	2990	1 4 10 20 35 53 73 93 112 128 142 152 160 163 164 160 156 148 141 131 123 112 103 92 83 72 64 54 47 39 34 27 23 18 15 11 9 6 5 3 2 1 1
(5,23)	44	3510	1 4 10 20 35 54 75 97 118 137 153 166 176 182 185 183 180 173 166 156 148 136 127 115 105 93 84 72 64 54 47 39 34 27 23 18 15 11 9 6 5 3 2 1 1
(5,24)	46	4095	1 4 10 20 35 55 77 101 124 146 164 180 192 201 206 207 205 199 193 183 174 163 153 140 130 117 106 94 84 72 64 54 47 39 34 27 23 18 15 11 9 6 5 3 2 1 1

(p,q)	δ_X	$e(X)$	$h^0 h^2 h^4 h^6 h^8 \ldots$
(6,5)	10	42	1 4 6 7 7 5 5 3 2 1 1
(6,7)	15	132	1 5 10 14 17 16 16 14 11 9 7 5 3 2 1 1
(6,11)	25	728	1 5 14 26 39 50 59 63 64 62 59 53 47 41 34 28 23 18 13 10 7 5 3 2 1 1
(6,13)	30	1428	1 5 15 30 49 67 85 97 106 108 109 104 99 90 82 71 63 53 45 36 30 23 18 13 10 7 5 3 2 1 1
(6,17)	40	4389	1 5 15 34 61 95 131 167 200 227 248 259 266 264 260 249 237 220 204 184 167 147 131 113 98 82 70 57 47 37 30 23 18 13 10 7 5 3 2 1 1
(6,19)	45	7084	1 5 15 35 65 105 151 199 247 290 328 355 376 385 389 385 376 361 344 322 299 275 250 226 202 179 157 137 117 100 84 70 57 47 37 30 23 18 13 10 7 5 3 2 1 1
(7,2)	3	4	1 1 1 1
(7,3)	6	12	1 2 3 2 2 1 1
(7,4)	9	30	1 3 5 5 5 4 3 2 1 1
(7,5)	12	66	1 4 7 9 10 9 8 6 5 3 2 1 1
(7,6)	15	132	1 5 10 14 17 16 16 14 11 9 7 5 3 2 1 1
(7,8)	21	429	1 6 15 25 35 40 43 44 40 37 32 28 22 18 13 11 7 5 3 2 1 1
(7,9)	24	715	1 6 16 29 43 54 62 66 66 63 58 51 45 37 31 24 19 14 11 7 5 3 2 1 1
(7,10)	27	1144	1 6 17 33 52 70 84 93 97 97 92 86 77 69 58 50 40 33 25 20 14 11 7 5 3 2 1 1
(7,11)	30	1768	1 6 18 37 61 86 108 124 135 139 138 132 124 112 101 88 76 63 53 42 34 26 20 14 11 7 5 3 2 1 1
(7,12)	33	2652	1 6 19 41 70 102 133 159 178 190 194 193 184 174 159 145 127 112 95 81 66 55 43 35 26 20 14 11 7 5 3 2 1 1
(7,13)	36	3876	1 6 20 45 80 120 162 199 229 249 262 265 261 251 237 218 200 179 159 137 119 100 84 68 56 44 35 26 20 14 11 7 5 3 2 1 1
(7,15)	42	7752	1 6 21 50 95 151 216 280 341 389 429 453 468 468 462 443 423 394 367 332 302 268 238 205 179 151 129 106 88 70 58 44 35 26 20 14 11 7 5 3 2 1 1
(8,3)	7	15	1 2 3 3 2 2 1 1
(8,5)	14	99	1 4 8 11 13 13 12 10 9 6 5 3 2 1 1
(8,7)	21	429	1 6 15 25 35 40 43 44 40 37 32 28 22 18 13 11 7 5 3 2 1 1
(8,9)	28	1430	1 7 21 41 65 86 102 115 118 118 113 106 96 85 73 63 53 42 34 26 20 15 11 7 5 3 2 1 1
(8,11)	35	3978	1 7 23 51 90 135 180 220 251 272 282 282 275 262 244 223 200 177 154 132 111 92 75 61 47 37 28 21 15 11 7 5 3 2 1 1
(8,13)	42	9690	1 7 25 61 117 190 273 357 435 501 551 584 600 600 588 566 535 498 458 415 372 329 288 249 213 179 150 123 100 80 64 49 38 28 21 15 11 7 5 3 2 1 1
(9,2)	4	5	1 1 1 1 1
(9,4)	12	55	1 3 6 7 8 7 7 5 4 3 2 1 1
(9,5)	16	143	1 4 9 13 16 17 17 15 13 11 9 6 5 3 2 1 1
(9,7)	24	715	1 6 16 29 43 54 62 66 66 63 58 51 45 37 31 24 19 14 11 7 5 3 2 1 1
(9,8)	28	1430	1 7 21 41 65 86 102 115 118 118 113 106 96 85 73 63 53 42 34 26 20 15 11 7 5 3 2 1 1
(9,10)	36	4862	1 8 28 63 112 167 219 268 303 326 338 338 331 314 293 268 245 215 190 162 139 116 97 77 63 48 38 28 22 15 11 7 5 3 2 1 1
(9,11)	40	8398	1 8 29 69 129 203 282 360 428 482 520 541 547 538 519 489 456 416 376 334 295 254 219 184 155 127 104 82 66 50 39 29 22 15 11 7 5 3 2 1 1

At least, we are able to describe their asymptotic behavior for $q \to \infty$. The following two theorems determine the first $\lfloor \frac{q}{p} \rfloor + 1$ and the last $q - \lceil \frac{q}{p} \rceil + 1$ of the $\delta_X + 1 = (p-1)(q-1)/2 + 1$ Betti numbers. In particular, all the Betti numbers for the singularities with characteristic Puiseux exponents $(2, q)$ or $(3, q)$ are described.

Theorem 5.3 *Let X be a unibranched plane singularity with Puiseux exponents (p, q) and J_X its Jacobi factor. Then the even Betti numbers $h^0(J_X), h^2(J_X), \ldots, h^{2\lfloor \frac{q}{p} \rfloor}(J_X)$ of the cohomology of J_X are the same as the first $\lfloor \frac{q}{p} \rfloor + 1$ coefficients of the power series*

$$P := \frac{1}{(1-t)^{p-1}}.$$

Proof. P is the Poincare series of the polynomial ring in the $p-1$ variables $t_1, \ldots, t_{p-1}$. Let Mon^d be the set of monomials of degree d in this ring. For $\Gamma = \langle p, q \rangle$ and $d \leq \lfloor \frac{q}{p} \rfloor$ we define the map

$$\begin{array}{rcl} \Phi_d : \mathrm{Mon}^d & \longrightarrow & \mathrm{Mod}^d(\Gamma) \\ \prod_{j=1}^{p-1} t_j^{r_j} & \longmapsto & \left\langle a_j := jq - \left(\sum_{i=1}^{j} r_i\right) p \,\middle|\, j = 0 \ldots p-1 \right\rangle. \end{array}$$

The theorem is proved when we have shown that Φ_d is well–defined and bijective. Note that (a_j) is a p–basis, thus the map is injective. To see that the map is well–defined, we need to show that a semi–module Δ with a p–basis like above has really codimension d. From $\sum_{i=1}^{p-1} r_i \leq \lfloor \frac{q}{p} \rfloor$ we see that $0 = a_0 < a_1 < a_2 < \ldots < a_{p-1}$ and $a_j + q < a_{j+2}$; hence, defining for any interval $I \subseteq \mathbb{N}$

$$S^j(I) := \{n \in I \mid n = iq \bmod p \text{ for some } i \in \{0, 1, \ldots, j\}\}$$

we find for $\Delta = \bigcup (a_j + p\mathbb{N})$

$$[a_j, a_j + q[\, \cap \Delta = S^j([a_j, a_j + q[) \cup \{a_j + q - kp \mid 0 < k \leq r_{j+1}\}$$

where the union is disjoint. We compare Δ with the semi–module Γ, which has the p–basis (jq). Here we have

$$[jq, jq + q[\, \cap \Gamma = S^j([qj, qj + q[).$$

Because $jq \equiv a_j \bmod p$, we obviously have $\#S^j([a_j, a_j + q[) = \#S^j([qj, qj + q[)$. Therefore,

$$\#\left([a_j, a_j + q[\, \cap \Delta\right) = \#\left([jq, jq + q[\, \cap \Gamma\right) + r_{j+1},$$

and the dimension formula implies that $\operatorname{codim} \Delta = \sum_{j=1}^{p-1} r_j = d$.

It remains to prove that the maps Φ_d are surjective, i.e., we need to show that the dimension of any semi–module not in the image of any $\Phi_0, \ldots, \Phi_{\lfloor \frac{q}{p} \rfloor}$ has codimension greater than $\lfloor \frac{q}{p} \rfloor$. Let Δ be any 0–normalized semi–module with p–basis $(a_j = jq - \alpha_j p)$. Set $r_0 := 0$ and $r_j := \alpha_j - \alpha_{j-1} \geq 0$ for $0 < j \leq p-1$. The semi–module Δ lies in the image of Φ_d for some $d \leq \lfloor \frac{q}{p} \rfloor$ iff $d = \sum_{j=1}^{p-1} r_j$. Therefore, for the surjectivity of the Φ_d, $d \leq \lfloor \frac{q}{p} \rfloor$, it is enough to show that

$$\dim \Delta \geq \min \left\{ \sum_{j=1}^{p-1} r_j, \left\lceil \frac{q}{p} \right\rceil \right\};$$

however, we will show the stronger statement

$$\dim\Delta \geq \sum_{j=1}^{p-1} \min\left\{r_j, \left\lceil \frac{q}{p} \right\rceil\right\}.$$

We prove this by successively reducing the vector $r = (r_j)$ to zero, where the statement is trivial. Let k be the least integer with $r_k \neq 0$. We define the semi–module Δ' to be the one that corresponds to the vector $r' = (0, \ldots, 0, r_k - 1, r_{k+1}, \ldots, r_{p-1})$, i.e., Δ' has the p–basis (a'_j) with $a'_j = a_j$ for $j < k$ and $a'_j = a_j + p$ for $j \geq k$. Our estimate is proven when we have shown that $\dim\Delta' \geq \dim\Delta$ with strict inequality when $r_k \leq \lceil \frac{q}{p} \rceil$. Set $I_j := [a_j, a_j + q[$ and $I'_j := [a'_j, a'_j + q[$. Then $I_j = I'_j = [jq, jq + q[$ for $j < k$ and their disjoint union is $[0, kq[$. Because $\Delta' \subset \Delta$, we have $\#([0, kq[\setminus\Delta') \geq \#([0, kq[\setminus\Delta)$ as a first indication of $\dim\Delta' \geq \dim\Delta$.

For $j \geq k$ we have $I'_j = p + I_j$, and there is the natural injective map

$$\Psi_j : I_j \cap \Delta \longrightarrow I'_j \cap \Delta', \quad n \longmapsto n + p.$$

Because $\Delta' \setminus (p + \Delta) = \{0, q, 2q, \ldots, (k-1)q\}$ and $\#I'_j = q$, we have that either Ψ_j is bijective or $(I'_j \cap \Delta') \setminus \operatorname{Im}\Psi_j = \{lq\}$ for some $l < k$. In the later case we get $a'_j < lq$, in particular $a'_j \in [0, kq[$ and $a_j = a'_j - p \in [0, kq[$ as well. It follows that $a_j \in [0, kq[\cap(\Delta \setminus \Delta')$. Summarizing we have shown that either

$$\begin{aligned} &\#(I'_j \cap \Delta') = \#(I_j \cap \Delta) && \text{or} \\ &\#(I'_j \cap \Delta') = \#(I_j \cap \Delta) + 1 && \text{and} \quad a_j \in [0, kq[\cap(\Delta \setminus \Delta'). \end{aligned}$$

Since

$$\begin{aligned} \dim\Delta\ &= \#([0, kq[\setminus\Delta) + \sum_{j=k}^{p-1} \#(I_j \setminus \Delta) \\ \dim\Delta' &= \#([0, kq[\setminus\Delta') + \sum_{j=k}^{p-1} \#(I'_j \setminus \Delta'), \end{aligned}$$

we conclude that $\dim\Delta' \geq \dim\Delta$.

Now assume that $r_k \leq \lceil \frac{q}{p} \rceil$. Then $a'_k = kq - (r_k - 1)p > (k-1)q$ and the interval I'_k cannot contain any of the $0, q, \ldots, (k-1)q$, thus $\#(I'_k \cap \Delta') = \#(I_k \cap \Delta)$. Since we have $a_k = kq - r_k p \in [0, kq[\cap(\Delta \setminus \Delta')$ as well, it follows that $\dim\Delta' > \dim\Delta$. $\square$

Theorem 5.4 *Let X be a unibranched plane singularity with Puiseux exponents (p, q) and J_X its Jacobi factor. Set $n := q - \lceil \frac{q}{p} \rceil$. Then the even Betti numbers $h_0(J_X), h_2(J_X), \ldots, h_{2n}(J_X)$ of the homology of J_X are the same as the first $n+1$ coefficients of the power series*

$$P := \frac{1}{\prod_{i=1}^{p-1}(1 - t^i)}.$$

Proof. P is the Poincare series of the weighted polynomial ring in the $p-1$ variables $t_1, \ldots, t_{p-1}$ where the weighted degree of t_i is i. Let Mon be the set of all monomials and Mon_d the monomials of weighted degree d in this ring. The strategy of this proof is to define an obviously surjective map from Mon into the set of $\langle p\rangle$–semi–modules, $\mathrm{Mod}(\langle p\rangle)$,

$$\Psi : \mathrm{Mon} \longrightarrow \mathrm{Mod}(\langle p\rangle),$$

and then show that it induces a bijection between Mon_d and $\mathrm{Mod}_d(\Gamma)$ for $d \le n$.

For a $\langle p \rangle$–semi–module Δ we have also a notion of a p–basis. It is the unique set $\{b_0 = 0, b_1, \ldots, b_{p-1}\}$ such that $\Delta = \bigcup_{j=0}^{p-1}(b_j + p\mathbb{N})$. Whenever possible we will assume that the b_j are ordered by $0 = b_0 < b_1 < \ldots < b_{p-1}$. Now the map Ψ is defined in the following way: Let $m = \prod_{j=1}^{p-1} t_j^{r_j}$ be a monomial of weighted degree $d = \sum_{j=1}^{p-1} r_j j$. Then $\Psi(m)$ is the unique $\langle p \rangle$–semi–module Δ which possesses an ordered p–basis $\{b_j\}$ with $\#([b_{j-1}, b_j] \setminus \Delta) = r_j$ for $j = 1 \ldots r-1$, i.e., there are r_j gaps in Δ between the basis elements b_{j-1} and b_j. A p–basis for such a Δ can be constructed inductively: Having found $b_0 = 0, b_1, \ldots, b_{j-1}$ let b_j be the position of the (r_j+1)–th gap in $\bigcup_{i=0}^{j-1}(a_i + p\mathbb{N})$ after b_{j-1}. Obviously, Ψ is bijective. The following table illustrates this map for $p = 3$. The module Δ is represented as a sequence of members of Δ, "$\bullet$", and gaps of Δ, "$\circ$"; the elements of the 3–basis are underlined.

wdeg	Mon	Δ									
0	1	$\underline{\bullet}$	$\underline{\bullet}$	$\underline{\bullet}$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
1	t_1	$\underline{\bullet}$	$\circ$	$\underline{\bullet}$	$\bullet$	$\underline{\bullet}$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
2	t_1^2	$\underline{\bullet}$	$\circ$	$\circ$	$\bullet$	$\underline{\bullet}$	$\underline{\bullet}$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
	t_2	$\underline{\bullet}$	$\underline{\bullet}$	$\circ$	$\bullet$	$\bullet$	$\underline{\bullet}$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
3	t_1^3	$\underline{\bullet}$	$\circ$	$\circ$	$\bullet$	$\circ$	$\underline{\bullet}$	$\bullet$	$\underline{\bullet}$	$\bullet$	$\bullet$
	$t_1 t_2$	$\underline{\bullet}$	$\circ$	$\underline{\bullet}$	$\bullet$	$\circ$	$\bullet$	$\bullet$	$\underline{\bullet}$	$\bullet$	$\bullet$
4	t_1^4	$\underline{\bullet}$	$\circ$	$\circ$	$\bullet$	$\circ$	$\circ$	$\bullet$	$\underline{\bullet}$	$\underline{\bullet}$	$\bullet$
	$t_1^2 t_2$	$\underline{\bullet}$	$\circ$	$\circ$	$\bullet$	$\underline{\bullet}$	$\circ$	$\bullet$	$\bullet$	$\underline{\bullet}$	$\bullet$
	t_2^2	$\underline{\bullet}$	$\underline{\bullet}$	$\circ$	$\bullet$	$\bullet$	$\circ$	$\bullet$	$\bullet$	$\underline{\bullet}$	$\bullet$

Several arguments of this proof are based on a comparison of an arbitrary $\langle p \rangle$–semi–module Δ with the $\langle p \rangle$–semi–modules $\Delta_r := \Psi(t_1^r)$. Note that

$$\Delta_r = \left\{0, p, 2p, \ldots, \left\lfloor \frac{r}{p-1} \right\rfloor p, r + \left\lfloor \frac{r}{p-1} \right\rfloor + 1, r + \left\lfloor \frac{r}{p-1} \right\rfloor + 2, \ldots \right\},$$

and the conductor of Δ_r is $r + \lceil \frac{r}{p-1} \rceil$. The most important case is the one for $r = n$. Here one finds $\Delta_n = p\mathbb{N} \cup (q + \mathbb{N})$ and the conductor is q or $q-1$. The essential comparison property of the Δ_r is

($\dagger$) *Let m be a monomial of weighted degree d and $\Delta = \Psi(m)$ be the corresponding $\langle p \rangle$–semi–module, then $c(\Delta) \le c(\Delta_d)$.*

We prove the claim ($\dagger$) by induction. Assuming it holds for $\Delta = \Psi(t_1^{r_1} \cdots t_\varrho^{r_\varrho})$ we will show that for $\Delta' = \Psi(t_1^{r_1} \cdots t_\varrho^{r_\varrho + 1})$ we have $c(\Delta') \le c(\Delta_{d+\varrho})$ as well. (Some or all of the r_i may be zero.) First we consider the modules Δ_d and $\Delta_{d+\varrho}$. Let lp be the smallest p–multiple with $lp > c(\Delta_d) =: c$. Then we have the following partition of Δ_d:

$$\begin{array}{ll}
\Delta_d = \Delta_{d+\varrho} \cup \{c+1, c+2, \ldots, c+\varrho\} & \text{if } c \in p\mathbb{N} \subset \Delta_d \\
\Delta_d = \Delta_{d+\varrho} \cup \{c, c+1, \ldots, lp-1\} \cup \{lp+1, lp+2, \ldots, c+\varrho\} & \text{if } c+\varrho \ge lp \\
\Delta_d = \Delta_{d+\varrho} \cup \{c, c+1, \ldots, c+\varrho-1\} & \text{else.}
\end{array}$$

Therefore, we have

$$c(\Delta_{d+\varrho}) = \begin{cases} c(\Delta_d) + \varrho & \text{for } [c(\Delta_d), c(\Delta_d) + \varrho] \cap p\mathbb{N} = \emptyset \\ c(\Delta_d) + \varrho + 1 & \text{else.} \end{cases}$$

The claim (†) is proved when we have shown that

$$c(\Delta') \leq \begin{cases} c(\Delta)+\varrho & \text{for } [c(\Delta), c(\Delta)+\varrho] \cap p\mathbb{N} = \emptyset \\ c(\Delta)+\varrho+1 & \text{else,} \end{cases}$$

because on the one hand if $c(\Delta) < c(\Delta_d)$ then $c(\Delta') \leq c(\Delta)+\varrho \leq c(\Delta_{d+\varrho})$ and on the other hand if $c(\Delta) = c(\Delta_d)$ then $c(\Delta') \leq c(\Delta_{d+\varrho})$ is obvious from the above.

Let $\{b_j\}$ be an ordered p–basis of Δ. We know for the ordered p–basis of Δ' that $b'_j = b_j$ for $j < \varrho$ and $b'_j > b_j$ for $j \geq \varrho$. By the definition of Δ', Δ and Δ' differ only by one element, an additional gap in Δ between the $(\varrho-1)$–th and ϱ–th element of the p–basis $\{b'_j\}$ of Δ' — the last gap in Δ' at all. By the definition of b_ϱ this must be b_ϱ, i.e., $\Delta = \Delta' \cup \{b_\varrho\}$. In particular, $c(\Delta') = b_\varrho + 1$. To get an estimate for $c(\Delta)$ from below, consider Δ in the interval between $b_\varrho - \varrho - 1$ and b_ϱ

$$[b_\varrho - \varrho - 1, b_\varrho[\setminus\Delta = [b_\varrho - \varrho - 1, b_\varrho[\setminus \bigcup_{j=0}^{p-1}(b_j + p\mathbb{N}) = [b_\varrho - \varrho - 1, b_\varrho[\setminus \bigcup_{j=0}^{\varrho-1}(b_j + p\mathbb{N}).$$

Because the interval $[b_\varrho - \varrho - 1, b_\varrho[$ are $\varrho + 1$ consecutive numbers, the above set is nonempty, thus there is a gap in Δ greater or equal to $b_\varrho - \varrho - 1$. Hence, $c(\Delta) \geq b_\varrho - \varrho$, and $c(\Delta') \leq c(\Delta) + \varrho + 1$. If this inequality is not strict, we find

$$[c(\Delta), c(\Delta)+\varrho[= [b_\varrho - \varrho, b_\varrho[\subset \bigcup_{j=0}^{\varrho-1}(b_j + p\mathbb{N}).$$

This can only happen if $[c(\Delta), c(\Delta)+\varrho[\cap (b_j + p\mathbb{N}) \neq \emptyset$ for all $j = 0, \ldots, \varrho - 1$; in particular, with $j = 0$ we find the claimed estimate $c(\Delta') \leq c(\Delta) + \varrho$ for $[c(\Delta), c(\Delta)+\varrho[\cap p\mathbb{N} = \emptyset$, and the statement (†) is proved.

The statement (†) has two immediate consequences. If $d \leq n$ then $c(\Delta) \leq c(\Delta_d) \leq c(\Delta_n) \leq q$ and thus the $\langle p\rangle$–semi–module is trivially a Γ–semi–module as well. Further, the dimension of any Γ–semi–module Δ with $c(\Delta) \leq q$ is

$$\dim \Delta = \sum_{j=0}^{p-1} g_\Delta(b_j) = \sum_{j=0}^{p-1}\sum_{i=j+1}^{p-1} r_i = \sum_{j=1}^{p-1} r_j j.$$

Hence, if m is the unique monomial with $\Psi(m) = \Delta$ then $\dim \Delta = \operatorname{wdeg} m$. This shows that the image of Mon_d under Ψ lies in $\mathrm{Mod}_d(\Gamma)$. Therefore, we obtain injective maps

$$\Psi_d : \mathrm{Mon}_d \longrightarrow \mathrm{Mod}_d(\Gamma).$$

The proof of the Theorem is finished when we have shown that they are surjective as well.

For the surjectivity of Ψ_d with $d \leq n$, we must show that for any $\Delta \in \mathrm{Mod}_d(\Gamma)$ the unique monomial m with $\Psi(m) = \Delta$ has weighted degree d. By the above argument this is clear if $c(\Delta) \leq q$. Thus to prove the surjectivity of the $\Psi_0, \ldots, \Psi_n$, it is enough to show that for any $\Delta \in \mathrm{Mod}(\Gamma)$ with $\dim \Delta \leq n$ we have $c(\Delta) \leq q$. We will prove that $c(\Delta) > q$ implies $\dim \Delta > n$ by an inductive process like above. We close the last gap in the semi–module Δ to obtain the semi–module Δ', thereby reducing the conductor. We will show that $\dim \Delta' \leq \dim \Delta$ and $\dim \Delta > n$ if $c(\Delta') \leq q < c(\Delta)$.

If $\{b_j\}$ is an ordered p–basis of Δ, then its conductor is $c := b_{p-1} - p + 1$. The semi–module Δ' has an unordered p–basis $\{b'_j\}$ with $b'_j = b_j$ for $j < p-1$ and $b'_{p-1} = b_{p-1} - p$. The conductor of Δ' is $c' := \max\{b_{p-2} - p + 1, b_{p-1} - 2p + 1\}$; in particular $c-p \le c' < c$. Since $b_{p-1} \ge c$ and $b'_{p-1} \ge c'$, we have $[b_{p-1}, b_{p-1}+q[\subset \Delta, \Delta'$ and $[b'_{p-1}, b'_{p-1}+q[\subset \Delta'$, and the dimensions of Δ and Δ' can be computed very similarly as

$$\dim \Delta = \sum_{j=0}^{p-1} \#(I_j \setminus \Delta) \text{ resp. } \dim \Delta = \sum_{j=0}^{p-1} \#(I_j \setminus \Delta') \text{ with } I_j = [b_j, b_j + q[\,.$$

Due to $\Delta' = \Delta \cup \{b'_{p-1}\}$, we get

$$\dim \Delta = \dim \Delta' + \#J \qquad \text{with} \quad J := \left\{ j \in \{0, \ldots, p-1\} \mid b'_{p-1} \in I_j \right\},$$

showing $\dim \Delta \ge \dim \Delta'$.

