Controlling the Number of False Rejections in Multiple Hypotheses Testing

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NATALIE, NIKOLAS UND MEINEN UNGEBORENEN SOHN

Contents

Abstract

The invention of computers has a massive impact on our everyday life and the work of scientists. Nowadays, we are able to collect, store, and process a huge amount of data. In order to analyze these data, it is necessary to adapt and extend the classical statistical theory. For instance, today it is possible to measure the expression pattern of all genes (approximately 23.000) from one person at an arbitrary fixed time point. Measuring the expression pattern for 2 groups with 500 persons per group results in approximately $2 \cdot 10^7$ measurements. And of course the aim is not only to state whether the expression pattern between both groups is different but to determine those genes that are differently expressed. From a more abstract point of view, for a set of null hypotheses the aim is to decide which null hypotheses are true and which are false. One part within the multiple testing framework is the development of procedures which make a decision for every single null hypothesis and at the same time control a predefined error criterion. The main topic of this thesis is to introduce a new error criterion and to develop procedures controlling this new criterion.

In Chapter 1 we state the general framework for this thesis and introduce some known error measures which are important for our theory. Additionally, we define a large class of test procedures that are commonly applied to control miscellaneous error rates.

In Chapter 2 we motivate and define a new error criterion based on the expected number of false rejections (ENFR). We investigate the ENFR with respect to least favorable parameter configurations. Furthermore, the asymptotic relations between ENFR and a large class of test procedures are investigated. Finally, a central limit theorem is formulated for the number of false rejections.

Chapter 3 is devoted to relationships and differences between ENFR and another important error measure the so-called false discovery rate (FDR). It turns out that under some regularity conditions control of the ENFR implies control of the FDR. And control of the FDR under independence may also imply control of the ENFR. We also state situations where the FDR is controlled and the ENFR is inflated and vice versa.

In Chapter 4 we investigate the ENFR behavior of different procedures which control the FDR for a finite number of null hypotheses or asymptotically.

Chapter 5 is devoted to exact control of the ENFR for arbitrary bounding functions. We present a recursive scheme that allows exact control of the ENFR. But, in general, the resulting solutions are not feasible. Therefore, an algorithm is developed that yields feasible solutions with good performance with respect to exact control.

Chapter 6 is concerned with an error measure based on the number of false rejections which is a probabilistic counterpart to the ENFR. Ordinary differential equations will play an important role in constructing procedures that asymptotically control this error measure. Moreover, the results are carried over to the false discovery exceedance (FDX), a probabilistic counterpart to the FDR.

Chapter 7 concludes the thesis with an outlook that presents some possible approaches for controlling the ENFR under dependence.

Zusammenfassung

Die Erfindung des Computers hat unser alltägliches Leben und die Arbeit von Wissenschaftlern gravierend verändert. Heutzutage ist man in der Lage, große Mengen an Daten zu sammeln, zu speichern und zu verarbeiten. Die Analyse solcher Datenensätze erfordert es, die klassische Statistik anzupassen und zu erweitern. Beispielsweise kann das Expressionsmuster aller Gene einer einzelnen Person zu einem beliebigen Zeitpunkt gemessen werden. Mißt man das Expressionsmuster aller Gene (ca. 23.000) bei 2 Gruppen mit je 500 Personen, erhält man circa $2 \cdot 10^7$ Messwerte. Selbstverständlich ist man nicht nur daran interessiert festzustellen, ob die Muster zwischen den Gruppen sich unterscheiden, sondern insbesondere daran, einzelne Gene zu bestimmen, die unterschiedlich exprimiert werden. Allgemeiner gesagt, für eine Menge von Nullhypothesen ist es das Ziel zu entscheiden, welche wahr bzw. falsch sind. Die Entwicklung von Prozeduren, die für jede einzele Nullhypothese eine Entscheidung treffen und gleichzeitig ein vorgegebenes Fehlerkriterium einhalten, ist ein Teil der Theorie des multiplen Hypothesentestens. Das Hauptthema dieser Dissertation ist die Einführung eines neuen Fehlerkriteriums und die Entwicklung von Prozeduren, welche dieses Fehlerkriterium einhalten.

In Kapitel 1 geben wir allgemeine Rahmenbedingungen für die Dissertation an und führen einige bekannte Fehlermaße ein, die für unsere Theorie maßgeblich sind. Zudem definieren wir eine große Klasse von Testprozeduren, welche üblicherweise zur Kontrolle diverser Fehlerraten eingesetzt werden.

In Kapitel 2 motivieren und definieren wir ein neues Fehlerkriterium, welches auf der erwarteten Anzahl falscher Ablehnungen (engl: expected number of false rejections, kurz ENFR) basiert. Wir untersuchen die ENFR hinsichtlich ungünstigster Parameterkonstellationen. Desweiteren wird der asymptotische Zusammenhang der ENFR mit einer großen Klasse von Testprozeduren untersucht. Abschließend wird ein zentraler Grenzwertsatz für die Anzahl falscher Ablehnung formuliert.

Kapitel 3 widmet sich den Beziehungen und Unterschieden zwischen der ENFR und einem anderen wichtigen Fehlermaß, der sogenannten "False Discovery Rate" (FDR). Unter gewissen Regularitätsbedingungen wird sich zeigen, dass die Kontrolle der ENFR die Kontrolle der FDR impliziert. Zudem kann die Kontrolle der FDR unter Unabhängigkeit auch die Kontrolle der ENFR implizieren. Wir werden jedoch Situationen darlegen, in denen die FDR aber nicht die ENFR kontrolliert wird und umgekehrt.

In Kapitel 4 untersuchen wir das Verhalten der ENFR für verschiedene Prozeduren, welche die FDR für eine feste Anzahl von Nullhypothesen oder asymptotisch kontrollieren.

Kapitel 5 widmet sich der exakten Kontrolle der ENFR zu beliebigen vorgegebenen Schranken. Wir stellen ein rekursives Schema dar, das die exakte Kontrolle der ENFR ermöglicht. Allerdings sind die resultierenden Lösungen im allgemeinen nicht zulässig. Daher wird ein Algorithmus entwickelt, der zulässige Lösungen liefert und gute Ergebnisse bzgl. der exakten Kontrolle zeigt.

Kapitel 6 befasst sich mit einem Fehlermaß, das auf der Anzahl falscher Ablehnungen beruht

und ein probabilistisches Gegenstück zur ENFR darstellt. Gewöhnliche Differentialgleichungen werden für die Konstruktion von Prozeduren, welche dieses Fehlermaß kontrollieren, eine wichtige Rolle spielen. Ferner übertragen wir die Resultate auf die "False Discovery Exceedance" (FDX), welche ein probabilistisches Gegenstück zur FDR darstellt.
Kapitel 7 schließt die Dissertation ab mit einem Ausblick über mögliche Ansätze die ENFR unter Abhängigkeit zu kontrollieren.

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List of Abbreviations and Symbols

a.s.	Almost surely
AORC	Asymptotically optimal rejection curve
$\text{argmax}_{x \in I} f(x)$	$\{x \in I : f(y) \leq f(x)$ for all $y \in I\}$
$B(n, p)$	Binomial distribution with parameters n and p
Cauchy$(0, 1)$	Cauchy distribution with location zero and scale one
cdf	Cumulative distribution function
$\lceil x \rceil$	$\min\{m \in \mathbb{Z} \mid m \geq x\}$
χ^2_ν	Chi-square distribution with ν degrees of freedom
A^c	The complement of a set A
Cov(X, Y)	Covariance of X and Y
DU	Dirac-uniform (p. 16)
ecdf	Empirical cumulative distribution function
ENFR	Expected number of false rejections (p. 16)
FDP	False discovery proportion (p. 8)
FDR	False discovery rate (p. 8)
FDX	False discovery exceedance (p. 10)
$\lfloor x \rfloor$	$\max\{m \in \mathbb{Z} \mid m \leq x\}$
$F_n(t)$	ecdf of $p_1, \ldots, p_n$
FWER	Family-wise error rate (p. 5)
id	$id(x) = x$ for all $x \in \mathbb{R}$

iff	If and only if
iid	Independent and identically distributed
$\mathbb{I}_{\{M\}}$	Indicator function of M
LFC	Least favorable parameter configuration (p. 16)
LSD	Linear step-down (p. 38)
LSDPI	Linear step-down plug-in (p. 59)
LSU	Linear step-up (p. 38)
LSUPI	Linear step-up plug-in (p. 59)
$\mathbb{N}$	{1, 2, ...}
$\mathbb{N}_0$	{0, 1, 2, ...}
NFRX	Number of false rejections exceedance (p. 85)
$\mathrm{N}_n(\mu, \Sigma)$	n-dimensional multivariate normal distribution with mean vector μ and covarinace matrix Σ
$\mathrm{N}(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2
$O(g(n))$	$\{f(n) : \exists\, K > 0, N \in \mathbb{N} :\mid f(n)/g(n) \mid \leq K, n \geq N\}$
$o(g(n))$	$\{f(n) : f(n)/g(n) \to 0\ (n \to \infty)\}$
ODE	Ordinary differential equation
$O_P(Y_n)$	$\{X_n : \forall\, \epsilon > 0\ \exists\, K, N \in \mathbb{N} : \mathbb{P}(\mid X_n/Y_n \mid \geq K) \leq \epsilon, n \geq N\}$
$o_P(Y_n)$	$\{X_n : X_n/Y_n \to 0$ in probability as $n \to \infty\}$
Φ	Cumulative distribution function of the $\mathcal{N}(0, 1)$ distribution
$p_{i:n}$	The ith ordered p-value
pmf	Probability mass function
SD	Step-down (p. 4)
SUD	Step-up-down (p. 5)
$\mid\mid f \mid\mid$	$\sup\{\mid f(x) \mid : x \in [0, 1]\}$
SU	Step-up (p. 4)

Θ	Parameter space
Θ_m	$\{\vartheta \in \Theta : n_1(\vartheta) = m\}$
u.i.	Uniform integrable
U$[a, b]$	Uniform distribution on the interval $[a, b]$
$\mathbb{Z}$	{0, $\pm$ 1, $\pm$ 2, ...}
w.l.o.g.	Without loss of generality
ζ	Proportion of true null hypotheses

Chapter 1

Introduction

1.1 For non-statisticians

The science of multiple hypotheses testing is a very broad field. The main topic of this thesis is only concerned with a small part of it. In order to introduce a reader, who is not statistically or even mathematically versed, to this main topic, we will first look at one specific problem, that is, testing the fairness of a coin within the classical hypotheses testing framework. Generalizing this problem will lead to questions addressed in this thesis.
In this subsection, the author deliberately avoids some mathematical terms and conditions like "independence". Therefore, mathematical accuracy is lost, but comprehensibility is hopefully gained for people having little knowledge in statistics and mathematics.
Imagine two persons X and Y play head or tail with a coin. Person X tosses the coin, and while the coin is in the air, person Y has to predict the outcome. In case of a correct prediction, person Y wins 1 EURO, and in case of a false prediction person Y loses 1 EURO. If the coin is fair, the probability to observe head equals the probability to observe tail, namely 1/2. A coin is not fair if the two mentioned probabilities differ. In this case, person Y can try to exploit the situation to earn some money. Thus, one may raise the question: “Is the coin fair?”. In classical hypotheses testing this question leads to the formulation of two hypotheses denoted by H_0 and H_A,

$$H_0 : \text{The coin is fair,} \qquad H_A : \text{The coin is not fair.}$$

We call H_0 the null hypothesis and H_A the alternative hypothesis.
Suppose the coin has been tossed a hundred times. Intuitively, if the outcome was head a hundred times, we would say “The coin is not fair.” A statistician would say that the null hypothesis is rejected. On the other hand, we probably would say that the coin is fair if the outcome was 50 times head and 50 times tail. Obviously, a subjective approach like this is inadvisable because if we observe 65 times tail, probably some people will state that the coin is fair and some state the opposite. Actually, one can expect that one person will judge the result of this experiment differently at two different time points.

Objective decisions can be made by using an appropriate statistical test. This means, the result of the test depends solely on the data, which we can gather by experiments, for example, by tossing the coin a hundred times and calculating the number of heads observed. Who applied the test at which time has no influence on the result. Two different errors are possible. A type I error occurs if the coin is fair and the statistical test (erroneously) rejects H_0 and a type II error occurs if the coin is not fair and the test (erroneously) accepts H_0. One tries to protect oneself against the type I error. But roughly spoken, detecting an unfair coin is solely possible if we take the risk to make a type I error. For instance, rejecting the null hypothesis (The coin is fair) only if we observe 100 times head or 100 times tail, the probability for a type I error is $2/2^{100} \approx 10^{-30}$. An upper bound for the risk we are willing to take is called significance level, commonly denoted by α. Standard values for α are 0.05 or 0.01.
Although it is not important for this introduction, we want to state one possible statistical test which tests the fairness of the coin at a significance level $\alpha = 0.05$. In order to keep it simple, a plain test out of many others is presented. Suppose we toss the coin again a hundred times. We reject the null hypothesis (The coin is fair) if the number of heads or tails is larger than $c = 60$ and accept the null hypothesis otherwise. The threshold c can not be diminished further because if c is smaller than 60 the probability of a type I error is larger than the significance level $\alpha = 0.05$.
In summary, we have one null hypothesis, two different types of errors that can occur, either type I error or type II error, and with a statistical test we control the (probability of a) type I error at a fixed significance level.
In order to give a meaningful example in the context of multiple hypotheses testing, we adapt the previous example. Imagine now an ill person X and a gene g on the human genome. How often a gene is expressed, roughly spoken, how often the gene is used to build proteins, can be crucial for the health status of a person. Let us now assume that the illness is caused by an over-expression of the gene g. Implicitly, we assumed that we know what normal-expression of the gene g means. Thus, we formulate the two hypotheses:

$$H_0 : \text{gene } g \text{ is normally expressed}, \quad H_A : \text{gene } g \text{ is over-expressed}.$$

If we gather data, for example, we measure the expression of the gene g from 100 different persons with the same illness as X, we can conduct a statistical test at significance level α to make an objective decision about H_0 and H_A. Discovering that this gene g is over-expressed in ill people is a step towards curing the disease or at least any secondary disorders. The human genome consists of thousands of genes and usually it is unclear which of those genes should be tested. Therefore, for every gene g_i of the genome a null hypothesis and an alternative hypothesis is formulated, that is $H_{0,i}$: gene g_i is normally expressed, and $H_{A,i}$: gene g_i is not normally expressed. Sometimes, it is possible to restrict oneself to a few "candidate" genes and formulate only hypotheses for these few "candidates". Nevertheless, instead of testing one null hypothesis we now have to test various null hypotheses at the same time in the sense that every null hypothesis must be rejected or accepted. Ideally, we reject all false null hypotheses and accept all true null hypotheses. Human beings have about 23.000 genes and suppose we have no "candidates" and hence have to test 23.000 null hypotheses. One might be inclined to test every gene g_i, or, to be more precise, every null hypothesis $H_{0,i}$ separately with a classical test and accept or reject $H_{0,i}$ according to the classical test. To illustrate that this ad-hoc method is not a good solution, suppose for a moment that all null hypotheses are true. This means, all genes are normally expressed. How many type 1 errors can we expect by applying this method? A statistical test that exhausts the significance level α will reject a true hypothesis with probability α. Therefore, we will erroneously reject roughly $\alpha \times 23.000$ null hypotheses. For the common significance level of $\alpha = 0.05$ this will result in approximately 1150 type I errors, which is unacceptable. In fact, testing more than one null hypothesis raises also another question: Which errors can be made? Of course, we can only make type I errors and type II errors, but in general we will make the errors simultaneously and every error type will occur multiple times. For example, let $H_{0,1}$ and $H_{0,2}$ be false and all other null hypotheses be true. Rejecting $H_{0,2}$, $H_{0,3}$, $H_{0,42}$ and accepting all other hypotheses will yield one type II error and two type I errors. At this point a suitable error rate is needed together with a procedure controlling this error rate. These are the main challenges of multiple hypotheses testing. In this thesis, a new error rate is introduced and related to other existing error rates. Furthermore, procedures are provided to control the new error rate.

1.2 General framework for multiple testing

In this section, we introduce the basic notation and the general setup for this thesis. Let Θ denote the parameter space and $(\Omega, \mathcal{A}, \mathbb{P}_\vartheta)$ a parameterized statistical experiment, where $\vartheta \in \Theta$ is the underlying true parameter. Any non-empty subset $H \subset \Theta$ is regarded as a null hypothesis which is called a *true null hypothesis* if $\vartheta \in H$ and a *false null hypothesis* otherwise. We always assume that null hypotheses $H_1, \ldots, H_n$ are given with $n > 1$ and $H_i \neq H_j$ for all $i \neq j$. In general, it will be convenient to subsume the indices of true null hypotheses and false null hypotheses under $I_0 = I_0(\vartheta)$ and $I_1 = I_1(\vartheta) = \{1, \ldots, n\} \setminus I_0$, respectively. Hence, $n_0 = |I_0|$ null hypotheses are true and $n_1 = n - n_0$ are false. A p-value for testing the null hypothesis H_i will be denoted by p_i, with $p_i : (\Omega, \mathcal{A}) \to ([0,1], \mathcal{B})$, where $\mathcal{B}$ denotes the Borel-σ-field over $[0,1]$. A reasonable assumption for all p-values corresponding to true null hypotheses is $0 < \mathbb{P}_\vartheta (p_i \leq x) \leq x$ for all $x \in (0,1]$. The empirical cumulative distribution function (ecdf) of the p-values $p_1, \ldots, p_n$, will be denoted by F_n, that is

$$F_n(t) = \sum_{i=1}^{n} \mathbb{I}_{\{p_i \leq t\}}.$$

A function $\varphi = (\phi_1, \ldots, \phi_n) : [0,1]^n \to \{0,1\}^n$ is called a *non-randomized multiple test procedure*, and we reject H_i iff $\phi_i = 1$. For the sake of simplicity, we usually speak of multiple test procedures or simply procedures.

Part of this thesis are asymptotic considerations ($n \to \infty$). In that case, we assume that the parameter space Θ is large enough such that $\vartheta \in \Theta$ can be chosen fixed for all $n \in \mathbb{N}$. For example, if $X_i \sim \mathcal{N}(\mu_i, 1)$ are iid random variables, we can model this with $\vartheta = (\mu_i)_{i \in \mathbb{N}} \in \Theta = \mathbb{R}^{\aleph_0}$, where $\aleph_0$ is the cardinality of $\mathbb{N}$ and $\mathbb{P}_\vartheta(X_i \leq x) = \Phi(x - \mu_i)$, where $\Phi(x)$ is the cumulative distribution function (cdf) of a standard normal random variable.
Note that, under asymptotic considerations, $I_0(\vartheta)$ and $I_1(\vartheta)$ also depend on n, but for notational simplicity this will be suppressed.

The very essential random variables for this dissertation are the number of rejected null hypotheses $R_n = R_n(\varphi) = \sum_{i=1}^n \phi_i$ and the number of false rejections $V_n = V_n(\varphi) = \sum_{i \in I_0} \phi_i$. In realistic scenarios only R_n is observable and V_n as well as n_0 are unknown. Further interesting and in general unknown random variables are the number of rejected false null hypotheses $S_n = S_n(\varphi) = \sum_{i \in I_1} \phi_i$, the number of accepted true null hypotheses $\sum_{i \in I_0}(1 - \phi_i)$, and the number of accepted false null hypotheses $\sum_{i \in I_1}(1 - \phi_i)$. In this thesis, these objects play only a minor or no role.
Frequently, we will work with ordered p-values $p_{1:n} \leq \ldots \leq p_{n:n}$ and, for simplicity, we denote by $H_{1:n}$, ..., $H_{n:n}$ the corresponding null hypotheses. For technical reasons, we define $p_{0:n} \equiv 0$ and $p_{n+1:n} \equiv 1$. Before we define our new error rate in chapter 2, we introduce some important error rates that are relevant for this thesis. Furthermore, we introduce a large class of multiple test procedures, the so-called step-wise tests.

1.3 Step-wise tests

Step-wise tests constitute a large class of multiple test procedures. We distinguish between step-down (SD), step-up (SU), and step-up-down (SUD) tests. Common to all step-wise procedures are critical values $0 \leq c_1 \leq ... \leq c_n \leq 1$. For instance, a very classical set of critical values is given by $c_i = \alpha i/n$ for $i = 1, \ldots, n$. For technical reasons, we define $c_0 = 0$ and $c_{n+1} = 1$ if not stated otherwise. A step-down (SD) test works as follows. If $p_{1:n} \leq c_1$, then $H_{1:n}$ is rejected, otherwise all null hypotheses are accepted. Suppose $H_{1:n}, \ldots, H_{i-1:n}$ have already been rejected, then $H_{i:n}$ will be rejected if $p_{i:n} \leq c_i$, otherwise $H_{i:n}, \ldots, H_{n:n}$ will be accepted. Thus, a step-down test rejects H_i iff $i \leq \max\{j | p_{k:n} \leq c_k \text{ for all } k \leq j\}$, with $\max\{\emptyset\}$ defined as zero; or equivalently, H_i is rejected if $p_i \leq \max\{c_j | p_{k:n} \leq c_k \text{ for all } k \leq j\}$. If one accepts the phrase: "The significance of a null hypothesis increases if the corresponding p-value decreases.", one has the following memory hook. The procedure is called step-down test because it starts with the most significant null hypothesis, that is the null hypothesis with the smallest p-value, and at each step the significance of the null hypothesis is non-increasing.
Similar, a step-up (SU) test starts with the least significant null hypothesis, that is the null hy-

pothesis with the largest p-value, and at every step the significance of the null hypothesis is non-decreasing. To be more precise, if $p_{n:n} > c_n$, then $H_{n:n}$ is accepted, otherwise, all null hypotheses are rejected. Suppose $H_{n:n}, \ldots, H_{i+1:n}$ have already been accepted, then $H_{i:n}$ will be accepted if $p_{i:n} > c_i$, otherwise $H_{1:n}, \ldots, H_{i:n}$ will be rejected. Thus, a step-up test rejects H_i iff $i \leq \max\{j | p_{j:n} \leq c_j\}$; or equivalently, H_i is rejected iff $p_i \leq \max\{c_j | p_{j:n} \leq c_j\}$.
A step-up-down (SUD) test is a generalization of both a step-down and a step-up test. A fixed index $1 \leq k \leq n$ must be chosen in advance. If $p_{k:n} > c_k$, then $H_{k:n}, \ldots, H_{n:n}$ are accepted, and if $k > 1$, the null hypotheses $H_{1:n}, \ldots, H_{k-1:n}$ are processed by a step-up test with critical values $c_1, \ldots, c_{k-1}$. Otherwise, $H_{1:n}, \ldots, H_{k:n}$ are rejected, and if $k < n$, the null hypotheses $H_{k+1:n}, \ldots, H_{n:n}$ are processed by a step-down test with critical values $c_{k+1}, \ldots, c_n$. We call such a procedure a SUD(k) test. Obviously, a SUD(1) and a SUD(n) test is a regular SD and SU test, respectively.
Let φ be a SUD(k) test. Then there exits a random variable τ (depending on $p_1, \ldots, p_n$) such that H_i is rejected iff $p_i \leq \tau$. We define $V_n(t) = \sum_{i \in I_0} \mathbb{I}_{\{p_i \leq t\}}$, $S_n(t) = \sum_{i \in I_1} \mathbb{I}_{\{p_i \leq t\}}$, and $R_n(t) = \sum_{i=1}^n \mathbb{I}_{\{p_i \leq t\}}$. Hence, $V_n(\varphi) = V_n(\tau)$, $S_n(\varphi) = S_n(\tau)$, and $R_n(\varphi) = R_n(\tau)$. Working with τ is very fruitful and this kind of notation will be used frequently in this thesis.

In the following sections we introduce well-studied error measures that are important for this thesis. Every section contains a subsection for the corresponding literature. Instead of presenting a complete overview about the existing literature, which would comprise hundreds of publications, we give a short overview of key papers, publications relevant for this thesis, or publications that contain a "nice" idea.

1.4 Family-wise error rate (FWER)

The family-wise error rate (FWER) is probably the oldest error measure in the context of multiple hypotheses testing. It is defined as the probability to reject at least one true null hypothesis, that is

$$\text{FWER}_\vartheta(\varphi) = \mathbb{P}_\vartheta\left(V_n(\varphi) > 0\right).$$

A procedure φ controls the FWER at level α (in the strong sense) if

$$\text{FWER}_\vartheta(\varphi) \leq \alpha \quad \text{for all } \vartheta \in \Theta.$$

The FWER is applied in many different fields, for instance, clinical trails, functional neuroimaging, and model selection. Furthermore, if the aim is not to reject a true null hypothesis with high probability, the FWER is an appropriate candidate for an error measure. But it is frequently criticized that procedures controlling the FWER are too conservative. For instance, assume we have 1.000.000 null hypotheses and only 10 of them are false. Of course, p-values corresponding to these 10 false null hypotheses typically tend to be smaller than p-values for true null hypotheses. Assuming that these "false" p-values will be the ten smallest ones is unrealistic. Depending on the "power" of the experiment, the smallest p-value corresponds to a true null hypothesis with high probability. This shows that a null hypothesis has to be rejected with great caution in order to control the FWER which often results in no rejection at all.

1.4.1 Literature

The Bonferroni procedure is the most famous procedure that corrects for simultaneously testing more than one null hypothesis. It is often called the Bonferroni correction. For fixed $\alpha \in (0,1)$ this procedure simply rejects a null hypothesis if the corresponding p-value is less than α/n, cf. [7]. This guarantees under general dependence of the p-values that the FWER is less than or equal to α. In 1979, Holm presented a SD procedure, with critical values $c_i = \alpha/(n-i+1)$, which is uniformly more powerful than the Bonferroni correction and controls the FWER under general dependence, cf. [33]. Hence, instead of the Bonferroni correction one should preferably use Holm's procedure. In 1986, Simes proposed a SU test with critical values $c_i = i\alpha/n$. He showed that this SU test controls the FWER if all p-values are independent and uniformly distributed on (0,1), cf. [56]. Although the procedure does not make any statements about the individual null hypotheses he proposed for exploratory analysis to reject a null hypothesis if the corresponding p-value is less than or equal to $\max\{j\alpha/n | p_{j:n} \leq j\alpha/n\}$. Nowadays, this procedure is known as the BH-procedure, due to the pioneering work [1] of Benjamini and Hochberg, see Section 1.6.1. It will frequently appear in this thesis. Further, very classical FWER controlling procedures, which were developed between 1967 and 1988, are Sidak's test, Hommel's test and Hochberg's SU test, cf. [55], [34], [31]. None of the so far mentioned procedures exploits the dependence structure between the p-values. Usually, the p-values are calculated from some other test statistics. It is sometimes possible to bootstrap the *joint* distribution of these test statistics. In 1993, Westfall and Young [76] developed resampling strategies to (asymptotically) control the FWER under an assumption called subset pivotality. Further resampling strategies without the subset pivotality assumption were developed in 2003 by van der Laan, Dudoit, and Pollard, cf. [14] and [73].
A striking simple idea for controlling the FWER is the closure principle. The backbone of the principle is a set $\mathcal{H}$ of null hypotheses that is closed under intersection and a corresponding set of statistical tests such that every null hypothesis $H \in \mathcal{H}$ can be tested at level α. The idea is that any true null hypothesis is at most rejected if the intersection of *all* true null hypotheses is rejected. By assumption, the intersection of all true null hypotheses is tested with a level α test and thus is rejected only with probability less than or equal to α. Therefore, any true null hypothesis is only rejected with probability less than or equal to α, cf. [41; 59; 60].

1.5 k-Family-wise error rate (k-FWER)

The k-family-wise error rate (k-FWER) is a generalization of the FWER. It is defined as the probability to reject at least k true null hypothesis, that is

$$k\text{-FWER}_\vartheta(\varphi) = \mathbb{P}_\vartheta\left(V_n(\varphi) \geq k\right).$$

A procedure φ controls the k-FWER at level α if

$$k\text{-FWER}_\vartheta(\varphi) \leq \alpha \quad \text{for all } \vartheta \in \Theta.$$

Obviously, $k\text{-FWER}_\vartheta(\varphi) = \text{FWER}_\vartheta(\varphi)$ for $k = 1$. It may be appropriate to use the k-FWER whenever the FWER is appropriate but additionally one is willing to tolerate k false rejections. An advantage of this criteria is that it is not as conservative as the FWER. However, it seems odd that one *always* allows to reject k true null hypotheses irrespective of how many null hypotheses are actually false. Chapter 6 will be devoted to this issue.

1.5.1 Literature

It seems that the k-FWER was already introduced by Victor [75] in 1982, but he did not provide any procedure that controls the k-FWER. In 1988, a first procedure controlling the k-FWER was given by Hommel and Hoffmann [35]. Among other procedures, they showed that a generalized version of Holm's procedure, which is still a SD procedure, controls the k-FWER under general dependence. In 2004, Korn et al. [38] reinvented this criterion and presented a permutation based test procedure that controls the k-FWER. In the same year Dudoit et al. [14; 73] proposed bootstrap methods that control the k-FWER asymptotically. Further, they presented an approach in [72] called augmentation. The basic idea is to start with a procedure controlling the FWER and then "augmenting" the rejected null hypotheses by additionally rejecting k further null hypotheses. One year later, Lehmann and Romano [40] introduced the term k-FWER and showed, unaware of the publication of Hommel and Hoffmann [35], that a generalized version of Holm's procedure controls the k-FWER under general dependence. Furthermore, they showed that under general dependence one cannot increase any of the critical values of the generalized Holm procedure without violating the k-FWER criterion. This optimality property was improved by Gordon and Salzmann two years later. In [28] they considered a class $\mathcal{C}$ of multiple test procedures being a strict superset of the class of SD tests considered by Lehmann and Romano in [40]. Given a set of p-values, they showed that all hypotheses rejected by any multiple test procedure $\mathcal{M} \in \mathcal{C}$ are also rejected by the generalized Holm procedure. One year later, Romano and Shaikh [68] introduced a SU test that controls the k-FWER under general dependence. In 2007, Gordon [27] provided explicit formulas for the k-FWER under general dependence for SD procedures. This paper also has a key role in deriving formulas for the expected number of false rejections $\mathbb{E}V_n(\varphi)$ under general dependence for SUD tests φ, cf. [29]. In the same year, Romano and Wolf [69] proposed a bootstrap method that controls the k-FWER asymptotically. In 2008, Korn and Freidlin [39] performed a simulation study indicating that the permutation based test introduced in 2004 by Korn et al. [38] outperforms the generalized Holm procedure. Somerville and Hemmelmann had the idea to restrict the number of steps in a SD/SU test, cf. [58]. Given an arbitrary set of critical values they tried to determine the maximal number of steps such that the "restricted test" controls the k-FWER. Recently, Romano and Wolf [71] developed another resampling procedure controlling the k-FWER asymptotically. They generalize Beran's idea in [5] of balanced confidence intervals. By using Beran's balanced confidence intervals it is possible to control the FWER. Romano and Wolf [71] generalized the definition of the balanced confidence intervals and applied the bootstrap methodology to derive procedures that control the k-FWER asymptotically.

1.6 False discovery rate (FDR) and false discovery proportion (FDP)

The false discovery proportion (FDP) is defined by

$$\text{FDP}(\varphi) = \frac{V_n(\varphi)}{R_n(\varphi) \vee 1}$$

and its expectation is called the false discovery rate (FDR), that is

$$\text{FDR}_\vartheta(\varphi) = \mathbb{E}_\vartheta\,[\text{FDP}(\varphi)].$$

A procedure φ controls the FDR at level α if

$$\text{FDR}_\vartheta(\varphi) \le \alpha \quad \text{for all } \vartheta \in \Theta.$$

The FDR criterion is more liberal than the FWER criterion, that is $\text{FDR}_\vartheta(\varphi) \le \text{FWER}_\vartheta(\varphi)$, cf. [1]. Genome-wide association studies are prime examples where FDR procedures are applied. For example, the aim may be to find the genes associated with a disease. To prevent oneself of making a type I error is not of primary interest. At a first stage, it is important to reduce the number of potential "candidate genes" considerably. Suppose some procedure has rejected some null hypotheses, then it does not matter if a small fraction of the corresponding "candidate genes" are not associated with the disease. By controlling the FDR at level α, we allow on average a fraction of α of the rejected null hypotheses to be true.

1.6.1 Literature

The most important and probably the most famous paper with respect to the FDR was published in 1995 by Benjamini and Hochberg, cf. [1]. This is the first publication that provides a procedure that controls the FDR for independent test statistics. This procedure is now often called the BH-procedure. Neither the idea to relate V and R nor the procedure is new. For instance, Soric [61] bounds the ratio of expected false discoveries divided by the number of rejected null hypotheses for a very simple test under independence. According to [53] Eklund proposed the same procedure as Benjamini and Hochberg for large exploratory analysis in the 1960's. It was also discussed in the 1960's and 1970's in [16], [52] and [66]. In 1986, Simes showed that this procedure controls the FWER if all null hypotheses are true. Note, under this condition the FWER is identical to the FDR. He also proposed this procedure for exploratory analysis. In 1997, it was proved that this test controls the FWER under a type of positive dependence and the assumption that all null hypotheses are true, cf. [48]. Four years later, Benjamini and Yekutieli [3] showed that the procedure also controls the FDR under some specific dependence structures. They also state a rather conservative procedure that controls the FDR under general dependence. One year later, Sarkar [49] extended the results obtained in [3]. He showed that a SUD test based on the critical values of the BH-procedure controls the FDR under the same dependence structure considered in [3].

A popular approach to improve the power of the BH-procedure is to estimate the number of true null hypotheses by $\hat{n}_0$ and incorporate this into the BH-procedure. Such procedures are sometimes called adaptive BH-procedures. Benjamini and Hochberg [2] proposed an adaptive BH-procedure in 2000, but did not provide a proof that it controls the FDR. A first proof for one specific adaptive BH-procerdure was given in 2004 by Storey, Taylor and Siegmund in [65]. In 2008, a thorough answer to the question for which estimator $\hat{n}_0$ such adaptive BH-procedures still control the FDR at a prespecified level is given in [50]. In the same year, resampling strategies for controlling the FDR were developed by Romano, Shaikh, and Wolf, cf. [70]. Basically, they bootstrap the distribution of the order statistics under the global null hypotheses and utilize this estimate to sequentially calculate a set of critical values. The procedure is a SD test that controls the FDR asymptotically.

Finner, Dickhaus, and Roters [21] developed a SUD test that controls the FDR asymptotically at level α under the assumption of independence. They also proposed a method called β-adjustment in order to correct the SUD test and achieve control of the FDR for finite n. This adjusted SUD test will also play a major role in this thesis.

Finally, we mention that Neuvial calculated the asymptotic distribution of the FDP for SU tests under some regularity conditions in 2008, cf. [44] and [45].

1.7 False discovery exceedance (FDX)

For any FDR controlling procedure φ, the case where FDP(φ) is close to one is unpleasant, because most of the rejected null hypotheses are true. Genovese and Wasserman [25] note, "Because the distribution of the FDP need not be concentrated around its expected value, controlling the FDR does not necessarily offer high confidence that the FDP will be small.". They proposed the following error measure, paying more attention to the tails of the FDP. We define the $c-$false discovery exceedance ($c-$FDX) of a procedure φ by

$$\text{FDX}(\varphi, c) = \mathbb{P}_\vartheta \left(\text{FDP}(\varphi) > c \right).$$

A procedure φ controls the $c-$FDX at level α if

$$\text{FDX}(\varphi, c) \leq \alpha \quad \text{for all } \vartheta \in \Theta.$$

1.7.1 Literature

Meanwhile, many FDR related measures have been proposed. The FDX is the only one relevant for this thesis. It is motivated by the fact that controlling the FDR does not necessarily offer high confidence that the FDP will be small. In 2004, Genovese and Wasserman [25] considered the FDP as a stochastic process and constructed two confidence envelopes for the whole FDP process under the assumption of independent test statistics. One envelope holds only asymptotically by invariance principles. The other is valid in finite samples by inverting hypothesis tests. From these envelopes control of the FDX is achieved asymptotically and in finite samples. At the same time, Korn et al. [38] developed a permutation based test that controls the FDX. Basically, this procedure is an iterative application of their permutation based test that controls the k-FWER, which was also presented in that paper. Van der Laan et al. [72] presented, as for the k-FWER, an "augmentation" procedure that controls the FDX. The following three papers, that were already cited in Section 1.5.1, also investigated the k-FWER and FDX at the same time. In 2005, Lehmann and Romano [40] provide two SD tests that control the FDX. One controls the FDX under mild dependence conditions and the other under general dependence. One year later, Romano and Shaikh [68] introduced a SU test that control the FDX under general dependence. Somerville and Hemmelmann [58] also applied their idea to restrict the number of steps in order to control the FDX. The idea to iteratively apply a procedure that controls the k-FWER in order to obtain a FDX controlling procedure was presented by Korn et al. [38] in 2004. Recently, Romano and Wolf [71] used this idea to develop a procedure controlling the FDX asymptotically utilizing a procedure which asymptotically control the k-FWER developed by Romano and Wolf [69].

1.8 Rejection curves

Once a rejection curve is defined, we are able to conduct a step-wise test graphically (cf. Figure 1.1). Sometimes, it is technically easier to work with rejection curves, but it also helps to develop an intuitive understanding of such tests.

By definition, a rejection curve is a continuous and strictly increasing function $r : [0, 1] \to [0, \infty)$ with $r(0) = 0$ and $r(1) \geq 1$, and the critical values induced by this curve for fixed n are $c_i = \rho(i/n)$ for $1 \leq i \leq n$, where $\rho : [0, 1] \to [0, 1]$ is the inverse of r. We call ρ the critical value curve. For instance, the very classical critical values $c_i = \alpha i/n$ are induced by the so-called Simes line, which is $r(t) = t/\alpha$. Sometimes, the rejection curve itself is data driven and thus random. Of course, this entails that the critical values are also random variables.

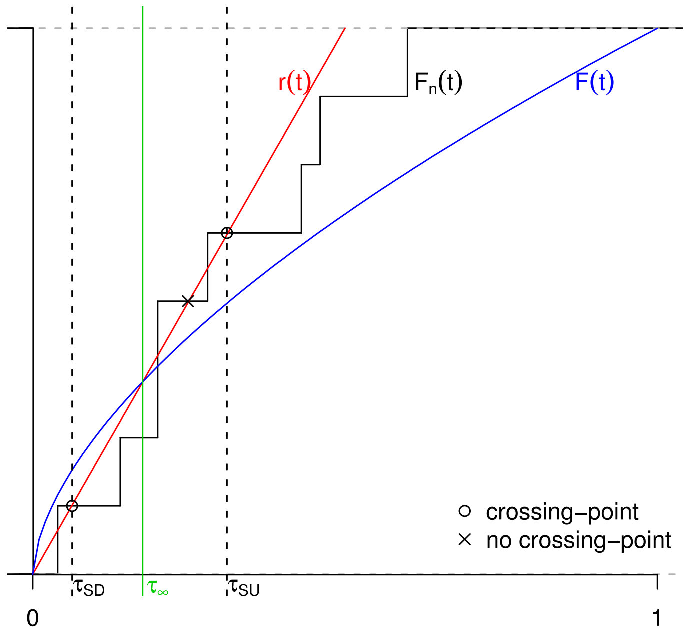

Figure 1.1: *The SD (SU) test induced by the rejection curve r rejects a null hypothesis H_i iff $p_i \le \tau_{SD}$ ($p_i \le \tau_{SU}$). Thereby, τ_∞ denotes the asymptotic crossing-point between $F_n(t)$ and r for $n \to \infty$ assuming that $F_n(t) \to F(t)$ almost surely.*

Remark 1.1

In the literature, sometimes the critical value curve $\rho : [0,1] \to [0,1]$ is defined first. Assuming that ρ is continuous and non-decreasing with $\rho(0) = 0$ and positive values on $(0,1]$, the rejection curve r is defined by $r(t) = \inf\{u \in [0,1] : \rho(u) = t\}$ for $t \in [0,1]$, where $\inf \emptyset = \infty$. Hence, r is strictly increasing until it equals ∞ and it also entails that r may be discontinuous.

Definition 1.2

For a rejection curve r we define

$$\mathcal{E} = \mathcal{E}(r, p_1, \ldots, p_n) = \{c_i : r(p_{i:n}) \le i/n, i \in \{1, \ldots, n\}\} \cup \{0\}$$

and

$$\bar{\mathcal{E}} = \bar{\mathcal{E}}(r, p_1, \ldots, p_n) = \{c_i : r(p_{j:n}) \le j/n \textit{ for all } j = 1, \ldots, i, i \in \{1, \ldots, n\}\} \cup \{0\}.$$

Observing that

$$\{p_{j:n} \le c_j\} \Leftrightarrow \{p_{j:n} \le \rho(j/n)\} \Leftrightarrow \{r(p_{j:n}) \le j/n\},$$

the connection to a SD/SU test is obvious. A SD test rejects H_i iff $p_i \leq \max \bar{\mathcal{E}}$ and a SU test rejects H_i iff $p_i \leq \max \mathcal{E}$.

It is interesting to note that $r(\max \bar{\mathcal{E}}) = F_n(\max \bar{\mathcal{E}})$ and $r(\max \mathcal{E}) = F_n(\max \mathcal{E})$. Additionally, $\max \bar{\mathcal{E}}$ and $\max \mathcal{E}$ are crossing-points in the following sense.

Definition 1.3

Let c_i, $i = 1, \ldots, n$, be induced by a rejection curve r. We call a critical value c_i, $i = 0, \ldots, n-1$ a crossing-point between $F_n(t)$ and $r(t)$ if

$$r(p_{i:n}) \leq F_n(p_{i:n}) \text{ and } F_n(p_{i+1:n}) < r(p_{i+1:n})$$

holds true. We call c_n a crossing-point between $F_n(t)$ and $r(t)$ if

$$r(p_{n:n}) \leq F_n(p_{n:n})$$

holds true.

Recall, $c_0 = 0$ and $p_{0:n} \equiv 0$. Note, if c_i is a crossing-point, then $F_n(c_i) = r(c_i)$ holds, but the opposite does not hold in general (cf. Figure 1.1). By plotting r and F_n, it is very easy to recognize whether a point is a crossing-point or not. Therefore, determining the smallest or largest crossing-point graphically is easy. These points will be of special interest, see below.

Lemma 1.4

$\max \bar{\mathcal{E}}$ and $\max \mathcal{E}$ are crossing-points between F_n and r.

Proof: Let $c_i \in \{\max \bar{\mathcal{E}}, \max \mathcal{E}\}$. If $c_i = c_n$, then $r(p_{n:n}) \leq n/n = F_n(p_{n:n})$. Otherwise, it holds that $r(p_{i:n}) \leq i/n$ and $(i+1)/n < r(p_{i+1:n})$ which imply that $p_{i:n} < p_{i+1:n}$. Hence, $r(p_{i:n}) \leq i/n = F_n(p_{i:n})$ and $F_n(p_{i+1:n}) = (i+1)/n < r(p_{i+1:n})$. □

Lemma 1.5

If for some $k = 0, \ldots, n$ it holds that k p-values are zero and the remaining are larger than zero and pairwise different, then $\max \bar{\mathcal{E}}$ is the smallest crossing-point between F_n and r.

Proof: The case $k = n$ is trivial, therefore we assume that $k < n$. Denote by c_s the smallest crossing-point.

Assume that $s < n$. Since k p-values are zero and $r(0) = 0$, we have $r(p_{j:n}) \leq j/n$ for $j = 0, \ldots, k$ and $k \leq s$. Because $0 = p_{k:n} < p_{k+1:n} < \ldots < p_{n:n}$, we have $F_n(p_{j:n}) = j/n$ for all $j = k, \ldots, n$. It follows that $r(p_{j:n}) \leq j/n$ for $j = 0, \ldots, s < n$ and $r(p_{s+1:n}) > (s+1)/n$. Hence $c_s = \max \bar{\mathcal{E}}$.

If $s = n$, then by definition, $r(p_{n:n}) \leq n/n = F_n(p_{n:n})$ and from $r(0) = r(p_{0:n}) \leq F_n(p_{0:n})$ we conclude $r(p_{j:n}) \leq j/n$ for all $j = 1, \ldots, n-1$. In sum, $c_s = \max \bar{\mathcal{E}}$. □

Corollary 1.6

Under the assumption of Lemma 1.5, a SD test induced by a rejection curve r rejects H_i iff $p_i \leq \tau$, where τ is the smallest crossing-point between F_n and r.

Lemma 1.7

The largest crossing-point between F_n and r is given by $\max \mathcal{E}$.

Proof: Denote by c_l the largest crossing-point. If $l < n$, then $r(p_{l:n}) \leq F_n(p_{l:n})$ and $j/n \leq F_n(p_{j:n}) < r(p_{j:n})$ for all $j = l+1, \ldots, n$. This entails that $p_{l:n} < p_{l+1:n}$. Together we get $r(p_{l:n}) \leq l/n$ and $j/n < r(p_{j:n})$ for $j = l+1, \ldots, n$. Hence $c_l = \max \mathcal{E}$. If $l = n$, then $r(p_{n:n}) \leq n/n = F_n(p_{n:n})$ directly entails $c_l = \max \mathcal{E}$. □

Corollary 1.8

A SU test induced by a rejection curve r rejects H_i iff $p_i \leq \tau$, where τ is the largest crossing-point between F_n and r.

Remark 1.9

The assumption in Lemma 1.5 that p-values different from zero have to be pairwise different is annoying with respect to the graphical determination of τ, see Corollary 1.6. Suppose we choose $\tilde{p}_1, \ldots, \tilde{p}_n$ such that $p_i \in (c_{j-1}, c_j]$ iff $\tilde{p}_i \in (c_{j-1}, c_j]$ for all $i, j = 1, \ldots, n$ and $\tilde{p}_{1:n} < \ldots < \tilde{p}_{n:n}$. For a SUD($k$) test φ ($\tilde{\varphi}$) let τ ($\tilde{\tau}$) denote the random variable such that H_i is rejected iff $p_i \leq \tau$ ($\tilde{p}_i \leq \tilde{\tau}$). Then, $\tau = \tilde{\tau}$, $\varphi = \tilde{\varphi}$, and $\tilde{\tau}$ is a crossing-point between r and the ecdf of $\tilde{p}_1, \ldots, \tilde{p}_n$. Recall, a SUD(1) test is a SD test.

Remark 1.10

In the view of the last corollary the result obtained by Simes in 1989 is not new. He showed that a SU test induced by $r(t) = t/\alpha$ controls the FWER if all p-values are independent and uniformly distributed on (0,1), cf. [56]. But the FWER for this test can also be expressed as

$$1 - \mathbb{P}_{n,n}(V_n(t) \leq t/\alpha \text{ for } 0 \leq t \leq 1).$$

This expression was already investigated by Daniels in 1949, cf. [11], or Theorem 2, p. 345 in [54]. And it holds that $\mathbb{P}_{n,n}(V_n(t) \leq t/\alpha \text{ for } 0 \leq t \leq 1) = 1 - \alpha$.

It is not surprising that a SUD test can also be conducted graphically. By Remark 1.9 we assume w.l.o.g. that the p-values are pairwise different. We choose an index k in advance, and if $r(p_{k:n}) \leq F_n(k/n)$, then τ is the first crossing-point after $p_{k:n}$, otherwise, the largest crossing-point smaller than $p_{k:n}$. Again, a null hypothesis H_i is rejected iff $p_i \leq \tau$. Roughly spoken, the difference between SD-, SU- and SUD-test is which crossing-point is chosen as τ if there are more then one.

Chapter 2

A new criterion based on the expected number of false rejections

2.1 Background

Spjøtvoll investigated the ENFR in [62]. The author states his motivation very clearly in that publication, see page 398 and 399 in [62]:

"Instead of using the constraint that the probability of at least one false rejection is smaller than a certain number α, an upper bound γ on the expected number of false rejections is used.
... if he uses $\alpha = 0.05$, then in average for every twentieth problem he makes false rejections, but he does not know how many false rejections he makes. The author feels that it is important to know this."

Spjøtvoll considered a multidimensional random variable X with probability distribution depending on $\vartheta \in \Theta$ and a family of hypotheses testing problems

$$H_i : \vartheta \in \Theta_{0i} \quad \text{against} \quad K_i : \vartheta \in \Theta_{1i}, \quad i \in \{1, \dots, n\},$$

where $\Theta_{0,i}$ and $\Theta_{1,i}$ are subsets of Θ. For $X = x$, let $\phi_i(x)$ be the probability to reject H_i. Spjøtvoll's intention was to find $(\phi_1, \dots, \phi_n)$ such that

$$\sum_{i=1}^{n} \mathbb{E}_\vartheta \phi_i(X) \leq \gamma \quad \text{for all} \quad \vartheta \in \bigcap_{i=1}^{n} \Theta_{0i} \tag{2.1}$$

holds, and at the same time to maximize for certain subsets $\tilde{\Theta}_{1i}$ of Θ_{1i} the following expression

$$\inf_{i=1,\dots,n} \inf_{\vartheta \in \tilde{\Theta}_{1i}} \mathbb{E}_\vartheta \phi_i(X) \tag{2.2}$$

or alternatively,

$$\sum_{i=1}^{n} \inf_{\vartheta \in \tilde{\Theta}_{1i}} \mathbb{E}_\vartheta \phi_i(X). \tag{2.3}$$

The main tool in deriving theoretical results concerning this issue was to assume that the multiple test defined by

$$\phi_i(x) = \begin{cases} 1, & \text{if } c_i f_i(x) > f_{0i}(x) \\ a_i, & \text{if } c_i f_i(x) = f_{0i}(x), \quad i \in \{1,\ldots,n\}, \\ 0, & \text{if } c_i f_i(x) < f_{0i}(x) \end{cases}$$

fulfills (2.1), for appropriate constants $a_1,\ldots,a_n$, $c_1,\ldots,c_n$ and integrable functions $f_1,\ldots,f_n$ and $f_{01},\ldots,f_{0n}$. In [62], in Sections 3 and 4, Spjøtvoll used these assumptions to develop procedures for the comparison of means of normal random variables with common known and unknown variance. The findings result in rejecting H_i if and only if the corresponding p-value is less or equal γ/n. This is the well-known *Bonferroni correction* which is frequently criticized of being too conservative.

In the following, we will also use the ENFR to define a new, more liberate, criterion. First, we make some heuristic considerations. In general, the p-values corresponding to false null hypotheses are not zero. These p-values hide between the p-values of the true null hypotheses. Usually, if we know that n_1 is small, we are willing to allow only a few true null hypotheses to be rejected. On the other hand, if we know that n_1 is large, we are inclined to allow a large number of true null hypotheses to be rejected. In other words, we want to increase the risk of rejecting a true null hypothesis if n_1 increases. Suppose we have 100.000 null hypotheses and we know that 1.000 null hypotheses are false. How many true null hypotheses are we willing to reject in order to find some of the 1.000 false null hypotheses? 5 or 10 or more? Let us further assume that we also know that there is "lack of power". In such situation there is probably only one way to get hold of some false null hypotheses. We have to increase the risk of rejecting true null hypotheses.

The FDR criterion relies on a similar philosophy. Recall, FDR control of a multiple test $\varphi_n = (\phi_1,\ldots,\phi_n)$ requires

$$\text{FDR}_\vartheta(\varphi_n) = \mathbb{E}_\vartheta\left[\frac{V_n}{(S_n + V_n) \vee 1}\right] \le \alpha,$$

where S_n is the number of correct rejections and V_n is the number of false rejections. Loosely formulated, one can say, if S_n is small, then on average V_n must be small in order to assure that the FDR is smaller than α. And if S_n is large, we allow on average more false rejections.

In contrast to Spjøtvoll, we will discard the optimality criteria (2.2) and (2.3). And instead of controlling the ENFR at a fixed level γ under the global null hypothesis, see (2.1), we require that the ENFR is controlled at different levels depending on n_1. Thus, the following definition is reasonable.

