On taut singularities in arbitrary characteristics

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Summary

Let (S, s) be a normal surface singularity over an algebraically closed field k. For a desingularization of S the reduced exceptional locus E induces a dual graph. Now one calls such a singularity (S, s) taut if all other singularities with isomorphic dual graph are equivalent to (S, s). For $k = \mathbb{C}$ Henry Laufer gave a full classification of all taut surface singularities using analytical methods. For $\mathrm{Char}(k) = p > 0$ the general question is still open.

Our main results are the following: If a singularity is taut over $\mathbb{C}$, then the singularities with isomorphic dual graph over k with $\mathrm{Char}(k) = p > 0$ are taut for all but finitely many p. Also we can reprove a result of Michael Artin on the tautness of rational double points using our methods.

To get this results, we first reduce the question of equivalence for singularities to the question whether direct systems $\{Z_i\}$ of infinitesimal neighbourhoods of E are isomorphic. Some parts of this reduction should be well known, but we found no references for them.

After this reduction we go through Laufer's work and try to carry as much of it as possible to $p > 0$. Sometimes we just have to give adapted proofs, sometimes we have to use new ways.

Let $\{Z_i\}$ and $\{\widetilde{Z}_i\}$ be two such direct systems with Z_i and $\widetilde{Z}_i$ combinatorially equivalent. The first step is to find an obstruction when an isomorphism between Z_i and $\widetilde{Z}_i$ can be extended to an isomorphism between Z_{i+1} and $\widetilde{Z}_{i+1}$. For this we show first that we can extent the isomorphism locally. This is much harder in our setting than in Laufer's. But after we have this, we get, as Laufer, that the local extensions give an element in the $\check{H}^1$ of some non-abelian sheaf, and we can glue them to a global extension iff this element is the distinguished point. Then we can adapt Laufer's proof to get numerical conditions for the triviality of the $\check{H}^1$. From this we get that $\{Z_i\}$ and $\{\widetilde{Z}_i\}$ are isomorphic if and only if $Z_{i_0} \cong \widetilde{Z}_{i_0}$ for one $i_0 \gg 0$.

For the question whether Z_{i_0} and $\widetilde{Z}_{i_0}$ are isomorphic we need a different idea: There is a special scheme P for Z_{i_0}, and if $h^1(P, \Theta_P) = 0$ then we have $h^1(Z_{i_0}, \Theta_{Z_{i_0}}) = 0$ and $Z_{i_0} \cong P \cong \widetilde{Z}_{i_0}$. Then we reduce the calculation of $h^1(P, \Theta_P)$ to the calculation of the rank of a matrix M. For $k = \mathbb{C}$ this M has integer entries and if we look at the same singularity over k with $p > 0$, then we have to calculate the rank of M reduced modulo p. The last argument is the last step we need to prove the first result we mentioned above. The basic idea of using P is again due to Laufer, but some of his proofs are incompatible with our setting.

For rational double points, we reduce the question whether $h^1(P, \Theta_P) = 0$ to cohomology with support of some other sheaf. Then we can use a result of Jonathan Wahl to show the vanishing of this, depending only on the dual graph of Z_i. With this we get exactly Artin's tautness result. Finally we calculate $h^1(P, \Theta_P)$ for all dual graphs of the non-taut rational double points with the help of some computer algebra system, and get that this dimension agrees with the number of non-equivalent rational double points with this dual graph minus one.

Zusammenfassung

Ist (S, s) eine normale Flächensingularität über einem algebraisch abgeschlossenen Körper k und wählen wir eine Desingularisierung von S, dann induziert der reduzierte exzeptionelle Ort E einen dualen Graphen. Wir nennen eine solche Singularität taut, falls alle anderen Singularitäten mit isomorphen dualen Graphen schon äquivalent zu (S, s) sind. Für $k = \mathbb{C}$ gibt es eine komplette Klassifizierung von tauten Singularitäten, erstellt von Henry Laufer mittels analytischer Methoden. Für $\mathrm{Char}(k) = p > 0$ ist eine allgemeine Klassifizierung nicht bekannt.

Die Hauptresultate dieser Arbeit sind: Falls eine Singularität taut über $\mathbb{C}$ ist, dann sind die Singularitäten mit isomorphen dualen Graphen über k mit $\mathrm{Char}(k) = p > 0$ taut für alle bis auf endlich viele p. Ausserdem können wir ein Resultat von Michael Artin über die Tautheit von rationalen Doppelpunkten mit unseren Methoden neu beweisen.

Dafür gehen wir wie folgt vor: Zuerst reduzieren wir die Frage nach der Äquivalenz von Singularitäten auf die Frage ob direkte Systeme $\{Z_i\}$ von infinitesimalen Umgebungen von E isomorph sind. Einige Teile dieser Reduktion sollten bekannt sein, aber wir fanden keine Quellen dazu.

Nach dieser Reduktion gehen wir durch Laufers Arbeiten und übertragen soviel wie möglich davon nach $p > 0$. Manchmal reicht es dazu die Beweise anzupassen, manchmal müssen wir neue Wege gehen.

Seien $\{Z_i\}$ und $\{\widetilde{Z}_i\}$ zwei solcher direkte Systeme mit Z_i und $\widetilde{Z}_i$ kombinatorisch äquivalent. Der erste Schritt ist es, eine Obstruktion zu finden, wann man einen Isomorphismus zwischen Z_i und $\widetilde{Z}_i$ zu einem Isomorphismus zwischen Z_{i+1} und $\widetilde{Z}_{i+1}$ erweitern kann. Dazu zeigen wir zuerst, dass dies lokal immer möglich ist. Dies ist in unserem Setting deutlich schwieriger als in Laufers. Wenn wir dies haben, dann erhalten wir, wie Laufer, dass die lokalen Erweiterungen ein Element in $\check{H}^1$ einer nicht abelschen Garbe ergeben, und wir können sie zu einer globalen Erweiterung genau dann verkleben, wenn dieses Element der ausgezeichnete Punkt ist. Hiernach können wir Laufers Beweise anpassen, und erhalten numerische Bedingungen für das Verschwinden von $\check{H}^1$. Daraus folgert man, dass $\{Z_i\}$ und $\{\widetilde{Z}_i\}$ genau dann isomorph sind, wenn $Z_{i_0} \cong \widetilde{Z}_{i_0}$ für ein $i_0 \gg 0$ gilt.

