The p-adic Zeta Functions of Chevalley Groups

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Introduction

Let p be a prime and $\mathbb{Q}_p$ the field of p-adic numbers with the ring of p-adic integers $\mathbb{Z}_p$. An easy computation shows

$$\int_{\mathbb{Z}_p} |x|_p^s \, d\mu(x) = \frac{p-1}{p} \cdot \frac{1}{1 - \frac{1}{p^{1+s}}}, \tag{1}$$

where $|\cdot|_p$ denotes the p-adic valuation and μ is the (additive) Haar measure on $\mathbb{Q}_p$, normalized such that $\mu(\mathbb{Z}_p) = 1$.

This formula occurred for the first time as an important ingredient of Tate's theory of the Riemann Zeta function. Its success gave rise to A. Weil's generalization:

Definition 0.1. *Let k be a finite extension field of $\mathbb{Q}_p$, let G be a linear algebraic group over k and let $\rho : G \to \mathrm{GL}_n$ be a k-rational representation. We define the **Zeta function of the algebraic group G at the representation ρ** to be*

$$Z_{G(k),\rho}(s) = \int_{G^+} |\det(\rho(g))|_p^s \; \mu_G(g),$$

where $G^+ = \rho^{-1}(\rho(G(k)) \cap M_n(\vartheta_k))$, ϑ_k is the ring of integers of k and μ_G denotes the right Haar measure on $G(k)$, normalized so that $\mu_G(G(\vartheta_k)) = 1$.

Many attempts to evaluate this integral for other groups than just $\mathbb{Q}_p$, hoping to find similar formulas as (1), have been made, for example by Weil, Macdonald, Igusa, Lubotzky, du Sautoy and Grunewald.

This work is based on a formula, that has been developed by M. du Sautoy and A. Lubotzky (see [8]) and a bit earlier in a slightly different setting by J.I. Igusa (see [7]). For Chevalley groups the result of du Sautoy and Lubotzky is:

Proposition 0.2. *Let G be a Chevalley group over $\mathbb{Q}_p$, let $\rho : G \to \mathrm{GL}_n$ be an irreducible $\mathbb{Q}_p$-rational representation and let ω be the dominant weight of the contragredient representation $\rho^* = {}^t\rho^{-1}$. Let $\{\alpha_1, \ldots, \alpha_\ell\}$ be a basis for* *the root system Φ of the derived group G' of $G(\mathbb{Q}_p)$. Define two polynomials $P_{G,\rho}(X,Y)$, $Q_{G,\rho}(X,Y) \in \mathbb{Z}[X,Y]$ by*

$$P_{G,\rho}(X,Y) = \sum_{w\in\mathcal{W}} X^{-\lambda(w)} \prod_{\alpha_k\in w(\Phi^-)} X^{a_k} Y^{b_k(\omega)} \tag{2}$$

$$\text{and} \quad Q_{G,\rho}(X,Y) = (1-Y^m)\prod_{k=1}^{\ell}(1-X^{a_k}Y^{b_k(\omega)}), \tag{3}$$

where m is the order of G', $\mathcal{W}$ denotes the Weyl group of Φ and λ is the length function of $\mathcal{W}$. ℓ is the rank of the root system and Φ^- is the set of negative roots. The a_k are certain natural numbers associated to Φ and the b_k are certain linear maps from the set of dominant weights to $\mathbb{N}_0$. For a precise description of the a_k and b_k see chapter 2.

Then

$$Z_{G,\rho,p}(s) = \frac{P_{G,\rho}(p,p^{-(n/m)s})}{Q_{G,\rho}(p,p^{-(n/m)s})}.$$

Remark. *(a) Unless in chapter 4, the particular choice of the representation will play no role in this thesis; thus we will often be interested in the more general objects*

$$P_{G,\vec{b}}(X,Y) = \sum_{w\in\mathcal{W}} X^{-\lambda(w)} \prod_{\alpha_k\in w(\Phi^-)} X^{a_k} Y^{b_k}$$

and $Q_{G,\vec{b}}(X,Y)$ respectively, where $\vec{b} = (b_1,\ldots,b_k)$ is allowed to be any element of $\mathbb{N}^\ell$. We will leave the subscribt ρ or $\vec{b}$ out, if any choice for $\vec{b}$ is allowed; an additional subscript ρ will express explicitly, that the polynomial associated to a special representation is meant.

(b) Note that for any irreducible representation ρ both $P_{G,\rho}(X,Y)$ and $Q_{G,\rho}(X,Y)$ are independent of p, thus we have $Z_{G,\rho,p}(s) = \frac{P_{G,\rho}(p,p^{-s})}{Q_{G,\rho}(p,p^{-s})}$ for all primes p. J.I. Igusa calls Zeta functions with this quality ***universal****.*

(c) M. du Sautoy and A. Lubotzky formulated this result not only for $\mathbb{Q}_p$ but for every finite extension of $\mathbb{Q}_p$. We confine ourselves to $\mathbb{Q}_p$, because we are only interested in the resulting polynomials P_G and Q_G, and these are independent of the particular extension of $\mathbb{Q}_p$.

The numerator polynomials $Q_G(X,Y)$ obviously lead to factors as in (1), and for the special case, where G is the Chevalley group $\mathrm{GL}_{\ell+1}$ ($\ell \in \mathbb{N}$) in its natural representation ρ, also the numerator polynomial $P_{G,\rho}(X,Y)$ has this property. This is an immediate consequence of a formula from I.G. Macdonald (see [2]). The derived group of $\mathrm{GL}_{\ell+1}$ is $\mathrm{SL}_{\ell+1}$ and has the root system A_ℓ. The Weyl group is isomorphic to the symmetric group $\mathcal{S}_{\ell+1}$. Macdonald's formula for the numerator polynomial is

$$P_{G,\rho}(X,Y) = \prod_{k=0}^{\ell} \Big(\sum_{j=0}^{k} X^{\ell-k} Y \Big)^{j}.$$

We see, that $P(X,Y)$ has one factor which is one monomial, one factor which is the sum of two monomials, one factor which is the sum of three monomials and so on. The question comes up, whether it might be possible to find formulae for other Chevalley groups, in which the structure of the associated Weyl group becomes similarly visible. Unfortunately this is impossible; in most cases the polynomials have an irreducible factor which is the sum of $\ell!$ many monomials.

The groups we will consider in this thesis are the group of orthogonal similitudes $\mathrm{GO}_{2\ell+1}$, the group of symplectic similitudes $\mathrm{GSp}_{2\ell}$ and the group of orthogonal similitudes whose Spinor-norm is 1, $\mathrm{GO}^{+}_{2\ell}$.

We will refer to these groups as "groups of type B_ℓ, C_ℓ and D_ℓ", since the root systems of their commutator subgroups (which are also their derived groups) are of type B_ℓ, C_ℓ and D_ℓ. Note that the root systems of the entire groups have an additional summand A_1.

The Weyl groups of the root systems B_ℓ and C_ℓ are isomorphic to $\mathcal{S}_\ell \ltimes \mathcal{P}(\{1,\dots,\ell\})$ (via ι, say), where the group operation of the power set $\mathcal{P}(\{1,\dots,\ell\})$ is the symmetric difference. The Weyl group of the root system D_ℓ is isomorphic to $\mathcal{S}_\ell \ltimes \mathcal{P}_{\mathrm{even}}(\{1,\dots,\ell\})$, where $\mathcal{P}_{\mathrm{even}}(\{1,\dots,\ell\})$ denotes the subgroup of $\mathcal{P}(\{1,\dots,\ell\})$, which consists of the subsets of $\{1,\dots,\ell\}$ with even cardinality.

Under the conditions of proposition 0.2 we define for $w \in \mathcal{W}$

$$M_w(X,Y) := X^{-\lambda(w)} \prod_{\alpha_k \in w(\Phi^-)} X^{a_k} Y^{b_k}.$$

In these terms we can formulate the first of our two most important results:

Theorem 0.3. *(a) Let G be the Chevalley group $\mathrm{GO}_{2\ell+1}$ or $\mathrm{GSp}_{2\ell}$. Then the numerator polynomial (2) is*[1]

$$P_G(X,Y) = \sum_{\sigma \in \mathcal{S}_\ell} M_{\iota^{-1}(\sigma,\emptyset)}(X,Y) \cdot \prod_{k=1}^{\ell} \left(1 + \frac{M_{\iota^{-1}(\sigma,\{k\})}(X,Y)}{M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)} \right). \quad (4)$$

[1] We omit one pair of brackets, writing $\iota^{-1}(\sigma,S)$ instead of $\iota^{-1}((\sigma,S))$.

(b) *Let G be the Chevalley group $\mathrm{GO}_{2\ell}^{+}$. Then the numerator polynomial (2) is*

$$P_G(X,Y) = \sum_{\sigma\in\mathcal{S}_\ell} M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)\cdot\prod_{k=1}^{\ell-1}\left(1+\frac{M_{\iota^{-1}(\sigma,\{k,\ell\})}(X,Y)}{M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)}\right). \quad (5)$$

The structure of the Weyl groups becomes partly visible in these formulas; in particular we see the $\mathcal{P}(\{1,\ldots,\ell\})$ component (the $\mathcal{P}_{\text{even}}(\{1,\ldots,\ell\})$ component, respectively).

For groups of type B_ℓ we are now going to outline briefly our derivation of formula (4) from formula (2). The root system B_ℓ is the subset of $\mathbb{R}^\ell$, that consists of all vectors with integer coordinates and squared length 1 or 2. A basis is given by $\alpha_1,\ldots,\alpha_\ell$, where $\alpha_k = e_k - e_{k+1}$ for $k<\ell$ and $\alpha_\ell = e_\ell$. The Weyl group $\mathcal{W}$ is defined as the group of linear maps $\mathbb{R}^\ell\to\mathbb{R}^\ell$ generated (via composition) by the reflections at the hyperplanes orthogonal to the elements of our basis. These generators ς_{α_k} (or ς_k) are thus called "fundamental reflections" and they act on a vector $x\in\mathbb{R}^\ell$ as follows:

- for $k<\ell$: ς_{α_k} exchanges the k-th and the $(k+1)$-st coordinate.
- ς_{α_ℓ} changes the sign of the ℓ-th coordinate.

So the elements of the Weyl group act on a vector $x\in\mathbb{R}^\ell$ by permuting the coordinates and changing the signs of some of them.

For any element $w\in\mathcal{W}$ of the Weyl group this action can obviously be represented by a permutation $\sigma\in\mathcal{S}_\ell$ and a set $S\subset\{1,\ldots,\ell\}$ if we let for $(x_k)_{k\in\{1,\ldots,\ell\}}\in\mathbb{R}^\ell$

$$(\sigma,S)(x_k)_{k\in\{1,\ldots,\ell\}} = \big((-1)^{\chi_S(\sigma^{-1}(k))}x_{\sigma^{-1}(k)}\big)_{k\in\{1,\ldots,\ell\}}.$$

There exists a group isomorphism

$$\iota:\mathcal{W}\to\mathcal{S}_\ell\ltimes\mathcal{P}(\{1,\ldots,\ell\}),$$

where the group operation $\bullet$ in $\mathcal{S}_\ell\ltimes\mathcal{P}(\{1,\ldots,\ell\})$ is described by

$$(\sigma_1,S_1)\bullet(\sigma_2,S_2) = (\sigma_1\circ\sigma_2,\sigma_2^{-1}(S_1)\triangle S_2)$$

for $(\sigma_1,S_1),(\sigma_2,S_2)\in\mathcal{S}_\ell\ltimes\mathcal{P}(\{1,\ldots,\ell\})$. It is induced by the composition on the one hand and the symmetric difference on the other.

The images of the fundamental reflections under this isomorphism are

$$\iota(\varsigma_k) = \big((k,k+1),\emptyset\big)\quad\text{for } k<\ell\quad\text{and}\quad\iota(\varsigma_\ell) = \big(\mathrm{id}\,,\{\ell\}\big).$$

We seek for a better understanding of the formula for the numerator polynomials, therefore we shall develop an easy to handle criterion for deciding the question, which primitive roots lie in the image of the positive roots and which lie in the image of the negative roots under a given element of the Weyl group. The set of positive roots has the following explicit description:

Lemma 0.4. *Choose* $\alpha_k = e_k - e_{k-1}$ $(k \in \{1, \ldots, \ell-1\})$ *and* $\alpha_\ell = e_\ell$ *as a basis for the root system* B_ℓ*. Then the set of positive roots is*

$$\Phi^+ = \{e_i \,|\, 1 \leq i \leq \ell\} \cup \{e_i - e_j \,|\, 1 \leq i < j \leq \ell\} \cup \{e_i + e_j \,|\, 1 \leq i < j \leq \ell\}.$$

If we restrict ourselves to the elements of the Weyl subgroup which do only permute coordinates, it is due to the last lemma easy to decide, whether the preimage of a simple root lies in Φ^+ or Φ^-. The result is:

Proposition 0.5. *Let* $\mathcal{W}$ *be the Weyl group of the root system* B_ℓ *and let* $w \in \mathcal{W}$ *with* $\iota(w) = (\sigma, \emptyset)$ *for a suitable* $\sigma \in \mathcal{S}_\ell$*. Then*

- $\alpha_\ell \in w(\Phi^+)$
- $\alpha_k \in w(\Phi^+) \Leftrightarrow \sigma^{-1}(k) < \sigma^{-1}(k+1)$ *for* $k \in \{1, \ldots, \ell-1\}$*.*

Under our restriction the length can be described in a similar manner:

Lemma 0.6. *Let* $\mathcal{W}$ *be the Weyl group of the root system* B_ℓ *and let* $w \in \mathcal{W}$ *with* $\iota(w) = (\sigma, \emptyset)$ *for a suitable* $\sigma \in \mathcal{S}_\ell$*. Then*

$$\begin{aligned} \lambda(w) &= \sharp\{(i,j) \in \{1, \ldots, \ell\}^2 \,|\, i < j \,\wedge\, \sigma(i) > \sigma(j)\} \\ &= \sharp\{(i,j) \in \{1, \ldots, \ell\}^2 \,|\, i < j \,\wedge\, \sigma^{-1}(i) > \sigma^{-1}(j)\}. \end{aligned}$$

We will refer to the cardinalities of the above sets as $\mathrm{inv}(\sigma)$ *(and* $\mathrm{inv}(\sigma^{-1})$*, respectively) (*inv *for "number of inversions").*

The search for such statements for the rest of $\mathcal{W}$ took a long time. The key to this was then found in expressing the elements of $\mathcal{W}$ as a special type of products of fundamental reflections, which seems to be the canonical description in our context. First we define

Definition 0.7. *For* $s \in \{1, \ldots, \ell\}$ *we define*

$$\tau_s := \varsigma_s \circ \cdots \circ \varsigma_\ell.$$

In the language of $\mathcal{S}_\ell \ltimes \mathcal{P}(\{1, \ldots, \ell\})$ this means

$$\iota(\tau_s) = \left(\begin{pmatrix} 1 & \cdots & s-1 & s & s+1 & \cdots & \ell \\ 1 & \cdots & s-1 & \ell & s & \cdots & \ell-1 \end{pmatrix}, \{s\} \right).$$

τ_s denotes the unique Weyl group element of shortest length, that changes the sign of the s-th coordinate when applied to an ℓ-vector (more precisely, $-x_s$ becomes the ℓ-th coordinate).

Additionally we make the following

Definition 0.8. *Let $\mathcal{W}$ denote the Weyl group of the root system B_ℓ. Let $w \in \mathcal{W}$ with $\iota(w) = (\sigma, S)$, $S = \{s_1, \ldots, s_m\}$ with $s_1 < \cdots < s_m$ and $S^c = \{t_1, \ldots, t_n\}$ with $t_1 < \cdots < t_n$. Then we define a permutation $\pi(w)$ by*

$$\begin{aligned} \sigma(s_k) &= \pi(w)(\ell + 1 - k) \quad \textit{for } k \in \{1, \ldots, m\} \\ \textit{and} \quad \sigma(t_k) &= \pi(w)(k) \quad \textit{for } k \in \{1, \ldots, n\}. \end{aligned}$$

This induces a map $\pi : \mathcal{W} \to \mathcal{S}_\ell$.

With some combinatorics it is then possible to prove

Proposition 0.9. *Let $\mathcal{W}$ be the Weyl group of the root system B_ℓ and let $w \in \mathcal{W}$ with $\iota(w) = (\sigma, S)$ where $S = \{s_1, \ldots, s_m\}$ with $s_1 < \cdots < s_m$. Then*

$$(\sigma, S) = (\pi(w), \emptyset) \bullet \iota(\tau_{s_1} \circ \cdots \circ \tau_{s_m})$$

Note that the function π is not a group isomorphism; it does not preserve the group operation. Anyway, it plays the key role in getting the information about the elements of the Weyl group we need for our formula much faster. That is because of

Proposition 0.10. *Let $\mathcal{W}$ be the Weyl group of the root system B_ℓ and let $w \in \mathcal{W}$ with $\iota(w) = (\sigma, S)$ where $S = \{s_1, \ldots, s_m\}$ with $s_1 < \cdots < s_m$.*

a) The length of w equals the sum of the length of the factors in formula (4), that is:

$$\lambda(w) = \mathrm{inv}(\pi(w)) + \sum_{k=1}^{m} (\ell - s_k + 1)$$

b)

$$\alpha_\ell \in w(\Phi^+) \;\Leftrightarrow\; \sigma^{-1}(\ell) \in S^c.$$

And for $k < \ell$:

$$\alpha_k \in w(\Phi^+) \;\Leftrightarrow\; \alpha_k \in (\pi(w), \emptyset)(\Phi^+)$$

The proof of the following result requires 0.5–0.10:

Proposition 0.11. *Let $\mathcal{W}$ be the Weyl group of the root system B_ℓ and $w \in \mathcal{W}$ with $\iota(w) = (\sigma, S)$ and $k \in S^c$. Then*

$$M_{\iota^{-1}(\sigma, S \cup \{k\})}(X, Y) = \frac{M_w(X, Y) \cdot M_{\iota^{-1}(\sigma, \{k\})}(X, Y)}{M_{\iota^{-1}(\sigma, \emptyset)}(X, Y)}.$$

In the proof of this proposition we will see, that each pair $(i,j) \in \{1,\ldots,\ell\}^2$ appears the same number of times as an inversion of $\pi(w)$ and of $\pi(\iota^{-1}(\sigma,\{k\}))$ as of $\pi(\iota^{-1}(\sigma,\emptyset))$ and $\pi(\iota^{-1}(\sigma,S\cup\{k\}))$.

Applying this proposition inductively we obtain

Corollary 0.12. *Let $\mathcal{W}$ be the Weyl group of the root system B_ℓ and $w \in \mathcal{W}$ with $\iota(w) = (\sigma, S)$. Then*

$$M_w(X,Y) = M_{\iota^{-1}(\sigma,\emptyset)}(X,Y) \prod_{s\in S} \frac{M_{\iota^{-1}(\sigma,\{s\})}(X,Y)}{M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)}.$$

This corollary offers a new sort of natural objects:

Definition 0.13. *1. For the root systems B_ℓ and C_ℓ let for $\sigma \in \mathcal{S}_\ell$*

$$\begin{aligned} P_\sigma(X,Y) &:= \sum_{S\subset\{1,\ldots,\ell\}} M_{\iota^{-1}(\sigma,S)}(X,Y) \\ &= M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)\cdot\prod_{k=1}^{\ell}\left(1+\frac{M_{\iota^{-1}(\sigma,\{k\})}(X,Y)}{M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)}\right). \end{aligned}$$

2. For the root system D_ℓ let for $\sigma \in \mathcal{S}_\ell$

$$\begin{aligned} P_\sigma(X,Y) &:= \sum_{S\subset\{1,\ldots,\ell\},\ \sharp S \text{ even}} M_{\iota^{-1}(\sigma,S)}(X,Y) \\ &= \sum_{\sigma\in\mathcal{S}_\ell} M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)\cdot\prod_{k=1}^{\ell}\left(1+\frac{M_{\iota^{-1}(\sigma,\{k,\ell\})}(X,Y)}{M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)}\right). \end{aligned}$$

At this point our formula (3) is clear, because obviously

$$P_G(X,Y) = \sum_{\sigma\in\mathcal{S}_\ell} P_\sigma(X,Y).$$

We should point out, that one typical feature of Zeta functions becomes easily visible through formula (3): The existence of a functional equation. For our type of Zeta functions this feature was first discovered in [7], and was then further investigated in [8]. In terms of our two variable polynomials the functional equation is

$$\frac{P(X^{-1},Y^{-1})}{Q(X^{-1},Y^{-1})} = (-1)^{\ell+1} X^{\mathrm{card}(\Phi^+)} Y^m \frac{P(X,Y)}{Q(X,Y)}.$$

We can use definition 0.1 to define Zeta functions of algebraic groups over global fields. We restrict the definition from M. duSautoy and F. Grunewald made in [11] to the case of $\mathbb{Q}$.