Definition 2.1

Let $g : \{0,\ldots,n-1\} \to [0,n]$. *The ENFR is said to be* g-controlled *by* φ_n *if*

$$\sup_{\vartheta\in\Theta_{n_1}} \mathit{ENFR}_\vartheta(\varphi_n) \le g(n_1) \quad \text{for all } n_1 = 0,\ldots,n-1, \tag{2.4}$$

where ENFR$_\vartheta(\varphi_n) = \mathbb{E}_\vartheta V_n(\varphi_n)$*. We refer to* g *as an ENFR bounding function. If* $g(n_1) = \gamma n_1 + \gamma_0$ *for some* $\gamma_0 > 0$*,* g *is referred to as a linear ENFR bounding function. The ENFR is said to be (linearly) controlled at level* γ *if* (2.4) *holds for* $g(n_1) = (n_1 + 1)\gamma$*.*

Remark 2.2
For $n_1 = n$, equation (2.4) always holds because $E_\vartheta V_n = 0$.

2.1.1 Literature

Although Spjøtvoll proposed a criterion based upon the ENFR nearly 40 years ago, it seems that not many scientists investigated this measure in the meantime. At the first glance there seem to be some publications which concern the ENFR. The title of many of these publications contain the phrases like "type 1 error", "control of false positives" or even "control of expected false discoveries" but the topic is only related to the k-FWER or the FDR and their derivatives. Hence, only a handful of publications remain which are concerned with the ENFR.
Thorough investigation of the ENFR was conducted by Finner and Roters in [17], [18] and [19]. In [17] they showed under some regularity conditions the existence of multiple test procedures φ_n such that $\lim_{n\to\infty}$ FWER$_\vartheta(\varphi_n) = \alpha$ but $\lim_{n\to\infty}$ ENFR$_\vartheta(\varphi_n) = \infty$. In [18] and [19] they investigated the ENFR of procedures controlling the FWER or the FDR and partly provide astonishing simple formulas for the ENFR. A very interesting result was obtained by Gordon in 2011 [29]. Given a SUD procedure he provides sharp upper bounds for the ENFR under general dependence. Other publications usually provide minor results on the ENFR but focus on other measures. For instance, in [71] a half page out of 36 pages is dedicated to the ENFR. Some only conduct heuristic considerations. For instance, in [46] a mixture distribution is fitted to the observed p-values. This mixture distribution contains a part representing the p-values corresponding to true null hypotheses. From this part of the fitted model they extract their information about the number of false rejections and especially the ENFR.

2.2 Least favorable configurations

In view of the ENFR bounding condition (2.4), a desirable situation is that $\sup_{\vartheta\in\Theta_{n_1}} E_\vartheta V_n = E_{\vartheta^*} V_n$ for some known $\vartheta^* = \vartheta^*(n_1)$ for $n_1 = 0, \ldots, n-1$. Such a ϑ^*, which is not necessarily an element of Θ, is called a *least favorable configuration* (LFC). Important candidates are the so-called *Dirac-uniform (DU) configurations*. A DU configuration denoted by DU(n, n_0) appears if $p_i = 0$ almost surely (a.s.) for all $i \in I_1$ and the remaining p-values are iid uniformly distributed on [0,1]. DU configurations are often LFC for the ENFR. This can be seen as follows. If the "false" p-values decrease, then, as can be seen from Figure 1.1, all crossing-points increase. Hence, the probability of a type I error also increases and thus the ENFR. Note, we write ENFR$_{n,n_0}(\varphi_n)$ for short if the ENFR is calculated under DU(n, n_0) and ENFR$_{\vartheta,n_1}(\varphi_n)$ for the ENFR calculated after the p-values corresponding to false null hypotheses have been set to zero. We refer to the *basic independence assumptions (BIA)* if $p_i \sim U[0,1]$ iid for $i \in I_0$, and $(p_i)_{i\in I_0}$ and $(p_i)_{i\in I_1}$ are mutually independent.

Remark 2.3

At this point, we want to distress that $\text{ENFR}_{\vartheta,n_1}(\varphi_n)$ *only* depends on φ_n and the joint distribution of the p-values corresponding to true null hypotheses under $\mathbb{P}_\vartheta$. Assuming that $I_1 = \{n_0 + 1, \ldots, n\}$ we have

$$\begin{aligned}\text{ENFR}_{\vartheta,n_1}(\varphi_n) &= \sum_{i=1}^{n} \mathbb{E}_\vartheta[\phi_i(p_1,\ldots,p_{n_0},0,\ldots,0)]\\ &= \sum_{i=1}^{n} \int \phi_i(x_1,\ldots,x_{n_0},0,\ldots,0)dP_\vartheta^{(p_1,\ldots,p_{n_0})}(x_1,\ldots,x_{n_0}),\end{aligned}$$

with $\varphi_n = (\phi_1,\ldots,\phi_n)$. Note, for step wise procedures as defined in Section 1.3 it holds that $\phi_n(p_1,\ldots,p_n) = \phi_n(p_{\Pi(1)},\ldots,p_{\Pi(n)})$, where Π maps $\{1,\ldots,n\}$ bijective into $\{1,\ldots,n\}$.

The following lemma summarizes under which circumstances the DU(n,n_0) configurations are LFCs for the ENFR or where at least the ENFR increases by setting the p-values corresponding to false null hypotheses to zero.

Lemma 2.4

(a) If $\varphi_n = (\phi_1,\ldots,\phi_n)$ is non-increasing in each $p_i, i \in I_1$, then

$$\textit{ENFR}_\vartheta(\varphi_n) \leq \textit{ENFR}_{\vartheta,n_1}(\varphi_n).$$

(b) If (BIA) applies and if $\varphi_n = (\phi_1,\ldots,\phi_n)$ is non-increasing in each $p_i, i \in I_1$, then

$$\textit{ENFR}_\vartheta(\varphi_n) \leq \textit{ENFR}_{n,n_0}(\varphi_n),$$

that is, DU(n,n_0) is least favorable for ENFR.

Proof: W.l.o.g. assume that $I_1 = \{n_0+1,\ldots,n\}$. Denote by $(p_1,\ldots,p_{n_0})$ the vector of p-values corresponding to true null hypotheses. The assertion (a) follows from

$$\text{ENFR}_\vartheta(\varphi_n) = \sum_{i=1}^{n} \mathbb{E}_\vartheta[\phi_i(p_1,\ldots,p_n)] \leq \sum_{i=1}^{n} \mathbb{E}_\vartheta[\phi_i(p_1,\ldots,p_{n_0},0,\ldots,0)] = \text{ENFR}_{\vartheta,n_1}(\varphi_n).$$

Under BIA we have $\text{ENFR}_{\vartheta,n_1}(\varphi_n) = \text{ENFR}_{n,n_0}(\varphi_n)$ and thus the assertion (b). □

A similar argumentation fails for the FDR because $V_n(t)/\max\{R_n(t),1\}$, in general, is not increasing a.s. if p_i decreases for $i \in I_1$. So far, it is still difficult to determine LFCs for the FDR. Only two results concerning this issue seem to exist. If a SU test is defined in terms of a critical value curve ρ satisfying

- $\rho(t)/t$ non-decreasing in t,
- $\mathbb{P}_\vartheta\left(p_i \le t\right) = t$ for all $t \in [0,1]$, $i \in I_0$,
- $\{p_i\}_{i \in I_0}$ is an independent sequence,
- $(p_i)_{i\in I_0}$ and $(p_i)_{i \in I_1}$ are mutually independent,

then DU(n, n_0) is LFC for FDR. This follows by Theorem 5.3 in [3], cf. [21], p. 605. Recently, in [47] it was proven that DU configurations are LFCs with respect to the FDR for SD procedures under some restrictive assumptions.

Remark 2.5

If we assume that $\varphi_n = (\phi_1, \ldots, \phi_n)$ is non-increasing in each p_i, $i = 1, \ldots, n$, the BIA in Lemma 2.4 (b) can be weakened to $\mathbb{P}_\vartheta\left(p_i \le t\right) \le t$ for all $t \in [0,1]$ and $i \in I_0$, $\{p_i\}_{i\in I_0}$ is an independent sequence, with possible dependence between the two vectors $(p_i)_{i\in I_0}$ and $(p_i)_{i\in I_1}$. This follows from Proposition 17.A.1 in [42], which states that the condition

$$\mathbb{E}f(Y) \le \mathbb{E}f(X) \text{ for all non-increasing functions } f \text{ such that the expectations exist}$$

is equivalent to $P(Y \le t) \le P(X \le t)$ for all $t \in \mathbb{R}$. Since p_i, $i \in I_0$, is an independent sequence, the proposition can be applied separately to each p_j with $j \in I_0$ given $(p_k)_{k \in I_0 \setminus \{j\}}$. Assuming that $I_1 = \{n_0 + 1, \ldots, n\}$ we get

$$\begin{aligned}\text{ENFR}_\vartheta(\varphi_n) &\le \sum_{i=1}^n \mathbb{E}_{\vartheta,n_1}[\phi_i(p_1, \ldots, p_{n_0}, 0, \ldots, 0)] \\ &\le \sum_{i=1}^n \mathbb{E}_{n,n_0}[\phi_i(p_1, \ldots, p_{n_0}, 0, \ldots, 0)] \\ &= \text{ENFR}_{n,n_0}(\varphi_n).\end{aligned}$$

Remark 2.6

Setting the p-values corresponding to false null hypotheses to zero is a first helpful step in bounding the ENFR. As already mentioned in Remark 2.3, in order to calculate $\text{ENFR}_{\vartheta,n_1}(\varphi_n)$, only the joint distribution of the p-values corresponding to true null hypotheses has to be considered. In the case of dependence this distribution is sometimes known. In the many-one problem, test statistics T_i of the type

$$T_i = \sqrt{1-\rho}Z_i - \sqrt{\rho}Z_0$$

may appear, where $(Z_k)_{k\in\mathbb{N}}$ and Z_0 are iid standard normally distributed random variables and $\rho \in (0,1)$ is a known constant. Further information on the test statistic or the many-one problem can be found in Section 3.3. Obviously, $(T_i)_{i=1,\ldots,n}$ is a multivariate normally distributed random variable with known equicorrelation ρ. Hence, for given n_1 and φ_n we are able to calculate $\text{ENFR}_{\vartheta,n_1}(\varphi_n)$ exactly.

2.3 Asymptotic rejection curves

Finner, Dickhaus, and Roters [21] used asymptotic considerations to motivate a rejection curve called the *asymptotically optimal rejection curve* (AORC). The AORC is defined as $f_\alpha(t) = t/[t(1-\alpha)+\alpha]$ for $t \in [0,1]$ and $\alpha \in (0,1)$. This rejection curve has the important property that for a large class of SUD tests induced by f_α we obtain that the FDR converges under DU configurations to α. It is the basis of powerful FDR controlling procedures with prespecified fixed rejection curve, see [21]. This rejection curve will appear frequently in this thesis. In this section we use similar techniques to investigate the asymptotic relations between rejection curves and ENFR bounding functions. Thereby, we consider a sequence of multiple tests $(\varphi_n)_{n\in\mathbb{N}}$. The following definition is in accordance with Definition 2.1.

Definition 2.7
Let $g : [0,1] \to [0,1]$. The ENFR is said to be asymptotically g-controlled by $(\varphi_n)_{n\in\mathbb{N}}$ if

$$\limsup_{n\to\infty} \mathit{ENFR}_\vartheta(\varphi_n)/n \le g(\zeta) \tag{2.5}$$

for all $\vartheta \in \Theta$ with $n_0/n \to \zeta$. We refer to g as an asymptotic ENFR bounding function.

Remark 2.8
An alternative for the expression $\mathrm{ENFR}_\vartheta(\varphi_n)/n$ would be $\mathrm{ENFR}_\vartheta(\varphi_n)/n_1$. The latter expression may be fruitful if $n_0/n \to 1$ is considered. But in this section we only investigate $n_0/n \to \zeta \in [0,1)$ and thus $\limsup_{n\to\infty} \mathrm{ENFR}_\vartheta(\varphi_n)/n = (1-\zeta)\limsup_{n\to\infty} \mathrm{ENFR}_\vartheta(\varphi_n)/n_1$.

All calculations in this section are under DU configurations. In the first part we consider SUD tests φ_n induced by a given rejection curve r and pursue the aim to state sufficient conditions such that

$$\lim_{n\to\infty} \mathrm{ENFR}_{n,n_0}(\varphi_n)/n$$

exists for $n_0/n \to \zeta \in [0,1)$ and calculate this limit provided that it exists. If this limit exists for all $\zeta \in [0,1]$ and DU is LFC for the ENFR, then setting $g(\zeta) = \lim_{n\to\infty} \mathrm{ENFR}_{n,n_0}(\varphi_n)/n$ obviously yields that the ENFR is asymptotically g-controlled and inequality (2.5) holds with equality.

In the second part we consider a given asymptotic ENFR bounding function g. Then the aim is to construct SUD tests φ_n such that $\lim_{n\to\infty} \mathrm{ENFR}_{n,n_0}(\varphi_n)/n = g(\zeta)$ under some regularity conditions. By our construction, we will see that these SUD tests will be induced by a fixed rejection curve r, which of course depends on g.

Consider now a fixed rejection curve r as given. As stated in Section 1.3, if φ_n is a SUD test induced by r, then there exists a random variable τ such that H_i is rejected iff $p_i \le \tau$. By Section 1.8, this τ is a crossing-point between r and F_n. In our asymptotic settings we consider a sequence of SUD(k_n) tests $(\varphi_n)_{n\in\mathbb{N}}$ induced by r. The corresponding crossing-points are denoted by τ_n. A

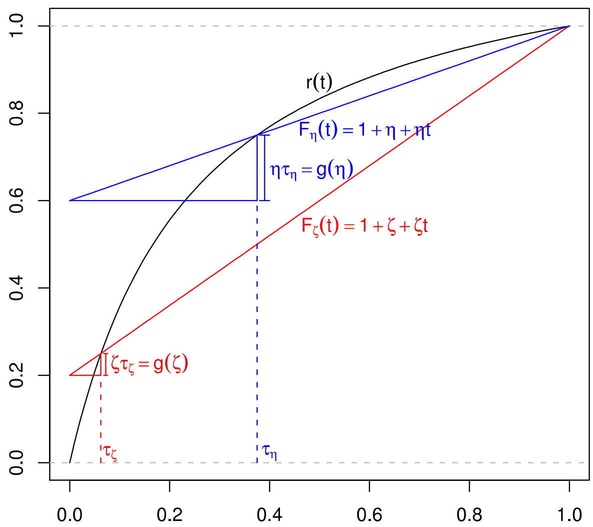

Figure 2.1: *The figure shows (asymptotically/heuristically) the expected fraction of false rejections (ζt_ζ and ηt_η) for two different fractions of true null hypotheses (ζ and η) under DU configurations.*

first good impression of how r and g are connected is provided by Figure 2.1. Note, under DU configurations and $\zeta_n = n_0/n \to \zeta \in [0,1]$ it holds by the Theorem of Glivenko-Cantelli that

$$F_n(t) \to F_\zeta(t) = 1 - \zeta + \zeta t \tag{2.6}$$

a.s. and uniformly in $t \in [0,1]$. The backbone of our investigations is the assumption that τ_n converges a.s. to a constant $t_\zeta \in (0,1]$. The following lemma states sufficient conditions such this holds true.

Lemma 2.9

Let $r(t)$ be a rejection curve, $(\varphi_n)_{n\in\mathbb{N}}$ a sequence of SUD(k_n) tests induced by r with a corresponding sequence of random variables $(\tau_n)_{n\in\mathbb{N}}$ such that H_i is rejected by φ_n iff $p_i \le \tau_n$, $i = 1,\dots,n$, $n \in \mathbb{N}$. Further, let $n_0/n \to \zeta \in [0,1)$ and $T = \{t \in [0,1] : r(t) = 1 - \zeta + \zeta t\}$ be a finite set. Suppose $T\backslash\{1\}$ is a disjoint union of T^+ and T^-, where

$$T^+ = \{t \in T\backslash\{1\} | \exists\, \epsilon > 0 : r(s) < F_\zeta(s) \text{ for } s \in [t-\epsilon, t) \text{ and } r(s) > F_\zeta(s) \text{ for } s \in (t, t+\epsilon]\}$$

and

$$T^- = \{t \in T\backslash\{1\} | \exists\, \epsilon > 0 : r(s) > F_\zeta(s) \text{ for } s \in [t-\epsilon, t) \text{ and } r(s) < F_\zeta(s) \text{ for } s \in (t, t+\epsilon]\}.$$

Let further $F_\zeta^{-1}(k_n/n) \to \kappa$. If one of the following conditions holds,

1. $\kappa \in [0,1)\backslash T^-$,

2. *$\kappa = 1$ and $r(1) > 1$,*

3. *$\kappa = 1$ and there exists an $\epsilon > 0$ such that $r(s) < F_\zeta(s)$ for $s \in (1-\epsilon, 1)$,*

then under DU(n, n_0) configurations, τ_n converges a.s. to a constant $t_\zeta \in T^+ \cup \{1\}$.

Proof: If $\kappa \in T^+$, the uniform a.s. convergence in (2.6) directly entails $\tau_n \to \kappa$ almost surely. If $\kappa \in [0,1]\backslash T$, then $r(\kappa) > F_n(\kappa)$ or $r(\kappa) < F_n(\kappa)$ eventually for all n. Therefore, it holds that $r(p_{k_n:n}) > F_n(p_{k_n:n})$ or $r(p_{k_n:n}) < F_n(p_{k_n:n})$ a.s. and eventually for all n, because $p_{k_n:n} = F_n^{-1}(k_n/n) = F_\zeta^{-1}(k_n/n) + o(1) = \kappa + o(1)$ a.s. by the uniform convergence of F_n. In the first case, $r(\kappa) > F_\zeta(\kappa)$, τ_n converges a.s. to $t_\zeta = \max\{T \cap [0,\kappa]\}$. Note, t_ζ is well-defined because $r(0) = 0 < F_\zeta(0)$ and $r(\kappa) > F_\zeta(\kappa)$. Furthermore, by $r(\kappa) > F_\zeta(\kappa)$, the continuity of r and the definition of t_ζ there exists an $\epsilon > 0$ with $r(s) < F_\zeta(s)$ for $s \in [t_\zeta - \epsilon, t_\zeta)$ and $r(s) > F_\zeta(s)$ for $s \in (t_\zeta, t_\zeta + \epsilon]$, which entails $t_\zeta \in T^+$. In the second case, $r(\kappa) < F_\zeta(\kappa)$, we have that τ_n converges a.s. to $t_\zeta = \min\{T \cap [\kappa, 1]\}$. In the same way we conclude that $t_\zeta \in T^+$ if $T^+ \cap [\kappa, 1] \neq \emptyset$. Otherwise, if $T^+ \cap [\kappa, 1] = \emptyset$, then $r(s) < F_\zeta(s)$ for $s \in [\kappa, 1)$, which entails $\tau_n \to 1$ almost surely.
Finally, consider the case $\kappa = 1$. If $r(1) > 1$, then $\kappa = 1 \in [0,1]\backslash T$. This case has just been investigated. If $r(s) < F_\zeta(s)$ on $(1-\epsilon, 1)$, then it follows by the Theorem of Glivenko-Cantelli that $\tau_n \to 1$ almost surely. □

Note, if the rejection curve r is tangential to F_ζ for at least one point in $(0,1]$, then the last lemma is not applicable. We say that r is tangential to F_ζ at a point $t \in (0,1]$ if there exists a $\epsilon > 0$ such that $r(s) > F_\zeta(s)$ for $s \in I = [t-\epsilon, t+\epsilon]\backslash\{t\}$ and $r(t) = F_\zeta(t)$ or $r(s) < F_\zeta(s)$ for $s \in I$ and $r(t) = F_\zeta(t)$.
The next theorem states sufficient conditions for the existence of $\lim_{n\to\infty} \mathbb{E}_{n,n_0} V_n(\varphi_n)/n$ and provides a formula for the limit, given that it exists. This concludes our first part.

Theorem 2.10
Under the assumptions of Lemma 2.9 and DU(n, n_0) configurations it holds that

$$r(t_\zeta) = 1 - \zeta + \zeta t_\zeta, \tag{2.7}$$

$$\lim_{n\to\infty} \mathbb{E}_{n,n_0} V_n(\varphi_n)/n = \zeta t_\zeta, \tag{2.8}$$

where t_ζ is the constant from Lemma 2.9.

Proof: By virtue of Lemma 2.9, we have $\tau_n \to t_\zeta$ a.s. where τ_n is the random variable from Lemma 2.9 and t_ζ fulfills (2.7). Observing that

$$V_n(\varphi_n)/n = \frac{n_0}{n}\frac{1}{n_0}\sum_{i\in I_0} \mathbb{I}_{\{p_i \le \tau_n\}} = F_n(\tau_n) - \left(1 - \frac{n_0}{n}\right),$$

equation (2.8) follows from the Theorem of Glivenko-Cantelli and the dominated convergence theorem. □

For the second part suppose that the asymptotic ENFR bounding function g is given. The aim is to construct SUD tests φ_n such that

$$\lim_{n\to\infty} \mathbb{E}_{n,n_0} V_n(\varphi_n)/n = g(\zeta) \tag{2.9}$$

under some regularity conditions. Suppose that φ_n is induced by an unknown rejection curve r. If Theorem 2.10 is applicable for all $\zeta \in (0,1)$, then combining the equations (2.7), (2.8), and (2.9) yields

$$r(g(\zeta)/\zeta) = 1 - \zeta + g(\zeta) \tag{2.10}$$

for $\zeta \in (0,1)$. In the following, we will show that φ_n induced by r, which is implicitly defined by equation (2.10), will fulfill equation (2.9) under some regularity conditions.

Remark 2.11

If $g(\zeta)/\zeta = 1$, then (2.10) becomes $r(1) = 1$. Obviously, if $g(\zeta)/\zeta > 1$, then (2.10) is not of interest because the domain of a rejection curve is $[0,1]$. Hence, (2.10) is only relevant for $\{\zeta \in (0,1] : g(\zeta)/\zeta < 1\}$.

Remark 2.12

Similar investigations have already been conducted by Gontscharuk in [26] for the FDR. For a given function $g^* : [0,1] \to [0,\alpha]$ she considered the equation

$$\lim_{n\to\infty} \mathrm{FDR}_{n,n_0}(\varphi_n) = g^*(\zeta), \quad \zeta \in (0,1)$$

and derived an implicit equation for r, i.e.

$$r\left(\frac{g^*(\zeta)(1-\zeta)}{\zeta(1-g^*(\zeta))}\right) = \frac{1-\zeta}{1-g^*(\zeta)}, \quad \zeta \in (0,1). \tag{2.11}$$

In order to ensure that DU is LFC for the FDR using a SU test, it is necessary that $\rho(t)/t$ is non-decreasing, which is implied by equation (2.11) and the additional assumptions that $h^*(\zeta) = g^*(\zeta)/\zeta$ is non-increasing and $\lim_{\zeta\to 0} g^*(\zeta)/\zeta \in (0,1]$, cf. Lemma 3.12 in [26]. In the ENFR setting it is also essential that $h(\zeta) = g(\zeta)/\zeta$ is non-increasing; additionally we need that $H(\zeta) = 1 - \zeta + g(\zeta)$ is non-increasing to ensure that r is non-decreasing and therefore DU is LFC for the ENFR. This set of assumptions is weaker than for the FDR because h non-increasing and $\lim_{\zeta\to 0} g(\zeta)/\zeta \in (0,1]$ obviously imply that $H(\zeta) = 1 - \zeta(1 - g(\zeta)/\zeta)$ is strictly decreasing. Although the assumptions on g and g^* are the same, that is $g(\zeta)/\zeta$ is non-increasing and $\lim_{\zeta\to 0} g(\zeta)/\zeta \in (0,1]$ in the ENFR setting and $g^*(\zeta)/\zeta$ is non-increasing and $\lim_{\zeta\to 0} g^*(\zeta)/\zeta \in (0,1]$ in the FDR setting, the different implicit equations yield different properties for ρ. For instance, in the FDR setting we get that $\rho(t)/t$ is non-decreasing and in the ENFR setting we get the weaker property that r and thus ρ is non-decreasing. It is intuitively clear that the assumptions on the rejection curve in an ENFR setting with respect to LFCs can be more liberal compared with the assumptions in the FDR setting because the LFCs issue is much more easier for the ENFR than for the FDR.

Remark 2.13

If we assume that both $h(\zeta) = g(\zeta)/\zeta$ and $1 - \zeta + g(\zeta)$ are non-increasing and equation (2.10) holds true, then h is strictly decreasing on $\mathcal{I} = \{\zeta \in (0, 1] : h(\zeta) < 1\}$. This can be seen as follows. Suppose $0 < \zeta < \eta$ and $h(\zeta) = h(\eta)$. By (2.10) we get $1 - \zeta + g(\zeta) = 1 - \eta + g(\eta)$ which is equivalent to $g(\eta) = g(\zeta) + \eta - \zeta$. Thus, by $h(\zeta) = h(\eta)$, we get that

$$\frac{g(\zeta)}{\zeta} = \frac{g(\eta)}{\eta} = \frac{g(\zeta) + \eta - \zeta}{\eta} = \frac{g(\zeta)}{\zeta}\frac{\zeta}{\eta} + 1 - \zeta/\eta$$

is equivalent to

$$\frac{g(\zeta)}{\zeta}(1 - \zeta/\eta) = (1 - \zeta/\eta).$$

Since $\zeta/\eta \neq 1$ we have $h(\zeta) = 1$ and hence h is strictly decreasing on $\mathcal{I}$. Note by Remark 2.11, $\mathcal{I}$ consists of the relevant ζ's for the equation (2.10).

The following remark shows how r and ρ can be written as composition of functions of g and the inverse functions of h and H.

Remark 2.14

Suppose that g is continuous. As we elucidated in Remark 2.11 the relevant ζ's are given by the set $\mathcal{I} = \{\zeta \in (0, 1] : g(\zeta)/\zeta < 1\}$. By Remark 2.13 we can assume that $h(\zeta) = g(\zeta)/\zeta$ is strictly decreasing on $\mathcal{I}$. Then also the function $H(\zeta) = 1 - \zeta + g(\zeta) = 1 - \zeta(1 - h(\zeta))$ is strictly decreasing on $\mathcal{I}$. Suppose equation (2.10), that is $r(h(\zeta)) = H(\zeta)$, holds for $\zeta \in \mathcal{I}$. From this equation it is possible to reobtain r and ρ. Since $h(\zeta)$ is invertible we have

$$r(t) = r\left(\frac{g(h^{-1}(t))}{h^{-1}(t)}\right) = 1 - h^{-1}(t) + g(h^{-1}(t)) \quad \text{for all } t \in h(\mathcal{I}) \tag{2.12}$$

and because H is strictly decreasing we get that r is strictly increasing. This also entails that ρ is strictly increasing. Further, we have for $t = H(\zeta)$, with $\zeta \in \mathcal{I}$, that

$$\begin{aligned} & r(h(\zeta)) = H(\zeta) \\ \Leftrightarrow\ & r(h(H^{-1}(t))) = t \\ \Leftrightarrow\ & \frac{g(H^{-1}(t))}{H^{-1}(t)} = \rho(t). \end{aligned} \tag{2.13}$$

By the Sections 1.3 and 1.8, a step-wise test can be conducted with r or ρ solely. Hence, once either h^{-1} or H^{-1} is explicitly known one can conduct the step-wise test. If both h^{-1} and H^{-1} are not explicitly given, the test can be conducted graphically by hand, cf. Remark 2.15 and Figure 2.2.

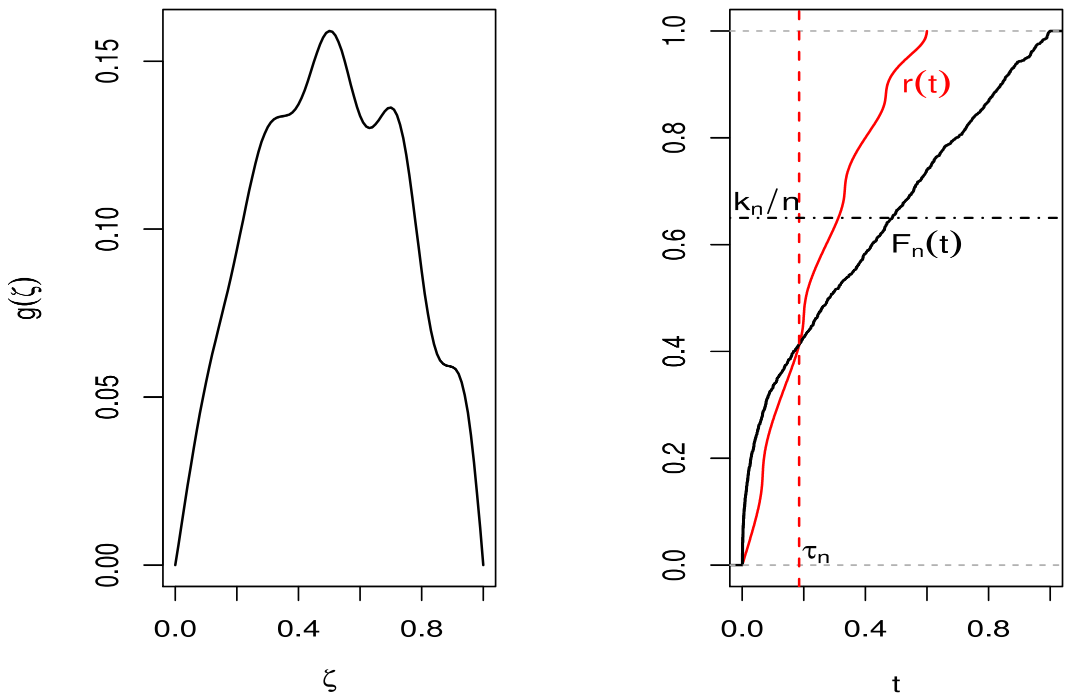

Figure 2.2: *The left figure shows* $g(\zeta) = 0.6\zeta(1 - \zeta + 0.03\sin(9\pi\zeta))$. *The right figure shows the corresponding rejection curve* $r(t)$ *with an ecdf of a set of* p*-values and the* τ_n *for a SUD(*k_n*) test.*

Remark 2.15

Equation (2.10) also has a nice graphical application. Irrespective of the existence of explicit inverse functions in Remark 2.14 we can easily plot the rejection curve r for $t \in \{g(\zeta)/\zeta : \zeta \in (0,1]\} \cap [0,1]$, cf. Figure 2.2. Of course, for conducting a step-wise test automatically at least r or the critical values must be given. But if the aim is to analyze only one set of p-values, it is possible to do this by hand with little effort by plotting the rejection curve and the ecdf of the p-values, cf. Figure 2.2 and see Section 1.8 for details.

Now we are able to answer the question under which circumstances step-wise tests φ_n induced by a function r that is implicitly defined by (2.10) fulfills $\lim_{n\to\infty} \mathbb{E}_{n,n_0} V_n(\varphi_n)/n = g(\zeta)$. Lemma 2.9 is an important tool for answering this question. Therefore, we must state conditions on g such that r fulfills the conditions of Lemma 2.9. Obviously, if $g(1) = 0$, then equation (2.10) yields that $r(0) = 0$. The assumption on g means that if all null hypotheses are true, then the ENFR divided by n must converge to zero. Further, if $h(\zeta) = g(\zeta)/\zeta$ is strictly decreasing, then r defined by (2.10) is strictly increasing, cf. Remark 2.14. Later on we will see that the assumption on h has the interpretation that the crossing-point between the resulting rejection curve r and $F_\zeta(t) = 1-\zeta+\zeta t$ is strictly decreasing in ζ. Recall, it is important that ρ is at least non-decreasing because we want to ensure that DU is LFC.

Theorem 2.16

Let $n_0/n \to \zeta$, $g : [0,1] \to [0,1]$ be continuous with $g(1) = 0$. Suppose $\mathcal{I} = \{\zeta \in (0,1] : g(\zeta)/\zeta < 1\}$ is an interval and $h : \bar{\mathcal{I}} \to [0,1]$ with $h(\zeta) = g(\zeta)/\zeta$ is strictly decreasing, where $h(0) = \lim_{\zeta\to 0} g(\zeta)/\zeta$ if $0 \in \bar{\mathcal{I}}$ and $\bar{\mathcal{I}}$ denotes the topological closure. Let $c = h(\min \bar{\mathcal{I}})$. For $H(\zeta) = 1 - \zeta + g(\zeta)$ it holds that

$$r(t) = \begin{cases} H(h^{-1}(t)), & \text{if } t \in [0,c), \\ t/c, & \text{if } t \in [c,1], \end{cases}$$

is strictly increasing and continuous. Further, for all $\zeta \in (0,1)$ the set $T^+ = T^+(\zeta) = \{t \in [0,1] : r(t) = F_\zeta(t)\} \setminus \{1\}$ contains only one element denoted by t_ζ. For SUD(k_n) tests φ_n induced by r with $\lim F_\zeta^{-1}(k_n/n) = \kappa$ and all $\zeta \in [0,1)$, it holds

$$r(t_\zeta) = 1 - \zeta + \zeta t_\zeta, \tag{2.14}$$

$$\lim_{n\to\infty} \mathit{ENFR}_{n,n_0}(\varphi_n)/n = \zeta t_\zeta \tag{2.15}$$

if (i) $\kappa \in [0,1)$ or (ii) $\kappa = 1$ and $r(1) > 1$ or (iii) $\kappa = 1$ and there exists an $\epsilon > 0$ such that $r(s) < F_\zeta(s)$ for $s \in (1-\epsilon, 1)$. Moreover,

$$\zeta t_\zeta = g(\zeta) \tag{2.16}$$

for all ζ with $0 < g(\zeta) < \zeta c$.

Proof: Since g is continuous, so are h and h^{-1}. It is easy to see that in both cases, $0 \notin \bar{\mathcal{I}}$ and $0 \in \bar{\mathcal{I}}$, we have $r(c) = 1$. In the first case we have $h^{-1}(c) \neq 0$ and $c = 1$, thus

$$r(c) = 1 - h^{-1}(c) + g(h^{-1}(c)) = 1 - h^{-1}(c)(1 - \frac{g(h^{-1}(c))}{h^{-1}(c)}) = 1 - h^{-1}(c)(1-c) = 1.$$

In the second case we obtain from $1 \geq h(0) = \lim_{\zeta\to 0} g(\zeta)/\zeta$ that $g(0) = 0$ and hence $r(c) = 1 - 0 + g(0) = 1$. Therefore, r is continuous. Since $g(1) = h(1) = 0$ we also have $r(0) = 1 - h^{-1}(0) + g(h^{-1}(0)) = 0$. Note, $H(\zeta) = 1 - \zeta(1 - h(\zeta))$ is strictly decreasing on $\bar{\mathcal{I}}$ because h is strictly decreasing and $h(\bar{\mathcal{I}}) = [0,c] \subset [0,1]$. Therefore, we conclude that r is strictly increasing on $[0,c]$; and hence r is strictly increasing on $[0,1]$. Thus, r is a rejection curve. We now show that equation (2.16) holds true. Observe that $r([0,c]) = [0,1]$. Hence, $(0,c) = \cup_{\zeta\in(0,1)} T^+(\zeta)$. For $t_\zeta \in (0,c)$ we have

$$\begin{aligned} 1 - \zeta + \zeta t_\zeta = r(t_\zeta) &= 1 - h^{-1}(t_\zeta) + g(h^{-1}(t_\zeta)) \\ \Leftrightarrow \frac{\zeta(t_\zeta - 1)}{h^{-1}(t_\zeta)} &= \frac{g(h^{-1}(t_\zeta))}{h^{-1}(t_\zeta)} - 1 \\ \Leftrightarrow \frac{\zeta(t_\zeta - 1)}{h^{-1}(t_\zeta)} &= t_\zeta - 1 \\ \Leftrightarrow \zeta &= h^{-1}(t_\zeta). \end{aligned} \tag{2.17}$$

By definition of $h(\zeta)$, equation (2.17) provides $g(\zeta) = \zeta t_\zeta$ if $t_\zeta \in (0, c)$. This shows (2.16).

It remains to show that T^+ contains only one element. Suppose that $t_\zeta < t' < c$ and $t_\zeta, t' \in T^+(\zeta)$. Then by (2.17) and the monotonicity of h, we have $\zeta = h^{-1}(t_\zeta) > h^{-1}(t') = \eta$ and thus $r(t') = F_\eta(t')$. We also assumed that $t' \in T^+(\zeta)$ and hence $F_\zeta(t') = r(t') = F_\eta(t')$ which is equivalent to $\zeta = \eta$ because $t' < c\ (\leq 1)$, but this contradicts $\zeta > \eta$. Applying Lemma 2.9 yields the assertions. □

Remark 2.17

Using the notation from Theorem 2.16, the assumption that $g(1) = 0$ in Theorem 2.16 was important to ensure that $r(0) = 0$. If $g(1) = a > 0$, then $h(\bar{\mathcal{I}}) \subset [a, c]$ and therefore if h is invertible the function $1 - h^{-1}(t) + g(h^{-1}(t))$ is not defined for $t < a$. But one can define a new bounding curve by

$$\tilde{g}(\zeta) = \begin{cases} g(\zeta), & \text{if } \zeta \in [0, 1-\epsilon] \\ (1-\zeta)g(1-\epsilon)/\epsilon, & \text{if } \zeta \in (1-\epsilon, 1], \end{cases}$$

where $\epsilon \in (0, 1)$. Theorem 2.16 can be applied to $\tilde{g}$ and $\zeta t_\zeta = \tilde{g}(\zeta) = g(\zeta)$ will hold for $\zeta \leq 1-\epsilon$ if $0 < g(\zeta) < \zeta c$.

We now apply the technique explained in Remark 2.17 to a concrete example with $g(1) > 0$ and show how Theorem 2.16 can be applied in this case.

Example 2.18

An interesting asymptotic ENFR bounding function is $g(\zeta) \equiv \alpha \in (0, 1)$. This means that we roughly allow αn true null hypotheses to be rejected. As in Remark 2.17, we have $h(\zeta) \geq \alpha$ for $\zeta \in \bar{\mathcal{I}} = [\alpha, 1]$ and therefore, if h is invertible the function $h^{-1}(t)$ is not defined for $t < \alpha$. For $\epsilon \in (0, 1-\alpha)$ we set $\tilde{g}(\zeta) = \min\{\alpha, \alpha(1-\zeta)/\epsilon\}$ which leads to

$$\tilde{h}(\zeta) = \min\{\alpha/\zeta, \alpha(1-\zeta)/(\zeta\epsilon)\},$$

$$\tilde{h}^{-1}(t) = \min\{\alpha/t, \alpha/(\alpha + \epsilon t)\},$$

$$\tilde{r}(t) = \begin{cases} 1 - \alpha/t + \alpha, \text{if } t \in (\alpha/(1-\epsilon), 1] \\ 1 - \frac{\alpha(1-t)}{\alpha+\epsilon t}, \text{if } t \in [0, \alpha/(1-\epsilon)]. \end{cases}$$

Since α/ζ and $\alpha(1-\zeta)/(\zeta\epsilon)$ are strictly decreasing in ζ on $(0, 1]$, $\tilde{g}(\zeta)/\zeta$ is strictly decreasing on $(0, 1]$. Further, we have $\{\zeta : \tilde{g}(\zeta)/\zeta < 1\} = (\alpha, 1]$. Hence, Theorem 2.16 is applicable. For instance for a SUD($\lfloor kn \rfloor$) test φ_n induced by $\tilde{r}$ with $k \in [0, 1)$, we get $\text{ENFR}_{n,n_0}(\varphi_n) \to \tilde{g}(\zeta)$ for $n_0/n \to \zeta \in [\alpha, 1)$. Note $\tilde{g}(\zeta) = g(\zeta) = \alpha$ for $\zeta \in [0, 1-\epsilon]$.

In the next two examples, we reproduce the Simes line and the AORC. The assumptions of Theorem 2.16 can easily be verified.

Example 2.19

As stated in Remark 4.15, we have $\lim_{n\to\infty} \text{ENFR}_{n,n_0}(\varphi_n)/n_1 = \zeta\alpha/(1-\zeta\alpha)$ for the LSD and LSU test with $\alpha \in (0,1)$. Thus, by changing the norming variable from n_1 to n, we get

$$g(\zeta) = \text{ENFR}_{n,n_0}(\varphi_n)/n = (1-\zeta)\zeta\alpha/(1-\zeta\alpha),$$
$$h(\zeta) = g(\zeta)/\zeta = (1-\zeta)\alpha/(1-\zeta\alpha),$$

and

$$h^{-1}(t) = (t-a)/(a(t-1)).$$

Obviously, $g(1) = 0$ and $h(0) = \alpha$. Hence, by Theorem 2.16, we have

$$r(t) = 1 - h^{-1}(t) + g(h^{-1}(t)) = t/\alpha \quad \text{for all } t \in [0,1]$$

and $\lim_{n\to\infty} \text{ENFR}_{n,n_0}(\varphi_n)/n = g(\zeta)$ for SUD(k_n) tests φ_n induced by r for $n_0/n \to \zeta \in [0,1)$ and $\kappa \in [0,1]$.

Example 2.20

We pursuit to control the ENFR asymptotically at level

$$g(\zeta) = \lim_{n\to\infty} \frac{1}{n}(n_1+1)\frac{\alpha}{1-\alpha} = (1-\zeta)\frac{\alpha}{1-\alpha},$$

where $\alpha \in (0,1)$. Thus, we get

$$h(\zeta) = g(\zeta)/\zeta = (1-\zeta)\alpha/(\zeta(1-\alpha)) \text{ and } h^{-1}(t) = \alpha/(t(1-\alpha)+\alpha).$$

Again, by Theorem 2.16

$$r(t) = 1 - h^{-1}(t) + g(h^{-1}(t)) = \frac{t}{t(1-\alpha)+\alpha} \quad \text{for all } t \in [0,1],$$

with $\lim_{n\to\infty} \text{ENFR}_{n,n_0}(\varphi_n)/n = g(\zeta)$ for SUD(k_n) tests φ_n induced by r with $n_0/n \to \zeta \in [0,1)$, $g(\zeta) < \zeta$, and $\kappa \in [0,1)$. Note that $g(\zeta) < \zeta$ is equivalent to $\alpha < \zeta$.

2.4 Asymptotic normality of the number of false rejections

In this section we state a few results with respect to the asymptotic distribution of $V_n(\varphi_n)$, where φ_n is a SU test induced by r. These results are simple by-products of the considerations we conduct in Chapter 6, cf. Remark 6.10. Nevertheless, for sake of completeness of this chapter, we already state the assertions in this section.

The following function

$$s(x,\zeta,t) = \frac{\sqrt{(x'(t)-(1-\zeta)f_1(t))^2\zeta F_0(t)(1-F_0(t)) + \zeta^2 f_0^2(t)(1-\zeta)F_1(t)(1-F_1(t))}}{x'(t)-(1-\zeta)f_1(t)-\zeta f_0(t)},$$

where x' is the first derivative of a differentiable function $x : [0,1] \to \mathbb{R}$ will appear in the next theorem and frequently in Chapter 6.

Corollary 2.21

Let φ_n be a SU test induced by a continuously differentiable rejection curve r. Further, assume that f_0 and f_1 are density functions on $[0,1]$ which are continuous on $(0,1)$. Denote by F_0 and F_1 the corresponding cdfs. If

1. *p_i are iid with density function f_0 for $i \in I_0$,*
2. *p_i are iid with density function f_1 for $i \in I_1$,*
3. *$(p_i)_{i\in I_0}$ and $(p_i)_{i\in I_1}$ are mutually independent,*
4. *$n_0/n = \zeta + o(n^{-1/2})$,*
5. *there exists a $\tau^* \in (0,1)$ such that τ^* is the only point in $(0,1]$ with $r(\tau^*) = (1-\zeta)F_1(\tau^*) + \zeta F_0(\tau^*)$,*
6. *$r'(\tau^*) > (1-\zeta)f_1(\tau^*) + \zeta f_0(\tau^*)$ for the first derivative r' of r,*

then

$$\sqrt{n}(V_n/n - \zeta F_0(\tau^*)) \to V$$

in distribution, where V is a normally distributed random variable with zero mean and standard deviation $s(r, \zeta, \tau^*)$.

Proof: Follows directly from Corollary 6.9, cf. Remark 6.10. □

A slightly more general version of Corollary 2.21, where a sequence of rejection curves r_n is considered, is Corollary 6.9. The special case where we assume DU(n, n_0) configurations in Corollary 2.21 is summarized in the following remark.

Remark 2.22

Under DU configurations and the assumptions of Corollary 2.21 we get

$$\sqrt{n}(V_n(\varphi_n)/n - \zeta\tau^*) \to V \tag{2.18}$$

in distribution, where V is a normally distributed random variable with zero mean and standard deviation

$$\frac{r'(\tau^*)\sqrt{\zeta\tau^*(1-\tau^*)}}{r'(\tau^*) - \zeta},$$

where τ^* is the point defined in 5. of Corollary 2.21.

Alternatively, the limit distribution from Remark 2.22 could be easily obtained from the results in [44]. In that paper, the asymptotic distribution of FDP(φ_n) was calculated for SU tests φ_n induced by a fixed rejection curve r; And under DU configurations we have FDP$(\varphi_n) = V_n(\varphi_n)/(n_1 +$

$V_n(\varphi_n))$. Hence, $\mathrm{FDP}(\varphi_n)/(1-\mathrm{FDP}(\varphi_n)) = V_n/n_1$ holds and Remark 2.22 follows by applying the delta method.

Note that the formulas for the variances given in [44] are not correct but a corrigendum exists, cf. [45].

Remark 2.23

Equation (2.18) entails that $\mathbb{E}_{n,n_0} V_n(\varphi_n)/n \to \zeta\tau^*$. This result is in accordance with (2.8) from Theorem 2.10.

2.5 Summary

Actually, our ENFR criterion is a generalized version of the criterion that Spjøtvoll introduced about 40 years ago. Instead of controlling the ENFR at a fixed level γ, we allow that γ depends on the number of false hypotheses (n_1). For instance, this provides us the flexibility to demand that the ENFR should be small only if n_1 is small. A very useful property of a test procedure with respect to the LFC issue is that the ENFR increases by setting the p-values corresponding to false null hypotheses to zero. We have shown that this holds true under quite general assumptions. Furthermore, we investigated the asymptotic relation between rejection curves and bounding curves. This asymptotic relation will be advantageously applied in Chapter 5. Finally, we formulated a central limit theorem for the number of false rejections in the non-sparsity case, that is $n_0/n \to \zeta \in (0, 1)$, and for a fixed rejection curve.

Chapter 3

Relationships between ENFR and FDR

In the following, we will see that the ENFR and the FDR are naturally connected, especially under some kind of independence. Furthermore, we show that under dependence it is possible that ENFR control implies FDR control. However, we point out situations under dependence where the FDR is controlled but the ENFR tends to infinity.

3.1 Asymptotic relation between ENFR and FDR

We now consider sequences of null hypotheses $(H_i)_{i\in\mathbb{N}}$, p-values $(p_i)_{i\in\mathbb{N}}$ and multiple test procedures $(\varphi_n)_{n\in\mathbb{N}}$. Suppose there exists a τ_n such that φ_n rejects H_i iff $p_i \leq \tau_n$ $(i = 1, \ldots, n)$. As a first assumption we suppose that the ecdf of the true (false) null hypotheses converges uniformly and a.s. to a distribution function $F_0(t)$ $(F_1(t))$. Sometimes, this relation is called weak dependence, see for instance [64] or [65]. Furthermore, we assume that $F_0(t), F_1(t) \in (0,1)$ for $t \in (0,1)$, $\zeta_n = n_0/n \to \zeta$, and τ_n converges a.s. to a constant $t_\zeta \in [0,1]$, say.
Let us first consider the case where $t_\zeta > 0$. Due to the uniform convergence of the ecdf we have $\lim_{n\to\infty} V_n(\tau_n)/n_0 = F_0(t_\zeta)$ a.s. and $\lim_{n\to\infty} S_n(\tau_n)/n_1 = F_1(t_\zeta)$ almost surely. The theorem of dominating convergence yields

$$\lim_{n\to\infty} \text{FDR}_\vartheta(\varphi_n) = \lim_{n\to\infty} \mathbb{E}_\vartheta \left[\frac{V_n(\tau_n)}{R_n(\tau_n) \vee 1}\right] = \mathbb{E}_\vartheta \lim_{n\to\infty} \left[\frac{V_n(t_\zeta)}{R_n(t_\zeta)}\right] = \frac{\lim_{n\to\infty} V_n(t_\zeta)/n_1}{F_1(t_\zeta) + \lim_{n\to\infty} V_n(t_\zeta)/n_1}$$

almost surely which is equivalent to

$$\lim_{n\to\infty} V_n(t_\zeta)/n_1 = \lim_{n\to\infty} \text{ENFR}_\vartheta(\varphi_n)/n_1 = F_1(t_\zeta) \lim_{n\to\infty} \frac{\text{FDR}_\vartheta(\varphi_n)}{1 - \text{FDR}_\vartheta(\varphi_n)} \tag{3.1}$$

almost surely. Under DU configurations or simply if $F_1(t_\zeta) = 1$ this becomes

$$\lim_{n\to\infty} \text{ENFR}_{n,n_0}(\varphi_n)/n_1 = \lim_{n\to\infty} \frac{\text{FDR}_{n,n_0}(\varphi_n)}{1 - \text{FDR}_{n,n_0}(\varphi_n)},$$

where $\text{FDR}_{n,n_0}(\varphi_n)$ is the FDR calculated under $\text{DU}(n, n_0)$. Thus under DU, it is not surprising that a sequence of multiple test procedures with $\lim_{n\to\infty} \text{FDR}_{n,n_0}(\varphi_n) = \alpha$ fulfills

$$\text{ENFR}_{n,n_0}(\varphi_n) = n_1(1 + o(1))\alpha/(1 - \alpha)$$

and vice versa. As we have seen, t_ζ determines the asymptotic ENFR and FDR. This means, if we use two different procedures but t_ζ is the same for both, then both procedures have the same asymptotic ENFR and the same asymptotic FDR.

In contrast, the case $t_\zeta = 0$ causes serious problems. Usually, $V_n(\tau_n)$ does not converge to $\lim_{n\to\infty} V_n(t_\zeta) = 0$ and hence similar argumentations as for $t_\zeta > 0$ will fail. However, in general, $V_n(\tau_n)$ has a non-degenerated limiting distribution depending on $(\varphi_n)_{n\in\mathbb{N}}$. For instance, Theorem 4.8 and 4.11 state the asymptotic distribution and expectation of $V_n(\tau_n)$ for a SD/SU test induced by the rejection curve $s(t) = t/\alpha$ under DU(n, n_0) and n_1 fixed. For a SD test φ_n induced by the rejection curve $s(t) = t/\alpha$ we have

$$\lim_{n\to\infty} \text{ENFR}_{n,n_0}(\varphi_n) = (n_1 + 1)\frac{\alpha}{1-\alpha},$$

but for a SU test φ_n induced by $s(t)$ we have

$$\lim_{n\to\infty} \text{ENFR}_{n,n_0}(\varphi_n) = \left(n_1 + \frac{1}{1-\alpha}\right)\frac{\alpha}{1-\alpha}.$$

Since t_ζ equals zero for the SD test and SU test, this shows how sensitive the ENFR is with respect to other factors besides t_ζ.

3.2 Simultaneous control of ENFR and FDR

We will now investigate the appealing case where g is a linear ENFR bounding function. For this section let $\mathcal{F}_t = \sigma(\mathbb{I}_{\{p_i \le s\}}, t \le s \le 1, i = 1, \ldots, n)$ be a backward filtration. Note, as before, $\mathbb{E}_{\vartheta,n_1}[V_n(\varphi_n)] = \text{ENFR}_{\vartheta,n_1}(\varphi_n)$ is the ENFR calculated after the p-values corresponding to false null hypotheses have been set to zero.