Um zu klären, wann Z_{i_0} und $\widetilde{Z}_{i_0}$ isomorph sind, brauchen wir einen anderen Ansatz: Es gibt ein spezielles Schema P für Z_{i_0} und $h^1(P, \Theta_P) = 0$ impliziert $h^1(Z_{i_0}, \Theta_{Z_{i_0}}) = 0$ sowie $Z_{i_0} \cong P \cong \widetilde{Z}_{i_0}$. Dann reduzieren wir die Berechnung von $h^1(P, \Theta_P)$ auf die Berechnung des Ranges einer Matrix M. Im Fall $k = \mathbb{C}$ ist M ganzzahlig, und wenn wir eine Singularität mit isomorphen dualen Graphen über k mit $p > 0$, betrachten, dann müssen wir nur die Einträge von M modulo p reduzieren. Daraus folgern wir das oben zuerst genannte Resultat. Die Grundidee, P zu nutzen, stammt wieder von Laufer, aber einige seiner Beweise sind inkompatibel mit unserem Setting.

Für rationale Doppelpunkte reduzieren wir die Frage, ob $h^1(P, \Theta_P) = 0$ gilt, auf die Kohomologie mit Support einer anderen Garbe. Daraufhin können wir ein Ergebnis von Jonathan Wahl nutzen, um das Verschwinden, nur vom dualen Graphen von Z_i abhängend, zu erhalten. Damit bekommen wir genau Artins Tautheits-Aussage. Abschließend berechnen wir $h^1(P, \Theta_P)$ für alle dualen Graphen von den nicht tauten rationale Doppelpunkten mithilfe eines Computer-Algebra-Systems, und wir erhalten, dass diese Dimensionen plus 1 mit der Anzahl der nicht äquivalenten rationalen Doppelpunkte mit dem jeweiligen dualen Graphen übereinstimmen.

Contents

1 Introduction

Let k be an algebraically closed field and S a two-dimensional k-scheme such that S is normal, then S has only isolated singularities. For such schemes one has always a desingularization, that is a regular scheme X and a proper morphism $f : X \to S$ such that f is an isomorphism over the non-singular part of S. If $s \in S$ is a singular point, then the fibre $f^{-1}(s)$ is an one-dimensional subscheme of X. By further modifications of X on $f^{-1}(s)$ we may achieve that the reduction of $f^{-1}(s)$ consists of n regular curves E_l which intersect transversally. From now on we assume that S has only one singular point s.

Let now $E = \sum_{l=1}^{n} E_l$. We can assign some numerical and combinatorial data to E. First for every E_l we have the data how it intersects with the other E_i, if we just want the number of intersections, then this is $E_l \cdot E_j$. Then by the theory of intersection numbers on regular surfaces we also get the self-intersection number E_l^2. We can encode these values in a matrix $(E_i \cdot E_j)$. And indeed some properties of this matrix are very useful later, but we have more numbers that we can assign to E. For every E_l we have also the arithmetic genus $p_a(E_l)$. It turns out that the best way to present this data is a decorated graph Γ. For every E_l we add a vertex to Γ, and we add $E_l \cdot E_j$ edges between the vertex of E_l and of E_j. At each vertex we put E_l^2 and $[p_a(E_l)]$ as decoration. Also we decorate every vertex with (1), because later we also want to assign a graph to $Z = \sum_{l=0}^{n} n_l E_l$ and then we simply take the graph of E, and add (n_l) instead of (1) at every vertex.

We call the Γ of E the *dual graph of* (S, s). Also we can define morphism of graphs and get a notion of isomorphism. Now if (S', s') is a second normal, two-dimensional k-scheme with unique singular point s', and if the singularities are equivalent, that is we have open neighbourhoods U_s and $U_{s'}$ of s and s' such that U_s is isomorphic to $U_{s'}$, then (S, s) and (S', s') have isomorphic dual graphs. Because this definition of equivalence is hard to test, we later say that two normal, two-dimensional singularities are equivalent, if the completions of there local rings are isomorphic.

But what about the other direction? If (S, s) and (S', s') have isomorphic dual graphs, can we find such neighbourhoods U_s and $U_{s'}$? Suppose that one of the E_l has $p_a(E_l) = 1$, then by the theory of elliptic curves one knows that we have many other curves E_l' with $p_a(E_l') = 1$ but not isomorphic to E_l. So after solving some problems one can replace E_l in E by E_l', embed this into a regular, two-dimensional scheme, contract E' and gets a new singularity (S', s') with an isomorphic dual graph which cannot be equivalent to (S, s).

On the other hand, we have the so-called ADE-singularities, or rational double points. One possible definition of this class of singularities is that we have $\operatorname{edim}(\mathcal{O}_{S,s}) = 3$ and for all l we have $p_a(E_l) = 0$ and $E_l^2 = -2$. This singularities where first examined by Patrick Du Val. Later many other mathematicians contributed many equivalent characterisations of this singularities. The most important one for us is the following, which also explains the name: A normal, two-dimensional singularity (S, s) is a rational double point if and only if the dual graph of (S, s) is one of the Dynkin diagrams A_n, D_n, E_6, E_7 or E_8 known from the classification of semisimple Lie algebras, and if $k = \mathbb{C}$ for each of this diagrams there is exactly one singularity up to equivalence.

Now we call a normal, two-dimensional singularity (S, s) *taut* if all other singularities with isomorphic dual graph are already equivalent to (S, s). The ADE-singularities over $\mathbb{C}$ are the first examples of taut singularities, other examples where discovered by Hans Grauert ([Gra62]), Galina Tjurina ([Tju68]) and Philip Wagreich ([Wag70]). Finally in 1973 Henry Laufer classified all taut normal, two-dimensional singularities over $\mathbb{C}$. The result of this classification is a description of the dual graphs of taut singularities which needs several pages ([Lau73b]).

For algebraically closed fields k with $\mathrm{Char}(k) = p > 0$ the general question of the classification of all taut singularities is still open. For the ADE-singularities we have a result of Michael Artin ([Art77], Page 16) using explicit calculations: The A_n-singularities are still taut, the D_n singularities are taut if $p > 2$ and for the E_n-singularities we have to exclude $p = 2, 3$ for all and $p = 5$ for E_8 to get tautness. But even in the non-taut case, there are only finitely many equivalence classes. A recent result on tautness of a special class of singularities for $p > 0$ is a paper of Yongnam Lee and Noboru Nakayama: They show the tautness of Hirzebruch-Jung singularities ([LN12], Theorem 2.6) for every p.

We revise Laufer's general proof over $\mathbb{C}$ to make a step towards a general classification of taut normal, two-dimensional singularities for $p > 0$. We cannot give a full classification, but at least we can prove that if a singularity (S_0, s_0) over $\mathbb{C}$ is taut, then the singularity (S_p, s_p) over k with $\mathrm{Char}(k) = p > 0$ with isomorphic dual graph is taut for all but finitely many p. This is Theorem 5.1. Also we have a way of computing all this p for a given S_0, but we cannot prove whether for the "bad" p the (S_p, s_p) are taut or not.