Now, let us assume additionally that $c' \le q < c$. We need to show that $\dim \Delta > n$. We claim that

$$(\dagger\dagger) \qquad \#J \ge \#([c', q] \setminus p\mathbb{N}).$$

Knowing this we can easily finish the proof. Choose $l \in \mathbb{N}$ such that $c(\Delta_l) = c'$, then $\dim \Delta_l \le \dim \Delta'$ by (†). Further, $\Delta_n \setminus \Delta_l = [c', q[\setminus p\mathbb{N}$; hence, $\dim \Delta_n = \dim \Delta_l + \#([c', q[\setminus p\mathbb{N})$. Putting this together, we get

$$\dim \Delta = \dim \Delta' + \#J \ge \dim \Delta_l + \#([c', q] \setminus p\mathbb{N}) = \dim \Delta_n + 1 = n + 1.$$

Finally, it remains to prove (††). For each of the $k \in [c', b'_{p-1}[$ find the index j_k with $b_{j_k} \equiv k \bmod p$. Then $j_k \in J$ is equivalent to $b'_{p-1} \in [b_{j_k}, b_{j_k} + q[$ or to $b_{j_k} > b'_{p-1} - q$. Since $b'_{p-1} \notin \Delta$, we find $b'_{p-1} - q \notin \Delta$ and $b'_{p-1} - q \notin \Delta'$ as well. Therefore, $j_k \in J$ is in fact equivalent to $b_{j_k} \ge b'_{p-1} - q$. This implies that only $b'_{p-1} - q$ of the $b'_{p-1} - c'$ integers in $[c', b'_{p-1}[$ can fail to have a corresponding j index that lies in J; in particular, $\#J \ge q - c'$, nearly proving (††). If we actually have $\#J = q - c'$ then $b'_{p-1} - q$ of the b_{j_k} must be less than $b'_{p-1} - q$; thus we must have $b_j = j$ for $j = 0 \ldots b'_{p-1} - q - 1$. Let $l \in [c', b'_{p-1}[$ be the integer with $l \equiv b_0 = 0 \bmod p$. If l were greater than q, we would have $b_{b'_{p-1}-l} = b'_{p-1} - l \equiv b'_{p-1} \bmod p$, contradicting the definition of a p–basis. Therefore, $l \le q$ and $[c', q] \cap p\mathbb{N} \ne \emptyset$, proving (††). $\square$

We turn to the singularities with characteristic Puiseux exponents $(4, 2q, s)$. With the help of a computer program one obtains the following list of Betti numbers:

$(4,2q,s)$	δ_X	$e(X)$	$h^0\,h^2\,h^4\,h^6\,h^8\ \dots$
(4,6,7)	8	23	1 3 4 4 4 3 2 1 1
(4,6,9)	9	27	1 3 4 4 4 4 3 2 1 1
(4,6,11)	10	31	1 3 4 4 4 4 4 3 2 1 1
(4,6,13)	11	35	1 3 4 4 4 4 4 4 3 2 1 1
(4,10,11)	14	76	1 3 6 8 9 9 9 8 7 5 4 3 2 1 1
(4,10,13)	15	85	1 3 6 8 9 9 9 9 8 7 5 4 3 2 1 1
(4,10,15)	16	94	1 3 6 8 9 9 9 9 9 8 7 5 4 3 2 1 1
(4,10,17)	17	103	1 3 6 8 9 9 9 9 9 9 8 7 5 4 3 2 1 1
(4,14,15)	20	178	1 3 6 10 13 15 16 16 16 15 14 12 10 8 7 5 4 3 2 1 1
(4,14,17)	21	194	1 3 6 10 13 15 16 16 16 16 15 14 12 10 8 7 5 4 3 2 1 1
(4,14,19)	22	210	1 3 6 10 13 15 16 16 16 16 16 15 14 12 10 8 7 5 4 3 2 1 1
(4,14,21)	23	226	1 3 6 10 13 15 16 16 16 16 16 16 15 14 12 10 8 7 5 4 3 2 1 1
(4,18,19)	26	345	1 3 6 10 15 19 22 24 25 25 25 24 23 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,18,21)	27	370	1 3 6 10 15 19 22 24 25 25 25 25 24 23 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,18,23)	28	395	1 3 6 10 15 19 22 24 25 25 25 25 25 24 23 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,18,25)	29	420	1 3 6 10 15 19 22 24 25 25 25 25 25 25 24 23 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,22,23)	32	593	1 3 6 10 15 21 26 30 33 35 36 36 36 35 34 32 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,22,25)	33	629	1 3 6 10 15 21 26 30 33 35 36 36 36 36 35 34 32 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,22,27)	34	665	1 3 6 10 15 21 26 30 33 35 36 36 36 36 36 35 34 32 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,22,29)	35	701	1 3 6 10 15 21 26 30 33 35 36 36 36 36 36 36 35 34 32 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,26,27)	38	938	1 3 6 10 15 21 28 34 39 43 46 48 49 49 49 48 47 45 43 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,26,29)	39	987	1 3 6 10 15 21 28 34 39 43 46 48 49 49 49 49 48 47 45 43 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,26,31)	40	1036	1 3 6 10 15 21 28 34 39 43 46 48 49 49 49 49 49 48 47 45 43 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,26,33)	41	1085	1 3 6 10 15 21 28 34 39 43 46 48 49 49 49 49 49 49 48 47 45 43 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,30,31)	44	1396	1 3 6 10 15 21 28 36 43 49 54 58 61 63 64 64 64 63 62 60 58 55 52 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,30,33)	45	1460	1 3 6 10 15 21 28 36 43 49 54 58 61 63 64 64 64 64 63 62 60 58 55 52 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,30,35)	46	1524	1 3 6 10 15 21 28 36 43 49 54 58 61 63 64 64 64 64 64 63 62 60 58 55 52 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,30,37)	47	1588	1 3 6 10 15 21 28 36 43 49 54 58 61 63 64 64 64 64 64 64 63 62 60 58 55 52 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,34,35)	50	1983	1 3 6 10 15 21 28 36 45 53 60 66 71 75 78 80 81 81 81 80 79 77 75 72 69 65 61 56 52 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,34,37)	51	2064	1 3 6 10 15 21 28 36 45 53 60 66 71 75 78 80 81 81 81 81 80 79 77 75 72 69 65 61 56 52 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,34,39)	52	2145	1 3 6 10 15 21 28 36 45 53 60 66 71 75 78 80 81 81 81 81 81 80 79 77 75 72 69 65 61 56 52 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,34,41)	53	2226	1 3 6 10 15 21 28 36 45 53 60 66 71 75 78 80 81 81 81 81 81 81 80 79 77 75 72 69 65 61 56 52 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1

The table leads to the following

Conjecture 5.5 *For a unibranched plane singularity with characteristic Puiseux exponents* $(4, 2q, s)$ *the* $\delta_X + 1 = 2q + (s-1)/2$ *even Betti numbers of the cohomology of its Jacobi factor are as follows:*

1. *the first* $(q+1)/2$ *even Betti numbers* $h^0(J_X), h^2(J_X), \dots, h^{q-1}(J_X)$ *are the same as the first* $(q+1)/2$ *coefficients of the power series* $(1-t)^{-3}$.
2. *the last* $(3q+1)/2$ *even Betti numbers* $h^{2\delta_x}(J_X), h^{2\delta_x - 2}(J_X), \dots, h^{s+q-2}(J_X)$ *are the same as the first* $(3q+1)/2$ *coefficients of the power series* $\prod_{j=1}^{3}(1-t^j)^{-1}$.
3. $h^{2(q-1)}(J_X) = h^{2q}(J_X) = \dots = h^{s+1}(J_X) = (q+1)^2/4$.
4. *For* $l = 1, \dots, (q-3)/2:\ h^{2(q-1-l)}(J_X) = h^{2(q-l)}(J_X) - l$.
5. *For* $l = 1, \dots, (q-5)/2:\ h^{s+1+2l}(J_X) = h^{s-1+2l}(J_X) - \lceil \frac{l}{2} \rceil$.

Part 1 and Part 2 of the conjecture are proven in Theorems 5.6 and 5.7. They describe $2q+1$ of the $2q+(s-1)/2$ Betti numbers. Unfortunately, we are not able to prove the remaining parts of the Conjecture. However, the conjecture implies that the sequence of Betti numbers of $J_{X'}$ for a singularity X' with Puiseux exponents $(4, 2q, s+2)$ can be obtained from the sequence for a singularity X with Puiseux exponents $(4, 2q, s)$ by inserting $(q+1)^2/4 = e(J_{X'}) - e(J_X)$ after $h^{2(q-1)}(J_X)$. We prove this partially by showing in Theorems 5.8 and 5.9 that the first $(s-q)/2+2$ and the last $(q+s)/2+1$ numbers of the above two sequences are the same, thus determining all or at least $s+3$ of the Betti numbers.

Theorem 5.6 *Let* X *be a unibranched plane singularity with Puiseux exponents* $(4, 2q, s)$ *and* J_X *its Jacobi factor. Then the even Betti numbers* $h^0(J_X), h^2(J_X), \dots, h^{2\lfloor \frac{q}{2} \rfloor}(J_X)$ *of the cohomology of* J_X *are the same as the first* $\lfloor \frac{q}{2} \rfloor + 1$ *coefficients of the power series*

$$P := \frac{1}{(1-t)^3}.$$

Proof. The proof is similar to the proof of Theorem 5.3. For $d \leq \lfloor \frac{q}{2} \rfloor$ we construct a bijection between the monomials of degree d of the polynomial ring in the variables t_1, t_2, t_3 and the admissible semi–modules of $\Gamma = \langle 4, \gamma_1 = 2q, \gamma_2 = \gamma_1 + s \rangle$ of codimension d

$$\begin{array}{llll} \Phi_d : & \mathrm{Mon}^d & \longrightarrow & \mathrm{Mod}^d(\Gamma) \\ & t_1^{r_1} t_2^{r_2} t_3^{r_3} & \longmapsto & \left\langle \begin{array}{l} a_{00} = 0, a_{01} = \gamma_1 - 4r_1, a_{10} = \gamma_2 - 4(r_1 + r_2), \\ a_{11} = \gamma_2 + \gamma_1 - 4(r_1 + r_2 + r_3) \end{array} \right\rangle . \end{array}$$

The maps Φ_d are well–defined if we can show that a semi–module Δ with a 2×2–basis like the one on the right hand side is admissible and of correct dimension. Admissibility is obvious as $a_{01} + s = a_{10} + 4r_2 \in \Delta$. We compute its codimension by comparing it with the Γ–semi–module Γ itself. Due to $r_1 + r_2 + r_3 < q/2$, we have the following ordering

$$a_{00} = 0 < a_{01} \leq \gamma_1 < s < a_{10} \leq a_{01} + s \leq \gamma_2 < a_{11}.$$

As $a_{10}, a_{11} > s$, we find $n = \min((\{s, a_{10}, a_{11}\} + 4\mathbb{N}) \cap [\gamma_1, \infty[) = s$, and the dimension of Δ can be computed as

$$\dim \Delta = \#([0, \gamma_1[\setminus \Delta) + \#([a_{01}, a_{01} + s[\setminus \Delta) + \#([a_{10}, \infty[\setminus \Delta).$$

The semi–module Γ is $\Phi_0(1)$; hence, an analogous formula holds for it as well. Because of the above ordering we get the following partitions

$$\begin{aligned} [0,\gamma_1[\,\setminus\Gamma &= [0,\gamma_1[\,\setminus\Delta \qquad \cup\{a_{01}+4k \mid 0\le k<r_1\}\\ -4r_1+([\gamma_1,\gamma_1+s[\,\setminus\Gamma) &= [a_{01},a_{01}+s[\,\setminus\Delta\cup\{a_{10}+4k \mid 0\le k<r_2\}\\ -4(r_1+r_2)+([\gamma_2,\infty[\,\setminus\Gamma) &= [a_{10},\infty[\,\setminus\Delta \qquad \cup\{a_{11}+4k \mid 0\le k<r_3\}. \end{aligned}$$

Therefore, $\operatorname{codim}\Delta = \dim\Gamma - \dim\Delta = r_1+r_2+r_3 = d$ as desired.

The maps Φ_d are clearly injective, thus it remains to show that they are surjective, too. We must prove that the modules which are not in the image of some $\Phi_0,\dots,\Phi_{\lfloor\frac{q}{2}\rfloor}$ have codimension greater than $\lfloor\frac{q}{2}\rfloor$. Since there are two types of admissible modules, this falls naturally into two parts.

Let us assume that we have an admissible module Δ with $a_{01}+s\in\Delta$. Since $a_{01}+s\equiv a_{10} \bmod 4$, we get $a_{01}+s\ge a_{10}$. By the relation between the elements of a 2×2–basis — see the paragraph below Definition 2.5 — we find r_1,r_2,r_3 with

$$a_{00}=0,\quad a_{01}=\gamma_1-4r_1,\quad a_{10}=\gamma_2-4(r_1+r_2),\quad a_{11}=\gamma_2+\gamma_1-4(r_1+r_2+r_3).$$

We claim the following rough estimate

$$(\star)\qquad \operatorname{codim}\Delta \ge r_1+\min\{r_2,q+1-r_3\}+\min\left\{r_3,\left\lceil\tfrac{q}{2}\right\rceil\right\}.$$

This implies in particular that if $r_1+r_2+r_3>\lfloor\frac{q}{2}\rfloor$ then $\operatorname{codim}\Delta>\lfloor\frac{q}{2}\rfloor$, i.e., any admissible semi–module Δ with $a_{01}+s\in\Delta$ that is not in the image of some $\Phi_0,\dots,\Phi_{\lfloor\frac{q}{2}\rfloor}$ has a codimension greater than $\lfloor\frac{q}{2}\rfloor$.

We prove the claim $(\star)$ by modifying Γ into Δ in three steps. The first step consists of the remark that the module $\Delta'=\Phi_{r_1}(t_1^{r_1})$ was described above in detail. In particular, we found $\operatorname{codim}\Delta'=r_1$ and for its 2×2–basis (a'_{ij}) the ordering

$$a'_{00}=0<a'_{01}=a_{01}<\gamma_1<s<a'_{10}=a_{01}+s=a_{01}+n'\le\gamma_2<a'_{11}=a'_{10}+\gamma_1.$$

In the second step we consider the semi–module $\tilde\Delta$ with 2×2–basis $\tilde a_{00}=0$, $\tilde a_{01}=a_{01}$, $\tilde a_{10}=a'_{10}-4r_2=a_{10}$, and $\tilde a_{11}=a'_{11}-4r_2=a_{10}+\gamma_1$, i.e., $\tilde\Delta$ is obtained from Δ' by closing the $2r_2$ gaps $a'_{10}-4k, a'_{11}-4k$ for $k=1,\dots,r_2$. We write the dimension formulas as

$$\begin{aligned} \dim\Delta' &= (g_{\Delta'}(0)+g_{\Delta'}(a_{01})-g_{\Delta'}(\gamma_1))+g_{\Delta'}(a'_{10})-g_{\Delta'}(a_{01}+n')\\ \dim\tilde\Delta &= (g_{\tilde\Delta}(0)+g_{\tilde\Delta}(a_{01})-g_{\tilde\Delta}(\gamma_1)) \quad +g_{\tilde\Delta}(\tilde a_{10})-g_{\tilde\Delta}(a_{01}+\tilde n). \end{aligned}$$

Since $0<a_{01}\le\gamma_1$ the closing of any $2r_2$ gaps in Δ' decreases the term in the brackets for Δ' to the terms for $\tilde\Delta$ by at least $2r_2$. Next note that $a_{01}+n'\ge a'_{10}$ by the definition of n', thus the only gaps greater than or equal to $a_{01}+n'$ in Δ' are those which are a'_{11} mod 4 and analogously for $\tilde\Delta$. Since $\tilde n\le n'=s$, we can estimate the length of the intervals $[a_{01}+\tilde n,\tilde a_{11}[$ and $[a_{01}+n',a'_{11}[$ by $\tilde a_{11}-(a_{01}+\tilde n)\ge a'_{11}-4r_2-(a_{01}+s)=\gamma_1-4r_2$ and $a'_{11}-(a_{01}+s)=\gamma_1$; hence, $g_{\tilde\Delta}(a_{01}+\tilde n)\ge g_{\Delta'}(a_{01}+n')-\min\{r_2,\lfloor\frac{\gamma_1}{4}\rfloor\}$. Summing up, we get as an intermediate result

$$\dim\Delta'\ge\dim\tilde\Delta+r_2+\max\left\{0,r_2-\left\lfloor\frac{q}{2}\right\rfloor\right\}+g_{\Delta'}(a_{10})-g_{\tilde\Delta}(\tilde a_{10}).$$

The only gaps in Δ' after a'_{10} are the $\lfloor\frac{q}{2}\rfloor$ gaps which are equal to a'_{11} mod 4. $\tilde\Delta$ has also $\lfloor\frac{q}{2}\rfloor$ gaps equal to $\tilde a_{11}$ mod 4 after $\tilde a_{10}$, but may have in addition some that are equal to a_{01} mod 4 if $\tilde a_{10}<a_{01}-4$. In this case set $\tilde r_2=\lceil\frac{s}{4}\rceil$ then

$a'_{10} - 4\tilde{r}_2 \in]a_{01} - 4, a_{01}[$ and thus $a_{01} - 4(r_2 - \tilde{r}_2), a_{01} - 4(r_2 - \tilde{r}_2 - 1), \ldots, a_{01} - 4$ are the addition gaps. Therefore, $g_{\tilde{\Delta}}(\tilde{a}_{10}) - g_{\Delta'}(a_{10}) = \max\{0, r_2 - \lceil \frac{s}{4} \rceil\}$ and in the whole $\operatorname{codim} \tilde{\Delta} \geq \operatorname{codim} \Delta' + r_2 = r_1 + r_2$.

In the final step we compare the codimensions of $\tilde{\Delta}$ and Δ itself. The only difference in the 2×2–bases of $\tilde{\Delta}$ and Δ is that $a_{11} = \tilde{a}_{11} - 4r_3$, i.e., we are closing the r_3 gaps $\tilde{a}_{11} - 4, \ldots, \tilde{a}_{11} - 4r_3 = a_{11}$ in $\tilde{\Delta}$. By the same argument as before, the term $g_{\tilde{\Delta}}(0) + g_{\tilde{\Delta}}(a_{01}) - g_{\tilde{\Delta}}(\gamma_1)$ is at least r_3 greater than $g_\Delta(0) + g_\Delta(a_{01}) - g_\Delta(\gamma_1)$. Due to $a_{10} \leq a_{01} + \tilde{n}$, all closed gaps equal to or after $a_{01} + \tilde{n}$ are closed gaps after a_{10} as well, thus using $n \leq \tilde{n}$

$$g_{\tilde{\Delta}}(a_{10}) - g_{\tilde{\Delta}}(a_{01} + \tilde{n}) \geq g_\Delta(a_{10}) - g_\Delta(a_{01} + \tilde{n}) \geq g_\Delta(a_{10}) - g_\Delta(a_{01} + n).$$

However, for the first time there may be gaps in Δ after a_{11}, and we obtain as an intermediate result only

$$\operatorname{codim} \Delta \geq r_1 + r_2 + r_3 - g_\Delta(a_{11}).$$

We can count the gaps after a_{11} precisely. There $\max\{0, r_3 - \lceil \frac{q}{2} \rceil\}$ equal to a_{10} modulo 4 and $\max\{0, r_3 + r_2 - \lceil \frac{\gamma_2}{4} \rceil\}$ equal to a_{01} modulo 4. Using $\lceil \frac{\gamma_2}{4} \rceil = \lceil \frac{2q+s}{4} \rceil \geq q + 1$, we obtain $(\star)$ by

$$\begin{aligned} \operatorname{codim} \Delta &\geq r_1 + (r_2 - \max\{0, r_2 + r_3 - (q+1)\}) + \left(r_3 - \max\left\{0, r_3 - \left\lceil \tfrac{q}{2} \right\rceil\right\}\right) \\ &= r_1 + \min\{r_2, q + 1 - r_3\} + \min\left\{r_3, \left\lceil \tfrac{q}{2} \right\rceil\right\}. \end{aligned}$$

The second type of admissible modules are those with $s \in \Delta$. Let us assume in addition that $a_{10} > a_{01} + s$ for a semi–module Δ, otherwise we have $a_{01} + s \in \Delta$ and Δ is admissible of first type as well. We show that for all these semi–modules $\operatorname{codim} \Delta \geq \lceil \frac{q}{2} \rceil$ by comparing Δ with the following simple semi–module

$$\overline{\Delta} = \langle \overline{a}_{00} = 0, \overline{a}_{01} = \gamma_1, \overline{a}_{10} = \gamma_2, \overline{a}_{11} = s = \gamma_2 + \gamma_1 - 4q \rangle .$$

We find $\overline{n} = s$ and $\dim \overline{\Delta} = g_{\overline{\Delta}}(0) + g_{\overline{\Delta}}(s) = g_\Gamma(0) - q + \lfloor \frac{q}{2} \rfloor = \dim \Gamma - \lceil \frac{q}{2} \rceil$; hence $\operatorname{codim} \overline{\Delta} = \lceil \frac{q}{2} \rceil$. We will modify $\overline{\Delta}$ in three steps into Δ and show that the codimension does not decrease during these modifications.

First let $\Delta' = \langle 0, a_{01}, \overline{a}_{10}, \overline{a}_{11} \rangle$. If α_{01} is chosen such that $a_{01} = \gamma_1 - 4\alpha_{01}$, then we are closing the α_{01} gaps $a_{01}, a_{01} + 4, \ldots, \gamma_1 - 4$ in $\overline{\Delta}$. Hence, $g_{\Delta'}(k) = g_{\overline{\Delta}}(k)$ for $k \geq \gamma_1$. Further $\overline{n} = n' = s$ and $g_{\Delta'}(0) = g_{\overline{\Delta}}(0) - \alpha_{01}$, $g_{\Delta'}(a_{01}) = g_{\overline{\Delta}}(\overline{a}_{01}) + 2\alpha_{01}$, $g_{\Delta'}(a_{01} + s) = \alpha_{01}$, as well as $g_{\overline{\Delta}}(\overline{a}_{01} + s) = 0$. Plugging this into the dimension formulas yields $\dim \overline{\Delta} = \dim \Delta'$.

Next we modify Δ' into $\tilde{\Delta} = \langle 0, a_{01}, a_{10}, \overline{a}_{11} \rangle$. Setting $\alpha_{10} = (\overline{a}_{10} - a_{10})/4$, this means that we are closing the α_{10} gaps $a_{10}, a_{10} + 4, \ldots, \overline{a}_{10} - 4$ of Δ'. We have the ordering $a_{10} > a_{01} + s > s = \overline{a}_{11}$ thus $\tilde{n} = s$ and $g_{\tilde{\Delta}}(k) = g_{\Delta'}(k) - \alpha_{10}$ for $k \leq a_{10}$. In addition $g_{\tilde{\Delta}}(a_{10}) = g_{\Delta'}(\overline{a}_{10}) = 0$, because a_{10} resp. $\overline{a}_{10}$ are the greatest elements of the 2×2–bases of $\tilde{\Delta}$ resp. Δ'. From the dimension formula we obtain $\operatorname{codim} \tilde{\Delta} = \operatorname{codim} \Delta' + \alpha_{10} \geq \lceil \frac{q}{2} \rceil$.

Finally, the semi–module Δ is obtained from $\tilde{\Delta}$ by closing the $(s - a_{11})/4 =: \beta$ gaps $a_{11}, a_{11} - 4, \ldots, s - 4 = \overline{a}_{11} - 4$. By definition $n \leq \tilde{n} = s$ and because no gaps after s were closed, we obtain $g_\Delta(a_{01} + n) \geq g_\Delta(a_{01} + \tilde{n}) = g_{\tilde{\Delta}}(a_{01} + \tilde{n})$. As usual, $g_{\tilde{\Delta}}(0) + g_{\tilde{\Delta}}(a_{01}) - g_{\tilde{\Delta}}(\gamma_1)$ is at least β greater than the corresponding term for Δ. From the semi–module property $a_{11} \geq a_{10} - \gamma_1$ we conclude $a_{11} > a_{01} + s - \gamma_1 > a_{01}$. Thus the only gaps after a_{11} resp. $\overline{a}_{11}$ are those which are equal to a_{10} modulo 4, and we find $g_\Delta(a_{11}) = g_{\tilde{\Delta}}(\overline{a}_{11}) + \beta$. Summing up, we obtain $\operatorname{codim} \Delta \geq \operatorname{codim} \tilde{\Delta} \geq \lceil \frac{q}{2} \rceil$ again. $\square$

Theorem 5.7 *Let X be a unibranched plane singularity with Puiseux exponents $(4, 2q, s)$ and J_X its Jacobi factor. Set $k := (3q-1)/2$. Then the even Betti numbers $h_0(J_X), h_2(J_X), \ldots, h_{2k}(J_X)$ of the homology of J_X are the same as the first $k+1$ coefficients of the power series*

$$P := \frac{1}{(1-t)(1-t^2)(1-t^3)}.$$

Proof. The beginning of this proof is the same as the one of the proof of Theorem 5.4 with p replaced by 4 and q replaced by $\gamma_1 = 2q$. Of all the modules $\Delta_r = \psi(t_1^r)$ the following two will be of special importance at the end:

$$\begin{aligned} \Delta_k &= 4\mathbb{N} \cup (\gamma_1 + \mathbb{N}) && \text{with } c(\Delta_k) = \gamma_1 \quad \text{and} \\ \Delta_{k-1} &= 4\mathbb{N} \cup (\gamma_1 - 2 + \mathbb{N}) && \text{with } c(\Delta_{k-1}) = \gamma_1 - 2. \end{aligned}$$

The last step of the proof, where one proves that the maps $\Psi_d : \mathrm{Mon}_d \to \mathrm{Mod}_d(\Gamma)$ are surjective has to be modified due to the different dimension formula. As before we show that $c(\Delta) > \gamma_1$ implies $\dim \Delta > k$ by an inductive process. Let Δ' be the Γ–semi–module obtained from Δ by closing the last gap. If $\{b_0 = 0, b_1, b_2, b_3\}$ is an ordered 4–basis of Δ then its conductor is $c := b_3 - 3$. Because $c > \gamma_1$, b_3 is the element a_{10} or a_{11} is a 2×2–basis of Δ. The module Δ' has the unordered 4–basis $\{b'_j\}$ with $b'_0 = 0$, $b'_1 = b_1$, $b'_2 = b_2$, and $b'_3 = b_3 - 4$, and its conductor is $c' = \max\{b_2 - 3, b_3 - 3\}$. The dimension formula says

$$\dim \Delta = (g_\Delta(0) + g_\Delta(a_{01}) - g_\Delta(\gamma_1)) + g_\Delta(a_{10}) + g_\Delta(a_{11}) - g_\Delta(a_{01} + n)$$

and analogously for Δ'. Since we are closing one gap in Δ greater than γ_1, the term in the brackets decreases by one for Δ'. Because $b_3 = a_{10}$ or $b_3 = a_{11}$, $g_\Delta(a_{11})$ resp. $g_\Delta(a_{10})$ decreases by one or stays the same. By definition $b_3 > c$ and $b'_3 \geq c'$, hence $g_\Delta(b_3) = g_{\Delta'}(b'_3) = 0$. The number n may stay the same or be reduced by at most 4. If $n' = n$, then obviously $g_\Delta(a_{01} + n)$ decreases by one or stays the same. If $n' < n$, then b'_3 must be the smallest odd number in Δ'. Hence, the smallest odd number in Δ is $b'_3 + 2 = b_3 - 2 = b_2 = n$. Due to $a_{01} \geq 2$ and $\gamma_1 + 2\mathbb{N} \subset \Delta, \Delta'$, we find $0 \leq g_\Delta(a_{01} + n) \leq g_\Delta(b_3) = 0$ and $0 \leq g_{\Delta'}(a_{01} + n') \leq g_{\Delta'}(b'_3) = 0$, showing $g_\Delta(a_{01} + n) = g_{\Delta'}(a_{01} + n') = 0$. Summing up the changes, we obtain $\dim \Delta \geq \dim \Delta'$.