Theorem 3.1

Let φ_n be a multiple test such that the FDR increases by setting the p-values corresponding to false null hypotheses to zero. If FWER$_\vartheta(\varphi_n) \le \alpha$ under the global null hypothesis ($\vartheta \in \cap_{i=1}^n H_i$) and

$$\sup_{\vartheta\in\Theta_{n_1}} \textit{ENFR}_{\vartheta,n_1}(\varphi_n) \le \frac{\alpha}{1-\alpha} n_1 \quad \textit{for all } n_1 = 1, \ldots, n-1, \tag{3.2}$$

then

$$\sup_{\vartheta\in\Theta} \textit{FDR}_\vartheta(\varphi_n) \le \alpha. \tag{3.3}$$

Proof: Let $\vartheta \in \Theta$ with $n_0 = n_0(\vartheta) \ne n$. We get from (3.2) and Jensen's inequality that

$$\begin{aligned} \text{FDR}_\vartheta(\varphi_n) &\le \mathbb{E}_{\vartheta,n_1}\left[\frac{V_n}{n_1 + V_n}\right] \\ &\le \frac{\mathbb{E}_{\vartheta,n_1}[V_n]}{n_1 + \mathbb{E}_{\vartheta,n_1}[V_n]} \\ &\le \alpha. \end{aligned}$$

For $\vartheta \in \cap_{i=1}^{n} H_i$, i.e., $n_0 = n_0(\vartheta) = n$, we have $\text{FDR}_\vartheta(\varphi_n) = \text{FWER}_\vartheta(\varphi_n) \leq \alpha$. □

More restrictive assumptions even yield equivalence between "FDR control" and "ENFR control".

Theorem 3.2
Let $\alpha \in (0,1)$ and suppose BIA applies. If a SU test φ_n is defined in terms of the critical values induced by $f_{\alpha,\beta_n} = (1+\beta_n/n)t/[t(1-\alpha)+\alpha]$ for some $\beta_n > 0$, then

$$\forall \vartheta \in \Theta : \quad [\mathit{FDR}_\vartheta(\varphi_n) \leq \alpha \iff \mathit{ENFR}_\vartheta(\varphi_n) \leq \frac{\alpha}{1-\alpha}(n_1(\vartheta)+\beta_n)]. \tag{3.4}$$

Moreover,

$$\sup_{\vartheta \in \Theta_{n_1}} \mathit{ENFR}_\vartheta(\varphi_n) \leq \frac{\alpha}{1-\alpha}(n_1+\beta_n) \quad \text{for all } n_1 = 0,\ldots,n-1, \tag{3.5}$$

is equivalent to

$$\sup_{\vartheta \in \Theta} \mathit{FDR}_\vartheta(\varphi_n) \leq \alpha, \tag{3.6}$$

where $\Theta_k = \{\vartheta \in \Theta : |I_1(\vartheta)| = k\}$, $k = 0,\ldots,n-1$.

Note, $f_{\alpha,0}$ is the well-known AORC. In order to construct procedures which control the ENFR linearly at a fixed level, it may be fruitful, in view of Theorems 3.1 and 3.2, to investigate conventional FDR procedures. For instance, it is known that a SD test induced by the β_n-adjusted asymptotically optimal rejection curve f_{α,β_n}, with $\beta_n = 1$, controls the FDR at level α under BIA. The FDR level is nearly exhausted under DU configurations, especially for large n. In Section 4.2 we show that this procedure also yields nearly perfect ENFR control at the level $\alpha/(1-\alpha)$. On the other hand, in Section 4.3 we try to employ plug-in techniques to develop an ENFR procedure under BIA and end up with a known FDR procedure.

The three following auxiliary lemmas will simplify the proof of Theorem 3.2.

Lemma 3.3
Under (BIA), $\{V_n(t)/t, \mathcal{F}_t\}_{t\in(0,1]}$ is a backward martingale.

Proof: Confer [54], p. 136, Proposition 3.6.2. □

Lemma 3.4
Let r denote a rejection curve. Then

$$\tau = \max\{t \in [0,1] : F_n(t) \vee 1/n = r(t)\} \tag{3.7}$$

is a stopping time with respect to the backward filtration $\{\mathcal{F}_t\}_{t\in(0,1]}$. Further it holds that $\tau \in \{c_1,\ldots,c_n\}$ almost surely.

Proof: Since $c_i = r^{-1}(i/n)$ for $i = 1, \ldots, n$ and $F_n(t) \vee 1/n \in \{1/n, 2/n, \ldots, 1\}$ a.s. for all $t \in [0,1]$ we have $\tau \in \{c_1, \ldots, c_n\}$ almost surely. If $c_n < t \leq 1$, then $\{\tau \geq t\} = \emptyset \in \mathcal{F}_t$. Suppose $t \in (0, c_n]$. Let $i(t) = \min\{i : t \leq c_i\}$. Then,

$$\{\tau \geq t\} = \{\tau \geq c_{i(t)}\} = \bigcup_{j=i(t)}^{n} \{p_{j:n} \leq c_j\} \subseteq \mathcal{F}_{c_{i(t)}} \subseteq \mathcal{F}_t. \qquad \square$$

Applying the optional sampling theorem to the time-continuous process $V_n(t)/t$ is crucial for the proof of Theorem 3.2. But this process is a right-continuous backward martingale. Thus, we have to invert the time to get a martingale, but the resulting martingale is left-continuous. The (time-continuous) optional sampling theorem is not applicable because of the lack of right-continuity of the time-continuous martingale. Therefore, we resort to the time-discrete version of the optional sampling theorem where the right-continuity is not important. The following remark elucidates how the optional sampling theorem of the time-discrete theory can be applied to a time-continuous backward martingale if the stopping time has only a finite number of values.

Remark 3.5
Let M_t^b be a backward martingale with respect to the backward filtration $\mathcal{A}_t^b$ on $0 < t \leq 1$. Suppose we want to stop M_t^b at τ, with $\mathbb{P}_\vartheta(\tau \in \{t_1, \ldots, t_n\}) = 1$ and $1 = t_1 \geq \ldots \geq t_n > 0$. Then $M_i = M_{t_i}^b$ is a time-discrete martingale with respect to the filtration $\mathcal{A}_i = \mathcal{A}_{t_i}^b$, where $i = 1, \ldots, n$. Note that at this point the time has already been inverted because t_i decreases as i increases. Defining ν by $\{\nu = i\}$ if and only if $\{\tau = t_i\}$ for $i = 1, \ldots, n$ yields $M_\nu = M_\tau^b$ almost surely. Obviously, ν is a stopping time with respect to $\mathcal{A}_i$ if and only if $\{\tau \geq t_i\} \in \mathcal{A}_{t_i}^b$. By definition, $\nu \leq n$ almost surely. Hence, the optional sampling theorem of the time-discrete theory (cf. Theorem 10.10.(b) [77]) provided $\{\tau \geq t_i\} \in \mathcal{A}_{t_i}^b$ for $i = 1, \ldots, n$ yields

$$\mathbb{E}_\vartheta M_\tau^b = \mathbb{E}_\vartheta M_\nu = \mathbb{E}_\vartheta M_1 = \mathbb{E}_\vartheta M_{t_1}^b = \mathbb{E}_\vartheta M_1^b.$$

Lemma 3.6
Under BIA and the assumption of Lemma 3.4 we have

$$\mathbb{E}_\vartheta \left[\frac{V_n(\tau)}{\tau}\right] = \mathbb{E}_\vartheta \left[\frac{V_n(1)}{1}\right] = n_0,$$

with τ defined in (3.7).

Proof: The assertion follows immediately from Lemmas 3.3, 3.4, and Remark 3.5 □

Proof of Theorem 3.2. By virtue of Lemma 1.7, the SU test rejects H_i iff $p_i \leq \tilde{\tau}$, where $\tilde{\tau}$ is the largest crossing-point between F_n and f_{α,β_n}. Noting that, $V_n(\tilde{\tau})/\tilde{\tau}$ is not defined for $\tilde{\tau} = 0$, we define

$$\tau = \max\{t \in [0,1] : F_n(t) \vee 1/n = f_{\alpha,\beta_n}(t)\} = \max\{t \in [0,1] : R_n(t) \vee 1 = n f_{\alpha,\beta_n}(t)\},$$

which is a stopping time, confer Lemma 3.4. In general, $\mathbb{P}_\vartheta(\tau \neq \tilde{\tau}) > 0$, but nevertheless

$$p_i \leq \tau \Leftrightarrow p_i \leq \tilde{\tau}$$

still holds true. This can be seen as follows. If $\tilde{\tau} = c_i$ for some $i = 1, \ldots, n$, then obviously $\tilde{\tau} = \tau$. If $\tilde{\tau} = 0$, then $p_{i:n} > c_i$ for all $i = 1, \ldots, n$ and therefore $\tau = c_1$. Hence, $p_i > \tilde{\tau} = 0$ for all $i = 1, \ldots, n$ and also $p_i > \tau = c_1$ for all $i = 1, \ldots, n$.
With Lemma 3.6 we immediately get for all $\vartheta \in \Theta$ that

$$\begin{aligned}
\mathrm{FDR}_\vartheta(\varphi_n) &= \mathbb{E}_\vartheta\left[\frac{V_n(\tau)}{R_n(\tau) \vee 1}\right] \\
&= \frac{1}{n}\mathbb{E}_\vartheta\left[\frac{V_n(\tau)}{f_{\alpha,\beta_n}(\tau)}\right] \\
&= \frac{1}{n+\beta_n}\left((1-\alpha)\mathbb{E}_\vartheta V_n(\tau) + \alpha\mathbb{E}_\vartheta\left[\frac{V_n(\tau)}{\tau}\right]\right) \\
&= \frac{1}{n+\beta_n}\left((1-\alpha)\mathbb{E}_\vartheta V_n(\tau) + \alpha n_0(\vartheta)\right).
\end{aligned}$$

Hence, (3.4) and the equivalence of (3.5) and (3.6) follow immediately. □

3.3 Differences between ENFR and FDR

The FDR and the ENFR seem to be closely linked for conventional multiple test procedures under independence. But there are situations where a procedure controls the ENFR linearly and the FDR gets inflated. It is also possible that the FDR is controlled at a fixed level and the ENFR gets inflated.
Let $\gamma = \alpha/(1-\alpha)$. Suppose we reject all null hypotheses H_i with $p_i \leq \gamma/n$, then the ENFR equals $\gamma n_0/n$ under BIA, but under the global null hypothesis (all null hypotheses are true) and BIA we obtain from equation (3.11) in [19] that FDR $\to 1 - \exp(-\gamma) > \alpha$ for $n \to \infty$. Note that $1 - \exp(-\gamma) < \gamma$.
A more serious issue is the fact that controlling the FDR under dependence may lead to an undesirable inflation of the ENFR. In order to illustrate this, we first consider an extreme example. Suppose for a moment that all null hypotheses are true and all p-values are totally dependent, that is, $p_1 = \cdots = p_n$. Then a multiple test which rejects all H_i if $p_i \leq \alpha$ controls the FWER at level α while the ENFR equaling αn can become very large. In order to ensure ENFR control at level γ in this case, we have to replace the threshold α by γ/n.
More realistic and important examples are many-one comparisons and pairwise comparisons. Many-one comparison means that one compares many different groups with one control group. For instance, let μ_i, $i = 0, \ldots, n$, be the mean effect of some treatment in group G_i. Testing the n null hypotheses $H_i : \mu_0 = \mu_i$, $i = 0, \ldots, n$, is the so-called many-one problem. Finner and

Distr. Z_i	Distr. Z_0	Test statistic types
N(0,1)	N(0,1)	$\bar{\rho}^{1/2}Z_i - \rho^{1/2}Z_0$; $\lvert\bar{\rho}^{1/2}Z_i - \rho^{1/2}Z_0\rvert$; $(\bar{\rho}^{1/2}Z_i - \rho^{1/2}Z_0)/S$; $\lvert\bar{\rho}^{1/2}Z_i - \rho^{1/2}Z_0\rvert/S$; Z_i/S; $\lvert Z_i\rvert/S$
Exp(1)	Exp(1)	$Z_i - Z_0$; $\lvert Z_i - Z_0\rvert$; $(Z_i - Z_0)/S^2$; $\lvert Z_i - Z_0\rvert/S^2$
χ^2_ν	χ^2_μ	$\log(Z_i) - \log(Z_0)$
χ^2_ν	χ^2_μ $(\nu > \mu)$	$\lvert\log(Z_i) - \log(Z_0) - \vartheta\rvert$ $(\vartheta \in \mathbb{R})$

Table 3.1: *Many-One: $(Z_k)_{k\in\mathbb{N}}$, Z_0, and S are independent random variables, with $\nu S^2 \sim \chi^2_\nu$, $\nu \in \mathbb{N}$, and $\rho = 1-\bar{\rho} \in (0,1)$. Let T_i be a sequence of test statistics of a fixed type with the corresponding distributions for Z_i. Then there exists a sequence of $d_n \in \mathbb{R}$ such that FWER $= \mathbb{P}(\max_{i=1,\dots,n} T_i > d_n) \to \alpha$ but $\mathbb{E}(V_n) = \mathbb{E}(\sum_{i=1}^n \mathbb{I}_{\{T_i > d_n\}}) \to \infty$ under the global null hypothesis.*

Roters [17] showed the existence of multiple test procedures φ_n for the many-one problem such that $\text{FWER}_\vartheta(\varphi_n) \to \alpha$ but $\text{ENFR}_\vartheta(\varphi_n) \to \infty$ as $n \to \infty$. For instance, let $X_{ij} \sim N(\mu_i, \sigma^2)$, $j = 1,\dots,m_i$, $i = 0,\dots,n$, and $\nu S^2/\sigma^2 \sim \chi^2_\nu$ be independently distributed. A common test statistic T_i for testing H_i is

$$T_i = \sqrt{\frac{m_i m_0}{m_i + m_0}} \left(\frac{1}{m_i} \sum_{j=1}^{m_i} X_{ij} - \frac{1}{m_0} \sum_{j=1}^{m_0} X_{0j} \right) /S$$

$$= \left((1-\rho_i)^{1/2} \frac{1}{\sqrt{m_i}} \sum_{j=1}^{m_i} X_{ij} - \rho_i^{1/2} \frac{1}{\sqrt{m_0}} \sum_{j=1}^{m_0} X_{0j} \right) /S,$$

where $\rho_i = m_i/(m_i + m_0)$. If $m_1 = \ldots = m_n$ and $\mu_0 = \mu_1 = \ldots = \mu_n$, then $\rho_1 = \ldots = \rho_n$ and $(T_1,\dots,T_n)$ has the same distribution as $(((1-\rho_1)^{1/2}Z_1 - \rho_1^{1/2}Z_0)/\tilde{S},\dots,((1-\rho_1)^{1/2}Z_n - \rho_1^{1/2}Z_0)/\tilde{S})$, where $Z_i \sim N(0,1)$, $i = 0,\dots,n$ and $\nu\tilde{S} \sim \chi^2_\nu$ are independently distributed. This corresponds to the first row and third type of test statistics in Table 3.1. This table summarizes constellations for the many-one problem such that there exists multiple test procedures φ_n with $\text{FWER}_\vartheta(\varphi_n) \to \alpha$ and $\text{ENFR}_\vartheta(\varphi_n) \to \infty$ as $n_0 \to \infty$, for details cf. [17].

Pairwise comparison means that one compares every group with every other group. Abusing the notation introduced for the many-one comparison, the following $\binom{n}{2}$ null hypotheses $H_{ij} : \mu_i = \mu_j$, $1 \le i < j \le n$ is a classical set of null hypotheses occurring in pairwise comparisons. Table 3.2 summarizes constellations for the pairwise comparison such that there exists multiple test procedures φ_n with $\text{FWER}_\vartheta(\varphi_n) \to \alpha$ and $\text{ENFR}_\vartheta(\varphi_n) \to \infty$ as $n_0 \to \infty$, for details cf. [17].

We briefly investigate the ENFR inflation of an FDR controlling procedure in the classical many-one multiple testing problem, that is, multiple comparisons with a control. Such models were extensively studied in [20] with respect to the asymptotic behavior of the FDR and the "expected error rate EER $=$ ENFR $/n$" for the linear step-up (LSU) procedure under the global null hy-

Distr. Z_i	Test statistic types
N(0,1)	$\lvert Z_i/2^{1/2} - Z_j/2^{1/2}\rvert$; $\lvert Z_i/(2^{1/2}S) - Z_j/(2^{1/2}S)\rvert$
Exp(1)	$\lvert Z_i - Z_j\rvert$; $\lvert Z_i/S^2 - Z_j/S^2\rvert$
χ^2_ν	$\lvert \log(Z_i) - \log(Z_j)\rvert$
Cauchy$(0,1)$	$\lvert Z_i/2 - Z_j/2\rvert$

Table 3.2: *Pairwise comparison:* $(Z_k)_{k\in\mathbb{N}}$ *and* S *are independent random variables, with* $\nu S^2 \sim \chi^2_\nu$, $\nu \in \mathbb{N}$. *Let* T_{ij} *be a sequence of test statistics of a fixed type with the corresponding distributions for* Z_i. *Then there exists a sequence of* $d_n \in \mathbb{R}$ *such that FWER* $= \mathbb{P}(\max_{1\leq i<j\leq n} T_{ij} > d_n) \to \alpha$ *but* $\mathbb{E}(\sum_{1\leq i<j\leq n} \mathbb{I}_{\{T_{ij}>d_n\}}) \to \infty$.

pothesis. This procedure is a SU test induced by the rejection curve $r(t) = t/\alpha$. In order to illustrate the impact of dependence on the ENFR in this case, we consider a small simulation study. Let $X_i \sim N(0,1)$, $i \in \mathbb{N}_0$, be a sequence of independent random variables and $p_i = (1 - \Phi(\sqrt{1-\rho}X_i - \sqrt{\rho}X_0))\mathbb{I}_{\{H_i \text{ is true}\}}$ for $i \in \mathbb{N}$. Since p_i equals zero if H_i is false, the p-values corresponding to the true null hypotheses are independent of the p-values corresponding to the false null hypotheses. It is well known that an LSU procedure controls the FDR under this type of dependence (PRDS: *positive regression dependence on subsets*; cf. [3]) at a fixed level for all $\rho \in [0,1]$.

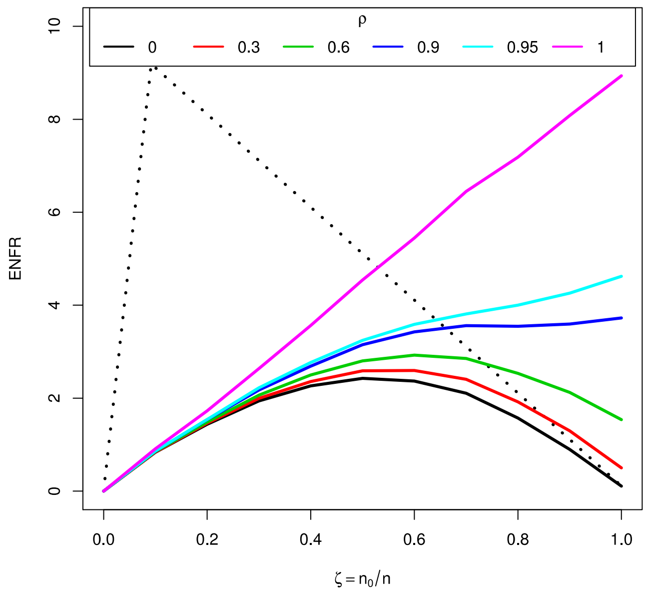

Figure 3.1: *The ENFR as a function of* ζ *for different (equi-)correlation coefficients* ρ *for the LSU procedure with* $\alpha/(1-\alpha) = \gamma = 0.1$. *The dotted line is the ENFR bounding function* $g(n_1) = (n_1 + 1 + \gamma)\gamma$.

In Section 4.1 we show that this procedure controls the ENFR under DU configurations with bounding function $g(n_1) = (n_1 + 1 + \gamma)\gamma$. But as Figure 3.1 illustrates for $n = 100$, control of the ENFR at level $\gamma = 0.1$ is typically far from being achieved under dependence. Other values of n lead to similar pictures.

3.4 Summary

Assuming weak dependence and converging crossing-points, an asymptotic relation between the ENFR and the FDR has been established. For fixed n, we proved that control of the ENFR combined with weak control of the FWER implies control of the FDR, also under dependence, if the FDR increases by setting the p-values corresponding to false null hypotheses to zero. Moreover, for fixed n, we showed for a SU test induced by the β-adjusted AORC under BIA that control of the FDR is equivalent to control of the ENFR. Nevertheless, under dependence it is possible that a procedure controls the FWER but the ENFR tends to infinity. We pointed out that this may occur in the many-one problem and for pairwise comparisons for very common distributions like the standard normal distribution.

Chapter 4

ENFR of some common multiple test procedures

4.1 ENFR of linear step-up and linear step-down procedures

Benjamini and Hochberg [1] proved in 1995 that a SU test induced by $r(t) = t/\alpha$, sometimes called the Simes test, controls the FDR under BIA at level $\alpha n_0/n$. Two years later, it was proven that the Simes test controls the FWER under a type of positive dependence and the assumption that all null hypotheses are true, cf. [48]. In 2001, Benjamini and Yekutieli [3] showed that the procedure also controls the FDR under some specific dependence structure. Nowadays, this is a standard procedure and often called the *BH-procedure*. It is well investigated as an FDR controlling procedure. In this section we will further investigate this procedure with respect to the ENFR. Some very useful properties for the BH-procedure with respect to the ENFR are already known, cf. [18] and [19]. The original BH-procedure uses $c_i = \alpha i/n$ as critical values in a SU manner. Since c_i depends linearly on i, we call the corresponding SD/SU test the *linear SD/SU test*, LSD/LSU test for short. Further in this section, the number of false rejections for LSD and LSU will be denoted by $V_{n,SD}$ and $V_{n,SU}$. The results in this section will always refer to both LSD and LSU, but since the proofs are very similar, usually only one proof will be presented.

Remark 4.1
We assume BIA and therefore, w.l.o.g., by Lemma 2.4 we state all theorems with respect to DU configurations. Although there are different DU configurations, we simplify the notation by suppressing the subscript that indicates DU configurations, that is $\mathbb{P} = \mathbb{P}_{n,n_0}$ and $\mathbb{E} = \mathbb{E}_{n,n_0}$.

The following theorem is very helpful in finding the asymptotic distribution and expectation of $V_{n,SD}$ and $V_{n,SU}$.

Theorem 4.2 (cf. Finner and Roters [18], p. 990 and 991)
Under DU(n, n_0) *and* $\alpha \in (0,1)$*, it holds that*

$$\mathbb{P}(V_{n,SD} = i) = \binom{n_0}{i} \frac{n_1+1}{n}\alpha \left(\frac{n_1+i+1}{n}\alpha\right)^{i-1} \left(1 - \frac{n_1+i+1}{n}\alpha\right)^{n_0-i}, \tag{4.1}$$

$$\mathbb{P}(V_{n,SU} = i) = (1-\alpha)\binom{n_0}{i} \left(\frac{n_1+i}{n}\alpha\right)^{i} \left(1 - \frac{n_1+i}{n}\alpha\right)^{n_0-i-1},$$

$$\mathbb{E}V_{n,SD} = (n_1+1)\sum_{i=1}^{n-n_1} \binom{n-n_1}{i} i! \left(\frac{\alpha}{n}\right)^i,$$

$$\mathbb{E}V_{n,SU} = \sum_{i=1}^{n-n_1} (n_1+i)\binom{n-n_1}{i} i! \left(\frac{\alpha}{n}\right)^i,$$

where $0 \le i \le n_0$.

Since the formulas of Theorem 4.2 are stated in the publication without a proof we shortly elucidate how these formulas can be obtained. The main tools are Lemma 4.1 for the SD case and Lemma 4.2 for the SU case from [19]. These Lemmas give explicit formulas for the distribution/expectation of $V_{n,SD}$ and $V_{n,SU}$ under the global null hypothesis, that is all p-values are iid $U[0,1]$ distributed, and for critical values of the form $c_{n-i+1} = \beta - (i-1)\tau$, with constants $\beta, \tau \in [0,1]$ such that $\beta \ge (n-1)\tau$. Note, under DU(n, n_0) we have

$$\{p_{1:n} \le \alpha/n, \ldots, p_{n_1:n} \le n_1\alpha/n, p_{n_1+1:n} \le (n_1+1)\alpha/n, \ldots, p_{n:n} \le \alpha\}$$
$$= \{p_{1:I_0} \le (n_1+1)\alpha/n, \ldots, p_{n_0:I_0} \le \alpha\},$$

where $p_{i:I_0}$ denotes the ith order statistic of $\{p_i\}_{i \in I_0}$. Hence, we can calculate under DU(n_0, n_0) with critical values $\tilde{c}_{n_0-i+1} = \alpha - (i-1)\alpha/n$, $i = 1, \ldots, n_0$. Note, $\tilde{c}_1 = (n_1+1)\alpha/n$. Setting $\beta = \alpha$ and $\tau = \alpha/n$ we immediately get from Lemma 4.1. in [19] that

$$\mathbb{P}(V_{n,SD} = i) = \binom{n_0}{i}(\beta - (n_0-1)\tau)(\beta - (n_0-i-1)\tau)^{i-1}(1 - \beta + (n-i-1)\tau)^{n_0-i}$$
$$= \binom{n_0}{i} \frac{n_1+1}{n}\alpha \left(\frac{n_1+i+1}{n}\alpha\right)^{i-1} \left(1 - \frac{n_1+i+1}{n}\alpha\right)^{n_0-i}.$$

In the same way, also with $\beta = \alpha$ and $\tau = \alpha/n$, the other formulas of Theorem 4.2 are obtained.

Remark 4.3
Equation (4.1) was already obtained by Dempster, cf. [12] or Proposition 1, p. 344 in [54]. Considering DU(n_0, n_0) and choosing $a = (n_1+1)\alpha/n$ and $b = \alpha/n$ in Proposition 1, p. 344 from [54] directly yields (4.1).

The decisive property for finding the asymptotic distribution and expectation of $V_{n,SD}$ and $V_{n,SU}$ is the uniform integrability (u.i.) of $\{V_{n,SD}^{\rho}\}_{n \ge n_1}$ and $\{V_{n,SU}^{\rho}\}_{n \ge n_1}$ for all $\rho > 0$, as stated in the following theorem.

Lemma 4.4

Under DU configurations with n_1 fixed and $\alpha \in (0,1)$, the sequences $\{V^\rho_{n,SD}\}_{n\geq n_1}$ and $\{V^\rho_{n,SU}\}_{n\geq n_1}$ are u.i. for all $\rho > 0$, that is, for every $\epsilon > 0$ there exists a $c > 0$ such that

$$\sup_{n\geq n_1} \int_{\{|V^\rho_{n,SD}|>c\}} V^\rho_{n,SD} d\mathbb{P} \leq \epsilon,$$

$$\sup_{n\geq n_1} \int_{\{|V^\rho_{n,SU}|>c\}} V^\rho_{n,SU} d\mathbb{P} \leq \epsilon.$$

In order to proof the uniform integrability we will apply the following auxiliary lemma.

Lemma 4.5

It holds that

$$\forall\, k \in \mathbb{N}_0 : \exists\, C_k > 0 : \forall\, n \in \mathbb{N}, p \in [0,1] : \mathbb{E}\left(Z_{n,p} - np\right)^k \leq C_k \max\{1, (np)^{\lfloor k/2 \rfloor}\},$$

where $Z_{n,p} \sim B(n,p)$.

Remark 4.6

We want to distress that the inequality in Lemma 4.5 holds uniformly in p. This fact is essential for the argumentation in the proof of Lemma 4.4 and will also be used in Section 4.3.
A similar inequality can be easily obtained by normal approximation, cf. Corollary 9.3.1. from [8], but the inequality lacks to hold uniformly in p.

We are going to present two proofs for Lemma 4.5. The first proof is short and bases upon a recursive formula for binomial moments. The second proof is somewhat longer but self-contained.

First proof of Lemma 4.5. We will proof the result by complete induction. Obviously, the inequality holds for $k = 0$ and $k = 1$. Assume now that the assertion is true for all $i \leq k$, where $k \geq 2$. Following the notation of [36], let $\mu_k = \mathbb{E}\left(Z_{n,p} - np\right)^k$ and $q = 1 - p$. Then (3.13) from [36], p. 108, states that

$$\mu_{k+1} = npq \sum_{j=0}^{k-1} \binom{k}{j} \mu_j - p \sum_{j=0}^{k-1} \binom{k}{j} \mu_{j+1}.$$

Hence, for suitable $M_1, M_2 > 0$ we obtain that

$$\begin{aligned}
|\mu_{k+1}| &\leq np \sum_{j=0}^{k-1} \binom{k}{j} |\mu_j| + \sum_{j=0}^{k-1} \binom{k}{j} |\mu_{j+1}| \\
&\leq npM_1 \sum_{j=0}^{k-1} \max\{1, (np)^{\lfloor j/2 \rfloor}\} + M_2 \sum_{j=0}^{k-1} \max\{1, (np)^{\lfloor (j+1)/2 \rfloor}\} \\
&\leq C_{k+1} \max\{1, (np)^{\lfloor (k-1)/2 \rfloor + 1}\} \quad \text{(say)} \\
&= C_{k+1} \max\{1, (np)^{\lfloor (k+1)/2 \rfloor}\}.
\end{aligned}$$

This establishes the formula. □

Second proof of Lemma 4.5. It is well known that the moment generating function for $Z_{n,p} - np$ is $M(t) = (1 - p + p\exp(t))^n \exp(-tnp)$. Now

$$\left.\frac{d^k M(t)}{dt^k}\right|_{t=0} = M^{(k)}(0) = \mathbb{E}\left(Z_{n,p} - np\right)^k$$

holds. By defining $\eta = np(1-p)$, $g(t) = 1 - p + p\exp(t)$, and $h(t) = (\exp(t) - 1)/g(t)$, we see that $M(t) = g^n(t)\exp(-tnp)$ and

$$\begin{aligned}
M^{(1)}(t) &= ng^{n-1}(t)pe^t e^{-tnp} - g^n(t)e^{-tnp}np \\
&= M(t)np\left(\frac{e^t}{g(t)} - 1\right) \\
&= M(t)np\left(\frac{(1-p)e^t - 1 + p}{g(t)}\right) \\
&= M(t)np(1-p)\frac{e^t - 1}{g(t)} \\
&= \eta M(t)h(t).
\end{aligned}$$

We prove the result by complete induction. Obviously, the inequality holds for $k = 0$ and $k = 1$. Assume now that the assertion is true for all $i \le k$, where $k \ge 2$. Additionally, we also assume for now that $h^{(i)}(0) \le L$ for all $1 \le i \le k$ (this will be shown later). Setting $C_0 = 1$, due to $h(0) = 0$ we get

$$\begin{aligned}
M^{(k+1)}(0) &= (\eta M h)^{(k)}(0) \\
&= \eta \sum_{i=0}^{k} \binom{k}{i} M^{(i)}(0)h^{(k-i)}(0) \\
&= \eta \sum_{i=0}^{k-1} \binom{k}{i} M^{(i)}(0)h^{(k-i)}(0) \\
&\le np(1-p)Lk \max_{0\le i\le k-1} \binom{k}{i} C_i \max\{1, (np)^{\lfloor i/2 \rfloor}\} \\
&\le C_{k+1} \max\{1, (np)^{\lfloor (k-1)/2 \rfloor + 1}\} \ \text{(say)} \\
&= C_{k+1} \max\{1, (np)^{\lfloor (k+1)/2 \rfloor}\}.
\end{aligned}$$

It remains to show that $h^{(i)}(0) \le L$ for all $1 \le i \le k$. First note, suppressing t, that for every differentiable function f and $m \in \mathbb{N}$ we have

$$\left(\frac{f}{g^m}\right)' = \frac{f'}{g^m} - \frac{mfg'}{g^{m+1}}. \tag{4.2}$$

Setting $f(t) = \exp(t) - 1$, we see that $h^{(i)} = (f/g)^{(i)} = \sum_{j=1}^{i+1} f_j/g^j$ for appropriate functions $f_1, \ldots, f_{k+1}$. Because $g(0) = 1$, we get $h^{(i)}(0) = \sum_{j=1}^{i+1} f_j(0)$. By (4.2), any f_j can be written as

$$f_j = P_j(f, f^{(1)}, \ldots, f^{(j)}, g, g^{(1)}, \ldots, g^{(j)}),$$

where P_j denotes a polynomial in several variables. In our case $g(t) = 1-p+p\exp(t)$, $g^{(j)}(t) = p\exp(t)$, $f(t) = \exp(t) - 1$, and $f^{(j)}(t) = \exp(t)$ for $j \in \mathbb{N}$. Since $g(0), g^{(j)}(0), f(0), f^{(j)}(0) \in [0,1]$ for all $p \in [0,1]$, and P_j is a polynomial for all $j = 1, \ldots, k+1$, we see that there must exist a $L \in \mathbb{R}$ such that $h^{(i)}(0) \leq L$ for $1 \leq i \leq k$. □

Now, we are able to prove the uniform integrability of $V^{\rho}_{n,SD}$ and $V^{\rho}_{n,SU}$ for $\rho > 0$.

Proof of Lemma 4.4. Let $\epsilon > 0$ and $\rho > 0$ be arbitrary. For the step-down case let $p_i = \alpha(n_1 + i + 1)/(n_0 + n_1)$ with $1 \leq i \leq n_0$ and $Z_i \sim B(n_0, p_i)$. Then it holds for $m = \lceil \rho \rceil$ that

$$\begin{aligned}
\mathbb{E}V^m_{n,SD}\mathbb{I}_{\{V^m_{n,SD}>c\}} &= \sum_{c<i^m \leq n_0^m} i^m \frac{n_1+1}{n}\alpha \binom{n_0}{i} \left(\frac{n_1+i+1}{n}\alpha\right)^{\mathbf{i-1}} \left(1 - \frac{n_1+i+1}{n}\alpha\right)^{n_0-i} \\
&= \sum_{c<i^m \leq n_0^m} \frac{i^m(n_1+1)}{n_1+i+1}\binom{n_0}{i} p_i^{\mathbf{i}}(1-p_i)^{n_0-i} \\
&\leq \sum_{c<i^m \leq n_0^m} i^m \mathbb{P}\left(Z_i = i\right).
\end{aligned}$$

For the step-up case let $\tilde{p}_i = \alpha(n_1 + i)/(n_0 + n_1)$ with $1 \leq i \leq n_0$ and $\tilde{Z}_i \sim B(n_0, \tilde{p}_i)$. Then it holds that

$$\begin{aligned}
\mathbb{E}V^m_{n,SU}\mathbb{I}_{\{|V^m_{n,SU}|>c\}} &= \sum_{c<i^m \leq n_0^m} i^m(1-\alpha)\binom{n_0}{i}\left(\frac{n_1+i}{n}\alpha\right)^i \left(1 - \frac{n_1+i}{n}\alpha\right)^{\mathbf{n_0-i-1}} \\
&\leq \sum_{c<i^m \leq n_0^m} i^m \binom{n_0}{i}\left(\frac{n_1+i}{n}\alpha\right)^i \left(1 - \frac{n_1+i}{n}\alpha\right)^{\mathbf{n_0-i}} \\
&\leq \sum_{c<i^m \leq n_0^m} i^m \mathbb{P}\left(\tilde{Z}_i = i\right).
\end{aligned}$$

Obviously, $\tilde{p}_i < p_i = \alpha(n_1 + i + 1)/n \leq \alpha(1 + (n_1 + 1))i/n = Ci/n$ holds. Thus, by Markov inequality and Lemma 4.5 for $p_i^* \in \{p_i, \tilde{p}_i\}$, $Z_i^* \sim B(n_0, p_i^*)$, and $k \in \mathbb{N}$ we get

$$\begin{aligned}
i^m\mathbb{P}\left(Z_i^* = i\right) &\leq i^m\mathbb{P}\left(|Z_i^* - n_0p_i^*| \geq i - n_0p_i^*\right) \\
&\leq \frac{i^m\mathbb{E}\left(Z_i^* - n_0p_i^*\right)^{2k}}{\left(i - n_0p_i\right)^{2k}} \\
&\leq \frac{i^m C_{2k}\max\{1, (Ci)^k\}}{i^{2k}\left(1 - n_0\frac{(n_1/i+1+1/i)}{n}\alpha\right)^{2k}}.
\end{aligned}$$

Note, the Markov inequality is only applicable if $i - n_0 p_i^* > 0$. This holds true for fixed n_1 if i is large enough because $i - n_0 p_i^* \geq i - \alpha(n_1 + i + 1) = i(1-\alpha) - \alpha(n_1+1)$. Hence, c must be at least $(\alpha(n_1+1)/(1-\alpha))^m$, which is a constant because n_1, m, and α are fixed. Further note that the constant C_{2k} does not depend on i or p_i^* since the inequality in Lemma 4.5 holds uniformly in p. Choosing $k = m+2$ yields

$$\sup_{n \geq n_1} \sum_{n_0^m > i^m > c} i^m \mathbb{P}\left(Z_i^* = i\right) \leq K \sup_{n \geq n_1} \sum_{i^m > c} 1/i^2 \leq \epsilon,$$

where $K > 0$. Since $\rho \leq m$, we get uniform integrability of $\left\{V_{n,SU}^{\rho}\right\}_n$ and $\left\{V_{n,SD}^{\rho}\right\}_n$. □

Because $\{V_{n,SD}\}_{n \geq n_1}$ and $\{V_{n,SU}\}_{n \geq n_1}$ are uniformly integrable we also get the following result on tightness.

Corollary 4.7
Under DU(n, n_0) for n_1 fixed, the sequences of probability measures $\left\{\mathbb{P}^{V_{n,SD}}\right\}_{n \geq n_1}$ and $\{\mathbb{P}^{V_{n,SU}}\}_{n \geq n_1}$ are tight.

Proof: By Lemma 4.4, for every $\epsilon > 0$ we can choose a $c > 0$ such that

$$\mathbb{P}^{V_{n,SD}}\left(\mathbb{N}_0 \backslash [0, c]\right) = \int_{\{|V_{n,SD}| > c\}} d\mathbb{P} \leq \int_{\{|V_{n,SD}| > c\}} V_{n,SD} d\mathbb{P} \leq \epsilon$$

holds for all $n \geq n_1$. The tightness of $\left\{\mathbb{P}^{V_{n,SU}}\right\}_{n \geq n_1}$ is established in the same way. □

Calculating the limits of the probabilities in Theorem 4.2 will give us a possible limit distribution for $V_{n,SD}$ $(V_{n,SU})$. These limit distributions appearing in (4.3) and (4.4) will be denoted by LSD(n_1, α) and LSU(n_1, α), respectively. The uniform integrability of $V_{n,SD}$ $(V_{n,SU})$ or to be more precise the tightness of the corresponding distribution will ensure that the calculated limits are actually probability distributions.

Theorem 4.8 (Asymptotic distribution)
Under DU(n, n_0) with fixed $n_1 = n - n_0$, the limiting probability mass function (pmf) of $V_{n,SD}$ is given by

$$\lim_{n \to \infty} \mathbb{P}\left(V_{n,SD} = i\right) = q_{SD}(i) = \frac{\alpha}{i!}(n_1+1)((n_1+i+1)\alpha)^{i-1} \exp(-(n_1+i+1)\alpha) \quad (4.3)$$

and the limiting pmf of $V_{n,SU}$ is given by

$$\lim_{n \to \infty} \mathbb{P}\left(V_{n,SU} = i\right) = q_{SU}(i) = \frac{1-\alpha}{i!}((n_1+i)\alpha)^i \exp(-(n_1+i)\alpha). \quad (4.4)$$

Proof: Obviously, by $\binom{n_0}{i}/n^i = (1/i!)\prod_{k=0}^{i-1}(n-n_1-k)/n$ we have

$$\begin{aligned}\mathbb{P}\left(V_{n,SD}=i\right) &= \binom{n_0}{i}\frac{n_1+1}{n}\alpha\left(\frac{n_1+i+1}{n}\alpha\right)^{i-1}\left(1-\frac{n_1+i+1}{n}\alpha\right)^{n_0-i} \\ &= \frac{\alpha}{i!}(n_1+1)((n_1+i+1)\alpha)^{i-1}\frac{\left(1-\frac{n_1+i+1}{n}\alpha\right)^n}{\left(1-\frac{n_1+i+1}{n}\alpha\right)^{n_1+i}}\prod_{k=0}^{i-1}\frac{n-n_1-k}{n} \\ &\to \frac{\alpha}{i!}(n_1+1)((n_1+i+1)\alpha)^{i-1}\exp(-(n_1+i+1)\alpha) \\ &= q_{SD}(i)\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}\left(V_{n,SU}=i\right) &= (1-\alpha)\binom{n_0}{i}\left(\frac{n_1+i}{n}\alpha\right)^{i}\left(1-\frac{n_1+i}{n}\alpha\right)^{n_0-i-1} \\ &= \frac{1-\alpha}{i!}((n_1+i)\alpha)^{i}\frac{\left(1-\frac{n_1+i}{n}\alpha\right)^n}{\left(1-\frac{n_1+i}{n}\alpha\right)^{n_1+i+1}}\prod_{k=0}^{i-1}\frac{n-n_1+k}{n} \\ &\to \frac{1-\alpha}{i!}((n_1+i)\alpha)^{i}\exp(-(n_1+i)\alpha) \\ &= q_{SU}(i).\end{aligned}$$

At the moment it is not clear whether q_{SD} and q_{SU} are *probability* measures. By Corollary 4.7 and Prohorov's Theorem [6], we know that $\left\{\mathbb{P}^{V_{n,SD}}\right\}_{n\geq n_1}$ contains a subsequence $\left\{\mathbb{P}^{V_{n_j,SD}}\right\}_j$ converging to a probability measure. But we also know that

$$\lim_{j\to\infty}\mathbb{P}\left(V_{n_j,SD}=i\right)=\lim_{n\to\infty}\mathbb{P}\left(V_{n,SD}=i\right)=q_{SD}(i)$$

for all $i\in\mathbb{N}_0$. Thus, the limiting probability measure must be q_{SD}. In the same way, we conclude that q_{SU} is a probability measure. □

Remark 4.9

The distribution LSD(n_1,α) belongs to the class of so-called generalized Poisson distributions. In [10], Consul defined for $\lambda_1>0$ and $\lambda_2\in[0,1)$ the generalized Poisson distribution by

$$p(x|\lambda_1,\lambda_2)=\lambda_1(\lambda_1+x\lambda_2)^{x-1}e^{-(\lambda_1+x\lambda_2)}/x!,\qquad x\in\mathbb{N}_0.$$

For $\lambda_1=(n_1+1)\alpha$ and $\lambda_2=\alpha$ we obtain $p(x|\lambda_1,\lambda_2)=q_{SD}(x)$. According to [9] the first moment and the variance of LSD(n_1,α) are $\lambda_1/(1-\lambda_2)=(n_1+1)\alpha/(1-\alpha)$ and $(n_1+1)\alpha/(1-\alpha)^3$.

We want to remark that Consul et al. originally introduced the generalized Poisson distribution in [9] and allowed $\lambda_2\in(-1,1)$. In the next remark we will see that this assumption is to liberal.

Remark 4.10

Let L be implicitly defined by $Le^{1-L} = -1$. We have $L \approx -0.27$. Further, let $S(k, \lambda_1, \lambda_2) = \sum_{x=0}^{\infty} (\lambda_1 + x\lambda_2)^{x+k-1} e^{-(\lambda_1 + x\lambda_2)}/x!$. We now show that $S(k, \lambda_1, \lambda_2) < \infty$ for $(k, \lambda_1, \lambda_2) \in \{0\} \times \mathbb{R}\backslash\{0\} \times (L, 1) \cup \mathbb{N} \times \mathbb{R} \times (L, 1)$. Suppose $k \in \mathbb{N}$ and denote by s_x the xth summand of $S(k, \lambda_1, \lambda_2)$. We have

$$\begin{aligned} \frac{s_{x+1}}{s_x} &= \frac{(\lambda_1 + (x+1)\lambda_2)^{x+k} e^{-(\lambda_1 + (x+1)\lambda_2)}/(x+1)!}{(\lambda_1 + x\lambda_2)^{x+k-1} e^{-(\lambda_1 + x\lambda_2)}/x!} \\ &= \frac{(\lambda_1 + x\lambda_2 + \lambda_2)^{x+k}}{(\lambda_1 + x\lambda_2)^{x+k}} \frac{(\lambda_1 + x\lambda_2) e^{-\lambda_2}}{x+1} \\ &= \left(1 + \frac{\lambda_2}{\lambda_1 + x\lambda_2}\right)^{x+k} \frac{e^{-\lambda_2}(\lambda_1 + x\lambda_2)}{x+1} \\ &\to e^{1-\lambda_2}\lambda_2. \end{aligned}$$

Since $|e^{1-\lambda_2}\lambda_2| < 1$ for all $\lambda_2 \in (L, 1)$, the ratio test yields that $S(k, \lambda_1, \lambda_2) = \sum_{x=0}^{\infty} s_x$ is absolute convergent. For $k = 0$ we conduct the same calculation in order to show that $\sum_{x=1}^{\infty} s_x$ is absolute convergent. Note, that $s_0 = \lambda_1^{-1} e^{-\lambda_1} \in \mathbb{R}$. Thus, $S(0, \lambda_1, \lambda_2) < \infty$ for $\lambda_1 \in \mathbb{R}\backslash\{0\}$ and $\lambda_2 \in (L, 1)$. Furthermore, by the ratio test we get that $\sum_{x=1}^{\infty} s_x$ is divergent if $\lambda_2 \in \mathbb{R}\backslash[L, 1]$. For $\lambda_2 \in \{L, 1\}$ nothing is known. The fact that it is possible to consider $\lambda_2 \in (L, 1)$ instead of $\lambda_2 \in [0, 1)$, as Consul did in [10], is not new. Tuenter showed this in [67] for $k = 0, 1$.
Extending the definition of the generalized Poisson distribution to

$$p(x|k, \lambda_1, \lambda_2) = \frac{1}{S(k, \lambda_1, \lambda_2)} (\lambda_1 + x\lambda_2)^{x+k-1} e^{-(\lambda_1 + x\lambda_2)}/x!, \qquad x \in \mathbb{N}_0,$$

for $(k, \lambda_1, \lambda_2) \in \{0\} \times \mathbb{R}\backslash\{0\} \times (L, 1) \cup \mathbb{N} \times \mathbb{R} \times (L, 1)$ we get $q_{SD}(x) = p(x|0, (n_1+1)\alpha, \alpha)$ and $q_{SU}(x) = p(x|1, n_1\alpha, \alpha)$. By this identification it is possible to apply the techniques developed in [9] in order to calculate the first moment and the variance of LSU(n_1, α). Nevertheless, in the proof of Theorem 4.11 we show a different technique to (re)obtain the first two (centered) moments of LSD(n_1, α) and LSU(n_1, α).
We want to note that both techniques, the one developed by Consul et al. and the one presented in the proof of Theorem 4.11 can be used to calculate arbitrary moments of LSD(n_1, α) and LSU(n_1, α). Both techniques have the disadvantage that the kth moment can only be calculated if all previous moments are known. The author prefers neither of the two techniques because both involve tedious calculations.

The uniform integrability of $V_{n,SD}^{\rho}$ ($V_{n,SU}^{\rho}$) will also ensure that the moments of $V_{n,SD}^{\rho}$ ($V_{n,SU}^{\rho}$) will converge to the moments of the corresponding limit distribution.

Theorem 4.11 (Asymptotic expectation)
Let $V_{SD} \sim LSD(n_1, \alpha)$ and $V_{SU} \sim LSU(n_1, \alpha)$. Under $DU(n, n_0)$ for n_1 fixed and $\alpha \in (0,1)$ it holds for all $\rho > 0$ that

$$\lim_{n\to\infty} \mathbb{E}V_{n,SD}^{\rho} = \mathbb{E}V_{SD}^{\rho},$$
$$\lim_{n\to\infty} \mathbb{E}V_{n,SU}^{\rho} = \mathbb{E}V_{SU}^{\rho}.$$

In particular, it holds for the first moments

$$\mathbb{E}V_{n,SD} \uparrow \mathbb{E}V_{SD} = (n_1+1)\frac{\alpha}{1-\alpha},$$
$$\mathbb{E}V_{n,SU} \uparrow \mathbb{E}V_{SU} = \left(n_1 + 1 + \frac{\alpha}{1-\alpha}\right)\frac{\alpha}{1-\alpha},$$

and the variance

$$\lim_{n\to\infty} \mathit{Var}V_{n,SD} = \mathit{Var}V_{SD} = \frac{(n_1+1)\alpha}{(1-\alpha)^3},$$
$$\lim_{n\to\infty} \mathit{Var}V_{n,SU} = \mathit{Var}V_{SU} = \frac{\alpha(n_1+1) - \alpha^2(n_1-1)}{(1-\alpha)^4} = \frac{(n_1+1)\alpha}{(1-\alpha)^3} + \frac{2\alpha^2}{(1-\alpha)^4}.$$

Proof: Since $V_{n,SD}$ converges in distribution to V_{SD} and Lemma 4.4 (uniform integrability) holds, we have by Theorem 5.4, p. 32 in [6], $\lim_{n\to\infty} \mathbb{E}V_{n,SD}^{\rho} = \mathbb{E}V_{SD}^{\rho}$ for $\rho > 0$. The same arguments yield $\lim_{n\to\infty} \mathbb{E}V_{n,SU}^{\rho} = \mathbb{E}V_{SU}^{\rho}$ for $\rho > 0$. For the first moments we conclude by the monotone convergence theorem that

$$\begin{aligned}
\mathbb{E}V_{n,SD} &= (n_1+1)\sum_{i=1}^{n-n_1}\binom{n-n_1}{i} i! \left(\frac{\alpha}{n}\right)^i \\
&= (n_1+1)\sum_{i\in\mathbb{N}}\left[\prod_{k=0}^{i-1}\frac{n-n_1-k}{n}\right]\alpha^i \mathbb{I}_{\{i\le n-n_1\}} \\
&\uparrow (n_1+1)\sum_{i\in\mathbb{N}}\alpha^i \\
&= (n_1+1)\frac{\alpha}{1-\alpha}
\end{aligned}$$

and

$$
\begin{aligned}
\mathbb{E}V_{n,SU} &= \sum_{i=1}^{n-n_1} (n_1+i)\binom{n-n_1}{i} i! \left(\frac{\alpha}{n}\right)^i \\
&= \sum_{i\in\mathbb{N}} (n_1+i)\left[\prod_{k=0}^{i-1} \frac{n-n_1-k}{n}\right] \alpha^i \mathbb{I}_{\{i\le n-n_1\}} \\
&\uparrow \sum_{i\in\mathbb{N}} (n_1+i)\alpha^i \\
&= n_1\frac{\alpha}{1-\alpha} + \frac{\alpha}{(1-\alpha)^2} \\
&= \frac{\alpha}{1-\alpha}\left(n_1 + \frac{1}{1-\alpha}\right),
\end{aligned}
$$

where the third equality follows from the last equality in [63]. We will now calculate the variance and start with the step-down case. For $\eta = \exp(-(n_1{+}1)\alpha)$, $c_i(x) = (n_1{+}1)(n_1{+}i{+}1)^{i-1}x^i/i!$, and $g(x) = x\exp(-x)$ we get

$$
\begin{aligned}
q_{SD}(i) &= \frac{\alpha}{i!}(n_1+1)((n_1+i+1)\alpha)^{i-1}\exp(-(n_1+i+1)\alpha) \\
&= \exp(-(n_1+1)\alpha)\frac{n_1+1}{i!}(n_1+i+1)^{i-1}(\alpha\exp(-\alpha))^i \\
&= \eta c_i(g(\alpha)).
\end{aligned}
$$

Since $\eta\sum_{i=1}^{\infty} c_i(g(\alpha)) = 1 - q_{SD}(0)$ for all $\alpha \in (0,1)$, the radius of convergence of $h(x) = \sum_{i=1}^{\infty} c_i(x)$ is at least $g(1) = \exp(-1)$. Due to $xc_i'(x) = ic_i(x)$and $x^2c_i''(x) = i(i-1)c_i(x)$ we get

$$
\begin{aligned}
&\eta h(g(\alpha)) = 1 - q_{SD}(0) = 1 - \exp(-(n_1+1)\alpha), \\
&\eta g(\alpha)h'(g(\alpha)) = \sum_{i=1}^{\infty} i\eta c_i(g(\alpha)) = \mathbb{E}V_{SD} = (n_1+1)\frac{\alpha}{1-\alpha}, \\
&\eta g^2(\alpha)h''(g(\alpha)) = \sum_{i=1}^{\infty} i(i-1)\eta c_i(g(\alpha)) = \mathbb{E}[V_{SD}(V_{SD}-1)].
\end{aligned}
$$

By the chain rule $(h \circ g)'' = (h' \circ g')' = (h'' \circ g) \cdot (g')^2 + (h' \circ g) \cdot g''$, we see that

$$\begin{aligned}
\mathbb{E}V_{SD}^2 - \mathbb{E}V_{SD} &= \eta g^2(\alpha) h''(g(\alpha)) \\
&= \eta g^2(\alpha) \frac{(h \circ g)''(\alpha) - h'(g(\alpha)) g''(\alpha)}{(g'(\alpha))^2} \\
&= \eta g^2(\alpha) \frac{\frac{d^2}{d\alpha^2}(e^{(n_1+1)\alpha} - 1) - \frac{(n_1+1)\alpha}{(1-\alpha)\eta g(\alpha)} g''(\alpha)}{(g'(\alpha))^2} \\
&= \alpha^2 e^{-(n_1+3)\alpha} \frac{(n_1+1)^2 e^{(n_1+1)\alpha} - \frac{(n_1+1)e^{-\alpha}(\alpha-2)}{(1-\alpha)e^{-(n_1+1)\alpha} e^{-\alpha}}}{e^{-2\alpha}(1-\alpha)^2} \\
&= \alpha^2 \frac{(n_1+1)^2 - \frac{(n_1+1)(\alpha-2)}{(1-\alpha)}}{(1-\alpha)^2} \\
&= \alpha^2 \frac{(n_1+1)^2(1-\alpha) - (n_1+1)(\alpha-2)}{(1-\alpha)^3}
\end{aligned}$$

and therefore

$$\begin{aligned}
\mathrm{Var} V_{SD} &= \mathbb{E}V_{SD}^2 - (\mathbb{E}V_{SD})^2 \\
&= \eta g^2(\alpha) h''(g(\alpha)) + \mathbb{E}V_{SD} - (\mathbb{E}V_{SD})^2 \\
&= \alpha^2 \frac{(n_1+1)^2(1-\alpha) - (n_1+1)(\alpha-2)}{(1-\alpha)^3} + (n_1+1)\frac{\alpha}{1-\alpha} - \left((n_1+1)\frac{\alpha}{1-\alpha}\right)^2 \\
&= \frac{(n_1+1)\alpha}{(1-\alpha)^3}.
\end{aligned}$$

Calculating the variance for the step-up case follows the same scheme. The only difference is that now $\eta = (1-\alpha)\exp(-n_1\alpha)$ and $c_i(x) = (n_1+i)^i x^i / i!$. Again, we get

$$\begin{aligned}
\eta h(g(\alpha)) &= 1 - q_{SU}(0) = 1 - (1-\alpha)\exp(-n_1\alpha), \\
\eta g(\alpha) h'(g(\alpha)) &= \mathbb{E}V_{SU} = (n_1 + \frac{1}{1-\alpha})\frac{\alpha}{1-\alpha}, \\
\eta g^2(\alpha) h''(g(\alpha)) &= \mathbb{E}[V_{SU}(V_{SU}-1)].
\end{aligned}$$

As before, we conclude from the chain rule

$$\begin{aligned}
\mathbb{E}V_{SU}^2 - \mathbb{E}V_{SU} &= \eta g^2(\alpha) \frac{(h \circ g)''(\alpha) - h'(g(\alpha)) g''(\alpha)}{(g'(\alpha))^2} \\
&= \eta g^2(\alpha) \frac{\frac{d^2}{d\alpha^2}(e^{n_1\alpha}/(1-\alpha) - 1) - \frac{(n_1+1/(1-\alpha))\alpha}{(1-\alpha)\eta g(\alpha)} g''(\alpha)}{(g'(\alpha))^2} \\
&= \frac{\alpha^2(n_1^2(1-\alpha)^2 + n_1(4-\alpha)(1-\alpha) + 4 - \alpha)}{(1-\alpha)^4}
\end{aligned}$$

and

$$\begin{aligned}
\text{Var}V_{SU} &= \mathbb{E}V_{SU}^2 - (\mathbb{E}V_{SU})^2 \\
&= \eta g^2(\alpha)h''(g(\alpha)) + \mathbb{E}V_{SU} - (\mathbb{E}V_{SU})^2 \\
&= \frac{\alpha^2(n_1^2(1-\alpha)^2 + n_1(4-\alpha)(1-\alpha) + 4 - \alpha)}{(1-\alpha)^4} + \mathbb{E}V_{SU} - (\mathbb{E}V_{SU})^2 \\
&= \frac{\alpha(n_1+1) - \alpha^2(n_1-1)}{(1-\alpha)^4}.
\end{aligned}$$

This is our claim. □

Remark 4.12
Note, that the techniques applied in the proof of Theorem 4.11 can be used to calculate $\mathbb{E}V_{n,SD}^{\rho}$ ($\mathbb{E}V_{n,SU}^{\rho}$) for arbitrary $\rho \in \mathbb{N}$.