For the ADE-singularities we can use some results of Daniel Bruns and Jonathan Wahl to reprove Artin's tautness result with our methods without explicit calculations (Corollary 5.7).

Content of this work At first we note that the terminology of equivalence of singularities used above is very unhandy to verify, because one has to prove something on all open neighbourhoods. So after recalling some definitions we start Section 2 by discussing that two singularities are equivalent if and only if their completed local rings $\widehat{\mathcal{O}}_{S,s}$ are isomorphic. So for the classification of singularities we can assume that those are spectra of some complete local ring, and we build this into the definition. Now the question of equivalence is just the question of isomorphism.

Now take a desingularization of $f : X \to S$ and look at the exceptional fibre $f^{-1}(s)$ respectively at its reduction E. Then E is an one-dimensional subscheme of X. We want to look at chains of infinitesimal neighbourhoods of E in X, that is non-reduced subschemes $Z_i \subset X$ with $(Z_i)_{\mathrm{red}} = E$ and $E \subset Z_0 \subset Z_1 \subset \cdots$. One example of such a chain is $\widetilde{Z}_i = X \times \mathrm{Spec}(\mathcal{O}_{S,s}/m_s^{i+1})$, there we have $Z_0 = f^{-1}(s)$. If E_l are the irreducible components of E, then $Z_i = \sum_{l=1}^{n} n_l^i E_l$ with n_l^i non-decreasing and not bounded is another system. We show that two singularities are isomorphic if and only if their chains of the form $\widetilde{Z}_i$ or Z_i are isomorphic as direct systems. In particular it is enough to understand systems of the form Z_i which are effective divisors on a regular surface, whereas the non-reduced structure of the $\widetilde{Z}_i$ is unknown. This should be well known to the experts, but we could find no reference for it.

If we now try to understand the Z_i we hit a new problem: In general $\widehat{\mathcal{O}}_{S,s}$ is not a k-algebra of finite type, and with this also X is not of finite type over k. So we lose the equivalence of regularity and smoothness for X, but for some arguments we need the infinitesimal lifting property. But by a result of Artin every local, noetherian, normal, complete ring of dimension two is the completion of a local ring of a point on a two dimensional k-scheme of finite type. We call such a scheme A an algebraization of S. And the nice point is: We have a mapping between desingularizations of S and A, and if we build the $\widetilde{Z}_i$ or Z_i on a desingularization of S or on the corresponding desingularization of A, then they are isomorphic as direct systems of schemes.

After this we can prove that a singularity is taut if and only if all Z_i are defined by their dual graph. The problem proving this Theorem is the "only if" part. For this we need to contract a given negative definite divisor on a regular surface, and this is in general only possible if we allow the contraction to be an algebraic space. But we can show that maybe after changing the surface away from the divisor, we get a contraction that is a scheme. Again we think this is known to the experts, but we could find no references for it.

The last result from Section 2 we want to mention here is the following necessary condition for a singularity to be taut: A singularity cannot be taut if one of the E_l is not isomorphic to $\mathbb{P}^1_k$ or if one E_l intersects with more than 3 others. In Laufer's proof this is the result of some non-vanishing of a cohomology group, but we need this result even before we can show that this vanishing implies tautness.

In Section 3 we reprove some results of Laufer and Tjurina from [Lau71] and [Tju68]. The main result is that if we want to know whether all Z_i are defined by their dual graph, we only have to know this up to an i_0, then for all $i \geq i_0$ this follows automatically. To prove this result Laufer works mostly in local coordinates, and because he also works in the analytical category, he can always choose these coordinates to be $k[x,y]/(y^{n_l})$ or $k[x,y]/(x^{n_j}y^{n_l})$. But in the algebraic category we cannot do this, so we have to give adapted proofs. Also with this choice of coordinates it is clear that an isomorphism between Z_i and some combinatorial equivalent scheme C always extends locally to isomorphisms of open subsets of Z_{i+1} and the respectively infinitesimal neighbourhood of subsets of C. But in our setting we have to work to get this result.

In Section 4 we take another result of Laufer, adapt it to our situation and push it a little bit further: If the Z_i fulfil the necessary conditions for tautness, we can construct a combinatorially equivalent scheme P_i and show that if $H^1(P_i, \mathcal{H}om_{\mathcal{O}_{P_i}}(\Omega^1_{P_i/k}, \mathcal{O}_{P_i})) = 0$, then Z_i is defined by its dual graph. Now Laufer gets even "if and only if" for this, but this is in general false for $\mathrm{Char}(k) > 0$. We discuss this at the end of Section 4. Also

Laufer defines the P_i without the necessary conditions and then deduces these conditions from the non-vanishing of $H^1(P_i, \mathcal{H}om_{\mathcal{O}_{P_i}}(\Omega^1_{P_i/k}, \mathcal{O}_{P_i}))$.

To show that $H^1(P_i, \mathcal{H}om_{\mathcal{O}_{P_i}}(\Omega^1_{P_i/k}, \mathcal{O}_{P_i})) = 0$ implies that Z_i is defined by its dual graph, we cannot simply adapt Laufer's proof. We can reprove the basic ideas of his proof, namely that $H^1(P_i, \mathcal{H}om_{\mathcal{O}_{P_i}}(\Omega^1_{P_i/k}, \mathcal{O}_{P_i})) = 0$ implies $H^1(Z_i, \mathcal{H}om_{\mathcal{O}_{Z_i}}(\Omega^1_{Z_i/k}, \mathcal{O}_{Z_i})) = 0$ and that for every $\widetilde{Z}_i$ combinatorially equivalent to Z_i we have a family over an affine scheme with one closed fibre isomorphic to $\widetilde{Z}_i$ and one isomorphic to P_i. Then we use the algebraic deformation theory (which differs much from the Kodaira-Spencer deformation theory which Laufer uses) to get the result we wanted.

Now if the Z_i fulfil the necessary conditions for (S, s) to be a taut singularity, Laufer gives an "algorithm to calculate whether $H^1(P_i, \mathcal{H}om_{\mathcal{O}_{P_i}}(\Omega^1_{P_i/k}, \mathcal{O}_{P_i})) = 0$". This algorithm is the reduction to the question whether a map between two huge, but finitely dimensional k-vector spaces is surjective. Again we have to give adapted proofs, in particular we have to prove some kind of Mayer–Vietoris sequence for sheaf cohomology. Also we go one step further: The question whether this map is surjective is really the question whether the rank of some matrix over the integers is maximal. But if we construct P_i for $\mathbb{C}$ and for $\mathrm{Char}(k) = p > 0$, the resulting matrix for $p > 0$ is just the one for $\mathbb{C}$ reduced modulo p. From this we get that $H^1(P_i, \mathcal{H}om_{\mathcal{O}_{P_i}}(\Omega^1_{P_i/k}, \mathcal{O}_{P_i})) = 0$ for $\mathbb{C}$ implies vanishing for all but finitely many $p > 0$.