Definition 0.14. *Let G be a linear algebraic group over $\mathbb{Q}$, let p be a prime and let $\rho : G \to \mathrm{GL}_n$ be a $\mathbb{Q}$-rational representation. Then we define the* ***local Zeta function of the algebraic group G at the representation ρ and the prime p*** *to be*

$$Z_{G(\mathbb{Q}),\rho,p}(s) = Z_{G(\mathbb{Q}_p),\rho}(s).$$

Secondly we define the ***global Zeta function of the algebraic group G at the representation ρ*** *to be the Euler product*

$$Z_{G(\mathbb{Q}),\rho}(s) = \prod_{p \text{ prime}} Z_{G(\mathbb{Q}),\rho,p}(s).$$

It has become a matter of common understanding, that such an Euler product has to offer the possibility of meromorphic continuation to deserve the "Zeta" label. Unfortunately, the Euler products appearing in our context are not meromorphically continuable to the whole of $\mathbb{C}$. Therefore the question arose, which rational functions lead to Euler products that do have this property. In this context we will use the following terminology:

Definition 0.15. *a) $W(X,Y) \in \mathbb{C}(X,Y)$ is called* ***friendly*** *if there exist cyclotomic polynomials $g_k(U) \in \mathbb{C}[U]$, $k = 1,\dots,n$ and integers u_k, v_k, such that $W(X,Y) = \prod_{k=1}^{n} g_k(X^{u_k}Y^{v_k})$. Otherwise $W(X,Y)$ is called* ***unfriendly****.*

b) Let $W(X,Y) \in \mathbb{C}(X,Y)$. The Euler product

$$Z(s) = \prod_{p \text{ prime}} W(p, p^{-s})$$

is called ***friendly*** *if it is meromorphically continuable to the whole of $\mathbb{C}$. Otherwise it is called* ***unfriendly****.*

Using these terms we can formulate a result from M. du Sautoy (see [10]) as:

Proposition 0.16. *Let $W(X,Y) \in \mathbb{C}(X,Y)$ be friendly. Then the corresponding Euler product $Z(s) = \prod_{p \text{ prime}} W(p,p^{-s})$ is friendly as well.*

The denominator polynomials (formula (3)) are obviously friendly and the $P_\sigma(X,Y)$ defined in 0.13 are friendly polynomials, which are shifted by some monomial factor. But the numerators as a whole are unfriendly and restrict the possibility of meromorphic continuation of the Euler product.

The reverse direction of the upper proposition still has the status of a conjecture. In [10] the author explains how to continue such an Euler product of an unfriendly polynomial up to a certain natural boundary. In most cases this natural boundary can be read off from the so-called **ghost polynomial**.

The ghost polynomial $\tilde{P}(X,Y)$ of a polynomial $P(X,Y)$ is constructed from the algebraic curve $P(X,Y)=0$ as $X\to\infty$, such that the algebraic curve $\tilde{P}(X,Y)=0$ approximates $P(X,Y)=0$.

A precise description of this construction requires the Puiseux power series expansion, but one stops the approximation of each branch after the first step. Since the ghost polynomial depends on the Newton polygon and the coefficients related to the points on it only, the location of the natural boundary for meromorphic continuation of the Euler product depends on these information only, too. The ghost polynomials for all polynomials that appear in the context of the Chevalley groups of type A_ℓ–D_ℓ in their natural representations are constructed in [11]. For a table of their results see proposition 4.2.

Such ghost polynomials are – if certain coefficients, i.e. the constant coefficient, are 1; a condition that is satisfied for the polynomials that come up with our universal Zeta functions – in many cases polynomials like the denominator polynomials of the explicit formula, that is they are products of factors of type $(1+X^{u_k}Y^{v_k})$ or $(1-X^{u_k}Y^{v_k})$. This means in particular, that they are friendly in most of our cases.

The denominator polynomials are in every case identical with their own ghost. Surprisingly, the ghost polynomials of our numerator polynomials seem to have a much deeper connection to the original polynomials than one would expect from the process that generates them. The reason why we believe this, and the starting point of the research leading to the result of the work on hand was an examination of the singularities of the algebraic curves associated to our two variable polynomials. The interested reader is referred to chapter 5 for some explanations, why we believe that investigating the singularities of such polynomials may be a suitable method for finding symmetries.

In particular, we found that in case of the natural representation the ghost polynomial can be expressed as in formula (3) by summing up not over all of $\mathcal{S}_\ell$, but only over a certain subset. Setting for $k<\ell$

$$\rho_k = \begin{pmatrix} 1 & \dots & k-1 & k & k+1 & \dots & \ell \\ 1 & \dots & k-1 & \ell & \ell-1 & \dots & k \end{pmatrix}$$

we will prove in chapter 4 our second important result:

Proposition 0.17. *(a) Let $P(X,Y)$ be the numerator polynomial associated to the Chevalley group* $\mathrm{GO}_{2\ell+1}$ *(type* B_ℓ*) in its natural representation. Then its ghost is*

$$\tilde{P}(X,Y) = P_{\mathrm{id}}(X,Y).$$

(b) Let $P(X,Y)$ be the numerator polynomial associated to the Chevalley group $\mathrm{GSp}_{2\ell}$ *(type* C_ℓ*) or* $\mathrm{GO}_{2\ell}^{+}$ *(type* D_ℓ*) in its natural representation. Then its ghost is*

$$\tilde{P}(X,Y) = \sum_{(v_1,\ldots,v_{\ell-1})\in\{0,1\}^{\ell-1}} P_{\rho_1^{v_1}\circ\cdots\circ\rho_{\ell-1}^{v_{\ell-1}}}(X,Y).$$

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Chapter 1

Root Systems and Weyl Groups

This thesis deals with the Zeta functions of algebraic groups defined (as in the introduction) by p-adic integration via the Haar measure of the considered group.

For our purpose, however, this very definition will not be of great interest; our starting point is in fact a particular description of these Zeta functions in terms of the associated root system. It was developed in [7] and [8]. This description was already mentioned in the introduction, and for writing it down as well as for the following research it is necessary to have some knowledge about root systems and Weyl groups. For a detailed treatise on this topic see [2] and [3].

1.1 Basic Definitions and Examples

Throughout this chapter E will always denote some $\mathbb{R}^\ell$ with a scalar product $(\cdot\,,\cdot) : E\times E \to E$. For each nonzero $\alpha \in E$ we define a corresponding **reflection** $\varsigma_\alpha : E \to E$ by $\varsigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$. It is an orthogonal automorphism of E, which leaves the **reflecting hyperplane** $P_\alpha = \{\beta \in E | (\beta, \alpha) = 0\}$ pointwise fixed.

In the abstract sense a **root system** is a subset Φ of E which satisfies the following properties:

(R1) Φ is finite, spans E and does not contain 0.

(R2) If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.

(R3) If $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant.

(R4) If α, $\beta \in \Phi$, then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

The **Weyl group** $\mathcal{W}$ of the root system Φ is then defined to be the subgroup of $\mathrm{GL}(E)$ generated by the reflections ς_α, where $\alpha \in \Phi$.

There is only one root system of rank 1, which is clear by (R2). Given any nonzero real number α the root system consists of nothing but $\pm\alpha$. It is called A_1 and its Weyl group is isomorphic to $\mathcal{S}_2$.

For rank 2 there are four different root systems. We can describe them by drawing the following pictures:

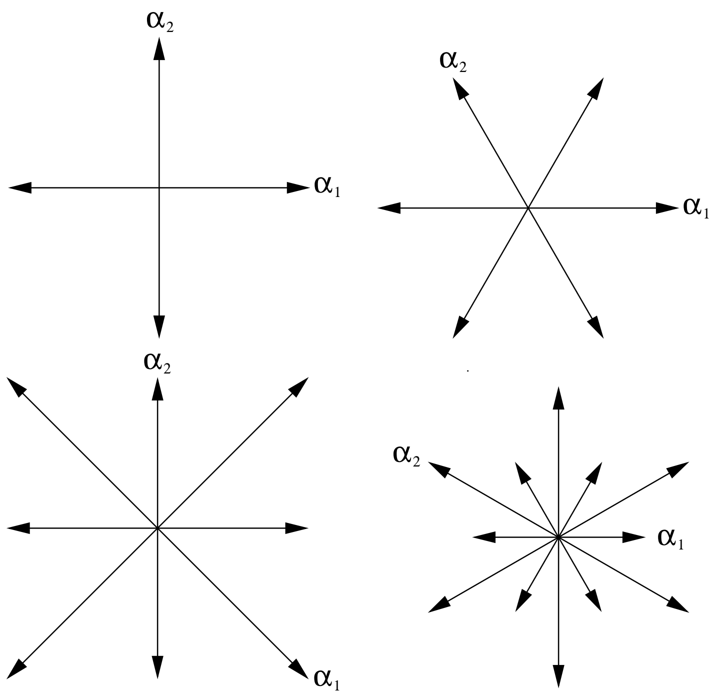


Figure 1.1: Root systems of rank 2: $\mathrm{A}_1 \times \mathrm{A}_1$, A_2, B_2, G_2

The proof of the statement, that there are no other root systems of rank 2 than these, is mainly based on the fact, that for a given nonorthogonal pair of roots only certain ratios of their squared lengths can occur, namely $1:1$, $2:1$ and $3:1$ (this follows immediately from the axioms, especially (R4) is important), which means that only angles of π, $\pi/2$, $\pi/3$, $2\pi/3$, $\pi/4$, $3\pi/4$, $\pi/6$ and $5\pi/6$ can occur. This observation can be used to classify the root systems of higher ranks as well (see section 1.3).

1.2 Bases and Weyl Chambers

To classify the root systems we make use of bases. For a proof of their existence see [2] again.

Definition 1.1. *A subset Δ of Φ is called a* ***basis*** *if:*

(B1) *Δ is a basis of E.*

(B2) *each root β can be written as $\beta = \sum m_\alpha \alpha$ $(\alpha \in \Delta)$ with integral coefficients m_α all nonnegative or nonpositive.*

The elements of Δ are called ***simple roots****.*

The roots which can be obtained as in (B2) as sums of simple roots with nonnegative coefficients are called positive roots, and the set consisting of all of them is denoted by Φ^+. On the other hand there is the set Φ^- of negative roots, and we have $\Phi^- = -\Phi^+$. For these sets to be sensible notations it has to be shown that the described linear combinations of the roots are unique.

Once we have a basis Δ, we can define

Definition 1.2. *Let $\mathcal{W}$ be a Weyl group and let $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ be a basis. Then we define the* ***length function*** *of $\mathcal{W}$ to be*

$$\begin{aligned} \lambda : \mathcal{W} &\rightarrow \mathbb{N}_0, \\ w &\mapsto \min\{n \in \mathbb{N}_0 : \exists k_1, \dots k_n \textit{ such that } \varsigma_{\alpha_{k_1}} \circ \dots \circ \varsigma_{\alpha_{k_n}} = w\}. \end{aligned}$$

The length of $w \in \mathcal{W}$ equals the number of positive roots, which w maps onto negative.

The bases of a given root system are in one-to-one correspondence with the so-called **Weyl chambers**. This correspondence can be described as follows:

Let Φ be a root system and let $\gamma \in E \backslash \bigcup_{\alpha\in\Phi} P_\alpha$. Then γ lies for each root either on the "positive" side ($(\gamma, \alpha) > 0$) or on the "negative" ($(\gamma, \alpha) < 0$) of the reflecting hyperplane P_α. $\gamma' \in E \setminus \bigcup_{\alpha\in\Phi} P_\alpha$ shall represent the same class as γ does, if for every root $\alpha \in \Phi$ both γ and γ' lie on the same side of P_α. The equivalence class of γ is called the **Weyl chamber** of γ. The Weyl chambers are the connected components of $E \setminus \bigcup_{\alpha\in\Phi} P_\alpha$, and each vector $x \in E \setminus \bigcup_{\alpha\in\Phi} P_\alpha$ is conjugate to exactly one vector per Weyl chamber under the action of the Weyl group.

Conversely, for each $\gamma \in E \setminus \bigcup_{\alpha\in\Phi} P_\alpha$ we define the set $\Phi^+(\gamma) := \{\alpha \in \Phi | (\gamma, \alpha > 0\}$. It turns out that the subset of the indecomposable roots of $\Phi^+(\gamma)$ (those elements that cannot be expressed as a sum of two elements of $\Phi^+(\gamma)$) is a base of Φ.

1.3 Classification of the Root Systems

Just as in many other cases, the starting point for the classification is a suitable decomposition of our objects of interest.

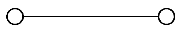
Definition 1.3. *Let Φ be a root system. Φ is called* ***reducible*** *if there exist root systems Ψ_1 and Ψ_2, where $(\cdot,\cdot)|_{\Psi_1\times\Psi_2} \equiv 0$ (the scalar product is that of the underlying vector space) and $\Phi = \Psi_1 \cup \Psi_2$.*

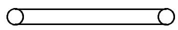
Otherwise Φ is called ***irreducible****.*

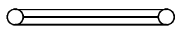
Now it suffices to examine the irreducible root systems and to consider the others as compositions of irreducibles.

One possibility to visualize the data of a root system which are relevant for the classification is to draw the so-called **Coxeter graph**. If we have a root system Φ and simple roots $\alpha_1, \ldots, \alpha_\ell$ we draw ℓ vertices and join the i-th and the j-th with $4\frac{(\alpha_j,\alpha_i)(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)(\alpha_j,\alpha_j)}$ lines. For example, the Coxeter graphs of the two-dimensional root systems are:

$A_1 \times A_1$ 

A_2 

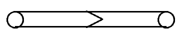
B_2 

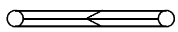
G_2 

In many cases the information about all numbers $2\frac{(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)}$ can be found in the Coxeter graph; at least this is possible if all roots have equal (squared) length, because then we have also $\frac{(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \frac{(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)}$ for all i, j.

If more than one root length occurs, this may fail. For that reason it seems to be sensible to add the information, which of two connected vertices belongs to the longer simple root, to the diagram. In these cases one adds an arrow pointing from the vertex representing the longer root to the one representing the shorter. This new structure is called **Dynkin diagram**[1].

The Dynkin diagrams of the root systems $A_1 \times A_1$ and A_2 do not differ from the Coxeter graphs, while the Dynkin diagrams of the root systems B_2 and G_2 look as follows:

B_2 

G_2 

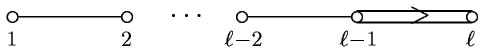
[1] It should be mentioned that the shape of both the Coxeter graph and the Dynkin diagram depend on the numbering of the simple roots. Anyway, for the computation of our Zeta functions the particular choice of the basis is irrelevant, see [7] or [8].

It is clear that a root system Φ is irreducible if and only if its Coxeter graph is connected. The classification theorem then states the following:

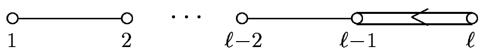
Proposition 1.4. *Let Φ be a root system in the vector space E. Then Φ decomposes uniquely as the union of irreducible root systems Φ_k (in subspaces E_k of E), such that $E = E_1 \oplus \cdots \oplus E_t$. Each Φ_k has one of the following Dynkin diagrams (where ℓ denotes the rank of Φ_k):*

A_ℓ $(\ell \geq 1)$

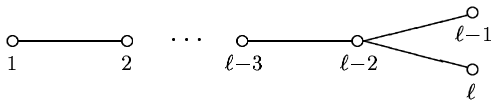
B_ℓ $(\ell \geq 2)$



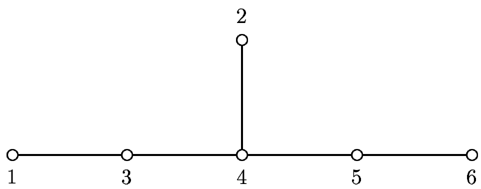
C_ℓ $(\ell \geq 3)$



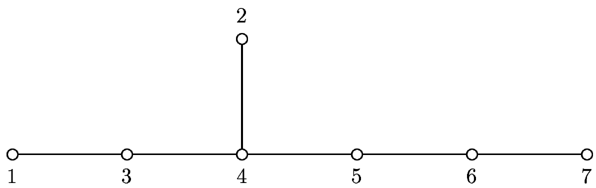
D_ℓ $(\ell \geq 4)$



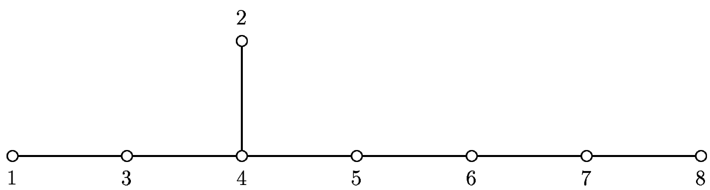
E_6



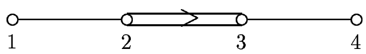
E_7



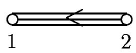
E_8



F_4



G_2



We will not be particularly interested in the root systems of exceptional type (that is E_6, E_7, E_8, F_4, G_2); our main goal will be to find some regularities in the series of polynomials that come up with the root systems A_ℓ–D_ℓ. We are now going to explain how these root systems can be imagined.

For the B_ℓ, C_ℓ and D_ℓ case we should recall, that $\mathcal{P}(\{1,\dots,\ell\})$ (the power set of $\{1,\dots,\ell\}$) together with the symmetric difference is a commutative group in which every element is self inverse and $\emptyset$ is the neutral element. It is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\ell$. The set $\mathcal{P}_{\text{even}}(\{1,\dots,\ell\})$ of subsets of $\{1,\dots,\ell\}$ with even cardinality is a subgroup of $\mathcal{P}(\{1,\dots,\ell\})$.

A_ℓ $(\ell \geq 1)$: $\Phi = \{e_i - e_j | i \neq j\} \subset \mathbb{R}^{\ell+1}$

All roots have the same length 2. As a basis one can choose the set of all $\alpha_k := e_k - e_{k+1}$ $(k \in \{1,\dots,\ell\})$. The fundamental reflection ς_k $(k \in \{1,\dots,\ell\})$ relative to α_k acts on a vector $x \in \mathbb{R}^{\ell+1}$ by exchanging the k-th and the $(k+1)$-st coordinate.

Thus the Weyl group $\mathcal{W}$, which is the group generated by these fundamental reflections, is isomorphic via $\iota : \mathcal{W} \to \mathcal{S}_{\ell+1}$ to $\mathcal{S}_{\ell+1}$, if we let $\sigma \in \mathcal{S}_{\ell+1}$ permute the coordinates of a vector $(x_k)_{k\in\{1,\dots,\ell\}} \in \mathbb{R}^{\ell+1}$ such that $(x_k)_{k\in\{1,\dots,\ell\}} \mapsto (x_{\sigma^{-1}(k)})_{k\in\{1,\dots,\ell\}}$.

The images of the fundamental reflections under our isomorphism are

$$\iota(\varsigma_k) = (k, k+1) \quad \text{for } k \in \{1,\dots,\ell\}.$$

B_ℓ $(\ell \geq 2)$: $\Phi = \{\pm e_i\} \cup \{\pm(e_i \pm e_j) | i \neq j\} \subset \mathbb{R}^\ell$

The elements of the first subset are the short roots of length 1 while the elements of the second subset are the long roots of length 2. As a basis one can choose the set of all $\alpha_k := e_k - e_{k+1}$ $(k \in \{1,\dots,\ell-1\})$, this time completed with $\alpha_\ell := e_\ell$. For $k \in \{1,\dots,\ell-1\}$ the fundamental reflection ς_k relative to α_k acts on a vector $x \in \mathbb{R}^{\ell+1}$ by exchanging the k-th and the $(k+1)$-st coordinate (as in A_ℓ), while α_ℓ changes the sign of the ℓ-th coordinate.