With the exact and explicit formulas for $\mathbb{E}V_{n,SD}$ ($\mathbb{E}V_{n,SU}$) given in Theorem 4.2 we are able to derive some results for the case where $n_1 \to \infty$ and $n_0/n \to \zeta$ including the case $\zeta = 1$.

Corollary 4.13
Under DU(n, n_0) for $\alpha_n \to \alpha \in (0,1)$ and $n_0/n \to \zeta \in [0,1]$ it holds for $n_1 \to \infty$ that

$$\lim_{n\to\infty} \mathbb{E}V_{n,SD}/n_1 = \lim_{n\to\infty} \mathbb{E}V_{n,SU}/n_1 = \frac{\zeta\alpha}{1-\zeta\alpha}$$

and if $\lim_{n\to\infty} n_1 = N_1 < \infty$, then

$$\begin{aligned}
\mathbb{E}V_{n,SD} &\to (N_1+1)\frac{\alpha}{1-\alpha}, \\
\mathbb{E}V_{n,SU} &\to \left(N_1 + 1 + \frac{\alpha}{1-\alpha}\right)\frac{\alpha}{1-\alpha},
\end{aligned}$$

where $V_{n,SD}$ ($V_{n,SU}$) corresponds to a SD (SU) test induced by $r_n(t) = t/\alpha_n$.

Proof: Let $h_n(i) = \alpha_n^i \mathbb{I}_{\{i\le n-n_1\}} \prod_{k=0}^{i-1}(n-n_1-k)/n$ and μ denote the counting measure. Then, as in the proof of Theorem 4.11, we have

$$\mathbb{E}V_{n,SD} = (n_1+1)\sum_{i\in\mathbb{N}} \alpha_n^i \mathbb{I}_{\{i\le n_0\}} \prod_{k=0}^{i-1} \frac{n-n_1-k}{n} = (n_1+1)\int_{\mathbb{N}} h_n(i)\mu(di).$$

Suppose n_0 is bounded, then $\mathbb{E}V_{n,SD}/n_1 \to 0$ which equals $\zeta\alpha/(1-\zeta\alpha)$ since $\zeta = 0$. Now we assume that n_0 is not bounded. There exists an $\epsilon > 0$ and an $N \in \mathbb{N}$ such that $\alpha_n \le \alpha + \epsilon < 1$ for $n > N$. Obviously, $h_n(i) \le (\alpha+\epsilon)^i$ for $n > N$, $\int_{\mathbb{N}}(\alpha+\epsilon)^i\mu(di) < \infty$, and $h_n(i) \to \zeta^i\alpha^i$ for all $i \in \mathbb{N}$. Hence, the dominated convergence theorem yields

$$\lim_{n\to\infty} \mathbb{E}V_{n,SD}/n_1 = \lim_{n\to\infty}(1+1/n_1)\int_{\mathbb{N}} h_n(i)\mu(di) = \lim_{n\to\infty}(1+1/n_1)\int_{\mathbb{N}}(\zeta\alpha)^i\mu(di),$$

which is the assertion. For $\mathbb{E}V_{n,SU}$ the argumentation is similar and omitted here. □

Remark 4.14
Of course, if $N_1^l = \liminf_{n\to\infty} n_1 < \limsup_{n\to\infty} n_1 < \infty = N_1^u$, then $\liminf_{n\to\infty} \mathbb{E}V_{n,SD} = (N_1^l + 1)\frac{\alpha}{1-\alpha}$ and $\limsup_{n\to\infty} \mathbb{E}V_{n,SD} = (N_1^u + 1)\frac{\alpha}{1-\alpha}$. A similar assertion is true for $\mathbb{E}V_{n,SU}$.

In Corollary 4.13 we have calculated the asymptotic first moment of $V_{n,SD}$ $(V_{n,SU})$ for $\zeta \in [0,1]$. We explicitly used the structure of $\mathbb{E}V_{n,SD}$ $(\mathbb{E}V_{n,SU})$. If $\zeta < 1$, the asymptotic first moments can be calculated more easily as elucidated in the next remark. Further, this simple technique may be applied to other rejection curves beside Simes line t/α.

Remark 4.15
Under DU(n, n_0) with $n_0/n \to \zeta \in [0,1)$ and $\alpha \in (0,1)$, we have

$$\lim_{n\to\infty} \mathbb{E}\frac{V_{n,SD}}{n_1} = \lim_{n\to\infty} \mathbb{E}\frac{V_{n,SU}}{n_1} = \frac{\zeta\alpha}{1-\zeta\alpha}.$$

Obviously, $t_\zeta/\alpha = 1 - \zeta + \zeta t_\zeta$ is equivalent to $t_\zeta = (1-\zeta)\alpha/(1-\zeta\alpha)$. Hence, the assertion is a direct consequence of Theorem 2.10.

If $\zeta < 1$ the asymptotic distribution of a LSU procedure can easily be obtained from Corollary 2.21. After the following corollary we will heuristically compare the asymptotic variances we obtained for n_1 fixed and for $\zeta < 1$.

Corollary 4.16
Under DU(n, n_0) for $n_0/n = \zeta + o(n^{-1/2})$ with $\zeta \in (0,1)$ it holds that

$$\sqrt{n}\left(V_{n,SU}/n_1 - \frac{\zeta\alpha}{1-\zeta\alpha}\right) \to V \tag{4.5}$$

in distribution, where $V \sim N(0,\sigma^2)$ with

$$\sigma^2 = \frac{\zeta\alpha(1-\alpha)}{(1-\zeta)(1-\zeta\alpha)^4}.$$

Proof: Following Remark 2.22 we have to determine τ^* such that $r(\tau^*) = 1-\zeta+\zeta\tau^*$, where r is the rejection curve of the LSU test, i.e. $r(t) = t/\alpha$. Obviously, we have $\tau^* = (1-\zeta)\alpha/(1-\zeta\alpha)$ and hence the asymptotic normality as stated in (4.5) follows from Remark 2.22. Note that in Remark 2.22 $V_n(\varphi_n)/n$ was considered but we calculate the asymptotic distribution for $V_{n,SU}/n_1$. It remains to calculate the variance of V. Since the first derivative $r'(\tau^*)$ equals $1/\alpha$, we get from

Remark 2.22 that

$$\begin{aligned}
\mathrm{Var}[\lim_{n\to\infty} \sqrt{n}V_n/n_1] &= \frac{1}{(1-\zeta)^2}\mathrm{Var}[\lim_{n\to\infty} V_n/\sqrt{n}] \\
&= \frac{1}{(1-\zeta)^2}\frac{(r'(\tau^*))^2\zeta\tau^*(1-\tau^*)}{(r'(\tau^*)-\zeta)^2} \\
&= \frac{1}{(1-\zeta)^2}\frac{1/\alpha^2\zeta\frac{(1-\zeta)\alpha}{1-\zeta\alpha}\frac{1-\alpha}{1-\zeta\alpha}}{(1/\alpha-\zeta)^2} \\
&= \frac{1}{(1-\zeta)^2}\frac{\zeta\frac{(1-\zeta)\alpha}{1-\zeta\alpha}\frac{1-\alpha}{1-\zeta\alpha}}{(1-\zeta\alpha)^2} \\
&= \frac{\zeta\alpha(1-\alpha)}{(1-\zeta)(1-\zeta\alpha)^4}.
\end{aligned}$$

This proves the theorem. □

Remark 4.17

We have calculated the variance of the ENFR of the LSU test for the two cases n_1 fixed and $n_0/n \to \zeta \in (0,1)$. Comparing heuristically these two variances for fixed n_1 and "large" n shows that the variances are similar. Theorem 4.11 yields

$$\begin{aligned}
\mathrm{Var}[\sqrt{n}V_n/n_1] &= n/n_1^2\mathrm{Var}(V_n) \\
&\approx n/n_1^2\frac{\alpha(n_1+1)-\alpha^2(n_1-1)}{(1-\alpha)^4} \\
&\approx \frac{\alpha-\alpha^2}{n_1/n(1-\alpha)^4} \\
&= \frac{\alpha(1-\alpha)}{n_1/n(1-\alpha)^4}.
\end{aligned}$$

Corollary 4.16 provides us

$$\mathrm{Var}[\lim_{n\to\infty} \sqrt{n}V_n/n_1] = \frac{\zeta\alpha(1-\alpha)}{(1-\zeta)(1-\zeta\alpha)^4}.$$

Note that under the assumption of Theorem 4.11 that n_1 is fixed, one can consider $1-\zeta \approx 0 \approx n_1/n$ and $1-\zeta\alpha \approx 1-\alpha$.

4.2 ENFR and asymptotically optimal rejection curve (AORC)

Finner et al. [21] developed the AORC to control the FDR at exact level $\alpha \in (0,1)$. A large class of SUD tests, but no SU test, induced by $f_\alpha(t) = t/[t(1-\alpha)+\alpha]$ yields that the FDR under DU configurations converges to α if $n_0/n \to \zeta > \alpha$. Thus, different methods have been proposed to achieve FDR control for finite n. One method is to consider $f_{\alpha,\beta_n} = (1+\beta_n/n)f_\alpha$, where $\beta_n > 0$.

This new rejection curve is called the β_n*-adjusted AORC*. Choosing $\beta_n = 1$ yields control of the FDR for finite n under BIA if a SD test induced by the β_n-adjusted AORC is used, see [24]. In this chapter, we will see that this procedure also controls the ENFR at level $\alpha/(1-\alpha)$. Again, we assume BIA and suppress the subscript that indicates a DU configuration, that is $\mathbb{P} = \mathbb{P}_{n,n_0}$ and $\mathbb{E} = \mathbb{E}_{n,n_0}$ The critical values of the β_n-adjusted AORC with $\beta_n = 1$ are denoted by

$$c_i = \frac{i\alpha}{n+1-i(1-\alpha)}, \quad i = 1, \dots, n.$$

Recall, we set $c_{n+1} = 1$.

Theorem 4.18

The SD test induced by the β_n*-adjusted AORC with* $\beta_n = 1$ *fulfills*

$$\mathbb{E}V_n \leq ((1-\mathbb{P}(V_n = n_0))(n_1+1) - n_0\mathbb{P}(V_n = n_0))\frac{\alpha}{1-\alpha} \leq (n_1+1)\frac{\alpha}{1-\alpha}$$

under DU(n, n_0) *with* $\alpha \in (0, 1)$.

Proof: The proof of the theorem is longish but the single steps are very elementary and the techniques are similar to those in [24]. We abuse the notation slightly by setting $c_{n_1} = 0$ in order to simplify the formulas. The case $n_0 = 0$ is trivial, since $V_n = 0$ almost surely in this case. Let $p_1, \dots, p_{n_0}$ denote the p-values corresponding to true null hypotheses and $\varphi_n = (\phi_1, \dots, \phi_n)$ the SD test induced by the β_n-adjusted AORC. For $n_0 = 1$, we note that $\mathbb{E}V_n = \mathbb{P}(V_n = n_0) = c_n = n\alpha/(1+n\alpha)$ and

$$\begin{aligned}
&[(1-\mathbb{P}(V_n = n_0))(n_1+1) - n_0\mathbb{P}(V_n = n_0)]\frac{\alpha}{1-\alpha} \\
&= ((1-c_n)n - c_n)\frac{\alpha}{1-\alpha} \\
&= (\frac{n}{1+n\alpha} - \frac{n\alpha}{1+n\alpha})\frac{\alpha}{1-\alpha} \\
&= c_n.
\end{aligned}$$

Thus, we assume $n_0 > 1$ for the rest of the proof.

It holds that

$$\mathbb{E}V_n = n_0\mathbb{E}\phi_{n_0} = n_0\sum_{k=1}^{n_0}\mathbb{P}(\phi_{n_0} = 1, V_n = k) = n_0\sum_{k=1}^{n_0}\mathbb{P}(p_{n_0} \leq c_{n_1+k}, V_n = k). \quad (4.6)$$

Further we have, using $c_{n+1} = 1$,

$$\sum_{k=1}^{n_0} \mathbb{P}\left(p_{n_0} \le c_{n_1+k},\ V_n = k\right)$$

$$\le \sum_{k=1}^{n_0} \mathbb{P}\left(p_{n_0} \le c_{n_1+k+1},\ V_n = k\right)$$

$$= \sum_{k=1}^{n_0} \mathbb{P}\left(p_{n_0} \le c_{n_1+k+1},\ V_n \ge k\right) - \sum_{k=1}^{n_0-1} \mathbb{P}\left(p_{n_0} \le c_{n_1+k+1},\ V_n \ge k+1\right)$$

$$= \sum_{k=0}^{n_0-1} \mathbb{P}\left(p_{n_0} \le c_{n_1+k+2},\ V_n \ge k+1\right) - \sum_{k=1}^{n_0-1} \mathbb{P}\left(p_{n_0} \le c_{n_1+k+1},\ V_n \ge k+1\right)$$

$$= \mathbb{P}\left(p_{n_0} \le c_{n_1+2},\ V_n \ge 1\right) + \sum_{k=1}^{n_0-1} \mathbb{P}\left(V_n \ge k+1,\ c_{n_1+k+1} < p_{n_0} \le c_{n_1+k+2}\right)$$

$$= \mathbb{P}\left(p_{n_0} \le c_{n_1+2},\ V_n \ge 1\right) + \sum_{k=2}^{n_0} \mathbb{P}\left(V_n \ge k,\ c_{n_1+k} < p_{n_0} \le c_{n_1+k+1}\right)$$

$$= \mathbb{P}\left(p_{n_0} \le c_{n_1+1}, V_n \ge 1\right) + \sum_{k=1}^{n_0} \mathbb{P}\left(V_n \ge k,\ c_{n_1+k} < p_{n_0} \le c_{n_1+k+1}\right)$$

$$= \sum_{k=0}^{n_0} \mathbb{P}\left(V_n \ge k,\ c_{n_1+k} < p_{n_0} \le c_{n_1+k+1}\right),$$

where the last equality is due to $c_{n_1} = 0$ and

$$\{0 < p_{n_0} \le c_{n_1+1}\} \cap \{V_n \ge 1\} = \{0 < p_{n_0} \le c_{n_1+1}\} \cap \{V_n \ge 0\}.$$

So we get from (4.6) the bound

$$\mathbb{E}V_n \le n_0 \sum_{k=0}^{n_0} \mathbb{P}\left(V_n \ge k,\ c_{n_1+k} < p_{n_0} \le c_{n_1+k+1}\right). \tag{4.7}$$

We now take a closer look at the events appearing in (4.7). Let V_n' be defined by $\{V_n' = k\}$ if and only if $\{p_{1:n_0-1} \le c_{n_1+1}, \dots, p_{k:n_0-1} \le c_{n_1+k},\ p_{k+1:n_0-1} > c_{n_1+k+1}\}$ for all $k = 1, \dots, n_0 - 1$ with $p_{n_0:n_0-1} = 1$, and $\{V_n' = 0\}$ if and only if $\{p_{1:n_0-1} > c_{n_1+1}\}$. By definition, V_n' represents the number of rejected true null hypotheses after discarding p_{n_0} but still using $c_1, \dots, c_{n-1}$ as critical values. For $k = 1, \dots, n_0$ we have

$$\{V_n \ge k,\ c_{n_1+k} < p_{n_0} \le c_{n_1+k+1}\}$$

$$= \{p_{1:n_0} \le c_{n_1+1}, \dots, p_{k:n_0} \le c_{n_1+k},\ c_{n_1+k} < p_{n_0} \le c_{n_1+k+1}\}$$

$$= \{p_{1:n_0-1} \le c_{n_1+1}, \dots, p_{k:n_0-1} \le c_{n_1+k},\ c_{n_1+k} < p_{n_0} \le c_{n_1+k+1}\}$$

$$= \left\{V_n' \ge k,\ c_{n_1+k} < p_{n_0} \le c_{n_1+k+1}\right\}. \tag{4.8}$$

By (4.8) and noting that $\{V_n \geq 0\} = \{V_n' \geq 0\}$, (4.7) can be reformulate as

$$\mathbb{E}V_n \leq n_0 \sum_{k=0}^{n_0} \mathbb{P}\left(V_n' \geq k,\ c_{n_1+k} < p_{n_0} \leq c_{n_1+k+1}\right).$$

Since $\{V_n' \geq k\}$ and $\{c_{n_1+k} < p_{n_0} \leq c_{n_1+k+1}\}$ are independent events, noting that $c_{n_1} = 0$ and $\{V_n' \geq n_0\} = \emptyset$, the sum can be further transformed to

$$\begin{aligned}
&\sum_{k=0}^{n_0} \mathbb{P}\left(V_n' \geq k,\ c_{n_1+k} < p_{n_0} \leq c_{n_1+k+1}\right) \\
&\quad = \sum_{k=0}^{n_0} \mathbb{P}\left(V_n' \geq k\right)\left(c_{n_1+k+1} - c_{n_1+k}\right) \\
&\quad = \sum_{k=0}^{n_0-1} \mathbb{P}\left(V_n' \geq k\right) c_{n_1+k+1} - \sum_{k=1}^{n_0} \mathbb{P}\left(V_n' \geq k\right) c_{n_1+k} \\
&\quad = \sum_{k=1}^{n_0} \mathbb{P}\left(V_n' \geq k-1\right) c_{n_1+k} - \sum_{k=1}^{n_0} \mathbb{P}\left(V_n' \geq k\right) c_{n_1+k} \\
&\quad = \sum_{k=1}^{n_0} \mathbb{P}\left(V_n' = k-1\right) c_{n_1+k}.
\end{aligned}$$

Note that $c_k = \alpha k(1-c_k)/(n-k+1)$ entails

$$\begin{aligned}
\mathbb{E}V_n &\leq n_0 \sum_{k=1}^{n_0} \mathbb{P}\left(V_n' = k-1\right) c_{n_1+k} \\
&= n_0 \sum_{k=1}^{n_0} \alpha \frac{(n_1+k)(1-c_{n_1+k})}{n-(n_1+k)+1} \mathbb{P}\left(V_n' = k-1\right) \\
&= \alpha n_0 \sum_{k=1}^{n_0} \frac{(n_1+k)}{n-(n_1+k)+1} \mathbb{P}\left(V_n' = k-1,\ p_{n_0} > c_{n_1+k}\right) \\
&= \alpha n_0 \sum_{k=1}^{n_0} \frac{(n_1+k)}{n-(n_1+k)+1} \mathbb{P}\left(V_n = k-1,\ p_{n_0} > c_{n_1+k}\right), \qquad (4.9)
\end{aligned}$$

where the second equality follows from independence of the events $\{V_n' = k-1\}$ and $\{p_{n_0} > c_{n_1+k}\}$, and the last equality follows from a similar argumentation as for (4.8). Considering now the probability in (4.9), we see that

$$\begin{aligned}
\mathbb{P}\left(V_n = k-1,\ p_{n_0} > c_{n_1+k}\right) &= \mathbb{P}\left(p_{n_0} > c_{n_1+k} | V_n = k-1\right) \mathbb{P}\left(V_n = k-1\right) \\
&= \mathbb{P}\left(\phi_{n_0} = 0 | V_n = k-1\right) \mathbb{P}\left(V_n = k-1\right) \\
&= \frac{n_0-(k-1)}{n_0} \mathbb{P}\left(V_n = k-1\right).
\end{aligned}$$

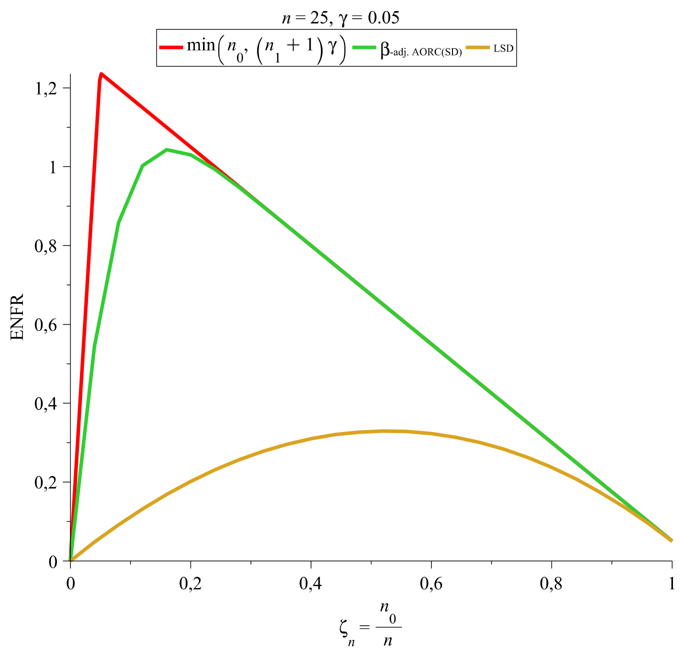

Figure 4.1: *Comparison of the ENFR (exactly calculated) under DU configurations for the LSD test and the SD test based on* β_n*-adjusted AORC with* $\beta_n \equiv 1$ *for* $n = 25$ *and* $\gamma = 0.05$.

Altogether, we get that

$$\begin{aligned}
\mathbb{E}V_n &\leq \alpha n_0 \sum_{k=1}^{n_0} \frac{(n_1+k)}{n-(n_1+k)+1} \mathbb{P}\left(V_n = k-1,\ p_{n_0} > c_{n_1+k}\right) \\
&= \alpha n_0 \sum_{k=1}^{n_0} \frac{(n_1+k)}{n-(n_1+k)+1} \frac{n_0-k+1}{n_0} \mathbb{P}\left(V_n = k-1\right) \\
&= \alpha \sum_{k=1}^{n_0} (n_1+1+k-1) \mathbb{P}\left(V_n = k-1\right) \\
&= \alpha \left((n_1+1)(1-\mathbb{P}\left(V_n = n_0\right)) + \mathbb{E}V_n - n_0 \mathbb{P}\left(V_n = n_0\right)\right).
\end{aligned}$$

This is equivalent to

$$\mathbb{E}V_n \leq ((1-\mathbb{P}\left(V_n = n_0\right))(n_1+1) - n_0 \mathbb{P}\left(V_n = n_0\right)) \frac{\alpha}{1-\alpha}. \qquad \square$$

As Figure 4.1 illustrates, the ENFR of this SD procedure nearly perfectly fits the ENFR bounding curve. But a SU test is preferred to a SD test because if $p_{1:n} > c_1$, the SD test would immediately stop without rejecting any null hypothesis. Therefore, it is also interesting to study how β_n has to be chosen for a SU test in order to achieve control of the ENFR for finite n.

Gontscharuk [26] studied the behavior of β_n for finite control of the FDR. She showed under DU configurations for SU tests and the optimal β_n^* that $\beta_n^* \to \infty$ and $\beta_n^*/n = o(1)$, cf. Lemma 3.24

and 3.25 in [26]. A β_n^* is optimal for the FDR if the SU test induced by the β_n^*-adjusted AORC is less or equal than α for all $0 \leq n_0 \leq n$ and equal α for at least one n_0. Theorem 3.2 together with Theorem 4.22 below provides an alternative proof of $\beta_n^*/n = o(1)$.
We say that β_n^* *is optimal with respect to linear ENFR control* if the SU test induced by the β_n^*-adjusted AORC is less or equal than $(n_1+1)\alpha/(1-\alpha)$ for all $0 \leq n_0 \leq n$ and equal $(n_1+1)\alpha/(1-\alpha)$ for at least one n_0. In contrast to the FDR, we will only obtain $M^* \geq \beta_n^*/n \geq M_* > 0$ and not $\beta_n^*/n = o(1)$. Obviously, we also have $\beta_n^* \to \infty$.
Till the end of this section, if not stated otherwise, we denote by ENFR$_{n,n_0}(r)$ the ENFR under DU configurations of a SU test induced by the rejection curve r. Let f_{α,β_n} denote the β_n-adjusted AORC.

Lemma 4.19
There exists a $k > 0$ such that ENFR$_{n,n_0}(f_{\alpha,nk}) \leq (n_1+1)\alpha/(1-\alpha)$ for all $n_1, n \in \mathbb{N}$ with $n_1 < n$ and $\alpha \in (0,1)$.

Proof: Let $\gamma = \alpha/(1-\alpha)$. Choosing $\tilde{\gamma} = \sqrt{\gamma + 1/4} - 1/2$ yields $(n_1+1+\tilde{\gamma})\tilde{\gamma} \leq (n_1+1)\gamma$ for all $0 \leq n_1 \leq n$. By virtue of Theorem 4.11, we conclude for the SU case that ENFR$_{n,n_0}(s^*) \leq (n_1+1)\gamma$, where $s^*(t) = t(1+\tilde{\gamma})/\tilde{\gamma}$. Setting $\tilde{\alpha} = \tilde{\gamma}/(1+\tilde{\gamma})$ we see that $f_{\alpha,nk}(t) \geq t/\tilde{\alpha}$ for $t \in [0,1]$ is equivalent to $f_{\alpha,nk}(1) \geq 1/\tilde{\alpha}$ which holds iff $k \geq (1-\tilde{\alpha})/\tilde{\alpha}$. For such k we have ENFR$_{n,n_0}(f_{\alpha,nk}) \leq$ ENFR$_{n,n_0}(s^*) \leq (n_1+1+\tilde{\gamma})\tilde{\gamma} \leq (n_1+1)\gamma$. □

Lemma 4.20
Let $\alpha_n \in (0,1)$ and $\beta_n > 0$ with $\alpha_n \to \alpha$ and $\beta_n/n \to 0$. Then, $\liminf_{n\to\infty}$ ENFR$_{n,n_0}(f_{\alpha_n,\beta_n}) > (n_1+1)\alpha/(1-\alpha)$ holds for fixed $n_1 \in \mathbb{N}$ and $\alpha \in (0,1)$.

Proof: Let $s_n(t) = t(1+\beta_n/n)/\alpha_n$. Since the derivative $f'_{\alpha_n,\beta_n}(0)$ equals $(1+\beta_n/n)/\alpha_n$, we have $s_n(t) \geq f_{\alpha_n,\beta_n}(t)$ for $t \geq 0$. By Corollary 4.13, we conclude

$$\text{ENFR}_{n,n_0}(f_{\alpha_n,\beta_n}) \geq \text{ENFR}_{n,n_0}(s_n) \to (n_1+1+\gamma)\gamma > (n_1+1)\gamma,$$

with $\gamma = \alpha/(1-\alpha)$. □

We summarize Lemma 4.19 and 4.20 as the following theorem.

Theorem 4.21
Let $\alpha \in (0,1)$. A SU test induced by the β_n-adjusted AORC, the optimal β_n^* with respect to linear ENFR control fulfills $0 < M_* \leq \beta_n^*/n \leq M^* < \infty$ for $n \geq N$ and appropriate constants N, M_*, and M^* possibly depending on α.

Proof: This is a direct consequence of Lemma 4.19 and 4.20. □

Changing the bounding function from $(n_1+1)\gamma$ to $(n_1+1+\gamma)\gamma$ yields $\beta_n^*/n \to 0$, where $\gamma = \alpha/(1-\alpha)$.

Theorem 4.22

Let $\alpha \in (0,1)$*,* $\gamma = \alpha/(1-\alpha)$*, and* $\beta_n^* > 0$ *such that ENFR*$_{n,n-n_1^*}(f_{\alpha,\beta_n^*}) = (n_1^*+1+\gamma)\gamma$ *for some* $n_1^* = n_1^*(n)$ *and ENFR*$_{n,n_0}(f_{\alpha,\beta_n^*}) \leq (n_1+1+\gamma)\gamma$ *for all* $n_1 \in \{0,\dots,n\}$*. Then* $\beta_n^*/n \to 0$*.*

Proof: Suppose $\beta_n^*/n \to b > 0$ and define $n_0^* = n - n_1^*$. In each case we will show that the assumption $b > 0$ contradicts the assertion that $\text{ENFR}_{n,n-n_1^*}(f_{\alpha,\beta_n^*}) = (n_1^*+1+\gamma)\gamma$ for all $n \in \mathbb{N}$.

Case 1: $\limsup_{n\to\infty} n_0^* < \infty$*:*

Suppose $\limsup_{n\to\infty} n_0^* < \infty$, then $\limsup_{n\to\infty} \text{ENFR}_{n,n_0^*}(f_{\alpha,\beta_n^*}) < \infty = \lim_{n\to\infty}(n_1^*+1+\gamma)\gamma$. Hence, we assume that n_0^* is not bounded in n.

Case 2: $\limsup_{n\to\infty} n_0^*/n \in [0,1)$, n_0^* *not bounded:*

If $\limsup_{n\to\infty} n_0^*/n = \zeta^* < 1$ we can assume w.l.o.g. that $n_0^*/n \to \zeta^*$. If $\zeta^* = 0$, then as before we get $\lim_{n\to\infty} \text{ENFR}_{n,n_0^*}(f_{\alpha,\beta_n^*})/n = 0 < 1 = \lim_{n\to\infty}(n_1^*+1+\gamma)\gamma/n$. Hence, we assume that $\zeta^* > 0$.

By the Theorem of Glivenko-Cantelli and uniform convergence of f_{α,β_n^*} to $f_{\alpha,b}$ we get

$$\lim_{n\to\infty} \text{ENFR}_{n,\tilde{n}_0^*}(f_{\alpha,\tilde{\beta}_n^*})/\tilde{n}_1^* = \lim_{n\to\infty} \text{ENFR}_{n,\tilde{n}_0^*}(f_{\alpha,b})/\tilde{n}_1^* = \frac{\zeta^*}{1-\zeta^*} t_{\zeta^*}, \tag{4.10}$$

where t_{ζ^*} is the unique point with $f_{\alpha,b}(t_{\zeta^*}) = 1 - \zeta^* + \zeta^* t_{\zeta^*}$, cf. Theorem 2.10. Note that the ENFR in (4.10) is the ENFR of the SU procedure induced by f_{α,β_n^*} but the limit in (4.10) does not change if we conduct a SD test instead of a SU test. For $t_{\zeta^*}^0 = \min(\gamma(1-\zeta^*)/\zeta^*, 1)$ we get by the Theorem of Glivenko-Cantelli

$$\lim_{n\to\infty} \text{ENFR}_{n,n_0^*}(f_{\alpha,0})/n_1^* = \frac{\zeta^*}{1-\zeta^*} t_{\zeta^*}^0 = \min(\gamma, \frac{\zeta^*}{1-\zeta^*}), \tag{4.11}$$

where $\text{ENFR}_{n,n_0^*}(f_{\alpha,0})$ is the ENFR of a *SD test* induced by $f_{\alpha,0}$. Note, a SU test induced by $f_{\alpha,0}$ will always reject all null hypotheses. Since $f_{\alpha,b} = (1+b)f_{\alpha,0}$, we get $t_{\zeta^*} < t_{\zeta^*}^0$ and thus

$$\lim_{n\to\infty} \text{ENFR}_{n,\tilde{n}_0^*}(f_{\alpha,\tilde{\beta}_n^*})/n_1^* < \lim_{n\to\infty} \text{ENFR}_{n,\tilde{n}_0^*}(f_{\alpha,0})/n_1^* \leq \gamma$$

by (4.10) and (4.11). Recall, (4.10) holds for the SU and SD case and (4.11) only for the SD case.

Case 3: $\limsup_{n\to\infty} n_0^*/n = 1$*:*

W.l.o.g. let $n_0^*/n \to 1$. Denote by $F_{n_0}^0$ the ecdf of $\{p_i\}_{i\in I_0}$. By the DKW inequality [54], p. 12 we have

$$\mathbb{P}\left(\sup_{t\in[0,1]} F_{n_0^*}^0(t) - t > \epsilon_n\right) \leq C\exp(-2n_0^*\epsilon_n^2) = C(1/n_0^*)^2, \tag{4.12}$$

where $\epsilon_n = \sqrt{\log(n_0^*)/n_0^*} = o(1)$ and $C > 0$. Obviously, we have $F_{n_0^*}^0(t) - t = (F_n(t) - n_1^*/n - tn_0^*/n)n/n_0^*$. Let $A_n = \{\sup_{t\in[0,1]}(F_n(t) - n_1^*/n - tn_0^*/n) \leq \epsilon_n n_0^*/n\}$ and $t_n \in (0,1)$ the unique point such that $f_{\alpha,\beta_n^*}(t_n) = n_1^*/n + (t_n+\epsilon_n)n_0^*/n$. Since $n_1^*/n \to 0$ and $\epsilon_n \to 0$

we have $t_n \to 0$. Further, let $a_n = (f_{\alpha,\beta_n^*}(t_n) - f_{\alpha,\beta_n^*}(0))/t_n$ and define $s_n(t) = a_n t$. Since $\beta_n^*/n \to b > 0$, we get that $a_n \to (1+b)/\alpha$, which is the derivative of $f_{\alpha,b}(t)$ at $t = 0$. Hence, $s_n(t)$ converges uniformly to $s(t) = (1+b)t/\alpha$. We have,

$$\begin{aligned}\text{ENFR}_{n,n_0^*}(f_{\alpha,\beta_n^*}) &= \mathbb{E}[V_n(f_{\alpha,\beta_n^*})\mathbb{I}_{\{A_n^c\}}] + \mathbb{E}[V_n(f_{\alpha,\beta_n^*})\mathbb{I}_{\{A_n\}}] \\ &\le C/n_0^* + \text{ENFR}_{n,n_0^*}(s_n),\end{aligned}$$

where $V_n(f_{\alpha,\beta_n^*})$ is the number of rejected true null hypotheses using a SU test induced by f_{α,β_n^*}. If $\limsup_{n\to\infty} n_1^* = \infty$, then Corollary 4.13 provides

$$\limsup_{n\to\infty} \text{ENFR}_{n,n_0^*}(s_n)/n_1^* = \limsup_{n\to\infty} \text{ENFR}_{n,n_0^*}(s)/n_1^* = \tilde{\gamma},$$

with $\tilde{\gamma} = (\alpha/(1+b))/(1-\alpha/(1+b)) < \alpha/(1-\alpha) = \gamma$.
If $\limsup_{n\to\infty} n_1^* < \infty$, then choose a subsequence $\tilde{n}_1^*$ ($\tilde{n}_0^*$) such that $\tilde{n}_1^* = \limsup_{n\to\infty} n_1^*$ holds true. This entails

$$\limsup_{n\to\infty} \text{ENFR}_{n,\tilde{n}_0^*}(s_n) = \limsup_{n\to\infty} \text{ENFR}_{n,\tilde{n}_0^*}(s) = (\tilde{n}_1^* + 1 + \tilde{\gamma})\tilde{\gamma},$$

with $\tilde{\gamma} = (\alpha/(1+b))/(1-\alpha/(1+b)) < \alpha/(1-\alpha) = \gamma$. □

4.3 Linear Plug-in test

The plug-in approach has a long tradition in statistics. Probably most famous in this context is the t-test, which evolves from a standardized test statistic $(\bar{x} - \mu)/\sigma$ by substituting the standard deviation σ by the empirical standard deviation.
Schweder and Spjøtvoll introduced and discussed in [51] an estimator $\hat{n}_0$ for n_0. In that paper they mentioned that the FWER is controlled at level α if a Bonferroni test with α/n_0 instead of α/n is applied. In general, n_0 is unknown and thus they proposed to use $\alpha/\hat{n}_0$ but did not give a proof that this procedure actually controls the FWER. Finner and Gontscharuk showed under some regularity conditions in [22] that a slightly different version of the estimator proposed by Schweder and Spjøtvoll controls the FWER at a predefined level.
Under some regularity conditions the FDR of the LSU procedure equals $\alpha n_0/n$. Hence, increasing α to $\alpha n/n_0$ is desirable. Procedures using $\hat{r}(t) = (\hat{n}_0/n)t/\alpha$ are sometimes called *adaptive BH-procedures*. A thorough answer to the question for which estimator $\hat{n}_0$ such adaptive BH-procedures still control the FDR at level α is given in [50].
In this section, we apply the plug-in methodology in order to define a stopping time τ_n such that $\mathbb{E}_\vartheta[V_n(\tau_n)] \le (n_1+1)\gamma$. We start with some heuristic considerations. Suppose we have an estimate $\hat{n}_0$ for n_0 and reject any null hypothesis with a p-value below the fixed threshold t. Thus, it is reasonable to estimate $\mathbb{E}V_n(t)$ by $\hat{n}_0 t$. After estimating n_1 by $\hat{n}_1$ we can try to choose t as large as possible, that is $\tau_n = \sup\{t \in [0,1] : \hat{n}_0 t \le (\hat{n}_1 + 1)\gamma\}$. One way of estimating

n_1 is $\hat{n}_1 = n - \hat{n}_0$, then $\tau_n = \gamma(n - \hat{n}_0 + 1)/\hat{n}_0$. Obviously, if we underestimate n_0, we will overestimate n_1. A little example illustrates this drawback. Suppose $n = n_0 = 100$, then $\tau_n = \gamma/100$ would be a correct threshold, but $\hat{n}_0 = 95$ will yield $\tau_n = 6\gamma/95$. This is more than 6 fold above $\gamma/100$. Evidently, one way to handle this problem is to underestimate n_1 and overestimate n_0. Suppose the p-values follow the mixture distribution $F(t) = tn_0/n + F_1(t)n_1/n$ and $\mathbb{E}\hat{n}_0 = n_0$. We now define a "conservative" estimator for n_1. Instead choosing $\hat{n}_1 = n - \hat{n}_0$, we define $\hat{n}_1(t) = R(t) - \hat{n}_0 t - 1$. Then for fixed t we have $\mathbb{E}(R(t) - \hat{n}_0 t) = n_1 F_1(t) \leq n_1$. Hence, $\hat{n}_1 = R(t) - \hat{n}_0 t - 1$ is a "conservative" estimator for n_1. Note, for $t = 1$ we get $R(1) - \hat{n}_0 = n - \hat{n}_0$. Defining τ_n as before we get

$$\begin{aligned}\tau_n &= \sup\{t \in [0,1] : \hat{n}_0 t \leq (\hat{n}_1(t) + 1)\gamma\} \\ &= \sup\{t \in [0,1] : \hat{n}_0 t \leq (R(t) - \hat{n}_0 t)\gamma\} \\ &= \sup\left\{t \in [0,1] : \frac{\hat{n}_0 t(1+\gamma)}{\gamma} \leq R(t)\right\} \\ &= \sup\left\{t \in [0,1] : \frac{\hat{\zeta}_n t(1+\gamma)}{\gamma} \leq F_n(t)\right\},\end{aligned}$$

where $\hat{\zeta}_n = \hat{n}_0/n$. Thus, rejecting any null hypothesis H_i with $p_i \leq \tau_n$ is equivalent to a linear step-up procedure with rejection curve $r(t|\hat{\zeta}_n) = \hat{\zeta}_n t/\alpha$, where $\alpha = \frac{\gamma}{1+\gamma}$. This is an adaptive BH-procedure. Recall that our aim is to create a procedure controlling the ENFR at a specific level, but our heuristic considerations also lead to procedures developed for FDR control. Since every SU test can also easily be converted to a SD test, we are going to investigate both variants and call them *linear step-down/step-up plug-in (LSDPI/LSUPI)* procedures.
One appealing estimator for n_0 is

$$(n - R(\lambda))/(1-\lambda),$$

where $\lambda \in [0, 1)$ is a fixed tuning parameter. This estimator was proposed by Schweder and Spjøtvoll [51] nearly 30 years ago. For a slightly different estimator, that is,

$$\hat{n}_0 = (n - R(\lambda) + 1)/(1-\lambda), \tag{4.13}$$

Storey, Taylor and Siegmund proved that a SU test induced by $r(t|\hat{n}_0/n) = (\hat{n}_0/n)t/\alpha$ controls the FDR at level α, cf. [65]. Although the estimator (4.13) is essentially the estimator proposed by Schweder and Spjøtvoll, the common name is Storey's estimator. It has the property that if $F_1(\lambda) = 1$, then

$$\mathbb{E}\hat{n}_0 = (n - n_1 - n_0\lambda)/(1-\lambda) + 1/(1-\lambda) = n_0 + 1/(1-\lambda).$$

Otherwise, we have $\mathbb{E}\hat{n}_0 > n_0 + 1/(1-\lambda)$. This means that on average we overestimate n_0 and underestimate n_1. In this chapter we will see that LSDPI and LSUPI using Storey's estimator, in general, only provide asymptotic ENFR control, but the violations are negligible for practical purpose. Again, by assuming BIA the DU configurations are LFC's for the ENFR because $R(\lambda)$ increases if we set the p-values corresponding to false null hypotheses to zero. Thus,

$$\tau_n = \sup\left\{t \in [0,1] : \frac{n - R(\lambda) + 1}{n(1-\lambda)}\frac{1+\gamma}{\gamma}t \leq F_n(t)\right\} \tag{4.14}$$

increases if $R(\lambda)$ increases. Hence, under BIA the DU configurations are LFC for the ENFR. Like before, we suppress the subscript that indicates DU configuration, that is $\mathbb{P} = \mathbb{P}_{n,n_0}$. In the following, we are going to investigate the ENFR-Plug-in-Algorithm which has four major steps:

1. Fix $\lambda \in (0,1)$.
2. Estimate n_0 by $\hat{n}_0 = (n - R(\lambda) + 1)/(1-\lambda)$.
3. Calculate τ_n as defined in (4.14).
4. Reject any null hypothesis H_i with $p_i \leq \tau_n$.

4.3.1 Exact formulas

We now derive formulas for $\mathbb{P}\left(V_n(\tau_n) \leq i\right)$ for the LSDPI and LSUPI procedure based on Storey's estimator.
A common choice for λ is $1/2$ or even larger and one hopes that $\{p_i > \lambda\}$ is a very rare event for $i \in I_1$. Thus it might be reasonable not to reject any null hypothesis with a p-value larger than λ. Therefore, we also derive formulas for $\mathbb{P}\left(V_n(\tau_n \wedge \lambda) \leq i\right)$ for the LSDPI and LSUPI.
Throughout this section denote the (random) critical values by

$$\hat{c}_i = \frac{\alpha}{\hat{n}_0/n}\frac{i}{n} = \frac{\alpha i(1-\lambda)}{n - R_n(\lambda) + 1} = \frac{\alpha i(1-\lambda)}{n_0 - V_n(\lambda) + 1} \tag{4.15}$$

and by $c_i^s = \alpha i(1-\lambda)/(n_0 - s + 1)$ the critical values given $V_n(\lambda) = s$. Since we consider n fixed in this section, we will suppress the index n of the stopping time τ_n. The following facts will be used later and are very elementary, so that proofs are omitted.
Let $U_1, \ldots, U_n$ iid $U[0,1]$, $U_1', \ldots, U_n'$ iid $U[0,\lambda]$, and $U_1'', \ldots, U_n''$ iid $U[\lambda,1]$, then

$$\mathbb{P}^{(U_1,\ldots,U_n)|U_1\leq\lambda,\ldots,U_n\leq\lambda} = \mathbb{P}^{(U_1',\ldots,U_n')} \tag{4.16}$$

$$\mathbb{P}^{(U_1,\ldots,U_n)|U_1>\lambda,\ldots,U_n>\lambda} = \mathbb{P}^{(U_1'',\ldots,U_n'')}. \tag{4.17}$$

A very helpful notation is the following. Let $J \subset \{1,\ldots,n\}$. We denote by $X_{i:J}$ the ith order statistic of $\{X_j\}_{j\in J}$.

Remark 4.23

Given that $X_1, \ldots, X_s \le \lambda$ and $X_{s+1}, \ldots, X_n > \lambda$ we have

$$X_{i:n} = \begin{cases} X_{i:J_s}, & \text{if } i \le s, \\ X_{i-s:J_s'}, & \text{if } i > s, \end{cases}$$

where $J_s = \{1, \ldots, s\}$ and $J_s' = \{s+1, \ldots, n_0\}$.

The following theorem is stated in terms of the joint cdf of order statistics. Let $U_{1:n} \le \ldots \le U_{n:n}$ denote the order statistics of n independent and uniformly distributed random variables $U_1, \ldots, U_n$ on $[0,1]$. We define for $n \in \mathbb{N}$ and $0 \le k \le n-1$,

$$F_n^k(x_1, \ldots, x_{n-k}) = \mathbb{P}\left(U_{1:n} \le x_1, \ldots, U_{n-k:n} \le x_{n-k}\right), \tag{4.18}$$

$$F_0^0 \equiv 1. \tag{4.19}$$

Remark 4.24

Bolshev´s recursion can be applied to evaluate (4.18), see Shorack and Wellner [54], p. 366-367.

Remark 4.25

Let $U_1, \ldots, U_n \sim U[0,1]$ be independent, then

$$\begin{aligned} F_n^k(1-x_n, \ldots, 1-x_{k+1}) &= \mathbb{P}\left(U_{1:n} \le 1-x_n, \ldots, U_{n-k:n} \le 1-x_{k+1}\right) \\ &= \mathbb{P}\left(U_{n:n} > x_n, \ldots, U_{k+1:n} > x_{k+1}\right). \end{aligned}$$

In the proofs of this section we assume that $p_1, \ldots, p_{n_0}$ are the p-values corresponding to the true null hypotheses.

Theorem 4.26

In a DU(n, n_0)-model we have for the LSUPI test that $\mathbb{P}\left(V_n(\tau) \le n_0\right) = 1$ and for $0 \le i < n_0$ we get

$$\begin{aligned} \mathbb{P}\left(V_n(\tau) \le i\right) = &\sum_{s=0}^{i} b_s F_{n_0-s}^{i-s}\left(d_n^s, \ldots, d_{n-(n_0-i)+1}^s\right) \\ &+ \sum_{s=i+1}^{n_0} b_s F_{n_0-s}^0\left(d_n^s, \ldots, d_{n-(n_0-s)+1}^s\right) F_s^i\left(a_{n-(n_0-s)}^s, \ldots, a_{n-(n_0-i)+1}^s\right) \end{aligned}$$

with $b_s = \binom{n_0}{s}\lambda^s(1-\lambda)^{n_0-s}$, $a_i^s = 1 - \frac{c_i^s}{\lambda}$ and $d_i^s = 1 - \frac{c_i^s - \lambda}{1-\lambda}$.