Now, let us assume additionally that $c' \leq \gamma_1 < c$, i.e., we are closing the last gap, $\gamma_1 + 1$ or $\gamma_1 + 3$, greater than γ_1. Choose the index J such that $a_{1J} = b_3$. We first consider the case where the last gap is $\gamma_1 + 3$. Here we have $a_{1,1-J} \leq \gamma_1 + 1$, thus $g_\Delta(a_{1,1-J})$ decreases by one during this process. The above discussion yields $\dim \Delta > \dim \Delta'$. Due to $c(\Delta') = \gamma_1$, we get $\dim \Delta > \dim \Delta' \geq \dim \Delta_k = k$ by (†).

Finally, we consider the other case, where the last gap of Δ is $\gamma_1 + 1$. Because we always have $n \geq \gamma_1 + 1$ and $a_{01} + n \geq \gamma_1 + 3$, we get $g_\Delta(a_{01} + n) = g_{\Delta'}(a_{01} + n') = 0$. Therefore, by the above discussion $\dim \Delta > \dim \Delta'$ and if $c(\Delta') = \gamma_1$, we can finish the proof like above. However, $c(\Delta')$ may as well be $\gamma_1 - 2$. Here $a_{1,1-J} < \gamma_1 < b'_3$ thus $g_{\Delta'}(a_{1,1-J}) = g_\Delta(a_{1,1-J}) - 1$ and the above discussion yields $\dim \Delta > \dim \Delta' + 1$. Using $c(\Delta_{k-1}) = \gamma_1 - 2$ and (†), we obtain $\dim \Delta > \dim \Delta' + 1 \geq \dim \Delta_{k-1} + 1 = k$. □

Theorem 5.8 *Let X and X' be unibranched plane singularities with Puiseux exponents $(4, 2q, s)$ resp. $(4, 2q, s')$ with $s' \geq s$ and J_X resp. $J_{X'}$ their Jacobi factors. Set $k := (s-q)/2 + 1$. Then the first $k+1$ even Betti numbers of the cohomology of J_X and $J_{X'}$ are the same, i.e., $h^{2d}(J_X) = h^{2d}(J_{X'})$ for $d = 0, \ldots, k$.*

Proof. Let Γ and Γ' be the semi–groups corresponding to the singularities. By induction we may assume $s' = s + 2$. We are going to show that the following map is well–defined and bijective for $d \leq k$

$$\begin{array}{rccc} \Phi_d : & \mathrm{Mod}^d(\Gamma) & \longrightarrow & \mathrm{Mod}^d(\Gamma') \\ & \Delta = \langle 0, a_{01}, a_{11}, a_{11} \rangle & \longmapsto & \Delta' = \langle 0, a_{01}, a'_{10} = a_{10} + 2, a'_{11} = a_{11} + 2 \rangle \end{array}$$

If the 2×2–basis of Δ is written as $a_{00} = 0$, $a_{01} = \gamma_1 - 4\alpha_{01}$, $a_{10} = \gamma_2 - 4\alpha_{10}$, $a_{11} = \gamma_2 + \gamma_1 - 4\alpha_{11}$ then Δ' is the Γ'–semi–module whose 2×2–basis has the same α_{ij}. Δ' is admissible, because $s = a_{11} + 4l \in \Delta$ or $a_{01} + s = a_{10} + 4l \in \Delta$ implies $s' = a'_{11} + 4l \in \Delta'$ or $a_{01} + s' = a'_{10} + 4l \in \Delta'$. The injectivity of the map is trivial, its well–definedness and surjectivity will follow from the statement

($\circledast$) Let Δ be an admissible Γ–semi–module and J the index with $a_{1J} = \min\{a_{10}, a_{11}\}$ then

$$\begin{array}{lll} a) & \operatorname{codim} \Delta \leq k \quad \Longrightarrow a_{10}, a_{11} > \gamma_1 & \text{or } (a_{1J}, a_{1,1-J}) = (\gamma_1 - 1, 2\gamma_1 - 1) \\ b) & \operatorname{codim} \Delta \leq k - 1 \Longrightarrow a_{10}, a_{11} > \gamma_1 + 2 & \text{or } (a_{1J}, a_{1,1-J}) = (\gamma_1 + 1, 2\gamma_1 + 1) \end{array}$$

Assume we have proven ($\circledast$). For the well–definedness of Φ_d we need to show that $\operatorname{codim} \Delta' = \operatorname{codim} \Delta$ or equivalently $\dim \Delta' = \dim \Delta + 1$. If $a_{10}, a_{11} > \gamma_1$, then we obtain Δ' from Δ by inserting a gap and nongap after γ_1, more precisely $\Delta' = (\Delta \cap [0, \gamma_1]) \cup (2 + (\Delta \cap [\gamma_1, \infty[))$. Here $n = \min\{s, a_{10}, a_{11}\}$ and $n' = \min\{s', a'_{10}, a'_{11}\}$. In the dimension formula for Δ' the term $g_{\Delta'}(0) + g_{\Delta'}(a_{01}) - g_{\Delta'}(\gamma_1)$ is by one greater than the corresponding term for Δ because of the extra gap after γ_1. In contrast $g_{\Delta'}(a_{10}) + g_{\Delta'}(a_{11}) - g_{\Delta'}(a_{01} + n')$ is the same as the term for Δ, because everything is shifted by 2. Hence, $\dim \Delta' = \dim \Delta + 1$ as desired.

If $(a_{1J}, a_{1,1-J}) = (\gamma_1 - 1, 2\gamma_1 - 1)$ then $(a'_{1J}, a'_{1,1-J}) = (\gamma_1 + 1, 2\gamma_1 + 1)$, $n = \gamma_1 + 3$, and $n' = \gamma_1 + 1$. Obviously, $\#([a_{01}, \gamma_1[\setminus \Delta) = \#([a_{01}, \gamma_1[\setminus \Delta') + \varepsilon$ where $\varepsilon = 0$ if $a_{01} = \gamma_1$ and $\varepsilon = 1$ otherwise. Further, $g_\Delta(0) = g_{\Delta'}(0) + 1$, $g_\Delta(a_{10}) = g_{\Delta'}(a'_{10})$, and $g_\Delta(a_{11}) = g_{\Delta'}(a'_{11})$. Thus the interesting terms are $g_\Delta(a_{01} + n) = g_\Delta(2\gamma_1 - 4\alpha_{01} + 3)$ and $g_{\Delta'}(a_{01} + n') = g_{\Delta'}(2\gamma_1 - 4\alpha_{01} + 1)$. As shifting by 2 gives a bijection between $[2\gamma_1 - 4\alpha_{01} + 3, \infty[$ and $[2\gamma_1 - 4\alpha_{01} + 5, \infty[$, which respects membership in Δ resp. Δ', we have $g_{\Delta'}(a_{01} + n') = g_\Delta(a_{01} + n) + \#([2\gamma_1 - 4\alpha_{01} + 1, 2\gamma_1 - 4\alpha_{01} + 5[\setminus \Delta')$. From $a_{01}, a'_{1J} \leq \gamma_1 + 1$ we see that the only possible gap in in the above interval must be equal to $a'_{1,1-J} = 2\gamma_1 + 1$ modulo 4, i.e., it can only be $2\gamma_1 - 4\alpha_{01} + 1$. For this to be a gap, we must have $\alpha_{01} > 0$, hence $g_{\Delta'}(a_{01} + n') = g_\Delta(a_{01} + n) + \varepsilon$. This shows that we always have $\dim \Delta' = \dim \Delta + 1$.

The surjectivity follows now, too. Let $\Delta' \in \mathrm{Mod}^d(\Gamma')$ with $d \leq k$. As $k = k' - 1$ we may apply *b)* to Δ' to obtain $a'_{10}, a'_{11} > \gamma_1 + 2$ or $(a'_{10}, a'_{11}) = (\gamma_1 + 1, 2\gamma_1 + 1)$. Thus Δ' is the image of $\Delta = \langle 0, a_{01}, a'_{10} - 2, a'_{11} - 2 \rangle$ under Φ_d — that Δ has the correct dimension was shown above.

We prove the statement ($\circledast$) by first considering two special types of semi–modules and then compare the other modules with them. Define

$$\begin{array}{ll} \Delta^{10}_\alpha := \langle 0, a_{01} = \gamma_1, a_{10} = \gamma_2 - 4\alpha, a_{11} = \gamma_2 + \gamma_1 - 4\alpha \rangle & \text{for } \left\lceil \frac{q}{2} \right\rceil \leq \alpha \leq \left\lfloor \frac{\gamma_2}{4} \right\rfloor \\ \Delta^{11}_\alpha := \langle 0, a_{01} = \gamma_1, a_{10} = \gamma_2 - 4\alpha, a_{11} = s - 4\alpha \rangle & \text{for } 0 \leq \alpha \leq \left\lfloor \frac{s}{4} \right\rfloor \end{array}$$

the definition is such that in Δ^{1J}_α the minimum of a_{10} and a_{11} is a_{1J} and $a_{1,1-J} = a_{1J} + \gamma_1$. Their dimension is computed easily: Using $n \geq a_{1J} \Rightarrow n + \gamma_1 \geq a_{1,1-J} \geq$

$c(\Delta_\alpha^{1J})$, we find $g_{\Delta_\alpha^{1J}}(\gamma_1+n)=0$

$$\dim\Delta_\alpha^{10} = g_{\Delta_\alpha^{10}}(0)+g_{\Delta_\alpha^{10}}(a_{10}) = \dim\Gamma - 2\alpha + \left\lfloor \tfrac{q}{2}\right\rfloor \Longrightarrow \operatorname{codim}\Delta_\alpha^{10} = 2\alpha - \left\lfloor \tfrac{q}{2}\right\rfloor$$

$$\dim\Delta_\alpha^{11} = g_{\Delta_\alpha^{11}}(0)+g_{\Delta_\alpha^{11}}(a_{11}) = \dim\Gamma - (2\alpha+q) + \left\lfloor \tfrac{q}{2}\right\rfloor \Longrightarrow \operatorname{codim}\Delta_\alpha^{11} = 2\alpha + \left\lceil \tfrac{q}{2}\right\rceil$$

We claim that for these two types of semi–modules $a_{1J}\le\gamma_1-1$ implies $\operatorname{codim}\Delta_\alpha^{1J}\ge k$. If $s\equiv 1 \bmod 4$, then $a_{1J}\le\gamma_1-1$ is equivalent to $\alpha\ge(s+3)/4$ for Δ_α^{10} and $\alpha\ge(s-2q+1)/4$ for Δ_α^{11} and their codimension is bounded by

$$\operatorname{codim}\Delta_\alpha^{1J}\ge\min\left\{\tfrac{s+3}{2}-\left\lfloor\tfrac{q}{2}\right\rfloor, \tfrac{s-2q+1}{2}+\left\lceil\tfrac{q}{2}\right\rceil\right\} = \tfrac{s+1}{2}-\left\lfloor\tfrac{q}{2}\right\rfloor = k.$$

An analogous consideration for $s\equiv 3 \bmod 4$ yields the same result. Obviously, $a_{1J}\le\gamma_1+1$ implies $\operatorname{codim}\Delta_\alpha^{1J}\ge k-1$ in the same way. Finally, note that the codimension is strictly increasing in α.

Now ($\circledast$) follows from this and the following comparison statement, which we prove in a moment:

($\circledast\circledast$) Let Δ be an admissible Γ–semi–module with 2×2–basis (a_{ij}). Let J be the index with $a_{1J}=\min\{a_{10},a_{11}\}$. Assume that $a_{1J}\le\gamma_1+1$. Let Δ^{1J} be the unique special semi–module like above with the same a_{1J}. Then $\operatorname{codim}\Delta\ge\operatorname{codim}\Delta^{1J}$ with strict inequality if $a_{1J}+\gamma_1>a_{1,1-J}$.

For example, we show ($\circledast$ a). Let Δ be a semi–module as in ($\circledast\circledast$) with $a_{1J}\le\gamma_1-1$ then $\Delta^{1J}=\langle 0,\gamma_1,a_{1J},a_{1J}+\gamma_1\rangle$. Now ($\circledast\circledast$) implies $\operatorname{codim}\Delta\ge\operatorname{codim}\Delta^{1J}\ge k$ and equality holds only for $a_{1,1-J}=a_{1J}+\gamma_1$ and $a_{1J}=\gamma_1-1$. This is the statement ($\circledast$ a).

To prove the claim ($\circledast\circledast$) we modify Δ^{1J} in two steps into Δ and watch for the dimension changes. The 2×2–basis of Δ^{1J} is by definition $(0,\gamma_1;a_{1J},a_{1J}+\gamma_1)$ — up to the oder of the last two elements. Let $\alpha_{01}:=(\gamma_1-a_{01})/4$ and define $\tilde\Delta=\Delta^{1J}\cup\{a_{01},\ldots,\gamma_1-4\}$, i.e., $\tilde\Delta$ has a 2×2–basis $(0,a_{01};a_{1J},a_{1J}+\gamma_1)$. We compare its dimension

$$\dim\tilde\Delta = g_{\tilde\Delta}(0)+\#([a_{01},\gamma_1[\,\setminus\tilde\Delta)+g_{\tilde\Delta}(a_{1J})-g_{\tilde\Delta}(a_{01}+\tilde n)$$

with the dimension $g_{\Delta^{1J}}(0)+g_{\Delta^{1J}}(a_{1J})$ of Δ^{1J}. Since we are closing α_{01} gaps, we find $g_{\tilde\Delta}(0)=g_{\Delta^{1J}}(0)-\alpha_{01}$ and $g_{\tilde\Delta}(a_{1J})\le g_{\Delta^{1J}}(a_{1J})$. In the interval $[a_{01},\gamma_1[$ there are α_{01} gaps in $\tilde\Delta$ equal to $a_{1J}+\gamma_1$ modulo 4. The only other possible gaps in this interval have to be equal to a_{1J} modulo 4. Let l be one of them. Then $l+\gamma_1\equiv a_{1J}+\gamma_1 \bmod 4$ and $a_{01}<l<\min\{\gamma_1,a_{10}\}$ implies $a_{01}+\gamma_1<l+\gamma_1<\min\{a_{1J}+\gamma_1,2\gamma_1\}$; hence, $l+\gamma_1$ is also a gap in $\tilde\Delta$. Now we have either $\tilde n=\gamma_1+1$ or $\tilde n$ is the smallest number greater than γ_1 and equal to a_{1J} mod 4. In the first case we have trivially $a_{01}+\tilde n\le\gamma_1+l$; in the second case $a_{01}+\tilde n$ is the smallest number greater than $a_{01}+\gamma_1$ and equal to $a_{1J}+\gamma_1$ mod 4, and we get again $a_{01}+\tilde n\le l+\gamma_1$. Therefore, we found for any of the gaps in $([a_{01},\gamma_1[\,\setminus\Delta)\cap(a_{1J}+4\mathbb{Z})$ a gap that contributes to $g_{\tilde\Delta}(a_{01}+\tilde n)$. Summing up the changes, we obtain $\dim\tilde\Delta\le\dim\Delta^{1J}$.

We obtain Δ from $\tilde\Delta$ by closing the $\eta:=(a_{1J}+\gamma_1-a_{1,1-J})/4$ gaps $\{a_{1,1-J},a_{1,1-J}+4,\ldots,a_{1J}+\gamma_1-4\}$. Due to our assumption $a_{1J}<a_{1,1-J}$, the computation of the dimension of Δ is easy. Obviously, $g_\Delta(0)=g_{\tilde\Delta}(0)-\eta$, $g_\Delta(a_{1J})=g_{\tilde\Delta}(a_{1J})-\eta$, and $\#([a_{01},\gamma_1[\,\setminus\Delta)\le\#([a_{01},\gamma_1[\,\setminus\tilde\Delta)$. Because $n\le\tilde n$, we

find $g_\Delta(a_{01}+n) \geq g_\Delta(a_{01}+\tilde{n}) \geq g_{\tilde{\Delta}}(a_{01}+\tilde{n}) - \eta$. Finally, $g_\Delta(a_{1,1-J})$ maybe nonzero this time, but there can only be gaps equal to a_{01} modulo 4 after $a_{1,1-J}$, thus $g_\Delta(a_{1,1-J}) \leq \eta$. In fact, $g_\Delta(a_{1,1-J}) \leq \max\{\eta-1, 0\}$, using $a_{01} < a_{1J} + \gamma_1$. Summation yields $\dim\Delta + \min\{\eta, 1\} \leq \dim\tilde{\Delta} \leq \dim\Delta^{1J}$. □

Theorem 5.9 *Let X and X' be unibranched plane singularities with Puiseux exponents $(4, 2q, s)$ resp. $(4, 2q, s')$ with $s' \geq s$ and J_X resp. $J_{X'}$ their Jacobi factors. Set $k := (q+s)/2$. Then the first $k+1$ even Betti numbers of the homology of J_X and $J_{X'}$ are the same, i.e., $h_{2d}(J_X) = h_{2d}(J_{X'})$ for $d = 0, \ldots, k$.*

Proof. We will prove that $\mathrm{Mod}_d(\Gamma) = \mathrm{Mod}_d(\Gamma')$ for $d = 0, \ldots, k$, where Γ and Γ' are the semi–groups of the singularities X resp. X'. By induction we may restrict to the case $s' = s+2$. We claim the following:

$$(\ddagger) \qquad \begin{array}{lcl} a)\text{ For } \Delta \in \mathrm{Mod}_d(\Gamma) \text{ with } d \leq k & : & c(\Delta) \leq s+1. \\ b)\text{ For } \Delta \in \mathrm{Mod}_d(\Gamma) \text{ with } d \leq k-1 & : & c(\Delta) \leq s-1. \end{array}$$

The obvious consequence of a) is that any such Γ–semi–module Δ is a Γ'–semi–module as well. In fact, its dimension as a Γ–semi–module and Γ'–semi–module must be the same. Namely, the terms of the dimension formula depend only on the the 4–basis of Δ as a $\langle 4\rangle$–semi–module — with the exception of the computation of n. However, if n differs for Δ as a Γ–semi–module and Γ'–semi–module then n must be s for Δ as a Γ–semi–module and even bigger for Δ as a Γ'–semi–module; hence, $a_{01}+s \geq 2+s > c(\Delta)$ shows that $g_\Delta(a_{01}+n) = 0$ in both cases. Therefore, we have an inclusion $\mathrm{Mod}_d(\Gamma) \subseteq \mathrm{Mod}_d(\Gamma')$.

To prove equality, apply b) to a $\Delta \in \mathrm{Mod}_d(\Gamma')$. We find $c(\Delta) \leq s'-1 = s+1$, thus Δ is also a Γ–semi–module, and we have just shown that it has the same dimension d as a Γ–semi–module.

The claim $(\ddagger)$ is proven by comparing Δ with simpler semi–modules. For $c \in \mathbb{N}\setminus(1+\Gamma)$ define Δ_c as the Γ–semi–module $\Delta_c = \Gamma \cup (c+\mathbb{N})$, then $c(\Delta_c) = c$. The dimension of $\Delta_{s-1} = \Gamma + s\mathbb{N} = \langle 0, \gamma_1, s+2, s\rangle = \left\langle 0, \gamma_1, \gamma_2 - 4\left\lfloor \frac{q}{2}\right\rfloor, \gamma_2+\gamma_1-4q\right\rangle$ is

$$\dim\Delta_{s-1} = g_{\Delta_{s-1}}(0) = g_\Gamma(0) - q - \left\lfloor\frac{q}{2}\right\rfloor = \frac{\gamma_2+\gamma_1-3}{2} - q - \left\lfloor\frac{q}{2}\right\rfloor = k-1.$$

Clearly, the dimensions of $\Delta_{s+1} = \Delta_{s-1}\setminus\{s\}$ and $\Delta_{s+3} = \Delta_{s-1}\setminus\{s, s+2\}$ are k resp. $k+1$.

Now $(\ddagger)$ follows from the obvious fact that $\dim\Delta_c$ is monotone increasing and

$(\ddagger\ddagger)$ for $\Delta \in \mathrm{Mod}(\Gamma)$ with $c := c(\Delta) \geq s$ we have $\dim\Delta \geq \dim\Delta_c$.

For example, assume $c \geq s+2$. Since the even number $s+1$ lies in $\Gamma \subset \Delta$ we find $c \geq s+3$, thus $\dim\Delta \geq \dim\Delta_c \geq \dim\Delta_{s+3} = k+1$.

It remains to prove $(\ddagger\ddagger)$ by modifying Δ_c in two steps into Δ. Let $\{b_0 = 0, b_1, b_2, b_3\}$ be an ordered 4–basis of Δ_c as a $\langle 4\rangle$–semi–module. By the definition of Δ_c we find $b_1 = \gamma_1$, $c = b_3 - 3$, and $b_2 = \gamma_2$ or $b_2 = b_3 - 2$. Let $\{e_0 = 0, e_1, e_2, e_3\}$ be a 4–basis of Δ which we order such that $e_i \equiv b_i \bmod 4$. Since the greatest element of the 4–basis of Δ as well as of Δ' is $c+3$, one gets $e_3 = b_3 = c+3 > s$.

Setting $\beta := (e_2 - b_2)/4$, we define $\tilde{\Delta}$ to be the semi–module obtained by closing the β gaps $e_2, e_2+4, \ldots, e_2+4(\beta-1) = b_2-4$ in Δ_c. Let $\hat{\beta}$ be the number of these gaps that are less than γ_1. The dimensions of Δ_c and $\tilde{\Delta}$ are

$$\begin{aligned} \dim\Delta_c &= g_{\Delta_c}(0) + g_{\Delta_c}(b_2) - g_{\Delta_c}(\gamma_1+s), \\ \dim\tilde{\Delta} &= g_{\tilde{\Delta}}(0) + g_{\tilde{\Delta}}(e_2) - g_{\tilde{\Delta}}(\gamma_1+\tilde{n}), \end{aligned}$$

making use of $c \geq s \Rightarrow n_c = s$ for Δ_c. Clearly, $g_{\tilde{\Delta}}(0) = g_{\Delta_c}(0) - \beta$. Between e_2 and b_2 in $\tilde{\Delta}$ there are β gaps equal to b_3 mod 4 and $\max\{0, \hat{\beta} - 1\}$ gaps equal to $b_1 = \gamma_1$ mod 4, i.e., $g_{\tilde{\Delta}}(e_2) = g_{\Delta_c}(b_2) + \beta + \max\{0, \hat{\beta} - 1\}$. For the last term $g_{\tilde{\Delta}}(\gamma_1 + \tilde{n})$ we observe the following: If $\tilde{n} = s$ then $g_{\tilde{\Delta}}(\gamma_1 + \tilde{n}) \leq g_{\Delta_c}(\gamma_1 + s)$. If $\tilde{n} < s$ then $e_2 \leq \tilde{n}$ and from the semi–module property $e_3 \leq e_2 + \gamma_1 \leq \tilde{n} + \gamma_1$; hence, $s + \gamma_1 > \tilde{n} + \gamma_1 \geq c$ and $g_{\Delta_c}(\gamma_1 + s) = g_{\tilde{\Delta}}(\gamma_1 + \tilde{n}) = 0$. Summing up the above terms we get $\dim \tilde{\Delta} \geq \dim \Delta_c + \max\{0, \hat{\beta} - 1\}$.