The Weyl group is isomorphic to $\mathcal{S}_\ell \ltimes \mathcal{P}(\{1,\dots,\ell\})$, if we let (σ, S) act on a vector $(x_k)_{k\in\{1,\dots,\ell\}}$ by first changing the sign of the k-th coordinate if and only if $k \in S$ and then permuting the coordinates as in the case of A_ℓ.

Group multiplication on $\mathcal{S}_\ell \ltimes \mathcal{P}(\{1,\dots,\ell\}$ can be described explicitly by

$$(\sigma_1, S_1) \bullet (\sigma_2, S_2) = \left(\sigma_1 \circ \sigma_2, \sigma_2^{-1}(S_1) \triangle S_2\right)$$

for $(\sigma_1, S_1), (\sigma_2, S_2) \in \mathcal{S}_\ell \ltimes \mathcal{P}(\{1,\dots,\ell\}$.

The images of the fundamental reflections under our isomorphism are

$$\begin{aligned} \iota(\varsigma_k) &= \big((k,k+1),\emptyset\big) \quad \text{for } k \in \{1,\dots,\ell-1\} \\ \text{and} \quad \iota(\varsigma_\ell) &= \big(\mathrm{id},\{\ell\}\big). \end{aligned}$$

C_ℓ ($\ell \geq 3$): $\Phi = \{\pm 2e_i\} \cup \{\pm(e_i \pm e_j) | i \neq j\} \subset \mathbb{R}^\ell$

The first subset contains the long roots of length 4, while the second subset is that of the short roots with length 2. The root system is dual to B_ℓ. The vectors $\alpha_k := e_k - e_{k+1}$ ($k \in \{1,\dots,\ell-1\}$) together with $\alpha_\ell := 2e_\ell$ form a base.

The action of the fundamental reflections is obviously the same as in the case of C_ℓ, such that the Weyl groups of C_ℓ and B_ℓ coincide.

D_ℓ ($\ell \geq 4$): $\Phi = \{\pm(e_i \pm e_j) | i \neq j\} \subset \mathbb{R}^\ell$

All roots have the same length 2. The vectors $\alpha_k := e_k - e_{k+1}$ ($k \in \{1,\dots,\ell-1\}$ and $\alpha_\ell := e_{\ell-1} + e_\ell$ form a base. Again, the fundamental reflection relative to α_k ($k \in \{1,\dots,\ell-1\}$) exchanges the k-th and the $(k+1)$-st coordinate, but ς_ℓ exchanges $(\ell-1)$-st and the ℓ-th coordinate and changes the sign of both $x_{\ell-1}$ and x_ℓ.

The Weyl group is isomorphic to $\mathcal{S}_\ell \ltimes \mathcal{P}_{\text{even}}(\{1,\dots,\ell\})$, if we let (σ, S) act on a vector $(x_k)_{k\in\{1,\dots,\ell\}}$ as in the case of B_ℓ and C_ℓ.

The images of the fundamental reflections under our isomorphism are

$$\begin{aligned} \iota(\varsigma_k) &= \big((k,k+1),\emptyset\big) \quad \text{for } i \in \{1,\dots,\ell-1\} \\ \text{and} \quad \iota(\varsigma_\ell) &= \big((\ell-1,\ell),\{\ell-1,\ell\}\big). \end{aligned}$$

An explicit description of the set of positive roots of a root system X_ℓ will be useful in chapter 3:

Lemma 1.5. *1. The set of positive roots of the root system* A_ℓ *is*

$$\Phi^+ = \{e_i - e_j \,|\, 1 \leq i < j \leq \ell\}.$$

2. The set of positive roots of the root system B_ℓ *is*

$$\Phi^+ = \{e_i \,|\, 1 \leq i \leq \ell\} \cup \{e_i - e_j \,|\, 1 \leq i < j \leq \ell\} \cup \{e_i + e_j \,|\, 1 \leq i < j \leq \ell\}.$$

3. The set of positive roots of the root system C_ℓ *is*

$$\Phi^+ = \{2e_i \,|\, 1 \leq i \leq \ell\} \cup \{e_i - e_j \,|\, 1 \leq i < j \leq \ell\} \cup \{e_i + e_j \,|\, 1 \leq i < j \leq \ell\}.$$

4. The set of positive roots of the root system D_ℓ *is*

$$\Phi^+ = \{e_i - e_j \,|\, 1 \leq i < j \leq \ell\} \cup \{e_i + e_j \,|\, 1 \leq i < j \leq \ell\}.$$

Proof. In the A_ℓ case we have for $1 \le i < j \le \ell + 1$

$$\begin{aligned} e_i - e_j &= (e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + \cdots + (e_{j-1} - e_j) \\ &= \sum_{k=i}^{j-1} \alpha_k. \end{aligned}$$

In the B_ℓ case we can construct $e_i - e_j$ for $1 \le i < j \le \ell$ the same way. Additionally we get for $1 \le i < j \le \ell$

$$\begin{aligned} e_i + e_j &= (e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + \cdots + (e_{j-1} - e_j) \\ &\quad +2(e_j - e_{j+1}) + 2(e_{j+1} - e_{j+2}) + \cdots + 2e_\ell \\ &= \sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{\ell} \alpha_k \end{aligned}$$

and for $1 \le i \le \ell$

$$\begin{aligned} e_i &= (e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + \cdots + e_\ell \\ &= \sum_{k=i}^{\ell} \alpha_k. \end{aligned}$$

For root systems C_ℓ the proof works the same, with additional factors 2 occurring frequently in the coefficients.

For root systems D_ℓ we construct for $1 \le i < j \le \ell$

$$\begin{aligned} e_i - e_j &= (e_i - e_{i+1}) + \cdots + (e_{j-1} - e_j) \\ &= \sum_{k=i}^{j-1} \alpha_k, \end{aligned}$$

and for $1 \le i < j < \ell$:

$$\begin{aligned} e_i + e_j &= (e_i - e_{i+1}) + \cdots + (e_{j-1} - e_j) \\ &\quad +2(e_j - e_{j+1}) + \cdots + 2(e_{\ell-2} - e_{\ell-1}) \\ &\quad +(e_{\ell-1} - e_\ell) + (e_{\ell-1} + e_\ell) \\ &= \sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{\ell-2} \alpha_k + \alpha_{\ell-1} + \alpha_\ell \end{aligned}$$

and finally for $1 \le i < \ell$:

$$\begin{aligned} e_i + e_\ell &= (e_i - e_{i+1}) + \cdots + (e_{\ell-2} - e_{\ell-1}) + (e_{\ell-1} + e_\ell) \\ &= \sum_{k=i}^{\ell-2} \alpha_k + \alpha_\ell \end{aligned}$$

In all three cases it is clear, that the set of all roots consists of these sets Φ^+ together with their negatives (for a description of the set of roots see section 1.3). This finishes the proof. $\square$

In chapter 3 we will see, that this characterization of the root systems A_ℓ–D_ℓ and the associated Weyl groups allows us to give combinatorial proofs of our propositions about the Zeta functions.

1.4 Weights

Let Λ be the set of all $\lambda \in E$ for which $2\frac{(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha \in \Phi$ and call its elements **weights**. Once we have fixed a base $\{\alpha_1, \dots, \alpha_\ell\}$ we call $\lambda \in \Lambda$ **dominant** if $2\frac{(\lambda,\alpha_j)}{(\alpha_j,\alpha_j)} \geq 0$ for all primitive roots α_j. Obviously all roots and all linear combinations of them (with integral coefficients) are weights. They form a sublattice of Λ, called the **root lattice** Λ_r.

The Weyl group $\mathcal{W}$ preserves the inner product on E, and hence leaves Λ invariant. Orbits of weights under $\mathcal{W}$ occur frequently in the study of representations. That is because each weight is conjugate under $\mathcal{W}$ to exactly one dominant weight.

Let $\{\lambda_1, \dots, \lambda_\ell\}$ be the basis of E dual to $\{\frac{2\alpha_1}{(\alpha_1,\alpha_1)}, \dots, \frac{2\alpha_\ell}{(\alpha_\ell,\alpha_\ell)}\}$. Then $2\frac{(\lambda_k,\alpha_j)}{(\alpha_j,\alpha_j)} = \delta_{jk}$ and the λ_k are dominant weights, called **fundamental dominant weights** (relative to Δ). It is an easy exercise to prove that Λ is a lattice and that $\{\lambda_1, \dots, \lambda_\ell\}$ is a basis of it.

These fundamental dominant weights are particularly interesting for our purpose, because for a group of type X_ℓ the irreducible representations can be parametrized by the set of all $\sum_{k=1}^{\ell} n_k\lambda_k$, where $n_1, \dots, n_k \in \mathbb{N}_0$.

A list of the fundamental dominant weights for the cases of A_ℓ–D_ℓ is:

A_ℓ:

$$\lambda_k = \frac{1}{\ell+1}\left((\ell-k+1)\sum_{i=1}^{k-1} i\alpha_i + k\sum_{i=k}^{\ell}(\ell-i+1)\alpha_i\right)$$

B_ℓ:

$$\lambda_k = \sum_{i=1}^{k-1} i\alpha_i + k\sum_{i=k}^{\ell}\alpha_i \quad (k<\ell) \quad \text{and} \quad \lambda_\ell = \frac{1}{2}\sum_{i=1}^{\ell} i\alpha_i$$

C_ℓ:

$$\lambda_k = \sum_{i=1}^{k-1} i\alpha_i + k\left(\sum_{i=k}^{\ell-1}\alpha_i + \frac{1}{2}\alpha_\ell\right)$$

D_ℓ:

$$\lambda_k = \sum_{i=1}^{k-1} i\alpha_i + k\left(\sum_{i=k}^{\ell-2}\alpha_i + \frac{1}{2}\alpha_{\ell-1} + \frac{1}{2}\alpha_\ell\right) \quad (k < \ell - 1),$$

$$\lambda_{\ell-1} = \frac{1}{2}\left(\sum_{i=1}^{\ell-2} i\alpha_i + \frac{1}{2}\ell\alpha_{\ell-1} + \frac{1}{2}(\ell-2)\alpha_\ell\right)$$

and $$\lambda_\ell = \frac{1}{2}\left(\sum_{i=1}^{\ell-2} i\alpha_i + \frac{1}{2}(\ell-2)\alpha_{\ell-1} + \frac{1}{2}\ell\alpha_\ell\right)$$

Chapter 2

Zeta Functions of Classical Algebraic Groups

2.1 Definition and the Connection to Root Systems

When Mathematicans associate Zeta functions to their objects of research they hope, that these Zeta functions make certain "hidden" properties of their objects better visible.

It seems evident, that one expects from a "proper" Zeta function, that it contains exactly as much information as the object it is associated to. One tries to get close to this requirement by searching for an in some way canonical generating process. It is clear that the shape of this process must be dependent on the type of mathematical object one is considering.

In our case we have algebraic groups, which can be identified (via representation) with groups of certain matrices. For the case of algebraic groups over a finite extension of $\mathbb{Q}_p$, A. Weil suggested to define Zeta functions of these groups as Euler products of certain p-adic integrals. Later on M. du Sautoy and F. Grunewald used this definition to define Zeta functions of such algebraic groups over finite extensions of $\mathbb{Q}$ as Euler products of the former.

Definition 2.1. *Let G be a linear algebraic group over a field k and let $\rho : G \to \mathrm{GL}_n$ be a k-rational representation.*

1. *If k is a finite extension of $\mathbb{Q}_p$ we define the* ***Zeta function of the algebraic group G at the representation ρ*** *to be*

$$Z_{G(k),\rho}(s) = \int_{G^+} |\det(\rho(g))|^s \mu_G(g)$$

where $G^+ = \rho^{-1}(\rho(G(k)) \cap M_n(\vartheta_k))$, ϑ_k is the ring of integers of k and μ_G denotes the right Haar measure on $G(k)$, normalized so that $\mu_G(G(\vartheta_k)) = 1$.

2. *If k is a finite extension of $\mathbb{Q}$ and $\mathfrak{p}$ is a prime of k we define the* ***local Zeta function of the algebraic group G at the representation ρ and the prime $\mathfrak{p}$*** *to be*

$$Z_{G(k),\rho,\mathfrak{p}}(s) := Z_{G(k_\mathfrak{p}),\rho}(s).$$

In the latter case we define the ***global Zeta function of G at the representation ρ*** *as the Euler product*

$$Z_{G(k),\rho}(s) := \prod_{\mathfrak{p}} Z_{G(k),\rho,\mathfrak{p}}(s).$$

The problem with these Zeta functions is that – apart from the case of $\mathrm{GL}_{\ell+1}$ in its natural representation – they are not meromorphically continuable to the whole of $\mathbb{C}$. This is one important reason, why group theorists lost their interest in them for some time.

But then Grunewald, Segal and Smith found out, that in case of the symplectic group these Zeta functions have an interpretation as a Dirichlet series, where the coefficients count the subalgebras of the Lie algebra associated to the group. A similar discovery had been made by K. Hey in 1929 (see [1]). She found out, that the Zeta function of $\mathrm{GL}_{\ell+1}$ counts ideals in central simple algebras.

Furthermore, in [8] M. du Sautoy and A. Lubotzky managed to find formulae for these Zeta functions in terms of combinatorial data of the buildings associated to these groups. Similar work had been done by J.I. Igusa in [7] before, only that he used a slightly different definition of the Zeta functions in a less general setting. In the context of Grunewald's, Segal's and Smith's observation the definition from [8] seems to be more canonical.

In both cases it turned out, that the shape of the Zeta function of a Chevalley group depends only on the root system of the group and – since the irreducible representations can be parametrized by the weights – on the choice of some element of $\mathbb{N}_0^\ell$, where ℓ denotes the rank of the root system.

Perhaps we should recall the exact explanation of the formula as it is done in [8] and [11]. To understand the results of chapter 3 it is absolutely sufficient to keep proposition 0.2 in mind. Only in chapter 4 the choice of the representation will come into play; for that reason it might be helpful to understand how the values a_i and $b_j(\omega)$ are connected to the root system and the representation whose contragredient has weight ω. We may restrict their result to the case of $k = \mathbb{Q}$; the resulting polynomials are identical for any other global field.

Definition 2.2. *Let G be a Chevalley group over $\mathbb{Q}$, and let $\rho : G \to \mathrm{GL}_n$ be an irreducible rational representation. We may consider G as an almost direct product of the derived group G' and a one dimensional central torus $\mathbb{G}_m 1_n$. The derived group is a Chevalley group of rank ℓ, say. Let X_ℓ denote its Dynkin diagram. Choose a basis $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ for the root system Φ of G'. We shall think of the root system in the abstract sense as explained in chapter 1. Roots and weights are thus vectors in $\mathbb{R}^n$ for us.*

Let ω denote the dominant weight of the contragredient representation $\rho^* = {}^t\rho^{-1}$. Let m denote the order of the centre of G' (in the abstract sense m can be described as the smallest multiple of the vector ω that lies in the root lattice). Then there exist uniquely determined natural numbers $b_1(\omega), \ldots, b_\ell(\omega)$ for which

$$m\omega = \sum_{k=1}^{\ell} b_k(\omega)\alpha_k$$

and natural numbers $a_1, \ldots, a_\ell$ for which

$$\sum_{\alpha\in\Phi^+} = \sum_{k=1}^{\ell} a_k\alpha_k.$$

In particular, for the natural representation the values a_k, $b_k(\omega)$ and m are:

- *for A_ℓ:*
$$m = \ell + 1, \quad a_k = k(\ell - k + 1), \quad b_k(\omega) = k$$
- *for B_ℓ:*
$$m = 1, \quad a_k = k(2\ell - k), \quad b_k(\omega) = 1$$
- *for C_ℓ: $m = 2$,*
$$a_k = \begin{cases} k(2\ell - k + 1) & , \text{ for } k < \ell \\ \frac{\ell(\ell+1)}{2} & , \text{ for } k = \ell \end{cases}, \quad b_k(\omega) = \begin{cases} 2 \ , & \text{for } k < \ell \\ 1 \ , & \text{for } k = \ell \end{cases}$$
- *for D_ℓ: $m = 2$,*
$$a_k = \begin{cases} k(2\ell - k - 1) & , \text{ for } k < \ell - 1 \\ \frac{\ell(\ell-1)}{2} & , \text{ for } k \geq \ell - 1 \end{cases}, \quad b_k(\omega) = \begin{cases} 2 \ , & \text{for } k < \ell - 1 \\ 1 \ , & \text{for } k \geq \ell - 1 \end{cases}$$

Let $\mathcal{W}$ be the Weyl group of the root system Φ and let λ be the length function of $\mathcal{W}$. Then we define $P_{G,\rho}(X,Y), Q_{G,\rho}(X,Y) \in \mathbb{Z}[X,Y]$ by

$$P_{G,\rho}(X,Y) = \sum_{w\in\mathcal{W}} X^{-\lambda(w)} \prod_{\alpha_k\in w(\Phi^-)} X^{a_k}Y^{b_k(\omega)} \tag{2.1}$$

$$Q_{G,\rho}(X,Y) = (1-Y^m)\prod_{k=1}^{\ell}(1-X^{a_k}Y^{b_k(\omega)}). \tag{2.2}$$

In these terms the result from [8] can be formulated as

Proposition 2.3. *Consider the conditions of definition 2.2. Let p be a prime. Then the local Zeta function of G at the representation ρ and the prime p is*

$$Z_{G(\mathbb{Q}),\rho,p}(s) = \frac{P_{G,\rho}(p,p^{-\frac{n}{m}s})}{Q_{G,\rho}(p,p^{-\frac{n}{m}s})}.$$

Because of these observations the question, why these functions are not meromorphically continuable to $\mathbb{C}$ became interesting. This question was the motivation for the definition of the so-called ghost polynomials.

2.2 Ghosts

Due to the observations made by Grunewald, Segal, Smith and Igusa the Zeta functions defined in the last section seem to be quite good candidates for Zeta functions, though they are not meromorphically continuable to $\mathbb{C}$. Thus M. du Sautoy and F. Grunewald in [11] sought for a better understanding of the problems concerning the possibility of meromorphic continuation.

The reason for this problem is, that as p tends to infinity, the zeros of $P(p,Y)$ lie dense on a line $\{y\in\mathbb{C} : \Re(y) = \frac{u}{v}\}$ with suitable integers u and v. In [10] it is explained, how to continue such a Zeta function up to this boundary, and why no meromorphic continuation beyond this boundary is possible.

Thus the question came up, which polynomials $P(X,Y)\in\mathbb{C}[X,Y]$ (with constant coefficient 1) have an Euler product $\prod_{p \text{ prime}} P(p,p^{-s})$, that is meromorphically continuable to the whole of $\mathbb{C}$. F. Grunewald and M. du Sautoy defined a class of polynomials, for which it is easy to verify that they have this property.

Definition 2.4. *a) $W(X,Y)\in\mathbb{C}(X,Y)$ is called* ***friendly*** *if there exist cyclotomic polynomials $g_k(U)\in\mathbb{C}[U]$, $k=1,\ldots,n$ and integers u_k, v_k, such that $W(X,Y)=\prod_{k=1}^{n} g_k(X^{u_k}Y^{v_k})$. Otherwise $W(X,Y)$ is called* ***unfriendly****.*

b) Let $W(X,Y) \in \mathbb{C}(X,Y)$. The Euler product

$$Z(s) = \prod_{p\ prime} W(p, p^{-s})$$

is called ***friendly*** *if it is meromorphically continuable to the whole of $\mathbb{C}$. Otherwise it is called* ***unfriendly****.*

A result from [10] is then

Proposition 2.5. *Let $W(X,Y) \in \mathbb{C}(X,Y)$ be friendly. Then the corresponding Euler product $Z(s) = \prod_{p\ prime} W(p, p^{-s})$ is friendly as well.*

The authors believe, that the reverse direction is also true, but this has not been proved yet.

The "ghost polynomial" $\tilde{P}(X,Y)$ is then constructed from $P(X,Y)$ as an approximation, which turns out to have the quality of being friendly in most cases where $P(X,Y)$ is one of the polynomials that are connected to the Zeta functions via proposition 2.3. The construction can be done for any other two variable polynomial as well, but the result is not friendly in the general case. The principle reminds one of the "Puiseux power series expansion".