Proof: First note that $\{V_n(\lambda) = s\}$ is the disjoint union $\bigcup_{I\subset\{1,\ldots,n_0\},|I|=s} A_I$ where

$$A_I = \{p_i \le \lambda, i \in I,\ p_j > \lambda, j \in \{1,\ldots,n_0\}\backslash I\}.$$

We write $A_0 = \{p_i > \lambda, i = 1,\ldots,n_0\}$ and $A_s = A_{\{1,\ldots,s\}}$ for short. We immediately get that

$$\begin{aligned}
\mathbb{P}\left(V_n(\tau) \le i\right) &= \sum_{s=0}^{n_0} \mathbb{P}\left(V_n(\tau) \le i, V_n(\lambda) = s\right) \\
&= \sum_{s=0}^{n_0} \sum_{\substack{I\subset\{1,\ldots,n_0\},\\ |I|=s}} \mathbb{P}\left(V_n(\tau) \le i, A_I\right) \\
&= \sum_{s=0}^{n_0} \binom{n_0}{s} \mathbb{P}\left(A_s\right)\ \mathbb{P}\left(V_n(\tau) \le i | A_s\right) \\
&= \sum_{s=0}^{n_0} b_s \mathbb{P}\left(V_n(\tau) \le i | A_s\right).
\end{aligned}$$

We now investigate $\mathbb{P}\left(V_n(\tau) \le i|A_s\right)$. Setting

$$A_s^{\le} = \{p_1 \le \lambda,\ldots,p_s \le \lambda\} \text{ and } A_s^{>} = \{p_{s+1} > \lambda,\ldots,p_{n_0} > \lambda\}$$

we get $A_s = A_s^{\le} \cap A_s^{>}$. Further, we define $J_s = \{1,\ldots,s\}$ and $J_s' = \{s+1,\ldots,n_0\}$.
First consider the case $i < s < n_0$. This means that the null hypotheses corresponding to all p-values greater than λ and some that are smaller than λ will be accepted. Let

$$B_s^{>} = \{p_{n_0-s:J_s'} > c_n^s,\ldots,p_{1:J_s'} > c_{n-(n_0-s)+1}^s\}$$

and

$$B_s^{\le} = \{p_{s:J_s} > c_{n-(n_0-s)}^s,\ldots,p_{i+1:J_s} > c_{n-(n_0-i)+1}^s\}.$$

Note, $p_{n_0-s:J_s'},\ldots,p_{1:J_s'}$ and $p_{s:J_s},\ldots,p_{i+1:J_s}$ are the p-values larger than and less than λ corresponding to the $n_0 - i$ true null hypotheses that will be accepted for sure. With Remark 4.23 we get that

$$\begin{aligned}
\mathbb{P}\left(V_n(\tau) \le i|A_s\right) &= \mathbb{P}\left(p_{n_0:n_0} > c_n^s,\ldots,p_{i+1:n_0} > c_{n-(n_0-i)+1}^s | A_s^{\le} \cap A_s^{>}\right) \\
&= \mathbb{P}\left(B_s^{>} \cap B_s^{\le} | A_s^{\le} \cap A_s^{>}\right) \\
&= \frac{\mathbb{P}\left(B_s^{>} \cap A_s^{>} \cap B_s^{\le} \cap A_s^{\le}\right)}{\mathbb{P}\left(A_s^{>} \cap A_s^{\le}\right)}.
\end{aligned}$$

Note that $A_s^{>}$ and $B_s^{>}$ depend only on $p_{s+1},\ldots,p_{n_0}$, and $A_s^{\le}$ and $B_s^{\le}$ depend only on $p_1,\ldots,p_s$. It holds that

$$\mathbb{P}\left(V_n(\tau) \le i|A_s\right) = \frac{\mathbb{P}\left(B_s^{>} \cap A_s^{>}\right)\mathbb{P}\left(B_s^{\le} \cap A_s^{\le}\right)}{\mathbb{P}\left(A_s^{>}\right)\mathbb{P}\left(A_s^{\le}\right)} = \mathbb{P}\left(B_s^{>}|A_s^{>}\right)\mathbb{P}\left(B_s^{\le}|A_s^{\le}\right).$$

Till the end of this proof let $U'_1, \dots, U'_s$ and $U''_1, \dots, U''_{n_0-s}$ be iid $U[0, \lambda]$ and $U[\lambda, 1]$, respectively. From equations (4.17) and (4.16) we get

$$\mathbb{P}\left(B_s^{>}|A_s^{>}\right) = \mathbb{P}\left(U''_{n_0-s:n_0-s} > c_n^s, \dots, U''_{1:n_0-s} > c^s_{n-(n_0-s)+1}\right)$$

$$\mathbb{P}\left(B_s^{\leq}|A_s^{\leq}\right) = \mathbb{P}\left(U'_{s:s} > c^s_{n-(n_0-s)}, \dots, U'_{i+1:s} > c^s_{n-(n_0-i)+1}\right).$$

Note that $\frac{U'_i}{\lambda} \sim U[0, 1]$ and $\frac{U''_i-\lambda}{1-\lambda} \sim U[0, 1]$. With Remark 4.25 we get

$$\mathbb{P}\left(B_s^{>}|A_s^{>}\right) = F^0_{n_0-s}\left(d_n^s, \dots, d^s_{n-(n_0-s)+1}\right)$$

$$\mathbb{P}\left(B_s^{\leq}|A_s^{\leq}\right) = F^i_s\left(a^s_{n-(n_0-s)}, \dots, a^s_{n-(n_0-i)+1}\right).$$

This yields

$$\mathbb{P}\left(V_n(\tau) \leq i|A_s\right) = F^0_{n_0-s}\left(d_n^s, \dots, d^s_{n-(n_0-s)+1}\right) F^i_s\left(a^s_{n-(n_0-s)}, \dots, a^s_{n-(n_0-i)+1}\right).$$

We now consider the case $i < s = n_0$, that is, all p-values are below λ and therefore $A_s = A^{\leq}_{n_0}$ and $\{V_n(\tau) \leq i\} = B^{\leq}_{n_0}$. Similar as before, we get

$$\begin{aligned}\mathbb{P}\left(V_n(\tau) \leq i|A_s\right) &= \mathbb{P}\left(B^{\leq}_{n_0}|A^{\leq}_{n_0}\right)\\ &= \mathbb{P}\left(U'_{n_0:n_0} > c_n^s, \dots, U'_{i+1:n_0} > c^s_{n-(n_0-i)+1}\right)\\ &= F^i_{n_0}\left(a_n^s, \dots, a^s_{n-(n_0-i)+1}\right).\end{aligned}$$

Finally, for $s \leq i < n_0$ note that p-values corresponding to the null hypotheses that are accepted for sure lie above λ. Therefore, we have

$$\{V_n(\tau) \leq i\} = \{p_{n_0-s:J'_s} > c_n^s, \dots, p_{i-s+1:J'_s} > c^s_{n-(n_0-i)+1}\}.$$

Since $A^{\leq}_s$ is independent of $\{V_n(\tau) \leq i\}$ and independent of $A^{>}_s$, we obtain

$$\begin{aligned}\mathbb{P}\left(V_n(\tau) \leq i|A_s\right) &= \mathbb{P}\left(V_n(\tau) \leq i|A^{>}_s\right)\\ &= \mathbb{P}\left(U''_{n_0-s:n_0-s} > c_n^s, \dots, U''_{i-s+1:n_0-s} > c^s_{n-(n_0-i)+1}\right)\\ &= F^{i-s}_{n_0-s}\left(d_n^s, \dots, d^s_{n-(n_0-i)+1}\right).\end{aligned}$$

This establishes the formula. □

Theorem 4.27

In a DU(n, n_0)-model we have for the LSDPI test that $\mathbb{P}\left(V_n(\tau) \geq 0\right) = 1$ and for $0 < i \leq n_0$ we get

$$\begin{aligned}\mathbb{P}\left(V_n(\tau) \geq i\right) = &\sum_{s=0}^{i-1} b_s F^0_s\left(h^s_{n-n_0+1}, \dots, h^s_{n-n_0+s}\right) F^{n_0-i}_{n_0-s}\left(k^s_{n-n_0+s+1}, \dots, k^s_{n-n_0+i}\right)\\ &+ \sum_{s=i}^{n_0} b_s F^{s-i}_s\left(h^s_{n-n_0+1}, \dots, h^s_{n-n_0+i}\right)\end{aligned}$$

with $b_s = \binom{n_0}{s}\lambda^s(1-\lambda)^{n_0-s}$, $h_i^s = \frac{c_i^s}{\lambda}$ *and* $k_i^s = \frac{c_i^s-\lambda}{1-\lambda}$.

Proof: The calculations and arguments for the SD case are quite similar to those for the SU case. Like in the proof of Theorem 4.26 we conclude

$$\mathbb{P}\left(V_n(\tau_n) \geq i\right) = \sum_{s=0}^{n_0} b_s \mathbb{P}\left(V_n(\tau) \geq i | A_s\right)$$

with $A_s = \{p_1, \ldots, p_s \leq \lambda, p_{s+1}, \ldots, p_{n_0} > \lambda\}$. We first consider $s < i$. As in the proof of Theorem 4.26, we define $J_s = \{1, \ldots, s\}$, $J'_s = \{s+1, \ldots, n_0\}$, $A_s^{\leq} = \{p_1, \ldots, p_s \leq \lambda\}$, and $A_s^{>} = \{p_{s+1}, \ldots, p_{n_0} > \lambda\}$. Additionally, we define

$$B_s^{\leq} = \{p_{1:J_s} \leq c^s_{n-n_0+1}, \ldots, p_{s:J_s} \leq c^s_{n-n_0+s}\}$$

and

$$B_s^{>} = \{p_{1:J'_s} \leq c^s_{n-n_0+s+1}, \ldots, p_{i-s:J'_s} \leq c^s_{n-n_0+i}\}.$$

Now, $p_{1:J_s}, \ldots, p_{s:J_s}$ and $p_{1:J'_s}, \ldots, p_{i-s:J'_s}$ are the p-values corresponding to the i true null hypotheses that are rejected for sure. By the same arguments as in the proof of Theorem 4.26, we see for $s < i$ that

$$\begin{aligned}
&\mathbb{P}\left(V_n(\tau_n) \geq i | A_s\right) \\
&= \mathbb{P}\left(B_s^{\leq} | A_s^{\leq}\right) \mathbb{P}\left(B_s^{>} | A_s^{>}\right) \\
&= \mathbb{P}\left(U_{1:J_s} \leq \frac{c^s_{n-n_0+1}}{\lambda}, \ldots, U_{s:J_s} \leq \frac{c^s_{n-n_0+s}}{\lambda}\right) \\
&\quad \times \mathbb{P}\left(U_{1:J'_s} \leq \frac{c^s_{n-n_0+s+1} - \lambda}{1-\lambda}, \ldots, U_{i-s:J'_s} \leq \frac{c^s_{n-n_0+i} - \lambda}{1-\lambda}\right) \\
&= F_s^0\left(\frac{c^s_{n-n_0+1}}{\lambda}, \ldots, \frac{c^s_{n-n_0+s}}{\lambda}\right) F_{n_0-s}^{n_0-i}\left(\frac{c^s_{n-n_0+s+1} - \lambda}{1-\lambda}, \ldots, \frac{c^s_{n-n_0+i} - \lambda}{1-\lambda}\right),
\end{aligned}$$

where $U_1, \ldots, U_{n_0} \sim U[0,1]$ are independent.
For $s \geq i$ we conclude

$$\begin{aligned}
&\mathbb{P}\left(V_n(\tau_n) \geq i | A_s\right) \\
&= \mathbb{P}\left(p_{1:J_s} \leq c^s_{n_1+1}, \ldots, p_{i:J_s} \leq c^s_{n_1+i} | A_s^{\leq}\right) \\
&= \mathbb{P}\left(U_{1:J_s} \leq \frac{c^s_{n-n_0+1}}{\lambda}, \ldots, U_{i:J_s} \leq \frac{c^s_{n-n_0+i}}{\lambda}\right) \\
&= F_s^{s-i}\left(\frac{c^s_{n-n_0+1}}{\lambda}, \ldots, \frac{c^s_{n-n_0+i}}{\lambda}\right).
\end{aligned}$$

This completes the proof. □

One of the heuristic arguments behind Storey's estimator $\hat{n}_0$ was that for a well-chosen λ virtually all p-values in $(\lambda, 1]$ correspond to true null hypotheses . Therefore, a natural restriction seems to reject only hypotheses H_i if $p_i \leq \tau \wedge \lambda$. The next goal is to determine the distribution function of $V_n(\tau \wedge \lambda)$. Fortunately, we can again use the expressions evaluated in the proof of Theorems 4.26 and 4.27.

Lemma 4.28

Under DU configurations we have for the LSDPI/LSUPI test that $\mathbb{P}\left(V_n(\tau \wedge \lambda) \leq n_0\right) = 1$ *and*

$$\mathbb{P}\left(V_n(\tau \wedge \lambda) \leq i\right) = \mathbb{P}\left(V_n(\lambda) \leq i\right) + \sum_{s=i+1}^{n_0} \mathbb{P}\left(V_n(\tau) \leq i, V_n(\lambda) = s\right) \tag{4.20}$$

$$= 1 - \sum_{s=i+1}^{n_0} \mathbb{P}\left(V_n(\tau) \geq i+1, V_n(\lambda) = s\right) \tag{4.21}$$

for $0 \leq i < n_0$.

Proof: Obviously, $V_n(\lambda) = s$ implies $V_n(\tau \wedge \lambda) \leq s$. Hence,

$$\{V_n(\tau \wedge \lambda) \leq i, V_n(\lambda) = s\} = \{V_n(\lambda) = s\} \text{ for all } i \geq s.$$

Since $\{V_n(\tau \wedge \lambda) \leq i, V_n(\lambda) = s\} \subset \{\tau < \lambda\}$ for all $i < s$, we get

$$\{V_n(\tau \wedge \lambda) \leq i, V_n(\lambda) = s\} = \{V_n(\tau) \leq i, V_n(\lambda) = s\} \text{ for all } i < s.$$

Altogether, we conclude that

$$\mathbb{P}\left(V_n(\tau \wedge \lambda) \leq i\right) = \sum_{s=0}^{n_0} \mathbb{P}\left(V_n(\tau \wedge \lambda) \leq i, V_n(\lambda) = s\right)$$

$$= \sum_{s=0}^{i} \mathbb{P}\left(V_n(\lambda) = s\right) + \sum_{s=i+1}^{n_0} \mathbb{P}\left(V_n(\tau) \leq i, V_n(\lambda) = s\right).$$

This is (4.20) and can be further transformed to

$$\mathbb{P}\left(V_n(\tau \wedge \lambda) \leq i\right) = \mathbb{P}\left(V_n(\lambda) \leq i\right) + \sum_{s=i+1}^{n_0} \mathbb{P}\left(V_n(\lambda) = s\right) - \mathbb{P}\left(V_n(\tau) > i, V_n(\lambda) = s\right)$$

$$= 1 - \sum_{s=i+1}^{n_0} \mathbb{P}\left(V_n(\tau) > i, V_n(\lambda) = s\right).$$

This finishes the proof. □

Remark 4.29

The assumptions of Lemma 4.28 can be weakend. It would be sufficient to assume that $V_n(t)$ is a.s. non-decreasing in $t \in [0, 1]$, τ is a random variable from Ω to $[0, 1]$, and $\lambda \in [0, 1]$.

Alternative expressions for the right hand terms in (4.20) and (4.21) were obtained in the proof of Theorems 4.26 and 4.27. Plugging in these alternative expressions immediately leads to the following formulas for $\mathbb{P}\left(V_n(\tau \wedge \lambda) \leq i\right)$.

Corollary 4.30
Under DU configurations we have for the LSUPI test that $\mathbb{P}\left(V_n(\tau \wedge \lambda) \leq n_0\right) = 1$ *and for* $0 \leq i < n_0$ *we get*

$$\mathbb{P}\left(V_n(\tau \wedge \lambda) \leq i\right) = \sum_{s=0}^{i} b_s + \sum_{s=i+1}^{n_0} b_s F^0_{n_0-s}\left(d^s_n, \ldots, d^s_{n-(n_0-s)+1}\right) F^i_s\left(a^s_{n-(n_0-s)}, \ldots, a^s_{n-(n_0-i)+1}\right)$$

with $b_s = \binom{n_0}{s}\lambda^s(1-\lambda)^{n_0-s}$, $a^s_i = 1 - \frac{c^s_i}{\lambda}$ *and* $d^s_i = 1 - \frac{c^s_i - \lambda}{1-\lambda}$.

Corollary 4.31
Under DU configurations we have for the LSDPI test that $\mathbb{P}\left(V_n(\tau \wedge \lambda) \leq n_0\right) = 1$ *and for* $0 \leq i < n_0$ *we get*

$$\mathbb{P}\left(V_n(\tau \wedge \lambda) \leq i\right) = 1 - \sum_{s=i+1}^{n_0} b_s F^{s-i-1}_s\left(h^s_{n-n_0+1}, \ldots, h^s_{n-n_0+i+1}\right)$$

with $b_s = \binom{n_0}{s}\lambda^s(1-\lambda)^{n_0-s}$ *and* $h^s_i = \frac{c^s_i}{\lambda}$.

Applying these formulas in numerical calculations demonstrates that the LSDPI test and LSUPI test do not control the ENFR under DU configurations at level $\gamma = \alpha/(1+\alpha)$. But the violation is very weak. For instance, it would not be possible to distinguish the ENFR of the LSDPI/LSUPI test in Figure 4.1 from the ENFR of the β_n-adjusted AORC.

4.3.2 Asymptotic behavior

In order to investigate the asymptotic behavior of the plug-in method, we assume for this section that $n_0/n \to \zeta \in [0,1]$ for $n \to \infty$. For the special case $\zeta = 1$ we assume that n_1 is fixed.

Theorem 4.32 (Asymptotic distribution)
Let $V_{SD} \sim LSD(n_1, \alpha)$, $V_{SU} \sim LSU(n_1, \alpha)$, *and* $\hat{\zeta}_n \to 1$ *in distribution. Then, for* n_1 *fixed and* $\alpha \in (0,1)$ *we have under* $DU(n, n_0)$ *that* $V_n(\tau_n) \to V_{SD}$ *in distribution for the LSDPI and* $V_n(\tau_n) \to V_{SU}$ *in distribution for the LSUPI.*

Proof: Let $\epsilon > 0$ be arbitrary and $B_n = \{1 - \epsilon \leq \hat{n}_0/n \leq 1 + \epsilon\}$. Since $\hat{n}_0/n \to 1$ in distribution, there exists a $N \in \mathbb{N}$ such that $\mathbb{P}\left(B^c_n\right) \leq \epsilon$ for $n > N$. Denote by $V_{n,\alpha}$ the number of false rejections if one uses t/α as a rejection curve in a step-up manner. From Theorem 4.8 and 4.2 we already know the asymptotic and finite distribution of $V_{n,\alpha}$. We conclude for $n > N$ that

$$\begin{aligned}
\mathbb{P}\left(V_n(\tau_n) \leq k\right) &= \mathbb{P}\left(V_n(\tau_n) \leq k, B_n\right) + \mathbb{P}\left(V_n(\tau_n) \leq k, B_n^c\right) \\
&\leq \mathbb{P}\left(V_{n,\frac{\alpha}{1+\epsilon}} \leq k, B_n\right) + \epsilon \\
&\leq \sum_{i=0}^{k} \binom{n_0}{i} \left(1 - \frac{\alpha}{1+\epsilon}\right) \left(\frac{n_1+i}{n}\frac{\alpha}{1+\epsilon}\right)^i \left(1 - \frac{n_1+i}{n}\frac{\alpha}{1+\epsilon}\right)^{n_0-i-1} + \epsilon \\
&\longrightarrow \sum_{i=0}^{k} \frac{1}{i!} \left(1 - \frac{\alpha}{1+\epsilon}\right) \left((n_1+i)\frac{\alpha}{1+\epsilon}\right)^i \exp\left(-(n_1+i)\frac{\alpha}{1+\epsilon}\right) + \epsilon.
\end{aligned}$$

Because $\epsilon > 0$ can be chosen arbitrary small, we get

$$\limsup_{n\to\infty} \mathbb{P}\left(V_n(\tau_n) \leq k\right) \leq \sum_{i=0}^{k} \frac{1-\alpha}{i!}((n_1+i)\alpha)^i \exp\left(-(n_1+i)\alpha\right) = \lim_{n\to\infty} \mathbb{P}\left(V_{n,\alpha} \leq k\right).$$

On the other hand, we also have

$$\begin{aligned}
\mathbb{P}\left(V_n(\tau_n) \leq k\right) &\geq \mathbb{P}\left(V_n(\tau_n) \leq k, B_n\right) \\
&\geq \mathbb{P}\left(V_{n,\frac{\alpha}{1-\epsilon}} \leq k, B_n\right) \\
&\geq \mathbb{P}\left(V_{n,\frac{\alpha}{1-\epsilon}} \leq k, B_n\right) + \mathbb{P}\left(V_{n,\frac{\alpha}{1-\epsilon}} \leq k, B_n^c\right) - \epsilon \\
&= \sum_{i=0}^{k} \binom{n_0}{i} \left(1 - \frac{\alpha}{1-\epsilon}\right) \left(\frac{n_1+i}{n}\frac{\alpha}{1-\epsilon}\right)^i \left(1 - \frac{n_1+i}{n}\frac{\alpha}{1-\epsilon}\right)^{n_0-i-1} - \epsilon \\
&\longrightarrow \sum_{i=0}^{k} \frac{1}{i!} \left(1 - \frac{\alpha}{1-\epsilon}\right) \left((n_1+i)\frac{\alpha}{1-\epsilon}\right)^i \exp\left(-(n_1+i)\frac{\alpha}{1-\epsilon}\right) - \epsilon.
\end{aligned}$$

With the same argumentation as before, this yields

$$\liminf_{n\to\infty} \mathbb{P}\left(V_n(\tau_n) \leq k\right) \geq \lim_{n\to\infty} \mathbb{P}\left(V_{n,\alpha} \leq k\right).$$

This means that the asymptotic distributions of $V_n(\tau_n)$ and $V_{n,\alpha}$ are equal.
Similar arguments for the SD case together with the results from Theorems 4.8 and 4.2 entail

$$\lim_{n\to\infty} \mathbb{P}\left(V_n(\tau_n) \leq k\right) = \sum_{i=0}^{k} \frac{\alpha}{i!}(n_1+1)((n_1+i+1)\alpha)^{i-1} \exp(-(n_1+i+1)\alpha).$$

This is our assertion. □

Remark 4.33

Under BIA and for n_1 fixed, by the law of large numbers, Storey's estimator $\hat{n}_0/n$ converges a.s. to 1. Furthermore, under some regularity conditions the $p_{k:n}$-estimator

$$\hat{\zeta}_n = \frac{1}{n}\frac{n-k+1}{1-p_{k:n}}$$

also converges in distribution to 1, cf. Theorem 4.37.

Theorem 4.34 (Asymptotic expectation)
For n_1 fixed, $m \in \mathbb{N}_0$, and $\alpha \in (0,1)$ we have under DU(n, n_0) that

$$\lim_{n\to\infty} \mathbb{E}V_n^m(\tau_n) = \mathbb{E}V_{SU}^m, \quad \textit{for LSDPI},$$

$$\lim_{n\to\infty} \mathbb{E}V_n^m(\tau_n) = \mathbb{E}V_{SD}^m, \quad \textit{for LSUPI},$$

if $n_0^m \mathbb{P}(\hat{\zeta}_n \leq 1-\epsilon) \to 0$ holds for all $\epsilon > 0$ and additionally $\hat{\zeta}_n \to 1$ weakly, where V_{SD} and V_{SU} are the random variables from Theorem 4.11. In particular, if $n_0 \mathbb{P}(\hat{\zeta}_n \leq 1-\epsilon) \to 0$ holds for all $\epsilon > 0$ and additionally $\hat{\zeta}_n \to 1$ weakly, then

$$\lim_{n\to\infty} \mathbb{E}V_n(\tau_n) = \begin{cases} (n_1+1)\gamma & \textit{, for LSDPI}, \\ (n_1+1+\gamma)\gamma & \textit{, for LSUPI}, \end{cases}$$

where $\gamma = \alpha/(1-\alpha)$.

Before we proof Theorem 4.34 we investigate the conditions appearing in Theorem 4.34. Theorems 4.35 and 4.36 are concerned with sufficient conditions for $n_0^m \mathbb{P}(\hat{\zeta}_n \leq 1-\epsilon) \to 0$ for Storey's estimator and the $p_{k:n}$-estimator. In Corollary 4.38 we summarize when Theorems 4.32 and 4.34 hold true.

Theorem 4.35
Under DU(n, n_0) Storey's estimator for $\lambda \in [0,1)$ fulfills for all $m \in \mathbb{N}$, $\epsilon > 0$, and $n_1 \in \mathbb{N}$ fixed the condition $n_0^m \mathbb{P}(\hat{\zeta}_n \leq 1-\epsilon) \to 0$ for $n \to \infty$.

Proof: Let $\zeta_n = n_0/n$. We have

$$\begin{aligned}
\mathbb{P}\left(\hat{\zeta}_n \leq 1-\epsilon\right) &= \mathbb{P}\left(\frac{n_0 - V(\lambda)+1}{n(1-\lambda)} \leq 1-\epsilon\right) \\
&\leq \mathbb{P}\left(1 - F_{n_0}^0(\lambda) \leq \frac{(1-\epsilon)(1-\lambda)}{\zeta_n}\right) \\
&= \mathbb{P}\left(F_{n_0}^0(\lambda) - \lambda \geq 1-\lambda - \frac{(1-\epsilon)(1-\lambda)}{\zeta_n}\right) \\
&\leq c\exp\left(-2n_0\left(1-\lambda-\frac{(1-\epsilon)(1-\lambda)}{\zeta_n}\right)^2\right),
\end{aligned}$$

where the last inequality is due to the DKW inequality [54], p. 12 and $c > 0$ is a constant. Since $\zeta_n \to 1$, we get $n_0^m \mathbb{P}(\hat{\zeta}_n \leq 1-\epsilon) \to 0$ for $n \to \infty$. □

Theorem 4.36
Let $m \in \mathbb{N}_0$ be fixed and $k(n) = k \in \mathbb{N}$ with $k < n$. If there exists an $r \in \mathbb{N}$ with $n^{m-r}/(1-k/n)^{2r} \to 0$, then the $p_{k:n}$ estimator fulfills under DU(n, n_0) for all $\epsilon \in (0,1)$ and $n_1 \in \mathbb{N}$ fixed the condition $n_0^m \mathbb{P}(\hat{\zeta}_n \leq 1-\epsilon) \to 0$ for $n \to \infty$.

Proof: First note that $o(1) = n^{m-r}/(1-k/n)^{2r} = n^{m+r}/(n-k)^{2r}$ implies that $n-k \to \infty$. Denote by $p_{i:I_0}$ the ith order statistic of $(p_i)_{i \in I_0}$. Obviously, for $x \in [0,1]$ we have

$$\mathbb{P}\left(p_{i:I_0} \leq x\right) = \mathbb{P}\left(\sum_{j \in I_0} \mathbb{I}_{\{p_j \leq x\}} \geq i\right) = \mathbb{P}\left(B \geq i\right),$$

where $B \sim B(n_0, x)$. In general, we have

$$\begin{aligned}\mathbb{P}\left(\hat{\zeta}_n \leq 1-\epsilon\right) &= \mathbb{P}\left(\frac{n-k+1}{n(1-p_{k:n})} \leq 1-\epsilon\right)\\ &\leq \mathbb{P}\left(\frac{1-k/n}{1-\epsilon} \leq 1-p_{k:n}\right)\\ &= \mathbb{P}\left(p_{k:n} \leq \frac{k/n-\epsilon}{1-\epsilon}\right).\end{aligned}$$

First, we examine the simple case where $\limsup k/n \in [0,\epsilon)$. Obviously, we have $\mathbb{P}(\hat{\zeta}_n \leq 1-\epsilon) = 0$ eventually for all n and thus $n_0^m \mathbb{P}\left(\hat{\zeta}_n \leq 1-\epsilon\right) \to 0$.
Now, let $q_n = (k/n-\epsilon)/(1-\epsilon)$ and $B_n \sim B(n_0, q_n)$. Note, w.l.o.g. we can assume in the following that $\liminf k/n > \epsilon$. Since n_1 is fixed and $n-k \to \infty$, we can further assume that $k > n_1 + n_0 q_n$ eventually for all n because

$$k > n_1 + n_0 q_n \Leftarrow k > n_1 + n q_n \Leftrightarrow k(1-\epsilon) > n_1(1-\epsilon) + k - \epsilon n \Leftrightarrow \epsilon(n-k) > n_1(1-\epsilon).$$

Thus, we get

$$\begin{aligned}\mathbb{P}\left(\hat{\zeta}_n \leq 1-\epsilon\right) &\leq \mathbb{P}\left(p_{k:n} \leq q_n\right)\\ &= \mathbb{P}\left(p_{k-n_1:I_0} \leq q_n\right)\\ &= \mathbb{P}\left(B_n - n_0 q_n \geq k - n_1 - n_0 q_n\right)\\ &\leq \frac{\mathbb{E}(B_n - n_0 q_n)^{2r}}{(k-n+n_0(1-q_n))^{2r}}.\end{aligned}$$

Note, since $\zeta_n \to 1$, the inequality $\zeta_n/(1-\epsilon) - 1 \geq \epsilon/(2(1-\epsilon))$ holds eventually for all n. Furthermore, we have $\liminf q_n > 0$ because $\liminf k/n > \epsilon$ and thus $n_0 q_n \to \infty$. By Lemma 4.5

we get

$$\begin{aligned}
\mathbb{P}\left(\hat{\zeta}_n \leq 1-\epsilon\right) &\leq \frac{C_{2r}\max\{1,(n_0q_n)^r\}}{(k-n+n_0(1-q_n))^{2r}} \\
&= \frac{C_{2r}(n_0q_n)^r}{(k-n+n_0\frac{1-k/n}{1-\epsilon})^{2r}} \\
&= \frac{C_{2r}(n_0q_n)^r}{(k-n+\zeta_n\frac{n-k}{1-\epsilon})^{2r}} \\
&= \frac{C_{2r}(\zeta_n n q_n)^r}{((n-k)(\zeta_n/(1-\epsilon)-1))^{2r}} \\
&\leq \frac{C_{2r}}{(\zeta_n/(1-\epsilon)-1)^{2r}}\frac{(\zeta_n n q_n)^r}{(n-k)^{2r}} \\
&= \frac{C_{2r}}{(\zeta_n/(1-\epsilon)-1)^{2r}}\left(\zeta_n\frac{k/n-\epsilon}{1-\epsilon}\right)^r\frac{n^r}{n^{2r}(1-k/n)^{2r}} \\
&\leq \frac{K}{n^r(1-k/n)^{2r}},
\end{aligned}$$

where $K \geq 0$. □

Theorem 4.37 (Lemma 2.15 in Gontscharuk [26])
Let $k = k(n) \in \mathbb{N}$ and $\hat{n}_0$ be the $p_{k:n}$ estimator with $k < n$ and

$$\liminf_{n\to\infty}\frac{k-n_1}{n_0} \geq 0 \text{ and } \limsup_{n\to\infty}\frac{k-n_1}{n_0} < 1.$$

Then, under DU(n, n_0), for all $\epsilon > 0$ there exist constants $C_1, C_2 > 0$ such that for all $n \in \mathbb{N}$

$$\mathbb{P}_{n,n_0}\left(\left|\frac{\hat{n}_0}{n_0}-1\right| \geq \epsilon\right) \leq C_1 e^{-n_0C_2}$$

holds true.

Corollary 4.38
Theorems 4.32 and 4.34 hold true for

1. *Storey's estimator*

$$\hat{n}_{0,1} = \frac{n-R(\lambda)+1}{1-\lambda}$$

   if $\lambda \in [0, 1)$,

2. *the $p_{k:n}$ estimator*

$$\hat{n}_{0,2} = \frac{n-k+1}{1-p_{k:n}}$$

   if $\limsup_{n\to\infty}\frac{k(n)-n_1}{n_0} < 1$, *3. the truncated $p_{k:n}$ estimator*

$$\hat{n}_{0,3} = \min\left\{n, \frac{n-k+1}{1-p_{k:n}}\right\}$$

if $k < n$ and for all $m \in \mathbb{N}$ there exists an $r = r(m) \in \mathbb{N}$ with $n^{m-r}/(1-k/n)^{2r} \to 0$.

Proof: The first assertion follows directly from Remark 4.33 and Theorem 4.35.
Since n_1 is considered fixed we have $\liminf_{n\to\infty}(k-n_1)/n \geq 0$. Theorem 4.37 yields for all $m \in \mathbb{N}$ and $\epsilon > 0$ that

$$n_0^m \mathbb{P}_{n,n_0}\left(|\hat{n}_{0,2}/n - 1| \geq \epsilon\right) \to 0,$$

which entails the second assertion.
The last assertion follows from Theorem 4.36 and $\mathbb{P}_{n,n_0}\left(\hat{n}_{0,3}/n \geq 1+\epsilon\right) = 0$ for all $\epsilon > 0$. □

Remark 4.39
Note that the case $k/n \to 1$ is possible for the truncated $p_{k:n}$ estimator from Corollary 4.38.

The key in the proof of Theorem 4.34 is the uniform integrability of $\{V_n^m(\tau_n)\}_{n\geq n_1}$ for LSDPI/LSUPI test. The next theorem will establish this by using the property that $n_0^m \mathbb{P}\left(\hat{\zeta}_n \leq 1-\epsilon\right)$ converges to zero to reduce the uniform integrability of $\{V_n^m(\tau_n)\}_{n\geq n_1}$ to the uniform integrability of $\{V_n^m(\varphi_n)\}_{n\geq n_1}$, where φ_n is LSD or LSU procedure. But by Lemma 4.4 the sequence $\{V_n^m(\varphi_n)\}_{n\geq n_1}$ is uniform integrable.

Theorem 4.40
Suppose for fixed $m, n_1 \in \mathbb{N}$ and all $\epsilon > 0$ that $n_0^m \mathbb{P}(\hat{\zeta}_n \leq 1-\epsilon) \to 0$ for $n \to \infty$ under DU(n, n_0). Then $\{V_n^m(\tau_n)\}_{n\geq n_1}$ is uniformly integrable under DU(n, n_0).

Proof: Let $\epsilon > 0$ be arbitrary and $\beta > 1$ such that $\beta\alpha < 1$. Denote by φ_n the LSU test induced by the rejection curve $t/(\beta\alpha)$. The argumentation for the step-up case is the same as for the step-down case, so in the proof we only consider the step-up case. For all $c > 0$ we have

$$\begin{aligned}\int_{\{V_n^m(\tau_n)>c\}} V_n^m(\tau_n)d\mathbb{P} &= \int_{\{V_n^m(\tau_n)>c,\, \hat{\zeta}_n>1/\beta\}} V_n^m(\tau_n)d\mathbb{P} + \int_{\{V_n^m(\tau_n)>c,\, \hat{\zeta}_n\leq 1/\beta\}} V_n^m(\tau_n)d\mathbb{P} \\ &= I_1 + I_2.\end{aligned}$$

There exits a $M > 0$ such that for all $n_0 > M$ we have

$$I_2 \leq \int_{\{V_n^m(\tau_n)>c,\, \hat{\zeta}_n\leq 1/\beta\}} n_0^m d\mathbb{P} \leq n_0^m \mathbb{P}(\hat{\zeta}_n \leq 1/\beta) \leq \epsilon/2.$$

Since $r(t|\hat{\zeta}_n) = \hat{\zeta}_n t/\alpha \geq t/(\beta\alpha)$ on $\{\hat{\zeta}_n > 1/\beta\}$, we also have

$$I_1 \leq \int_{\{V_n^m(\varphi_n)>c,\, \hat{\zeta}_n>1/\beta\}} V_n^m(\varphi_n)d\mathbb{P} \leq \int_{\{V_n^m(\varphi)>c\}} V_n^m(\varphi_n)d\mathbb{P} \leq \epsilon/2$$

for c large enough because $\{V_n^m(\varphi_n)\}_{n\geq n_1}$ is uniformly integrable, cf. Lemma 4.4. Altogether, we have

$$\sup_{n\geq n_1}\int_{\{V_n^m(\tau_n)>\max(c,M^m)\}} V_n^m(\tau_n) \leq \epsilon$$

because for all $K \in \mathbb{N}$ and $n_0 \leq K$ the set $\{|V_n^m(\tau_n)| > K^m\}$ is empty. □

Proof of Theorem 4.34. Theorem 4.32 shows $V_n(\tau_n) \to V_{SU}$ in distribution and with Theorem 4.40 (uniform integrability) we have by Theorem 5.4, [6], p. 32 $\lim_{n\to\infty} \mathbb{E}V_n^m(\tau_n) = \mathbb{E}V_{SU}^m$. In particular, for the first moment we directly conclude from Theorem 4.11 for the step-up case

$$\lim_{n\to\infty} \mathbb{E}V_n(\tau_n) = \left(n_1 + \frac{1}{1-\alpha}\right)\frac{\alpha}{1-\alpha} = (n_1+1+\gamma)\gamma,$$

and with the same argumentation we conclude for the step-down case

$$\lim_{n\to\infty} \mathbb{E}V_n(\tau_n) = (n_1+1)\frac{\alpha}{1-\alpha} = (n_1+1)\gamma.$$ □

Theorem 4.41 (Asymptotic expectation)
If we assume $\zeta_n = n_0/n \to \zeta \in [0,1)$, then under DU$(n, n_0)$ we get for the LSDPI test and LSUPI test that

$$\lim_{n\to\infty} \mathbb{E}_{n,n_0}\left[\frac{V_n(\tau_n)}{n_1}\right] = \min\{\gamma, \zeta/(1-\zeta)\}$$

if $\hat{\zeta}_n \to \zeta$ in distribution, where $\gamma = \alpha/(1-\alpha)$.

Proof: For $\zeta < 1$ the argumentation, irrespective of step-up or step-down, is a little easier than for $\zeta = 1$. Since ζ is constant, we have $\hat{\zeta}_n \to \zeta$ in probability. Thus, for every subsequence n'_k there exists another subsequence n_k of n'_k such that $\hat{\zeta}_{n_k} \to \zeta$ almost surely. It holds that:

1. $V_{n_k}(t)/n_0 \to t$ almost surely and uniformly,
2. $F_{n_k}(t) \to F_\zeta(t) = (1-\zeta) + \zeta t$ almost surely and uniformly,
3. $\hat{\zeta}_{n_k} t/\alpha \to \zeta t/\alpha$ almost surely and uniformly.

In fact, n_0 in 1. depends now on subsequence n_k, but for notational convenience we suppress this fact. The first step is to calculate the asymptotic intersection point t_ζ of $F_{n_k}(t)$ and $\zeta t/\alpha$, that is, the point for which $\tau_{n_k} \to t_\zeta$ almost surely. Obviously, by 2. and 3., t_ζ is characterized by $F_\zeta(t_\zeta) = \zeta t_\zeta/\alpha$. Solving for t_ζ yields $t_\zeta = \gamma(1-\zeta)/\zeta$. We have $V_{n_k}(\tau_{n_k})/n_0 \to \min\{t_\zeta, 1\}$ a.s., and thus we obtain

$$V_{n_k}(\tau_{n_k})/n_1 = \frac{n_0/n}{n_1/n} V_{n_k}(\tau_{n_k})/n_0 \to \frac{\zeta}{1-\zeta}\min\{t_\zeta, 1\} = \min\{\gamma, \zeta/(1-\zeta)\}$$

almost surely. Thus, for every subsequence n'_k exists another subsequence n_k of n'_k such that $V_{n_k}(\tau_{n_k})/n_1 \to \min\{\gamma, \zeta/(1-\zeta)\}$ almost surely. This entails that $V_n(\tau_n) \to \min\{\gamma, \zeta/(1-\zeta)\}$ in probability. The assertion follows by the dominated convergence theorem. □

4.4 Summary

In this chapter, we have investigated different procedures that control the FDR for finite n with respect to their ENFR behavior. Since for all considered procedures DU is LFC for the ENFR, all calculations were conducted under DU. The first two procedures investigated were the LSD and LSU. We computed the finite and asymptotic distribution of V_n, where in the asymptotic setting n_1 is kept fixed or $n_0/n \to \zeta \in (0,1)$. Since the ENFR for finite n was already calculated by Finner and Roters [18], the asymptotic ENFR was determined for the two cases where the fraction of true null hypotheses converges to a constant smaller than one (non-sparsity) and for n_1 fixed. For the non-sparsity situation we established that V_n is asymptotically normally distributed. Additionally, for n_1 fixed, we computed the asymptotic variance. A heuristic comparison shows that both expressions are similar. Further, we calculated the ENFR for LSD/LSU test for the "intermediate" case, that is for $n_1 \to \infty$ but $n_0/n \to 1$.
Then, we focused on the AORC. It is well known that a SD test induced by the β_n-adjusted AORC with $\beta_n = 1$ controls the FDR and that the level α is nearly exhausted. We showed that this procedure also controls the ENFR with level function $\min\{n_0, (n_1+1)\alpha/(1-\alpha)\}$ and nearly exhausts this level. The purpose of β_n is to achieve finite control of the FDR. The behavior of β_n in the FDR context was already investigated in [26]. It turned out that for a SU test β_n tends to infinity but β_n/n converges to zero. In the ENFR context, using $\min\{n_0, (n_1+1)\gamma\}$ as the bounding function, we proved that β_n/n is bounded but does not converge to zero unless the bounding function is $\min\{n_0, (n_1+1+\gamma)\gamma\}$.
Attempting to develop a procedure controlling the ENFR at level $(n_1+1)\gamma$ using plug-in techniques we rediscovered plug-in procedures which are nowadays known as adaptive BH-procedures. Because there is a tight relation to LSD and LSU, we call them LSDPI and LSUPI, respectively. The finite distribution of V_n in the SD and SU case was established for a popular estimator sometimes known as Storey's estimator, which estimates the fraction of true null hypotheses. Although the LSDPI and LSUPI procedure controls the FDR at level α for finite n, it slightly violates the bounding function $(n_1+1)\alpha/(1-\alpha)$. Asymptotic considerations have been conducted under a more general setting. Assuming that the estimator converges in distribution to the fraction of true null hypotheses, the LSDPI and LSUPI exhaust the level asymptotically for the non-sparsity case. And if n_1 is kept fixed, even the asymptotic distribution coincides with the corresponding asymptotic distribution of LSD and LSU. With further regularity conditions on the estimator we showed that also *all* asymptotic moments of V_n are equal to the corresponding moments calculated for LSD and LSU. It was proven that all required conditions are fulfilled by Storey's estimator. Additionally, we stated sufficient conditions for another popular estimator, sometimes called $p_{k:n}$-estimator, such that the asymptotic results also hold true for this estimator.

Chapter 5

Exact SU procedures

In the latter chapter we have investigated well-known procedures, controlling the FDR, with respect to the ENFR. This means, that the rejection curve was given and then for instance we determined the corresponding ENFR. Now, we consider the ENFR bounding function g fixed. The aim is to determine a rejection curve such that for finite n the corresponding ENFR equals or at least approximately is g.

5.1 Exact solving

For a prespecified ENFR bounding function g we try to determine critical values c_i such that

$$\text{ENFR}_{n,n_0}(\varphi_n) = g(n_1), \quad \text{for all } n_0 = 1, \ldots, n, \tag{5.1}$$

where φ_n is a SU test with critical values c_i. But φ_n is only a useful procedure if the DU configurations are indeed LFCs for $\text{ENFR}_\vartheta(\varphi_n)$. Otherwise, there exist $\vartheta \in \Theta$ such that

$$g(n_1(\vartheta)) = \text{ENFR}_{n,n_0}(\varphi_n) < \text{ENFR}_\vartheta(\varphi_n).$$

For example, suppose an algorithm returned critical values c_i which are decreasing in i, then also the corresponding rejection curve is decreasing. As it can be seen from Figure 5.1, in this case the crossing-points move to the left after setting the p-values corresponding to false null hypotheses to zero. Hence, the DU configurations are not LFCs for the ENFR. Of course, if the critical values are increasing in i, then the DU configurations are LFCs. Under $\text{DU}(n, n_0)$, given the critical values, we are able to calculate the ENFR exactly. Therefore, for a fixed ENFR bounding curve g we can try to determine the critical values by these exact formulas. Unfortunately, we will see that usually (5.1) can not be achieved and we also can not state simple and useful conditions on g which guarantee that critical values fulfilling (5.1) exist. All issues of the "determining process"

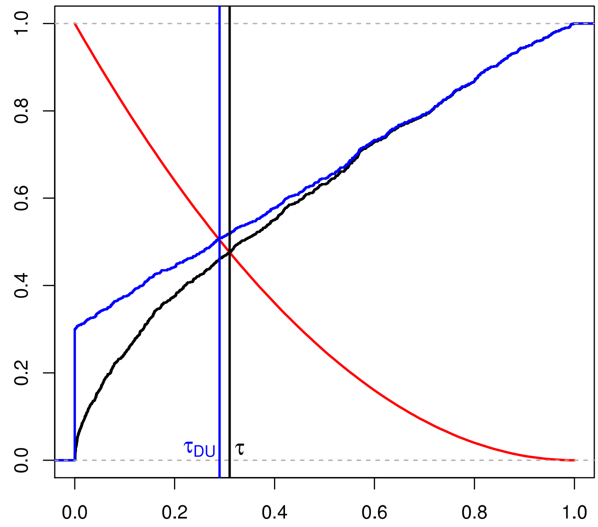

Figure 5.1: *The crossing-points decrease for a decreasing rejection curve (red line) if p-values are set to zero. The black line is the ecdf of all p-values and the blue line is the ecdf of all p-values with $p_i = 0$ for $i \in I_1$.*

can be illustrated for $n = 2$. For this simple setting the ENFR can be calculated as follows,

$$\begin{aligned}
\mathbb{E}_{2,0}V_n &= 0,\\
\mathbb{E}_{2,1}V_n &= \mathbb{P}_{2,1}(V_n = 1) = c_2,\\
\mathbb{E}_{2,2}V_n &= \mathbb{P}_{2,2}(V_n = 1) + 2\mathbb{P}_{2,2}(V_n = 2)\\
&= 2\mathbb{P}_{2,2}(p_1 \le c_1, p_2 > c_2) + 2\mathbb{P}_{2,2}(p_1 \le c_2, p_2 \le c_2)\\
&= 2c_1(1 - c_2) + 2c_2^2.
\end{aligned}$$

Thereby, (5.1) is equivalent to

$$c_2 = g(n-1) = g(1) \tag{5.2}$$

$$c_1 = \frac{g(0)/2 - c_2^2}{1 - c_2}. \tag{5.3}$$

In order to recursively solve these equations, one starts with the first equation (5.2). A problem already occurs at this first step. If $c_2 = g(1) \ge 1$, a SU test *always* rejects all null hypotheses. Especially, for large n it is not unusual that $g(n-1)$ is larger than one. For instance, consider the bounding function $g(n_1) = (n_1 + 1)\gamma$ for $n_1 = 100$ and $\gamma = 0.05$. Again, let $n = 2$. Suppose for a moment that $g(n-1) = g(1) < 1$, then we directly encounter the next problem. Obviously, if $g(0)/2 < c_2^2 = g^2(1)$, then $c_1 < 0$. Under the global null hypothesis $\{p_{1:2} \leq c_1\}$ equals $\{p_{1:2} \leq 0\}$ and the SU test with critical values c_1 and c_2 will exceed the bounding function g under the global null hypothesis, that is

$$\mathbb{E}_{2,2}V_n = 2(0(1-c_2)+c_2^2) > 2(c_1(1-c_2)+c_2^2) = g(0)$$

if $c_2 \in (0,1)$. As mentioned at the beginning, it is essential that the resulting SU test has the DU configurations as LFCs. Thus, we also must ensure that $c_1 \leq c_2$. In our simple example, this leads to

$$\frac{g(0)/2 - c_2^2}{1-c_2} = c_1 \leq c_2 \Longleftrightarrow g(0)/2 \leq c_2.$$

Furthermore, we have seen that we need $g(0)/2 > c_2^2$ in order to ensure $c_1 > 0$. Assuming $c_2 = g(1)$ we get

$$g^2(1) \leq g(0)/2 \leq g(1) \tag{5.4}$$

or in words, g may not increase or decrease too fast. In sum, for $n = 2$ we need $g(n-1) = g(1) < 1$ and (5.4) in order to avoid any problems.

We now turn to the general case where $n \geq 2$. In principle, conditions like (5.4) can be formulated for larger n, but in general we will have $n-1$ of such conditions. Every of these conditions would be very complicated and useless for practical purpose. Nevertheless, we now formulate the recursive schema for arbitrary $n \geq 2$. By virtue of Lemma 3.21 in [13], we have

$$\mathbb{P}_{n,n_0}\left(V_n(\varphi_n) = j\right) = \frac{n_0}{j} c_{n_1+j}\mathbb{P}_{n,n_0-1}(V_n(\varphi_n) = j-1) \tag{5.5}$$

for all $n_0 = 1,\ldots,n$ and $j = 1,\ldots,n$, where φ_n is a SU test with critical values $0 \leq c_1 \leq \ldots \leq c_n \leq 1$. This is the main tool for our recursive schema. For $n_0 = 1,\ldots,n$ we have

$$\begin{aligned}
\min\{n_0, g(n_1)\} &= \mathbb{E}_{n,n_0}V_n \\
&= \sum_{j=1}^{n_0} j\mathbb{P}_{n,n_0}\left(V_n = j\right) \\
&= \sum_{j=1}^{n_0} j\frac{n_0}{j} c_{n_1+j}\mathbb{P}_{n,n_0-1}(V_n = j-1) \\
&= n_0 \sum_{j=0}^{n_0-1} c_{n_1+j+1}\mathbb{P}_{n,n_0-1}(V_n = j) \\
&= n_0\left(c_{n_1+1}\mathbb{P}_{n,n_0-1}(V_n = 0) + \sum_{j=1}^{n_0-1} c_{n_1+j+1}\mathbb{P}_{n,n_0-1}(V_n = j)\right).
\end{aligned}$$

We use $\min\{n_0, g(n_1)\}$ as the bounding function to indicate that there are the natural restrictions that $\text{ENFR}_{n,n_0}(\varphi_n) \leq n_0$ for all $n_0 = 0, \ldots, n$. Solving for c_{n_1+1} yields

$$c_{n_1+1} = \frac{\min\{1, g(n_1)/n_0\} - \sum_{j=1}^{n_0-1} c_{n_1+j+1}\mathbb{P}_{n,n_0-1}(V_n = j)}{1 - \sum_{j=1}^{n_0-1} c_{n_1+j+1}\mathbb{P}_{n,n_0-1}(V_n = j)}. \tag{5.6}$$

Similar to the case with $n = 2$, equation (5.6) becomes $c_n = \min\{1, g(n-1)\}$ for $n_0 = 1$. Choosing $c_k \leq c_{k+1} \leq \ldots \leq c_n < 1$ in advance and conducting the recursion for $n_1 = 0, \ldots, k-2$, starting with $n_1 = k-2$ can solve the problem that the last critical values are larger than one. Another idea is to approximate $\min\{n_0, g(n_1)\}$ by $g^*(n_1)$ such that $g^*(n_1) < n_0$ and $g^*(n_1) \leq g(n_1)$ and solve (5.1) for g^*. Nevertheless, the relations between the critical values and the bounding curve stay very complex. Both ideas lead to the same problem, we realize that $c_m > c_{m+1}$ or $c_m < 0$ solely at the very moment when we compute c_m. In the first case we could set $c_m = c_{m+1}$, but then for $n_1 = m-1$ we may not exhaust the level specified by the bounding function. The second case is also unpleasant, in particular if m is large. If $c_m < 0$, then $c_1, \ldots, c_m < 0$ by monotonicity. Such a SU test will reject no null hypothesis or at least $m+1$, that is $R_n \in \{0, m+1, \ldots, n\}$ almost surely. Furthermore, at least for $n_1 = m-1$ the procedure may exceed the bounding curve. It seems that by the complexity of the conditions one has to be lucky to find critical values such that (5.1) is fulfilled. Gontscharuk [26] pursued a similar aim for the FDR with a similar unsatisfactory result.

A slightly different way to consider the problem is to minimize

$$(c_1, \ldots, c_n) \mapsto ||(\mathbb{E}_{n,1}[V_n], \ldots, \mathbb{E}_{n,n}[V_n]) - (g(n-1), \ldots, g(0))|| \tag{5.7}$$

with respect to $0 \leq c_1 \leq \ldots \leq c_n \leq 1$ for some $\mathbb{R}^n$-norm $||\cdot||$. Since we already know that exact solving in general is not achievable, we are now interested in minimizing the function in (5.7).

5.2 An algorithm for computing exact critical values

In this section we present an algorithm for the construction of critical values $0 \leq c_1 \leq \ldots \leq c_n < 1$ such that

$$\max_{n_0=k,\ldots,n} |\mathbb{E}_{n,n_0} V_n(\varphi_n) - \min\{g_n(n-n_0), n_0\}| \tag{5.8}$$

is "minimal", where $k \geq 1$ is a predetermined constant, φ_n is SU test with critical values $(c_i)_{i=1,\ldots,n}$ and $g_n : \{0, \ldots, n-1\} \to [0, n]$. Indeed, it is a minimization problem in an n-dimensional space and we will not provide any mathematical statement about the optimality of our algorithm, but at least the performance in practice is admissible. The rough structure of the algorithm for a given ENFR bounding curve is the following.

1. Fix k between 1 and n and a constant $c \in (0, 1)$. Determine critical values with the techniques from Section 2.3 and set $n_0 = k$.

2. Calculate/approximate ENFR_{n,n_0} for the critical values.

3. Determine/approximate the critical value that has the "largest impact" on ENFR_{n,n_0}.

4. Update that critical value found in Step 3 and increase n_0 by one.

5. Go to Step 2 until $n_0 = n$.

This was one iteration of the algorithm. Of course, a new iteration can be initiated by starting at Step 2 and setting again $n_0 = k$.