We obtain the semi–module Δ from $\tilde{\Delta}$ by closing the $\eta := (e_1 - \gamma_1)/4$ gaps $e_1, e_1 + 4, \ldots, \gamma_1 - 4$. Again we need to compute the dimension

$$\dim \Delta = g_\Delta(0) + \#([e_1, \gamma_1[\, \setminus \Delta) + g_\Delta(e_2) - g_\Delta(e_1 + n).$$

Obviously, $g_\Delta(0) = g_{\tilde{\Delta}}(0) - \eta$. In the interval $[e_1, \gamma_1[$ there are η gaps equal to b_3 modulo 4 and $\max\{0, \eta - \hat{\beta}\}$ equal to e_2 modulo 4. The number of gaps after e_2 decreases from $\tilde{\Delta}$ to Δ by less than $\max\{0, \hat{\beta} - 1\}$. Therefore,

$$\begin{aligned} \dim \Delta &\geq \dim \tilde{\Delta} - \max\{0, \hat{\beta} - 1\} + \max\{0, \eta - \hat{\beta}\} - g_\Delta(e_1 + n) + g_{\tilde{\Delta}}(\gamma_1 + \tilde{n}) \\ &\geq \dim \Delta_c + \max\{0, \eta - \hat{\beta}\} - g_\Delta(e_1 + n) + g_{\tilde{\Delta}}(\gamma_1 + \tilde{n}). \end{aligned}$$

Note that $n = \tilde{n}$, since Δ and $\tilde{\Delta}$ have the same odd numbers. Further, $g_\Delta(\gamma_1 + n) = g_{\tilde{\Delta}}(\gamma_1 + n)$ because Δ and $\tilde{\Delta}$ differ only in numbers less than γ_1. Therefore, we need to count the gaps of Δ in the interval $[e_1 + n, \gamma_1 + n[$. The gaps in this interval are b_3 modulo 4, because Δ being admissible implies $b_2 \leq e_1 + s$ and $b_2 \leq e_1 + n$ using $n = \min((\{s\} \cup (b_2 + 4\mathbb{N})) \cap [\gamma_1, \infty[)$. In particular, $g_\Delta(e_1 + n) \leq g_{\tilde{\Delta}}(\gamma_1 + n) + \eta$ and the estimate $\dim \Delta \geq \dim \Delta_c$ is obvious for $\hat{\beta} = 0 \Leftrightarrow b_2 > \gamma_1$. However, for any of the $\min\{\eta, \hat{\beta}\}$ numbers $e_2 + 4l \in [e_1, \gamma_1[\cap \Delta$ we find $e_2 + \gamma_1 + 4l \in [e_1 + \gamma_1, 2\gamma_1[\cap \Delta \subset [e_1 + \gamma_1, \gamma_1 + n[\cap \Delta$. Since $e_2 + \gamma_1 + 4l \equiv b_3$ mod 4 at least $\min\{\eta, \hat{\beta}\}$ of the positions $[e_1 + n, \gamma_1 + n[\cap (b_3 + 4\mathbb{Z})$ are not gaps. This implies $g_\Delta(e_1 + n) \leq g_{\tilde{\Delta}}(\gamma_1 + \tilde{n}) + \eta - \min\{\eta, \hat{\beta}\}$ showing $\dim \tilde{\Delta} \geq \dim \Delta_c$ even in this case. $\square$

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Part II

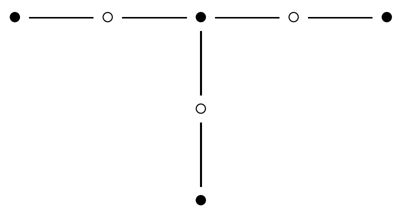
Linear Symmetric Determinantal Hypersurfaces

Introduction

The question which equations of hypersurfaces in the complex projective space can be expressed as the determinant of a matrix whose entries are linear forms is classical. In 1844 Hesse proved that a smooth plane cubic has three essentially different linear symmetric representations [He]. Dixon showed in 1904 that for smooth plane curves linear symmetric determinantal representations correspond to ineffective theta–characteristics, i.e., ineffective divisor classes whose double is the canonical divisor [Di]. Barth proved the corresponding statement for singular plane curves [B]. The general case for any hypersurface was treated by Catanese [C], Meyer–Brandis [M–B], and Beauville [Be].

Any plane curve has a linear symmetric determinantal representation [Be, 4.4], but every linear symmetric determinantal surface is singular. Already Salmon knew that such a surface of degree n possesses in general $\binom{n+1}{3}$ nodes [S, p. 495] and Cayley examined the position of these [Ca]. Catanese studied these surfaces with only nodes in a more general context [C]. Here we are dealing mainly with the question which combinations of singularities can occur on a linear symmetric determinantal cubic or quartic surface. For the cubics we find all their linear symmetric representations and obtain in particular the following theorem.

Theorem. *There are four types of linear symmetric determinantal cubic surfaces with isolated singularities. The combinations of their singularities are given by the subgraphs of* $\widetilde{E}_6$



which are obtained by removing some of the white dots. In addition, all nonnormal cubics with the exception of the union of a smooth quadric with a transversal plane are linear symmetric determinantal cubics.

The combination of isolated singularities which occur on a linear symmetric determinantal quartic can be described similarly — only with more Dynkin diagrams as starting points for the splitting process.

The author's original motivation for this study was the desire to understand linear maps from a vector space V into the space of symmetric matrices, which occur for example in the examination of focal varieties, see for example [FP, 2.2.4]. Such a map can be understood as a symmetric matrix M whose entries are linear forms on V and $\det M$ describes the locus of V which is mapped to symmetric matrices of reduced rank. For $\dim V = 2$ such maps are classified up to the choice of coordinates classically [G, 12.6]. The case of $n = 2$ and arbitrary dimension of V is easy and the case of $n = 3$ is treated in course of proving the above Theorem. For $n = 4$ and $\dim V = 3$ the classification can be obtained with the methods used here if the linear symmetric determinantal quartic is a normal rational surface. However, if the quartic has only rational singularities, the below methods are not constructive because Torelli type theorems are used. This corresponds to the fact that while every possible combination of rational double points on a quartic is known by the work of Urabe and Yang [U1, U2, Y], equations for most of these surfaces are unknown.

The author is indebted to Y.-G. Yang who sent his program for the enumeration of the combinations of rational singularities on quartics to the author. Further, the author thanks J. Nagel and D. van Straten for several discussions.

1 General definitions and statements

Definition 1.1 *Let $M \in \mathrm{Sym}(n, V^*)$ be a symmetric $n \times n$–matrix whose entries are linear forms on a vector space V over $\mathbb{C}$. If $F := \det M$ is not zero, then it determines a* linear symmetric (determinantal) hypersurface *of degree n in $\mathbb{P}(V)$. Two matrix representations M and M' of F are* equivalent *if there is a $T \in \mathrm{GL}(n, \mathbb{C})$ with $M' = T^t M T$. A matrix representation M will be called* nondegenerate *if the induced map $V \to \mathrm{Sym}(n, \mathbb{C})$, $v \mapsto M(v)$, is injective.*

We note that the hypersurface F of a degenerate matrix representation M will be a cone over the kernel of the induced map.

Often M will be obtained by choosing some matrices $A_0, \ldots, A_N$ and setting $M := \sum_{i=0}^{N} x_i A_i$, where the x_i are a basis of $(\mathbb{C}^{N+1})^*$. The representation M will be *nondegenerate* if the matrices $A_0, \ldots, A_N$ are linearly independent. Choosing different generators $A'_0, \ldots, A'_N$ of the space $\mathrm{span}\{A_0, \ldots, A_N\}$ corresponds to a projective transformation of $\mathbb{P}^N$. Thus the hypersurface $F = \det M$ is determined up to projective equivalence by the choice of the linear space $\mathcal{A} := \mathrm{span}\{A_i\} \subseteq \mathrm{Sym}(n, \mathbb{C})$. In fact, we may view F as the intersection of $\mathbb{P}(\mathcal{A}) \subseteq \mathbb{P}(\mathrm{Sym}(n, \mathbb{C}))$ with the general determinantal hypersurface $V(\det)$ or a cone over such a construction, in case we started with a degenerate representation.

One might expect that the linear symmetric hypersurfaces form a Zariski–closed subset of all hypersurfaces of degree n. However, this may be false because the map

$$\mathbb{P}(\mathrm{Sym}(n, V^*)) - - \to \mathbb{P}(\text{polynomials of degree } n), \; [M] \mapsto [\det M],$$

is only a rational map and not regular for $n \geq 2$; hence, the set of linear symmetric hypersurfaces is only constructible.

As it is well known, the locus of corank 1 matrices is precisely singular along the locus of corank ≥ 2 matrices. Therefore, singularities of F appear if either $\mathbb{P}(\mathcal{A})$ intersects $V(\det)$ at a corank ≥ 2 matrix or tangentially at a corank 1 matrix. We use this in the following definition.

Definition 1.2 *A singular point $x \in F$ is called an* essential *singularity if the corank of $M(x)$ is greater or equal to 2, otherwise an* accidental *singularity.*

The accidental singularities are difficult to control. Luckily, we can prove that for small sizes of the matrix M only certain types of singularities can occur.

Proposition 1.3 *Let F be a linear symmetric determinantal hypersurface of degree n in $\mathbb{P}^N$. Then the isolated accidental singularities of F are of corank less or equal to $n - N - 1$. (Here corank denotes the corank of the Hesse matrix of F at the singular point.)*

In particular, a linear symmetric cubic in $\mathbb{P}^3$ has no isolated accidental singularities, a quartic only nodes, and a quintic only A_k–singularities.

Before we start with the proof of the proposition, we show the following lemma, which enables us to identify some of the nonisolated singularities of a linear symmetric hypersurface. This statement was already known to Salmon [S, p. 495].

Lemma 1.4 *Let $M = (m_{ij})$ be a linear symmetric $n \times n$–matrix with $m_{11} = 0$. Then the hypersurface $F = \det M$ is singular along $V(m_{12}, \ldots, m_{1n})$.*

Proof. We expand the determinant F by the Leibniz formula. Then each summand of

$$\frac{\partial F}{\partial x_j} = \sum_{\sigma \in S(n)} \sum_{i=1}^{n} \operatorname{sgn} \sigma \cdot m_{1\sigma(1)} \cdot \ldots \frac{\partial m_{i\sigma(i)}}{\partial x_j} \ldots \cdot m_{n\sigma(n)},$$

contains $\frac{\partial m_{11}}{\partial x_j} = 0$, $m_{1\sigma(1)}$, or $m_{\sigma^{-1}(1)1} = m_{1\sigma^{-1}(1)}$; hence, it vanishes on $V(m_{12}, \ldots, m_{1n})$. □

Proof of Proposition 1.3. Assume that we are examining the point $p = (1 : 0 : \ldots : 0)$. Because p is an accidental singularity, corank $A_0 = 1$ and we can choose coordinates on the $\mathbb{C}^r$ such that

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We set $x_0 = 1$ and write

$$M = \begin{pmatrix} f_{11} & f_{12} & f_{13} & \cdots & f_{1n} \\ f_{12} & 1 + f_{22} & f_{23} & \cdots & f_{2n} \\ f_{13} & f_{23} & \ddots & & f_{3n} \\ \vdots & \vdots & & \ddots & \vdots \\ f_{1n} & f_{2n} & f_{3n} & \cdots & 1 + f_{nn} \end{pmatrix} \qquad \text{with } f_{ij} \in \mathbb{C}[x_1, \ldots, x_N].$$

Obviously, the linear part of $F = \det M$ is f_{11}, which has to vanish for p to be singular. Looking at

$$F = \det M = \sum_{\sigma \in S(n)} \operatorname{sgn} \sigma \cdot m_{1\sigma(1)} \cdot \ldots \cdot m_{n\sigma(n)},$$

we see that the quadratic part of F is due to the summands where $n - 2$ of the $m_{i\sigma(i)}$ are of order 0, i.e., where σ is a transposition of 1 with $i \in \{2, \ldots, n\}$; hence the quadratic part of F is

$$-\sum_{i=2}^{n} (f_{1i})^2.$$

The Hessian of F in p is the associated symmetric $N \times N$–matrix S of this quadric. Our task is to show that the rank of S is at least $2N - n + 1$. By Lemma 1.4 there are N linearly independent forms among $f_{12}, \ldots, f_{1n}$ because the point p is an isolated singularity. Let us assume that $f_{12}, \ldots, f_{1N+1}$ are linearly independent, then the associated symmetric $N \times N$–matrix $\tilde{S}$ of the quadric $-\sum_{i=2}^{N+1} f_{1i}^2$ has rank N. The symmetric matrices $S_{N+2}, \ldots, S_n$ associated to $f_{1N+2}^2, \ldots, f_{1n}^2$ have rank 1 or 0. From $\tilde{S} = S + \sum_{i=N+2}^{n} S_i$ we find

$$N = \operatorname{rank} \tilde{S} \leq \operatorname{rank} S + \sum_{i=N+2}^{n} \operatorname{rank} S_i \leq \operatorname{rank} S + n - N - 1,$$

thus $\operatorname{rank} S \geq 2N - n + 1$. □

Remark. We will see soon that an essential singularity of a linear symmetric hypersurface can never be an A_{2k}–singularity, but an accidental singularity may as well be one. For example the quintic given as the determinant of the matrix

$$\begin{pmatrix} 0 & x & y & z & \sqrt{-1}z \\ x & w+z & 0 & 0 & 0 \\ y & 0 & w+y & 0 & 0 \\ z & 0 & 0 & w+x & \sqrt{-1}z \\ \sqrt{-1}z & 0 & 0 & \sqrt{-1}z & w \end{pmatrix}$$

has an A_2–singularity at $(1:0:0:0)$. It seems likely that as the size of the matrix increases all types of singularities will occur as accidental singularities.

We turn to the examination of the essential singularities. First, we will count them. The following statement was already known to Salmon [S, p. 495].

Proposition 1.5 *The general linear symmetric determinantal hypersurface F of degree n has only essential singularities and its singular locus has codimension 2 and degree $\binom{n+1}{3}$.*

In particular, a general linear symmetric surface $F \subset \mathbb{P}^3$ has $\binom{n+1}{3}$ essential A_1–singularities.

Proof. For the first statement we view F as $V(\det) \cap \mathbb{P}(\mathcal{A}) \subseteq \mathbb{P}(\mathrm{Sym}(n,\mathbb{C}))$. A general linear space $\mathbb{P}(\mathcal{A}) \subseteq \mathbb{P}(\mathrm{Sym}(n,\mathbb{C}))$ intersects $V(\det)$ transversally, thus there are no accidental singularities. The locus of corank ≥ 2 matrices has codimension 3 in $\mathbb{P}(\mathrm{Sym}(n,\mathbb{C}))$ and degree $\binom{n+1}{3}$ [HT]. As its intersection with $\mathbb{P}(\mathcal{A})$ consists of the essential singularities of F, the first statement follows.

For the second statement we prove that a general essential singularity is a node with arguments similar to the ones in the proof of Proposition 1.3. We assume that the essential singularity is at $p = (1:0:0:0)$. Because of the generality of the linear space $\mathcal{A}$, the matrix A_0 has corank 2, thus we may choose coordinates such that A_0 is the diagonal matrix with $(0,0,1,\ldots,1)$ in the diagonal. Thus at $w=1$ we can write

$$M = \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} & \cdots & f_{1n} \\ f_{12} & f_{22} & f_{23} & f_{24} & \cdots & f_{2n} \\ f_{13} & f_{23} & 1+f_{33} & f_{34} & \cdots & f_{3n} \\ f_{14} & f_{24} & f_{34} & \ddots & & f_{4n} \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ f_{1n} & f_{2n} & f_{3n} & f_{4n} & \cdots & 1+f_{nn} \end{pmatrix} \qquad \text{with } f_{ij} \in \mathbb{C}[x,y,z].$$

The quadric part of $f = \det M$ is $f_{11}f_{22} - f_{12}^2$. The generality of the choice of $\mathcal{A}$ still implies that the linear forms f_{11}, f_{12}, f_{22} are linearly independent; hence, the quadric $f_{11}f_{22} - f_{12}^2$ has rank 3, and F has an A_1–singularity in p. □

In order to examine the essential singularities further we localize our definitions.

Definition 1.6 *A* local symmetric matrix representation *of a power series $f \in \mathbb{C}[[x_1,\ldots,x_N]]$ is a symmetric matrix $M \in \mathrm{Sym}(r, \mathbb{C}[[x_1,\ldots,x_N]])$ with $\det M = f$. Two matrix representations M and M' are* equivalent *if there exists a $T \in \mathrm{GL}(r, \mathbb{C}[[x_1,\ldots,x_N]])$ such that $M' = T^tMT$. A matrix representation M is* essential *if $\operatorname{corank} M(0) \geq 2$ and* reduced *if $M(0) = 0$.*

If one considers the equation of the power series f only up to a choice of holomorphic coordinates, it is convenient to extend the above definition of equivalence by allowing changes of coordinates as well. It is enough to consider only reduced matrix representations due to the following well–known lemma.

Lemma 1.7 *Any local symmetric matrix representation M of a power series $f \in \mathbb{C}[[x_1, \dots, x_N]]$ is equivalent to*

$$\begin{pmatrix} \tilde{M} & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix},$$

where $\tilde{M}$ is a reduced local symmetric matrix representation of f.

Not every singularity has an essential local symmetric matrix representation. For the ADE–singularities we have the following

Theorem 1.8 *The surface singularities A_{2k}, E_6, E_8 have no essential local symmetric matrix representation. The reduced essential symmetric matrix representations for A_{2k+1}, D_{2k}, D_{2k+1}, E_7 are up to equivalence:*

singularity X	*equation*	*matrix representation M*	$l(X)$
A_{2k+1} ($A^{\bullet}_{2k+1}$)	$-x^2+y^2-z^{2k+2}$	$\begin{pmatrix} y+z^{k+1} & x \\ x & y-z^{k+1} \end{pmatrix}$	$k+1$
D_{2k} ($D^{\bullet}_{2k}$)	$-x^2+y^2z-z^{2k-1}$	$\begin{pmatrix} z & x \\ x & y^2-z^{2k-2} \end{pmatrix}$	2
D_{2k} ($D^{\pm}_{2k}$)		$\begin{pmatrix} y \pm z^{k-1} & x \\ x & z(y \mp z^{k-1}) \end{pmatrix}$	k
D_{2k+1} ($D^{\bullet}_{2k+1}$)	$-x^2+y^2z-z^{2k}$	$\begin{pmatrix} z & x \\ x & y^2-z^{2k-1} \end{pmatrix}$	2
E_7 ($E^{\bullet}_7$)	$-x^2+z^3+zy^3$	$\begin{pmatrix} z & x \\ x & z^2+y^3 \end{pmatrix}$	3

The symbols in brackets in the first column denote the specific matrix representation of the singularity from now on. The last column gives the length of the first Fitting ideal, F_1M, of the matrix representation of the singularity, which is here the ideal generated by the entries of the matrix.

Proof. Let M be a local symmetric matrix representation of an ADE–singularity, which is given by the equation $f = \det M$. We set $R = \mathbb{C}[[x,y,z]]/(f)$. Then $\hat{M} = \operatorname{coker} M$ is a maximal Cohen–Macaulay module of rank 1 [Yo, Chap. 7]. Due to the symmetry of M, we obtain a surjection $\hat{M} \twoheadrightarrow \operatorname{Hom}_R(\hat{M}, R)$. Such a module $\hat{M}$ is called a contact module. It determines the matrix M up to equivalence ([KU, §2] or [M–B, 3.34]). Over the local ring of an ADE–surface singularity there exists only a finite number of irreducible modules. This was proven by Auslander as follows: Recall that for each of the ADE–surface singularities there exists a group $G \subset \mathrm{GL}(2, \mathbb{C})$ such that the invariant subring $\mathbb{C}[[x,y]]^G$ is isomorphic to the local ring R of the singularity. Auslander exhibited a bijection between these irreducible modules and the irreducible representations of G [Yo, Chap. 10].

Since a contact module has rank 1, we are only interested in the irreducible rank 1 modules, not isomorphic to R. There are k for A_k, 3 for D_k, 2 for E_6, 1 for E_7, and none for E_8 [Yo, p. 95]. This already proves the claim for D_{2k}, E_7, and E_8. For the other singularities one uses the Auslander's bijection to work out the following representation matrices for the irreducible modules of rank 1:

singularity	standard equation	representation matrix
A_k	$-xy + z^{k+1}$	$M_i = \begin{pmatrix} z^i & y \\ x & z^{k+1-i} \end{pmatrix}$ for $1 \le i \le k$
D_{2k+1}	$-x^2 + zy^2 + z^{2k}$	$M_0 = \begin{pmatrix} z & x \\ x & y^2 + z^{2k-1} \end{pmatrix}$
		$M_\pm = \begin{pmatrix} z^k \pm x & yz \\ y & z^k \mp x \end{pmatrix}$
E_6	$x^2 - y^3 - z^4$	$M_\pm = \begin{pmatrix} x \pm z^2 & y^2 \\ y & x \mp z^2 \end{pmatrix}$

Kleiman and Ulrich showed that if an R–module of rank 1 represented by an $r \times r$–matrix M is a contact module then there exists a matrix $T \in \mathrm{GL}(r, \mathbb{C}[[x,y,z]])$ such that TM is symmetric [KU, 2.2]. As we are dealing only with 2×2-matrices, this condition is very easy to check. Let

$$T = \begin{pmatrix} g_1 & g_2 \\ f_2 & f_1 \end{pmatrix} \qquad \text{with } \; g_1 f_1 - g_2 f_2 \in \mathbb{C}[[x,y,z]]^*.$$

For A_k the symmetry of TM_i is equivalent to

$$x f_1 + z^i f_2 = y g_1 + z^{k+1-i} g_2.$$

Clearly, we have $f_1(0) = g_1(0) = 0$, which implies $f_2(0) \cdot g_2(0) \neq 0$ for T to be invertible. Therefore, $i = k + 1 - i$, i.e., $k + 1$ is even and $i = (k + 1)/2$. Hence, there can be no contact module for A_{2k} and only one for A_{2k+1}. In case of D_{2k+1} we need to check the modules represented by $M_\pm$. The matrix $TM_\pm$ is symmetric iff

$$y f_1 + (z^k \pm x) f_2 = yz g_1 + (z^k \mp x) g_2.$$

Again, we see that $f_1(0) = 0$, which implies $f_2(0) \cdot g_2(0) \neq 0$. Looking at the x– and z^k–term, we see that this is impossible. A completely analogous argument works for E_6.

The computation of the length of the Fitting ideals is simple. Denoting $S := \mathbb{C}[[x,y,z]]$ we have

$$\begin{aligned}
l(A^\bullet_{2k+1}) &= \dim S/(x, y + z^{k+1}, y - z^{k+1}) &&= \dim S/(x, y, z^{k+1}) &&= k+1 \\
l(D^\bullet_{2k}) &= \dim S/(x, z, y^2 - z^{2k-2}) &&= \dim S/(x, z, y^2) &&= 2 \\
l(D^\pm_{2k}) &= \dim S/(x, y \pm z^{k-1}, z(y \mp z^{k-1})) &&= \dim S/(x, z^k, y \pm z^{k-1}) &&= k \\
l(D^\bullet_{2k+1}) &= \dim S/(x, z, y^2 - z^{2k-1}) &&= \dim S/(x, z, y^2) &&= 2 \\
l(E^\bullet_7) &= \dim S/(x, z, z^2 + y^3) &&= \dim S/(x, z, y^3) &&= 3
\end{aligned}$$

□

Often it does not make much sense to distinguish between the representations D^+_{2k} and D^-_{2k} because the automorphism of the local ring of the singularity induced by $x \mapsto -x$, $y \mapsto -y$ swaps them. The above theorem restricts the possible combinations of essential singularities on a linear symmetric surface severely:

Corollary 1.9 *Let F be a linear symmetric determinantal surface of degree n in $\mathbb{P}^3$ whose essential singularities $X_1, \ldots, X_t$ are ADE–singularities. Then $X_i \in \{A^{\bullet}_{2k+1},\ D^{\bullet}_{2k+1},\ D^{\bullet}_{2k},\ D^{\pm}_{2k},\ E^{\bullet}_{7}\}$ and*

$$\sum_{i=1}^{t} l(X_i) = \binom{n+1}{3}.$$

Proof. This follows similarly to Proposition 1.5. We view F as $\mathbb{P}(\mathcal{A}) \cap V(\det) \subset \mathbb{P}(\mathrm{Sym}(n, \mathbb{C}))$. Let I_i be the vanishing ideal of symmetric matrices of corank $\geq i$. Then we find the essential singularities as the intersection of $\mathbb{P}(\mathcal{A})$ and $V(I_2)$, thus the sum of their intersection multiplicity is $\deg V(I_2) = \binom{n+1}{3}$. From the above theorem we see that the essential ADE–singularities appear only at corank 2 matrices and never at matrices of higher corank. As $V(I_2)$ is smooth outside $V(I_3)$, the local intersection multiplicities of $\mathbb{P}(\mathcal{A})$ and $V(I_2)$ can be found by computing locally the length of the sum of the ideal I_2 and the vanishing ideal of $\mathbb{P}(\mathcal{A})$, i.e., by computing locally the length of the first Fitting ideal of matrix representation. This was done for the various singularities in the above theorem. $\square$

Remark. We will see later that linear symmetric cubics and quartics cannot have D_{2k+1}–singularities. However, here is a quintic with an essential D_5–singularity at $x = y = z = 0$, showing that essential D_{2k+1}–singularities are in fact possible.

$$\begin{pmatrix} 0 & 665x & -2y+z & 3y+z & 2y+4z \\ 665x & 2y & -2771x & 6606x & 7138x \\ -2y+z & -2771x & 26y-6z & 0 & 4z+w \\ 3y+z & 6606x & 0 & w & 0 \\ 2y+4z & 7138x & 4z+w & 0 & 224y+136z \end{pmatrix}$$

A linear symmetric representation of $F \subset \mathbb{P}^3$ is closely related to the contact surfaces of F; a surface G is a contact surface if the intersection $G \cap F$ is twice a curve C. These partially classical ideas, which are connected with the Hilbert–Burch theorem, were recently refined by Beauville [Be], Catanese [C], Eisenbud [Ei2], Kleiman and Ulrich [KU], and Meyer–Brandis [M–B]. The next few pages are devoted to extend Catanese's results for even sets of nodes to sets of ADE–singularities.

While studying contact surfaces one also encounters nonlinear symmetric matrices, thus the following definition will be useful.

Definition 1.10 *A symmetric matrix $M = (m_{ij}) \in \mathrm{Sym}(r, \mathbb{C}[x_0, \ldots, x_N])$ is* homogeneous *if all its entries are homogeneous polynomials and* $\deg m_{ii} + \deg m_{jj} = 2 \deg m_{ij}$ *for all $i, j = 1, \ldots, r$. The degree of M is $\deg M := (d_1, d_2, \ldots, d_r)$, where $d_i := \deg m_{ii}$. By permutation of the rows and columns one can achieve that $d_1 \leq d_2 \leq \ldots \leq d_r$. For a homogeneous matrix M the determinant $F = \det M$ is a homogeneous polynomial of degree $n = \sum_{i=1}^{r} d_i$. Such an F is called a* symmetric (determinantal) hypersurface.

M is linear if and only if $d_1 = \ldots = d_r = 1$. A consequence of the homogeneity of M is that the adjoint matrix, $\operatorname{adj} M$, of M is homogeneous, too. From the adjoint matrix one obtains contact surfaces. Various versions of the following well–known lemma have appeared in the literature, we repeat the proof for the reader's convenience.