Let $P(X,Y) = \sum_{\mu,\nu} c_{\mu,\nu} X^\mu Y^\nu \in \mathbb{C}[X,Y]$ with $c_{0,0} \neq 0$. In an $\mathbb{N}_0 \times \mathbb{N}_0$-lattice, we mark for each nonzero coefficient $c_{\mu,\nu}$ the point (μ,ν). The result is called the **Newton diagram** of $P(X,Y)$. The particular shape of the ghost polynomial will depend on the lower convex hull of this diagram, called the **Newton polygon**, only.

Then, from all lines through $(0,0)$ and (μ,ν) with $c_{\mu,\nu} \neq 0$, we choose the particular one that has smallest slope. We can represent this line with a pair of nonnegative coprime integers (u_1, v_1), so that the points on the line are parametrized by (mu_1, mv_1), $m \in \mathbb{N}_0$. With M_1 we denote the largest integer with $c_{M_1u_1,M_1v_1} \neq 0$. We set

$$\tilde{P}_1(X,Y) := \sum_{m=0}^{M_1} c_{mu_1,mv_1} X^{mu_1} Y^{mv_1}.$$

Afterwards we repeat these steps recursively: Under all lines through (M_1u_1, M_1v_1) and (μ,ν) with $\mu \geq M_1u_1$, $\nu \geq M_1v_1$ and $c_{\mu,\nu} \neq 0$ we choose the one with smallest slope and represent it with (u_2, v_2), so that the points on the line are $(M_1u_1 + mu_2, M_1v_1 + mv_2)$, $m \in \mathbb{N}_0$. With M_2 we denote the largest integer with $c_{M_1u_1+M_2u_2,M_1v_1+M_2v_2} \neq 0$.

We continue as long as such lines exist and define recursively

$$\tilde{P}_{k+1}(X,Y) := \sum_{m=0}^{M_{k+1}} c_{mu_{k+1}+\sum_{i=1}^{k} M_k u_k, mv_{k+1}+\sum_{i=1}^{k} M_k v_k} X^{mu_{k+1}} Y^{mv_{k+1}}.$$

The ghost polynomial is then defined as the product of these $P_k(X,Y)$. If it is friendly, a boundary for the possibility of meromorphic continuation of the Euler product $\prod P(p,p^{-s})$ can directly be read off from it:

If

$$k_0 := \min\{k \in \mathbb{N} : \tilde{P}_k(X,Y) \text{ is unfriendly or is not a factor of } P(X,Y)\}$$

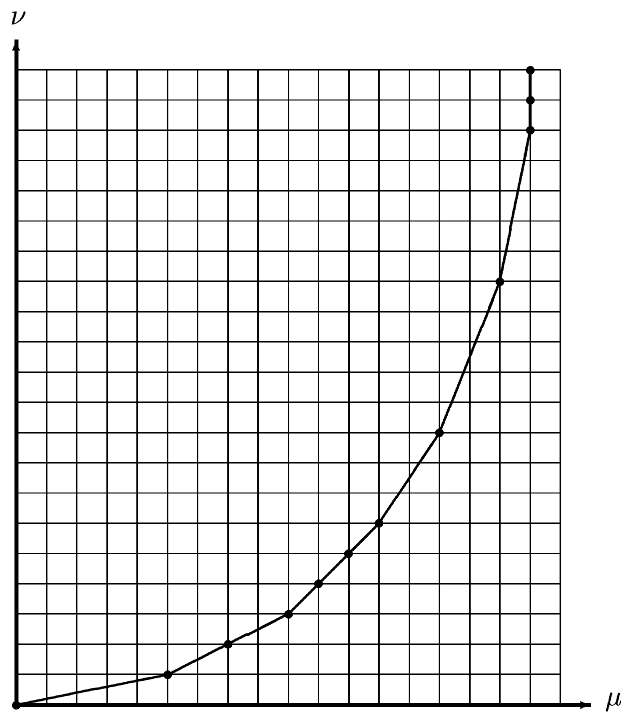
exists, then the Euler product can be meromorphically continued up to $\frac{u_{k_0}}{v_{k_0}}$. Otherwise the Euler product is obviously continuable to the whole of $\mathbb{C}$ (see proposition 2.1). This observation was made by M. du Sautoy. He also proved, that in all cases we are considering in this thesis no meromorphic continuation beyond this boundary is possible.

Let us consider an example.

Example 2.6. *We will determine the ghost polynomial of*

$$\begin{aligned} P(X,Y) &= 1+X^5Y+2X^7Y^2+X^9Y^3+2X^{10}Y^4+X^{11}Y^5+X^{12}Y^6 \\ &\quad +2X^{14}Y^9+X^{16}Y^{14}+X^{17}Y^{19}+X^{17}Y^{20}+X^{17}Y^{21}. \end{aligned}$$

The Newton diagram is



Note, that the reason we chose a polynomial where all points of the Newton diagram lie on the lower convex hull of the Newton diagram is, that (as already mentioned) all points beyond the lower convex hull (i.e. the corresponding coefficients) are irrelevant for the computation of the ghost polynomial. So if there were more nonzero coefficients, they would be nothing else but confusing.

The ghost polynomial is thus

$$\begin{aligned}\tilde{P}(X,Y) &= (1+X^5Y)(1+2X^2Y+X^4Y^2)(1+2XY+X^2Y^2+X^3Y^3)\\ &\quad \cdot(1+2X^2Y^3)(1+X^2Y^5)(1+X^3Y)(1+Y+Y^2).\end{aligned}$$

The factor $(1+X^5Y)$ *is friendly, because* $(1+U)$ *is a cyclotomic polynomial (substitute* $U = X^5Y$*).*

The factor $(1+2X^2Y+X^4Y^2)$ *is friendly as well, since* $(1+2U+U^2) = (1+U)^2$ *is a product of cyclotomic polynomials (substitute* $U = X^2Y$*).*

In a similar manner it turns out, that the factors $(1+X^2Y^5)$, $(1+X^3Y)$ *and* $(1+Y+Y^2)$ *are friendly, whilst the rest is not.*

As this example demonstrates, given an arbitrary polynomial one can neither expect it to be friendly, nor can one expect that the ghost is. In [11] it was proved, that those polynomials coming up with the Zeta functions of Chevalley groups of types A_ℓ–D_ℓ and their natural representations do have friendly ghost polynomials. In fact they have friendly ghosts for most of the irreducible representations.

The reason for this is, that the coefficients of these polynomials have their origin in counting Weyl group elements with certain properties. The points marking the convex hull correspond to in some way extremal properties. In particular for a given set of primitive roots $\Delta_0 \subset \Delta$ there exists a uniquely determined Weyl group element w_0 of shortest length, which satisfies $w_0(\alpha) \in \Phi^-$ for all $\alpha \in \Delta_0$. Let Φ_0 be the subroot system of Φ that is generated by Δ_0. Then $w_0(\alpha) \in \Phi^-$ if and only if $\alpha \in \Phi_0^+$, thus the length of w_0 is $\lambda(w_0) = \mathrm{card}(\Phi^+)$.

The points on the lower convex hull of the Newton diagram belong to such w_0, so in our case it seems, that unfriendly ghost polynomials are an exception. We will illustrate this view in chapter 4 with the example B_3.

Chapter 3

A Formula for the $P(X, Y)$ Dotting the Group Structure

In all what follows X_ℓ denotes a root system of type B_ℓ, C_ℓ or D_ℓ with Weyl group $\mathcal{W}$. As bases for the root system we will use those from section 1.3.

The phrase "group of type X_ℓ" will be used for a group G, whose derived group G' has the root system X_ℓ. Note that the root system of G differs from X_ℓ; in case of a simple G by a summand A_1.

For determining the numerator polynomials we need to know two things for each element w of the Weyl group. The first question is, which of the primitive roots lie in $w(\Phi^+)$ and which lie in $w(\Phi^-)$. The second is that for the length. For both issues we concentrate on the elements of the permutation subgroup first.

Proposition 3.1. *Let $\mathcal{W}$ be the Weyl group of a root system X_ℓ and $w \in \mathcal{W}$ with $\iota(w) = \sigma$, if $X_\ell = A_\ell$, and $\iota(w) = (\sigma, \emptyset)$, if $X_\ell = B_\ell$, C_ℓ or D_ℓ. Then*

- *if $X_\ell = A_\ell$:*

$$\alpha_k \in w(\Phi^+) \Leftrightarrow \sigma^{-1}(k) < \sigma^{-1}(k+1) \quad \textit{for } k \in \{1, \ldots, \ell\}.$$

- *if $X_\ell = B_\ell$, C_ℓ or D_ℓ:*

$$\begin{aligned} &\alpha_\ell \in w(\Phi^+) \quad \textit{and} \\ &\alpha_k \in w(\Phi^+) \Leftrightarrow \sigma^{-1}(k) < \sigma^{-1}(k+1) \quad \textit{for } k \in \{1, \ldots, \ell - 1\}. \end{aligned}$$

Proof. If $X_\ell = A_\ell$ and $\sigma^{-1}(k) < \sigma^{-1}(k+1)$ for $k \in \{1, \ldots, \ell\}$, the root $u := e_{\sigma^{-1}(k)} - e_{\sigma^{-1}(k+1)}$ is positive by lemma 1.2. Therefore

$$\alpha_k = e_k - e_{k+1} = e_{\sigma(\sigma^{-1}(k))} - e_{\sigma(\sigma^{-1}(k+1))} = \sigma(u) = w(u) \in w(\Phi^+).$$

The same argument holds in the B_ℓ case for $k \in \{1, \dots, \ell - 1\}$. Additionally the root $e_{\sigma^{-1}(\ell)}$ is positive in this case, so that

$$\alpha_\ell = e_\ell = e_{\sigma(\sigma^{-1}(\ell))} = (\sigma, \emptyset)(e_{\sigma^{-1}(\ell)}) = w(e_{\sigma^{-1}(\ell)}) \in w(\Phi^+).$$

The proof of the opposite direction works analogous and so does the C_ℓ case. In the D_ℓ case one little difference occurs when proving, that $\alpha_\ell \in w(\Phi^+)$. Here we set $u := e_{\sigma^{-1}(\ell-1)} + e_{\sigma^{-1}(\ell)} \in \Phi^+$ and thus

$$\alpha_\ell = e_{\ell-1} + e_\ell = e_{\sigma(\sigma^{-1}(\ell-1))} + e_{\sigma(\sigma^{-1}(\ell))} = (\sigma, \emptyset)(u) = w(u) \in w(\Phi^+).$$

□

The following terminology will make formulating the next lemma easier:

Definition 3.2. *Let $\sigma \in \mathcal{S}_\ell$ be a permutation. Then we define the* ***number of inversions*** *of σ as*

$$\operatorname{inv}(\sigma) = \sharp\{(i,j) \in \{1, \dots, \ell\} \mid i < j \wedge \sigma(i) > \sigma(j)\}.$$

Lemma 3.3. *Let $\mathcal{W}$ denote the Weyl group of the root system X_ℓ. Let $w \in \mathcal{W}$ with $\iota(w) = (\sigma, \emptyset)$, if $X_\ell = B_\ell$, C_ℓ or D_ℓ and $\iota(w) = \sigma$ if $X_\ell = A_\ell$. Then*

$$\lambda(w) = \operatorname{inv}(\sigma) = \operatorname{inv}(\sigma^{-1}).$$

Proof. We will perform the proof for $X_\ell = B_\ell$, C_ℓ or D_ℓ, keeping in mind, that the only difference in the A_ℓ case is, that ℓ has to be replaced by $\ell + 1$.

Let $\sigma \in \mathcal{S}_\ell$. We have to investigate which of the primitive roots are sent to negative by $(\sigma, \emptyset)$ sends to negative. For a list of the positive roots see Lemma 1.5. Since we have no signchange at all, the only positive roots that are into question are those of type $e_i - e_j$ $(i < j)$.

Let $1 \leq i < j \leq \ell$. Then $e_i - e_j$ is a positive root. But

$$\begin{aligned} & (\sigma, \emptyset)(e_i - e_j) = e_{\sigma(i)} - e_{\sigma(j)} \in (\sigma, \emptyset)(\Phi^+) \\ \iff & \sigma(i) < \sigma(j) \\ \iff & (i, j) \text{ is an inversion.} \end{aligned}$$

This proves the first equality. But the second is clear, anyway. □

At this point we have all tools needed to determine the monomial associated to any element of the permutation subgroups of our Weyl groups. In particular we have everything required for the A_ℓ case. But of course we want tools for the rest as well. For that purpose we will construct the element $w \in \mathcal{W}$ with $\iota(w) = (\sigma, S)$ in two steps:

We begin with constructing the Weyl group element of shortest length, which has S as its set of sign changes. Afterwards we will "sort" this provisional result with a suitable permutation $\pi(w)$, such that the result of this sorting is our w.

It turns out that the information, which of the primitive roots lie in $w(\Phi^-)$, can be read off as in Proposition 3.1, substituting σ with $\pi(w)$ in the general case. We will also see that our construction requires only the smallest number of fundamental reflections that is possible to generate w. Thus it suffices to add the length of $(\pi(w), \emptyset)$ (which can be calculated with Lemma 3.3 again) and the length of our provisional result (which will be very simple) to calculate the length of w.

Let us describe this formally.

Definition 3.4. *(a) Let $\mathcal{W}$ be the Weyl group of the root system of type B_ℓ or C_ℓ. For $s \in \{1, \dots, \ell\}$ we define*

$$\tau_s := \varsigma_\ell \circ \cdots \circ \varsigma_s.$$

(b) Let $\mathcal{W}$ be the Weyl group of the root system of type D_ℓ. For $s_1, s_2 \in \{1, \dots, \ell\}$ with $s_1 < s_2$ we define

$$\tau_{s_1,s_2} := \varsigma_\ell \circ (\varsigma_{\ell-2} \circ \cdots \circ \varsigma_{s_1}) \circ (\varsigma_{\ell-1} \circ \cdots \circ \varsigma_{s_2}).$$

Remark. *Carried over by ι the τ_s and τ_{s_1,s_2} are*

$$\iota(\tau_s) = \left(\begin{pmatrix} 1 & \cdots & s-1 & s & s+1 & \cdots & \ell \\ 1 & \cdots & s-1 & \ell & s & \cdots & \ell-1 \end{pmatrix}, \{s\} \right).$$

and $\iota(\tau_{s_1,s_2}) =$

$$\left(\begin{pmatrix} 1 & \cdots & s_1-1 & s_1 & s_1+1 & \cdots & s_2-1 & s_2 & s_2+1 & \dots & \ell \\ 1 & \cdots & s_1-1 & \ell & s_1 & \cdots & s_2-2 & \ell-1 & s_2-1 & \dots & \ell-1 \end{pmatrix}, \{s_1, s_2\} \right).$$

In other words, τ_s is shortest Weyl group element which changes the sign of the s-th coordinate, while τ_{s_1,s_2} is the shortest one that changes the sign of both the s_1-st and the s_2-nd. For an arbitrary set of signchanges $S = \{s_1, \dots, s_m\} \subset \{1, \dots, \ell\}$ with $s_1 < \cdots < s_m$ the Weyl group element of shortest length, whose set of sign changes is S, is just $\tau_{s_1} \circ \cdots \circ \tau_{s_m}$ or $\tau_{s_1,s_2} \circ \cdots \circ \tau_{s_{m-1},s_m}$, respectively.

Next we will define the permutation which will turn out to be the one sorting this composition the right way.

Definition 3.5. *Let $\mathcal{W}$ denote the Weyl group of the root system B_ℓ, C_ℓ or D_ℓ. For $w \in \mathcal{W}$ with $\iota(w) = (\sigma, S)$ where $S = \{s_1, \ldots, s_m\}$ with $s_1 < \cdots < s_m$ and $S^c = \{t_1, \ldots, t_n\}$ with $t_1 < \cdots < t_n$ we define $\pi(w) \in \mathcal{S}_\ell$ by*

$$\begin{aligned} \sigma(t_k) &= \pi(w)(k) \quad \text{for } k \in \{1, \ldots, n\} \\ \text{and} \quad \sigma(s_k) &= \pi(w)(\ell - k + 1) \quad \text{for } k \in \{1, \ldots, m\}. \end{aligned}$$

This definition induces a map $\pi : \mathcal{W} \to \mathcal{S}_\ell$.

Remark. *The map π is not a group homomorphism!*

The meaning of this definition becomes clear through

Proposition 3.6. *Let $\mathcal{W}$ be the Weyl group of the root system B_ℓ, C_ℓ or D_ℓ and $w \in \mathcal{W}$ with $\iota(w) = (\sigma, \{s_1, \ldots, s_m\})$ and $s_1 < \ldots, s_m$. Then $\pi(w)$ satisfies*

- *in the B_ℓ and C_ℓ case:*
$$(\sigma, \{s_1, \ldots, s_m\}) = \big(\pi(w), \emptyset\big) \bullet \iota(\tau_{s_1} \circ \cdots \circ \tau_{s_m}) \tag{3.1}$$
- *and in the D_ℓ case*
$$(\sigma, \{s_1, \ldots, s_m\}) = \big(\pi(w), \emptyset\big) \bullet \iota(\tau_{s_1,s_2} \circ \cdots \circ \tau_{s_{m-1},s_m}). \tag{3.2}$$

Proof. (B_ℓ and C_ℓ case):

The proposition is verified, if for all $(x_1, \ldots, x_\ell) \in \mathbb{R}^\ell$

$$(\tau_{s_1} \circ \cdots \circ \tau_{s_m})(x_1, \ldots, x_\ell) = (x_{t_1}, \ldots, x_{t_n}, -x_{s_m}, \ldots, -x_{s_1}),$$

since afterwards $(\pi(w), \emptyset)$ sends x_{t_k} from the k-th to the to the $\sigma(t_k)$-th position ($k \in \{1, \ldots, n\}$) and sends $-x_{s_k}$ from the $(\ell - k + 1)$-st to the $\sigma(s_k)$-th position ($k \in \{1, \ldots, m\}$) by definition of $\pi(w)$.

We prove our assertion by induction on $m = \sharp S$. It starts with the trivial case $S = \emptyset$, where the product is empty, hence equals the identity.

Induction step: (B_ℓ and C_ℓ case) Let $S = \{s_0, \ldots, s_m\}$ with $s_0 < \cdots < s_m$. For $S' := S \backslash \{s_0\}$ we can tell (since $\{1, \ldots, s_0\} \subset (S')^c$) by the induction hypothesis

$$\begin{aligned} & (\tau_{s_0} \circ \tau_{s_1} \circ \cdots \circ \tau_{s_m})(x_1, \ldots, x_\ell) \\ = \; & \tau_{s_0}(x_{t_1}, \ldots, x_{t_n}, -x_{s_m}, \ldots, -x_{s_1}) \\ = \; & \left(\begin{pmatrix} 1 & \cdots & s_0 - 1 & s_0 & s_0 + 1 & \cdots & \ell \\ 1 & \cdots & s_0 - 1 & \ell & s_0 & \cdots & \ell - 1 \end{pmatrix}, \{s_0\} \right) \\ & \qquad\qquad\qquad\qquad (x_{t_1}, \ldots, x_{t_n}, -x_{s_m}, \ldots, -x_{s_1}) \\ = \; & (x_{t_1}, \ldots, \not{x}_{s_0}, \ldots, x_{t_n}, -x_{s_m}, \ldots, -x_{s_1}, -x_{s_0}), \end{aligned}$$

($\not{x}_{s_0}$ shall indicate the disappearance of x_{s_0}) and we are done.

In the D_ℓ case the proof works analogous, since τ_{s_1,s_2} acts as $\tau_{s_2} \circ \tau_{s_1}$. $\square$

The next two propositions state, that most of the information about $w \in \mathcal{W}$ we need for our Zeta functions can be found in $\pi(w)$:

Proposition 3.7. *Let $\mathcal{W}$ be the Weyl group of the root system* B_ℓ, C_ℓ *or* D_ℓ *and let $w \in \mathcal{W}$ with $\iota(w) = (\sigma, S)$. Then the length of w equals the sum of the length of the factors in formula (3.1) (or (3.2), respectively), so that*

- *in the* B_ℓ *and* C_ℓ *case*

$$\lambda(w) = \mathrm{inv}(\pi(w)) + \sum_{s \in S} (\ell - s + 1),$$

- *while in the* D_ℓ *case*

$$\lambda(w) = \mathrm{inv}(\pi(w)) + \sum_{s \in S} (\ell - s).$$

Proof. We shall outline the proof only for the B_ℓ and C_ℓ case. In principle, the proof for the D_ℓ works the same.