5.2.1 The first step

With the first step we try to find a point in the n-dimensional space, that is a set of critical values, that serves as a "good" starting value for the minimization. At this point, the asymptotic relation between bounding functions and rejection curves, investigated in Section 2.3, are very helpful. In order to find an asymptotic critical value curve, we reformulate the problem. We assume that a function $g : [0, 1] \to [0, 1]$ exists such that

$$\lim_{n\to\infty} g_n(n - n_0)/n = g(\zeta)$$

holds for $n_0/n \to \zeta$ and $g(\zeta)/\zeta$ and $H(\zeta) = 1 - \zeta + g(\zeta)$ are strictly decreasing in ζ. If we know the inverse of $H(\zeta)$ explicitly, then we can directly determine the critical values c_i for $i/n \in H([0, 1])$ by $g(H^{-1}(i/n))/H^{-1}(i/n)$, cf. Section 2.3. As in Example 2.18, it is possible that $H(1) = \alpha > 0$. Then $H^{-1}(i/n)$ is not defined for $i/n < \alpha$. This can be solved by defining $\tilde{g}(\zeta) = \min\{g(\zeta), (1 - \zeta)g(1 - \epsilon)/\epsilon\}$ like in Remark 2.17. Recall that we are just interested in critical values in order to go to Step 2. Thus, we are somewhat free in choosing the missing critical values.
Any critical value equaling one, must be changed before going to the next step. For instance, two simple methods that seem to work well is to choose $\min\{c_i, c\}$ or cc_i as critical values for Step 2, where $c \in (0, 1)$.

5.2.2 The second and third step

In order to calculate the ENFR_{n,n_0}, we must evaluate $\mathbb{P}_{n,n_0}(V_n = j)$ for all j, which can be done by the recursion we apply in (5.9), confer (5.5).
From Section 5.1 we know that

$$\mathbb{E}_{n,n_0} V_n = \sum_{j=1}^{n_0} j\mathbb{P}_{n,n_0}(V_n = j) = \sum_{j=1}^{n_0} j\frac{n_0}{j} c_{n_1+j}\mathbb{P}_{n,n_0-1}(V_n = j - 1). \tag{5.9}$$

For a fixed n_0 we say $c_{n_1+j^*}$ has the largest impact on ENFR_{n,n_0}, where

$$j^* = \min \text{argmax}_{k=1,\dots,n_0} \mathbb{P}_{n,n_0-1}(V_n = k - 1).$$

Obviously, we ignore that $\mathbb{P}_{n,n_0'}(V_n = j)$ may depend on $c_{n_1+j^*}$ for some n_0' and j. The minimum in the definition of j^* saves computation time if argmax is not unique.

At this point we want to make some remarks with respect to the computational effort. In the first step we set $n_0 = k$ and before we enter Step 2 for the first time, $\mathbb{P}_{n,m}(V_n = j)$ has not been calculated for any m or j. Sometimes, it may be possible to calculate directly $\mathbb{P}_{n,m}(V_n = j)$. For instance, if $c_{n-m+1} = \ldots c_n = c$, then $\mathbb{P}_{n,m}(V_n = j) = \binom{m}{j} c^j (1-c)^{m-j}$ for $j = 0, \ldots, m (\leq n)$. But in general, we have to use the recursion and we see that, in order to calculate ENFR_{n,n_0}, we must calculate $\mathbb{P}_{n,m}(V_n = j)$ for all $m = 0, \ldots, k-1$ and $j = 0, \ldots, m$. Fortunately, this recursion starts with $m = 0$ only at the first time we enter the second step. This can be seen as follows. Suppose we did not alter any critical value in the fourth step. Then we can directly calculate $\mathbb{P}_{n,n_0+1}(V_n = \cdot)$ from $\mathbb{P}_{n,n_0}(V_n = \cdot)$. And even if we alter a critical value in the fourth step, and this can only be $c_{n_1+j^*}$, then this affects only $\mathbb{P}_{n,m}(V_n = \cdot)$ with $m > n_0 - j^*$ because $\mathbb{P}_{n,m}(V_n = \cdot)$ depends only on $c_{n-m+1}, \ldots, c_n$. Therefore, the recursion can start using $\mathbb{P}_{n,n_0-j^*}(V_n = \cdot)$ to update the probability mass functions of $\mathbb{P}_{n,m}(V_n = \cdot)$ for $m = n_0 - j^* + 1, \ldots, n_0$ instead of starting from scratch, that is $\mathbb{P}_{n,0}(V_n = \cdot)$.

5.2.3 The fourth step

Suppose $c_{n_1+j^*}$ is a critical value with the largest impact on ENFR_{n,n_0} found in the foregoing step. The update rule is

$$c_{n_1+j^*} \leftarrow \min\{\max\{c_{n_1+j^*} + \frac{\min\{g_n(n-n_0), n_0\} - \text{ENFR}_{n,n_0}}{n_0 \mathbb{P}_{n,n_0-1}(V_n = j^* - 1)}, c_{n_1+j^*-1}\}, c_{n_1+j^*+1}\},$$

where $c_0 = 0$ and $c_{n+1} = c < 1$; c is the predetermined constant from Step 1. The idea behind this rule is the following heuristic equation

$$g_n(n-n_0) = n_0 c_{n_1+j^*} \mathbb{P}_{n,n_0-1}(V_n = j^* - 1) + n_0 \sum_{\substack{j=1,\ldots,n_0,\\ j \neq j^*}} c_{n_1+j} \mathbb{P}_{n,n_0-1}(V_n = j-1),$$

where the right-hand side is the ENFR for the critical values $0 \leq c_1 \leq \ldots \leq c_{n_1+j^*-1} \leq c_{n_1+j^*} \leq c_{n_1+j^*+1} \leq \ldots \leq c_n$. Solving this heuristic equation for $c_{n_1+j^*}$ yields the essential part of the update rule by observing that

$$n_0 \sum_{\substack{j=1,\ldots,n_0,\\ j \neq j^*}} c_{n_1+j} \mathbb{P}_{n,n_0-1}(V_n = j-1) = \text{ENFR}_{n,n_0} - n_0 c_{n_1+j^*} \mathbb{P}_{n,n_0-1}(V_n = j^* - 1).$$

Obviously, we simplify the complex functional relation of ENFR_{n,n_0} and $c_{n_1+j^*}$ to a linear relation. It is possible to apply the recursion (5.5) many times to get a more sophisticated relation between the ENFR and the critical values and therefore to improve the update rule, but it seems that the simple update rule already yields good results. Furthermore, a more sophisticated relation will also lead to an update rule that would be more time consuming.

In order to ensure monotonicity of the critical values, other update rules may be possible at this step. For instance, one can first set

$$c_{n_1+j^*} \leftarrow \min\left\{c, c_{n_1+j^*} + \frac{\min\{g_n(n-n_0), n_0\} - \mathrm{ENFR}_{n,n_0}}{n_0 \mathbb{P}_{n,n_0-1}(V_n = j^* - 1)}\right\}$$

and then $c_k \leftarrow \min\{c_{n_1+j^*}, c_k\}$ for $k = 1, \ldots, n_1 + j^* - 1$ and $c_k \leftarrow \max\{c_{n_1+j^*}, c_k\}$ for $k = n_1 + j^* + 1, \ldots, n$. In general, this will update more than one critical value but the algorithm is not very stable in the sense that the resulting ENFR curve does not "converge".

Remark 5.1
It may seem odd that the algorithm does not search an n_0 such that the difference between the ENFR and g_n is maximal and tries to correct this as much as possible. The reason is that in order to find this largest difference, we have to calculate ENFR_{n,n_0} for all n_0 and by our recursive formulas we must start with $n_0 = 1$. Updating more than one critical value seems to lead in many situations to unstable ENFR curves. But calculating ENFR_{n,n_0} for all n_0 only to update one critical value and then again calculating ENFR_{n,n_0} for all n_0 to find the new largest difference is very time consuming.

Remark 5.2
The computational effort is considerable. One important factor is the accuracy of the computations. As n grows the accuracy of the calculations had to be increased because $\mathbb{P}_{n,n_0}(V_n = j)$ gets very small or even tends to zero. But as can be seen from Figure 5.6, at least for this example, the asymptotic rejection curve obtained in Step 1 is already suitable for $n = 200$. The computation time for Figure 5.6 was only a few minutes. But for $n = 2000$ the computation time would be a few hours. Unfortunately, the algorithm is not suitable for parallelization because of its recursive nature. An idea to save computation time could be to determine and update $c_{n_1+j^*}$ in Step 3 and 4 only if the ENFR differs from $g_n(n_1)$ by a certain amount.

5.2.4 Examples

Now, we apply the algorithm for fixed n in order to obtain critical values c_i, such that

$$\max_{n_0=k,\ldots,n} \{|\mathrm{ENFR}_{n,n_0}(\varphi_n) - \min\{g_n(n-n_0), n_0\}|\}$$

is "minimal" for SU test φ_n with critical values c_i. Figure 5.2 shows three different asymptotic ENFR bounding curves and the corresponding asymptotic rejection curves. The next three examples will refer to theses functions.
For all examples we choose $k = 1$. In the first example, with $n = 50$, we consider the ENFR bounding function $g_{1,n}(n_1) = (n_1 + 1)0.05/(1 - 0.05)$ which leads to the asymptotic ENFR bounding function $g_1(\zeta) = (1 - \zeta)0.05/(1 - 0.05)$ with the corresponding asymptotic rejection

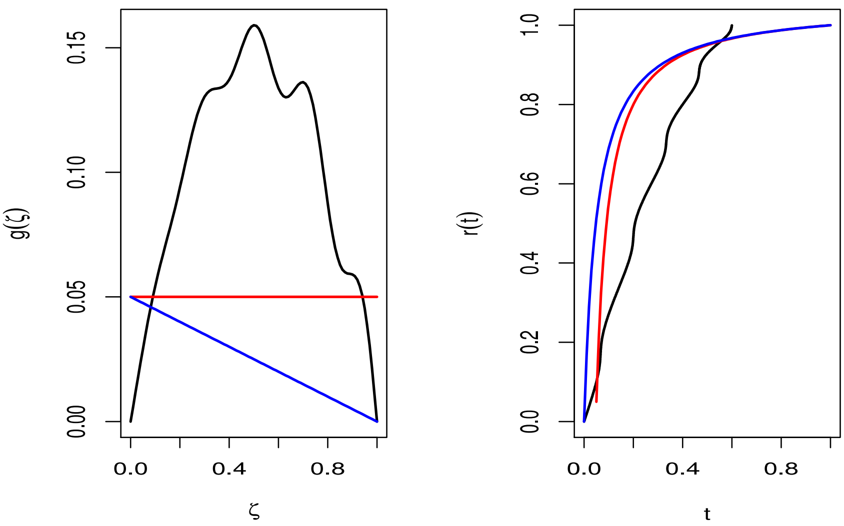

Figure 5.2: *The left figure shows the asymptotic ENFR bounding functions* $g_1(\zeta) = (1-\zeta)0.05/(1-0.05)$ *(blue),* $g_2(\zeta) = 0.05$ *(red) and* $g_3(\zeta) = 0.6\zeta(1-\zeta+0.03\sin(9\pi\zeta))$ *(black). The right figure shows the corresponding asymptotic rejection curves, cf. Remarks 2.14 and 2.15,* $r_2(t) = t/(t(1-0.05)+0.05)$ *and* $r_1(t) = 1-0.05/t+0.05$ *(red) defined only for* $t \in [0.05, 1]$*. Note,* $r_3(t)$ *has no explicit representation.*

curve $r_1(t) = t/(t(1-0.05)+0.05)$. The initial critical values for the algorithm are $c_i = \min\{c, r_1^{-1}(i/n)\}$ and we choose $c = 0.7$. Figure 5.3 visualize the ENFR after Step 1, after one iteration and after three iterations.

In the next example, again with $n = 50$, we consider the ENFR bounding function $g_{2,n}(n_1) = 0.05n$, which leads to $g_2(\zeta) = 0.05$. In this case, following Remark 2.14, the corresponding asymptotic rejection curve is $r_2(t) = 1-0.05/t+0.05$ for $t \in [0.05, 1]$. Actually, r_2 is not a rejection curve in our sense because the domain of r_2 is $[0.05, 1]$ instead of $[0, 1]$. Nevertheless, in order to create a set of critical values, we define $c_i = cr_2^{-1}(i/n)$ for $i = 1, \ldots, n$ and set $c = 0.98$. Note, $1-0.05/t+0.05$ is injective on $(0, 1]$. Hence, c_i is defined for $i = 1, \ldots, n$. Figure 5.4 shows the ENFR again after Step 1, after one iteration and after three iterations.

The last example contains the "odd" ENFR bounding function $g_{3,n}(n_1) = 0.6(n-n_0)(1-n_0/n+0.03\sin(9\pi n_0/n))$ where no explicit representation of the asymptotic rejection curve or the asymptotic critical value curve exists. This bounding function is not of practical relevance, but it demonstrates that the algorithm can not cope fast changes in the bounding function. But this seems to hold only for small sample sizes. One reason may be that the critical values were obtained by asymptotic considerations and the update rule is too "stiff". Another reason may be an effect we already observed in Section 5.1, where we have seen that g may not increase or decrease too fast in the case $n = 2$, cf. (5.4). For $n = 50$ only three iterations were conducted, but more iterations did not improve the result, cf. Figure 5.5. For $n = 200$ we directly see that the critical

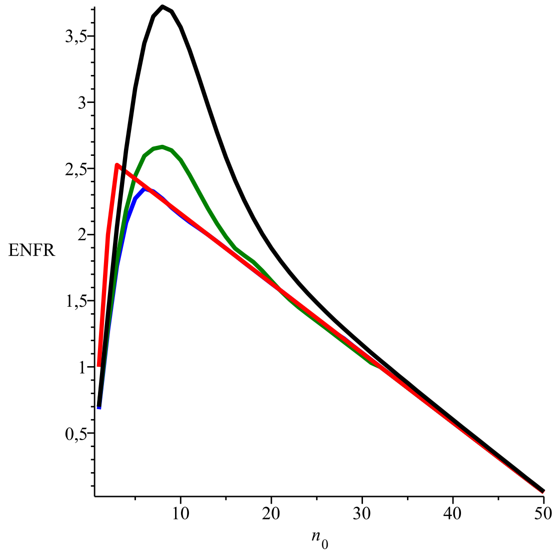

Figure 5.3: *The ENFR of a SU test with critical values* $\min\{c, r_1^{-1}(i/n)\}(i = 1, \ldots, 50)$ *(black), where* $r_1(t) = t/(t(1-0.05)+0.05)$, $k = 1$, $c = 0.7$ *and the ENFR of the SU test after one (green) and after three (blue) iterations. The red line is the ENFR bounding function* $\min\{n_0, (n_1+1)0.05/(1-0.05)\}$.

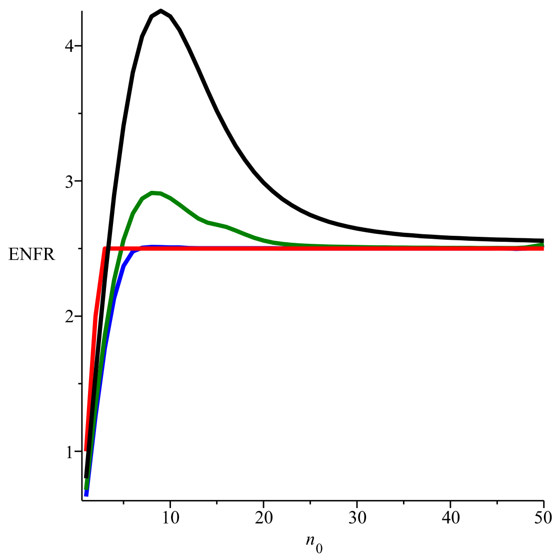

Figure 5.4: *The ENFR of a SU test with critical values* $cr_2^{-1}(i/n)(i = 1, \ldots, 50)$ *(black),* $k = 1$, $c = 0.98$, *where* $r_2(t) = 1 - 0.05/t + 0.05$, *and the ENFR of the SU test after one (green) and after three (blue) iterations. The red line is the ENFR bounding function* $\min\{n_0, 0.05n\}$.

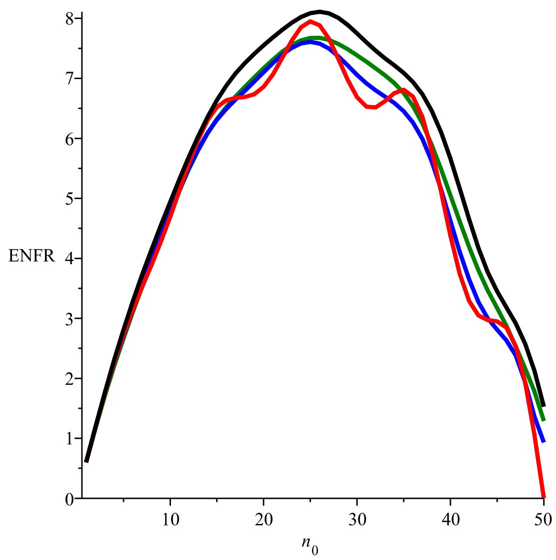

Figure 5.5: *The ENFR of a SU test (black) with critical values that are implicitly defined by equation* (2.10), *with* $g(\zeta) = g_3(\zeta) = 0.6\zeta(1 - \zeta + 0.03\sin(9\pi\zeta))$, $k = 1$, $c = 0.98$ *and the ENFR of the SU test after one (green) and after three (blue) iterations. The red line is the ENFR bounding function* $\min\{n_0, 0.6n_0(1 - n_0/n + 0.03\sin(9\pi n_0/n))\}$.

values from Step 1 already provide a SU test that is close to the ENFR bounding curve. Step 1 of the algorithm seems to be very important. It seems that the asymptotic rejection curve provides critical values that are beneficial for the iterations. Using other critical values, for instance induced by the Simes line, yields for $n = 200$ even after 10 iterations results that are worse, cf. Figure 5.7. The reason for this is the update rule, because the updated critical value $c_{n_1+j^*}$ is constrained by the critical values in the neighborhood to ensure monotonicity. An algorithm which decreases or increases the critical values around $c_{n_1+j^*}$ is also possible but it yields worse results.

5.3 Summary

This chapter started with the aim to construct a SU procedure that fulfills perfect ENFR control, that is (5.1), for fixed n and for arbitrary bounding functions. But even the case $n = 2$ shows that this aim is not achievable in general. Therefore, we developed an algorithm in order to achieve that the ENFR is as close as possible to the bounding function. This algorithm showed good performance in different situations but no mathematical results concerning its performance are derived.

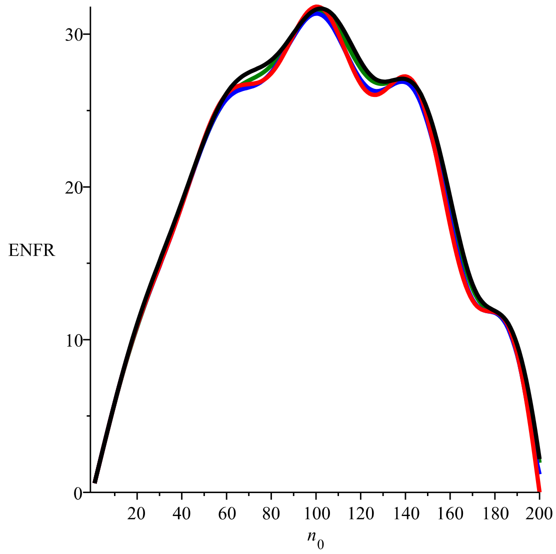

Figure 5.6: *The ENFR of a SU test (black) with critical values that are implicitly defined by equation* (2.10), *with* $g(\zeta) = g_3(\zeta) = 0.6\zeta(1 - \zeta + 0.03\sin(9\pi\zeta))$, $k = 1$, $c = 0.98$ *and the ENFR of the SU test after one (green) and after three (blue) iterations. The red line is the ENFR bounding function* $\min\{n_0, 0.6n_0(1 - n_0/n + 0.03\sin(9\pi n_0/n))\}$.

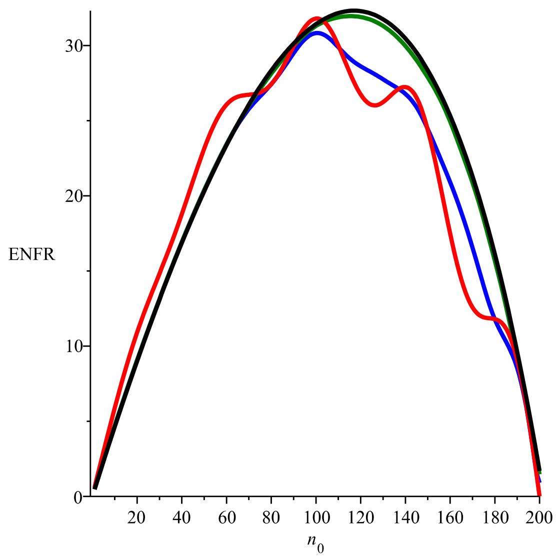

Figure 5.7: *The ENFR of a SU test (black) with critical values* $c_i = 0.5i/n$, $k = 1$, $c = 0.98$ *and the ENFR of the SU test after one (green) and after ten (blue) iterations. The red line is the ENFR bounding function* $\min\{n_0, 0.6n_0(1 - n_0/n + 0.03\sin(9\pi n_0/n))\}$.

Chapter 6

Excess probabilities and differential equations

As already mentioned in Section 1.7, Genovese and Wasserman proposed

$$\mathbb{P}_\vartheta\left(\text{FDP}(\varphi_n) > c\right) = \mathbb{P}_\vartheta\left(\frac{V_n(\varphi_n)}{R_n(\varphi_n) \vee 1} > c\right) \leq \alpha$$

as an error criterion instead of

$$\text{FDR}_\vartheta(\varphi_n) = \mathbb{E}_\vartheta\left(\frac{V_n(\varphi_n)}{R_n(\varphi_n) \vee 1}\right) \leq \alpha.$$

Their argument was that controlling the FDR does not necessarily offer high confidence that the FDP will be small. For the ENFR this argument leads to the following error criterion,

$$\mathbb{P}_\vartheta\left(V_n(\varphi_n) > g_n(n_1(\vartheta))\right) \leq \alpha \text{ for all } \vartheta \in \Theta, \tag{6.1}$$

instead of $\mathbb{E}_\vartheta V_n(\varphi_n) \leq g_n(n_1(\vartheta))$ for all $\vartheta \in \Theta$. We say that a SU procedure φ_n controls the g_n-*NFRX* at level α if (6.1) holds. NFRX is an abbreviation for the number of false rejections exceedance. In this chapter, we show for a sequence of suitable SU tests φ_n that

$$\lim_{n\to\infty} \mathbb{P}_\vartheta\left(V_n(\varphi_n) > g_n(n_1)\right) = \alpha \tag{6.2}$$

is connected to a sequence of ordinary differential equations (ODEs). As far as the author knows this is the first result that relates a specific probabilistic error measure to the field of differential equations. The aim is to construct SU tests φ_n such that (6.2) holds true for a given sequence of functions g_n. We start by investigating the crossing-point of φ_n and establish that the crossing-point is asymptotically normal. Then we investigate $\lim_{n\to\infty} \mathbb{P}_\vartheta\left(V_n(\varphi_n) > g_n(n_1)\right)$ and state sufficient conditions such that (6.2) holds. These conditions will lead to a sequence of ODEs that implicitly define a sequence of rejection curves. The existence of solutions is investigated and it is shown that an interval $I \subset (0, 1)$ (independent of n) exists such that the sequence of ODEs are solvable eventually for all n on I. Solutions of the ODEs will provide, under some regularity conditions, a sequence of rejection curves r_n such that (6.2) holds for SU tests φ_n induced by r_n. In our considerations $n_0/n \to \zeta \in (0,1)$ is kept fixed. Nevertheless, solutions of the ODE on the interval I will provide rejection curves such that (6.2) holds for $\zeta \in Z$, where $Z = (a,b)$ with $0 < a < b < 1$. For the important case $g_n(n_1) = (n_1+1)\gamma$, with $\gamma \in (0,1)$, we exemplify three different methods to obtain a sequence of rejection curves such that

$$\lim_{n\to\infty} \mathbb{P}_\vartheta\left(V_n(\varphi_n) > g_n(n_1)\right) \approx \alpha \tag{6.3}$$

holds for $\zeta \in Z$. At the end of this chapter, we discuss how our findings can be adapted if the error measure is the $c-$FDX.

The key property for the connection between (6.2) and ODEs is the fact that the crossing-point between a rejection curve and an ecdf asymptotically follows a normal distribution. This asymptotic normality was first observed by Neuvial in [44]. In the aforementioned publication a SU test induced by a fixed rejection curve and LSUPI tests were considered. In our case, a sequence of rejection curves has to be considered which is not covered by the theorems developed in [44].

6.1 General assumptions

In this chapter we assume that $(p_i)_{i\in\mathbb{N}}$ is an independent sequence with $p_i \sim F_0$ if $i \in I_0$ and $p_i \sim F_1$ if $i \in I_1$, with $F_0(t) < F_1(t)$ for all $t \in (0,1)$. There is a little slip in the notation because in this dissertation we assumed that F_n denotes the ecdf of the p-values $p_1,\dots,p_n$. For the sake of simplicity we ignore that F_n is ambiguous for $n=0$ and $n=1$.

We consider $\vartheta \in \Theta$ and $\lim_{n\to\infty} n_0/n = \zeta \in (0,1)$ fixed and define $F(t) = \zeta F_0(t) + (1-\zeta)F_1(t)$. Denote by $f(t) = \zeta f_0(t) + (1-\zeta) f_1(t)$ the corresponding density function and assume that f_0 and f_1 are continuous on $(0,1)$.

In the following, we always consider a sequence of continuously differentiable rejection curves r_n such that r_n converges uniformly to a rejection curve r. Further, we always assume that r_n' converges uniformly to r', where r_n' (r') denotes the first derivative of r_n (r). Denote by $C[0,1]$ the set of all continuous functions $x : [0,1] \to \mathbb{R}$ and by $D[0,1]$ the set of all functions $x : [0,1] \to \mathbb{R}$ that are right continuous and whose limits from the left exist everywhere in $[0,1]$. Both spaces are endowed with the supremum norm, that is $||x||_{[0,1]} = \sup\{|x(t)| : t \in [0,1]\}$. Further, define

$$D_n = \{x \in D[0,1] : F + n^{-1/2}x \text{ is a distribution function on [0,1]}\},$$

which we also endow with the supremum norm. We denote the largest crossing-point by

$$T_n(G) = \sup\{t \in [0,1] : G(t) = r_n(t)\},$$

where G is an arbitrary distribution function on $[0,1]$. Note, $T_n(G)$ is well-defined because r_n is assumed to be continuous, $r_n(0) = 0$, and $r_n(1) \geq 1$.

For technical reasons we also assume the following condition for the entire chapter.
Unique crossing-point condition: There exists a $\tau^* \in (0,1)$ such that

1. τ^* is the only point in $(0,1]$ with $r(\tau^*) = F(\tau^*)$,

2. $r'(\tau^*) > f(\tau^*)$.

We want to distress that by condition 1 of the unique crossing-point condition we exclude the case that $r(1) = 1$. SU tests induced by such a rejection curve will always reject all null hypotheses.

Remark 6.1
From $r_n(T_n(F)) = F(T_n(F)) = \zeta F_0(T_n(F)) + (1-\zeta)F_1(T_n(F))$ it follows that

$$\zeta = \frac{F_1(T_n(F)) - r_n(T_n(F))}{F_1(T_n(F)) - F_0(T_n(F))}$$

for $T_n(F) \in (0,1)$. Define

$$Q_n(T_n(F)) = \frac{F_1(T_n(F)) - r_n(T_n(F))}{F_1(T_n(F)) - F_0(T_n(F))}$$

for short. Similar, we have

$$\zeta = Q(\tau^*) = \frac{F_1(\tau^*) - r(\tau^*)}{F_1(\tau^*) - F_0(\tau^*)}.$$

Note, Q_n depends on r_n and Q depends on r. These two relations will be used frequently. The main purpose of these expressions is to obtain results that are not formulated in ζ but solely in terms of crossing-points $T_n(F)$ or τ^*.

As usual, we denote by $\Phi(z)$ the standard normal distribution function.

6.2 Asymptotic normality of crossing-points

The aim is to proof that $\sqrt{n}(T_n(F_n) - T_n(F))$ converges to a normally distributed random variable. In order to sketch the proof, we start with a similar but simpler case. Let X_n be real-valued random variables and $\theta \in \mathbb{R}$ with $\sqrt{n}(X_n - \theta) \to X$ in distribution, where $X \sim N(0, \sigma^2)$. Suppose we have a differentiable function $h : \mathbb{R} \to \mathbb{R}$, then the delta method, see Chapter 3 in [74], directly yields $\sqrt{n}(h(X_n) - h(\theta)) \to h'(\theta)X$ in distribution, where h' denotes the derivative of h.
For our purpose, we have to allow that X_n is a stochastic process and h is a statistical functional. Of course, if it is possible to establish a similar result as above, it is natural to ask what the meaning of h' is if h is a functional. The decisive term in this context is the Hadamard differentiability, which is defined as follows. Let D be a metric space. A functional $T : D_T \to \mathbb{R}$, $D_T \subset D$, is Hadamard differentiable at $\theta \in D_T$ if there exists a linear map $T'_\theta : D \to \mathbb{R}$ such that

$$\frac{T(\theta + tH_t) - T(\theta)}{t} \to T'_\theta(H), \quad \text{as } t \downarrow 0,\ H_t \to H \in D, \text{ and } \theta + tH_t \in D_T. \tag{6.4}$$

Suppose now that $X_n : \Omega \to D_T$ is a stochastic process, $\theta \in D_T$, and $\sqrt{n}(X_n - \theta) \to X$ in distribution, then the *functional* delta method yields $\sqrt{n}(T(X_n) - T(\theta)) \to T'_\theta(X)$ in distribution. Now, we are almost done. In our case, $X_n = F_n$, $\theta = F$ and $\sqrt{n}(F_n - F) \to B$ in distribution, where B denotes a Brownian bridge. However, we have a sequence of statistical functionals, namely T_n. But following the proof of the functional delta method (Theorem 20.8 in [74]) will allow to establish the asymptotic normality of $T_n(F_n)$ as follows. We have

$$n^{1/2}(T_n(F_n) - T_n(F)) = \frac{T_n\left(F + n^{-1/2}\left[n^{1/2}(F_n - F)\right]\right) - T_n(F)}{n^{-1/2}}.$$

Denote by

$$\Lambda_n(x) = \frac{T_n(F + n^{-1/2}x) - T_n(F)}{n^{-1/2}}$$

a functional that maps D_n to $\mathbb{R}$. Recall,

$$D_n = \{x \in D[0,1] : F + n^{-1/2}x \text{ is a distribution function on [0,1]}\}.$$

First, in Lemma 6.2 we show the existence of a functional $\Lambda : C[0,1] \to \mathbb{R}$ such that $\Lambda_n(H_n) \to \Lambda(H)$ if $H_n \to H$ in $D[0,1]$, where $H_n \in D_n$ and $H \in C[0,1]$. Second, assuming that the stochastic process $n^{1/2}(F_n - F)$ converges to B in distribution, we apply the extended continuous-mapping theorem (Theorem B.4) to obtain $\Lambda_n(n^{1/2}(F_n - F)) \to \Lambda(B)$ in distribution and conclude by establishing the normality of $\Lambda(B)$. To sum up,

$$n^{1/2}(T_n(F_n) - T_n(F)) = \Lambda_n(n^{1/2}(F_n - F)) \to \Lambda(B).$$

Note, Λ_n is very similar to the ratio in (6.4) and thus Λ is some kind of Hadamard derivative. Virtually, we follow the standard technique: calculate the (Hadamard) derivative and apply the (functional) delta method.

Lemma 6.2

Let $H \in C[0,1]$ and $H_n \in D_n \subset D[0,1]$ be an arbitrary sequence such that H_n converges to H in $D[0,1]$. Then

$$\Lambda_n(H_n) = \frac{T_n(F + n^{-1/2}H_n) - T_n(F)}{n^{-1/2}} \to \frac{H(\tau^*)}{r'(\tau^*) - f(\tau^*)}.$$

Furthermore, for all $n \in \mathbb{N}$ it holds that

$$\Lambda_n(H_n) = \frac{H_n(\tau_n)}{r'_n(t_n) - f(t_n)},$$

where $\tau_n = T_n(F + n^{-1/2}H_n)$ and t_n lies between τ_n and $T_n(F)$.

Proof: For short, let $\tau_n^* = T_n(F)$. Obviously, τ_n and τ_n^* converge to τ^* by the uniform convergence of H_n and r_n to H and r, respectively. We define $\Psi_n(t) = r_n(t) - F(t)$. By the mean value theorem we have

$$\Psi_n(\tau_n) - \Psi_n(\tau_n^*) = \Psi'_n(t_n)(\tau_n - \tau_n^*),$$

where t_n is between τ_n and τ_n^*. Since f and r' are continuous and r'_n converges uniformly to r', we have that $\Psi'_n(t_n)$ converges to $r'(\tau^*) - f(\tau^*)$. Note that $\Psi_n(\tau_n^*) = r_n(\tau_n^*) - F(\tau_n^*) = 0$ and hence

$$\tau_n - \tau_n^* = \frac{\Psi_n(\tau_n)}{r'_n(t_n) - f(t_n)}.$$

For the numerator it follows that

$$\begin{aligned}\Psi_n(\tau_n) &= r_n(\tau_n) - F(\tau_n)\\ &= r_n(\tau_n) - (F + n^{-1/2}H_n)(\tau_n) + (F + n^{-1/2}H_n)(\tau_n) - F(\tau_n)\\ &= n^{-1/2}H_n(\tau_n).\end{aligned}$$

Altogether we have

$$\Lambda_n(H_n) = n^{1/2}[\tau_n - \tau_n^*] = \frac{H_n(\tau_n)}{r'_n(t_n) - f(t_n)}.$$

We finally conclude that $H_n(\tau_n) \to H(\tau^*)$ by the uniform convergence of H_n and the continuity of H. □

Theorem 6.3 (Asymptotic normality of the crossing-points)
If $n_0/n = \zeta + o(n^{-1/2})$, then

$$\lim_{n\to\infty} \mathbb{P}_\vartheta\left(n^{1/2}\frac{T_n(F_n) - T_n(F)}{\tilde{s}(r,\zeta,\tau^*)} \leq z\right) = \Phi(z)$$

for all $z \in \mathbb{R}$, where

$$\tilde{s}(x,\zeta,t) = \frac{\sqrt{\zeta F_0(t)(1-F_0(t)) + (1-\zeta)F_1(t)(1-F_1(t))}}{x'(t) - f(t)},$$

with x' denoting the first derivative of x. Furthermore, the representation

$$n^{1/2}[T_n(F_n) - T_n(F)] = n^{1/2}\frac{F_n(T_n(F_n)) - F(T_n(F_n))}{r'_n(\tau^*) - f(\tau^*)} + o(1) \tag{6.5}$$

holds almost surely.

Proof: We first determine the limit process of $H_n = n^{1/2}(F_n - F)$ and then apply Lemma 6.2 together with the extended continuous-mapping theorem. We decompose the ecdf in

$$F_n(t) = \frac{1}{n}\sum_{i=1}^n \mathbb{I}_{\{p_i \leq t\}} = \frac{1}{n}\sum_{i=1}^n \mathbb{I}_{\{p_i \leq t, i \in I_0\}} + \frac{1}{n}\sum_{i=1}^n \mathbb{I}_{\{p_i \leq t, i \in I_1\}} = \frac{n_0}{n}F^0_{n_0}(t) + \frac{n_1}{n}F^1_{n_1}(t) \text{ (say)}.$$

Note,

$$
\begin{aligned}
H_n &= n^{1/2}[\frac{n_0}{n}F^0_{n_0} - \zeta F_0 + \frac{n_1}{n}F^1_{n_1} - (1-\zeta)F_1] \\
&= F^0_{n_0} n^{1/2}(\frac{n_0}{n} - \zeta) + \sqrt{\frac{n}{n_0}}\zeta n_0^{1/2}(F^0_{n_0} - F_0) \\
&\quad + F^1_{n_1} n^{1/2}(\frac{n_1}{n} - 1 + \zeta) + \sqrt{\frac{n}{n_1}}(1-\zeta) n_1^{1/2}(F^1_{n_1} - F_1) \\
&= \sqrt{\frac{n}{n_0}}\zeta n_0^{1/2}(F^0_{n_0} - F_0) + \sqrt{\frac{n}{n_1}}(1-\zeta) n_1^{1/2}(F^1_{n_1} - F_1) + o(1).
\end{aligned}
$$

Since $(p_i)_{i\in I_0}$ and $(p_i)_{i\in I_1}$ are independent, we obtain by Donsker's Theorem from [74], p. 266, that $H_n \to X = \sqrt{\zeta}B_{F_0} + \sqrt{1-\zeta}B_{F_1}$ in distribution in $D[0,1]$, where B_F denotes an F-Brownian bridge, that is a Gaussian process with zero mean and covariances

$$\mathrm{Cov}(B_F(t_1), B_F(t_2)) = F(\min\{t_1, t_2\}) - F(t_1)F(t_2).$$

By Theorem B.4 (extended continuous-mapping theorem) and Lemma 6.2 we conclude

$$n^{1/2}(T_n(F_n) - T_n(F)) \to \frac{X(\tau^*)}{r'(\tau^*) - f(\tau^*)}$$

in distribution. Obviously, $X(\tau^*) \sim N(0, \zeta F_0(\tau^*)(1-F_0(\tau^*)) + (1-\zeta)F_1(\tau^*)(1-F_1(\tau^*))$ by the independence of B_{F_0} and B_{F_1}.
The representation (6.5) directly follows by the definition of H_n, continuity of f and r' at τ^*, the uniform convergence of r'_n, and Lemma 6.2. □

Remark 6.4
Consider a fixed rejection curve r and the corresponding SU tests φ_n induced by $r_n = r$. Denote by $g(\zeta) = \lim_{n\to\infty} \mathrm{ENFR}_\vartheta(\varphi_n)/n$. If Theorem 6.3 is applicable, then

$$\lim_{n\to\infty} \mathbb{P}_\vartheta\left(n^{1/2}\frac{T_n(F_n) - T_n(F)}{\tilde{s}(r,\zeta,\tau^*)} \le 0\right) = \frac{1}{2}.$$

This means that with a probability of 1/2 the random crossing-point is asymptotically larger than the asymptotic crossing-point. Thereby, it is reasonable to expect that

$$\lim_{n\to\infty} \mathbb{P}_\vartheta\left(V_n > g_n(n_1)\right) = \frac{1}{2},$$

where $g_n(n_1) = ng(1 - n_1/n)$. Theorem 6.5 and Remark 6.17 will show that this loosely formulated conjecture is true.

6.3 Quantile equation

We are now able to prove that under some regularity conditions equation (6.2), that is,

$$\lim_{n\to\infty} \mathbb{P}_\vartheta\left(V_n(\varphi_n) > g_n(n_1)\right) = \alpha$$

holds true for suitable SU tests φ_n. Although the proof is longish and technical, the idea is very simple and based on

$$V_n(\varphi_n) > g_n(n_1) \Leftrightarrow W_n(T_n(F_n)) > d_n,$$

for suitable W_n and d_n. Applying Taylor's theorem to W_n shows that $W_n(T_n(F_n))$ is asymptotically normal since $T_n(F_n)$ is asymptotically normal. Finally, we show that $V_n(\varphi_n) > g_n(n_1)$ is equivalent to

$$Y_n > n^{1/2}\frac{(1-Q_n(T_n(F)))F_1(T_n(F)) + g(Q_n(T_n(F))) - r_n(T_n(F))}{r_n'(T_n(F))}, \tag{6.6}$$

where $Q_n(t) = (F_1(t)-r_n(t))/(F_1(t)-F_0(t))$ and $g(\zeta) = \lim_{n\to\infty} g_n(n_1)/n$. Thereby, $(Y_n)_{n\in\mathbb{N}}$ is a sequence of random variables which will will be shown to converge in distribution to a normally distributed random variable with mean zero and standard deviation $s(r,\zeta,\tau^*)/r'(\tau^*)$, where s is the function already introduced in Section 2.4, i.e.

$$s(x,\zeta,t) = \frac{\sqrt{(x'(t)-(1-\zeta)f_1(t))^2\zeta F_0(t)(1-F_0(t)) + \zeta^2 f_0^2(t)(1-\zeta)F_1(t)(1-F_1(t))}}{x'(t)-\zeta f_0(t)-(1-\zeta)f_1(t)}.$$

In the following, a helpful notation will be

$$q_n(T_n(F)) = n^{1/2}\frac{(1-Q_n(T_n(F)))F_1(T_n(F)) + g(Q_n(T_n(F))) - r_n(T_n(F))}{s(r_n, Q_n(T_n(F)), T_n(F))}, \tag{6.7}$$

which is the right-hand side of (6.6) divided by $s(r_n, Q_n(T_n(F)), T_n(F))/r_n'(T_n(F))$.

Theorem 6.5 (Quantile equation theorem)
If φ_n is a SU test induced by r_n, $n_0/n = \zeta + o(n^{-1/2})$, $g_n(n_1)/n = g(\zeta) + o(n^{-1/2})$, and

$$q_n(T_n(F)) = q + o(1) \tag{6.8}$$

for $q = \Phi^{-1}(1-\alpha)$, $\alpha \in (0,1)$, then (6.2) *holds, that is,*

$$\lim_{n\to\infty} \mathbb{P}_\vartheta\left(V_n(\varphi_n) > g_n(n_1)\right) = \alpha.$$

We call equation (6.8) the quantile equation. Before we present the proof, we want to make two remarks with respect to this theorem.

Remark 6.6
As stated at the beginning of this chapter we aim for equation (6.2), that is,

$$\lim_{n\to\infty} \mathbb{P}_\vartheta\left(V_n(\varphi_n) > g_n(n_1)\right) = \alpha,$$

where φ_n are SU tests and $\alpha \in (0,1)$. Suppose we have a sequence of rejection curves r_n such that $q_n(T_n(F)) = \Phi^{-1}(1-\alpha)+o(1)$. Theorem 6.5 guarantees that (6.2) holds true. Of course, we have no information about $\mathbb{P}_\vartheta\left(V_n(\varphi_n) > g_n(n_1)\right)$ for fixed n. So, if r_n fulfills $q_n(T_n(F)) = \Phi^{-1}(1-\alpha)$, both $\mathbb{P}_\vartheta\left(V_n(\varphi_n) > g_n(n_1)\right) < \alpha$ or $\mathbb{P}_\vartheta\left(V_n(\varphi_n) > g_n(n_1)\right) > \alpha$ is possible. Nevertheless, later in the thesis we try to obtain r_n such that $q_n(T_n(F))$ is as close as possible to $\Phi^{-1}(1-\alpha)$. This is in accordance with the usual approach in asymptotic statistic. For instance, if the distribution of some test statistic X_n converges to Φ and we aim for $\lim_{n\to\infty} \mathbb{P}\left(X_n > c\right) = \alpha$, then it is common to choose $c = \Phi^{-1}(1-\alpha)$.

Remark 6.7

Asymptotic tests are always unsatisfactory in the sense that they provide no information for fixed n. Of course, if $\mathbb{P}_\vartheta\left(V_n(\varphi_n) > g_n(n_1)\right) = 1 - (\Phi(q) + \Psi(q)/\sqrt{n}) + o(1)$, we would try to determine r_n such that $q_n(T_n(F)) = q$ and $1 - (\Phi(q) + \Psi(q)/\sqrt{n}) = \alpha$. Although this would not provide any information about $\mathbb{P}_\vartheta\left(V_n(\varphi_n) > g_n(n_1)\right)$ for fixed n, it could be expected that now the expression is closer to α than it would be if only $q_n(T_n(F)) = \Phi^{-1}(1-\alpha)$ holds true. Determining such a Ψ in Theorem 6.5 may be a field of future research.

Proof of Theorem 6.5. Denote by $F^0_{n_0}(t)$ ($F^1_{n_1}(t)$) the ecdf of the p-values corresponding to true (false) null hypotheses. Then,

$$\begin{aligned}
&V_n(\varphi_n) > g_n(n_1)\\
&\Leftrightarrow n_0 F^0_{n_0}(T_n(F_n)) > g_n(n_1)\\
&\Leftrightarrow \frac{n_0 F^0_{n_0}(T_n(F_n)) + n_1 F^1_{n_1}(T_n(F_n))}{n} - \frac{n_1 F^1_{n_1}(T_n(F_n))}{n} > \frac{g_n(n_1)}{n}\\
&\Leftrightarrow r_n(T_n(F_n)) - \frac{n_1 F^1_{n_1}(T_n(F_n))}{n} > \frac{g_n(n_1)}{n}.
\end{aligned}$$

We now investigate the last inequality and virtually collect all deterministic terms on the right-hand side and all terms on the left-hand side that depend on the p-values. After standardizing, the left-hand side will converge in distribution to a standard normal random variable. In order to simplify the notation, we suppress $T_n(F_n)$ and write solely m for $m(T_n(F_n))$. Hence, $V_n(\varphi_n) > g(n_1)$ is equivalent to $r_n - n_1 F^1_{n_1}/n > g_n(n_1)/n$. Substituting r_n by its Taylor expansion at $T_n(F)$ in the latter inequality, that is,

$$r_n(T_n(F_n)) = r_n(T_n(F)) + r_n'(T_n(F))(T_n(F_n) - T_n(F)) + o_P(n^{-1/2}),$$

and multiplying by $n^{1/2}$ yields

$$n^{1/2}[T_n(F_n) - T_n(F)] - \frac{n^{1/2} n_1 F^1_{n_1}}{n r_n'(T_n(F))} + o_P(1) > n^{1/2}\left[\frac{g_n(n_1)}{n r_n'(T_n(F))} - \frac{r_n(T_n(F))}{r_n'(T_n(F))}\right]. \quad (6.9)$$

For short, we rewrite (6.9) as

$$\Delta_n - \frac{A_n}{r_n'(T_n(F))} + o_P(1) > n^{1/2} D_{1,n}.$$

First, we center A_n which results in a deterministic expression, that is,

$$\begin{aligned}
A_n &= n^{1/2}\left[\frac{n_1}{n}F_{n_1}^1 - (1-\zeta)F_1\right] + n^{1/2}(1-\zeta)F_1 \\
&= n^{1/2}\left[(1-\zeta+o(n^{-1/2}))F_{n_1}^1 - (1-\zeta)F_1\right] + n^{1/2}(1-\zeta)F_1 \\
&= \sqrt{\frac{n}{n_1}}(1-\zeta)n_1^{1/2}\left[F_{n_1}^1 - F_1\right] + n^{1/2}(1-\zeta)(F_1 - F_1(T_n(F))) \\
&\quad + n^{1/2}(1-\zeta)F_1(T_n(F)) + o_P(1).
\end{aligned}$$

The Taylor expansion of F_1 at $T_n(F)$ yields

$$\begin{aligned}
A_n &= \sqrt{\frac{n}{n_1}}(1-\zeta)n_1^{1/2}\left[F_{n_1}^1 - F_1\right] + (1-\zeta)f_1(T_n(F))\Delta_n \\
&\quad + n^{1/2}(1-\zeta)F_1(T_n(F)) + o_P(1) \\
&= \Delta_{1,n} + D_{2,n}\Delta_n + n^{1/2}D_{3,n} + o_P(1) \text{ (say).}
\end{aligned}$$

Therefore, (6.9) is equivalent to

$$\left(1 - \frac{D_{2,n}}{r_n'(T_n(F))}\right)\Delta_n - \frac{\Delta_{1,n}}{r_n'(T_n(F))} + o_P(1) > n^{1/2}\left(\frac{D_{3,n}}{r_n'(T_n(F))} + D_{1,n}\right). \qquad (6.10)$$

In the following, we show that the left-hand side of (6.10) is asymptotically normally distributed. After introducing an appropriate standardizing factor in (6.10) the right-hand side will converge to q. This will finish the proof. We start with the left-hand side of (6.10) and common terms of Δ_n and $\Delta_{1,n}$. Using the representation (6.5) we see

$$\Delta_n = \frac{n^{1/2}[F_n - F]}{r_n'(\tau^*) - f(\tau^*)} + o_P(1).$$

The numerator was investigated at the beginning of the proof of Theorem 6.3 and equals

$$\sqrt{\frac{n}{n_0}}\zeta n_0^{1/2}(F_{n_0}^0 - F_0) + \sqrt{\frac{n}{n_1}}(1-\zeta)n_1^{1/2}(F_{n_1}^1 - F_1) + o(1) = \Delta_{0,n} + \Delta_{1,n} + o(1),$$

where $\Delta_{0,n} = \sqrt{\frac{n}{n_0}}\zeta n_0^{1/2}(F_{n_0}^0 - F_0)$. Denote by $D_{4,n} = (1 - D_{2,n}/r_n'(T_n(F))$, then

$$\begin{aligned}
D_{4,n}\Delta_n - \frac{\Delta_{1,n}}{r_n'(T_n(F))} &= \frac{D_{4,n}\Delta_{0,n}}{r_n'(\tau^*) - f(\tau^*)} + \frac{D_{4,n}\Delta_{1,n}}{r_n'(\tau^*) - f(\tau^*)} - \frac{\Delta_{1,n}}{r_n'(\tau^*)(1+o(1))} \\
&= \frac{D_{4,n}\Delta_{0,n}}{r_n'(\tau^*) - f(\tau^*)} + \frac{(D_{4,n}-1)r_n'(\tau^*) + f(\tau^*)}{r_n'(\tau^*)(r_n'(\tau^*) - f(\tau^*))}\Delta_{1,n} + o_P(1) \\
&= h_n(\Delta_{0,n}, \Delta_{1,n}) + o_P(1) \text{ (say).} \qquad (6.11)
\end{aligned}$$

We now show that $h_n(\Delta_{0,n}, \Delta_{1,n})$ is asymptotically normally distributed. Note, $h_n(\Delta_{0,n}, \Delta_{1,n})$ corresponds to Y_n in (6.6). So far, we have

$$V_n > g_n(n_1) \Leftrightarrow h_n(\Delta_{0,n}, \Delta_{1,n}) + o_P(1) > n^{1/2}\left(\frac{D_{3,n}}{r'_n(T_n(F))} + D_{1,n}\right).$$

What follows now is the same argumentation as in the proof of Theorem 6.3. Again by Donsker's Theorem, $(\Delta_{0,n}(\cdot), \Delta_{1,n}(\cdot)) \to (\sqrt{\zeta}B_{F_0}(\cdot), \sqrt{1-\zeta}B_{F_1}(\cdot))$ in distribution, where B_F denotes an F-Brownian bridge. By the extended continuous-mapping theorem (Theorem B.4) we obtain that $h_n(\Delta_{0,n}, \Delta_{1,n})$ converges in distribution to

$$X = \frac{D_4 X_0}{r'(\tau^*) - f(\tau^*)} + \frac{(D_4 - 1)r'(\tau^*) + f(\tau^*)}{r'(\tau^*)(r'(\tau^*) - f(\tau^*))}X_1,$$

with $D_4 = \lim_{n\to\infty} D_{4,n} = 1-(1-\zeta)\lim_{n\to\infty} f_1(T_n(F))/r'_n(T_n(F)) = 1-(1-\zeta)f_1(\tau^*)/r'(\tau^*)$, where $X_0 \sim N(0, \zeta F_0(\tau^*)(1-F_0(\tau^*)))$ is independent of $X_1 \sim N(0, (1-\zeta)F_1(\tau^*)(1-F_1(\tau^*)))$ by the independence of B_{F_0} and B_{F_1}. The variance of X is

$$\begin{aligned}\left(\frac{s(r,\zeta,\tau^*)}{r'(\tau^*)}\right)^2 &= \frac{(r'(\tau^*) - (1-\zeta)f_1(\tau^*))^2\zeta F_0(\tau^*)(1-F_0(\tau^*))}{[r'(\tau^*)(r'(\tau^*) - f(\tau^*))]^2} \\ &\quad + \frac{\zeta^2 f_0^2(\tau^*)(1-\zeta)F_1(\tau^*)(1-F_1(\tau^*))}{[r'(\tau^*)(r'(\tau^*) - f(\tau^*))]^2}.\end{aligned}$$

The right-hand side of (6.10) equals

$$n^{1/2}\left(\frac{(1-\zeta)F_1(T_n(F)) + g(\zeta)(1+o(n^{-1/2})) - r_n(T_n(F))}{r'_n(T_n(F))}\right), \tag{6.12}$$

which differs from the left-hand side of the quantile equation (6.8) by the factor $s(r_n, Q_n(T_n(F)), T_n(F))/r'_n(T_n(F))$. By Remark 6.1 we have $\zeta = Q_n(T_n(F))$ and hence by the quantile equation (6.8) we get

$$\begin{aligned}&V_n > g_n(n_1) \\ &\Leftrightarrow h_n(\Delta_{0,n}, \Delta_{1,n}) + o_P(1) > n^{1/2}\left(\frac{(1-\zeta)F_1(T_n(F)) + g(\zeta)(1+o(n^{-1/2})) - r_n(T_n(F))}{r'_n(T_n(F))}\right) \\ &\Leftrightarrow h_n(\Delta_{0,n}, \Delta_{1,n})r'_n(T_n(F))/s(r_n, Q_n(T_n(F)), T_n(F)) + o_P(1) > q + o(1).\end{aligned}$$

Since r_n (r'_n) converges uniformly to r (r'), $T_n(F) \to \tau^*$, and $Q_n(T_n(F)) = Q(\tau^*) = \zeta$, we have $s(r_n, Q_n(T_n(F)), T_n(F)) \to s(r, \zeta, \tau^*)$ and $r'_n(T_n(F)) \to r'(\tau^*)$. Altogether, we get

$$\begin{aligned}1 - \Phi(q) &= \mathbb{P}_\vartheta\left(Xr'(\tau^*)/s(r,\zeta,\tau^*) > q\right) \\ &= \lim_{n\to\infty} \mathbb{P}_\vartheta\left(h_n(\Delta_{0,n}, \Delta_{1,n})r'_n(T_n(F))/s(r_n, Q_n(T_n(F)), T_n(F)) > q + o_P(1)\right) \\ &= \lim_{n\to\infty} \mathbb{P}_\vartheta\left(V_n(\varphi_n) > g_n(n_1)\right).\end{aligned}$$

This completes the proof. □

Remark 6.8

The last theorem provides $\mathbb{P}_\vartheta\left(V_n(\varphi_n) > g_n(n_1)\right) = 1 - \Phi(q_n(T_n(F))) + o(1)$. For fixed ζ and a known rejection curve r_n we can calculate $1 - \Phi(q_n(T_n(F)))$. Note, in the cases considered in this thesis there will always be a one to one map from ζ to $T_n(F)$. For large n plotting $1 - \Phi(q_n(T_n(F)))$ against ζ can be interpreted as a plot of $\mathbb{P}_\vartheta\left(V_n(\varphi_n) > g_n(n_1)\right)$ against ζ. Such plots will appear a couple of times.