Lemma 1.11 *Let $F = \det M$ be a symmetric surface in $\mathbb{P}^3$ and m^{ii} a diagonal entry of* $\operatorname{adj} M$. *Assume that no component of* $\operatorname{div}_F(m^{ii})$ *is contained in the essential singular locus of F. Then $\operatorname{div}_F(m^{ii}) = 2C$, where C is a Cartier divisor outside the essential singularities of F.*

Proof. The proof is based on the Laplace identity ([KU, 2.4] or [C, (1.3)]) which states

$$Fm^{ik,jl} = m^{kj}m^{il} - m^{kl}m^{ij},$$

where m^{ij} are the entries of the adjoint matrix and $m^{ik,jl}$ is $(-1)^{i+j+k+l}$–times the determinant of the matrix M with the rows i,k and columns j,l deleted. Setting $k=j$, $l=i$, we have $m^{ii}m^{jj} = \left(m^{ij}\right)^2$ modulo F, thus

$$\mathrm{div}_F(m^{ii}) + \mathrm{div}_F(m^{jj}) = 2\mathrm{div}_F(m^{ij}). \qquad (*)$$

This formula also implies that at the zero locus of $m^{11} = \ldots = m^{rr} = 0$ on F all m^{ij} and with them $\mathrm{adj}\, M$ vanish. Therefore, the divisors $\mathrm{div}_F(m^{ii})$ cannot have a common component outside the essential singular locus, hence $(*)$ shows that all components in $\mathrm{div}_F(m^{ii})$ occur with even multiplicity. Finally, $m^{ii}m^{jj} = \left(m^{ij}\right)^2 \mod F$ shows that C is Cartier outside the essential singularities. □

If one uses instead of only M all equivalent matrix representations of F, one obtains a whole system of contact surfaces [M–B, §2.1]. From now on we restrict our attention to symmetric surfaces whose essential singularities are ADE–singularities. To understand their contact surfaces it is important to examine the local symmetric ADE–singularities, found in Theorem 1.8.

Definition 1.12 *Let $X \in \{A^{\bullet}_{2k+1}, D^{\bullet}_{k}, D^{\pm}_{2k}, E^{\bullet}_{7}\}$ be one of the essential symmetric surface singularities with equation $f = \det M$. The* Fitting cycle *of X on the minimal resolution $\pi : \tilde{X} \to X$ is defined as*

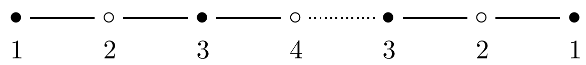
$$Z_X := \gcd\{\mathrm{div}_{\tilde{X}}(\pi^* g) \mid \textit{for all} \;\; g \in F_1 M\}.$$

Let g be a local contact surface induced by M, for example one of the main corank 1 minors. The parity diagram *of X is the minimal resolution graph G_X of X where the vertices are marked as follows: A vertex of G is drawn as $\bullet$ if the corresponding curve occurs with odd multiplicity in the total transform $\pi^* g$ of g, otherwise it is drawn as $\circ$.*

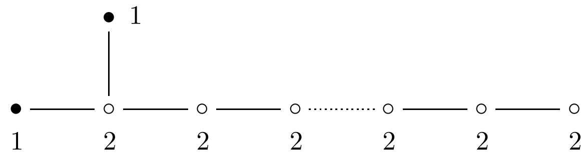
The generalized Laplace identity [M–B, 2.2] implies that the parity diagrams are the same for equivalent matrix representations and thus well–defined. Let us compute them.

Proposition 1.13 *The essential symmetric surface ADE–singularities have the following parity diagrams and Fitting cycles: (The multiplicity of an exceptional rational curve in the Fitting cycle is noted near the vertex representing this curve* *in the Dynkin diagram.)*

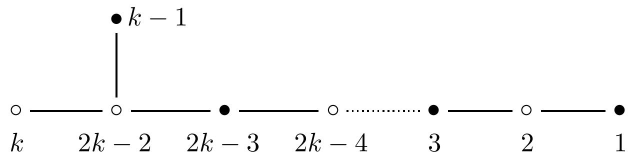
$A^\bullet_{2k+1}$



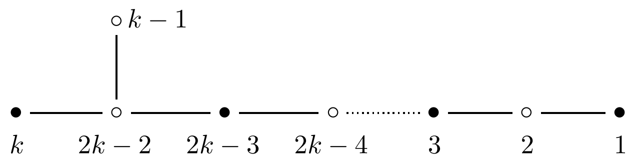
$D^\bullet_{2k+1}$, $D^\bullet_{2k}$



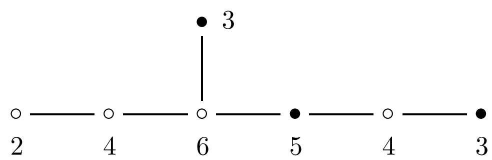
$D^\pm_{2k}$, *k even*



$D^\pm_{2k}$, *k odd*



$E^\bullet_7$



In particular, the number of the $\bullet$–vertices in the parity diagram is the length of the first Fitting ideal of the matrix representation and the self–intersection number of the Fitting cycle is -2 times the length of the Fitting ideal, i.e., $(Z_X)^2 = -2l(X)$. Further, $Z_X \cdot E \leq 0$ for any exceptional curve E.

Proof. Because of Theorem 1.8 we need only to resolve the singularities while keeping track of the divisors given by the matrix entries. Such a task is traditionally left to the interested reader. $\square$

We return to the global situation.

Definition 1.14 *Let $F \subset \mathbb{P}^3$ be a surface and $P = \{p_1, \ldots, p_t\} \subset F$ a set of singular points of type A_{2k+1}, D_k, or E_7 on F. To each of this singularities assign an essential symmetric surface ADE–singularity symbol of the same underlying type, i.e., for A_{2k+1}, D_{2k+1}, and E_7 one uses $A^\bullet_{2k+1}$, $D^\bullet_{2k+1}$, and $E^\bullet_7$ respectively, but for D_{2k} one may choose between $D^\bullet_{2k}$ and $D^\pm_{2k}$. Let $X = \{X_1, \ldots, X_t\}$ be the resulting set. Further, let $\pi : \tilde{F} \to F$ be the minimal resolution of F in these points and H the pullback of a hyperplane divisor of F to $\tilde{F}$.*

The set X is said to be even *if the divisor $\delta H + \sum_{i=1}^t Z_{X_i}$ for some $\delta \in \{0, 1\}$ is divisible by 2 in $\mathrm{Pic}(\tilde{F})$.*

X is strictly even *if $\delta = 0$, otherwise* weakly even*.*

The set X is called (linearly) symmetric if there is a (linearly) homogeneous symmetric matrix M with $F = \det M$ such that X is precisely the set of essential symmetric singularities of F.

Note that in the case of a symmetric set of ADE–singularities the pullbacks of the entries of the adjoint matrix of M define the cycle $\sum_{i=1}^t Z_{X_i}$ schemetheoretically by the definition of the Fitting cycles.

Proposition 1.15 *A symmetric set of ADE–singularities is even.*

Proof. Let $F = \det M$ be the surface which has X as essential singularities and G a contact surface given by a main corank 1 minor of M. Set $l = \deg G$ and $C = \frac{1}{2}\mathrm{div}_F(G)$. Pulling G back to the minimal resolution $\pi : \tilde{F} \to F$ we find $\pi^*G = 2\tilde{C} + D$, where $\tilde{C}$ is the strict transform of C and D a divisor supported on the exceptional set. By the definition of the Fitting cycle $D - \sum_{i=1}^t Z_{X_i}$ is effective as well. Further, from Proposition 1.13 we see that for all singularities the parity of the multiplicity of the exceptional rational curves in the Fitting cycle is the same as the one in the pullback of a contact surface, thus $D - \sum_{i=1}^t Z_{X_i}$ is divisible by 2, say $D - \sum_{i=1}^t Z_{X_i} = 2B$. Altogether we have with $\delta = l - 2\lfloor l/2 \rfloor$

$$\begin{aligned} \mathrm{div}_{\tilde{F}}\pi^*G = lH = \sum_{i=1}^t Z_{X_i} + 2(\tilde{C} + B) \\ \Longrightarrow \sum_{i=1}^t Z_{X_i} + \delta H = 2(\left\lceil \tfrac{l}{2} \right\rceil H - \tilde{C} - B), \end{aligned}$$

i.e., $\sum_{i=1}^t Z_{X_i} + \delta H$ is divisible by 2 in $\mathrm{Pic}(\tilde{F})$. □

We want to ensure the existence of contact surfaces for an even set of ADE–singularities with the same properties as G in the above proof.

Proposition 1.16 *Let X be an even set of ADE–singularities on a surface $F \subset \mathbb{P}^3$ and $\pi : \tilde{F} \to F$ the minimal resolution of F in these singular points. Then there exists a surface $G \subset \mathbb{P}^3$ such that its pullback divisor $\mathrm{div}_{\tilde{F}}\pi^*G$ on $\tilde{F}$ contains the Fitting cycles Z_{X_i} for $X_i \in X$ and the effective divisor $\mathrm{div}_{\tilde{F}}\pi^*G - \sum_{i=1}^t Z_{X_i} \in \mathrm{Pic}(\tilde{F})$ is divisible by 2.*

Proof. The proof is the same as the second half of [C, 2.6], we repeat it here because it is short and helps to understand the rest of the section. Let L be a divisor such that $2L = \sum_{i=1}^t Z_{X_i} + \delta H$. Choose l such that $lH - L$ is linearly equivalent to an effective divisor $\tilde{C}$. Then $(2l - \delta)H = 2\tilde{C} + \sum_{i=1}^t Z_{X_i}$; hence, there exists a surface of degree $2l - \delta$ with the required properties. □

From now on the theory of the even sets of ADE–singularities is the same as Catanese's theory of even nodes [C, 2.16–2.23]. We repeat the statements, but leave out the proofs if they are identical with the ones in the node case.

Definition 1.17 *The* order *of an even set of ADE–singularities of F is the smallest degree of a surface with the same properties as G in the above proposition.*

Definition 1.18 *Let X be an even set of ADE–singularities on F and $L \in \mathrm{Pic}(\tilde{F})$ such that $2L = \delta H + \sum_{i=1}^t Z_{X_i}$. Let S be the graded ring $\mathbb{C}[x, y, z, w]$. The associated graded S–module of X is*

$$R^- = \bigoplus_{l=0}^{\infty} H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) = \bigoplus_{l=0}^{\infty} H^0(F, (\pi_*\mathcal{O}_{\tilde{F}}(-L))(l)).$$

Note that if $w \in H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L))$ and $w' \in H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(l'H - L))$ then $ww' \in H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}((l+l')H - 2L)) = H^0(F, \mathcal{O}_F(l+l'-\delta))$. In particular, if l is the smallest number for which $R^-_l \neq 0$ then X has order $2l - \delta$ by Proposition 1.16.

Lemma 1.19 $H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) \cong H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}((l-\delta)H + L))$.

Proof. (Compare [C, 2.15].) From the long exact cohomology sequence associated to

$$0 \to \mathcal{O}_{\tilde{F}}(lH - L) \to \mathcal{O}_{\tilde{F}}((l-\delta)H + L) \to \bigoplus_{i=0}^{t} \mathcal{O}_{Z_{X_i}}(L) \to 0$$

we see that it is enough to show that the cohomology group $H^0(Z_{X_i}, \mathcal{O}_{Z_{X_i}}(L))$ vanishes. If there exists a section $s \in H^0(Z_{X_i}, \mathcal{O}_{Z_{X_i}}(L))$ then $s^2 \in H^0(Z_{X_i}, \mathcal{O}_{Z_{X_i}}(2L)) = H^0(Z_{X_i}, \mathcal{O}_{Z_{X_i}}(Z_{X_i} + \delta H + \sum_{j\neq i} Z_{X_j}))$, but the last homology group is zero by [R, Ex. 4.14]. $\square$

Theorem 1.20 *If X is a symmetric set of ADE–singularities on a reduced surface $F = \det M$, then the associated module R^- is a Cohen–Macaulay S–module.*

More precisely, if $\deg M = (d_1, \ldots, d_r)$, set $k_i = (n+\delta-d_i)/2$, $l_j = (n+\delta+d_j)/2$, where $n = \deg F$ and $\delta = n - d_i \bmod 2$. Then there exists a minimal set of generators $w_1, \ldots, w_r$ of R^- of degrees $k_1, \ldots, k_r$ such that $w_i w_j = m^{ij}$, where $(m^{ij}) = \operatorname{adj} M$. Moreover R^- admits the minimal free resolution

$$0 \to \bigoplus_{j=1}^{r} S[-l_j] \stackrel{(m_{ij})}{\to} \bigoplus_{i=1}^{r} S[-k_i] \stackrel{(w_j)}{\to} R^- \to 0.$$

The order of X is $n - \max\{d_i\}$.

Theorem 1.21 *Let F be an irreducible and reduced surface of degree n and X an even set of ADE–singularities on F. Then the following conditions are equivalent:*

- *X is symmetric.*
- *Let $w_1, \ldots, w_r$ be a minimal set of homogeneous generators for the S–module R^-. Set $m^{ij} = w_i w_j \in \bigoplus_{l=0}^{\infty} H(F, \mathcal{O}(l)) = S/(F)$. Then $\det(m^{ij})$ is a nonzero polynomial of degree $n(r-1)$.*
- *R^- is a Cohen–Macaulay S–module.*
- *$H^1(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) = 0 \quad \forall l \in \mathbb{Z}$.*

Catanese's construction of the symmetric matrix is such that none of the matrix entries is a nonzero constant, because the set of generators of R^- was chosen to be minimal.

Proposition 1.22 *Let F be a surface of degree n with an even set X of ADE–singularities.*

If $l(X) := \sum_{i=1}^{t} l(X_i) \leq \binom{n+1}{3}$, then X has order $\leq n-1$.

If $l(X) = \binom{n+1}{3}$ and $n = \delta \bmod 2$, then n is divisible by 8.

Proof. (Following [C, 2.21].) By the remark after the defintion 1.18 it suffices to show that $h^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) \neq 0$ for $2l - \delta \geq n - 1$ or even only for $2l \geq n-1$, observing that the order of X is an element of $2l - \delta + 2\mathbb{N}$. By Serre duality and Lemma 1.19

$$h^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH - L)) = h^0(\tilde{F}, \mathcal{O}_{\tilde{F}}((n-4-l)H + L)) = h^0(\tilde{F}, \mathcal{O}_{\tilde{F}}((n-4-l+\delta)H - L)).$$

Since $n-4-l+\delta \le l-3+\delta < l$, we get $h^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH-L)) \le h^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH-L))$, and it is enough to show $\chi(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH-L)) > 0$. Using $(\sum Z_{X_i})^2 = -2l(X)$ from Proposition 1.13 we get from Riemann–Roch:

$$\begin{aligned}
\chi(\tilde{F}, \mathcal{O}_{\tilde{F}}(lH-L)) &= \chi(\tilde{F}, \mathcal{O}_{\tilde{F}}) + \tfrac{1}{2}(lH-L)(lH-L-(n-4)H) \\
&= \chi(F, \mathcal{O}_F) + \tfrac{1}{2}\left((l-\tfrac{\delta}{2})H - \tfrac{1}{2}\textstyle\sum Z_{X_i}\right)\left((l-n+4-\tfrac{\delta}{2})H - \tfrac{1}{2}\textstyle\sum Z_{X_i}\right) \\
&= 1 + \binom{n-1}{3} + \tfrac{1}{2}\left((l-\tfrac{\delta}{2})\left(l-n+4-\tfrac{\delta}{2}\right)n - \tfrac{1}{2}l(X)\right).
\end{aligned}$$

It is not hard to see that this term is positive for $2r \ge n-1$ and $l(X) \le \binom{n+1}{3}$. For further reference we note that for $l(X) = \binom{n+1}{3}$ and $n-1 = \delta \bmod 2$ one finds

$$\chi(\tilde{F}, \mathcal{O}_{\tilde{F}}(\lfloor\tfrac{n}{2}\rfloor H - L)) = n \qquad \text{and} \qquad \chi(\tilde{F}, \mathcal{O}_{\tilde{F}}((\lfloor\tfrac{n}{2}\rfloor - 1)H - L)) = 0.$$

For $l(X) = \binom{n+1}{3}$ and $n = \delta \bmod 2$ we get

$$\chi(\tilde{F}, \mathcal{O}_{\tilde{F}}((\lceil\tfrac{n}{2}\rceil - 1)H - L)) = \frac{3n}{8} \in \mathbb{Z},$$

showing that n is divisible by 8. □

Theorem 1.23 *Let X be an even set of ADE–singularities on a reduced surface $F \subset \mathbb{P}^3$ of degree n: Then X is linearly symmetric if and only if X has length $\binom{n+1}{3}$ and order $n-1$.*

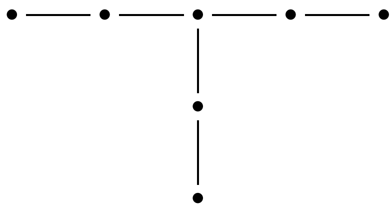
Proof. If X is linearly symmetric, its order is $n-1$ by Theorem 1.20 and its length was computed in Corollary 1.9. Alternatively, one can compute the length with the arguments in the proof of the above Proposition using Theorem 1.21. For the nontrivial reverse implication of the theorem we refer to Catanese's proof of [C, 2.23]. □

Catanese showed by example that in general the hypothesis on the order of X cannot be dropped.

2 Cubics

Before we study the determinantal cubics we recall the following beautiful theorem about cubics in $\mathbb{P}^3$, see Bruce and Wall [BW] or Looijenga [L].

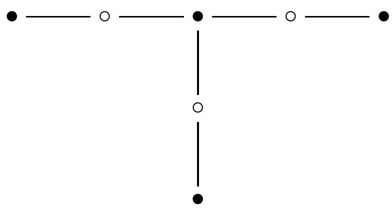
Theorem 2.1 *The combinations of singularities which can occur on a normal cubic surface in* $\mathbb{P}^3$ *are precisely the subgraphs of* $\tilde{E}_6$



which one obtains by removing some of the points. The nonnormal cubics are the cones over plane cubic curves, the reducible cubics, and two special irreducible types.

We want to prove a similar statement for linear symmetric cubics. As all plane cubics have a linear symmetric matrix representation (Appendix B), we focus on nondegenerate linear symmetric representations of the cubic surfaces first.

Theorem 2.2 *There are three nondegenerate linear symmetric determinantal cubics with isolated singularities. Their combinations of the singularities are given by the subgraphs of* $\widetilde{E}_6$



which are obtained by removing some — but at least one — of the white dots. They all have unique matrix representations up to equivalence.

In addition, of the nonnormal cubics both special irreducible types, the smooth quadric with a tangent plane, the quadric cone with a transversal plane, and the double plane with an additional plane are nondegenerate linear symmetric cubics. All their nondegenerate linear symmetric matrix representations are unique.

Including the degenerate matrix representations and with them all cubic cones, we immediately obtain the following corollary.

Corollary 2.3 *There are four types of linear symmetric determinantal cubics with isolated singularities. Their combinations of the singularities are given by the subgraphs of the above marked Dynkin diagram* $\widetilde{E}_6$ *which are obtained by removing some of the white dots. The cubics with an elliptic singularity have three matrix representations up to equivalence, the other cubics only one.*

In addition, all nonnormal cubics with the exception of the smooth quadric with a transversal plane are linear symmetric cubics.

Proof of the Theorem 2.2. A nondegenerate linear symmetric representation is determined up to equivalence by choosing a four–dimensional linear subspace $\mathcal{A} \subset$

$\mathrm{Sym}(3,\mathbb{C})$, see the discussion near Definition 1.1. Now there is a nondegenerate symmetric bilinear form on $\mathrm{Sym}(r,\mathbb{C})$ given by

$$< \, , \, >: \mathrm{Sym}(r,\mathbb{C}) \times \mathrm{Sym}(r,\mathbb{C}) \longrightarrow \mathbb{C}, (A,B) \longmapsto \mathrm{tr}(A \cdot B) = \sum_{i,j=1}^{r} a_{ij} b_{ij},$$

where tr denotes the trace. Therefore, instead of choosing a four–dimensional linear subspace $\mathcal{A} \subset \mathrm{Sym}(3,\mathbb{C})$, we may choose dually a two–dimensional linear subspace $\mathcal{A}^{\perp} \subset \mathrm{Sym}(3,\mathbb{C})$. There is only a finite number of these pencils of $\mathcal{A}^{\perp}$. This can be extracted from [G, 12.6] where these pencils together with a choice of basis are classified. However, using the identification of a symmetric 3×3–matrix modulo $\mathbb{C}^*$ with a quadric in $\mathbb{P}^2$, we can also view $\mathcal{A}^{\perp} \subset \mathrm{Sym}(3,\mathbb{C})$ as a pencil of quadrics in $\mathbb{P}^2$. Then one can see that prescribing the intersection type of two general members of this pencil determines the pencil up to a choice of coordinates. From there one can compute the corresponding determinantal cubic. We will give one example of this and summarize the remaining cases in a table.

Let us assume that two quadrics of the pencil intersect with multiplicities 1,1, and 2. We choose coordinates such that $(0:0:1)$ and $(0:1:0)$ are the simple intersection points and $(1:0:0)$ is the point where the quadrics intersect with multiplicity 2, i.e., they have a common tangent. This tangent cannot pass through $(0:0:1)$ or $(0:1:0)$, because otherwise it would intersect every quadric of the pencil with multiplicity $2+1=3$, i.e., it would be a component of every quadric by Bezout's theorem. Thus by a further adaption of coordinates we may assume that the tangent is spanned by $(1:0:0)$ and $(1:1:1)$. Let $(r:s:t)$ be the coordinates on $\mathbb{P}^2$. Then a quadric q passing through $(1:0:0)$, $(0:1:0)$, and $(0:0:1)$ has the form $ars+brt+cst$. Its tangent in the point $(1:0:0)$ is given by $\mathrm{grad}_{(1,0,0)} q = (0,a,b)$; hence, passing through $(1:1:1)$ implies $a=-b$. Therefore, the pencil of quadrics is spanned by $2r(s-t)$ and $2st$. These correspond to the symmetric matrices

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

which therefore span $\mathcal{A}^{\perp}$. From this a basis of $\mathcal{A}$ can be easily computed as

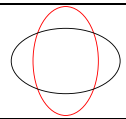
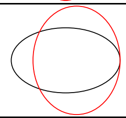
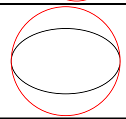
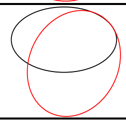
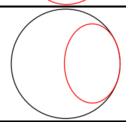
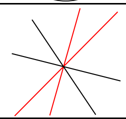
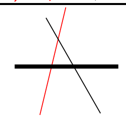
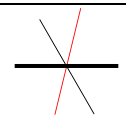
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and the equation of the cubic is

$$F = \det \begin{pmatrix} w & z & z \\ z & x & 0 \\ z & 0 & y \end{pmatrix} = wxy - xz^2 - yz^2.$$

It is easy to see that the singularities of F are the two A_1–singularities at $(0:1:0:0)$ and $(0:0:1:0)$ and an A_3–singularity at $(1:0:0:0)$.

We summarize all cases in the following table. Its first column describes the pencil of quadrics. If it contains only numbers, we consider the pencil whose general member is smooth and two of those intersect with multiplicities given by the numbers. $\square$

description of pencil	pencil of quadrics	two general members	$\mathcal{A}^\perp$	$\mathcal{A}$	F	description of cubic
$(1,1,1,1)$	$r(s-t)$ $t(r-s)$		$\begin{pmatrix} 0&1&-1\\ 1&0&0\\ -1&0&0 \end{pmatrix}\begin{pmatrix} 0&0&1\\ 0&0&-1\\ 1&-1&0 \end{pmatrix}$	$\begin{pmatrix} 1&0&0\\ 0&0&0\\ 0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 0&1&0\\ 0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&1 \end{pmatrix}\begin{pmatrix} 1&1&1\\ 1&1&1\\ 1&1&1 \end{pmatrix}$	$wxy+wxz+$ $wyz+xyz$	cubic with $4A_1$
$(2,1,1)$	$r(s-t)$ st		$\begin{pmatrix} 0&1&-1\\ 1&0&0\\ -1&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix}$	$\begin{pmatrix} 1&0&0\\ 0&0&0\\ 0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 0&1&0\\ 0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&1 \end{pmatrix}\begin{pmatrix} 0&1&1\\ 1&0&0\\ 1&0&0 \end{pmatrix}$	$wxy-xz^2$ $-yz^2$	cubic with $2A_1+A_3$
$(2,2)$	r^2 st		$\begin{pmatrix} 1&0&0\\ 0&0&0\\ 0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0\\ 0&1&0\\ 0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&1 \end{pmatrix}\begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&1\\ 0&0&0\\ 1&0&0 \end{pmatrix}$	$-xy^2-wz^2$	irreducible type I
$(3,1)$	$2r^2-2st$ rt		$\begin{pmatrix} 2&0&0\\ 0&0&-1\\ 0&-1&0 \end{pmatrix}\begin{pmatrix} 0&0&1\\ 0&0&0\\ 1&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0\\ 0&1&0\\ 0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&1 \end{pmatrix}\begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&0 \end{pmatrix}\begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix}$	$wxz-z^3$ $-xy^2$	cubic with A_5+A_1
(4)	$2s^2-2rt$ r^2		$\begin{pmatrix} 0&0&-1\\ 0&2&0\\ -1&0&0 \end{pmatrix}\begin{pmatrix} 1&0&0\\ 0&0&0\\ 0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&1\\ 0&1&0\\ 1&0&0 \end{pmatrix}\begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&1 \end{pmatrix}$	$-w^3+2wxy$ $-x^2z$	irreducible type II
all quadrics singular no fixed line	r^2 s^2		$\begin{pmatrix} 1&0&0\\ 0&0&0\\ 0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 0&1&0\\ 0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&1 \end{pmatrix}\begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&1\\ 0&0&0\\ 1&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix}$	$-x(wx-2yz)$	nondeg. quadric + tangent plane
fixed line; pencil with center outside line	rt st		$\begin{pmatrix} 0&0&1\\ 0&0&0\\ 1&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix}$	$\begin{pmatrix} 1&0&0\\ 0&0&0\\ 0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 0&1&0\\ 0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&1 \end{pmatrix}\begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&0 \end{pmatrix}$	$y(wx-z^2)$	quadric cone +transversal plane
fixed line; pencil with center on line	r^2 rs		$\begin{pmatrix} 1&0&0\\ 0&0&0\\ 0&0&0 \end{pmatrix}\begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0\\ 0&1&0\\ 0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&1 \end{pmatrix}\begin{pmatrix} 0&0&1\\ 0&0&0\\ 1&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix}$	$-wy^2$	double plane +plane

Looking at the table one notices that whenever the cubic F has only isolated singularities, these singularities are precisely the singularities of the quartic Q which is the union of the two smooth members of the pencil of the quadrics given by $\mathcal{A}$. We are going to explain this amazing fact.