Recall that for $k < \ell$ the fundamental reflection ς_k applied on an ℓ-vector $(x_1, \ldots, x_\ell)$ exchanges the k-th and the $(k+1)$-st coordinate, while ς_ℓ changes the sign of the ℓ-th. We will examine, how many ς_k are "caused" by a pair $(i, j) \in \{1, \ldots, \ell\}^2$ with $i < j$ to appear in our construction of w. It will turn out, that this number is for each pair just the smallest possible one. We have to distinguish whether (i, j) is an inversion or not:

1. $\sigma(i) > \sigma(j)$: This is the simpler case. (i, j) is an inversion, which means, that $\pm x_i$ has to appear on the right hand side of $\pm x_j$ in $w(x_1, \ldots, x_\ell)$. This means that in any product of fundamental reflections equalling w an odd number of them has to exchange the positions of $\pm x_i$ and $\pm x_j$.

2. $\sigma(i) < \sigma(j)$: $\pm x_i$ has to appear on the left hand side of $\pm x_j$ in $w(x_1, \ldots, x_\ell)$. Therefore in any product of fundamental reflections equalling w the positions of $\pm x_i$ and $\pm x_j$ have to be exchanged an even number of times.

   In case $i \in S$ it is furthermore necessary that at some point (that is after applying only a suitable part of the product) x_i has to be on the ℓ-th position, because only the sign of the ℓ-th coordinate can be changed by a fundamental reflection (namely by ς_ℓ). Therefore our even number we mentioned above cannot be zero in this case.

Additionally, for each element of S the reflection ς_ℓ has to appear an odd number of times in the product. So we get a lower bound for the length:

$$\begin{aligned}\lambda(w) \quad &\leq \quad \sharp\{(i,j) \in \{1,\ldots,\ell\}^2 \,|\, i < j \wedge \sigma(i) > \sigma(j)\} \\ &+2 \cdot \sharp\{(i,j) \in \{1,\ldots,\ell\}^2 \,|\, i < j \wedge \sigma(i) < \sigma(j) \wedge i \in S\} \\ &+ \quad \sharp S\end{aligned}$$

We will now determine the actual number of factors on the right hand side of formula (3.1). As usual, let $S = \{s_1, \ldots, s_m\}$ with $s_1 < \cdots < s_m$ and $S^c = \{t_1, \ldots, t_n\}$ with $t_1 < \cdots < t_n$. For $i \in \{1, \ldots, \ell\}$ the factor τ_i is defined as a product of $(\ell - i + 1)$ many fundamental reflections. The aggregate number of factors is thus $\sum_{i \in S}(\ell - i + 1)$.

By lemma 3.3 the length of $\pi(w)$ equals its inversion number. In the proof of Proposition 3.6 we had seen that

$$(\tau_{s_1} \circ \cdots \circ \tau_{s_m})(x_1, \ldots, x_\ell) = (x_{t_1}, \ldots, x_{t_n}, -x_{s_m}, \ldots, -x_{s_1}).$$

Therefore (i,j) with $i < j$ is in touch with an inversion[1] of $\pi(w)$, if $\pm x_i$ appears on the left hand side of $\pm x_j$ in $(x_{t_1}, \ldots, x_{t_n}, -x_{s_m}, \ldots, -x_{s_1})$ but on the right hand side in $w(x_1, \ldots, x_\ell)$, or vice versa. This is the case if:

- $i \in S^c \wedge j \in S^c \wedge \sigma(i) > \sigma(j)$
- $i \in S^c \wedge j \in S \wedge \sigma(i) > \sigma(j)$
- $i \in S \wedge j \in S^c \wedge \sigma(i) < \sigma(j)$
- $i \in S \wedge j \in S \wedge \sigma(i) < \sigma(j)$

The aggregate number of factors in our product is thus

$$\begin{aligned} &\sharp\{(i,j) \in \{1,\ldots,\ell\}^2 \,|\, i < j \wedge \sigma(i) > \sigma(j) \wedge i \in S^c\} \\ &+\sharp\{(i,j) \in \{1,\ldots,\ell\}^2 \,|\, i < j \wedge \sigma(i) < \sigma(j) \wedge i \in S^c\} \\ &+\sum_{i \in S}(\ell - i + 1) \\ = \quad &\sharp\{(i,j) \in \{1,\ldots,\ell\}^2 \,|\, i < j \wedge \sigma(i) > \sigma(j) \wedge i \in S^c\} \\ &+\sharp\{(i,j) \in \{1,\ldots,\ell\}^2 \,|\, i < j \wedge \sigma(i) < \sigma(j) \wedge i \in S^c\} \\ &+\sharp S + \sum_{i \in S}(\ell - i)\end{aligned}$$

[1]Note that we do not claim, that (i,j) is an inversion of $\pi(w)$ in these cases; in reality $(\sigma(i), \sigma(j))$ is an inversion of $\pi(w)^{-1}$ then. But as $\mathrm{inv}(\pi(w)) = \mathrm{inv}(\pi(w)^{-1})$ this doesn't matter at this point. We will treat this fact in detail in the proof of proposition 3.6, because it will be of particular relevance in this context.

$$\begin{aligned}
&= \sharp\{(i,j) \in \{1,\dots,\ell\}^2 \,|\, i<j \wedge \sigma(i) > \sigma(j) \wedge i \in S^c\} \\
&\quad +\sharp\{(i,j) \in \{1,\dots,\ell\}^2 \,|\, i<j \wedge \sigma(i) < \sigma(j) \wedge i \in S^c\} \\
&\quad +\sharp S + \sum_{i \in S} \sharp\{j \in \{1,\dots,\ell\} \,|\, i<j\} \\
&= \sharp\{(i,j) \in \{1,\dots,\ell\}^2 \,|\, i<j \wedge \sigma(i) > \sigma(j) \wedge i \in S^c\} \\
&\quad +\sharp\{(i,j) \in \{1,\dots,\ell\}^2 \,|\, i<j \wedge \sigma(i) < \sigma(j) \wedge i \in S^c\} \\
&\quad +\sharp S + \sharp\{(i,j) \in \{1,\dots,\ell\}^2 \,|\, i<j \wedge i \in S\} \\
&= \sharp\{(i,j) \in \{1,\dots,\ell\}^2 \,|\, i<j \wedge \sigma(i) > \sigma(j) \wedge i \in S^c\} \\
&\quad +\sharp\{(i,j) \in \{1,\dots,\ell\}^2 \,|\, i<j \wedge \sigma(i) < \sigma(j) \wedge i \in S^c\} \\
&\quad +\sharp S \\
&\quad +\sharp\{(i,j) \in \{1,\dots,\ell\}^2 \,|\, i<j \wedge \sigma(i) > \sigma(j) \wedge i \in S\} \\
&\quad +\sharp\{(i,j) \in \{1,\dots,\ell\}^2 \,|\, i<j \wedge \sigma(i) < \sigma(j) \wedge i \in S\} \\
&= \sharp\{(i,j) \in \{1,\dots,\ell\}^2 \,|\, i<j \wedge \sigma(i) > \sigma(j)\} \\
&\quad +2 \cdot \sharp\{(i,j) \in \{1,\dots,\ell\}^2 \,|\, i<j \wedge \sigma(i) < \sigma(j) \wedge i \in S\} \\
&\quad +\sharp S,
\end{aligned}$$

which is just our lower bound for the length of w. □

Proposition 3.8. *Let $\mathcal{W}$ be the Weyl group of the root system B_ℓ, C_ℓ or D_ℓ and let $w \in \mathcal{W}$ with $\iota(w) = (\sigma, S)$. Then*

for $k < \ell$:

$$\alpha_k \in w(\Phi^+) \;\Leftrightarrow\; \alpha_k \in (\pi(w), \emptyset)(\Phi^+)$$

and for $\mathrm{X}_\ell = \mathrm{B}_\ell$ or $\mathrm{X}_\ell = \mathrm{C}_\ell$:

$$\alpha_\ell \in w(\Phi^+) \;\Leftrightarrow\; \sigma^{-1}(\ell) \in S^c.$$

while for $\mathrm{X}_\ell = \mathrm{D}_\ell$:

$$\alpha_\ell \in w(\Phi^+) \;\Leftrightarrow\; \min\{\sigma^{-1}(\ell-1), \sigma^{-1}(\ell)\} \in S^c.$$

Proof. Let $\mathrm{X}_\ell = \mathrm{B}_\ell$ or C_ℓ. Recall that for $s \in \{1,\dots,\ell\}$:

$$\tau_s^{-1}(x_1,\dots,x_\ell) = (x_1,\dots,x_{s-1},-x_\ell,x_s,\dots,x_{\ell-1})$$

Thus for $1 \le i < j \le \ell$:

$$\tau_s^{-1}(e_i \pm e_j) = \begin{cases} e_i \pm e_j, & \text{if } j < s \\ e_i \pm e_{j+1}, & \text{if } i < s \le j < \ell \\ e_{i+1} \pm e_{j+1}, & \text{if } i \ge s \\ e_i \mp e_s, & \text{if } i < s \wedge j = \ell \\ e_{i+1} \mp e_s, & \text{if } i > s \wedge j = \ell \end{cases}$$

Let $w \in \mathcal{W}$ with $\iota(w) = (\sigma, S)$ and $\alpha_k \in (\pi(w), \emptyset)(\Phi^+)$.

We will prove by induction on the cardinality of S, that either

$$\exists\, 1 \le i < j \le \ell : \quad w^{-1}(\alpha_k) = e_i - e_j \in \Phi^+ \tag{3.3}$$

or

$$\exists\, 1 \le i \le \max(S) \wedge j > i : \quad w^{-1}(\alpha_k) = e_i + e_j \in \Phi^+. \tag{3.4}$$

For the induction start we recall that because of proposition 3.1 for $S = \emptyset$

$$\begin{aligned} &\exists\, 1 \le i < j \le \ell : \\ &w^{-1}(\alpha_k) = (\pi(w), \emptyset)^{-1}(\alpha_k) = (\pi(w)^{-1}, \emptyset)(e_k - e_{k+1}) = e_i - e_j. \end{aligned}$$

For the induction step let $S = \{s_1, \dots, s_{m+1}\}$ with $1 \le s_1 < \dots < s_{m+1} \le \ell$. By induction hypothesis we have either

$$\exists\, 1 \le i < j \le \ell : \iota^{-1}((\sigma, \{s_1, \dots, s_m\})^{-1})(\alpha_k) = e_i - e_j$$

or

$$\exists\, i \le s_m \wedge j > i : \iota^{-1}((\sigma, \{s_1, \dots, s_m\})^{-1})(\alpha_k) = e_i + e_j.$$

In the first case we continue with

$$\begin{aligned} w^{-1}(\alpha_k) &= (\sigma, S)^{-1}(\alpha_k) \\ &= \big((\pi(w), \emptyset) \circ \tau_{s_1} \circ \dots \circ \tau_{s_m} \circ \tau_{s_{m+1}}\big)^{-1}(\alpha_k) \\ &= \big(\tau_{s_{m+1}}^{-1} \circ \tau_{s_m}^{-1} \circ \dots \circ \tau_{s_1}^{-1} \circ (\pi(w)^{-1}, \emptyset)\big)(\alpha_k) \\ &= \tau_{s_{m+1}}^{-1}\big(\tau_{s_m}^{-1} \circ \dots \circ \tau_{s_1}^{-1} \circ (\pi(w)^{-1}, \emptyset)\big)(\alpha_k) \\ &= \tau_{s_{m+1}}^{-1}\big((\iota^{-1}((\sigma, \{s_1, \dots, s_m\})^{-1})\big)(\alpha_k) \\ &= \tau_{s_{m+1}}(e_i - e_j) \\ &= \begin{cases} e_i - e_j, & \text{if } j < s_{m+1} \\ e_i - e_{j+1}, & \text{if } i < s_{m+1} \le j < \ell \\ e_{i+1} - e_{j+1}, & \text{if } i \ge s_{m+1} \\ e_i + e_{s_{m+1}}, & \text{if } i < s_{m+1} \wedge j = \ell \\ e_{i+1} + e_{s_{m+1}}, & \text{if } i > s_{m+1} \wedge j = \ell \end{cases} \end{aligned}$$

and each of the five possibilities satisfies either (3.2) or (3.3) with suitable i', j' instead of i, j. In the second case we get

$$\begin{aligned} w^{-1}(\alpha_k) &= (\sigma, S)^{-1}(\alpha_k) \\ &= \dots \\ &= \tau_{s_{m+1}}(e_i + e_j) \quad (\text{with } i \le s_m) \\ &= \begin{cases} e_i + e_j, & \text{if } j < s_{m+1} \\ e_i + e_{j+1}, & \text{if } i < s_{m+1} \le j < \ell \\ e_{i+1} + e_{j+1}, & \text{if } i \ge s_{m+1} \text{ (impossible!)} \\ e_i - e_{s_{m+1}}, & \text{if } i < s_{m+1} \wedge j = \ell \\ e_{i+1} - e_{s_{m+1}}, & \text{if } i > s_{m+1} \text{ (impossible!)} \wedge j = \ell \end{cases} \end{aligned}$$

and the three possible cases are of the form described in (3.3) or (3.4) as well.

This proves, that for $k < \ell$

$$\alpha_k \in (\pi(w), \emptyset)(\Phi^+) \Rightarrow \alpha_k \in w(\Phi^+).$$

But, of course, the reverse direction can be obtained by considering those $k < \ell$ where $\alpha_k \in (\pi(w), \emptyset)(\Phi^-)$.

Since $w\big((x_i)_{i\in\{1,\ldots,\ell\}}\big) = \big((-1)^{\chi_S(\sigma^{-1}(i))} x_{\sigma^{-1}(i)}\big)_{i\in\{1,\ldots,\ell\}}$ we get for α_ℓ:

$$\alpha_\ell = (\delta_{i\ell})_{i\in\{1,\ldots,\ell\}} \in w(\Phi^+) \Leftrightarrow \sigma^{-1}(\ell) \in S^c.$$

For $\mathrm{X}_\ell = \mathrm{D}_\ell$ and $k < \ell$ the only thing we have to keep in mind is, that τ_{s_1,s_2} (with $1 \le s_1 < s_2 \le \ell$) acts on $(x_1, \ldots, x_\ell)$ exactly as $\tau_{s_1} \circ \tau_{s_2}$ does in the B_ℓ and C_ℓ case.

For α_ℓ we get the following:

$$\begin{aligned}
& \alpha_\ell \in w(\Phi^+) \\
\Leftrightarrow\ & \exists 1 \le i < j \le \ell : w(e_i + e_j) = e_{\ell-1} + e_\ell \vee w(e_i - e_j) = e_{\ell-1} + e_\ell \\
\Leftrightarrow\ & \big[\sigma^{-1}(\ell-1) \in S^c \wedge \sigma^{-1}(\ell) \in S^c\big] \\
& \vee \big[\min\{\sigma^{-1}(\ell-1), \sigma^{-1}(\ell)\} \in S^c \wedge \max\{\sigma^{-1}(\ell-1), \sigma^{-1}(\ell)\} \in S\big] \\
\Leftrightarrow\ & \min\{\sigma^{-1}(\ell-1), \sigma^{-1}(\ell)\} \in S^c
\end{aligned}$$

□

For the Weyl group $\mathcal{W}$ of the root system X_ℓ and $w \in \mathcal{W}$ we define

$$M_w(X,Y) := X^{-\lambda(w)} \prod_{\alpha_j \in w(\Phi^-)} X^{a_j} Y^{b_j},$$

so that the numerator polynomial associated to a Chevalley group of type X_ℓ is $P(X,Y) = \sum_{w\in\mathcal{W}} M_w(X,Y)$.

Let us consider an:

Example 3.9. *Consider the group* GO_{13}. *The root system* Φ *of the derived group* SO_{13} *is of type* B_6. *Let* $\mathcal{W}$ *be the Weyl group associated to this root system. We want to find the monomial* $M_w(X,Y)$ *where*

$$\iota(w) = (\sigma, S) = \left(\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 5 & 3 & 6 & 2 \end{pmatrix}, \{1,2,6\} \right).$$

The "old" algorithm would work as follows: First of all we would have to determine $w(\Phi^+)$ *(or* $w(\Phi^-)$*) and then count the number of positive roots* *sent to negative to calculate the length of* w. *Afterwards, we would have to check for the 6 primitive roots, which of them lie in* $w(\Phi^-)$.

Let us now perform our "new" algorithm in detail. First of all we have to calculate $\pi(w)$:

$$1 = \sigma(1) = \pi(w)(6), \quad 4 = \sigma(2) = \pi(w)(5), \quad 2 = \sigma(6) = \pi(w)(4)$$

and

$$5 = \sigma(3) = \pi(w)(1), \quad 3 = \sigma(4) = \pi(w)(2), \quad 6 = \sigma(5) = \pi(w)(3)$$

Thus we have

$$\pi(w) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 6 & 2 & 4 & 1 \end{pmatrix}, \quad \pi(w)^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 2 & 5 & 1 & 3 \end{pmatrix}$$

and

$$\begin{aligned} w &= \iota^{-1}\left(\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 6 & 2 & 4 & 1 \end{pmatrix}, \emptyset\right) \circ \tau_1 \circ \tau_2 \circ \tau_6 \\ &= (\varsigma_4\varsigma_5\varsigma_3\varsigma_2\varsigma_3\varsigma_4\varsigma_1\varsigma_2\varsigma_3\varsigma_4\varsigma_5) \circ (\varsigma_6\varsigma_5\varsigma_4\varsigma_3\varsigma_2\varsigma_1) \circ (\varsigma_6\varsigma_5\varsigma_4\varsigma_3\varsigma_2) \circ (\varsigma_6). \end{aligned}$$

The inversions of $\pi(w)$ *are*

$$(1,2),(1,4),(1,5),(1,6),(2,4),(2,6),(3,4),(3,5),(3,6),(4,6),(5,6),$$

so that $\mathrm{inv}(\pi(w)) = 11$.

By proposition 12a) the length of w *is*

$$\begin{aligned} \lambda(w) &= \mathrm{inv}(\pi(w)) + \lambda(\tau_1) + \lambda(\tau_2) + \lambda(\tau_6) \\ &= 11 + 6 + 5 + 1 \\ &= 23. \end{aligned}$$

Since

$$\pi(w)^{-1} = \begin{pmatrix} 1 & & 2 & & 3 & & 4 & & 5 & & 6 \\ 6 & (>) & 4 & (>) & 2 & (<) & 5 & (>) & 1 & (<) & 3 \end{pmatrix}$$

and $\sigma^{-1}(6) = 5 \in S^c$, *the propositions 7 and 12b) guarantee, that* $\alpha_3, \alpha_5, \alpha_6 \in w(\Phi^+)$ *while* $\alpha_1, \alpha_2, \alpha_4 \in w(\Phi^-)$.

Now we can write down our result:

$$\begin{aligned} M_w(X,Y) &= X^{-\lambda(w)} \prod_{\alpha_j \in w(\Phi^-)} X^{a_j} Y^{b_j} \\ &= X^{-23} \cdot X^{a_1}Y^{b_1} \cdot X^{a_2}Y^{b_2} \cdot X^{a_4}Y^{b_4} \\ &= X^{a_1+a_2+a_4-23}Y^{b_1+b_2+b_4} \end{aligned}$$

Now we have all tools which are necessary to prove

Proposition 3.10. *(a) Let $\mathcal{W}$ be the Weyl group of the root system* B_ℓ *or* C_ℓ*, $w \in \mathcal{W}$ with $\iota(w) = (\sigma, S)$ and $k \in S^c$. Then*[2]

$$M_{\iota^{-1}(\sigma, S\cup\{k\})}(X,Y) = \frac{M_w(X,Y)\cdot M_{\iota^{-1}(\sigma,\{k\})}(X,Y)}{M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)}.$$

(b) Let $\mathcal{W}$ be the Weyl group of the root system D_ℓ*, $w \in \mathcal{W}$ with $\iota(w) = (\sigma, S)$ and $k_1 < k_2 \in S^c$. Then*

$$M_{\iota^{-1}(\sigma, S\cup\{k_1,k_2\})}(X,Y) = \frac{M_w(X,Y)\cdot M_{\iota^{-1}(\sigma,\{k_1,k_2\})}(X,Y)}{M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)}.$$

Proof. B_ℓ and C_ℓ:

First of all, notice that our equation is equivalent to

$$\lambda(\iota^{-1}(\sigma,\emptyset)) + \lambda(\iota^{-1}(\sigma, S\cup\{k\})) = \lambda(w) + \lambda(\iota^{-1}(\sigma,\{k\})) \tag{3.5}$$

$$\wedge \sum_{\alpha_j\in\iota^{-1}(\sigma,\emptyset)(\Phi^-)} (a_j,b_j) + \sum_{\alpha_j\in\iota^{-1}(\sigma,S\cup\{k\})(\Phi^-)} (a_j,b_j)$$
$$= \sum_{\alpha_j\in w(\Phi^-)} (a_j,b_j) + \sum_{\alpha_j\in\iota^{-1}(\sigma,\{k\})(\Phi^-)} (a_j,b_j). \tag{3.6}$$

Condition (3.5) is thanks to Proposition 3.5a) equivalent to

$$\mathrm{inv}(\sigma) + \mathrm{inv}(\pi(\iota^{-1}(\sigma, S\cup\{k\}))) = \mathrm{inv}(\pi(w)) + \mathrm{inv}(\pi(\iota^{-1}(\sigma,\{k\}))).$$

Thanks to proposition 3.8 it is clear, that (a_ℓ, b_ℓ) either appears once on both sides of 3.6, or on none of it (we presupposed, that $k \in S^c$).