Corollary 6.9 (Asymptotic distribution of V_n)
Let φ_n be the SU test induced by r_n. If $n_0/n = \zeta + o(n^{-1/2})$, then

$$\sqrt{n}(V_n(\varphi_n)/n - \zeta F_0(\tau^*)) \to V$$

in distribution, where V is normally distributed with zero mean and standard deviation $s(r, \zeta, \tau^*)$.

Proof: Reflecting the proof of Theorem 6.5 it is obvious that

$$\sqrt{n}\frac{V_n(\varphi_n)/n - r_n(T_n(F)) + (1-\zeta)F_1(T_n(F))}{r_n'(T_n(F))} = h_n(\Delta_{0,n}, \Delta_{1,n}) + o_P(1),$$

where $h_n(\Delta_{0,n}, \Delta_{1,n})$ is defined in the proof of Theorem 6.5 (cf. (6.11)) and converges in distribution to a normal random variable with zero mean and standard deviation $s(r, \zeta, \tau^*)/r'(\tau^*)$. Since

$$r_n(T_n(F)) - (1-\zeta)F_1(T_n(F)) = \zeta F_0(T_n(F)) = \zeta F_0(\tau^*) + o(1)$$

the assertion follows. □

Remark 6.10

Obviously, if r_n equals r, then Corollary 6.9 reduces to Corollary 2.21.

Theorem 6.11

If $q_n(T_n(F)) = q + o(1)$ holds, then $r_n(T_n(F)) = r(T_n(F)) + O(n^{-1/2})$. If additionally $\sup\{r_n'(x) : n \in \mathbb{N},\ x \in [\tau^* - \epsilon, \tau^* + \epsilon]\} < \infty$ for some $\epsilon > 0$, then $r_n(\tau^*) = r(\tau^*) + O(n^{-1/2})$.

Proof: By the unique crossing-point condition it follows that $0 < s(r, \zeta, \tau^*) < \infty$. Since $s(r_n, Q_n(T_n(F)), T_n(F)) \to s(r, \zeta, \tau^*)$, we see for the numerator of $q_n(T_n(F))$ that

$$O(1) = \sqrt{n}((1-\zeta)F_1(T_n(F)) + g(\zeta) - r_n(T_n(F))) = \sqrt{n}(g(\zeta) - \zeta F_0(T_n(F))).$$

Therefore, $g(\zeta) = \zeta F_0(\tau^*)$. Furthermore, by Taylor's theorem and the continuity of f_0 it follows that

$$g(\zeta) - \zeta F_0(T_n(F)) = \zeta f_0(\tau^*)(T_n(F) - \tau^*) + o(1)(T_n(F)) - \tau^*).$$

Thus, $T_n(F) - \tau^* = O(n^{-1/2})$ holds. Similarly, it follows that $F_1(\tau^*) - F_1(T_n(F)) = O(n^{-1/2})$,

$r(\tau^*) - r(T_n(F)) = O(n^{-1/2})$, and

$$\begin{aligned} O(1) &= \sqrt{n}((1-\zeta)F_1(T_n(F)) + g(\zeta) - r_n(T_n(F))) \\ &= \sqrt{n}((1-\zeta)F_1(\tau^*) + \zeta F_0(\tau^*) - r_n(T_n(F))) + O(1) \\ &= \sqrt{n}(r(\tau^*) - r_n(T_n(F))) \\ &= \sqrt{n}(r(T_n(F)) - r_n(T_n(F))). \end{aligned}$$

This proves the first assertion.

Again, we obtain from Taylor's theorem that

$$r_n(T_n(F)) = r_n(\tau^*) + r_n'(\tau^*)(T_n(F) - \tau^*) + (r_n'(\eta(T_n(F))) - r_n'(\tau^*))(T_n(F) - \tau^*),$$

where $\eta(T_n(F))$ lies between $T_n(F)$ and τ^*. Since $T_n(F) \to \tau^*$ and we assume that $M = \sup\{r_n'(x) : n \in \mathbb{N},\ x \in [\tau^* - \epsilon, \tau^* + \epsilon]\} < \infty$ we get

$$|r_n(T_n(F)) - r_n(\tau^*)| \leq 2M(T_n(F) - \tau^*) = O(n^{-1/2}).$$

Similar as before, we conclude

$$O(1) = \sqrt{n}((1-\zeta)F_1(T_n(F)) + g(\zeta) - r_n(T_n(F))) = \sqrt{n}(r(\tau^*) - r_n(\tau^*)) + O(1).$$

This proves the second assertion. □

Obviously, Theorem 6.11 encourages one to determine a $\delta \in \mathbb{R}$ such that $r_n(\tau^*) = r(\tau^*) + \delta/\sqrt{n}$ holds. Ideally, we want to determine a differentiable function $\delta_1(t)$ such that $r_n(t) = r(t) + \delta_1(t)/\sqrt{n}$ holds for all $t \in (0,1)$. We start by investigating $q_n(t) = q$ for $r(t) + \delta_1(t)/\sqrt{n}$ and $t \in (0,1)$. Denote by $\delta_1'(t)$ the first derivative of $\delta_1(t)$ with respect to t. Assuming $\delta_1'(t) \in \mathbb{R}$ for all $t \in (0,1)$, we show that it is possible to obtain a $\delta_1(t)$ such that $\delta_1'(t) \in \mathbb{R}$ and $q_n(t) = q + o(1)$ for all $t \in (0,1)$.

Let the asymptotic ENFR bounding function $g(\zeta) = \lim_{n\to\infty} g_n(n_1)/n$ be known and fixed. Suppose that $r(t) = 1 - Q(t) + g(Q(t))$ holds true. For instance, if $g(\zeta) = (1-\zeta)\alpha/(1-\alpha)$, then $r(t)$ equals the AORC, that is $r(t) = f_\alpha(t) = t/(t(1-\alpha)+\alpha)$. The asymptotic relation between bounding functions and rejection curves was studied in Section 2.3. The Remarks 6.17 and 6.18 are also devoted to this asymptotic relation and their consequences for the NFRX. Next, we try to determine $\delta_1(t)$. Under DU configurations we have $F_0(t) = t$, $f_0(t) = 1$, $F_1(t) = 1$, and $f_1(t) = 0$ for $t \in (0,1)$ and thus we get

$$\begin{aligned} q = q_n(t) &= n^{1/2} \frac{(1-Q_n(t))F_1(t) + g(Q_n(t)) - r_n(t)}{s(r_n, Q_n(t), t)} \\ &= n^{1/2} \frac{[(1-Q_n(t))1 + g(Q_n(t)) - r_n(t)][r_n'(t) - Q_n(t)1 - (1-Q_n(t))0]}{\sqrt{(r_n'(t) - (1-Q_n(t))0)^2 Q_n(t)t(1-t) + Q_n(t)^2 1^2 (1-Q_n(t))1(1-1)}} \\ &= n^{1/2} \frac{(1-Q_n(t) + g(Q_n(t)) - r_n(t))(r_n'(t) - Q_n(t))}{r_n'(t)\sqrt{Q_n(t)t(1-t)}}. \end{aligned} \tag{6.13}$$

By definition, see Remark 6.1, we have

$$Q_n(t) = \frac{1 - r_n(t)}{1 - t} = \frac{1 - r(t)}{1 - t} - \frac{\delta_1(t)}{\sqrt{n}(1 - t)} = Q(t) - \frac{\delta_1(t)}{\sqrt{n}(1 - t)}.$$

Suppose g is continuously differentiable at $Q(t)$, then a Taylor expansion yields that

$$\begin{aligned} 1 - Q_n(t) + g(Q_n(t)) &= 1 - Q(t) + \frac{\delta_1(t)}{\sqrt{n}(1 - t)} + g(Q(t)) - [g'(Q(t)) + o(1)]\frac{\delta_1(t)}{\sqrt{n}(1 - t)} \\ &= r(t) + \delta_1(t)\frac{1 - g'(Q(t))}{\sqrt{n}(1 - t)} + \frac{\delta_1(t)}{\sqrt{n}(1 - t)} \cdot o(1). \end{aligned}$$

Hence, equation (6.13) becomes

$$\begin{aligned} q &= n^{1/2}\frac{(\delta_1(t)\frac{1-g'(Q(t))}{\sqrt{n}(1-t)} - \delta_1(t)/\sqrt{n} + \frac{\delta_1(t)}{\sqrt{n}(1-t)} \cdot o(1))(r_n'(t) - Q_n(t))}{r_n'(t)\sqrt{1 - r_n(t)}\sqrt{t}} \\ &= n^{1/2}\frac{\delta_1(t)}{\sqrt{n}}\frac{(\frac{t-g'(Q(t))}{1-t} + o(1))(r_n'(t) - Q_n(t))}{r_n'(t)\sqrt{1 - r_n(t)}\sqrt{t}} \qquad (6.14) \\ &= \delta_1(t)A_n(t) \text{ (say)}. \end{aligned}$$

Note, if g is linear in ζ or even constant, then the term $o(1)$ in (6.14) can be discarded. This fact will be explicitly used later when we generalize our calculations. By assumption $r_n'(t) = r'(t) + O(1/\sqrt{n})$ and this entails

$$q = \delta_1(t)\frac{t - g'(Q(t))}{1 - t}\frac{r'(t) - Q(t)}{r'(t)\sqrt{1 - r(t)}\sqrt{t}} = \delta_1(t)A(t) \text{ (say)}.$$

If $A(t) \neq 0$ for all $t \in (0, 1)$, then we obtain $\delta_1(t) = q/A(t)$. At the beginning we made the assumption that $\delta_1(t)$ is differentiable and $\delta_1'(t) \in \mathbb{R}$. Clearly, this is equivalent to the assumption that $A(t)$ is differentiable and $A'(t)/A^2(t) \in \mathbb{R}$. We summarize our findings in the next theorem in a slightly generalized version.

Theorem 6.12

Let g denote a continuously differentiable asymptotic ENFR bounding function with corresponding rejection curve r, that is $1 - Q(t) + g(Q(t)) = r(t)$ for all $t \in (a, b)$ with $0 < a < b < 1$. Suppose

$$A(t) = \frac{t - g'(Q(t))}{1 - t}\frac{r'(t) - Q(t)}{r'(t)\sqrt{1 - r(t)}\sqrt{t}}$$

is differentiable, nonzero, and $A'(t)/A^2(t) \in \mathbb{R}$ for $t \in (a, b)$. For $r_n(t) = r(t) + q/(A(t)\sqrt{n})$ it holds that $q_n(t) = q + o(1)$ for all $t \in (a, b)$.

Example 6.13

Let $g(\zeta) = (1 - \zeta)\zeta\alpha/(1 - \zeta\alpha)$. The corresponding rejection curve is the Simes line, that is $r(t) = t/\alpha$, cf. Example 2.19. We have

$$A(t) = \frac{\sqrt{\alpha(\alpha - t)}}{(1 - t)\sqrt{t}},$$

which is nonzero on $(0, \alpha)$, differentiable, and fulfills $A'(t)/A^2(t) \in \mathbb{R}$ for $t \in (0, \alpha)$. Theorem 6.12 states that $q_n(t) = q + o(1)$ for

$$r_n(t) = r(t) + \frac{q}{\sqrt{n}} \frac{(1-t)\sqrt{t}}{\sqrt{\alpha(\alpha - t)}}$$

for $t \in (0, \alpha)$.

In the next example we apply Theorem 6.12 to the important special case $g(\zeta) = (1-\zeta)\alpha/(1-\alpha)$.

Example 6.14
Consider the asymptotic ENFR bounding function $g(\zeta) = (1-\zeta)\alpha/(1-\alpha)$ with $\alpha \in (0, 1)$. Let r be the AORC, that is $r(t) = t/(t(1-\alpha) + \alpha)$. The equation $1 - Q(t) + g(Q(t)) = r(t)$ can easily be verified and the function $A(t)$ from Theorem 6.12 becomes

$$A(t) = \frac{(t(1-\alpha) + \alpha)^{3/2}}{\sqrt{\alpha t(1-t)}}.$$

This function is differentiable and nonzero on $(0, 1)$. Let $\delta_1(t) = q/A(t)$ for $t \in (0, 1)$. Elementary calculations show that $\delta_1'(t) \in \mathbb{R}$ for all $t \in (0, 1)$. Hence, Theorem 6.12 shows that for

$$r_n(t) = r(t) + q \frac{\sqrt{\alpha t(1-t)}}{(t(1-\alpha) + \alpha)^{3/2}\sqrt{n}}$$

we have that $q_n(t) = q + o(1)$. In Figure 6.1 the expression $1 - \Phi(q_n(T_n(F)))$ is visualized for $n = 10^3, 10^4$, and 10^5.

Figure 6.1 shows that $q_n(T_n(F)) \neq q$. As we explained in Remark 6.6, we aim for $q_n(T_n(F)) = q$ for fixed n. Instead of setting $r_n(t) = r(t) + \delta_1(t)/\sqrt{n}$ we consider a generalized version, that is $r_n(t) = r(t) + \Delta_n(t)$, where $\Delta_n(t)$ is a differentiable function. Then, the same steps that lead from (6.13) to (6.14) provide now that

$$q = \sqrt{n}\Delta_n(t) \frac{(\frac{t - g'(Q(t))}{1-t} + o(1))(r_n'(t) - Q_n(t))}{r_n'(t)\sqrt{1 - r_n(t)}\sqrt{t}} \tag{6.15}$$

It is reasonable to set $\Delta_n(t) = \sum_{i=1}^{k} \delta_i(t)/n^{i/2}$, where $k \in \mathbb{N}$ is fixed and $\delta_i(t)$ are differentiable functions. Obviously, for $k = 1$ we get the case we already treated and which led to Theorem 6.12. We restrict our attention to a special case and assume that $g(\zeta) = (1-\zeta)\gamma$ with $\gamma = \alpha/(1-\alpha)$, where $\alpha \in (0, 1)$. Thus, Equation 6.15 becomes

$$q = \sqrt{n}\Delta_n(t) \frac{(\frac{t+\gamma}{1-t})(r_n'(t) - Q_n(t))}{r_n'(t)\sqrt{1 - r_n(t)}\sqrt{t}} = \sqrt{n}\Delta_n(t)A_n(t) \text{ (say)}. \tag{6.16}$$

Note, $g'(Q(t))$ and $o(1)$ in $(t - g'(Q(t)))/(1-t) + o(1)$ in the numerator of (6.15) evolve from a Taylor expansion of g at $Q(t)$. Thus, $(t - g'(Q(t)))/(1-t) + o(1)$ can be replaced by $(t+\gamma)/(1-t)$ because g is linear in ζ and thus $g'' \equiv 0$ in the Taylor expansion of g at $Q(t)$. We exemplify

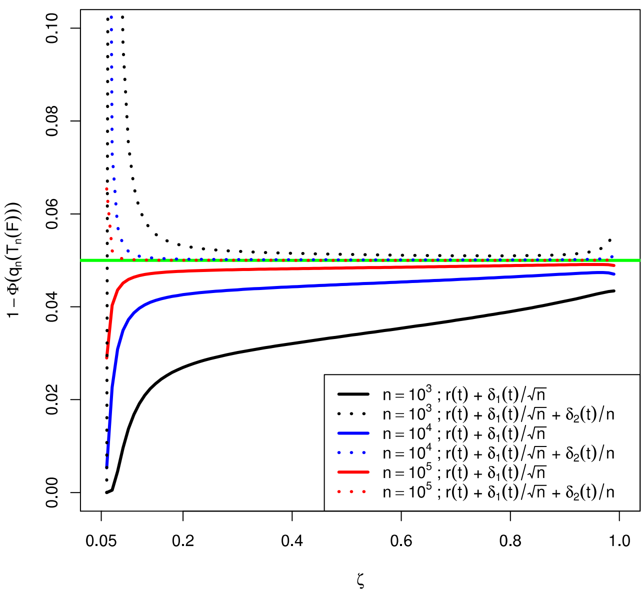

Figure 6.1: *The figure shows* $1 - \Phi(q_n(T_n(F)))$ *for* $r_n(t) = f_\alpha(t) + \delta_1(t)/\sqrt{n}$ *and* $r_n(t) = f_\alpha(t) + \delta_1(t)/\sqrt{n} + \delta_2(t)/n$ *for* $q = 1.645$, $\alpha = 0.05$ *and* $n = 10^3, 10^4$, *and* 10^5, *where* $\delta_1(t)$ *and* $\delta_2(t)$ *are defined in* (6.18) *and* (6.19), *respectively.*

now how $\delta_2(t)$ and $\delta_3(t)$ may be obtained. Similarly as before, we assume that $\delta_i'(t) \in \mathbb{R}$ for all $i = 1, \ldots, k$ and $t \in (0,1)$. Like before, $A_n(t) \to A(t)$ and thus Equation (6.16) directly provides us $\delta_1(t)$ by $q = \delta_1(t)A(t)$ if $A(t)$ is nonzero on $(0,1)$. Since t is considered fixed we suppress the dependence on t for simplicity. Again, from Equation (6.16) we get

$$(\delta_2 + O(1/\sqrt{n}))A_n = \sqrt{n}(q - \delta_1 A_n) = \sqrt{n}\left(q - q\frac{A_n}{A}\right) = \frac{q}{A}\sqrt{n}(A - A_n). \tag{6.17}$$

We now focus on $\sqrt{n}(A - A_n)$. First, define

$$B(x,y) = \frac{t+\gamma}{1-t}\,\frac{y - (1-x)/(1-t)}{y\sqrt{1-x}\sqrt{t}}.$$

Hence, $A = B(r(t), r'(t))$ and $A_n = B(r(t) + \Delta_n(t), r'(t) + \Delta_n'(t))$ and thus, suppressing t, $\sqrt{n}(A - A_n) = \sqrt{n}(B(r,r') - B(r + \Delta_n, r' + \Delta_n'))$. By assumption $\Delta_n, \Delta_n' = O(1/\sqrt{n})$. Since $B(x,y)$ is infinitely often differentiable with respect to x and y, it is obvious to apply the multivariate version of Taylor's Theorem, cf. Corollary 7.1. in [23], that is

$$\begin{aligned}
\sqrt{n}(A - A_n) &= \sqrt{n}(B(r,r') - B(r + \Delta_n, r' + \Delta_n')) \\
&= -\sqrt{n}\left(\frac{\partial B(r,r')}{\partial x}\Delta_n + \frac{\partial B(r,r')}{\partial y}\Delta_n' + o(1/\sqrt{n})\right) \\
&\to -\frac{\partial B(r,r')}{\partial x}\delta_1 - \frac{\partial B(r,r')}{\partial y}\delta_1' \\
&= L_1 \text{ (say)},
\end{aligned}$$

where $\partial B(r,r')/\partial x$ $(\partial B(r,r')/\partial y)$ denotes the partial derivative of B with respect to x (y) at the point $(r(t), r'(t))$. Altogether, we have $\delta_2 = qL_1/A^2$. The calculation of δ_3 is even more tedious. We have

$$
\begin{aligned}
(\delta_3 + O(1/\sqrt{n}))A_n &= \sqrt{n}(\sqrt{n}(q - \delta_1 A_n) - \delta_2 A_n) \\
&= \sqrt{n}(\sqrt{n}(q - \frac{q}{A}A_n) - q\frac{L_1}{A^2}A_n) \\
&= \frac{q}{A^2}\sqrt{n}(A\sqrt{n}(A - A_n) - L_1 A_n) \\
&= \frac{q}{A^2}\sqrt{n}(A[L_1 + \sqrt{n}(A - A_n) - L_1] - L_1 A_n) \\
&= \frac{q}{A^2}[\sqrt{n}(AL_1 - L_1 A_n) + \sqrt{n}A(\sqrt{n}(A - A_n) - L_1)] \\
&= \frac{q}{A^2}[L_1^2 + o(1) + \sqrt{n}A(\sqrt{n}(A - A_n) - L_1)].
\end{aligned}
$$

Now we focus on $\sqrt{n}(\sqrt{n}(A - A_n) - L_1)$. Again, Taylor's Theorem is the main tool. It holds that

$$
\begin{aligned}
A - A_n = -\Bigg(&\frac{\partial B(r,r')}{\partial x}\Delta_n + \frac{\partial B(r,r')}{\partial y}\Delta_n' + \frac{1}{2}\frac{\partial^2 B(r,r')}{\partial x^2}\Delta_n^2 + \frac{1}{2}\frac{\partial^2 B(r,r')}{\partial y^2}(\Delta_n')^2 \\
&+ \frac{\partial^2 B(r,r')}{\partial x \partial y}\Delta_n \Delta_n'\Bigg) + o(1/n).
\end{aligned}
$$

By definition of L_1 we have

$$
\begin{aligned}
-\frac{\partial B(r,r')}{\partial x}\Delta_n - \frac{\partial B(r,r')}{\partial y}\Delta_n' - L_1/\sqrt{n} &= -\frac{\partial B(r,r')}{\partial x}\sum_{i=2}^{k}\frac{\delta_i}{n^{i/2}} - \frac{\partial B(r,r')}{\partial y}\sum_{i=2}^{k}\frac{\delta_i'}{n^{i/2}} \\
&= L_{2,n} \text{ (say).}
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\sqrt{n}(\sqrt{n}(A - A_n) - L_1) &= \sqrt{n}(\sqrt{n}(A - A_n - L_1/\sqrt{n})) \\
&= nL_{2,n} - \Bigg(\frac{1}{2}\frac{\partial^2 B(r,r')}{\partial x^2}n\Delta_n^2 + \frac{1}{2}\frac{\partial^2 B(r,r')}{\partial y^2}n(\Delta_n')^2 \\
&\qquad + \frac{\partial^2 B(r,r')}{\partial x \partial y}\sqrt{n}\Delta_n\sqrt{n}\Delta_n'\Bigg) + o(1) \\
&\to -\Bigg(\frac{\partial B(r,r')}{\partial x}\delta_2 + \frac{\partial B(r,r')}{\partial y}\delta_2' + \frac{1}{2}\frac{\partial^2 B(r,r')}{\partial x^2}\delta_1^2 \\
&\qquad + \frac{1}{2}\frac{\partial^2 B(r,r')}{\partial y^2}(\delta_1')^2 + \frac{\partial^2 B(r,r')}{\partial x \partial y}\delta_1\delta_1'\Bigg) \\
&= L_2 \text{ (say).}
\end{aligned}
$$

Altogether, we conclude

$$(\delta_3 + O(1/\sqrt{n}))A_n = \frac{q}{A^2}[L_1^2 + \sqrt{n}A(\sqrt{n}(A - A_n) - L_1) + o(1)] \to \frac{q}{A^2}[L_1^2 + AL_2]$$

and hence $\delta_3 A = (q/A^3)[L_1^2 + AL_2]$. This leads to

$$\delta_1(t) = \frac{q\sqrt{\alpha t(1-t)}}{(t(1-\alpha)+\alpha)^{3/2}}, \tag{6.18}$$

$$\delta_2(t) = -\frac{q^2}{2}\frac{(1-\alpha)^2t^3 - (1-\alpha)^2t^2 + (1-\alpha^2)t + \alpha^2}{(t(1-\alpha)+\alpha)^2(1-t)(1-\alpha)}, \tag{6.19}$$

and

$$\delta_3(t) = \frac{q^3}{8}\frac{\sum_{i=0}^{5} a_i t^i}{(1-\alpha)^2(1-t)^2(t(1-\alpha)+\alpha)^{3/2}\sqrt{t(1-t)\alpha}}, \tag{6.20}$$

with $a_0 = 2\alpha^3$, $a_1 = 2\alpha^2(3 - 3\alpha + 4\alpha^2)$, $a_2 = (1-\alpha)(1 + 3\alpha - 4\alpha^2 + 32\alpha^3)$, $a_3 = -2(1 + 4\alpha)(1 - 6\alpha)(1-\alpha)^2$, $a_4 = (5 + 26\alpha - 32\alpha^2)(1-\alpha)^2$, and $a_5 = 2(3 - 4\alpha)(1-\alpha)^3$. We want to remind that we assumed that $\delta_i'(t) \in \mathbb{R}$ for $i = 1, \ldots, k$, $t \in (0,1)$. Now, after we obtained expressions for $\delta_1(t)$, $\delta_2(t)$, and $\delta_3(t)$ we need to verify that $\delta_i'(t) \in \mathbb{R}$ for $i = 1, 2, 3$ and $t \in (0, 1)$. This can be verified but is omitted here. Figure 6.1 shows $1 - \Phi(q_n(T_n(F)))$ for $r_n(t) = r(t) + \delta_1(t)/\sqrt{n}$ and $r_n(t) = r(t) + \delta_1(t)/\sqrt{n} + \delta_2(t)/n$. As can be seen from that figure, the benefit of the additional term $\delta_2(t)$ is considerable only for moderate number of null hypotheses, that is $n = 10^3$ or 10^4.

Remark 6.15

We conducted a simulation for $n = 10.000$, $\alpha = 0.05$, and $q = 1.645$ with $r_1(t) = r(t) + \delta_1(t)/\sqrt{n}$, $r_2(t) = r(t)+\delta_1(t)/\sqrt{n}+\delta_2(t)/n$, and $r_3(t) = r(t)+\delta_1(t)/\sqrt{n}+\delta_2(t)/n+\delta_3(t)/n^{3/2}$ and visualized the results in Figure 6.2. For a definition of δ_1, δ_2, and δ_3 see (6.18), (6.19), and (6.20), respectively. It is possible to conduct a SU test induced by a rejection curve r by determining the largest crossing-point between F_n and r, cf. Corollary 1.8. A SU test induced by r_1 will always reject all null hypotheses because $r_1(1) = 1$. Hence, we determined the largest crossing-point between F_n and r_1 (r_2, r_3) only on $[0, r_1^{-1}(0.98)] \approx [0, 0.673]$. Thus, null hypotheses with p-values larger than 0.673 are never rejected, which is not a severe restriction. It should be noted that r_2 and r_3 actually are no rejection curves. We have that $r_2(0) \neq 0$ and $r_3(0) \neq 0$. Furthermore, r_2 and r_3 are decreasing on $[0.93, 1]$ and $[0, 10^{-5}]$, respectively. But seeking the largest crossing-point only on $[0, 0.673]$ and considering only $\zeta \leq 0.95$ solve these problems for our specific situation.

Remark 6.16

On the right hand side of the quantile equation (6.8) we have $q + o(1)$. As explained in Remark

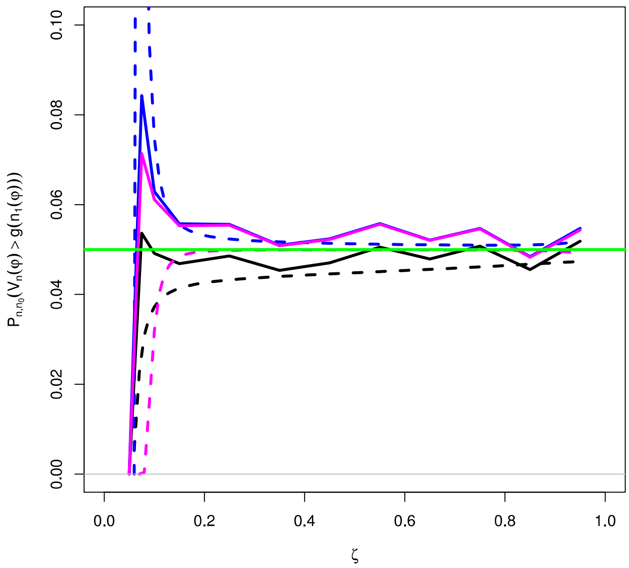

Figure 6.2: *The figure shows the empirical probabilities (solid lines; number of replications = 500.000) of $V_n(\varphi_n) > (n_1+1)\alpha/(1-\alpha)$ for $n = 10.000$ and $\alpha = 0.05$, where φ_n is a test based on r_1 (black), r_2 (blue), and r_3 (magenta), respectively, as explained in Remark 6.15. Additionaly the dashed lines show $1 - \Phi(q_n(T_n(F)))$ for r_1 (black), r_2 (blue), and r_3 (magenta), respectively.*

6.6, we conducted our calculation only with q discarding the term $o(1)$. Equation (6.16) defines $\delta_1(t)$ by $q = \sqrt{n}\Delta_n(t)A_n(t) \to \delta_1(t)A(t)$. Obviously, the definition of $\delta_1(t)$ is not affected irrespective of the presence or absence of the term $o(1)$, that is $q + o(1) = \sqrt{n}\Delta_n(t)A_n(t)$ will also yield $q = \delta_1(t)A(t)$. This is not true for $\delta_i(t)$ and $i > 1$. For instance, Equation (6.17) defines $\delta_2(t)$ by $(\delta_2(t) + O(1/\sqrt{n}))A_n(t) = \sqrt{n}(q - \delta_1(t)A_n(t)) = \sqrt{n}(q - qA_n(t)/A(t))$. Obviously, considering $q_n(t) = q + 1/\sqrt{n}$ instead of $q_n(t) = q$ we get $\delta_2(t) = 1/A(t) + qL_1(t)/A^2(t)$ instead of $\delta_2(t) = qL_1(t)/A^2(t)$. If further knowledge is available, for instance, $\mathbb{P}_\vartheta\left(V_n(\varphi_n) > g_n(n_1)\right) = 1-(\Phi(q)+\Psi_n(q))+o(1)$ it may be possible that the "quantile equation" reads like $q_n(T_n(F)) = q_0 + q_1/\sqrt{n} + O(1/n)$. In this case, it would hold that $q_0 = \delta_1(t)A(t)$ and $\delta_2 A(t) = q_1 + q_0 L_1(t)/A(t)$. Both $\delta_1(t)$ and $\delta_2(t)$ would be unique but $\delta_3(t)$ may not be unique.

In the next subsection we will see that it is also possible to solve an ordinary differential equation in order to achieve $q_n(T_n(F)) = q$ and we present two further techniques such that at least $q_n(T_n(F)) \approx q$ holds true. Now, we investigate a further consequence of the quantile equation. The following two remarks are devoted to the special case $r_n = r$ for all $n \in \mathbb{N}$ and show that the conjecture from Remark 6.4 is true.

Remark 6.17
Suppose $r_n = r$ for all $n \in \mathbb{N}$ and $n_0/n = \zeta + o(n^{-1/2})$. Further, suppose that $g(\zeta) =$

$\mathbf{g_n(n_1)}$	$\mathbf{g(\zeta)}$	$\mathbf{r(t)}$	$\mathbf{I}$
αn	α	$\max\{\tilde{r}(t), C(t)\}$	$(\frac{\alpha}{1-2\epsilon}, 1-\epsilon)$
$\frac{n_0(1-n_0/n)\alpha}{1-\alpha n_0/n}$	$\frac{\zeta(1-\zeta)\alpha}{1-\alpha\zeta}$	Simes line : t/α	$(0, 1)$
$\frac{(n_1+1)\alpha}{1-\alpha}$	$\frac{(1-\zeta)\alpha}{1-\alpha}$	AORC : $\max\{f_{\alpha,0}(t), C(t)\}$	$(\frac{\alpha}{1-2\epsilon+2\alpha\epsilon}, 1)$

Table 6.1: *At least for $\zeta \in I$ it holds for SU tests φ_n induced by r that $\mathbb{P}_{n,n_0}(V_n(\varphi_n) > g_n(n_1)) \to 1/2$ if $n_0/n = \zeta + o(n^{-1/2})$, where $\epsilon > 0$ is appropriate small and $C(t) = 2 + (t-1)/\epsilon$. For the definition of $\tilde{r}(t)$ see Example 2.18.*

$\lim_{n\to\infty} \mathrm{ENFR}_\vartheta(\varphi_n)/n$ for SU tests φ_n induced by r_n. Then, by Corollary 6.9 we have $\lim_{n\to\infty} \mathrm{ENFR}_\vartheta(\varphi_n)/n = \zeta F_0(\tau^*)$, that is $g(\zeta) = \zeta F_0(\tau^*)$. Since $r_n = r$, we have $T_n(F) = \tau^*$ for all $n \in \mathbb{N}$. Remark 6.1 states that $\zeta = Q_n(T_n(F))$. Therefore, we conclude for the numerator of $q_n(T_n(F))$ that

$$
\begin{aligned}
&(1 - Q_n(T_n(F)))F_1(T_n(F)) + g(Q_n(T_n(F))) - r_n(T_n(F)) \\
&= (1-\zeta)F_1(\tau^*) + g(\zeta) - r(\tau^*) \\
&= g(\zeta) - \zeta F_0(\tau^*) \\
&= 0,
\end{aligned}
$$

where the second equality holds because $r(\tau^*) = F(\tau^*)$. Altogether, the left-hand side of the quantile equation (6.8) is zero and therefore $q = 0$. Setting $g_n(n_1) = ng(n_0/n)$ and assuming $g_n(n_1)/n = g(\zeta) + o(n^{-1/2})$, we see that

$$\mathbb{P}_\vartheta(V_n(\varphi_n) > g_n(n_1)) \to 1 - \Phi(q) = 1/2.$$

Note, if g is differentiable and the derivative g' is continuous at ζ, then it easily follows that $g_n(n_1)/n = g(n_0/n) = g(\zeta) + o(n^{-1/2})$.

Remark 6.18

In Examples 2.18, 2.19 and 2.20 we considered three different bounding functions and derived the corresponding asymptotic rejection curves. In Example 2.19 we obtained the Simes line and in Example 2.20 we obtained the AORC. Unfortunately, the results from Remark 6.17 do not hold true for Example 2.18 and 2.20 because the corresponding rejection curves equal one at $t = 1$. Obviously, a rejection curve r with $r(1) = 1$ does not fulfill the unique crossing-point condition. And of course, a SU test induced by r always rejects all null hypotheses if $r(1) = 1$. But this can simply be solved by using a modified rejection curve defined as $\max\{r(t), C(t)\}$, where $C(t) = (t-1)/\epsilon + 2$ with $\epsilon \in (0, 1)$. Applying the results of Remark 6.17 to Examples 2.18, 2.19 and 2.20 with this modification in case of $r(1) = 1$, yields Table 6.1.

Asymptotic considerations in Section 2.3 led to the equation (2.10), which implicitly defines a rejection curve. In this chapter we are in a similar situation. Equation (6.8) implicitly defines the rejection curve of interest. But now also the derivative of the rejection curve is involved, which now leads to an ODE.

6.4 The ordinary differential equation

The quantile equation can be easily transformed into an ODE. In this section we will investigate the resulting ODE and numerically solve a transformed version of the ODE. Appendix A gives a short introduction to the field of ordinary differential equations.

Again, DU is LFC, that is, for $\vartheta \in \Theta$ and $n_1 = n_1(\vartheta)$ we have that

$$\mathbb{P}_\vartheta\left(V_n > g_n(n_1)\right) \leq \mathbb{P}_{n,n_0}\left(V_n > g_n(n_1)\right).$$

Under DU the quantile equation can easily be solved for r_n'. This will lead to the following differential equation (6.21) which is the basis for the construction of SU tests φ_n such that (6.2) holds. The two functions $Q_n(t) = (1 - r_n(t))/(1-t)$ and $H_n(t) = 1 - Q_n(t) + g(Q_n(t))$ will be used in the next corollary.

Corollary 6.19
Theorem 6.5 holds true under DU for $q \neq 0$, if equation (6.8) *is replaced by the assumption that*

$$r_n'(t) = Q_n(t)\frac{H_n(t) - r_n(t)}{H_n(t) - r_n(t) - n^{-1/2}q\sqrt{Q_n(t)t(1-t)}} \tag{6.21}$$

holds at $T_n(F)$ for $n > N$.

Proof: Under DU, we have $F_1(t) = 1$, $f_1(t) = 0$, $F_0(t) = t$, $f_0(t) = 1$ for $t \in (0,1)$. This simplifies the quantile equation (6.8) considerably. Recall, at $T_n(F)$ we have $Q_n(T_n(F)) = \zeta$. Thus, $f(t) = \zeta f_0(t) + (1-\zeta)f_1(t) = Q_n(t)$ at $t = T_n(F)$. Hence,

$$s(r_n, Q_n(t), t) = \frac{r_n'(t)\sqrt{Q_n(t)t(1-t)}}{r_n'(t) - Q_n(t)}$$

at $T_n(F)$. Discarding the $o(1)$ term in the quantile equation (6.8) and substituting $T_n(F)$ by t yields

$$q = n^{1/2}\frac{1 - Q_n(t) + g(Q_n(t)) - r_n(t)}{s(r_n, Q_n(t), t)} = n^{1/2}\frac{(H_n(t) - r_n(t))(r_n'(t) - Q_n(t))}{r_n'(t)\sqrt{Q_n(t)t(1-t)}}$$

Solving this equation with respect to $r_n'(t)$ provides (6.21). Hence, (6.21) at $t = T_n(F)$ implies the quantile equation (6.8). □

There seems to be no analytical solution of the ODE (6.21). Nevertheless, we are going to investigate the ODE from a mathematical point of view in the next subsection. Afterwards, we present two methods for constructing r_n such that (6.21) (approximately) holds. This is done for the important special case where $g_n(n_1) = (n_1 + 1)\gamma$.

6.4.1 Existence of solutions

We rewrite the ODE (6.21) under DU as

$$y' = w_n(x,y) = \bar{Q}(x,y)\frac{\bar{H}(x,y) - y}{\bar{H}(x,y) - y - n^{-1/2}q\bar{S}_0(x,y)}, \tag{6.22}$$

where $\bar{Q}(x,y) = (1-y)/(1-x), \bar{H}(x,y) = (1-\bar{Q}(x,y)) + g(\bar{Q}(x,y))$, and $\bar{S}_0(x,y) = \sqrt{\bar{Q}(x,y)x(1-x)}$.

In the next theorem we provide sufficient conditions such that an interval $I \subset (0,1)$ (independent of n) exists such that the sequence of ODEs (6.22) are solvable on I eventually for all n. The main tool for this is the Theorem of Peano.

Theorem 6.20

Let $g(\zeta) > 0$ for $\zeta \in (0,1)$, $q > 0$, and assume that $\bar{H}(x,y) - y$ is strictly increasing in y and that there exists a function $z : (0,1) \to (0,1)$ such that $\bar{H}(x, z(x)) - z(x) = 0$ for all $x \in (0,1)$. Then, there exist an interval $I = [x_0, x_0 + M] \subset (0,1)$, $M > 0$, $y_0 \in (0,1)$ and $N \in \mathbb{N}$ such that for all $n \geq N$ there exists a solution r_n of (6.22) on I with $r_n(x_0) = y_0$. Furthermore, there exists an $\epsilon > 0$ such that $r_n^{-1}(i/n)$ is defined for all $n \geq N$ if $i/n \in [y_0, y_0 + \epsilon]$.

Recall, a critical value is defined by $r_n^{-1}(i/n)$.

The proof comprises the following steps:

1. $\forall\, n \in \mathbb{N}\ \exists\, G_n \subset (0,1) \times (0,1) : w_n(x,y)$ is continuous on G_n.
2. $G_n \subset G_m$ if $n \leq m$.
3. $w_n(x,y) \geq w_m(x,y)$ if $n \leq m$ and $(x,y) \in G_n$.
4. For fixed N there exists an interval $I = [x_0, x_0 + M]$ such that (6.22) has at least one solution r_N with $r_N(x_0) = y_0$.
5. For $n > N$ there exists a solution r_n on I such that (6.22) holds true and $r_n(x_0) = y_0$.
6. $\inf_{n \geq N} \inf_{x \in I} r_n'(x) > 0$.

Remark 6.21

In Remark 6.17 we elucidated when $\bar{H}(x, z(x)) - z(x) = 0$. For instance, if $g(\zeta) = (1-\zeta)\alpha/(1-\alpha)$, then $z(x) = x/(x(1-\alpha)+\alpha)$, which is the AORC.

Step 1.
We are going to define G_n such that the numerator and denominator of $w_n(x,y)$ are positive and not zero. Note, $\bar{H}(x,y)$ is continuous on $(0,1)\times(0,1)$ because g and $\bar{Q}$ are continuous. By definition of z and since $\bar{H}(x,y)-y$ is strictly increasing in y, we see that

$$\bar{H}(x,z(x))-z(x)-n^{-1/2}q\sqrt{(1-z(x))x}<0 \text{ and } \bar{H}(x,1)-1-n^{-1/2}q\sqrt{(1-1)x}>0$$

because $q>0$ and $z(x)<1$ for $x\in(0,1)$ and therefore,

$$\bar{H}(x,d_{0,n}(x))-d_{0,n}(x)-n^{-1/2}q\sqrt{(1-d_{0,n}(x))x}=0$$

if and only if $d_{0,n}(x)\in(z(x),1)$. Let $G_n=\{(x,y): x\in(0,1),\ \ y\in(0,1),\ \ y>d_{0,n}(x)\}$. For all $(x,y)\in G_n$ we have w_n is continuous on G_n.

Remark 6.22
Although it is not important for the proof of Theorem 6.20 we want to note that for fixed x

$$w_n(x,y)\to\begin{cases}0, & \text{if } y\to 1 \text{ and } (x,y)\in G_n\\ \infty, & \text{if } y\to d_{0,n}(x) \text{ and } (x,y)\in G_n.\end{cases}$$

Confer Figure 6.3 for a schematic sketch of the subset G_n for $g(\zeta)=(1-\zeta)\alpha/(1-\alpha)$. It seems intuitively clear that a possible solution $y_n(x)$ may leave the set G_n at x^* because $y_n(x)\to 1$ when $x\to x^*$, but $y_n(x)\to d_{0,n}(x^*)$ for $x\to x^*$ does not seem to be possible.

Step 2.
We only have to show that $d_{0,n}(x)$ is pointwise decreasing in n. Note, by $q>0$ it holds true that

$$\bar{H}(x,d_{0,n}(x))-d_{0,n}(x)-m^{-1/2}q\sqrt{(1-d_{0,n}(x))x}>0$$

for $m>n$. Since $\bar{H}(x,y)-y$ is strictly increasing in y we have $d_{0,n}(x)>d_{0,m}(x)$ for $x\in(0,1)$.

Step 3.
Let $n,m\in\mathbb{N}$ be arbitrary and fixed. If $n\le m$, then $G_n\subset G_m$. By $q>0$ the denominator on the right-hand side of (6.22) is increasing in n for all $(x,y)\in G_n\subset G_m$. Hence, the assertion follows directly.

Step 4.
Let $N\in\mathbb{N}$, $(x_0,y_0)\in G_N$ be arbitrary and fixed. Choose $a,b>0$ such that the compact set $K=[x_0,x_0+a]\times[y_0-b,y_0+b]\subset G_N$. Peano's Theorem guarantees that there exists an r_N that solves (6.22) with $r_N(x_0)=y_0$ at least on $I=[x_0,x_0+M]$, where

$$M=\min\left\{a,\frac{b}{\sup_{(x,y)\in K}|w_n(x,y)|}\right\}>0.$$

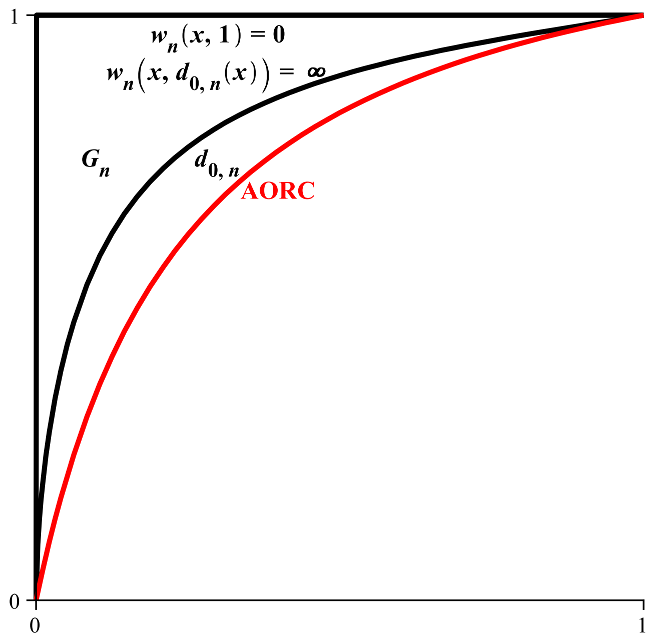

Figure 6.3: *The figure shows a schematic sketch of the set* $G_n = \{(x, y) : x \in (0, 1),\ \ y \in (0, 1),\ \ y > d_{0,n}(x)\}$ *if* $g(\zeta) = (1 - \zeta)\alpha/(1 - \alpha)$.

Step 5.

From Step 3 we conclude that $\sup_{(x,y)\in K} |w_N(x, y)| \geq \sup_{(x,y)\in K} |w_n(x, y)|$ for $n \geq N$. Therefore, again by Peano's Theorem, for $n \geq N$ it is guaranteed that an r_n solves (6.22) with $r_n(x_0) = y_0$ at least on I.

Step 6.

This is obvious because $w_n(x, y) > \bar{Q}(x, y) = (1 - y)/(1 - x)$ for all $n \in \mathbb{N}$. Therefore,

$$\inf_{n\in\mathbb{N}} \inf_{(x,y)\in K} |w_n(x, y)| \geq \inf_{(x,y)\in K} \bar{Q}(x, y) = (1 - y_0 - b)/(1 - x_0) > 0.$$

Step 6 ensures that $r_n'(x) > 0$ for $x \in I$ and $n \geq N$. Hence r_n is invertible on I and it exists an $\epsilon > 0$ independent of n such that $r_n^{-1}(i/n)$ is defined for all $n \geq N$ and $i/n \in [y_0, y_0 + \epsilon]$.

Remark 6.23

The Theorem of Peano guarantees the existence of a solution but not its uniqueness. If $g((1 - y)/(1 - x))$ is continuously differentiable with respect to y on G_N, then for all fixed $n \geq N$ the solution is also unique by virtue of the Theorem of Picard-Lindelöf.