Recall that a plane A_{2k+1}–singularity is defined to be the intersection of two smooth branches intersecting with multiplicity k. Thus knowing the singularities of the quartic Q, which is the union of the two smooth quadrics C_1 and C_2, is the same as knowing the intersection multiplicities of $C_1 \cap C_2$, whose sum is 4. Now we embed $\mathbb{P}^2$ via the Veronese embedding

$$v : \mathbb{P}^2 \longrightarrow \mathcal{V} \subset \mathbb{P}^5 = \mathbb{P}(\mathrm{Sym}(3, \mathbb{C})),\ [x] \longmapsto [x \cdot x^t].$$

as the Veronese surface $\mathcal{V}$ into $\mathbb{P}^5$. Then the quadrics C_1 and C_2 are pullbacks of two hyperplanes H_1 and H_2 of $\mathbb{P}^5$. The intersection multiplicities of $C_1 \cap C_2$ are the same as the intersection multiplicities of the curves $H_1 \cap \mathcal{V}$ and $H_2 \cap \mathcal{V}$ on the Veronese surface $\mathcal{V}$ by the projection formula. These are also the intersection multiplicities of the Veronese surface $\mathcal{V}$ and the 3–plane $H_1 \cap H_2 = \mathbb{P}(\mathcal{A})$. Denoting the affine coordinate ring of $\mathrm{Sym}^2(3, \mathbb{C})$ by $\mathbb{C}[x_0, \dots, x_5]$, they can be computed as the vector space dimensions of the ring

$$\mathbb{C}[x_0, \dots, x_5]/(I(\mathcal{V}) + I(H_1) + I(H_2))$$

localized at the corresponding points of $\mathbb{P}(\mathrm{Sym}(3, \mathbb{C}))$. Because H_1 and H_2 are linear and the ideal of the Veronese surface is given by the 2×2–minors of the general symmetric matrix, the above ring is isomorphic to

$$\mathbb{C}[w, x, y, z]/(2 \times 2\text{–minors of } M),$$

where M is the matrix representation of $F \in \mathbb{C}[w, x, y, z]$.

To determine the singularities of $F = \det M$, we project F from a general smooth point of F. Then it is classically known that the singularities of F are stably equivalent to the singularities of the branch curve of the projection. Let us recall the proof. If

$$F(w, x, y, z) = w^2 g_1(x, y, z) + w g_2(x, y, z) + g_3(x, y, w)$$

with $\deg g_i = i$ and $g_1 \neq 0$, then $(1 : 0 : 0 : 0)$ is a smooth point of F and the branch curve G of the projection is

$$G = {g_2}^2 - 4 g_1 g_3.$$

The stable equivalence between the points of F and G can be seen from

$$F/g_1 = \left(w + \frac{g_2}{2g_1}\right)^2 - \frac{1}{4{g_1}^2} G.$$

Dividing by g_1 does not cause problems for the following reasons: On the one hand we may assume by the generality of the projection point that F has no singularities on the tangent plane of the projection point, i.e., on $g_1 = 0$; on the other hand if we had a singularity on G in $p \in V(\{G, g_1\})$, then we would get $0 = G(p) = g_2(p)^2 - 4g_1(p)g_3(p) = g_2(p)^2$, thus $g_2(p) = 0$ and further $g_3(p) = 0$, in order for G to be singular in p. Therefore, F would have to contain the line spanned by p and $(1 : 0 : 0 : 0)$, which is impossible since the projection point was general and the cubic F contains only finitely many lines.

Now we apply this to our $F = \det M$. We know a priori that F has at least four A_1–singularities or worse in terms of the sum of the Milnor numbers; thus the branch curve G has these singularities as well and will be a reducible quartic. We will show that it is the union of two quadrics. We choose coordinates such that the general projection point is $(1:0:0:0)$ and

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{then} \quad M = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & w+f_{22} & f_{23} \\ f_{13} & f_{23} & w+f_{33} \end{pmatrix}$$

with $f_{ij} \in \mathbb{C}[x,y,z]$ linear. We denote the adjoint matrix of M by $\operatorname{adj} M = (m^{ij})$. Then

$$F = w^2 g_1 + w g_2 + g_3 \quad \text{with } g_1 = f_{11},\ g_2 = m^{22}_{w=0} + m^{33}_{w=0},\ g_3 = \det M_{w=0},$$

where the index $w=0$ stands for setting w equal to zero in the polynomial resp. matrix. By the determinantal formula of Laplace [KU, 2.4]

$$F \cdot f_{11} = m^{22}m^{33} - (m^{23})^2 \quad \Longrightarrow \quad g_1 g_3 = m^{22}_{w=0}m^{33}_{w=0} - (m^{23}_{w=0})^2$$

and

$$\begin{aligned} G = {g_2}^2 - 4g_1g_3 &= (m^{22}_{w=0} + m^{33}_{w=0})^2 - 4m^{22}_{w=0}m^{33}_{w=0} + 4(m^{23}_{w=0})^2 \\ &= (m^{22}_{w=0} - m^{33}_{w=0})^2 + (2m^{23}_{w=0})^2 \\ &= (m^{22}_{w=0} - m^{33}_{w=0} + 2\sqrt{-1}m^{23}_{w=0})(m^{22}_{w=0} - m^{33}_{w=0} - 2\sqrt{-1}m^{23}_{w=0}) \\ &= (m^{22} - m^{33} + 2\sqrt{-1}m^{23})(m^{22} - m^{33} - 2\sqrt{-1}m^{23}), \end{aligned}$$

where the last row follows from $m^{22}_{w=0} - m^{33}_{w=0} = m^{22} - m^{33}$ and $m^{23}_{w=0} = m^{23}$. Hence, G is the union of the quadric cones $\widetilde{C}_1 = V(m^{22} - m^{33} + 2\sqrt{-1}m^{23})$ and $\widetilde{C}_2 = V(m^{22} - m^{33} - 2\sqrt{-1}m^{23})$ with vertex $(1:0:0:0)$. We consider them as plane curves and compute their intersection multiplicities. They are given by the vector space dimensions of the ring

$$\mathbb{C}[x,y,z]/(m^{22} - m^{33} \pm 2\sqrt{-1}m^{23}) = \mathbb{C}[x,y,z]/(m^{22} - m^{33}, m^{23})$$

localized at the appropriate points. Since

$$(m^{22} - m^{33}, m^{23}) + (m^{11} = g_1 w + m^{11}_{w=0}) \subseteq (2 \times 2\text{–minors of } M)$$

and the sum of the intersection multiplicities is 4 in all cases, the intersection multiplicities of $C_1 \cap C_2$, $V \cap \mathbb{P}(\mathcal{A})$, and $\widetilde{C}_1 \cap \widetilde{C}_2$ are equal at corresponding points! Further, by what we said in the beginning the intersection multiplicities of $\widetilde{C}_1 \cap \widetilde{C}_2$ would determine the singularities of the branch curve if we knew that $\widetilde{C}_1$ and $\widetilde{C}_2$ are smooth, which we do not. However, the singularities will get only worse if $\widetilde{C}_1$ or $\widetilde{C}_2$ are singular, and we can at least conclude that the singularities of the branch curve, which are also the singularities of the cubic F, are equal to or worse than the singularities of the quartic $Q = C_1 + C_2$, that we started with. But the combination of the singularities for $C_1 + C_2$ are $4A_1$, $2A_1 + A_3$, $2A_3$, $A_5 + A_1$, and A_7 and these combinations are all extremal combinations of isolated singularities on a normal cubic — with the exception of $2A_3$ which is impossible — by the classification of cubics [BW]. Therefore, the singularities of F are in fact the singularities of $C_1 + C_2$ if F is normal.

3 Quartics

The methods of studying a normal quartic in $\mathbb{P}^3$ depend on whether its resolution is a $K3$–surface or a rational surface. If the quartic has only rational double points, its resolution is a $K3$–surface. In this case Urabe and Yang used Torelli type theorems for $K3$–surfaces to list all possible combinations of rational singularities. If the normal quartic surface possesses a nonrational double point or a triple point, the quartic is rational and can be examined by studying the projection of the quartic from this singular point. Degtyarev used this to list all possible combinations of singularities in this case. The proof also yields a method for producing equations of these quartics, in contrast to this for most possible combinations of rational singularities equations of the respective quartics are unknown. In the next three subsections we will adapt all this to the case of linear symmetric quartics.

If a quartic surface has a quadruple point, it is a cone over a plane curve. As any plane curve can be represented by a linear symmetric matrix [Be, 4.4], the same holds for any such quartic surface, and we will not discuss this case further.

3.1 Linear symmetric quartics with only rational double points

Urabe and Yang examined the question which combinations of rational double points can occur on a quartic at all [U1, U2, Y]. The general idea is not to study the quartic in $\mathbb{P}^3$ directly, but its minimal desingularization Y, which is a K3–surface. For general facts about K3–surfaces see [BPV, VIII]; we recall only the following: For all K3–surfaces the second cohomology group $\mathrm{H}^2(Y, \mathbb{Z})$ is a free abelian group of rank 22. Together with the intersection form it is the unique unimodular even lattice of signature $(3, 19)$, which is $Q(-E_8) \oplus Q(-E_8) \oplus \mathbb{H} \oplus \mathbb{H} \oplus \mathbb{H}$. Here, $\oplus$ denotes the orthogonal direct sum, $Q(-E_8)$ the rank 8 lattice whose bilinear form is given by the Dynkin graph E_8 with sign–reversed weights, and $\mathbb{H}$ the hyperbolic plane $\mathbb{H} = \mathbb{Z}u + \mathbb{Z}v$, where — writing the symmetric bilinear form as multiplication — $u^2 = v^2 = 0$ and $u \cdot v = 1$. Due to $H^1(Y, \mathcal{O}) = 0$, the Picard group $\mathrm{Pic}(Y)$ injects into $\mathrm{H}^2(Y, \mathbb{Z})$ and is in fact a primitive subgroup there, i.e. $\mathrm{H}^2(Y, \mathbb{Z})/\mathrm{Pic}(Y)$ is torsion free.

Using Torelli type theorems for K3–surfaces and work of Saint–Donat, Urabe proved the following

Theorem 3.1 ([U1, Theorem 1.15]) *Let $G = \sum a_k A_k + \sum b_l D_l + \sum c_m E_m$ be a Dynkin graph with components of type A, D, or E only. The following conditions are equivalent:*

1. *There is a quartic surface in $\mathbb{P}^3$ with only rational double points as singularities, the combination of singularities corresponding to G.*
2. *Let $Q = Q(G)$ be the lattice of type G. Let $\Lambda := Q(-E_8) \oplus Q(-E_8) \oplus \mathbb{H} \oplus \mathbb{H} \oplus \mathbb{H}$ denote the unimodular even lattice with signature $(3, 19)$. The lattice $S = \mathbb{Z}H \oplus Q$ ($H^2 = 4$, orthogonal direct sum) has an embedding $S \subseteq \Lambda$ satisfying the following conditions (a) and (b). Let $\tilde{S} = \{x \in \Lambda \mid mx \in S$ for some $m \in \mathbb{Z} \setminus \{0\}\}$ denote the primitive hull of S in Λ.*
   - *(a) If $\eta \in \tilde{S}$, $\eta \cdot H = 0$, and $\eta^2 = -2$, then $\eta \in Q$.*
   - *(b) $\tilde{S}$ does not contain any element u with $u^2 = 0$ and $u \cdot H = 2$.*

The sum $\mu := \sum a_k k + \sum b_l l + \sum c_m m$ is called the *Milnor number* of G or X. For quartic surfaces one always has $\mu \leq 19$.

Condition *(a)* ensures that there are only the expected singularities G on the quartic; condition *(b)* that the linear system given by H induces an embedding into $\mathbb{P}^3$.

By this theorem Urabe reduced the question of the existence of a quartic with a given combination of singularities to a purely lattice theoretic problem. We want a similar theorem for our situation and start by providing the Dynkin graph with additional information.

Definition 3.2 *A* parity Dynkin graph G *is a formal sum of the following marked Dynkin diagrams:*

- *The essential parity Dynkin diagrams, which are the marked Dynkin diagrams of Proposition 1.13. (We do not distinguish between* D_{2k}^+ *and* D_{2k}^- *or between* $D_4^\bullet$ *and* $D_4^\pm$*.)*
- *The accidental parity Dynkin diagrams, which are the Dynkin diagrams of* A_k*,* D_k*,* E_6*,* E_7*,* E_8 *with every vertex drawn as* $\circ$*. They are denoted by* A_k°*,* D_k°*,* E_6°*,* E_7°*,* E_8°*.*

The number of vertices of G *is the* Milnor number $\mu(G)$ *of* G *and the number of* $\bullet$*–vertices is the* length $l(G)$ *of* G*.*

To a linear symmetric surface with only rational singularities we assign a parity Dynkin diagram whose components correspond to the singularities in the obvious way: for the essential singularities we use the correspondence of Proposition 1.13 and to the accidental singularities we assign the corresponding accidental Dynkin diagrams.

Every parity Dynkin diagram comes with a special divisor in corresponding lattice:

Definition 3.3 *The lattice* $Q(G)$ *of a parity Dynkin graph has a canonical basis given by the vertices of the graph* G*. The* parity divisor D_G *is the sum of the* $\bullet$*–vertices.*

Now we can state the extension of Urabe's Theorem for linear symmetric quartics:

Theorem 3.4 *Let* G *be a parity Dynkin graph. The following conditions are equivalent:*

1. *There is a linear symmetric quartic in* $\mathbb{P}^3$ *with only rational double points as singularities, the combination of singularities corresponding to* G*.*
2. *Let* G *satisfy the condition 2 described in the above Theorem and in addition:*
   - *(c) The length of* G *is 10 and* $\frac{1}{2}H + \frac{1}{2}D_G \in \tilde{S}$*, where* D_G *is the parity divisor of* G*.*

Proof. In Urabe's correspondence between the lattices and the quartics, the primitive lattices $\tilde{S}$ correspond to the Picard group of the minimal resolution $\tilde{F}$ of the quartic. Now on a linear symmetric quartic F the essential singularities form an even set X of ADE–singularities of length 10 by Proposition 1.15 and Theorem 1.23. Let G be the parity Dynkin graph of F. Clearly, by the definitions $l(G) = l(X)$

and $H + D_G$ is divisible by 2 in $\mathrm{Pic}(\tilde{F}) = \tilde{S}$ precisely if $H + \sum Z_{X_i}$ is. Therefore, condition *(c)* holds.

Starting with a parity Dynkin graph with the properties *(a) – (c)* Urabe's Theorem yields a quartic F with an even set of ADE–singularities of length 10. Let $\tilde{F}$ be the minimal resolution of F. (This time for all singularities of F, but this makes no difference for the statements of Section 1.) By Theorem 1.23 the quartic F is linearly symmetric if the order of X is 3. By Proposition 1.22 X is weakly even and we only need to show that the order of X is not 1. Setting $L = \frac{1}{2}(H + \sum Z_{X_i}) \in \mathrm{Pic}(\tilde{F})$ and using the remark after Definition 1.18 this is equivalent to $H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}(H - L)) = 0$, i.e., we need to show that $H - L = \frac{1}{2}(H - \sum Z_{X_i}) \in \mathrm{Pic}(\tilde{F})$ is not effective.

Assume that $H - L$ is effective. Then we can decompose it into $\sum_{j=1}^{s} C_j + \sum_k B_k$ where the C_j, B_k are irreducible curves with $H \cdot C_j > 0$ and $H \cdot B_k = 0$. Recall that for any curve C on a $K3$–surface $C^2 \geq -2$ and C^2 is divisible by 2 [BPV, VIII (3.6)]. Because Q is a negative definite lattice and $B_k \in \mathbb{Q} \otimes Q$, we get $B_k^2 = -2$ and $B_k \in Q$ by condition *(a)*. We claim that there are at most two curves C_j, i.e., $s \leq 2$. Write $C_j = a_j H + \tilde{C}_j \in \mathbb{Q}H \oplus \mathbb{Q} \otimes Q$, then $H \cdot C_j = 4a_j \in \mathbb{N}$ thus $a_j = n_j/4$ for some $n_j \in \mathbb{N}$. From $\sum C_j = \frac{1}{2}H \mod \mathbb{Q} \otimes Q$ we find either $s = 1$ and $a_1 = \frac{1}{2}$ or $s = 2$ and $a_1 = a_2 = \frac{1}{4}$. It is not difficult to obtain contradictions for $s = 1$ or $s = 2$ and $C_1 \neq C_2$ by completely elementary calculations with divisors, but the $C_1 = C_2$ case seems inaccessible by these simple methods. Hence, we recall more lattice theory.

The primitive hull $\tilde{S}$ of S will always lie in $S^* = \mathrm{Hom}(S, \mathbb{Z}) \subset \mathbb{Q} \otimes S$, thus $\tilde{S}/S \subseteq S^*/S$. The finite group S^*/S is well known. If $G = \sum X_i$ is the decomposition of the parity Dynkin graph G into the parity Dynkin diagrams, then $S^*/S = \mathbb{Z}/4\mathbb{Z} \oplus \bigoplus_i Q(-X_i)^*/Q(-X_i)$, where the first summand is generated by $H/4$ and $Q(-X_i)^*/Q(-X_i)$ depends only on the underlying Dynkin diagram and is isomorphic to $\mathbb{Z}/(k+1)\mathbb{Z}$ for A_k, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for D_{2k}, $\mathbb{Z}/4\mathbb{Z}$ for D_{2k+1}, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, 0 for E_6, E_7, E_8 respectively [U3, 1.3]. Define for $D \in Q^*$

$$m(D) := \max\{(D + B)^2 \mid B \in Q\}.$$

Because the intersection form is negative definite, $m(D) < 0$ for $D \notin Q$. These numbers were computed by Urabe [U3, 1.3]. In particular, he found $m(\frac{1}{2}D_{X_i}) = -\frac{1}{2}l(X_i)$ for the parity divisors of the singularities X_i. Since $\mathbb{Q} \otimes Q$ is the orthogonal sum of the $\mathbb{Q} \otimes Q(X_i)$, we get $m(\frac{1}{2}D_G) = \sum m(\frac{1}{2}D_{X_i}) = -\frac{1}{2}\sum l(X_i) = -5$.

Now if $s = 1$ then $C_1 = H - L - \sum B_k = \frac{1}{2}H + \frac{1}{2}D_G + B$ for some $B \in Q$ and

$$C_1^2 = (\tfrac{1}{2}H + \tfrac{1}{2}D_G + B)^2 = (\tfrac{1}{2}H)^2 + (\tfrac{1}{2}D_G + B)^2 \leq 1 + m(\tfrac{1}{2}D_G) = 1 - 5 = -4,$$

contradicting $C_1^2 \geq -2$.

If $s = 2$ then write

$$C_j = \tfrac{1}{4}H + \sum\nolimits_i C_{j,X_i} \quad \text{with} \quad C_{j,X_i} \in \mathbb{Q} \otimes Q(X_i).$$

We see from $C_1 + C_2 = \frac{1}{2}H + \frac{1}{2}D_G \mod Q$ that $C_{1,X_i} + C_{2,X_i} = \frac{1}{2}D_{X_i} \mod Q$. Further, we find the estimates

$$\begin{aligned} C_j^2 \qquad &\leq (\tfrac{1}{4}H)^2 + \textstyle\sum_i m(C_{j,X_i}) = \tfrac{1}{4} + \sum_i m(C_{j,X_i}) \\ C_1^2 + C_2^2 &\leq \tfrac{1}{2} + \textstyle\sum_i \left(m(C_{1,X_i}) + m(C_{2,X_i})\right). \end{aligned}$$

A small computation using Urabe's values for the function m shows

$$m\left(C_{1,X_i}\right) + m\left(C_{2,X_i}\right) \leq m(\tfrac{1}{2}D_{X_i}) = -\tfrac{1}{2}l(X_i)$$

for any C_{j,X_i} with $C_{1,X_i}+C_{2,X_i}=\frac{1}{2}D_{X_i} \bmod Q(X_i)$. This implies $C_1^2+C_2^2 \le \frac{1}{2}-5=-4\frac{1}{2}$ and hence $C_1^2<-2$ or $C_2^2<-2$, which yields the required contradiction. □

From Urabe's Theorem it follows immediately that if there exists a quartic with Dynkin graph G then one can find a quartic for any complete subgraph $G' \subset G$. For linear symmetric quartics a similar statement holds:

Definition 3.5 *A parity Dynkin subgraph G' of a parity Dynkin graph G is a complete subgraph $G' \subset G$ which contains all the $\bullet$–vertices of G, i.e., $l(G') = l(G)$.*

Corollary 3.6 (Parity splitting principle) *If there exists a linear symmetric quartic with parity Dynkin graph G, then there exists a linear symmetric quartic for any parity Dynkin subgraph G' of G.*

Proof. Because of $D_{G'} = D_G \in \mathbb{Q}H \oplus \mathbb{Q} \otimes Q(G')$ we can use $\mathbb{Z}H \oplus Q(G') \subseteq \mathbb{Z}L \oplus Q(G) \subseteq \Lambda$ for the embedding required in the theorem. □

This parity splitting principle has amazing consequences which we state in the following summarizing theorem.

Theorem 3.7 *Let G be the parity Dynkin graph of a linear symmetric quartic with only rational double points, then the following holds:*

1. $10 \le \mu(G) \le 19$ *and* $l(G) = 10$.
2. *G is a union of the parity Dynkin diagrams $A^{\bullet}_{2k+1}$, $D^{\pm}_{2k}$, and A°_1.*

In particular, the parity Dynkin graph G is determined by its underlying Dynkin graph.

Proof. $l(G) = 10$ was stated in Theorem 3.4 and $\mu(G) \le 19$ holds for any quartic. By Proposition 1.3 the only possible accidental singularity on a linear symmetric quartic is an A_1–singularity. Proposition 1.13 already says that there are no essential A_{2k}, E_6, and E_8 singularities. Further, for the parity Dynkin diagrams $D^{\bullet}_{2k+1}$, $D^{\bullet}_{2k}$ — except for $D^{\bullet}_4 = D^{\pm}_4$ —there exist parity splittings which have an accidental A°_l, $l \ge 2$, parity Dynkin diagram as a component, contradicting the parity splitting principle. □

Urabe used his Theorem to give a short list of so–called basic Dynkin graphs and define two kinds of transformations for Dynkin graphs such that after applying two transformations to a basic Dynkin graph the resulting graph is a possible combination of rational singularities on a quartic [U1, U2]. This produced a long list of possible combinations of singularities on a quartic. Unfortunately, these operations are not compatible with our new condition (c). However, this long list of combinations of singularities was not complete as Urabe noted himself in [U2, 3]. There he also remarked that each Dynkin graph G can be checked individually by a tedious computation using Nikulin's lattice theory [N]. Yang wrote a computer program which precisely does this [Y]. Yang was so kind to send his program to the author. The modification to incorporate condition (c) is not difficult. The output of the program can be summarized as follows:

Theorem 3.8 *For linear symmetric quartics with only rational double points only the following parity Dynkin graphs or their parity splittings occur:*

(Only the underlying Dynkin graphs are listed as they determine the parity Dynkin diagrams by Theorem 3.7.)

$D_{18}+A_1$	$D_{14}+A_5$	$D_{14}+A_3+2A_1$
$D_{12}+D_6+A_1$	$D_{12}+A_5+2A_1$	$D_{10}+D_8+A_1$
$D_{10}+D_6+A_3$	$D_{10}+A_9$	$D_{10}+A_7+2A_1$
$D_{10}+A_5+A_3+A_1$	$2D_8+3A_1$	$D_8+D_6+D_4+A_1$
$D_8+D_6+A_5$	$D_8+D_6+A_3+2A_1$	$D_8+D_4+A_5+2A_1$
$D_8+A_9+2A_1$	$D_8+A_5+A_3+3A_1$	$3D_6+A_1$
$2D_6+D_4+3A_1$	$2D_6+A_7$	$2D_6+A_5+2A_1$
$D_6+2D_4+A_3+2A_1$	$D_6+D_4+A_5+A_3+A_1$	D_6+A_{13}
$D_6+A_9+A_3+A_1$	$D_6+A_7+A_5+A_1$	$4D_4+3A_1$
$D_4+A_9+A_5+A_1$	$D_4+2A_5+A_3+2A_1$	A_{19}
$A_{17}+2A_1$	$A_{15}+A_3+A_1$	$A_{13}+A_5+A_1$
$A_{11}+A_7+A_1$	$A_{11}+A_5+3A_1$	$A_{11}+2A_3+2A_1$
$2A_9+A_1$	$A_9+A_7+3A_1$	$2A_7+A_3+2A_1$
$3A_5+4A_1$	$4D_4+2A_1$	$16A_1$

The complete list of the underlying Dynkin graphs sorted by the Milnor number can be found in Appendix A. The Theorem shows that the possible combinations of singularities on a linear symmetric determinantal quartic are far less than the one of a general quartic, where one has 27 pages of combinations for the Milnor numbers 19, 18, and 17 alone and most combinations for the Milnor numbers 16 and 15 and below that all combinations are possible [Y]. However, one might have hoped for even less possible combinations.

Without the use of the program it is not clear why one needs only the parity diagrams of Milnor number 19 and $4D_4^{\pm}+2A_1^{\bullet}$ as well as $10A_1^{\bullet}+6A_1^{\circ}$ as starting points for the parity splitting process.