In Section 5 we first use the results of Section 4 to show that tautness over $\mathbb{C}$ implies tautness over all but finitely many p. Also we give the reprove of Artin's tautness result mentioned earlier, and we show that we have $h^1(P_i, \mathcal{H}om_{\mathcal{O}_{P_i}}(\Omega^1_{P_i/k}, \mathcal{O}_{P_i})) > 0$ in the non-taut cases. Furthermore we can show that this dimension plus one agrees with the number of non isomorphic singularities calculated by Artin.

For some proofs we need some local calculations. For better readability we have concentrated them in Section 6.

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2 From singularities to curves

2.1 Definitions and notations

First we want to set up some fixed notations for the rest of this work, and we want to recall some definitions we need later in this section.

2.1.1 Notation

For this work, if we write k or p, it is always an arbitrary algebraically closed field, and p its characteristic (so $p = 0$ or $p > 0$ prime).

All schemes are assumed to be noetherian and over $\operatorname{Spec}(k)$.

2.1.2 Regular and smooth points

Let X be a noetherian k-scheme, $x \in X$ a point and let $\mathfrak{m}_x$ be the maximal ideal of the local ring $\mathcal{O}_{X,x}$. We say that x is a *regular point*, if $\mathcal{O}_{X,x}$ is a regular ring, that is if $\dim(\mathcal{O}_{X,x}) = \dim_k(\mathfrak{m}_x/\mathfrak{m}_x^2)$. Else we call x a *singular point* or a *singularity*. We call X *regular* if every point is a regular point. We say that a singular point $x \in X$ is an *isolated singularity* if there is an open $U \subset X$ such that x is the only singular point in U.

This is a purely algebraic description of regularity, but if X is of finite type over a field k, we have also a geometric description: First, with Corollary 4.2.17 of [Liu02] we know that X is regular if and only if it is regular at all closed points. But for this points, we can use the Jacobian criterion: Locally around every rational point x we find a neighbourhood isomorphic to $V(I)$ in $\mathbb{A}_k^n = \operatorname{Spec}(k[x_1, \dots, x_n])$ with $I = (f_1, \dots, f_r)$. After choosing this, we can look at the Jacobian matrix

$$J_x = \left(\frac{\partial f_i}{\partial x_j}(x) \right)_{1 \le i \le r,\, 1 \le j \le n}$$

Now Theorem 4.2.19 of [Liu02] gives us that X is regular at x if and only if

$$\operatorname{rank}(J_x) = n - \dim(\mathcal{O}_{X,x}).$$

We also need the notion of a smooth morphism.

Definition 2.1. Let X, Y be schemes and $f : X \to Y$ be a morphism of finite type. We say that f is *smooth* if for all affine schemes Y' and all closed subschemes Y'_0 of Y' with nilpotent ideal $\mathcal{I} \subset \mathcal{O}_{Y'}$ and all morphism $Y' \to Y$ the map

$$\operatorname{Hom}_Y(Y', X) \longrightarrow \operatorname{Hom}_Y(Y'_0, X)$$

induced by the injection $Y'_0 \to Y'$ is surjective.

We call f *étale* if the map is bijective. We say that f is smooth or étale *at* $x \in X$ if we have an open $x \in U \subset X$ such that $f|_U$ is smooth or étale.

Now by Théorème 17.5.1 of [Gro67] we get the following local characterizations of smooth morphisms:

Lemma 2.2. *Let $f : X \to \operatorname{Spec}(k)$ be a morphism of finite type and k algebraically closed, $x \in X$ a point. Then f is smooth at x if and only if X is regular at x.*

2.1.3 Limits and completions

In this section we want to recall the definition and some facts about inverse and direct limits we need later. All omitted proofs can be found in [Bou68].

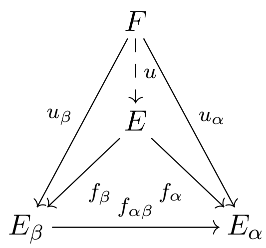
Let I be an ordered set. An *inverse system* $(E_\alpha, f_{\alpha\beta})$ is a family $(E_\alpha)_{\alpha\in I}$ of sets together with a set of morphisms

$$\{f_{\alpha\beta} : E_\beta \longrightarrow E_\alpha \mid \alpha, \beta \in I\ \alpha \leq \beta\}$$

such that:

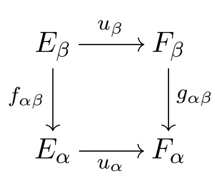
- For $\alpha \leq \beta \leq \gamma \in I$ we have $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$.
- For each $\alpha \in I$, $f_{\alpha\alpha}$ is the identity of E_α.

For an inverse system we get a new set $E = \varprojlim_I (E_\alpha, f_{\alpha\beta})$ with morphisms $f_\alpha : E \to E_\alpha$ for each $\alpha \in I$, which is defined by the following universal property: For each set F with morphism $u_\alpha : F \to E_\alpha$ for each $\alpha \in I$ such that for all $\alpha \leq \beta \in I$ the other triangle of the next diagram commutes, we have a unique morphism $u : F \to E$ such that the whole diagram commutes:



We call E the *inverse limit* of $(E_\alpha, f_{\alpha\beta})$, or just the inverse limit of $(E_\alpha)_{\alpha\in I}$ if there is no risk of ambiguity. In the latter case we write also $\varprojlim_I (E_\alpha, f_{\alpha\beta}) = \varprojlim E_\alpha$.

A *morphism of inverse systems* $(u_\alpha) : (E_\alpha, f_{\alpha\beta}) \to (F_\alpha, g_{\alpha\beta})$ is a family of morphism $u_\alpha : E_\alpha \to F_\alpha$, such that for all $\alpha \leq \beta \in I$ the following diagram commutes:



A morphism of inverse systems is an isomorphism if all u_α are isomorphism. Taking inverse limit is functorial, that is a morphism of inverse systems induces a morphism of the inverse limits, and this is compatible with composition.

If the sets E_α and F in the universal property are rings, modules or algebras, and all the morphisms are morphisms in the particular category, then so is the limit $\varprojlim E_\alpha$ and the f_α.

Later we need to know if two inverse systems have isomorphic limits. If J is a subset of I, we get an induced inverse system, and by abuse of notation we write $\varprojlim_J (E_\alpha, f_{\alpha\beta})$ for its inverse limit.