6.5 Practical considerations

The standard routines implemented in Maple for solving the ODE (6.22) numerically yield either no solutions or useless ones. In this section we present two different methods for obtaining rejection curves r_n such that

$$\lim_{n\to\infty} \mathbb{P}_{n,n_0}\left(V_n(\varphi_n) > (n_1+1)\alpha/(1-\alpha)\right) \approx \beta,$$

where φ_n is a SU test induced by r_n. Throughout this section we consider

$$g(\zeta) = \lim_{n\to\infty}(1/n)(n_1+1)\alpha/(1-\alpha) = (1-\zeta)\alpha/(1-\alpha).$$

6.5.1 Utilizing the quantile equation

This method is promising if the ODE cannot be solved analytically or numerically. Following Remark 6.18, we know that $\lim_{n\to\infty} P(V_n(\varphi_n) > (n_1+1)\alpha/(1-\alpha)) = 0.5$ holds for a SU test induced by a slightly adjusted AORC, see Remark 6.18 for details. Asymptotically, the solution must be near the AORC and above $d_{0,n}$ for fixed n. Therefore, it is reasonable to search for the solution near (above) $d_{0,n}$. In general, $d_{0,n}$ can easily be obtained numerically. Fortunately, in our case we can solve

$$\bar{H}(x, d_{0,n}(x)) - d_{0,n}(x) - n^{-1/2}q\sqrt{(1-d_{0,n}(x))x} = 0 \tag{6.23}$$

for $d_{0,n}(x)$, where $\bar{H}(x,y) = 1-(1-y)/(1-x) + g((1-y)/(1-x))$. This yields

$$d_{0,n}(x) = \frac{2n(1+\gamma)(x+\gamma)x - (x-2x^2+x^3)q^2 + q(1-x)W(x)}{2n(x+\gamma)^2}$$

with

$$W(x) = \sqrt{x(1-x)(4\gamma^2 n + (q^2+4n\gamma)x - q^2x^2)}$$

and $\gamma = \alpha/(1-\alpha)$. Note, the left-hand side of (6.23) is the denominator of the ODE (6.22). Hence, it is pointless to substitute y by $d_{0,n}(x)$ in (6.22). Recall that we are actually not interested in solving the ODE but to find a function r_n such that (6.8) holds. Therefore, it is interesting to calculate the left-hand side of (6.8),that is $q_n(T_n(F))$, for $r_n = d_{0,n}$. It is very astonishing that this expression virtually seems to depend linearly on $T_n(F) = \{t \in [0,1] : 1-\zeta+\zeta t = r_n(t)\}$ as n increases, cf. Figure 6.4. Below we argue that $T_n(F) \in (0, 0.1)$ corresponds roughly to $\zeta \in (1/3, 1)$. Looking at Figure 6.4 we notice that for $T_n(F) \in (0, 0.1)$ the left-hand side of (6.8) lies between $[1.7, 2]$. According to the quantile equation (6.8) this can be roughly interpreted as

$$\lim_{n\to\infty} \mathbb{P}_\vartheta\left(V_n(\varphi_n) > g_n(n_1)\right) \in 1-\Phi([1.7,2]) \approx [0.02, 0.05]$$

for $\zeta \in (1/3, 1)$, where φ_n is a SU test induced by $d_{0,n}(x)$ with $q = 2$. Thereby, we consider $d_{0,n}$ as a “good” starting point for an optimization step. Quite decent results can be obtained by modifying the parameter q in $d_{0,n}$. Hence, as a simple adjustment we choose for now $r_n(t) = d_{0,n}(t) + n^{-1/2}(a+bt)$ and consider (q, a, b) as parameters for an optimization process. This

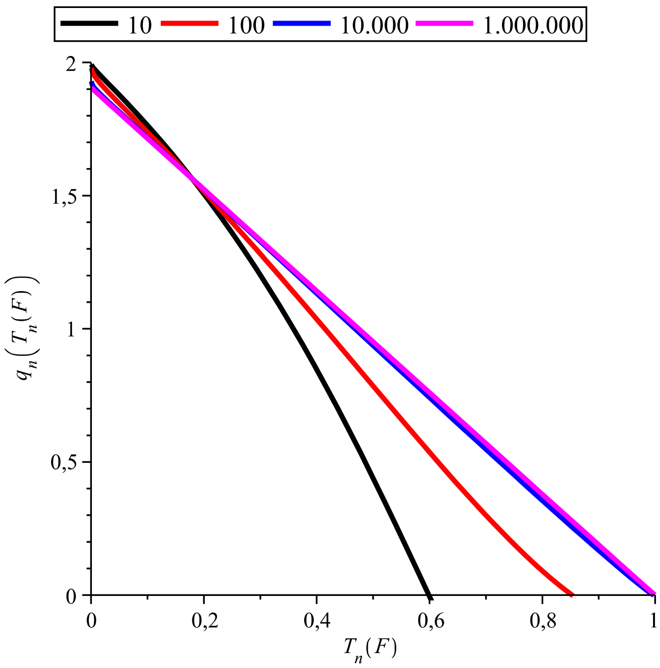

Figure 6.4: *The figure shows* $q_n(T_n(F))$ *for* $r_n = d_{0,n}$ *for* $n = 10,\ 10^2,\ 10^4,\ 10^6$ *with* $q = 2$.

choice is motivated by the following observation. Figure 6.4 indicates that for $r_n = d_{0,n}$ there exists $\tilde{a}$ and $\tilde{b}$ such that

$$\sqrt{n}\frac{1 - Q_n(T_n(F)) + g(Q_n(T_n(F))) - d_{0,n}(T_n(F))}{s(d_{0,n}(T_n(F)), Q_n(T_n(F)), T_n(F))} + \tilde{a} + \tilde{b}T_n(F) \approx q.$$

Ignoring the fact that the denominator and Q_n of the last expression is a complicated function of $d_{0,n}$ we assume that there exist a and b such that

$$\sqrt{n}\frac{1 - Q_n(T_n(F)) + g(Q_n(T_n(F))) - d_{0,n}(T_n(F))}{s(d_{0,n}(T_n(F)), Q_n(T_n(F)), T_n(F))} + \tilde{a} + \tilde{b}T_n(F)$$
$$\approx \sqrt{n}\frac{1 - Q_n(T_n(F)) + g(Q_n(T_n(F))) - d_{0,n}(T_n(F)) - n^{-1/2}(a + bT_n(F))}{s(r_n(T_n(F)), Q_n(T_n(F)), T_n(F))},$$

where $r_n(t) = d_{0,n}(t) + n^{-1/2}(a + bt)$ and $Q_n(t) = (1 - r_n(t))/(1 - t)$.
Instead of "minimizing"

$$||n^{1/2}\frac{1 - Q_n(t) + g(Q_n(t)) - r_n(t)}{s(r_n(t), Q_n(t), t)} - q_{1-\beta}||_{t\in I} \tag{6.24}$$

in (q, a, b), where $q_{1-\beta}$ is the $1 - \beta$ quantile of the standard normal distribution, we "minimize" the equivalent expression

$$||\Phi\left(n^{1/2}\frac{1 - Q_n(t) + g(Q_n(t)) - r_n(t)}{s(r_n(t), Q_n(t), t)}\right) - 1 - \beta||_{t\in I} \tag{6.25}$$

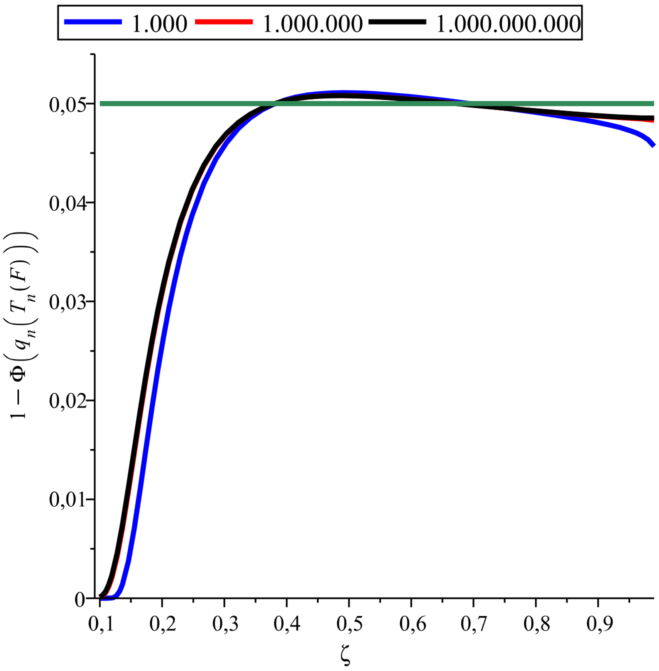

Figure 6.5: *The figure shows* $1-\Phi(q_n(T_n(F)))$ *for the rejection curves obtained by the minimization process for* $n=10^3$, $n=10^6$ *and* 10^9.

in (q,a,b), where I is some predefined interval. The only reason to work with (6.25) instead of (6.24) is because in practice it yields better results. In order to calculate concrete rejection curves, let $\alpha=\beta=0.05$. Now, we discuss a reasonable choice for the interval I. Obviously, $\mathbb{P}_{n,n_0}(V_n>n_1+1)=0$ if $n_1+1>n_0$. This happens asymptotically if $g(\zeta)>\zeta$, which is under our assumptions equivalent to $0.05=\alpha>\zeta$. Therefore, choosing $I=[0,1]$ is not a good idea. It is theoretically possible to state (6.25) in terms of ζ because $\zeta=Q_n(T_n(F))$. Ideally, we would specify $\tilde{I}$ as a set of $\zeta's$ and minimize (6.25) in terms of ζ for $\tilde{I}$, but in our case no explicit representation of $Q_n^{-1}(t)$ exists and therefore we use (6.25).

We set $I=(0,0.1]$ which virtually corresponds to $\zeta\in(1/3,1)$. This can be seen as follows. We remind that $1-\zeta+\zeta T_n(F)=r_n(T_n(F))$ is equivalent to $(1-r_n(T_n(F)))/(1-T_n(F))=Q_n(T_n(F))=\zeta$. We know that $r_n(t)$ will be near (and above) $r(t)=t/(t(1-\alpha)+\alpha)$. This means that the set of $\zeta's$ considered in our minimization will *contain* $\{(1-r(t))/(1-t):t\in I\}\approx(1/3,1)$. In our numerical example we minimized the following function

$$U(q,a,b)=\sum_{i=1}^{100}\left[\Phi\left(n^{1/2}\frac{1-Q_n(i/10^3)+g(Q_n(i/10^3))-r_n(i/10^3)}{s(r_n(i/10^3),Q_n(i/10^3),i/10^3)}\right)-1-\beta\right]^2.$$

Figure 6.5 shows the result of the minimization for $n=10^3$, 10^6, 10^9. Finite sample simulations have been conducted for $n=10.000$ and are visualized in Figure 6.6.

Extending I for instance to $(0,0.3]$, which corresponds to a set of $\zeta's$ that is roughly $[0.14,1)$, is possible, but in order to get a "good" solution one should raise the degree of freedom, for instance

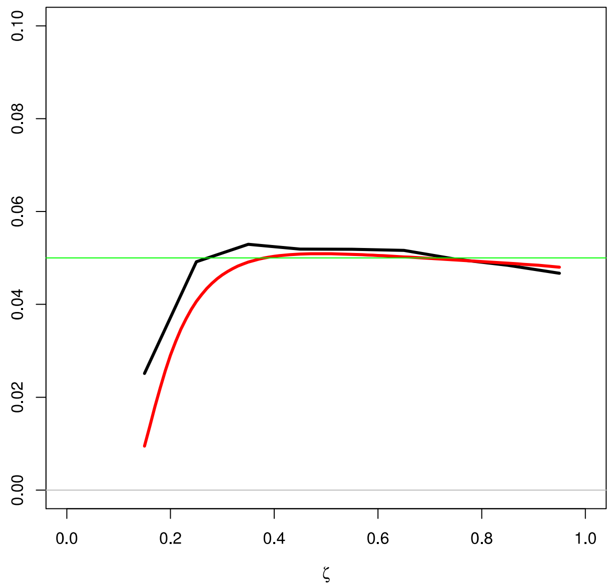

Figure 6.6: *The figure shows the empirical probability (solid black line; number of replications = 500.000) of* $V_n(\varphi_n) > (n_1+1)\alpha/(1-\alpha)$ *for* $n = 10.000$ *and* $\alpha = 0.05$, *where* φ_n *is a SU test induced by the rejection curve* r *obtained by the minimization process described in Section 6.5.1 The solid red line is* $1 - \Phi(q_n(T_n(F)))$ *for* r.

by setting $r_n(t) = d_{0,n}(t) + n^{-1/2}(a + bt + ct^2)$, with the simple restriction that $q, a, b + c \geq 0$ in order to ensure that r_n is strictly increasing.

Remark 6.24

In section 6.3 we elucidate how the quantile equation may be used to determine $\delta_i(t)$, $i = 1, \ldots, k$, $k \in \mathbb{N}$, such that $q_n(T_n(F)) \approx q$ for $r_n(t) = r(t) + \sum_{i=1}^k \delta_i(t)/n^{i/2}$. These rejection curves may provide better results than the approach presented in this section. In this case better means that $q_n(T_n(F))$ is closer to q for $r_n(t) = r(t) + \sum_{i=1}^k \delta_i(t)/n^{i/2}$ than for the rejection curves obtained in this section, compare Figure 6.1 and 6.5. The advantage of the approach in this section is that g has not to be continuously differentiable. This assumption is essential for the derivation of $\delta_i(t)$.

After the following subsection we have three methods for constructing rejection curves such that $q_n(T_n(F)) \approx q$. At the end of this section we compare all three methods briefly.

6.5.2 Solving by substitution

It turns out that substituting $(1-y)/(1-x)$ by z in (6.22) yields the possibility to solve the ODE numerically. This substitution yields the ODE

$$z - (1-x)z' = z\frac{1 - z + g(z) - (1 - (1-x)z)}{1 - z + g(z) - (1 - (1-x)z) - n^{-1/2}q\sqrt{x(1-x)z}}$$

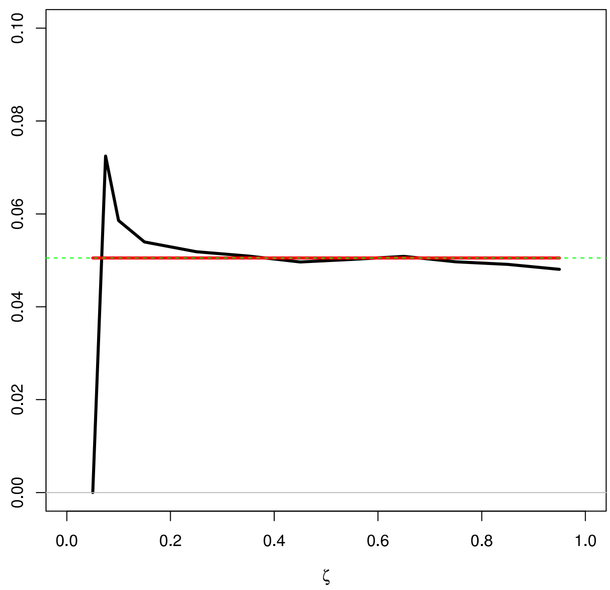

Figure 6.7: *The figure shows the empirical probability (solid black line; number of replications = 500.000) of* $V_n(\varphi_n) > (n_1+1)\alpha/(1-\alpha)$ *for* $n = 10.000$ *and* $\alpha = 0.05$, *where* φ_n *is a SU test induced by the rejection curve* r *obtained by numerically solving the ODE* (6.26). *The solid red line is* $1 - \Phi(q_n(T_n(F)))$ *for* r.

which is equivalent to

$$z' = -\frac{z}{1-x}\frac{n^{-1/2}q\sqrt{x(1-x)z}}{g(z) - xz - n^{-1/2}q\sqrt{x(1-x)z}}. \tag{6.26}$$

Ideally we would choose the initial value as $y(0) = 0$ which corresponds to $z(0) = 1$. But this entails that the denominator of the ODE (6.26) becomes zero. For the numerical solution we set $z(5/10^{10}) = 0.9998$ which corresponds to $y(5/10^{10}) \approx 0.0002$. Changing $5/10^{10}$ to $5/10^{11}$ or 0.9998 to 0.09999 results in no feasible solution. Of course, at this point further optimization is possible. Nevertheless, we conducted a similar simulation study as in the foregoing section for $n = 10.000$. In order to obtain the critical values for the simulation we calculated the two dimensional points $(i/10^6, 1-(1-i/10^6)z(i/10^6))$ for $i = 1, \ldots, 10^6 - 1$, where $z(x)$ denotes the numerical solution of the ODE (6.26). After a linear interpolation using the set of the two dimensional points extended by the points $(0,0)$ and $(1, 1+1/10^{10})$ the critical values $c_1, \ldots, c_{10^4}$ were obtained numerically from this linear interpolation. The critical values for the SU test used in the simulation study are $\min(c_1, c_{9800}), \ldots, \min(c_{10^4}, c_{9800})$. The reason for the truncation of the critical values is that a SU test with critical values $c_1, \ldots, c_{10^4}$ rejects all null hypotheses with high probability because the last critical values are near one. The chosen critical value c_{9800} is approximately 0.673. Hence, only null hypothesis with a corresponding p-value of less than 0.673 may be rejected. The simulation study is visualized in Figure 6.7. As can be seen from Figure 6.7 the empirical probability of $\mathbb{P}_{n,n_0}(V_n > g_n(n_1))$ increases if ζ tends to zero. If, for instance, $n_0/n = 0.075$, then $n_0 = n \cdot 0.075 = 750$. This is the point at which the empirical probability (black line in Figure 6.7) is maximal. For such constellations it is not surprising that the results of the asymptotic theory did not provide a good approximation. Of course, if $n_0/n < 0.05$, then $\mathbb{P}_{n,n_0}(V_n > g_n(n_1)) = 0$.

6.5.3 Discussion

Altogether, we have presented three techniques to obtain a sequence of rejection curves r_n such that $q_n(T_n(F)) = q$. In the first method we set $r_n(t) = r(t) + \sum_{i=0}^{k} \delta_i(t)/n^{i/2}$, $k \in \mathbb{N}$. By considering $q_n(t) = q$ for $t \in (0,1)$ it may be possible to determine $\delta_i(t)$ such that $q_n(t) \to q$ for all $t \in (0,1)$. The second method used the function $d_{0,n}(t)$ such that the numerator of the ODE (6.21) becomes zero. As a simple adjustment, cf. Subsection 6.5.1 for the motivation, we chose $r_n(t) = d_{0,n}(t) + n^{-1/2}(a + bt)$ and considered (a,b) and a further parameter that is part of $d_{0,n}$ as parameters for an optimization process. Of course, the three parameters are chosen such that $||q_n(t) - q||_{t\in I}$ is minimized for some prespecified interval I and norm $||\cdot||_{t\in I}$. In the third method we calculate a numerical solution for the ODE.

The validity of the following comparison of the three techniques is restricted to the important case where the asymptotic ENFR bounding function is $g(\zeta) = (1-\zeta)\alpha/(1-\alpha)$. SU tests $(\varphi_n)_{n\in\mathbb{N}}$ induced by rejection curves $(r_n)_{n\in\mathbb{N}}$ that fulfill $q_n(T_n(F)) = \Phi^{-1}(1-\alpha)$ for $\alpha \in (0,1)$ and $n \in \mathbb{N}$ will provide $\lim_{n\to\infty} \mathbb{P}_\vartheta(V_n(\varphi_n) > g_n(n_1)) = \alpha$. Hence, the main goal is to construct rejection curves r_n such that $q_n(T_n(F))$ is as close as possible to $\Phi^{-1}(1-\alpha)$. From that point of view, solving the ODE numerically seems to give the best result, cf. Figure 6.7. This procedure has two disadvantages. First, we were not able to solve the original ODE (6.21) numerically but a transformed version. Whether this transformation enables one to solve the ODE numerically for other g is not clear. Second, choosing $r(0) = 0$ as initial value for the ODE is not possible because the ODE is not defined at $(0,0)$. The first method yields also good results if k is large enough and $\delta_1(t), \ldots, \delta_k(t)$ exist, cf. Figure 6.1. The necessary condition that g has to be k-times continuously differentiable is a disadvantage compared with the second and the third method. The worst results are provided by the second method which is not surprising by its ad hoc approach. Of course, if $q_n(T_n(F)) = q$, no statement can be made for $\mathbb{P}_\vartheta(V_n(\varphi_n) > g_n(n_1))$ for fixed $n \in \mathbb{N}$. But it is also important that for fixed $n \in \mathbb{N}$ the probability $\mathbb{P}_\vartheta(V_n(\varphi_n) > g_n(n_1))$ is as close as possible to $1 - \Phi(q)$. Comparing Figures 6.2, 6.6 and 6.7, we can say that the second and the third method perform well because $1 - \Phi(q_n(T_n(F)))$ and the empirical probability $\mathbb{P}_\vartheta(V_n(\varphi_n) > g_n(n_1))$ are close at least for ζ not to small. With respect to this goal the simulation indicates that both the second and the third method perform well and the first method is somewhat unsatisfactory.

Again, we want to distress that the validity of this comparison is restricted since we considered only one asymptotic ENFR bounding function for a fixed number of null hypotheses and a few different fraction of true null hypotheses.

6.6 Asymptotic control of the FDX

The topic of this thesis is the number of false rejections. Nevertheless, from the results of the foregoing sections a "quantile equation" and an ODE can easily be derived for the FDP.

In general, DU is not LFC for the FDP, as can be seen from the next (counter) example or heuristically from Figure 6.8. It is not clear under which condition DU becomes LFC. In the following, we are interested in SU tests φ_n such that

$$\lim_{n\to\infty} \mathbb{P}_\vartheta \left(\frac{V_n(\varphi_n)}{\max\{S_n(\varphi_n) + V_n(\varphi_n), 1\}} > c \right) = \alpha. \tag{6.27}$$

One thing is very clear: if $n_0/(n_0+n_1) \leq c$, then $\mathbb{P}_{n,n_0}$ (FDP $> c) = 0$. This already shows that DU in general is not an LFC for the FDP because $\mathbb{P}_\vartheta$ (FDP $> c) > 0$ is possible.

Example 6.25 (Counter example)
DU is not LFC for a SD or SU test if one null hypothesis is false, one is true and $c > 1/2$. Denote by p (q) the p-value corresponding to the true (false) null hypothesis. We have

$$\mathbb{P}_\vartheta\,(\text{FDP} > c) = \mathbb{P}_\vartheta\,(p \leq c_1, q > c_2)$$

which in general is not zero, but becomes zero under DU configuration. A similar result can be obtained for a SU test.

Example 6.26
DU is a LFC for a SD test if one null hypothesis is false, one is true and $c < 1/2$. Denote by p (q) the p-value corresponding to the true (false) null hypothesis and by c_1 and c_2 the critical values of the SD test. We have

$$\begin{aligned}
&\mathbb{P}_\vartheta\,(\text{FDP} > c) \\
&= \mathbb{P}_\vartheta\,(p \leq c_1, q > c_2) + \mathbb{P}_\vartheta\,(\{p \leq c_1, q \leq c_2\} \cup \{p \leq c_2, q \leq c_1\}) \\
&= \mathbb{P}_\vartheta\,(p \leq c_1, q > c_2) + \mathbb{P}_\vartheta\,(p \leq c_1, q \leq c_2) + \mathbb{P}_\vartheta\,(p \leq c_2, q \leq c_1) - \mathbb{P}_\vartheta\,(p \leq c_1, q \leq c_1) \\
&= \mathbb{P}_\vartheta\,(p \leq c_1) + \mathbb{P}_\vartheta\,(p \leq c_2, q \leq c_1) - \mathbb{P}_\vartheta\,(p \leq c_1, q \leq c_1)\,.
\end{aligned}$$

Note, it holds

$$\mathbb{P}_\vartheta\,(p \leq c_2, q \leq c_1) = \mathbb{P}_\vartheta\,(p \leq c_2) + \mathbb{P}_\vartheta\,(q \leq c_1) - \mathbb{P}_\vartheta\,(\{p \leq c_2\} \cup \{q \leq c_1\})\,.$$

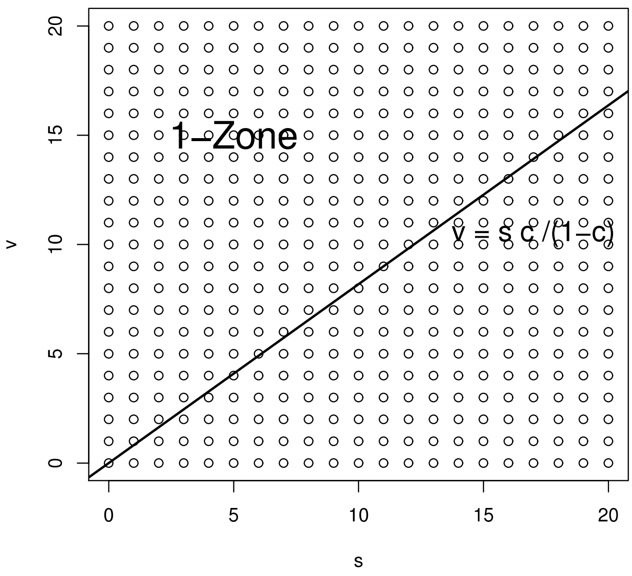

Figure 6.8: *The circles denote possible positions where* $\mathbb{P}_\vartheta(S = s, V = v)$ *may be positive. The 1-Zone denotes the region where* $\mathbb{I}_{\{v/(v+s)>c\}} = 1$ *for c = 0.45 and n=20 and thus only circles in this region may contribute something to* $\mathbb{P}_\vartheta(\text{FDP} > c)$.

Thus,

$$\begin{aligned}
\mathbb{P}_\vartheta(\text{FDP} > c) &= \mathbb{P}_\vartheta(p \le c_1) + \mathbb{P}_\vartheta(p \le c_2) + \mathbb{P}_\vartheta(q \le c_1) - \mathbb{P}_\vartheta(\{p \le c_2\} \cup \{q \le c_1\}) \\
&\quad - [\mathbb{P}_\vartheta(p \le c_1) + \mathbb{P}_\vartheta(q \le c_1) - \mathbb{P}_\vartheta(\{p \le c_1\} \cup \{q \le c_1\})] \\
&= \mathbb{P}_\vartheta(p \le c_2) - [\mathbb{P}_\vartheta(\{p \le c_2\} \cup \{q \le c_1\}) - \mathbb{P}_\vartheta(\{p \le c_1\} \cup \{q \le c_1\})] \\
&\le \mathbb{P}_\vartheta(p \le c_2) \\
&\le \mathbb{P}_{n,n_0}(\text{FDP} > c),
\end{aligned}$$

hence $\text{DU}(2,1)$ is LFC in this special case.

Remark 6.27

We conducted explicit calculations for two other scenarios, that is for a SU test with one true and one false null hypothesis and for a SU test with two true null hypotheses and one false null hypothesis. In both cases DU is a LFC if $c < 1/2$.

We now conduct some heuristic considerations. It holds that

$$\mathbb{P}(\text{FDP} > c) = \sum_v \sum_s \mathbb{I}_{\{v/(v+s)>c\}} \mathbb{P}(S = s, V = v) = \sum_v \sum_s \mathbb{I}_{\{v>sc/(1-c)\}} \mathbb{P}(S = s, V = v).$$

Considering Figure 6.8 we see that $\mathbb{P}(\text{FDP} > c)$ increases only if the sum of the probabilities of points in the 1-Zone increases. A point (s, v) is in the 1-Zone if $v > sc/(1-c)$. But under DU, all points with a positive probability are on the vertical at $S = n_1$. For large n_1, depending on c, only a few points lie in the 1-Zone. Hence, only a few points contribute to $\mathbb{P}\,(\text{FDP} > c)$. On the other hand, the figure indicates that $\mathbb{P}\,(\text{FDP} > c)$ increases if the p-values corresponding to false null hypotheses are set to 1, because then all points except $(v, s) = (0, 0)$ lie in the 1-Zone. Hence, many points contribute to $\mathbb{P}\,(\text{FDP} > c)$. Of course, assuming that a p-value corresponding to a false null hypothesis equals 1 a.s. is pointless, but it indicates that in order to increase $\mathbb{P}\,(\text{FDP} > c)$ one should stochastically increase the p-values corresponding to false null hypotheses instead of stochastically decrease these p-values. Nevertheless, for $c < 1/2$ and a few simple cases we were able to prove that DU is LFC for the FDX. However, we easily obtain the following theorem containing a "quantile equation" for the FDX.

Theorem 6.28 (Quantile equation for the $c-$FDX)
Assume that $n_0/n = \zeta + o(n^{-1/2})$ *and* $c \in (0, \zeta)$. *If*

$$n^{1/2}\frac{(1 - Q_n(T_n(F))F_1(T_n(F)) - (1-c)r_n(T_n(F))}{s(r_n, Q_n(T_n(F)), T_n(F))(1-c)r_n'(T_n(F))} = q + o(1) \tag{6.28}$$

and $S_n(\varphi_n)+V_n(\varphi_n) > 0$ *a.s. eventually for all* n, *where* $Q_n(t) = (F_1(t)-r_n(t))/(F_1(t)-F_0(t))$ *and*

$$s^2(r,\zeta,\tau^*) = \frac{[(1-c)r'(\tau^*) - (1-\zeta)f_1(\tau^*)]^2\zeta F_0(\tau^*)(1-F_0(\tau^*))}{[(1-c)r'(\tau^*)(r'(\tau^*) - f(\tau^*))]^2} + \frac{[cr'(\tau^*) - \zeta f_0(\tau^*)]^2(1-\zeta)F_1(\tau^*)(1-F_1(\tau^*))}{[(1-c)r'(\tau^*)(r'(\tau^*) - f(\tau^*))]^2},$$

then

$$\lim_{n\to\infty}\mathbb{P}_\vartheta\left(\frac{V_n(\varphi_n)}{\max\{S_n(\varphi_n)+V_n(\varphi_n), 1\}} > c\right) = 1 - \Phi(q),$$

where φ_n *is a SU test induced by* r_n.

Before we proof Theorem 6.28 we want to make two remarks.

Remark 6.29
Equation (6.28) can not hold if $c \in [\zeta, 1)$. This can be seen as follows. By Equation (6.28), we have

$$O(1) = n^{1/2}(1 - Q_n(T_n(F))F_1(T_n(F)) - (1-c)r_n(T_n(F)).$$

Hence, we get $(1 - Q(\tau^*))F_1(\tau^*) - (1-c)r(\tau^*) = 0$. Since $Q(\tau^*) = \zeta$, we obtain

$$\frac{1-\zeta}{1-c}F_1(\tau^*) = r(\tau^*).$$

Because $\zeta \le c$, this entails that $F_1(\tau^*) \le r(\tau^*) = (1-\zeta)F_1(\tau^*)+\zeta F_0(\tau^*) < F_1(\tau^*)$. Therefore, we assumed $c \in (0, \zeta)$ instead of $c \in (0, 1)$.

Remark 6.30

Note, $cr'(\tau^*) - \zeta f_0(\tau^*) = 0$ and $(1-c)r'(\tau^*) - (1-\zeta)f_1(\tau^*) = 0$ would entail $r'(\tau^*) = f(\tau^*)$ which contradicts the unique crossing-point condition. Hence, $s(r, \zeta, \tau^*) \neq 0$.

Proof of Theorem 6.28. The proof is very similar to the proof of Theorem 6.5. Denote by $F^0_{n_0}(t)$ and $F^1_{n_1}(t)$ the ecdf of the p-values corresponding to true and false null hypotheses, respectively. Since $S_n(\varphi_n) + V_n(\varphi_n) > 0$ a.s. eventually for all n, we directly consider

$$\frac{V_n}{S_n + V_n} > c \Leftrightarrow (1-c)n_0 F^0_{n_0} + (1-c)n_1 F^1_{n_1} - n_1 F^1_{n_1} > 0 \Leftrightarrow r_n - \frac{n_1 F^1_{n_1}}{n(1-c)} > 0,$$

suppressing the dependence on $T_n(F_n)$. In the proof of Theorem 6.5 we obtained a very similar expression, namely,

$$V_n > g_n(n_1) \Leftrightarrow r_n - \frac{n_1 F^1_{n_1}}{n} > \frac{g_n(n_1)}{n}.$$

Therefore, using the notation from the proof of Theorem 6.5, that is Δ_n, $\Delta_{0,n}$, $\Delta_{1,n}$, $D_{2,n}$ and $D_{3,n}$, and following the calculations in the proof of Theorem 6.5 we obtain

$$\begin{aligned}
&\frac{V_n}{S_n + V_n} > c \\
&\Leftrightarrow n^{1/2}(T_n(F_n) - T_n(F)) - \frac{n^{1/2} n_1 F_{n_1}}{n(1-c)r'_n(T_n(F))} > -n^{1/2}\frac{r_n(T_n(F))}{r'_n(T_n(F))} \\
&\Leftrightarrow \Delta_n - \frac{\Delta_{1,n} + D_{2,n}\Delta_n + n^{1/2}D_{3,n}}{(1-c)r'_n(T_n(F))} > -n^{1/2}\frac{r_n(T_n(F))}{r'_n(T_n(F))} \\
&\Leftrightarrow D_{4,n}(c)\Delta_n - \frac{\Delta_{1,n}}{(1-c)r'_n(T_n(F))} > n^{1/2}\left(\frac{D_{3,n}}{(1-c)r'_n(T_n(F))} - \frac{r_n(T_n(F))}{r'_n(T_n(F))}\right),
\end{aligned}$$

where $D_{4,n}(c) = 1 - D_{2,n}/((1-c)r'_n(T_n(F)))$. Again, using the calculation from the proof of Theorem 6.5, we see that

$$\begin{aligned}
&D_{4,n}(c)\Delta_n - \frac{\Delta_{1,n}}{(1-c)r'_n(T_n(F))} \\
&= \frac{D_{4,n}(c)\Delta_{0,n}}{r'(\tau^*) - f(\tau^*)} + \frac{D_{4,n}(c)\Delta_{1,n}}{r'(\tau^*) - f(\tau^*)} - \frac{\Delta_{1,n}}{(1-c)r'(\tau^*)(1+o(1))} \\
&= \frac{D_{4,n}(c)\Delta_{0,n}}{r'(\tau^*) - f(\tau^*)} + \frac{((1-c)D_{4,n}(c) - 1)r'(\tau^*) + f(\tau^*)}{(1-c)r'(\tau^*)(r'(\tau^*) - f(\tau^*))}\Delta_{1,n} + o_P(1) \\
&= h_n(\Delta_{0,n}, \Delta_{1,n}) + o_P(1) \text{ (say).}
\end{aligned}$$

The same argumentation as in the proof of Theorem 6.5 provides that $h_n(\Delta_{0,n}, \Delta_{1,n})$ converges in distribution to a random normal variable with zero mean and variance

$$\begin{aligned}
s^2(r, \zeta, \tau^*) = &\frac{[(1-c)r'(\tau^*) - (1-\zeta)f_1(\tau^*)]^2 \zeta F_0(\tau^*)(1 - F_0(\tau^*))}{[(1-c)r'(\tau^*)(r'(\tau^*) - f(\tau^*))]^2} \\
&+ \frac{[cr'(\tau^*) - \zeta f_0(\tau^*)]^2 (1-\zeta)F_1(\tau^*)(1 - F_1(\tau^*))}{[(1-c)r'(\tau^*)(r'(\tau^*) - f(\tau^*))]^2}.
\end{aligned}$$

Since we assumed that

$$n^{1/2}\left(\frac{D_{3,n}}{(1-c)r_n'(T_n(F))}-\frac{r_n(T_n(F))}{r_n'(T_n(F))}\right)/s(r_n,Q_n(T_n(F)),T_n(F))=q+o(1),$$

we conclude again as in the proof of Theorem 6.5 that

$$\lim_{n\to\infty}\mathbb{P}_\vartheta\left(\frac{V_n(\varphi_n)}{\max\{S_n(\varphi_n)+V_n(\varphi_n),1\}}>c\right)=1-\Phi(q).$$ □

Remark 6.31

Similar as in Remark 6.17, for $r=f_{\alpha,0}$ and $c=\alpha$ we have that the numerator of (6.28) equals zero. Hence, SU tests φ_n induced by $\tilde{r}(t)=\max\{f_{\alpha,0}(t),2+(t-1)/\epsilon\}$ yield

$$\mathbb{P}_{n,n_0}\left(\text{FDP}(\varphi_n)>\alpha\right)\to 1/2$$

for $n_0/n\to\zeta\in(\alpha/(1-2\epsilon+2\alpha\epsilon),1)$, where $\epsilon\in(0,\alpha)$. As in Remark 6.18 we have to consider $\tilde{r}$ instead of $f_{\alpha,0}$ because the AORC does not fulfill the unique crossing-point condition.

In general, DU is not LFC for the FDP. Nevertheless, we want to state the corresponding ODE. As before, we solve (6.28) for $r_n'(T_n(F))$ which provides the following corollary.

Corollary 6.32

Theorem 6.28 holds true under DU for $q\neq 0$, if equation (6.8) *is replaced by the assumption that*

$$r_n'(t)=Q_n(t)\frac{H_n(t)-(1-c)r_n(t)}{H_n(t)-(1-c)r_n(t)-n^{-1/2}q\sqrt{Q_n(t)t(1-t)}} \tag{6.29}$$

holds at $T_n(F)$ for $n>N$ with

$$H_n(t)=1-Q_n(t),$$
$$Q_n(t)=(1-r_n(t))/(1-t).$$

6.7 Summary

In this chapter we considered an error measure based on the number of false rejections that is very closely related to the ENFR criterion. All results were derived under independence assumptions for SU tests and exclusively for the non-sparsity case. First, we showed that the (random) crossing-point is asymptotically normally distributed. Then, we derived quantile equations guaranteeing that a sequence of rejection curves $(r_n)_{n\in\mathbb{N}}$ which fulfill the quantile equations and some further regularity conditions ensure (6.2) for SU tests φ_n induced by r_n. For a special case it turns out that the AORC fulfills all quantile equations. The quantile equations implicitly define rejection curves r_n and also contain the derivative r_n'. Solving the equations for r_n' yields ODEs. Since DU is in general LFC for this error measure, we investigated the ODEs under DU configurations

and showed that solutions exist such that for growing n the number of critical values that can be determined tends to infinity. Moreover, we exemplified with three different techniques how a sequence of rejection curves can be obtained (numerically) under DU. In a small simulation study these solutions show a "good" performance. Based on these results we briefly compared the three techniques.
In the final part of this chapter we elucidated how the results carry over to the FDP. Unfortunately, DU is in general not LFC for the FDP.

Chapter 7

Outlook

In this thesis we primarily focused on independent p-values. The independent case is an idealized situation and theorems developed under independence may serve as a reference point. But dependence between p-values is an important aspect that has to be taken into account because usually the test statistics are dependent. For instance, two adjacent SNPs will have similar test statistics compared to two SNPs with very different locations; and genes that encode enzyme cascades will probably provide similar expression patterns.
It is reasonable to investigate different types of dependence. If it is possible to exclude negative correlation one may loose "power" by using a procedure that controls some error rate under general dependence. In the following sections, we sketch some ideas/proofs with respect to the ENFR and dependent p-values.

7.1 Weak dependence

Roughly spoken, under weak dependence the ecdfs of the p-values corresponding to true and false null hypotheses converge to distribution functions F_0 and F_1, respectively. This can be interpreted that for large n the p-values behave like independent random variables. This dependence structure was thoroughly investigated in the thesis of Gontscharuk [26]. For example, weak dependence may hold if the p-values are "block dependent". She also elucidates procedures, which were developed under independence and control the FWER or FDR, still work under weak dependence. Of course, her results may be carried over to the ENFR. This probably will yield procedures that control the ENFR asymptotically if $n_0/n \to \zeta \in [0, 1)$.

7.2 Resampling

The resampling methodology is very well established and has already been applied to multiple testing situations, for instance cf. [15], [69], [70], [71], [76], and [78]. We sketch now how

resampling methods may be applied in order to ensure asymptotic ENFR control.
Suppose we have m persons and every person provides n measurements. We want to test if the mean of the jth measurement is larger than zero. More precisely, we assume that

$$X_i = (X_{i1}, \ldots, X_{in}) \sim F, \quad i = 1, \ldots, m$$

are iid and that the null hypotheses are given by $H_{0j} : \mu_j(F) = 0$ versus $H_{1j} : \mu_j(F) > 0$ for $j = 1, \ldots, n$, where $\mu_j(G)$ is the mean of the jth component under an arbitrary n-dimensional distribution function G. In the same way, $\mu(G)$ denotes the mean vector under an arbitrary n-dimensional distribution function G. Denote by F_m the n-dimensional ecdf of $\{X_i\}_{i=1,\ldots,m}$. Under the global null hypothesis we assume that

$$T_m = \sqrt{m}(\mu(F_m) - \vec{0}) \to T \quad \text{in distribution} \quad (m \to \infty),$$

where T denotes a centered multivariate normal random variable with covariance matrix Σ, that is $T \sim \mathrm{N}_n(\vec{0}, \Sigma)$. The bootstrap version of T_m is

$$T_m^* = \sqrt{m}(\mu(F_m^*) - \mu(F_m)),$$

where F_m^* is the ecdf of $\{X_i^*\}_{i=1,\ldots,m}$, which were sampled with replacement from $\{X_i\}_{i=1,\ldots,m}$. Then, given $\{X_i\}_{i=1,\ldots,m}$, the bootstrap version T_m^* converges in distribution to a random variable $T^* \sim \mathrm{N}_n(\vec{0}, \Sigma)$ as $m \to \infty$. Denote by $\mathbb{P}^*(A)$ and $\mathbb{E}^*(Y)$ the probability of an event A and the expectation of a random variable Y given $\{X_i\}_{i=1,\ldots,m}$. In a first step we construct an asymptotic upper confidence bound for n_1. Let $\vec{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$ be fixed and $V_n(\vec{c}) = \sum_{j:\mu_j(F)=0} \mathbb{I}_{\{T_{mj} > c_j\}}$. It holds that

$$\begin{aligned}
\lim_{m\to\infty} \mathbb{P}\left(V_n(\vec{c}) > k\right) &= \lim_{m\to\infty} \mathbb{P}\left(\sum_{j:\mu_j(F)=0} \mathbb{I}_{\{T_{mj} > c_j\}} > k\right) \\
&= \lim_{m\to\infty} \mathbb{P}^*\left(\sum_{j:\mu_j(F)=0} \mathbb{I}_{\{T^*_{mj} > c_j\}} > k\right) \\
&\leq \lim_{m\to\infty} \mathbb{P}^*\left(\sum_{j=1}^{n} \mathbb{I}_{\{T^*_{mj} > c_j\}} > k\right).
\end{aligned}$$

Choosing k such that $\mathbb{P}^*\left(\sum_{j=1}^{n} \mathbb{I}_{\{T^*_{mj} > c_j\}} > k\right) \leq \alpha/2$ provides that $\lim_{m\to\infty} \mathbb{P}\left(V_n(\vec{c}) > k\right) \leq \alpha/2$. Therefore,

$$\mathbb{P}\left(R_m(\vec{c}) - k > n_1\right) \leq \mathbb{P}\left(R_m(\vec{c}) - k > R_m(\vec{c}) - V_m(\vec{c})\right) \leq \mathbb{P}\left(V_m(\vec{c}) > k\right),$$

that is $\hat{n}_1 = R_m(\vec{c}) - k$ is asymptotically an $1 - \alpha/2$ upper confidence bound for n_1. A similar upper confidence bound was already proposed in [43] for permutation tests. With $\hat{n}_1$ it may be possible to achieve that either $\lim_{m\to\infty} \mathbb{P}(V_n \geq g(n_1)) \leq \alpha$ or $\lim_{m\to\infty} \mathbb{E}V_n \leq g(n_1)$. If g is non-decreasing in n_1, then it holds that

$$\begin{aligned}
\lim_{m\to\infty} \mathbb{P}\left(\frac{V_n(\tau)}{g(n_1)} \geq 1\right) &= \lim_{m\to\infty} \mathbb{P}\left(\frac{\sum_{j:\mu_j(F)=0} \mathbb{I}_{\{T_{mj}>\tau_j\}}}{g(n_1)} > 1\right) \\
&\leq \lim_{m\to\infty} \mathbb{P}^*\left(\frac{\sum_{j=1}^n \mathbb{I}_{\{T^*_{mj}>\tau_j\}}}{g(n_1)} > 1\right) \\
&\leq \lim_{m\to\infty} \mathbb{P}^*\left(\{\frac{\sum_{j=1}^n \mathbb{I}_{\{T^*_{mj}>\tau_j\}}}{g(n_1)} > 1\} \cap \{\hat{n}_1 \leq n_1\}\right) + \mathbb{P}^*(\hat{n}_1 > n_1) \\
&\leq \lim_{m\to\infty} \mathbb{P}^*\left(\frac{\sum_{j=1}^n \mathbb{I}_{\{T^*_{mj}>\tau_j\}}}{g(\hat{n}_1)} > 1\right) + \alpha/2.
\end{aligned}$$

Choosing $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n$ such that $\mathbb{P}^*\left(\sum_{j=1}^n \mathbb{I}_{\{T^*_{mj}>\tau_j\}}/g(\hat{n}_1) > 1\right) \leq \alpha/2$ provides that $\lim_{m\to\infty} \mathbb{P}(V_n(\tau) \geq g(n_1)) \leq \alpha$. At this point we also would have achieved a similar aim we pursued in Chapter 6, confer (6.2).

Let $V_n^* = \sum_{j=1}^n \mathbb{I}_{\{T^*_{mj}>\tau_j\}}$, then

$$\begin{aligned}
\lim_{m\to\infty} \mathbb{E}\frac{V_n(\tau)}{g(n_1)} &\leq \lim_{m\to\infty} \mathbb{E}\mathbb{E}^*\frac{V_n^*(\tau)}{g(n_1)} \\
&\leq \lim_{m\to\infty} \mathbb{E}\mathbb{E}^*\frac{V_n^*(\tau)}{g(\hat{n}_1)} + \frac{1}{g(0)}\mathbb{E}\mathbb{E}^* V_n^*(\tau)\mathbb{I}_{\{\hat{n}_1>n_1\}} \\
&\leq \lim_{m\to\infty} C_{m,1} + \frac{1}{g(0)}\mathbb{E}V_n^*(\tau)\mathbb{I}_{\{\hat{n}_1>n_1\}} \\
&\leq \lim_{m\to\infty} C_{m,1} + \frac{1}{g(0)}(\mathbb{E}\mathbb{I}_{\{\hat{n}_1>n_1\}})^{1/q}(\mathbb{E}V_n^*(\tau)^p)^{1/p} \\
&\leq \lim_{m\to\infty} C_{m,1} + \frac{1}{g(0)}(\alpha/2)^{1/q}(\mathbb{E}\mathbb{E}^*V_n^*(\tau)^p)^{1/p} \\
&\leq \lim_{m\to\infty} C_{m,1} + \frac{1}{g(0)}(\alpha/2)^{1/q}C_{m,2}^{1/p}.
\end{aligned}$$

Note,

$$C_{m,1} = \mathbb{E}^*\frac{\sum_{j=1}^n \mathbb{I}_{\{T^*_{mj}>\tau\}}}{g(\hat{n}_1)} \text{ and } C_{m,2} = \mathbb{E}^*\left(\sum_{j=1}^n \mathbb{I}_{\{T^*_{mj}>\tau\}}\right)^p$$

can be calculated/approximated for every fixed $\tau \in \mathbb{R}^n$. Therefore $\alpha, p, q = p/(1-p)$, and τ can be chosen such that $C_{m,1} + (\alpha/2)^{1/q}C_{m,2}^{1/p}/g(0) \leq 1$ which would establish asymptotic ENFR control, that is $\lim_{m\to\infty} \mathbb{E}V_n(\tau) \leq g(n_1)$.

Remark 7.1

We want to distress that the kind of asymptotic occurred in this section is very different from the asymptotic considerations done so far. Careful reading shows that m, the sample size, must tend

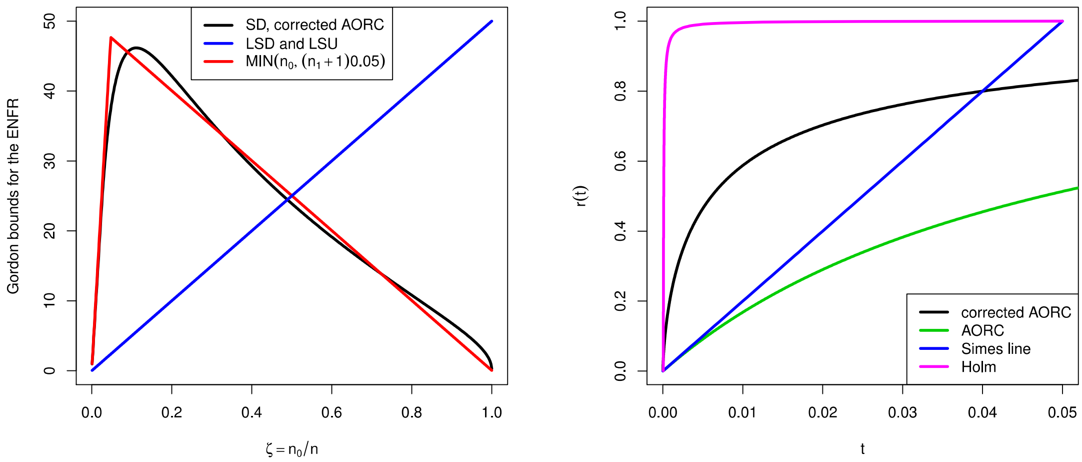

Figure 7.1: *The left figure shows the ENFR of a "corrected" SD, LSD, and LSU procedure under general dependence. The right figure shows the corresponding rejection curves on (0,0.05) with additional rejection curves (green and magenta) for comparative purpose.*

to infinity and n, the number of null hypotheses, must be kept fixed. This is also a disadvantage of this approach. For example, in SNP experiments one usually has $n \approx 10^6$ SNPs and only $m \approx 10^3$ persons. Of course, in such a situation it is doubtful whether resampling techniques are appropriate tools.

7.3 General dependence

Under general dependence any kind of dependence is allowed, that is

$$\Theta = \{\mathbb{P} : \mathbb{P} \text{ is a probability distribution on } [0,1]^n\}.$$

Of course, procedures that control the ENFR under general dependence will be conservative. Nevertheless, if nothing is known about the dependence structure or even if nothing can be guessed, then we have to retreat to general dependence. Furthermore, for a fixed procedure it is interesting to evaluate the worst case for the ENFR. Due to Alexander Gordon, cf. [29], we have formulas for the ENFR under general dependence for SUD tests. Here we briefly exemplify a possibility to correct a set of critical values to achieve nearly ENFR control under general dependence.
According to [29] we have for a SD test φ_n with critical values $0 \le c_1 \le \ldots \le c_n \le 1$ that

$$\sup_{\vartheta\in\Theta} \text{ENFR}_\vartheta(\varphi_n) = \max_{1\le n_0\le n} n_0^2 \min_{1\le i\le n_0} \frac{c_{n_1+j}}{j}.$$

Consider the critical values

$$c_i(a,b) = \left(\frac{ia}{n+1-i(1-a)}\right)^b \quad (i = 1,\ldots,n)$$

with parameters a and b. Note, $c_i(\alpha, 1)$ is the ith critical value of the β-adjusted AORC with $\beta = 1$. Figure 7.1 visualizes the resulting ENFR curve under general dependence for $a \approx 0.031$ and $b \approx 1.449$ which minimize

$$\sum_{n_0=200}^{1000} |n_0^2 \min_{1 \le i \le n_0} (c_{n_1+i}(a,b)/i) - (n_1+1)\alpha/(1-\alpha)|$$

for $\alpha = 0.05$ and $n = 1.000$. The resulting ENFR curve is not perfect but represents a good starting point for further correction steps. It should be mentioned that the resulting critical values are very small even compared to the critical values of Holm's procedure. For instance, $c_1(a, b) \approx 3 \cdot 10^{-7}$ and the first critical value of Holm's procedure is $\alpha \cdot 10^{-3} = 5 \cdot 10^{-5}$. See Figure 7.1 in order to compare the rejection curve obtained by the minimization process and the rejection curve that corresponds to Holm's procedure. A reason for this fact is that $\text{FWER}_\vartheta \le \text{ENFR}_\vartheta$ and in general we have that the ENFR_ϑ is much larger than the FWER_ϑ. But for instance for $n_1 = 0$ we require that $\text{ENFR}_\vartheta \le \alpha/(1-\alpha)$. Finally, we want to mention that by the simplicity of Gordon's formulas it would be worth to investigate them from a more analytical point of view.

Appendix A

Ordinary differential equations

Let $G \subset \mathbb{R}^2$ be open and

$$w : G \to \mathbb{R}, \quad (x, y) \mapsto w(x, y).$$

Then

$$y'(x) = w(x, y(x)) \quad \text{or simply} \quad y' = w(x, y) \tag{A.1}$$

is called an ordinary differential equation. A differentiable function

$$g : I \to \mathbb{R}$$

is called a solution of (A.1) if

$$\{(x, g(x)) : x \in I\} \subset G \quad \text{and} \quad g'(x) = w(x, g(x)) \text{ for all } x \in I.$$

Theorem A.1 (Peano's Theorem (cf. Theorem 2.1 in [30]))
Suppose $w(x, y)$ is continuous on $K = [x_0, x_0 + a] \times [y_0 - b, y_0 + b] \subset G$ and M is a bound for $|w(x, y)|$ on K. Then (A.1) *possesses at least one solution $y(x)$ with $y(x_0) = y_0$ on $I = [x_0, x_0 + \alpha]$, where $\alpha = \min(a, b/M)$.*

Definition A.2
A function $w(x, y)$ defined on G is uniformly Lipschitz continuous with respect to y on $D \subset G$ if there exists a constant C satisfying

$$|w(x, y_2) - w(x, y_1)| \leq C|y_1 - y_2| \quad \textit{for all } (x, y_1), (x, y_2) \in D.$$

Theorem A.3 (Theorem of Picard-Lindelöf (cf. Theorem 1.1 in [30]))
Suppose $w(x, y)$ is continuous on $K = [x_0, x_0 + a] \times [y_0 - b, y_0 + b] \subset G$ and uniformly Lipschitz continuous with respect to y on K. Let M be a bound for $|w(x, y)|$ on K. Then, (A.1) *has a unique solution $y(x)$ with $y(x_0) = y_0$ on $I = [x_0, x_0 + \alpha]$, where $\alpha = \min(a, b/M)$.*

Appendix B

Stochastic supplementary

Definition B.1
The space $D[-\infty, \infty]$ is the set of all functions $F : \mathbb{R} \to \mathbb{R}$ that are right continuous and whose limits from the left exist everywhere in $\mathbb{R}$ endowed with the supremums norm.

Definition B.2
Let F be a distribution function. An F-Brownian bridge B_F is a Gaussian process with zero mean and covariance

$$\mathbb{E}[B_F(t_1)B_F(t_2)] = F(\min\{t_1, t_2\}) - F(t_1)F(t_2).$$

Theorem B.3 (Donsker's Theorem (cf. Theorem 19.3 in [74])
If $X_1, X_2, \ldots$ are iid random variables with distribution function F, then the sequence of empirical processes $\sqrt{n}(F_n - F)$ converges in distribution in $D[-\infty, \infty]$ to an F-Brownian bridge.

Theorem B.4 (Extended continuous mapping theorem (cf. Theorem 18.11 in [74]))
Let D, E be arbitrary metric spaces, $D_n \subset D$ be arbitrary subsets and $g_n : D_n \to E$ be arbitrary maps $(n \geq 0)$ such that for every sequence $x_n \in D_n$: if $x_{n'} \to x$ along a subsequence and $x \in D_0$, then $g_{n'}(x_{n'}) \to g_0(x)$. Then, for arbitrary maps $X_n : \Omega_n \to D_n$ and every random element X with values in D_0 such that $g_0(X)$ is a random element in E it holds that $X_n \to X$ in distribution implies $g_n(X_n) \to g_0(X)$ in distribution.

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Bibliography

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Marsel Scheer