Example. In general it is difficult to find an explicit matrix representation for the combinations of rational singularities we determined above. However, with some tricks and enough computing power one finds the following matrix representation

$$\begin{pmatrix} x & iy & iy/2 & y/2-iz \\ iy & x & y/2+iz & iy/2 \\ iy/2 & y/2+iz & w+ix+3iy/2 & i(-2x+y-4z)/4 \\ y/2-iz & iy/2 & i(-2x+y-4z)/4 & w-ix-3iy/2 \end{pmatrix}$$

of the unique quartic with an A_{19}–singularity found by Kato and Naruki [KN].

3.2 Linear symmetric Quartics with a nonsimple double point

As soon as the normal quartic acquires a nonsimple double point, it is no longer a K3-surface, but a rational surface. Hence the techniques of the last section cannot be used to study this case. Degtyarev studies these quartics by projecting them from their worst singularity onto a plane [D]. We will use his extensive study of quartic equations to obtain the following

Theorem 3.9 *Only the following combinations of double points occur on a rational* *linear symmetric quartic with at most double points:*

$$\begin{aligned}
&X_{1,0}+A_3+\{X_{1,0}, D_6+A_1, D_4+3A_1, A_7, 2A_3+A_1, 6A_1\},\\
&X_{1,2}+A_3+\{D_6, D_4+2A_1\},\ \{X_{1,2}+A_1, X_{1,4}\}+\{D_6+A_1, A_7, 2A_3+A_1\},\\
&X_{1,4}+A_3+\{A_3+A_1, A_3\},\ X_{1,6}+A_3+A_1,\ X_{1,8}+A_3,\\
&Y^1_{2,2}+A_5+A_1,\ Y^1_{2,2}+2A_1+\{D_4, 4A_1\},\ Y^1_{2,4}+2A_1+\{2A_1, A_1\},\\
&Y^1_{4,4}+2A_1+\{A_1, \emptyset\},\ Y^1_{2,6}+2A_1.
\end{aligned}$$

Hereby, one has to choose one element out of the sets to get a valid expression, and the A_{2k+1} and D_{2k} singularities may be splitted in the same manner as $A^{\bullet}_{2k+1}$ and $D^{\pm}_{2k}$ in the section before.

Proof. To apply the results of Degtyarev, we need explicit equations. Let us assume that $M = wA_0 + xA_1 + yA_2 + zA_3$, $F = \det M$, and the worst singular point is at $p = (1:0:0:0)$. The rank of A_0 is two by Proposition 1.3 and the obvious fact that the multiplicity of F at p is equal or higher than the corank of A_0. We can choose a basis of $\mathbb{C}^4$ such that

$$A_0 = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{E}_2 \end{pmatrix} \qquad \text{with} \qquad \tilde{E}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If we use a 2×2–blocking for M,

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^t & M_{22} \end{pmatrix},$$

the quadric part of F in p is given by $-\det M_{11}$. Since we are still free to choose an arbitrary basis in span $\{e_1, e_2\}$ resp. span $\{x, y, z\}$, we may think of M_{11} as given by a linear subspace in $\mathbb{P}(\mathrm{Sym}(2,\mathbb{C})) \cong \mathbb{P}^2$. The matrices of rank 1 form a smooth conic C in this $\mathbb{P}^2$, and the linear spaces inside this $\mathbb{P}^2$ are characterized by their intersection with this conic [H, 10]. We get the following list:

subspace	normal form of M_{11}	$\det M_{11}$
$\mathbb{P}^2$	$\begin{pmatrix} x & z \\ z & y \end{pmatrix}$	$xy - z^2$
secant of C	$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$	xy
tangent to C	$\begin{pmatrix} 0 & x \\ x & y \end{pmatrix}$	$-x^2$
point outside C	$\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}$	$-x^2$
point on C	$\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$	0
$\emptyset$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0

In the last two cases we get nonisolated singularities by Lemma 1.4. In the first two cases we get A_k–singularities by the the classification of singularities [AGV, 16.2]. We want to show that in the third case only D_k–singularities occur. We write

$$M = \begin{pmatrix} 0 & x & f_{13} & f_{14} \\ x & y & f_{23} & f_{24} \\ f_{13} & f_{23} & f_{33} & w+f_{34} \\ f_{14} & f_{24} & w+f_{34} & f_{44} \end{pmatrix} \qquad \text{with } f_{ij} \in \mathbb{C}[x,y,z].$$

After a base change of the type $e_3 \mapsto e_3 - \lambda_3 e_1 - \mu_3 e_2$ and $e_4 \mapsto e_4 - \lambda_4 e_1 - \mu_4 e_2$, we may assume $f_{13}, f_{14}, f_{23}, f_{24} \in \mathbb{C}[y,z]$. Setting $w = 1$, computing the determinant and performing the substitution $x \to x - xf_{34} + f_{14}f_{23} + f_{13}f_{24}$, the equation of F starts with

$$x^2 + 2yf_{13}f_{14} + \dots .$$

Because of Lemma 1.4 the linear polynomials f_{13} and f_{14} are linear independent. Thus F has a D_k–singularity in p [AGV, 16.2].

Therefore, the fourth case is the only case where nonsimple double points may occur. We have

$$M = \begin{pmatrix} 0 & x & f_{13} & f_{14} \\ x & 0 & f_{23} & f_{24} \\ f_{13} & f_{23} & f_{33} & w + f_{34} \\ f_{14} & f_{24} & w + f_{34} & f_{44} \end{pmatrix} \qquad \text{with } f_{ij} \in \mathbb{C}[x,y,z].$$

The surface F is given as $F = w^2x^2 + wxP + Q$ with

$$\begin{aligned} P &= 2xf_{34} - 2(f_{13}f_{24} + f_{14}f_{23}) \\ Q &= x^2(f_{34}^2 - f_{33}f_{44}) + 2x(f_{13}f_{23}f_{44} + f_{14}f_{24}f_{33} - f_{13}f_{24}f_{34} - f_{14}f_{23}f_{34}) \\ &\quad + (f_{13}f_{24} - f_{14}f_{23})^2. \end{aligned}$$

The branch curve of the canonical projection of F from p is

$$D = P^2 - 4Q = 4(xf_{33} - 2f_{13}f_{23})(xf_{44} - 2f_{14}f_{24}) = 4C_1C_2,$$

a union of two conics C_1, C_2. Note that there is no restriction on the equation of the conics, since we can choose the f_{ij} arbitrary so far. Further, let the line L be the projected tangent cone $V(x)$ of F in p.

According to Degtyarev [D, §2], p will be an isolated singularity only if D is smooth at $L \cap Q$. Note that D cannot contain L because of the linear independence of the linear forms x, f_{13}, f_{14} resp. x, f_{23}, f_{24} (Lemma 1.4). This excludes singularities of the type N [D, §3]. F has the following singularities:

- To each singular point of D not lying on L there corresponds a singular point of F stably equivalent to it. In particular, the curve D cannot have a multiple component for a normal surface F.
- To each s–fold point of $Q \cap L$ not on D there corresponds an exceptional singular point of F of type A_{s-1}.
- The type of the singularity of F at p can be read off the following table. The first column describes the intersection configuration of L and D where 1 and 2 stands for a transversal resp. tangential intersection at a smooth point of D and A_k for a transversal intersection at an A_k–singularity of D. If $D \cap L$ and $Q \cap L$ have a common multiple point, its multiplicity in $Q \cap L$ is denoted by q, otherwise we set $q = 1$. The case of two common double points is written loosely as $q = (2,2)$.

$D \cap L$	q	
$(1,1,1,1)$	–	$X_{1,0}$
$(2,1,1)$	q	$X_{1,q}$
$(2,2)$	q	$Y_{1,q}$
$(2,2)$	$(2,2)$	$Y^1_{2,2}$
$(A_k,1,1)$	–	$X_{1,k+1}$
$(A_k,2)$	q	$Y^1_{k+1,q}$
(A_k,A_l)	–	$Y^1_{k+1,l+1}$

We apply this to our case. We have

$$D \cap L = V(f_{13} f_{14} f_{23} f_{24}, x).$$

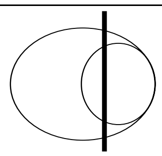
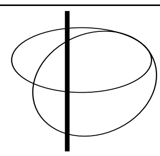
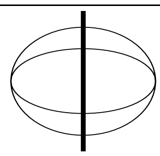
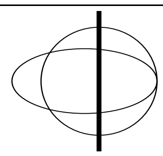
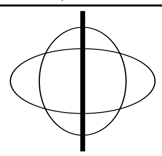
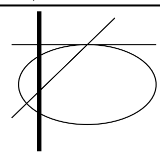
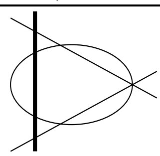
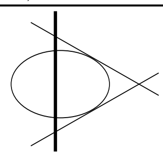
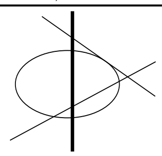
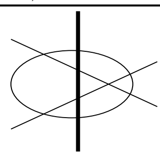
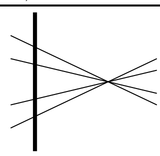
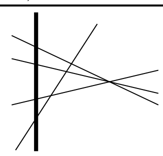
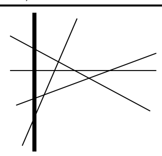
By Lemma 1.4 the linear forms x, f_{13}, f_{14} resp. x, f_{23}, f_{24} are linear independent; hence D and L may only intersect in the configuration given in the above table. Note that

$$Q \cap L = V\left((f_{13} f_{24} - f_{14} f_{23})^2, x\right);$$

thus the exceptional singularities are of type A_1 or A_3 if the corresponding multiple points do not lie on D. We treat the case of the different intersection configuration of D and L separately.

Case (1,1,1,1). Our main singularity is an $X_{1,0}$. Since f_{13}, f_{14}, f_{23}, f_{24} have pairwise distinct zeros on L, $Q \cap L$ and $D \cap L$ cannot have a common multiple zero, thus we can have either an A_3 or two A_1s as exceptional singularities. Further, we can change (f_{14}, f_{24}) to $(\lambda f_{14}, \lambda^{-1} f_{24})$ with $\lambda \in \mathbb{C}^*$ without changing the equation of D, but $Q \cap L$ changes to $V\left((f_{13} f_{24} - \lambda^2 f_{14} f_{23})^2, x\right)$. This restricted pencil for $\lambda^2 \in \mathbb{C}^*$ contains a quadruple point, because the complete pencil with $\lambda^2 \in \mathbb{P}^1$ does and we can exclude $\lambda = 0, \infty$. Therefore, we can always have two A_1 as well as an A_3 as exceptional singularities.

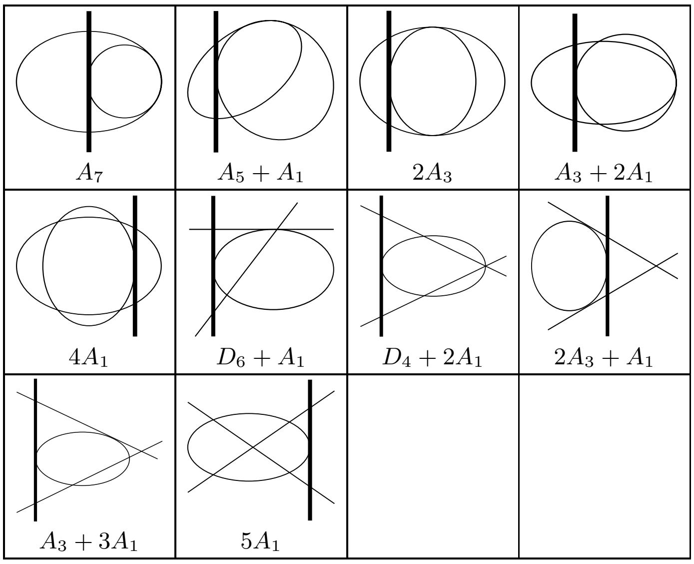
Finally, in the following table we sketch all singularities which can occur on a quartic D which is the union of two conics and list which singularities — apart from the exceptional ones — the surface F has in the corresponding case. The fat line represents L.

 $X_{1,0} + A_7$	 $X_{1,0} + A_5 + A_1$	 $X_{1,0} + 2A_3$	 $X_{1,0} + A_3 + 2A_1$
 $X_{1,0} + 4A_1$	 $X_{1,0} + D_6 + A_1$	 $X_{1,0} + D_4 + 2A_1$	 $X_{1,0} + 2A_3 + A_1$
 $X_{1,0} + A_3 + 3A_1$	 $X_{1,0} + 5A_1$	 $2X_{1,0}$	 $X_{1,0} + D_4 + 3A_1$
 $X_{1,0} + 6A_1$			

Case (2,1,1). Since L intersects D tangentially, one of the two conics C_1, C_2 —say C_1 — must be smooth and f_{13} and f_{23} are proportional modulo x, i.e., there exist $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{C}$ with $f_{23} = \alpha f_{13} + \beta x$. It follows that

$$D \cap L = V\left(f_{13}^2 f_{14} f_{24}, x\right) \qquad \text{and} \qquad Q \cap L = V\left(f_{13}^2 (f_{24} - \alpha f_{14})^2, x\right)$$

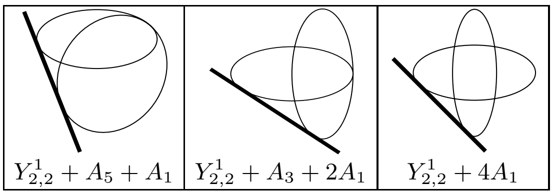
have a common double point. This point may become a quadruple point of $Q \cap L$ and thus the main singularity is either an $X_{1,4}$ or an $X_{1,2}$ in the latter case there exist an exceptional singularity of type A_1. With a similar argument as in the case before, the condition that $Q \cap L$ has a quadruple point is seen to be independent of the equation of D. We list the possible singularities F has apart from $X_{1,4}$ or $X_{1,2} + A_1$ in dependence of the shape of D:



Case (2,2). Because of the two tangential intersections of L and D, both conics C_1, C_2 must be smooth and f_{13} and f_{23} as well as f_{14} and f_{24} are proportional modulo x; hence

$$D \cap L = V\left(f_{13}^2 f_{14}^2, x\right) \qquad \text{and} \qquad Q \cap L = V\left(f_{13}^2 f_{14}^2, x\right)$$

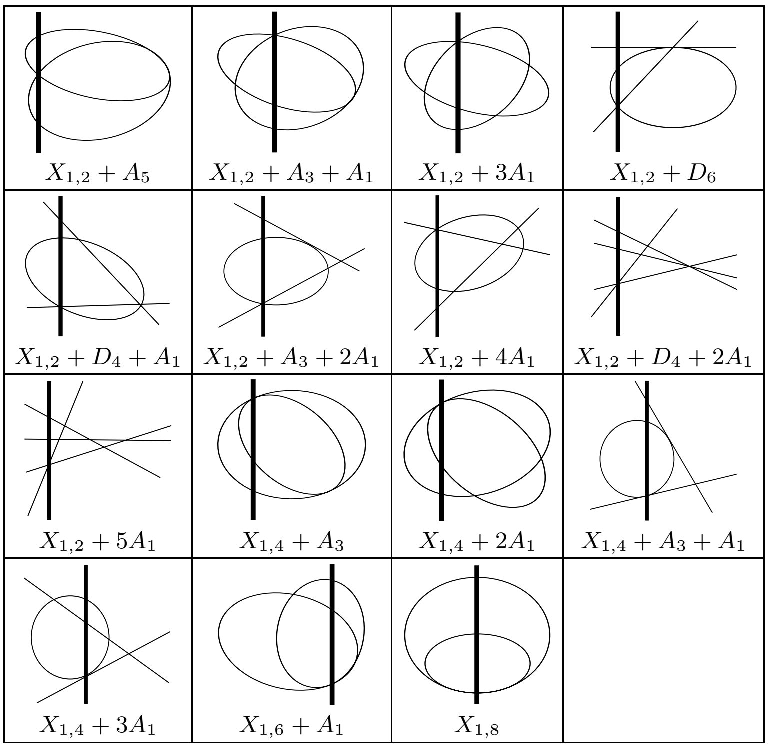
are the same divisor with two double points. Therefore, our main singularity is a $Y_{2,2}^1$ and there are no exceptional singularities. The singularities of F in dependence of the shape of D can be read off the following table:



Case (A_k,1,1). Here D has an A_k– singularity on L where k is necessarily odd. Its two branches belong to C_1 and C_2. Namely, if both branches belong to C_1, i.e., f_{13} and f_{23} are proportional modulo x, then the singular point of D would belong to $Q \cap L$, and F would have a nonisolated singularity. Remembering that f_{13} and f_{14} resp. f_{23} and f_{24} are also not proportional modulo x, we find that $f_{24} = \alpha f_{13} + \beta x$ for some $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{C}$ (or the same with the indices 1 and 2 exchanged) and

$$D \cap L = V\left(f_{13}^2 f_{14} f_{23}, x\right) \qquad \text{and} \qquad Q \cap L = V\left((\alpha f_{13}^2 - f_{14} f_{23})^2, x\right).$$

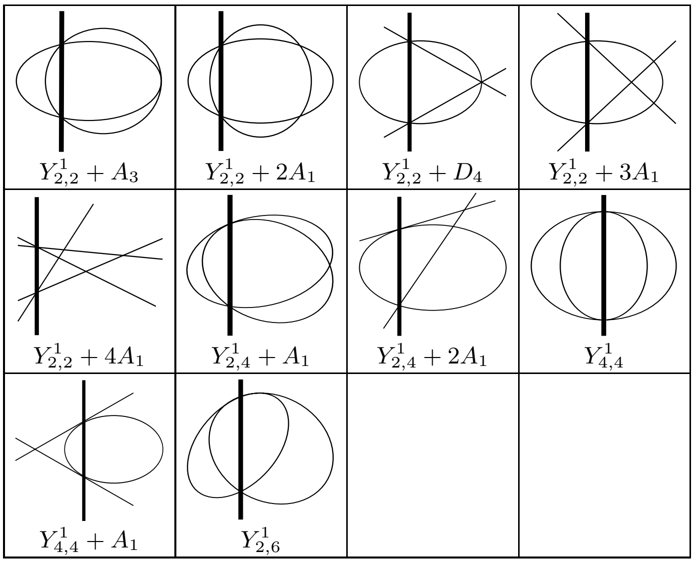
Thus $D \cap L$ and $Q \cap L$ cannot have a common multiple point. Our usual argument that $Q \cap L$ may have two double points as well as a quadruple point without changing the equation of D can be adapted to this case as well. Thus we can always have one A_3 as well as two A_1 as exceptional singularities. It remains to list all the singularities of F apart from the exceptional ones depending on the shape of D:



Cases (A_k,2) and (A_k,A_l). We have seen above that an A_k–singularity of D on L can only occur as the intersection of both C_1 and C_2; hence, no further tangential intersection of L and C_1 or C_2 is possible, i.e., the case $(A_k,2)$ does not occur. In the (A_k,A_l)–case we obtain $f_{23}=\alpha_1 f_{14}+\beta_1 x$ and $f_{24}=\alpha_2 f_{13}+\beta_2 x$ for some $\alpha_1,\alpha_2\in\mathbb{C}^*$ and $\beta_1,\beta_2\in\mathbb{C}$; thus

$$\begin{aligned}&D\cap L=V\left(f_{13}^2f_{14}^2,x\right) \qquad \text{and}\\ &Q\cap L=V\left((\sqrt{\alpha_2}f_{13}+\sqrt{\alpha_1}f_{14})^2(\sqrt{\alpha_2}f_{13}-\sqrt{\alpha_1}f_{14})^2,x\right).\end{aligned}$$

Therefore $Q\cap L$ has always two double points outside $D\cap L$, i.e., we have two exceptional A_1–singularities. We list the remaining singularities of F according to the shape of D:



□

3.3 Linear symmetric quartics with a triple point

Similar to the above case we obtain

Theorem 3.10 *Only the following combinations of singularities occur on a normal linear symmetric quartic with a triple point:*

$$T_{3,3,3}+A_{11}, T_{3,3,5}+A_9,\ T_{3,3,7}+A_7,\ T_{3,3,9}+A_5,\ T_{3,3,11}+A_3,\ T_{3,3,13}+A_1,$$
$$T_{3,3,15},\ T_{3,5,5}+A_5+A_1,\ T_{3,5,7}+A_5,\ T_{3,5,9}+2A_1,\ T_{3,5,11}+A_1,\ T_{3,7,7}+A_3,$$
$$T_{3,7,9}+A_1,\ T_{3,7,11},\ T_{5,5,5}+3A_1,\ T_{5,5,7}+2A_1,\ T_{5,7,7}+A_1,\ T_{7,7,7},$$
$$T_{3,3,4}+A_{11},\ T_{3,4,4}+A_3+A_7,\ T_{4,4,4}+3A_3,$$
$$Q_{11}+A_9,\ S_{1,0}+A_5+A_1,\ S^{\#}_{1,2}+A_5,\ S^{\#}_{1,4}+2A_1,\quad \textit{and}\ \ S^{\#}_{1,6}+A_1$$

Hereby, the A_{2k+1}–singularities can be splitted in the same manner as $A^{\bullet}_{2k+1}$ before.

Proof. Let $p=(1:0:0)$ be the triple point of the quartic F. From the first part of the proof of Theorem 3.9 it follows that the rank of A_0 is 1; hence, we choose a basis of $\mathbb{C}^4$ such that

$$A_0=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

in a (3,1)–blocking. Then the expansion of $F=\det(wA_0+xA_1+yA_2+zA_3)$ with respect to w is $F=wP+Q$ where Q is the determinant of the matrix $A_{123}=xA_1+yA_2+zA_3$ and P is the upper left 3×3–minor of the same matrix; hence, we consider A_{123} as a matrix representation of the curve Q, and P is one of the contact curves of Q, i.e., all intersection multiplicities are even. In order to determine the singularities of F we quote the following results of Degtyarev [D, §4]:

- The point p is an isolated singularity of F only if P and Q have no common singularities.
- Apart from the triple point the normal surface F has only A_{r-1}–singularities which are in one–to–one correspondence with points of r–fold intersection of P and Q at smooth points of P.
- The type of the double point of F is determined as follows: For a singular point S_i of P let r_i be the intersection multiplicity of P and Q in S_i, then we have the following correspondence

singularities of P	triple point of F
—	$P_8=T_{3,3,3}$
A_1	$P_{5+q_1}=T_{3,3,q_1}$, where $q_i=\max\{4,3+r_i\}$
$2A_1$	$R_{q_1,q_2}=T_{3,q_1,q_2}$
$3A_1$	T_{q_1,q_2,q_3}
A_2	Q_{10},Q_{11},Q_{12} for $r_1=0,2,$ and 3 respectively.
A_3	$S_{11},S_{12},S_{1,0}$ for $r_1=0,2,$ and 4 respectively.
	$S^{\#}_{1,r_1-4}$ for $r_1>4$.

Now we have to analyze our linear symmetric quartic F for the different possibilities of P. For the case of a smooth cubic P we can use abstract arguments involving the Jacobian of P and the theory of contact curves; for singular P we have to analyze the equations using the determinantal representations of P in Appendix B.

Case: P smooth cubic. Since P and Q have contact, the results of Degtyarev say that F has a $P_8=T_{3,3,3}$ singularity and a combination of A_k–singularities, which is a splitting of A_{11} in the usual way. To show that any such splitting is possible, we have to prove that for any partition $\sum m_i=6$, $m_i\in\mathbb{N}$, of 6, there exists a quartic

Q which intersects P with the multiplicities $(2m_i)$. We compute in the Jacobian of the smooth cubic P, which is isomorphic to P. We think of the Jacobian as a complex torus given as the warp around of a parallelogram inside $\mathbb{C}$ determined by the numbers $1, \tau \in \mathbb{C} \setminus \mathbb{R}$. Clearly, we can find pairwise distinct points q_i in the interior of the parallelogram given by $\frac{1}{2}$, $\frac{\tau}{2}$ such that $\sum m_i q_i = \frac{\tau+1}{2}$ in $\mathbb{C}$. Then $\sum 2m_i q_i = 0$ in the Jacobian of P, i.e., $\sum 2m_i q_i$ is a principal divisor. Due to our choice of the q_i, all proper nontrivial subcombinations $\sum n_i q_i$, $0 \leq n_i \leq m_i$, are not principal. Since a plane cubic is projectively normal and $\deg \sum 2m_i q_i = 3 \cdot 4$, there is a quartic Q with $P \cap Q = \sum 2m_i q_i$. Now it remains to show that there is a linear symmetric matrix representation of Q such that the top 3×3–minor of this matrix is P, because we can obtain a matrix representation of F from this matrix by adding w to the bottom right entry. Since P is smooth, we find a self–linked ideal I with respect to (P, Q), i.e., $(P, Q) : I = I$ [M–B, Prop. 4.3]. Note that I does not contain a quadric polynomial. Namely, if $G \in I$ with $\deg G = 2$ then $G^2 \in (P, Q)$, i.e., $G^2 = \lambda Q + LP$ with $\lambda \in \mathbb{C}^*$ and $L \in \mathbb{C}[x, y, z]_1$. Hence, we would have $2G \cap P = Q \cap P$ and $G \cap P$ would be a principle subdivisor of $Q \cap P$ on P, which is impossible by the construction of Q. By [Be, 2.4] or [M–B, §4] such a self–linked ideal induces a linear symmetric matrix representation of Q with a contact cubic P. After a change of basis we may assume that P is the upper left 3×3–minor of this matrix. In fact, knowing that such a matrix exists, it can be easily constructed by Dixon's method [Di].