A subset J of I is called *cofinal* if for each $\alpha \in I$ there exists a $\beta \in J$ with $\beta \geq \alpha$. An ordered set J is call *right directed* if for all $\alpha, \beta \in J$ exists a $\gamma \in J$ with $\alpha, \beta \leq \gamma$.

Then we have Proposition III 7.2.3 of [Bou68]:

Lemma 2.3. *Let I be an ordered set, let $(E_\alpha, f_{\alpha\beta})$ be an inverse system, and let J be a cofinal subset of I such that J is right directed. Then the induced morphism*

$$\varprojlim_I (E_\alpha, f_{\alpha\beta}) \longrightarrow \varprojlim_J (E_\alpha, f_{\alpha\beta})$$

is bijective.

Later, we need the following corollary:

Corollary 2.4. *Let I be an ordered set and $(E_\alpha, f_{\alpha\beta}), (F_\alpha, g_{\alpha\beta})$ be two inverse systems. Suppose that for every $\beta \in I$ we have $\alpha_\beta, \gamma_\beta \in I$ with $\alpha_\beta \geq \gamma_\beta$ and for every $\beta \geq \beta' \in I$ we have $\alpha_\beta \geq \alpha_{\beta'}$ and $\gamma_\beta \geq \gamma_{\beta'}$. Suppose further that we have maps*

$$E_{\alpha_\beta} \xrightarrow{u_{\alpha_\beta}} F_\beta \xrightarrow{u_{\gamma_\beta}} E_{\gamma_\beta}$$

such that for all $\beta \geq \beta'$ the diagram

$$\begin{array}{ccccc} & & \overset{f_{\gamma_\beta \alpha_\beta}}{\curvearrowright} & & \\ E_{\alpha_\beta} & \xrightarrow{u_{\alpha_\beta}} & F_\beta & \xrightarrow{u_{\gamma_\beta}} & E_{\gamma_\beta} \\ \downarrow f_{\alpha_{\beta'}\alpha_\beta} & & \downarrow g_{\beta'\beta} & & \downarrow f_{\gamma_{\beta'}\gamma_\beta} \\ E_{\alpha_{\beta'}} & \xrightarrow{u_{\alpha_{\beta'}}} & F_{\beta'} & \xrightarrow{u_{\gamma_{\beta'}}} & E_{\gamma_{\beta'}} \\ & & \underset{f_{\gamma_{\beta'}\alpha_{\beta'}}}{\curvearrowright} & & \end{array} \tag{2.1}$$

commutes, and that $\{\alpha_\beta\}$, $\{\gamma_\beta\}$ are right directed and cofinal in I. Then the induced morphism

$$\varprojlim_I (E_\alpha, f_{\alpha\beta}) \longrightarrow \varprojlim_I (F_\alpha, g_{\alpha\beta})$$

is bijective.

Proof. Taking limits we get

$$\varprojlim_I (E_{\alpha_\beta}, f_{\alpha_{\beta'}\alpha_\beta}) \xrightarrow{u_\alpha} \varprojlim_I (F_\beta, g_{\beta'\beta}) \xrightarrow{u_\gamma} \varprojlim_I (E_{\gamma_\beta}, f_{\gamma_{\beta'}\gamma_\beta}), \qquad u = u_\gamma \circ u_\alpha$$

but because $\{\alpha_\beta\}$, $\{\gamma_\beta\}$ are right directed and cofinal in I, u is a bijection and we have a bijection

$$\varprojlim_I (E_{\alpha_\beta}, f_{\alpha_{\beta'}\alpha_\beta}) \longrightarrow \varprojlim_I (E_\alpha, f_{\alpha\beta})$$

by the previous Lemma. Now by the Corollary of Proposition III 7.2.2 of [Bou68] we know that u_α and u_γ are injective, and so they are already bijective. □

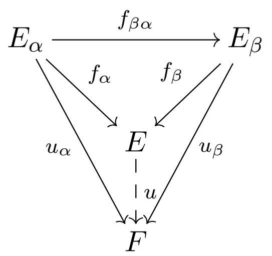
Analogous to the inverse system we define a *direct system* $(E_\alpha, f_{\beta\alpha})$ as the dual construction. That is, $(E_\alpha, f_{\beta\alpha})$ is a family $(E_\alpha)_{\alpha\in I}$ of sets together with a set of morphisms

$$\{f_{\beta\alpha} : E_\alpha \longrightarrow E_\beta \mid \alpha, \beta \in I \ \alpha \leq \beta\}$$

such that:

- For $\alpha \leq \beta \leq \gamma \in I$ we have $f_{\gamma\alpha} = f_{\gamma\beta} \circ f_{\beta\alpha}$.
- For each $\alpha \in I$, $f_{\alpha\alpha}$ is the identity of E_α.

Again we get a new set $E = \varinjlim_I (E_\alpha, f_{\beta\alpha})$, now with morphism $f_\alpha : E_\alpha \to E$ for each $\alpha \in I$, and the defining universal property is: For each set F with morphism $u_\alpha : E_\alpha \to F$ for each $\alpha \in I$ such that for all $\alpha \leq \beta \in I$ the other triangle of the next diagram commutes, we have a unique morphism $u : E \to F$ such that the whole diagram commutes:



We call E the *direct limit* of $(E_\alpha, f_{\beta\alpha})$, or just the direct limit of $(E_\alpha)_{\alpha\in I}$ if there is no risk of ambiguity. Again, in the latter case we write $\varinjlim_I (E_\alpha, f_{\beta\alpha}) = \varinjlim E_\alpha$.

Now we can define the completion of a local ring: Let $(R, \mathfrak{m})$ be a local ring. If we set $R_n = R/\mathfrak{m}^n$, then the R_n, together with the projections $\pi_{nn'} : R_{n'} \to R_n$ for $n' \geq n$ form an inverse system, and we define *the completion of* R as

$$\widehat{R} = \varprojlim R_n$$

which is again a ring. We say that R is *complete* if the natural morphism $R \to \widehat{R}$ is an isomorphism.