For $j < \ell$, applying the Propositions 3.8 and 3.1 shows that for $v \in \mathcal{W}$, α_j is in $v(\Phi^-)$, if and only if $\pi(v)^{-1}(j) > \pi(v)^{-1}(j+1)$, or in other words:

$$\alpha_j \in v(\Phi^-) \Leftrightarrow (j, j+1) \text{ is an inversion of } \pi(v)^{-1}.$$

Therefore both conditions hold, if each $(i,j) \in \{1,\dots,\ell\}^2$ appears equally often as an inversion of σ^{-1} and of $\pi(\iota^{-1}(\sigma, S\cup\{k\}))^{-1}$ as of $\pi(w)^{-1}$ and $\pi(\iota^{-1}(\sigma,\{k\}))^{-1}$.

For $1 \le s_1 < \cdots < s_m \le \ell$ let $\rho_{\{s_1,\dots,s_m\}}$ denote the permutation component of $\iota(\tau_{s_1}\circ\cdots\circ\tau_{s_m})$. Then proposition 3.6 implies:

- $\pi(w)^{-1} = \rho_S \circ \sigma^{-1}$

[2] For $(\sigma, S) \in \mathcal{S}_\ell \ltimes \mathcal{P}(\{1,\dots,\ell\}$ we write just $\iota^{-1}(\sigma, S)$ leaving one pair of brackets out.

- $\pi(\iota^{-1}(\sigma,\{k\}))^{-1} = \rho_{\{k\}} \circ \sigma^{-1}$
- $\pi(\iota^{-1}(\sigma, S \cup \{k\}))^{-1} = \rho_{S\cup\{k\}} \circ \sigma^{-1}$

Let $1 \leq i < j \leq \ell$. Then

- $$(\sigma(i), \sigma(j)) \text{ is an inversion of } \sigma^{-1} \text{ if and only if } \sigma(i) > \sigma(j).$$

- $$\pi(\iota^{-1}(\sigma,\{k\}))^{-1}(\sigma(i)) = \rho_{\{k\}}(i) = \begin{cases} i & \text{if } i < k \\ \ell & \text{if } i = k \\ i-1 & \text{if } i > k \end{cases}$$

  So $(\sigma(i), \sigma(j))$ is an inversion of $\pi(\iota^{-1}(\sigma, S \cup \{k\}))^{-1}$ if and only if

  $$\big[\sigma(i) < \sigma(j) \wedge i = k\big] \vee \big[\sigma(i) > \sigma(j) \wedge j = k\big].$$

- $$\pi(w)^{-1}(\sigma(i)) = \rho_S(i) = \begin{cases} 1 + \sharp\{t \in S^c \,|\, t < i\}, & \text{if } i \in S^c \\ \ell - \sharp\{s \in S \,|\, s < i\}, & \text{if } i \in S \end{cases}$$

  Thus $(\sigma(i), \sigma(j))$ is an inversion of $\pi(w)$ if and only if

  $$\big[\sigma(i) < \sigma(j) \wedge i \in S\big] \vee \big[\varsigma(i) > \varsigma(j) \wedge j \in S \cup \{k\}\big].$$

- $$\pi(\iota^{-1}(\sigma, S \cup \{k\}))^{-1}(\sigma(i)) = \rho_{S\cup\{k\}}(i)$$
  $$= \begin{cases} 1 + \sharp\{t \in (S \cup \{k\})^c \,|\, t < i\}, & \text{if } i \in (T \cup \{k\})^c \\ \ell - \sharp\{s \in (S \cup \{k\}) \,|\, s < i\}, & \text{if } i \in (S \cup \{k\}) \end{cases}$$

  So $(\varsigma(i), \varsigma(j))$ is an inversion of $\pi(\iota^{-1}(\sigma, S \cup \{k\}))^{-1}$ if and only if

  $$\big[\sigma(i) < \sigma(j) \wedge i \in (S \cup \{k\})\big] \vee \big[\sigma(i) > \sigma(j) \wedge j \in (S \cup \{k\})\big].$$

This implies exactly what we required. Additionally, we see that – as does (a_ℓ, b_ℓ) – any (a_j, b_j) can appear only once on each side of (3.6).

D_ℓ:

The proof works analogous. Again, for α_ℓ it suffices to refer to proposition 3.8. For the other simple roots the only difference is, that we have to consider $\pi(w)^{-1} = \rho_S \circ \sigma$, $\pi(\iota^{-1}(\sigma, \{k_1, k_2\}))^{-1} = \rho_{\{k_1,k_2\}}$ and $\pi(\iota^{-1}(\sigma, S \cup \{k_1, k_2\}))^{-1} = \rho_{S\cup\{k_1,k_2\}}$.

□

Applying this proposition once for each element of S we get:

Corollary 3.11. *Let $\mathcal{W}$ be the Weyl group of the root system B_ℓ or C_ℓ and $w \in \mathcal{W}$ with $\iota(w) = (\sigma, S)$. Then*

$$\frac{M_w(X,Y)}{M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)} = \prod_{k\in S} \frac{M_{\iota^{-1}(\sigma,\{k\})}(X,Y)}{M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)}.$$

And, finally our main result:

Theorem 3.12. *Let $P(X,Y)$ be the numerator defined in (2.1) for the group $\mathrm{GO}_{2\ell+1}$ (type B_ℓ) or $\mathrm{GSp}_{2\ell}$ (type C_ℓ). Then*

$$P(X,Y) = \sum_{\sigma\in\mathcal{S}_\ell} M_{\iota^{-1}(\sigma,\emptyset)}(X,Y) \cdot \prod_{k=1}^{\ell} \left(1 + \frac{M_{\iota^{-1}(\sigma,\{k\})}(X,Y)}{M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)}\right).$$

The products appearing on the right hand side of the above equation look very much like friendly ghost polynomials. Indeed, they are deeply connected to the ghost polynomial of P. An analysis of this effect will be subject of the next chapter.

Since $\sharp S$ is even for each element of $\mathcal{W}$ in the D_ℓ case, it is somewhat tricky to find formulas as in 3.10 and 3.11 for these groups.

Consider $\mathcal{S}_\ell \ltimes \mathcal{P}_{\text{even}}(\{1,\dots,\ell\})$, the image of the Weyl group under ι, as a subgroup of $\mathcal{S}_\ell \ltimes \mathcal{P}(\{1,\dots,\ell\})$. In $\mathcal{S}_\ell \ltimes \mathcal{P}(\{1,\dots,\ell\})$ we can proceed as we did in the B_ℓ and C_ℓ case. Each element of it can again be constructed as a product of $\bar{\varsigma}_k = \big((k,k+1),\emptyset\big)$ $(k \in \{1,\dots,\ell-1\})$ and $\bar{\varsigma}_\ell = (\mathrm{id}, \{\ell\})$. The length function λ of $\mathcal{W}$ can be continued to a function Λ on the larger group:

$$\Lambda(\sigma,S) = \min\{n \in \mathbb{N}_0 : \exists k_1,\dots,k_n \text{ such that } \bar{\varsigma}_{k_1} \circ \cdots \circ \bar{\varsigma}_{k_n} = (\sigma,S)\} - \sharp S$$

In other words: The length function counts as in the B_ℓ and C_ℓ case, but the signchanges ($\bar{\varsigma}_\ell$) are costless.

To each element $(\sigma,S) \in \mathcal{S}_\ell \ltimes \mathcal{P}(\{1,\dots,\ell\})$ we can then associate a monomial

$$M_{(\sigma,S)}(X,Y) = X^{-\Lambda(\sigma,S)} \prod_{\alpha_j\in(\sigma,S)(\Phi^-)} X^{a_j} Y^{b_j},$$

and for each element $w \in \mathcal{W}$ with $\iota(w) = (\sigma,S)$ we get

$$\frac{M_w(X,Y)}{M_{(\sigma,\emptyset)}(X,Y)} = \prod_{k\in S} \frac{M_{(\sigma,\{k\})}(X,Y)}{M_{(\sigma,\emptyset)}(X,Y)}.$$

But now, by the definition of the length function Λ of $\mathcal{S}_\ell \ltimes \mathcal{P}(\{1,\ldots,\ell\})$ it is immediately clear, that $\Lambda(\sigma,\{k\}) = \Lambda(\sigma,\{k,\ell\})$. Recalling the definition of π, it is furthermore obvious, that $\pi(\sigma,\{k\}) = \pi(\sigma,\{k,\ell\})$.

These two facts imply, that for $\sigma \in \mathcal{S}_\ell$ and $k \in \{1,\ldots,\ell-1\}$

$$M_{(\sigma,\{k\})}(X,Y) = M_{(\sigma,\{k,\ell\})}(X,Y) = M_{\iota^{-1}(\sigma,\{k,\ell\})}(X,Y).$$

This leads to

Corollary 3.13. *Let $\mathcal{W}$ be the Weyl group of the root system D_ℓ and let $w \in \mathcal{W}$ with $\iota(w) = \big(\sigma, S\big)$. Then*

$$\frac{M_w(X,Y)}{M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)} = \prod_{k\in S\setminus\{\ell\}} \frac{M_{\iota^{-1}(\sigma,\{k,\ell\})}(X,Y)}{M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)}.$$

And the main result is in this case

Theorem 3.14. *Let $P(X,Y)$ be the numerator defined in (2.1) for the group $\mathrm{GO}_{2\ell}^{+}$ (type D_ℓ). Then*

$$P(X,Y) = \sum_{\sigma\in\mathcal{S}_\ell} M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)\cdot\prod_{k=1}^{\ell-1}\left(1+\frac{M_{\iota^{-1}(\sigma,\{k,\ell\})}(X,Y)}{M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)}\right).$$

Chapter 4

The Shape of the Ghost

In this section we will demonstrate, that for all groups of type B_ℓ, C_ℓ and D_ℓ with their natural representation the ghost polynomial $\tilde{P}(X,Y)$ of the numerator $P(X,Y)$ can be expressed in the same way as $P(X,Y)$ in proposition 3.11 (or 3.13, respectively), except that the sum is not built over all of the symmetric group but only over certain subsets.

First of all let us give names to the individual summands:

Definition 4.1. *(a) Let* W *be the Weyl group of the root system* B_ℓ *or* C_ℓ *and let* $\sigma \in \mathcal{S}_\ell$. *Then we define*

$$P_\sigma(X,Y) := M_{\iota^{-1}(\sigma,\emptyset)}(X,Y) \cdot \prod_{k=1}^{\ell} \left(1 + \frac{M_{\iota^{-1}(\sigma,\{k\})}(X,Y)}{M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)}\right).$$

(b) Let W *be the Weyl group of the root system* D_ℓ *and let* $\sigma \in \mathcal{S}_\ell$. *Then we define*

$$P_\sigma(X,Y) := M_{\iota^{-1}(\sigma,\emptyset)}(X,Y) \cdot \prod_{k=1}^{\ell-1} \left(1 + \frac{M_{\iota^{-1}(\sigma,\{k,\ell\})}(X,Y)}{M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)}\right).$$

Before we begin we should list the results from [11]:

Proposition 4.2. *Let G be a Chevalley group of type* X_ℓ *over* $\mathbb{Q}$, *and let $P(X,Y)$ be the numerator polynomial associated to G and its natural representation by formula (2.1). Let $\tilde{P}(X,Y)$ be the ghost of $P(X,Y)$. Then*

- *for* $\mathrm{X}_\ell = \mathrm{A}_\ell$:

$$\tilde{P}(X,Y) = \prod_{k=0}^{\ell} \sum_{j=0}^{k} \left(X^{\ell-k}Y\right)^j$$

- *for* $\mathrm{X}_\ell = \mathrm{B}_\ell$*:*

$$\tilde{P}(X,Y) = \prod_{k=0}^{\ell-1}(1+X^{a_k}Y)$$

- *for* $\mathrm{X}_\ell = \mathrm{C}_\ell$*:*

$$\tilde{P}(X,Y) = \prod_{k=0}^{\ell-1}(1+X^{\frac{a_k}{2}}Y)\prod_{k=0}^{\ell-2}(1+X^{\frac{a_k}{2}+1}Y)$$

- *for* $\mathrm{X}_\ell = \mathrm{D}_\ell$*:*

$$\tilde{P}(X,Y) = \prod_{k=0}^{\ell-2}\left(1+X^{\frac{a_k}{2}}Y\right)^2.$$

Proposition 4.3. *Let* $P(X,Y)$ *be the numerator polynomial associated to a group of type* B_ℓ *and its natural representation. Then its ghost is*

$$\tilde{P}(X,Y) = P_{\mathrm{id}}(X,Y).$$

Proof. First of all, it is clear that $M_{\iota^{-1}(\mathrm{id},\emptyset)}(X,Y) = 1$. For $k \in \{1,\dots,\ell\}$ we abbreviate $\iota^{-1}(\mathrm{id},\{k\}) =: w_{\mathrm{id},k}$. To determine the monomial associated to $w_{\mathrm{id},k}$ we use the results of chapter 3.

First of all, since the set of signchanges is $\{k\}$, we can read off from definition 3.5, that $j = \mathrm{id}(j) = \pi(w_{\mathrm{id},k})(j)$ for $j < k$, $j+1 = \mathrm{id}(j+1) = \pi(w_{\mathrm{id},k})(j)$ for $k \le j < \ell$ and $k = \mathrm{id}(k) = \pi(w_{\mathrm{id},k})(\ell)$, or in other words

$$\pi(w_{\mathrm{id},k}) = \begin{pmatrix} 1 & \dots & k-1 & k & \dots & \ell-1 & \ell \\ 1 & \dots & k-1 & k+1 & \dots & \ell & k \end{pmatrix}.$$

By proposition 3.7 this implies

$$\begin{aligned} \lambda(w_{\mathrm{id},k}) &= \mathrm{inv}\left(\pi(w_{\mathrm{id},k})\right) + (\ell-k+1) \\ &= (\ell-k)+(\ell-k+1) \\ &= 2\ell-2k+1 \end{aligned}$$

Furthermore, the propositions 3.8 and 3.1 imply for $j < \ell$

$$\begin{aligned} \alpha_j \in w_{\mathrm{id},k}(\Phi^+) &\Leftrightarrow \alpha_j \in \left(\pi(w_{\mathrm{id},k},\emptyset)\right)(\Phi^+) \\ &\Leftrightarrow \pi(w_{\mathrm{id},k})^{-1}(j) < \pi(w_{\mathrm{id},k})^{-1}(j+1), \end{aligned}$$

so that $\alpha_k \in w_{\mathrm{id},k}(\Phi^-)$ and $\alpha_j \in w_{\mathrm{id},k}(\Phi^+)$ for $j \neq k$. Therefore

$$\begin{aligned} P_{\mathrm{id}}(X,Y) &= M_{\iota^{-1}(\mathrm{id},\emptyset)}(X,Y)\prod_{k=1}^{\ell}\left(1+\frac{M_{w_{\mathrm{id},k}}(X,Y)}{M_{\iota^{-1}(\mathrm{id},\emptyset)}(X,Y)}\right) \\ &= \prod_{k=1}^{\ell}\left(1+M_{w_{\mathrm{id},k}}(X,Y)\right) = \prod_{k=1}^{\ell}\left(1+X^{k(2\ell-k)-(2\ell-k+1)}Y\right) \\ &= \prod_{k=1}^{\ell}\left(1+X^{(k-1)(2\ell-k+1)}Y\right) = \prod_{k=0}^{\ell-1}\left(1+X^{k(2\ell-k)}Y\right) \\ &= \prod_{k=0}^{\ell-1}\left(1+X^{a_k}Y\right) = \tilde{P}(X,Y). \end{aligned}$$

□

For groups of type C_ℓ and D_ℓ the corresponding proposition is even more surprising; on the other hand it is much harder to prove it.

We will make use of the following terminology:

Definition 4.4. *Let $\ell \in \mathbb{N}$. For $k < \ell$ we define $\rho_k \in \mathcal{S}_\ell$ as the permutation, for which (i,j) (with $i < j$) is an inversion if and only if $i \geq k$.*

Remark. *(a) The explicit description of ρ_k is*

$$\rho_k = \begin{pmatrix} 1 & \dots & k-1 & k & k+1 & \dots & \ell \\ 1 & \dots & k-1 & \ell & \ell-1 & \dots & k \end{pmatrix}.$$

(b) The ρ_k are self-inverse.

Proposition 4.5. *Let G be a Chevalley group of type C_ℓ or D_ℓ in its natural representation. Let $S = \{s_1, \dots, s_m\}$ with $1 \leq s_1 < \dots < s_m < \ell$ and let $\sigma := \rho_{s_1} \circ \dots \circ \rho_{s_m}$. Then*

(a) if G is of type C_ℓ:

$$P_\sigma(X,Y) = \prod_{k=1}^{\ell-1}\left\{\begin{array}{ll} (1+X^{a_{k-1}+1}Y^2) & , \text{ if } k \notin S \\ (X^{\frac{a_{k-1}}{2}}Y + X^{\frac{a_{k-1}}{2}+1}Y) & , \text{ if } k \in S \end{array}\right\}\cdot(1+X^{\frac{a_{\ell-1}}{2}+1}Y)$$

(a) if G is of type D_ℓ:

$$P_\sigma(X,Y) = \prod_{k=1}^{\ell-1}\left\{\begin{array}{ll} (1+X^{a_{k-1}}Y^2) & , \text{ if } k \notin S \\ 2X^{\frac{a_{k-1}}{2}}Y & , \text{ if } k \in S \end{array}\right.$$

Proof. Let G be a group of type C_ℓ. First of all we have to determine $M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)$. For that purpose it is of course necessary to understand, which pairs (i,j) (with $i<j$) are inversions of σ^{-1} and which are not.

Since by the last remark

$$\sigma^{-1} = (\rho_{s_1}\circ\cdots\circ\rho_{s_m})^{-1} = \rho_{s_m}^{-1}\circ\cdots\circ\rho_{s_1}^{-1} = \rho_{s_m}\circ\cdots\circ\rho_{s_1}$$

and

$$\rho_s(i) = i < s \text{ for } i<s \quad\text{and}\quad \rho_s(i) = \ell - i + s \geq s \text{ for } i \geq s$$

we can conclude, that

$$(i,j) \text{ is an inversion of } \sigma^{-1} \iff \sharp\{s\in S : s\leq i\} \text{ is odd.} \tag{4.1}$$

By lemma 3.3 the length is thus

$$\begin{aligned} & \lambda((\sigma,\emptyset)) = \mathrm{inv}(\sigma) \\ = \; & \frac{1}{2}\big[(\ell-s_1)(\ell-s_1+1) - (\ell-s_2)(\ell-s_2+1) + - \ldots \\ & \cdots - (-1)^n(\ell-s_n)(\ell-s_n+1)\big]. \end{aligned}$$

¿From (4.1) and proposition 3.1 follows

$$\alpha_k \in \iota^{-1}(\sigma,\emptyset)(\Phi^+) \iff \sharp\{s\in S : s\leq k\} \text{ is odd.}$$

Hence

$$\begin{aligned} & M_{\iota^{-1}(\sigma,\emptyset)}(X,Y) \\ = \; & X^{-\frac{1}{2}[(\ell-s_1)(\ell-s_1+1)-(\ell-s_2)(\ell-s_2+1)+-\cdots-(\ell-s_n)(\ell-s_n+1)]} \\ \cdot \; & \prod_{k=s_1}^{s_2-1} X^{a_k}Y^{b_k} \cdot \prod_{k=s_3}^{s_4-1} X^{a_k}Y^{b_k} \cdot \ldots \cdot \prod_{k=s_{n-1}}^{s_n-1} X^{a_k}Y^{b_k}, \end{aligned} \tag{4.2}$$

if $n = \sharp S$ is even and

$$\begin{aligned} & M_{\iota^{-1}(\sigma,\emptyset)}(X,Y) \\ = \; & X^{-\frac{1}{2}[(\ell-s_1)(\ell-s_1+1)-(\ell-s_2)(\ell-s_2+1)+-\cdots+(\ell-s_n)(\ell-s_n+1)]} \\ \cdot \; & \prod_{k=s_1}^{s_2-1} X^{a_k}Y^{b_k} \cdot \prod_{k=s_3}^{s_4-1} X^{a_k}Y^{b_k} \cdot \ldots \cdot \prod_{k=s_n}^{\ell-1} X^{a_k}Y^{b_k}, \end{aligned} \tag{4.3}$$

if $n = \sharp S$ is odd.