Case: P nodal cubic. From Appendix B we know that up to a choice of basis there are only two different linear symmetric matrix representations of a nodal cubic $P = x^3 + y^3 + xyz$, i.e., for a matrix M with $F = \det M$ we may assume that

$$M = \begin{pmatrix} -y & 0 & x & a_y y + a_z z \\ 0 & -x & y & b_y y + b_z z \\ x & y & z & c_y y + c_z z \\ a_y y + a_z z & b_y y + b_z z & c_y y + c_z z & w + f \end{pmatrix} \qquad \text{or}$$

$$M' = \begin{pmatrix} -y & \frac{1}{2}z & x & a_y y + a_z z \\ \frac{1}{2}z & -x & y & b_y y + b_z z \\ x & y & 0 & c_y y + c_z z \\ a_y y + a_z z & b_y y + b_z z & c_y y + c_z z & w + f \end{pmatrix},$$

where $f \in \mathbb{C}[x, y, z]$. The variable x was eliminated from the last column by adding a suitable multiples of the first three columns. Now one can compute Q and Q' and obtains that $(a_z, b_z) \neq 0$ resp. $c_z \neq 0$ for Q resp. Q' to be smooth at the singular point $(0:0:1)$ of P. To compute the intersection multiplicities of P and the quartics, we choose the parameterization $(s:t) \mapsto (-s^2 t : st^2 : (t-s)(s^2+st+t^2))$ of P which maps $(1:0)$ and $(0:1)$ to the singular point of P. Plugging it into the quartics we get

$$st \left(a_z s^5 - c_z s^4 t + (b_z - a_y) s^3 t^2 + (c_y - a_z) s^2 t^3 + (c_z - b_y) s t^4 - b_z t^5\right)^2$$

resp.

$$\left(-\tfrac{1}{2} c_z s^6 + b_z s^5 t + (a_z + \tfrac{1}{2} c_y) s^4 t^2 - b_y s^3 t^3 - (b_z + a_y) s^2 t^4 \right.$$
$$\left. - (a_z - \tfrac{1}{2} c_y) s t^5 + \tfrac{1}{2} c_z t^6\right)^2.$$

In the first case the polynomial of degree 5 is arbitrary apart from $(a_z, b_z) \neq 0$, and we can distribute its zeros arbitrarily with the exception that we cannot have zeros at $(1:0)$ and $(0:1)$ at the same time. In the second case the sextic polynomial can also have any combination of multiple zeros, but none of the zeros can be at the points that map to the singular point of P because of $c_z \neq 0$. Therefore, by Degtyarev's results we get the following possible combinations of singularities together with the usual splitting of the A–singularities:

$$T_{3,3,5+2k} + A_{9-2k}, \text{ for } k \in \{0,\dots,4\},\ T_{3,3,15}, \text{ and } T_{3,3,4} + A_{11}.$$

Case: P smooth quadric+secant. Again there are two linear symmetric matrix representations of $P = x(x^2 + yz)$, thus we may assume that

$$M = \begin{pmatrix} y & 0 & x & a_y y + a_z z \\ 0 & -x & 0 & b_y y + b_z z \\ x & 0 & -z & c_y y + c_z z \\ a_y y + a_z z & b_y y + b_z z & c_y y + c_z z & w + f \end{pmatrix} \quad \text{or}$$

$$M' = \begin{pmatrix} 0 & y & x & a_y y + a_z z \\ y & -x & \frac{1}{2}z & b_y y + b_z z \\ x & \frac{1}{2}z & 0 & c_y y + c_z z \\ a_y y + a_z z & b_y y + b_z z & c_y y + c_z z & w + f \end{pmatrix}.$$

The singularities of P are $(0:1:0)$ and $(0:0:1)$. Since Q resp. Q' must be smooth at these points, we find that $(a_z, b_z) \neq 0$ and $(b_y, c_y) \neq 0$ resp. $a_z \neq 0$ and $c_y \neq 0$. We parameterize the secant of P by $(s:t) \mapsto (0:s:t)$ and the quadric by $(s:t) \mapsto (st : -s^2 : t^2)$, thus in both cases $(1:0)$ and $(0:1)$ map to the singular points of P. To compute the intersection multiplicities of P and the quartics Q and Q', we pull the quartics back via these parameterizations and obtain

$$st\,(b_y s + b_z t)^2 \ \text{ and } \ -st\left(c_y s^3 + a_y s^2 t - c_z s t^2 - a_z t^3\right)^2$$

resp.

$$\left(c_y s^2 + (c_z - \tfrac{1}{2}a_y)st - \tfrac{1}{2}a_z t^2\right)^2 \ \text{ and }$$

$$\left(c_y s^4 + b_y s^3 t - (c_z + \tfrac{1}{2}a_y)s^2t^2 - b_z s t^3 + \tfrac{1}{2}a_z t^4\right)^2.$$

In the first case we can distribute the zeros arbitrarily with the exception that not both, the linear and the cubic polynomial, have zeros at points which are mapped to the same singular point of P. In the second case we cannot have zeros at the points that are mapped to the singular points of P, but any combination of multiple zeros can occur; hence F can have the following combinations of singularities with the usual splitting of the A–singularities:

$$T_{3,5,5} + A_5 + A_1,\ T_{3,5,7} + A_5,\ T_{3,5,9} + 2A_1,\ T_{3,5,11} + A_1,\ T_{3,7,7} + A_3,$$
$$T_{3,7,9} + A_1,\ T_{3,7,11}, \text{ and } T_{3,4,4} + A_3 + A_7.$$

Case: P three noncongruent lines. This is the last case where there are two nonequivalent linear symmetric matrix representations of the cubic. We take P as xyz and may assume that

$$M = \begin{pmatrix} x & 0 & 0 & a_y y + a_z z \\ 0 & y & 0 & b_x x + b_z z \\ 0 & 0 & z & c_x x + c_y y \\ a_y y + a_z z & b_x x + b_z z & c_x x + c_y y & w + f \end{pmatrix} \quad \text{or}$$

$$M' = \begin{pmatrix} 0 & x & \frac{1}{2}y & a_z z \\ x & 0 & z & b_y y + b_z z \\ \frac{1}{2}y & z & 0 & c_x x + c_y y + c_z z \\ a_z z & b_y y + b_z z & c_x x + c_y y + c_z z & w + f \end{pmatrix}.$$

In order for Q resp. Q' to be smooth at the singular points of P, we must have $(a_z, b_z) \neq 0$, $(a_y, c_y) \neq 0$, and $(b_x, c_x) \neq 0$ resp. $a_z \neq 0$, $b_y \neq 0$, and $c_x \neq 0$. We use the parameterizations $(s:t) \mapsto (0:s:t)$, $(s:t) \mapsto (t:0:s)$, and $(s:t) \mapsto (s:t:0)$, which map $(1:0)$ and $(0:1)$ to the singular points of P. Pulling Q and Q' back via these mappings gives

$$-st\,(a_y s + a_z t)^2\,,\ -st\,(b_z s + b_x t)^2\,,\ \text{and}\ -st\,(c_x s + c_y t)^2$$

resp.

$$\left(\tfrac{1}{2}b_y s^2 + \tfrac{1}{2}b_z st - a_z t^2\right)^2,\ \left(a_z s^2 - c_z st - c_x t^2\right)^2,\ \text{and}\ \left(c_x s^2 + c_y st - \tfrac{1}{2}b_y t^2\right)^2.$$

The above inequalities imply in the first case that the linear forms can only contribute to the intersection multiplicities of $P \cap Q$ at different singular points of P and in the second case that the quadrics cannot have zeros at the points that map to the singular points of P. Therefore, F can have the following combinations of singularities with the usual splitting of the A–singularities:

$$T_{5,5,5} + 3A_1,\ T_{5,5,7} + 2A_1,\ T_{5,7,7} + A_1,\ T_{7,7,7},\ \text{and}\ T_{4,4,4} + 3A_3.$$

Case: P cuspidal cubic. Up to a choice of coordinates there is only one representation of $P = x^3 + yz^2$ as a linear symmetric determinant; hence we may assume that

$$M = \begin{pmatrix} -y & 0 & x & a_y y + a_z z \\ 0 & -x & z & b_y y + b_z z \\ x & z & 0 & c_y y + c_z z \\ a_y y + a_z z & b_y y + b_z z & c_y y + c_z z & w + f \end{pmatrix}.$$

The condition that Q has no singular point at the singular point $(0:1:0)$ of P turns out to be $c_y \neq 0$. Plugging the parameterization $(s:t) \mapsto (t^2 s : -s^3 : t^3)$ of P into Q gives

$$t^2\left(c_y s^5 + b_y s^4 t - a_y s^3 t^2 - c_z s^2 t^3 - b_z st^4 + a_z t^5\right)^2.$$

Because of $c_y \neq 0$, the intersection multiplicity of P and Q is always 2 in the singular point of P, otherwise the quintic is arbitrary; hence F can have $Q_{11} + A_9$ as singularities as well as any combination of singularities obtained by splitting A_9 in the usual way.

Case: P smooth quadric+tangent. As there is only one linear symmetric matrix representation of $P = z(x^2 + yz)$, we may assume that

$$M = \begin{pmatrix} y & x & 0 & a_y y + a_z z \\ x & -z & 0 & b_y y + b_z z \\ 0 & 0 & -z & c_x x + c_y y \\ a_y y + a_z z & b_y y + b_z z & c_x x + c_y y & w + f \end{pmatrix}.$$

The meeting point of the quartic and the tangent is the singular point $(0:1:0)$ of P. For Q to be nonsingular at this point means $b_y^2 + c_y^2 \neq 0$. To compute the intersection multiplicities of P and Q, we parameterize the line and the quadric of P by $(s:t) \mapsto (t:s:0)$ and $(s:t) \mapsto (st : s^2 : -t^2)$. Both parameterizations map $(1:0)$ to the singular point of P. Pulling Q back via these parameterizations yields

$$t^2\,(c_y s + c_x t)^2\ \text{and}\ -t^2\left(b_y s^3 - a_y s^2 t - b_z st^2 + a_z t^3\right)^2.$$

Since $b_y^2 + c_y^2 \neq 0$, not both terms can contribute further to the intersection multiplicities of P and Q at the singular point of F at the same time; therefore F

can have the following combinations of singularities with the usual splitting of the A–singularities:

$$S_{1,0} + A_5 + A_1,\ S^{\#}_{1,2} + A_5,\ S^{\#}_{1,4} + 2A_1,\ \text{ and } S^{\#}_{1,6} + A_1.$$

Case: P three congruent lines, double line+line, triple line, empty set. All these cases lead to nonnormal quartics. Since a linear symmetric matrix representation $\tilde{M}$ of three congruent lines involves only the variables x and y, the rank of a 4×4-matrix M with $\tilde{M}$ in the upper left corner is only 2 along the line $\{x = y = 0\}$. Therefore, F is singular along this line. For the remaining cases one can apply Lemma 1.4 after a reshuffling of coordinates. $\square$

A List of rational double points on a linear symmetric determinantal quartic

We list the combinations of rational double points that occur on a linear symmetric determinantal quartic sorted by their Milnor number:

Milnor number 19

$D_{18}+A_1$	$D_{14}+A_5$	$D_{14}+A_3+2A_1$
$D_{12}+D_6+A_1$	$D_{12}+A_5+2A_1$	$D_{10}+D_8+A_1$
$D_{10}+D_6+A_3$	$D_{10}+A_9$	$D_{10}+A_7+2A_1$
$D_{10}+A_5+A_3+A_1$	$2D_8+3A_1$	$D_8+D_6+D_4+A_1$
$D_8+D_6+A_5$	$D_8+D_6+A_3+2A_1$	$D_8+D_4+A_5+2A_1$
$D_8+A_9+2A_1$	$D_8+A_5+A_3+3A_1$	$3D_6+A_1$
$2D_6+D_4+3A_1$	$2D_6+A_7$	$2D_6+A_5+2A_1$
$D_6+2D_4+A_3+2A_1$	$D_6+D_4+A_5+A_3+A_1$	D_6+A_{13}
$D_6+A_9+A_3+A_1$	$D_6+A_7+A_5+A_1$	$4D_4+3A_1$
$D_4+A_9+A_5+A_1$	$D_4+2A_5+A_3+2A_1$	A_{19}
$A_{17}+2A_1$	$A_{15}+A_3+A_1$	$A_{13}+A_5+A_1$
$A_{11}+A_7+A_1$	$A_{11}+A_5+3A_1$	$A_{11}+2A_3+2A_1$
$2A_9+A_1$	$A_9+A_7+3A_1$	$2A_7+A_3+2A_1$
$3A_5+4A_1$		

Milnor number 18

$D_{16}+2A_1$	$D_{14}+A_3+A_1$	$D_{14}+4A_1$
$D_{12}+D_4+2A_1$	$D_{12}+A_5+A_1$	$D_{12}+A_3+3A_1$
$D_{10}+D_6+2A_1$	$D_{10}+D_4+A_3+A_1$	$D_{10}+A_7+A_1$
$D_{10}+A_5+A_3$	$D_{10}+A_5+3A_1$	$D_{10}+2A_3+2A_1$
$2D_8+2A_1$	$D_8+D_6+A_3+A_1$	$D_8+D_6+4A_1$
$D_8+2D_4+2A_1$	$D_8+D_4+A_5+A_1$	$D_8+D_4+A_3+3A_1$
$D_8+A_9+A_1$	$D_8+A_7+3A_1$	D_8+2A_5
$D_8+A_5+A_3+2A_1$	$D_8+A_5+5A_1$	$D_8+2A_3+4A_1$
$2D_6+D_4+2A_1$	$2D_6+A_5+A_1$	$2D_6+2A_3$
$2D_6+A_3+3A_1$	$2D_6+6A_1$	$D_6+2D_4+A_3+A_1$
$D_6+2D_4+4A_1$	$D_6+D_4+A_7+A_1$	$D_6+D_4+A_5+A_3$
$D_6+D_4+A_5+3A_1$	$D_6+D_4+2A_3+2A_1$	$D_6+D_4+A_3+5A_1$
$D_6+A_{11}+A_1$	$D_6+A_9+A_3$	$D_6+A_9+3A_1$
$D_6+A_7+A_5$	$D_6+A_7+A_3+2A_1$	$D_6+2A_5+2A_1$
$D_6+A_5+2A_3+A_1$	$D_6+A_5+A_3+4A_1$	$4D_4+2A_1$
$3D_4+A_3+3A_1$	$3D_4+6A_1$	$2D_4+A_5+A_3+2A_1$
$2D_4+2A_3+4A_1$	$D_4+A_{13}+A_1$	$D_4+A_9+A_5$
$D_4+A_9+A_3+2A_1$	$D_4+A_7+A_5+2A_1$	$D_4+2A_5+A_3+A_1$
$D_4+2A_5+4A_1$	$D_4+A_5+2A_3+3A_1$	$A_{17}+A_1$
$A_{15}+A_3$	$A_{15}+3A_1$	$A_{13}+A_5$
$A_{13}+A_3+2A_1$	$A_{11}+A_7$	$A_{11}+A_5+2A_1$
$A_{11}+2A_3+A_1$	$A_{11}+A_3+4A_1$	$2A_9$
$A_9+A_7+2A_1$	$A_9+A_5+A_3+A_1$	$A_9+A_5+4A_1$
$A_9+2A_3+3A_1$	$2A_7+A_3+A_1$	$2A_7+4A_1$
$A_7+2A_5+A_1$	$A_7+A_5+A_3+3A_1$	$A_7+3A_3+2A_1$
$3A_5+3A_1$	$2A_5+2A_3+2A_1$	$2A_5+A_3+5A_1$

Milnor number 17

$D_{14}+3A_1$	$D_{12}+A_3+2A_1$	$D_{12}+5A_1$
$D_{10}+D_4+3A_1$	$D_{10}+A_5+2A_1$	$D_{10}+2A_3+A_1$
$D_{10}+A_3+4A_1$	$D_8+D_6+3A_1$	$D_8+D_4+A_3+2A_1$
$D_8+D_4+5A_1$	$D_8+A_7+2A_1$	$D_8+A_5+A_3+A_1$
$D_8+A_5+4A_1$	$D_8+2A_3+3A_1$	$D_8+A_3+6A_1$
$2D_6+A_3+2A_1$	$2D_6+5A_1$	$D_6+2D_4+3A_1$
$D_6+D_4+A_5+2A_1$	$D_6+D_4+2A_3+A_1$	$D_6+D_4+A_3+4A_1$
$D_6+D_4+7A_1$	$D_6+A_9+2A_1$	$D_6+A_7+A_3+A_1$
$D_6+A_7+4A_1$	$D_6+2A_5+A_1$	$D_6+A_5+2A_3$
$D_6+A_5+A_3+3A_1$	$D_6+A_5+6A_1$	$D_6+3A_3+2A_1$
$D_6+2A_3+5A_1$	$D_6+A_3+8A_1$	$3D_4+A_3+2A_1$
$3D_4+5A_1$	$2D_4+A_7+2A_1$	$2D_4+A_5+A_3+A_1$
$2D_4+A_5+4A_1$	$2D_4+2A_3+3A_1$	$2D_4+A_3+6A_1$
$2D_4+9A_1$	$D_4+A_{11}+2A_1$	$D_4+A_9+A_3+A_1$
$D_4+A_9+4A_1$	$D_4+A_7+A_5+A_1$	$D_4+A_7+A_3+3A_1$
$D_4+2A_5+A_3$	$D_4+2A_5+3A_1$	$D_4+A_5+2A_3+2A_1$
$D_4+A_5+A_3+5A_1$	$D_4+3A_3+4A_1$	$D_4+2A_3+7A_1$
$A_{15}+2A_1$	$A_{13}+A_3+A_1$	$A_{13}+4A_1$
$A_{11}+A_5+A_1$	$A_{11}+2A_3$	$A_{11}+A_3+3A_1$
$A_{11}+6A_1$	$A_9+A_7+A_1$	$A_9+A_5+A_3$
$A_9+A_5+3A_1$	$A_9+2A_3+2A_1$	$A_9+A_3+5A_1$
$2A_7+A_3$	$2A_7+3A_1$	A_7+2A_5
$A_7+A_5+A_3+2A_1$	$A_7+A_5+5A_1$	$A_7+3A_3+A_1$
$A_7+2A_3+4A_1$	$3A_5+2A_1$	$2A_5+2A_3+A_1$
$2A_5+A_3+4A_1$	$2A_5+7A_1$	$A_5+3A_3+3A_1$
$A_5+2A_3+6A_1$	$5A_3+2A_1$	

Milnor number 16

$D_{12}+4A_1$	$D_{10}+A_3+3A_1$	$D_{10}+6A_1$
$D_8+D_4+4A_1$	$D_8+A_5+3A_1$	$D_8+2A_3+2A_1$
$D_8+A_3+5A_1$	D_8+8A_1	$2D_6+4A_1$
$D_6+D_4+A_3+3A_1$	$D_6+D_4+6A_1$	$D_6+A_7+3A_1$
$D_6+A_5+A_3+2A_1$	$D_6+A_5+5A_1$	$D_6+3A_3+A_1$
$D_6+2A_3+4A_1$	$D_6+A_3+7A_1$	D_6+10A_1
$3D_4+4A_1$	$2D_4+A_5+3A_1$	$2D_4+2A_3+2A_1$
$2D_4+A_3+5A_1$	$2D_4+8A_1$	$D_4+A_9+3A_1$
$D_4+A_7+A_3+2A_1$	$D_4+A_7+5A_1$	$D_4+2A_5+2A_1$
$D_4+A_5+2A_3+A_1$	$D_4+A_5+A_3+4A_1$	$D_4+A_5+7A_1$
$D_4+3A_3+3A_1$	$D_4+2A_3+6A_1$	$D_4+A_3+9A_1$
D_4+12A_1	$A_{13}+3A_1$	$A_{11}+A_3+2A_1$
$A_{11}+5A_1$	$A_9+A_5+2A_1$	$A_9+2A_3+A_1$
$A_9+A_3+4A_1$	A_9+7A_1	$2A_7+2A_1$
$A_7+A_5+A_3+A_1$	$A_7+A_5+4A_1$	A_7+3A_3
$A_7+2A_3+3A_1$	$A_7+A_3+6A_1$	$3A_5+A_1$
$2A_5+2A_3$	$2A_5+A_3+3A_1$	$2A_5+6A_1$
$A_5+3A_3+2A_1$	$A_5+2A_3+5A_1$	$A_5+A_3+8A_1$
$5A_3+A_1$	$4A_3+4A_1$	$3A_3+7A_1$
$2A_3+10A_1$	$16A_1$	

Milnor number 15

$D_{10} + 5A_1$	$D_8 + A_3 + 4A_1$	$D_8 + 7A_1$
$D_6 + D_4 + 5A_1$	$D_6 + A_5 + 4A_1$	$D_6 + 2A_3 + 3A_1$
$D_6 + A_3 + 6A_1$	$D_6 + 9A_1$	$2D_4 + A_3 + 4A_1$
$2D_4 + 7A_1$	$D_4 + A_7 + 4A_1$	$D_4 + A_5 + A_3 + 3A_1$
$D_4 + A_5 + 6A_1$	$D_4 + 3A_3 + 2A_1$	$D_4 + 2A_3 + 5A_1$
$D_4 + A_3 + 8A_1$	$D_4 + 11A_1$	$A_{11} + 4A_1$
$A_9 + A_3 + 3A_1$	$A_9 + 6A_1$	$A_7 + A_5 + 3A_1$
$A_7 + 2A_3 + 2A_1$	$A_7 + A_3 + 5A_1$	$A_7 + 8A_1$
$2A_5 + A_3 + 2A_1$	$2A_5 + 5A_1$	$A_5 + 3A_3 + A_1$
$A_5 + 2A_3 + 4A_1$	$A_5 + A_3 + 7A_1$	$A_5 + 10A_1$
$5A_3$	$4A_3 + 3A_1$	$3A_3 + 6A_1$
$2A_3 + 9A_1$	$A_3 + 12A_1$	$15A_1$

Milnor number 14

$D_8 + 6A_1$	$D_6 + A_3 + 5A_1$	$D_6 + 8A_1$	$2D_4 + 6A_1$
$D_4 + A_5 + 5A_1$	$D_4 + 2A_3 + 4A_1$	$D_4 + A_3 + 7A_1$	$D_4 + 10A_1$
$A_9 + 5A_1$	$A_7 + A_3 + 4A_1$	$A_7 + 7A_1$	$2A_5 + 4A_1$
$A_5 + 2A_3 + 3A_1$	$A_5 + A_3 + 6A_1$	$A_5 + 9A_1$	$4A_3 + 2A_1$
$3A_3 + 5A_1$	$2A_3 + 8A_1$	$A_3 + 11A_1$	$14A_1$

Milnor number 13

$D_6 + 7A_1$	$D_4 + A_3 + 6A_1$	$D_4 + 9A_1$	$A_7 + 6A_1$	$A_5 + A_3 + 5A_1$
$A_5 + 8A_1$	$3A_3 + 4A_1$	$2A_3 + 7A_1$	$A_3 + 10A_1$	$13A_1$

Milnor number 12

$D_4 + 8A_1$ $A_5 + 7A_1$ $2A_3 + 6A_1$ $A_3 + 9A_1$ $12A_1$

Milnor number 11

$A_3 + 8A_1$ $11A_1$

Milnor number 10

$10A_1$

B Linear symmetric matrix representations of plane cubics

Finding linear symmetric matrix representations of plane cubics is a classical problem. The three representations of a smooth cubic were found by Hesse [He]. The matrix representations of the singular cubics are scattered throughout the literature. Most of them were computed by Barth [B] and Meyer-Brandis [M–B]. The case of the empty cubic is a special case of Atkinson [A]. For the following complete list the representation matrices of the singular cubics were computed using the straight–forward method of Barth or Taussky [T]. The remarkable fact is that a reduced singular cubic has two nonequivalent representations if it has only A_1–singularities and only one representation if it has another singularity. In the last column the number of accidental singularities is written down, i.e., the number of points of $\mathbb{P}^2$ where the matrix has only rank 1. This number distinguishes the two representations of the cubics with A_1–singularities.

cubic	equation	representation	number of rank 1 / parameter ranges
smooth	$x^3+y^3+z^3-\lambda xyz$	$\frac{-1}{\mu}\begin{pmatrix}\mu x & z & y\\ z & \mu y & x\\ y & x & \mu z\end{pmatrix}$	0 / $\mu^2+2\mu^{-1}=\lambda$
nodal	x^3+y^3+xyz	$\begin{pmatrix}-y & \frac{1}{2}z & x\\ \frac{1}{2}z & -x & y\\ x & y & 0\end{pmatrix}$	0
		$\begin{pmatrix}-y & 0 & x\\ 0 & -x & y\\ x & y & z\end{pmatrix}$	1
quadric+secant	$x(x^2+yz)$	$\begin{pmatrix}0 & y & x\\ y & -x & \frac{1}{2}z\\ x & \frac{1}{2}z & 0\end{pmatrix}$	0
		$\begin{pmatrix}y & 0 & x\\ 0 & -x & 0\\ x & 0 & -z\end{pmatrix}$	2
3 lines	xyz	$\begin{pmatrix}0 & x & \frac{1}{2}y\\ x & 0 & z\\ \frac{1}{2}y & z & 0\end{pmatrix}$	0
		$\begin{pmatrix}x & 0 & 0\\ 0 & y & 0\\ 0 & 0 & z\end{pmatrix}$	3
cuspidal	x^3+yz^2	$\begin{pmatrix}-y & 0 & x\\ 0 & -x & z\\ x & z & 0\end{pmatrix}$	1
quadric+tangent	$z(x^2+yz)$	$\begin{pmatrix}y & x & 0\\ x & -z & 0\\ 0 & 0 & -z\end{pmatrix}$	1
3 congruent lines	$x(x^2+y^2)$	$\begin{pmatrix}0 & y & x\\ y & -x & \frac{1}{2}y\\ x & \frac{1}{2}y & 0\end{pmatrix}$	1
double line+line	x^2y	$\begin{pmatrix}az & x & bz\\ x & 0 & 0\\ bz & 0 & -y\end{pmatrix}$	1 or line / $a,b\in\mathbb{C}$
triple line	x^3	$\begin{pmatrix}az & by & x\\ by & -x & 0\\ x & 0 & 0\end{pmatrix}$	0 / $a,b\in\{0,1\}$
empty cubic	$\emptyset$	$\begin{pmatrix}* & * & 0\\ * & * & 0\\ 0 & 0 & 0\end{pmatrix}$	$---$
empty cubic	$\emptyset$	$\begin{pmatrix}* & * & *\\ * & 0 & 0\\ * & 0 & 0\end{pmatrix}$	$---$

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