If additionally R is noetherian, chapters 10 and 11 of [AM69] give us the following properties:

Lemma 2.5. *Let* $(R, \mathfrak{m})$ *be a noetherian local ring, then we have:*

1. *$\widehat{R}$ is a local ring with maximal ideal $\widehat{\mathfrak{m}}$.*

2. *$\widehat{R}/\widehat{\mathfrak{m}}^n \cong R/\mathfrak{m}^n$.*

3. *$\widehat{R}$ is noetherian.*

4. *$\dim(\widehat{R}) = \dim(R)$.*

5. *R is regular if and only if $\widehat{R}$ is regular.*

6. *$\widehat{R}$ is complete.*

Let X be a scheme, for $x \in X$ we define $k(x)$ as the residue field of $\mathcal{O}_{X,x}$. Then with Proposition 17.6.3 of [Gro67] we get the following local characterizations of étale morphisms:

Lemma 2.6. *Let $f : X \to Y$ be a morphism of finite type, $x \in X$ a point and $y = f(x)$. If $k(x) = k(y)$, then f is étale at x if and only if the induced morphism $\widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{X,x}$ is bijective.*

2.2 Isolated singularities

In this work, we want to study isolated singularities. So the first question we have to ask is simply: When are two singularities equivalent? We cannot simply call two singularities equivalent if they are isomorphic, because by passing to a smaller neighbourhood we get a non-isomorphic singularity which is essential the same as before. So intuitively one wants to call two singularities equivalent if and only if both have open neighbourhoods which are isomorphic. But this definition is not very practical, because one has to find these neighbourhoods or else has to prove that those do not exist. So we want an intrinsic definition of equivalent singularities.

If we change the category for a moment, we have the following motivation: Let X and Y be analytic spaces, $x \in X$ and $y \in Y$ points. Then by Corollary 1.6 of [Art68] the points x and y have isomorphic analytical neighbourhoods if and only if $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{Y,y}$.

If we go back to our situation, then we have $k \subset R = \mathcal{O}_{X,x}$ and thus we get $k \subset \widehat{\mathcal{O}}_{X,x}$. But then Corollary 28.2 of [Mat80] tells us that if $\widehat{\mathcal{O}}_{X,x}$ is regular then we have already

$$\widehat{\mathcal{O}}_{X,x} \cong K[[x_1, \dots x_d]]$$

where K is the residue field $\widehat{\mathcal{O}}_{X,x}/\widehat{m}_x$ and $d = \dim(\widehat{R})$. Or in other words: x is a regular point if and only if $\widehat{\mathcal{O}}_{X,x} \cong K[[x_1, \dots x_d]]$. These two considerations motivate the following definition:

Definition 2.7. Let X, Y be two noetherian k-schemes, $x \in X$ and $y \in Y$ the isolated singularities. We say that (X, x) is *equivalent* to (Y, y) if $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{Y,y}$.

Now we want to rephrase the term "isolated singularity" in algebraic terms. For this we need the following definition:

Definition 2.8. Let X be an irreducible scheme. We say that X is *normal* if for every point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a normal, integral domain, that is it is integral and integrally closed in its field of fractions.

Now by a criterion of Serre a normal scheme must be "regular in codimension 1", in particular, for a two-dimensional scheme, the singular points are a zero-dimensional subscheme, and thus isolated.

Not all two-dimensional schemes with isolated singularities are normal, a simple counterexample is the scheme we get if we glue two points of the affine plane. But suppose $\dim(X) = d$ and that locally around every point $x \in X$ we find an open $x \in U \subset X$ and a regular affine scheme $Y = \operatorname{Spec}(A)$ of dimension d_x and $d_x - d$ elements $f_i \in A$ such that the residue class f_{i+1} is not a zero-divisor in $A/(f_1, \ldots, f_i)$. That is, X is locally the intersection of $d_x - d$ distinct hypersurfaces in a regular scheme. We call such a scheme a *locally complete intersection*. But by the discussion in Section 8.2.2 of [Liu02], we know that a 2-dimensional scheme X which is a locally complete intersection with at most isolated singularities must be normal. So even if we restrict ourself to such singularities, we get a reasonably big class of isolated singularities, so we define:

Definition 2.9. We say (S, s) is a *normal, two-dimensional singularity* if S is the spectrum of a complete, normal, noetherian, local k-algebra $\mathcal{O}_{S,s}$ with closed point s, residue field k and $\dim(S) = 2$.

Note that we also include the regular scheme $\operatorname{Spec}(k[[x_1, x_2]])$ as a pathological case of a singularity.

We restrict ourselves to complete local rings. This has many advantages, but one disadvantage: $\mathcal{O}_{S,s}$ is not a k-algebra of finite type, so we do not have the description of Lemma 2.2 for smooth points on S or on desingularizations of S. That is, we may have a regular scheme which is not smooth. But sometimes we need explicit the smoothness. So sometimes we need to lose the completeness to gain a k-algebra of finite type:

Definition 2.10. Let (S, s) be a normal, two-dimensional singularity. Let A be a noetherian, normal, local k-algebra. We say that $\operatorname{Spec}(A)$ is an *algebraization* of S if A is of finite type and if $\mathcal{O}_{S,s} \cong \widehat{A}$.

Now we have Theorem 4.7 of [Art69]:

Theorem 2.11. *Let (S, s) be a normal, two-dimensional singularity. Then there exists an algebraization A of S.*

As seen in the motivation for the use of the completion, there is no hope for getting an unique/distinguished algebraization. Now we want to relate the desingularizations of an algebraization with the desingularizations of a normal, two-dimensional singularity. For this we need the following notation:

Notation. Let A be a k-algebra, B a A algebra and X a scheme over A. We define

$$X \otimes_A B = X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(B)$$

For reasons of readability we omit the A if there is no risk of ambiguity.

Now Lemma 4.2 of [Băd01] gives us:

Lemma 2.12. *Let (S, s) be a normal, two-dimensional singularity,* $\mathrm{Spec}(A)$ *an algebraization. Further let $f : X \to \mathrm{Spec}(A)$ be a desingularization. Then $X \otimes \mathcal{O}_{S,s}$ is a desingularization of S.*

Now we need a few more definitions:

Definition 2.13. Let X be an integral scheme with generic point η. We define the *function field* of X as $K(X) = \mathcal{O}_{X,\eta}$.

Let Y be a second integral scheme and $f : X \to Y$ a morphism. We say that f is a *birational morphism* if $f_\eta^\sharp : K(Y) \to K(X)$ is an isomorphism.

Now we know that we have a desingularization of the singularity, that is a proper, birational morphism $f : X \to S$ with X regular and f an isomorphism outside of $f^{-1}(s)$. Or, to be precise, Lipman [Lip69], §2 gives us:

Theorem 2.14. *Let S be a normal, two-dimensional singularity or an algebraization of one. Then S admits a desingularization by a finite sequence of blowups in closed points and normalisations.*

Note that the assumption "excellent" of Lipman is just a technical condition which ensures that some properties of the local rings transfer to the completion. In particular spectra of complete notherian rings are excellent. Also by Corollary 8.2.40 (a) of [Liu02] any scheme of finite type over a field is excellent.