Next we have to calculate the $\frac{M_{\iota^{-1}(\sigma,\{k\})}(X,Y)}{M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)}$ for $k\in\{1,\ldots,\ell\}$.

Let $i \in \{1, \ldots, \ell\}$. Then (see definition 3.5):

$$\pi(\iota^{-1}(\sigma, \{i\})) = \begin{pmatrix} 1 & \ldots & i-1 & i & \ldots & \ell-1 & \ell \\ \sigma(1) & \ldots & \sigma(i-1) & \sigma(i+1) & \ldots & \sigma(\ell) & \sigma(i) \end{pmatrix}$$

Now proposition 3.8 implies, that (independent of σ) $\alpha_{\sigma(i)} \in \iota^{-1}(\sigma, \{i\})(\Phi^-)$, $\alpha_{\sigma(i)-1} \in \iota^{-1}(\sigma, \{i\})(\Phi^+)$ (if $\sigma(i) > 1$) and the other simple roots behave just the way they did for $\iota^{-1}(\sigma, \emptyset)$.

How does the length change? We had seen, that if $\sharp\{s \in S : s \leq \sigma(i)\}$ is odd, then $(\sigma(i), j)$ is an inversion of σ^{-1} for every $j > \sigma(i)$. In this case the length increases only by 1.

If on the other hand $\sharp\{s \in S : s \leq \sigma(i)\}$ is even, then $(\sigma(i), j)$ is not an inversion of σ^{-1} for every $j > \sigma(i)$. Then each such pair $(\sigma(i), j)$ has to be exchanged twice in the construction of $\iota^{-1}(\sigma, \{i\})$, which means that the length increases by $2(\ell - \sigma(i)) + 1$.

We may substitute $k := \sigma(i)$. From the observations just made and the formulas for the a_k and b_k in case of the natural representation (see proposition 2.2) follows, that the k-th factor of $P_\sigma(X, Y)$ is

- for $k = 1$:
  - if $1 \notin S$:
  $$1 + X^{a_1 - 2(\ell-1)-1} Y^{b_1} = 1 + X^{a_0+1} Y^2 \tag{4.4}$$
  - if $1 \in S$:
  $$1 + X^{-1} \tag{4.5}$$
- for $1 < k < \ell$:
  - if $\sharp\{s \in S : s \leq k\}$ is even and $k \notin S$:
  $$1 + X^{a_k - 2(\ell-k)-1} Y^{b_k} = 1 + X^{a_{k-1}+1} Y^2 \tag{4.6}$$
  - if $\sharp\{s \in S : s \leq k\}$ is odd and $k \notin S$:
  $$1 + X^{-a_{k-1}-1} Y^{b_{k-1}} = 1 + X^{-(a_{k-1}+1)} Y^{-2} \tag{4.7}$$
  - if $\sharp\{s \in S : s \leq k\}$ is even and $k \in S$:
  $$1 + X^{a_k - a_{k-1} - 2(\ell-k)-1} Y^{b_k - b_{k-1}} = 1 + X \tag{4.8}$$
  - if $\sharp\{s \in S : s \leq k\}$ is odd and $k \in S$:
  $$1 + X^{-1} \tag{4.9}$$

- for $k = \ell$:

  - if $\sharp S$ is even:
$$1 + X^{a_\ell - 1} Y^{b_\ell} = 1 + X^{a_\ell - 1} Y \tag{4.10}$$

  - if $\sharp S$ is odd:
$$1 + X^{a_\ell - a_{\ell-1} - 1} Y^{b_\ell - b_{\ell-1}} = 1 + X^{-(a_\ell - 1)} Y^{-1} \tag{4.11}$$

The factors of type (4.4), (4.6) and (4.10) already have the predicted shape, the others require a multiplication with the factor

- (4.5) $X^{a_0+1}Y$
- (4.7) $X^{a_{k-1}-1}Y^2$
- (4.8) $X^{\frac{a_{k-1}}{2}}$
- (4.9) $X^{\frac{a_{k-1}+1}{2}}$
- (4.11) $X^{a_{\ell-1}-1}Y$.

It remains to be proved, that $M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)$ is just the product of these factors. We have already shown, that $P_\sigma(X,Y)$ can be described as follows

$$\begin{aligned}
&\frac{P_\sigma(X,Y)}{M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)} \\
= \;& \prod_{k=1}^{s_1-1}(1 + X^{a_{k-1}+1}Y^2) \cdot (1 + X^{-1}) \prod_{k=s_1+1}^{s_2-1}(1 + X^{-(a_{k-1}+1)}Y^{-2}) \cdot (1+X) \\
\cdot \;& \prod_{k=s_2+1}^{s_3-1}(1 + X^{a_{k-1}+1}Y^2) \cdot (1 + X^{-1}) \prod_{k=s_3+1}^{s_4-1}(1 + X^{-(a_{k-1}+1)}Y^{-2}) \cdot (1+X) \\
\cdot \;& \ldots \\
\cdot \;& \prod_{s_n+1}^{\ell-1}(1 + X^{a_{k-1}+1}Y^2) \cdot (1 + X^{a_\ell - 1}Y),
\end{aligned}$$

if $n = \sharp S$ is even,

and as

$$
\begin{aligned}
& \frac{P_\sigma(X,Y)}{M_{\iota^{-1}(\sigma,\emptyset)}(X,Y)} \\
= & \prod_{k=1}^{s_1-1}(1+X^{a_{k-1}+1}Y^2)\cdot(1+X^{-1})\prod_{k=s_1+1}^{s_2-1}(1+X^{-(a_{k-1}+1)}Y^{-2})\cdot(1+X) \\
\cdot & \prod_{k=s_2+1}^{s_3-1}(1+X^{a_{k-1}+1}Y^2)\cdot(1+X^{-1})\prod_{k=s_3+1}^{s_4-1}(1+X^{-(a_{k-1}+1)}Y^{-2})\cdot(1+X) \\
\cdot & \dots \\
\cdot & \prod_{k=s_{n-1}+1}^{s_n-1}(1+X^{a_{k-1}+1}Y^2)\cdot(1+X^{-1}) \\
\cdot & \prod_{k=s_n+1}^{\ell-1}(1+X^{-(a_{k-1}+1)}Y^{-2})\cdot(1+X^{-(a_\ell-1)}Y^{-1}),
\end{aligned}
$$

if $n=\sharp S$ id odd.

Recall the equations (4.2) and (4.3). Since

$$
\begin{aligned}
& X^{-\frac{1}{2}[(\ell-s_1)(\ell-s_1+1)-(\ell-s_2)(\ell-s_2+1)]}\prod_{k=s_1}^{s_2-1}X^{a_k}Y^2 \\
\cdot & \prod_{k=1}^{s_1-1}(1+X^{a_{k-1}+1}Y^2)\cdot(1+X^{-1})\prod_{k=s_1+1}^{s_2-1}(1+X^{-(a_{k-1}+1)}Y^{-2})\cdot(1+X) \\
= & X^{-\frac{1}{2}[(\ell^2-2\ell s_1+s_1^2-s_1)-(\ell^2-2\ell s_2+s_2^2-s_2)]}\prod_{k=s_1+1}^{s_2}X^{a_{k-1}}Y^2 \\
\cdot & \prod_{k=1}^{s_1-1}(1+X^{a_{k-1}+1}Y^2)\cdot(1+X^{-1})\prod_{k=s_1+1}^{s_2-1}(1+X^{-(a_{k-1}+1)}Y^{-2})\cdot(1+X) \\
= & X^{\ell s_1-\frac{1}{2}s_1^2+\frac{1}{2}s_1-\ell s_2+\frac{1}{2}s_2^2-\frac{1}{2}s_2}\cdot X^{a_{s_2-1}}Y^2 \\
\cdot & \prod_{k=1}^{s_1-1}(1+X^{a_{k-1}+1}Y^2)\cdot(1+X^{-1})\prod_{k=s_1+1}^{s_2-1}(X^{a_{k-1}}Y^2+X^{-1})\cdot(1+X) \\
= & X^{\ell s_1-\frac{1}{2}s_1^2+\frac{1}{2}s_1-\ell s_2+\frac{1}{2}s_2^2-\frac{1}{2}s_2}\cdot X^{a_{s_2-1}}Y^2\cdot X^{-(s_2-s_1)} \\
\cdot & \prod_{k=1}^{s_1-1}(1+X^{a_{k-1}+1}Y^2)\cdot(X+1)\prod_{k=s_1+1}^{s_2-1}(X^{a_{k-1}+1}Y^2+1)\cdot(1+X)
\end{aligned}
$$

$$
\begin{aligned}
= \;& X^{\ell s_1 - \frac{1}{2}s_1^2 + \frac{1}{2}s_1 - \ell s_2 + \frac{1}{2}s_2^2 - \frac{1}{2}s_2} \cdot X^{a_{s_2-1}} Y^2 \cdot X^{-(s_2-s_1)} \cdot X^{-\frac{a_{s_1-1}}{2} - \frac{a_{s_2-1}}{2}} Y^{-2} \\
\cdot \;& \prod_{k=1}^{s_1-1} (1 + X^{a_{k-1}+1} Y^2) \cdot (X^{\frac{a_{s_1-1}}{2}+1} Y + X^{\frac{a_{s_1-1}}{2}} Y) \\
\cdot \;& \prod_{k=s_1+1}^{s_2-1} (X^{a_{k-1}+1} Y^2 + 1) \cdot (X^{\frac{a_{s_2-1}}{2}} Y + X^{\frac{a_{s_2-1}}{2}+1} Y) \\
= \;& X^{\ell s_1 - \frac{1}{2}s_1^2 + \frac{3}{2}s_1 - \frac{1}{2}a_{s_1-1} - \ell s_2 + \frac{1}{2}s_2^2 - \frac{3}{2}s_2 + \frac{1}{2}a_{s_2-1}} \\
\cdot \;& \prod_{k=1}^{s_1-1} (1 + X^{a_{k-1}+1} Y^2) \cdot (X^{\frac{a_{s_1-1}}{2}+1} Y + X^{\frac{a_{s_1-1}}{2}} Y) \\
\cdot \;& \prod_{k=s_1+1}^{s_2-1} (X^{a_{k-1}+1} Y^2 + 1) \cdot (X^{\frac{a_{s_2-1}}{2}} Y + X^{\frac{a_{s_2-1}}{2}+1} Y) \\
= \;& X^{\ell s_1 - \frac{1}{2}s_1^2 + \frac{3}{2}s_1 - \frac{1}{2}(s_1-1)(2\ell - s_1 + 2) - \ell s_2 + \frac{1}{2}s_2^2 - \frac{3}{2}s_2 + \frac{1}{2}(s_2-1)(2\ell - s_2 + 2)} \\
\cdot \;& \prod_{k=1}^{s_1-1} (1 + X^{a_{k-1}+1} Y^2) \cdot (X^{\frac{a_{s_1-1}}{2}+1} Y + X^{\frac{a_{s_1-1}}{2}} Y) \\
\cdot \;& \prod_{k=s_1+1}^{s_2-1} (X^{a_{k-1}+1} Y^2 + 1) \cdot (X^{\frac{a_{s_2-1}}{2}} Y + X^{\frac{a_{s_2-1}}{2}+1} Y) \\
= \;& X^0 \cdot \prod_{k=1}^{s_1-1} (1 + X^{a_{k-1}+1} Y^2) \cdot (X^{\frac{a_{s_1-1}}{2}+1} Y + X^{\frac{a_{s_1-1}}{2}} Y) \\
\cdot \;& \prod_{k=s_1+1}^{s_2-1} (X^{a_{k-1}+1} Y^2 + 1) \cdot (X^{\frac{a_{s_2-1}}{2}} Y + X^{\frac{a_{s_2-1}}{2}+1} Y)
\end{aligned}
$$

the first s_2 factors behave as predicted. Of course, the same calculation can be made for the (s_2+1)-st – s_4-th factor and so on, and the calculation for the (s_n+1)-st – ℓ-th factor (or for the $(s_{n-1}+1)$-st – ℓ-th factor, respectively) works very similar.

For groups of type D_ℓ the proof works completely analogous. □

These two propositions imply immediately a formula for the ghost polynomial as a sum of a suitable set of $P_\sigma(X,Y)$:

Theorem 4.6. *Let $P(X,Y)$ be the numerator polynomial associated to a Chevalley group of type C_ℓ or D_ℓ in its natural representation. Then its ghost is*

$$\tilde{P}(X,Y) = \sum_{(v_1,\dots,v_{\ell-1}) \in \{0,1\}^{\ell-1}} P_{\rho_1^{v_1} \circ \dots \circ \rho_{\ell-1}^{v_{\ell-1}}}(X,Y).$$

Proof. Let $S \subset \{1, \ldots, \ell - 1\}$ with $k \notin S$ and let $S' := S \cup \{k\}$. Then the associated class-polynomials have all factors in common, except the $k - th$. In the sum of them, the k-th factor can be replaced by

$$(1 + X^{a_{k-1}+1}Y^2) + (X^{\frac{a_{k-1}}{2}}Y + X^{\frac{a_{k-1}}{2}+1}Y) = (1 + X^{\frac{a_{k-1}}{2}}Y)(1 + X^{\frac{a_{k-1}}{2}+1})$$

or

$$(1 + X^{a_{k-1}}Y^2) + 2X^{\frac{a_{k-1}}{2}}Y = (1 + X^{\frac{a_{k-1}}{2}}Y)^2,$$

respectively. Successive usage of this argument for each $k \in \{1, \ldots, \ell - 1\}$ proves the assertion. □

Example 4.7. *Consider the group* GO_7 *in the irreducible representation whose contragredient has weight* $\omega = n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3$ $(n_1, n_2, n_3 \in \mathbb{N}_0)$. *Then with* $Y = Z$ *if* n_3 *is even and* $Y = Z^{\frac{1}{2}}$ *if* n_3 *is odd, the numerator polynomial* $P(X, Y)$ *has the ghost* $\tilde{P}(X, Y)$*, where*

1. *if* $n_1 = n_2 = n_3 = 0$*:*

$$\tilde{P}(X, Z) = P(X, Z)$$

2. $6n_1 < 2n_2 + n_3 \wedge 2n_1 < n_3 \wedge 4n_2 < 6n_1 + n_3 \wedge n_1 > 0$*:*

$$\begin{aligned}\tilde{P}(X, Z) &= (1 + X^4Z^{b_1})(1 + X^3Z^{b_2-b_1}) \\ &\quad \cdot(1 + X^5Z^{b_1+b_3-b_2})(1 + XZ^{b_2-b_1})(1 + Z^{b_1}) \\ &= (1 + X^4Z^{2n_1+2n_2+n_3})(1 + X^3Z^{2n_2+n_3}) \\ &\quad \cdot(1 + X^5Z^{2n_1+2n_2+2n_3})(1 + XZ^{2n_2+n_3})(1 + Z^{2n_1+2n_2+n_3})\end{aligned}$$

3. $n_1 = 0 \wedge 4n_2 < n_3$*:*

$$\begin{aligned}\tilde{P}(X, Z) &= (1 + X^4Z^{b_1})(1 + X^3Z^{b_2-b_1} + X^6Z^{b_2}) \\ &\quad \cdot(1 + X^2Z^{b_3-b_2})(1 + XZ^{b_2-b_1})(1 + Z^{b_1}) \\ &= (1 + X^4Z^{2n_2+n_3})(1 + X^3Z^{2n_2+n_3} + X^6Z^{4n_2+2n_3}) \\ &\quad \cdot(1 + X^2Z^{n_3})(1 + XZ^{2n_2+n_3})(1 + Z^{2n_2+n_3})\end{aligned}$$

4. $6n_1 + n_3 = 4n_2 \wedge 2n_1 < n_3 \wedge n_1 > 0$*:*

$$\begin{aligned}\tilde{P}(X, Z) &= (1 + X^4Z^{b_1})(1 + X^3Z^{b_2-b_1} + X^8Z^{b_3}) \\ &\quad \cdot(1 + XZ^{b_2-b_1})(1 + Z^{b_1}) \\ &= (1 + X^4Z^{5n_1+\frac{3}{2}n_3})(1 + X^3Z^{3n_1+\frac{3}{2}n_3} + X^8Z^{8n_1+4n_3}) \\ &\quad \cdot(1 + XZ^{3n_1+\frac{3}{2}n_3})(1 + Z^{5n_1+\frac{3}{2}n_3})\end{aligned}$$

5. $n_1 = 0 \wedge n_3 = 4n_2 \wedge n_3 > 0$:

$$\begin{aligned}\tilde{P}(X,Z) &= (1+X^4Z^{b_1})(1+X^3Z^{b_2-b_1}+X^6Z^{b_2}+X^8Z^{b_3})\\ &\quad\cdot(1+XZ^{b_2-b_1})(1+Z^{b_1})\\ &= (1+X^4Z^{6n_2})(1+X^3Z^{6n_2}+X^6Z^{12n_2}+X^8Z^{16n_2})\\ &\quad\cdot(1+XZ^{6n_2})(1+Z^{6n_2})\end{aligned}$$

6. $6n_1 < 2n_2 + n_3 \wedge 2n_1 < n_3 \wedge 6n_1 + n_3 < 4n_2$:

$$\begin{aligned}\tilde{P}(X,Z) &= (1+X^4Z^{b_1})(1+X^8Z^{b_3})(1+XZ^{b_2-b_1})(1+Z^{b_1})\\ &= (1+X^4Z^{2n_1+2n_2+n_3})(1+X^8Z^{2n_1+4n_2+3n_3})\\ &\quad\cdot(1+XZ^{2n_2+n_1})(1+Z^{2n_1+2n_2+n_3})\end{aligned}$$

7. $2n_1 = n_3 \wedge n_3 < n_2$:

$$\begin{aligned}\tilde{P}(X,Z) &= (1+X^4Z^{b_1}+X^8Z^{b_3}+X^{12}Z^{b_1+b_3})\\ &\quad\cdot(1+XZ^{b_2-b_1})(1+Z^{b_1})\\ &= (1+X^4Z^{2n_2+2n_3}+X^8Z^{4n_2+4n_3}+X^{12}Z^{6n_2+6n_3})\\ &\quad\cdot(1+XZ^{2n_2+n_3})(1+Z^{2n_2+n_3})\end{aligned}$$

8. $6n_1 < 2n_2 + n_3 \wedge n_3 < 2n_1$:

$$\begin{aligned}\tilde{P}(X,Z) &= (1+X^8Z^{b_3})(1+X^4Z^{b_1})(1+XZ^{b_2-b_1})(1+Z^{b_1})\\ &= (1+X^8Z^{2n_1+4n_2+3n_3})(1+X^4Z^{2n_1+2n_2+n_3})\\ &\quad\cdot(1+XZ^{2n_2+n_3})(1+Z^{2n_1+2n_2+n_3})\end{aligned}$$