Now we want to transfer the question when two normal, two-dimensional singularities are isomorphic to their desingularizations. As a tool we need the following lemma:

Lemma 2.15. *Let R_1, R_2 be two normal rings and let $f_i : X_i \to \mathrm{Spec}(R_i)$ be two proper birational morphisms with $X_1 \cong X_2$. Then we have*

$$R_1 \cong H^0(X_1, \mathcal{O}_{X_1}) \cong H^0(X_2, \mathcal{O}_{X_2}) \cong R_2.$$

Proof. First we note that $H^0(X_1, \mathcal{O}_{X_1}) \cong H^0(X_2, \mathcal{O}_{X_2})$ follows directly because X_1 and X_2 are isomorphic. So by symmetry it suffices to show $R_1 \cong H^0(X_1, \mathcal{O}_{X_1})$. For every domain R let $\mathrm{Frac}(R)$ be its field of fractions, then by Proposition 2.4.18 of [Liu02] we have $K(X_1) = \mathrm{Frac}(\mathcal{O}_{X_1}(U))$ for every affine open $U \subset X_1$, in particular we have $H^0(X_1, \mathcal{O}_{X_1}) \subset K(X_1)$. But f is birational, so $K(X_1)$ is isomorphic to $K(\mathrm{Spec}(R_1)) = \mathrm{Frac}(R_1)$, so we get

$$H^0(X_1, \mathcal{O}_{X_1}) \subset \mathrm{Frac}(R_1).$$

Finally Proposition 3.3.18 of [Liu02] tells us that $H^0(X_1, \mathcal{O}_{X_1})$ over R_1 is an integral ring-extension, but R_1 is normal, in particular integrally closed in $\mathrm{Frac}(R_1)$, so we get $H^0(X_1, \mathcal{O}_{X_1}) \cong R_1$. □

Now we can transfer the question of isomorphism:

Theorem 2.16. *Let* $(S_1, s_1), (S_2, s_2)$ *be two normal, two-dimensional singularities.* (S_1, s_1) *is isomorphic to* (S_2, s_2) *if and only if there exist desingularizations* $f_i : X_i \to S_i$ *with* $X_1 \cong X_2$.

Proof. If we have $X_1 \cong X_2$, then the previous Lemma gives us the isomorphism. If on the other hand we have $S_1 \cong S_2$, and we have a desingularization $f_1 : X_1 \to S_1$, then we simply compose f_1 with the isomorphism between S_1 and S_2 to get a desingularization of S_2. □

In practise the previous theorem is not such a great help, because we have to check all desingularizations of (S_1, s_1) and $(S_2.s_2)$. But for normal, two-dimensional singularities we have a distinguished desingularization, the so-called minimal desingularization. Before we define this desingularization, we have to recall some facts about the exceptional set. If we have a desingularization $f : X \to S$ of a normal, two-dimensional singularity or of an algebraization of one, then by van der Waerden's purity theorem ([Liu02], Theorem 2.22 and Remark 2.24), the exceptional set has at least dimension 1. Now the desingularization is integral and thus irreducible, but then by [Liu02], Proposition 2.5.5 (b) the dimension of the exceptional set must be at most 1, because it is not everything. So the exceptional set is a curve. Now S is affine and f is proper, in particular separated and quasi-compact, so by [Liu02], Proposition 5.1.14 (c) $f_*(\mathcal{O}_X)$ is quasi-coherent, and because S is normal $f_*(\mathcal{O}_X)(S) = \mathcal{O}_S(S)$, but S is affine, thus we have already $f_*(\mathcal{O}_X) = \mathcal{O}_S$ (Taking global section is an equivalence of of categories, [Har77], Corollary 5.5). Now we use Zariski's connectness theorem ([Gro61], Corollaire 4.3.2) and see that the exceptional set must be connected.

Now f is proper, and thus also $f|_{f^{-1}(s)} : f^{-1}(s) \to \operatorname{Spec} k$, and because by [Har77], Proposition II 6.7 every proper scheme of dimension 1 over k is projective, it follows that $f^{-1}(s)$ is projective. Now we need some more notation:

Definition 2.17. Let X be a proper scheme of dimension r over k, and let $\mathcal{F}$ be a coherent sheaf on X. We define the *Euler characteristic* of $\mathcal{F}$ by

$$\chi(\mathcal{F}) = \sum_{i=0}^{r} (-1)^i \dim_k H^i(X, \mathcal{F}).$$

We define the *arithmetic genus* of X by

$$p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1).$$

If additionally X is an integral scheme of dimension 1, we have $H^0(X, \mathcal{O}_X) = k$, because $k = \overline{k}$, and thus

$$p_a(X) = (-1)(\dim_k H^0(X, \mathcal{O}_X) - \dim_k H^1(X, \mathcal{O}_X) - 1) = \dim_k H^1(X, \mathcal{O}_X).$$

Also we need:

Definition 2.18. Let C be a proper k-scheme of dimension 1, and $\mathcal{L} \in \operatorname{Pic}(C)$, then we define

$$\deg(\mathcal{L}) = \chi(\mathcal{L}) - \chi(\mathcal{O}_C) \in \mathbb{Z}$$

Let further (S, s) be a normal, two-dimensional singularity, $f : X \to S$ a desingularization and C an one-dimensional subscheme of X such that C is proper over k, then for any $\mathcal{L} \in \operatorname{Pic}(X)$ we define

$$\mathcal{L} \cdot C = \deg(\mathcal{L}|_C)$$

If C' is a closed subscheme of X such that its ideal sheaf $\mathcal{O}_X(-C')$ is invertible, then we define

$$C' \cdot C = \mathcal{O}_X(C') \cdot C = \deg(\mathcal{O}_X(C')|_C)$$

Now we have the following standard definition:

Definition 2.19. Let X be a regular, two-dimensional scheme, and let C be a regular, one-dimensional subscheme of X. We say that C is an *exceptional curve of the first kind* if C is projective over k, $p_a(C) = 0$ and $C^2 = -1$.

In our case, k is algebraically closed, so by [Liu02] Proposition 7.4.1 (b) $p_a(C) = 0$ is equivalent to $C \cong \mathbb{P}^1_k$. Then Castelnuovo's theorem ([Har77], Theorem 5.7) shows us that every exceptional curve of the first kind can be contracted, and the resulting scheme is still regular, which leads to:

Definition 2.20. Let (S, s) be a normal, two-dimensional singularity or an algebraization of one, $f : X \to S$ a desingularization of S. We say that f is a *minimal desingularization* if none of the integral components of $f^{-1}(s)$ is an exceptional curve of the first kind.