9. $2n_2 + n_3 < 6n_1 \wedge 2n_1 + 4n_2 < 5n_3 \wedge n_3 < 4n_1 + 4n_2$:

$$\begin{aligned}\tilde{P}(X,Z) &= (1+X^7Z^{b_2})(1+X^5Z^{b_1+b_3-b_2})(1+XZ^{b_2-b_1})(1+Z^{b_1})\\ &= (1+X^7Z^{2n_1+4n_2+2n_3})(1+X^5Z^{2n_1+2n_2+2n_3})\\ &\quad\cdot(1+XZ^{2n_2+n_3})(1+Z^{2n_1+2n_2+n_3})\end{aligned}$$

10. $2n_2 + n_3 < 6n_1 \wedge 4n_1 + 4n_2 < n_3$:

$$\begin{aligned}\tilde{P}(X,Z) &= (1+X^7Z^{b_2})(1+X^3Z^{b_1})(1+X^2Z^{b_3-b_2})\\ &\quad\cdot(1+XZ^{b_2-b_1})(1+Z^{b_1})\\ &= (1+X^7Z^{2n_1+4n_2+2n_3})(1+X^3Z^{2n_1+2n_2+n_3})(1+X^2Z^{n_3})\\ &\quad\cdot(1+XZ^{2n_2+n_3})(1+Z^{2n_1+2n_2+n_3})\end{aligned}$$

11. $3n_2 < n_1 \wedge 4n_1 + 4n_2 = n_3$:

$$\begin{aligned}\tilde{P}(X,Z) &= (1+X^7Z^{b_2})(1+X^3Z^{b_1}+X^5Z^{b_1+b_3-b_2})\\ &\quad\cdot(1+XZ^{b_2-b_1})(1+Z^{b_1})\\ &= (1+X^7Z^{10n_1+12n_2})(1+X^3Z^{6n_1+6n_2}+X^5Z^{10n_1+10n_2})\\ &\quad\cdot(1+XZ^{4n_1+6n_2})(1+Z^{6n_1+6n_2})\end{aligned}$$

12. $2n_2 + n_3 \leq 6n_1 \wedge 5n_3 < 2n_1 + 4n_2$:

$$\begin{aligned}\tilde{P}(X,Z) &= (1+X^8Z^{b_3})(1+X^5Z^{b_2})(1+Z^{b_1}) \\ &= (1+X^8Z^{2n_1+4n_2+3n_3})(1+X^5Z^{2n_1+4n_2+2n_3}) \\ &\quad \cdot(1+Z^{2n_1+2n_2+n_3})\end{aligned}$$

13. $n_2 < 2n_1 \wedge 2n_1 + 4n_2 = 5n_3$:

$$\begin{aligned}\tilde{P}(X,Z) &= (1+X^7Z^{b_2}+X^8Z^{b_3})(1+X^5Z^{b_2})(1+Z^{b_1}) \\ &= (1+X^7Z^{7n_3}+X^8Z^{8n_3})(1+X^5Z^{6n_3})(1+Z^{6n_3-2n_2})\end{aligned}$$

14. $2n_2 + n_3 = 6n_1 \wedge 2n_1 < n_3 \wedge 3n_3 < 16n_1$:

$$\begin{aligned}\tilde{P}(X,Z) &= (1+X^4Z^{b_1}+X^7Z^{b_2})(1+X^5Z^{b_1+b_3-b_2}) \\ &\quad \cdot(1+XZ^{b_2-b_1})(1+Z^{b_1}) \\ &= (1+X^4Z^{8n_1}+X^7Z^{14n_1})(1+X^5Z^{14n_1}) \\ &\quad \cdot(1+XZ^{6n_1-2n_3})(1+Z^{8n_1})\end{aligned}$$

15. $2n_2 + n_3 = 6n_1 \wedge 16n_1 < 3n_3$:

$$\begin{aligned}\tilde{P}(X,Z) &= (1+X^4Z^{b_1}+X^7Z^{b_2})(1+X^3Z^{b_1})(1+X^2Z^{b_3-b_2}) \\ &\quad \cdot(1+XZ^{b_2-b_1})(1+Z^{b_1}) \\ &= (1+X^4Z^{8n_1}+X^7Z^{14n_1})(1+X^3Z^{8n_1})(1+X^2Z^{n_3}) \\ &\quad \cdot(1+XZ^{6n_1-2n_3})(1+Z^{8n_1})\end{aligned}$$

16. $16n_1 = n_3 \wedge 16n_2 = n_3 \wedge n_3 > 0$:

$$\begin{aligned}\tilde{P}(X,Z) &= (1+X^4Z^{b_1}+X^7Z^{b_2})(1+X^3Z^{b_1}+X^5Z^{b_1+b_3-b_2}) \\ &\quad \cdot(1+XZ^{b_2-b_1})(1+Z^{b_1}) \\ &= (1+X^4Z^{\frac{3}{2}n_3}+X^7Y^{\frac{21}{8}n_3})(1+X^3Z^{\frac{3}{2}n_3}+X^5Z^{\frac{5}{2}n_3})\end{aligned}$$

17. $2n_1 = n_3 \wedge n_2 = n_3$:

$$\begin{aligned}\tilde{P}(X,Z) &= (1+X^4Z^{b_1}+X^7Z^{b_2}+X^8Z^{b_3}+X^{12}Z^{b_1+b_3}) \\ &\quad \cdot(1+XZ^{b_2-b_1})(1+Z^{b_1}) \\ &= (1+X^4Z^{4n_3}+X^7Z^{7n_3}+X^8Z^{8n_3}+X^{12}Z^{12n_3}) \\ &\quad \cdot(1+XZ^{3n_3})(1+Z^{4n_3})\end{aligned}$$

In particular, the ghost polynomial is friendly, if the following inequalities hold:

$$2n_2 + n_3 \neq 6n_1, \quad 2n_1 \neq n_3, \quad 6n_1 + n_3 \neq 4n_2,$$
$$2n_1 + 4n_2 \neq 5n_3, \quad 4n_1 + 4n_2 \neq n_3.$$

Proof. Following section 1.4 we have

$$\lambda_1 = \alpha_1 + \alpha_2 + \alpha_3, \quad \lambda_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3, \quad \lambda_3 = \frac{1}{2}\alpha_1 + \alpha_2 + \frac{3}{2}\alpha_3.$$

Thus

$$m\omega = m\Big(n_1 + n_2 + \frac{1}{2}n_3\Big)\alpha_1 + m\Big(n_1 + 2n_2 + n_3\Big)\alpha_2 + m\Big(n_1 + 2n_2 + \frac{3}{2}n_3\Big)\alpha_3,$$

so that (see definition 2.2)

$$m = \begin{cases} 1 & \text{, if } n_3 \text{ is even} \\ 2 & \text{, if } n_3 \text{ is odd} \end{cases}$$

and

$$b_1 = m\Big(n_1 + n_2 + \frac{1}{2}n_3\Big), \quad b_2 = m\Big(n_1 + 2n_2 + n_3\Big), \quad b_3 = m\Big(n_1 + 2n_2 + \frac{3}{2}n_3\Big).$$

Furthermore

$$a_1 = 5, \quad a_2 = 8, \quad a_3 = 9.$$

The resulting numerator polynomial is

$$\begin{aligned} P(X,Y) &= 1 + Y^{b_1} \\ &+ X(Y^{b_1} + Y^{b_2}) \\ &+ X^2(Y^{b_1} + Y^{b_2}) \\ &+ X^3(Y^{b_1} + 2Y^{b_2} + Y^{b_3}) \\ &+ X^4(Y^{b_1} + 2Y^{b_2} + Y^{b_3}) \\ &+ X^5(2Y^{b_2} + Y^{b_3} + Y^{b_1+b_3}) \\ &+ X^6(2Y^{b_2} + 2Y^{b_3} + Y^{b_1+b_2} + Y^{b_1+b_3}) \\ &+ X^7(Y^{b_2} + Y^{b_3} + 2Y^{b_1+b_2} + 2Y^{b_1+b_3}) \\ &+ X^8(Y^{b_3} + Y^{b_1+b_2} + 2Y^{b_1+b_3}) \\ &+ X^9(Y^{b_1+b_2} + 2Y^{b_1+b_3} + Y^{b_2+b_3}) \\ &+ X^{10}(Y^{b_1+b_2} + 2Y^{b_1+b_3} + Y^{b_2+b_3}) \\ &+ X^{11}(Y^{b_1+b_3} + Y^{b_2+b_3}) \\ &+ X^{12}(Y^{b_1+b_3} + Y^{b_2+b_3}) \\ &+ X^{13}(Y^{b_1+b_3} + Y^{b_1+b_2+b_3}). \end{aligned}$$

The points that are into question for connecting $(0,0)$ with the line having smallest slope are $(4, b_1)$, $(7, b_2)$, $(8, b_3)$, $(10, b_1+b_2)$, $(12, b_1+b_3)$ and $(13, b_2+b_3)$.

We will outline the proof only for case number 2. It is clear, that the other cases have to be handled similarly.

Therefore let $6n_1 < 2n_2 + n_3$, $2n_1 < n_3$, $4n_2 < 6n_1 + n_3$ and $n_1 > 0$. Then

$$
\begin{aligned}
\frac{b_1}{4} = \frac{14n_1 + 14n_2 + 7n_3}{28} &\stackrel{6n_1<2n_2+n_3}{<} \frac{8n_1 + 16n_2 + 8n_3}{28} = \frac{b_2}{7}\\
\frac{b_1}{4} = \frac{4n_1 + 4n_2 + 2n_3}{8} &\stackrel{2n_1<n_3}{<} \frac{2n_1 + 4n_2 + 3n_3}{8} = \frac{b_3}{8}\\
\frac{b_1}{4} = \frac{10n_1 + 10n_2 + 5n_3}{20} &\stackrel{2n_1<n_3}{<} \frac{8n_1 + 12n_2 + 6n_3}{20} = \frac{b_1 + b_2}{10}\\
\frac{b_1}{4} = \frac{6n_1 + 6n_2 + 3n_3}{12} &\stackrel{2n_1<n_3}{<} \frac{4n_1 + 6n_2 + 4n_3}{12} = \frac{b_1 + b_3}{12}\\
\frac{b_1}{4} = \frac{26n_1 + 26n_2 + 13n_3}{52} &\stackrel{2n_1<n_3}{<} \frac{16n_1 + 32n_2 + 18n_3}{52}\\
&\leq \frac{16n_1 + 32n_2 + 20n_3}{52} = \frac{b_2 + b_3}{13},
\end{aligned}
$$

which means that in the Newton diagram starting at $(0,0)$ the line with smallest slope is the one, which connects $(0,0)$ and $(4,b_1)$. The ghost polynomial has thus a factor $(1 + X^4 Y^{b_1})$.

In the next step we start at $(4,b_1)$ and we have to find out, which of the points $(7,b_2)$, $(8,b_3)$, $(10,b_1+b_2)$, $(12,b_1+b_3)$ and $(13,b_2+b_3)$ connected with $(4,b_1)$ gives the line with the smallest slope.

$$
\begin{aligned}
\frac{b_2 - b_1}{3} = \frac{16n_2 + 8n_3}{24} &\stackrel{4n_2<6n_1+n_3}{<} \frac{6n_1 + 12n_2 + 9n_3}{24}\\
&\stackrel{2n_1<n_3}{<} \frac{12n_2 + 12n_3}{24} = \frac{b_3 - b_1}{4}\\
\frac{b_2 - b_1}{3} = \frac{2n_2 + n_3}{3} &\stackrel{n_1>0}{<} \frac{n_1 + 2n_2 + n_3}{3} = \frac{b_2}{6}\\
\frac{b_2 - b_1}{3} = \frac{16n_2 + 8_3}{24} &\stackrel{4n_2<6n_1+n_3}{<} \frac{6n_1 + 12n_2 + 9n_3}{24} = \frac{b_3}{8}\\
\frac{b_2 - b_1}{3} = \frac{6n_2 + 3n_3}{9} &\stackrel{n_3>0}{<} \frac{2n_1 + 6n_2 + 4n_3}{9} = \frac{b_2 + b_3 - b_1}{9},
\end{aligned}
$$

so the second factor is $(1 + X^3 Y^{b_2 - b_1})$. Afterwards

$$
\begin{aligned}
\frac{b_1 + b_3 - b_2}{5} = \frac{4n_1 + 4n_2 + 4n_3}{10} &\stackrel{4n_2<6n_1+n_3}{<} \frac{10n_1 + 5n_3}{10}\\
&\stackrel{2n_1<n_3}{<} \frac{10n_3}{10} < n_3\\
&= b_3 - b_2\\
\frac{b_1 + b_3 - b_2}{5} = \frac{12n_1 + 12n_2}{30} &\stackrel{2n_1<n_3}{<} \frac{10n_1 + 12n_2 + n_3}{30}\\
&< \frac{2n_1 + 4n_2 + 3n_3}{6} = \frac{b_3}{6},
\end{aligned}
$$

so the third factor is $(1 + X^5 Y^{b_1 + b_3 - b_2})$ followed by $(1 + XY^{b_2 - b_1})$ and $(1 + Y^{b_1})$. □

Chapter 5

Concluding Remarks

In this final chapter I will explain, how I found the theorems proved in the last two chapters. Afterwards I will list some open questions.

The starting point for the research done in this thesis was the formula for the numerator polynomials from chapter 2. Since these polynomials are are constructed by a natural process from the groups, one might hope that certain group properties, which are "invisible" when looking at the group itself, can be seen when looking at the Zeta function.

As already explained in chapter 2, a proper Zeta function should be just as complicated as its origin, which is in our case the group it is associated to. It was clear that in this context there was still some work to be done, because the Zeta functions (namely the numerator polynomials) looked much more complicated than they should.

Two possible reasons for this are in question: One is, that the constructing process adds information. If this is the case, the shape of the function is not only dependent on the group but also on the construction. Under these circumstances the function should not be called "the" Zeta function of the group.

The other possibility is, that the function is only virtually complicated, which shall mean, that it has a symmetry, which makes it possible to express it entirely through a smaller set of data as well. For example, one would find a factorized polynomial less complicated than its expansion.

The key question seems to be: What is the shortest algorithm that calculates the Zeta function?

Unfortunately this question will never be answered. For any given algorithm it is impossible to prove that it is the shortest possible (G. Chaitin). So the best thing we can do is to find shorter and shorter algorithms that generate our Zeta functions, hoping to finallay gather one, from which it seems to be plausible, that it is the shortest possible one. For that purpuse it would certainly be an important hint, if one could see the group structure in it.

¿From the theory of algebraic curves it is well known, that the topological information about these curves is concentrated in its singularities. Examinations of multiple examples of curves $P_\rho(X,Y)=0$, where P_ρ is one of the numerator polynomials defined in chapter 2 (and ρ the natural representation) lead to the observation, that these polynomials and their ghosts behave very similar in this matter, namely:

1. Each considered curve $P_\rho(X,Y)=0$ has up to four singular points which are not just ordinary double points. Their (homogenized) coordinates are $[1:0:0]$, $[0:1:0]$, $[1:-1:1]$ and $[-1:-1:1]$.
2. At these four points the ghost curve $\tilde{P}_\rho(X,Y)=0$ is topologically equivalent to $P_\rho(X,Y)=0$.

Since two curves which have the same degree are topologically equivalent if their singularities are, this is a hint for a deeper connection between P_ρ and $\tilde{P}_\rho$ or even P and $\tilde{P}$.

In [9] the goal was a general proof of the equivalence of the mentioned singularities at $[1:0:0]$ and $[0:1:0]$. It turned out that these singularities are connected (via the coefficients of the polynomial) to a symmetry of the Weyl group. It is the same symmetry that causes the functional equation discovered by J.I. Igusa, A. Lubotzky and M. du Sautoy.

This observation lead to the idea, that a singularity might be a hint for some symmetry in the object the Zeta function is associated to. Indeed, through the new formulas for the numerator polynomials (theorems 3.12 and 3.14) we see that the singularities at $[1:-1:1]$ and $[-1:-1:1]$ do also appear not only by chance, but have a connection to the particular structure of the Weyl group. In particular, for groups of type B_ℓ in their natural representation it seems, that each of the P_σ has the same singularities in $[1:-1:1]$ and $[-1:-1:1]$ as $\tilde{P}$.

Perhaps we should dot the way we found our result 3.12 for the B_ℓ case:

A curve $P(X,Y)=0$ for $P(X,Y)=\sum_{\mu,\nu} c_{\mu,\nu}X^\mu Y^\nu$ has a singularity of multiplicity $m\in\mathbb{N}$ at a point (x_0,y_0), if all partial derivations of $P(X,Y)$ up to order $(m-1)$ vanish at (x_0,y_0). In our particular case $(x_0,y_0)=(-1,-1)$ this means

- if $m\geq 1$

$$\sum_{\mu,\nu}(-1)^{\mu+\nu}c_{\mu,\nu}=0 \quad \text{and}$$

- if $m \geq 2$

$$\sum_{\mu,\nu} \mu(-1)^{\mu+\nu} c_{\mu,\nu} = 0 \quad \text{and} \quad \sum_{\mu,\nu} \nu(-1)^{\mu+\nu} c_{\mu,\nu} = 0 \quad \text{and}$$

- if $m \geq 3$

$$\sum_{\mu,\nu} \mu(\mu-1)(-1)^{\mu+\nu} c_{\mu,\nu} = 0 \quad \text{and}$$
$$\sum_{\mu,\nu} \mu\nu(-1)^{\mu+\nu} c_{\mu,\nu} = 0 \quad \text{and} \quad \sum_{\mu,\nu} \nu(\nu-1)(-1)^{\mu+\nu} c_{\mu,\nu} = 0$$

and so on.

Unfortunately, checking these equations directly for the general case is impossible. But in the examples the ghost has the same singularity at this point. Thus I believed, that the singularity has something to do with a hidden connection between the polynomial and its ghost. Let us first write down the above equations for polynomials of type "friendly ghost". More precisely, let $\tilde{P}(X,Y) = \prod_k (1 + X^{u_k} Y^{v_k})$. In this case our equations are

- $m \geq 1$

$$\prod_i (1 + (-1)^{u_k+v_k}) = 0 \quad \text{and}$$

- $m \geq 2$

$$\sum_k u_k (-1)^{u_k+v_k} \prod_{j \neq k} (1 + (-1)^{u_j+v_j}) = 0 \quad \text{and}$$
$$\sum_k v_k (-1)^{u_k+v_k} \prod_{j \neq k} (1 + (-1)^{u_j+v_j}) = 0$$

and so on.

Obviously, the derivations vanish if for sufficiently many k the sum u_k+v_k is odd, and luckily this is indeed the case (for groups of type B_ℓ, C_ℓ, D_ℓ and their natural representations). If the equivalence of the singularities is no accident, this should be a hint for a deeper connection between the ghost polynomial and the corresponding original polynomial.

One aspect that turned out for all groups of type B_ℓ, C_ℓ and D_ℓ and their natural representations was, that all monomials the ghost consists of do also appear in the original polynomial. In particular, in the case of B_ℓ the ghost polynomial consists exactly of those monomials, that belong (in the formula for the numerator polynomial) to the subgroup isomorphic to $\mathcal{S}_\ell$ (proposition 4.3).

As already mentioned the ghost polynomial is topologically equivalent to the original polynomial at $(-1,-1)$; due to our last observation this means that in $(-1,-1)$ the whole polynomial is equivalent to a part of itself, that belongs to a subgroup of the Weyl group. Thus we constructed the polynomials that belong to the other classes and it turned out, that all of them also behave the same way in $(-1,-1)$.

Finally, since in case of the natural representation the ghost polynomial is a product of factors of type (1+monomial), I tried to find a similar description for the polynomials belonging to the other classes and found out, that each of them can be described as a product of such factors as well, this time translated by another monomial (proposition 3.11)

For the C_ℓ and D_ℓ cases similar results were achieved, except that the ghost polynomial is the sum of several class-polynomials then (proposition 4.6).

Due to these results the following questions arise:

- Is it possible to simplify the formula for the numerator polynomials even further?
- Can the ghost zeta function be expressed as a p-adic integral?
- What is the meaning of the way the ghost construction divides the set of dominant weights? It seems that the topological types of the algebraic curves $P_{\rho_1}(X,Y)=0$ and $P_{\rho_2}(X,Y)=0$ are essentially different only if the shapes of the ghost polynomials $\tilde{P}_{\rho_1}(X,Y)$ and $\tilde{P}_{\rho_2}(X,Y)$ differ.
- Will the method of examining the singularities help to find symmetries in Zeta functions associated to other than Chevalley groups as well?

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