Now by [Băd01] Proposition 4.5 we know that a minimal desingularization always exists, and they have a nice universal property:

Theorem 2.21. *Let (S, s) be a normal, two-dimensional singularity or an algebraization of one, then there exists a minimal desingularization $f : X \to S$. Furthermore, for any other desingularization $f' : X' \to S$ there exists a unique morphism $u : X' \to X$ such that $f \circ u = f'$. In particular, any two minimal desingularizations are canonically isomorphic.*

The universal property of the minimal desingularization is a great help, but in general, the minimal desingularization is not the desingularization we want to work with. The reason for this is that we have not enough control over the exceptional divisor. We know that it is an one-dimensional connected scheme, but its integral components may be rather singular. Also we may have points where more than two components meet. This makes the combinatorics more difficult, so we want to avoid this. Now we first define this combinatorially nicer desingularization, and then we give some examples.

Definition 2.22. Let (S, s) be a normal, two-dimensional singularity or an algebraization of one, $f : X \to S$ a desingularization of S. We say that f is a *good desingularization* if for $f^{-1}(s)_{\text{red}} = \sum_{i=1}^{n} E_i$ the following three conditions hold:

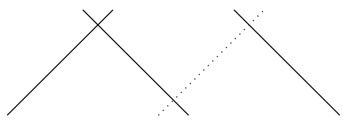
- Each E_i is regular.
- If $E_i \cap E_j \neq \emptyset$ $(i \neq j)$, then E_i and E_j intersect transversely, that is for all $a \in E_i \cap E_j$ we have $\widehat{\mathcal{O}}_{E_i+E_j,a} \cong k[[x,y]]/(xy)$.
- No three distinct E_i meet.

A good desingularization $f : X \to S$ is called *minimal* if the number of integral components of $f^{-1}(s)$ is minimal for all good desingularizations.

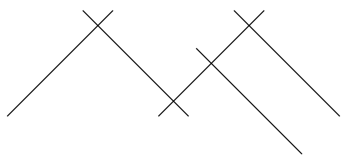
In this situation, we can say something more about E: Because X is regular and E_l integral, we know by the identification of Cartier divisors, and Weil divisors that E_l is defined by the invertible sheaf $\mathcal{O}_X(-E_l)$. So E_l is a local complete intersection, which means locally we have $E_l \cong \operatorname{Spec}(A/f_l)$ and with that $E \cong \operatorname{Spec}(A/\prod_l f_l)$ where A is a regular ring and the $f_l \in A$ are regular elements.

Examples Now we want to give a few examples of minimal and minimal good desingularization. Or to be precise: Examples of the combinatorics of the exceptional divisors. Writing down the full desingularization is of course possible, but one needs many charts and gets no additional insight. All examples are taken from [Ném99], 1.20 and 1.22.

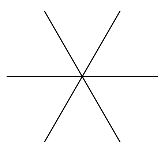
1. Let $S_1 = \operatorname{Spec}(\mathbb{C}[[x,y,z]]/(x^2+y^2+z^n))$, then the minimal desingularization is already good. The exceptional set consists of n projective lines E_i with $E_i^2 = -2$. They intersect in the following way:



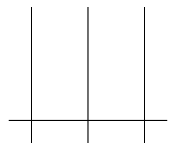
2. Let $S_2 = \operatorname{Spec}(\mathbb{C}[[x,y,z]]/(x^4+xy^2+z^2))$. Here also the minimal desingularization is already good. The exceptional set consists of 5 projective lines E_i with $E_i^2 = -2$ which intersect in the following way:



3. Let $S_3 = \operatorname{Spec}(\mathbb{C}[[x,y,z]]/(x^3+y^3+z^4))$. Then the minimal desingularization is not good, because the exceptional set consists of three projective lines E_i with $E_i^2 = -3$ which intersect in one point:

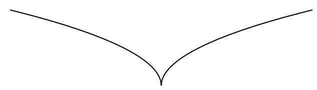


We get the minimal good desingularization via blowup at the intersection point:

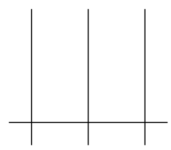


Because we have blown up in a smooth point of the ambient scheme, we have $E_4^2 = -1$. So, for the proper transformed $\widetilde{E}_i$ of the E_i we have $E_i^2 = (\widetilde{E}_i + E_4)^2$ and thus $\widetilde{E}_i^2 = -4$.

4. Let $S_4 = \operatorname{Spec}(\mathbb{C}[[x,y,z]]/(x^2 + y^3 + z^7))$, then the reduction of the exceptional locus of the minimal desingularization E_1 is an one-dimensional scheme with a cusp singularity and $E_1^2 = -1$:



So the minimal desingularization is not good. After embedded desingularization by blowing up in closed points we get first a $\mathbb{P}_k^1$ touching a parabola at the vertex, then a picture like the minimal desingularization of S_3 and finally the exceptional locus of the minimal good desingularization looks like:



Again we have $E_4^2 = -1$, but this time the three others have $\widetilde{E}_1^2 = -7$, $\widetilde{E}_2 = -3$ and $\widetilde{E}_3 = -2$. The calculation for this is analogously to the one for S_3.

Like the minimal desingularization, the minimal good desingularization always exists, and they have the same universal property:

Theorem 2.23. *Let (S, s) be a normal, two-dimensional singularity or an algebraization of one, then there exists a minimal good desingularization $f : X \to S$. Furthermore,for any other good desingularization $f' : X' \to S$ there exists a unique morphism $u : X' \to X$ such that $f \circ u = f'$. In particular, any two minimal good desingularizations are canonically isomorphic.*

Proof. We give a slightly modified and extended version of the proof of [Lau71], Theorem 5.2. The existence of a minimal good desingularization $f : X \to S$ follows simply by taking the minimal desingularization and blow up finitely many times at all the finitely many points, where the conditions (i), (ii) or (iii) are not fulfilled. Because every step is an isomorphism away from the centre, every choice we make just changes of the order of the blowups, but not the of resulting good desingularization, so this is really a minimal good desingularization.

It remains to show that every other good desingularization factorizes over this desingularization. For this let $f' : X' \to S$ be another good desingularization. Let X_{min} be the minimal desingularization of S. Then we have morphisms g, g' from X and X' to X_{min}, and $u = g^{-1} \circ g'$ is a birational map from X' to X. We want to show that u is actually a morphism, that is it is defined on all of X'.

By [Sha66], Page 45 we know that g and g' are sequences of blowups in closed points. Because the construction of X was unique up to order every component of the exceptional divisor of g in X must then already be contained in X', and u is just the blow down of additional components, in particular a morphism. □

This leads to the following corollary:

Corollary 2.24. *Let $(S_1, s_1), (S_2, s_2)$ be two normal, two-dimensional singularities and $f_i : X_i \to S_i$ their minimal desingularization. Then (S_1, s_1) is isomorphic to (S_2, s_2) if and only if $X_1 \cong X_2$.*

The same is true for the minimal good desingularization.

Proof. If X_1 and X_2 are isomorphic, then (S_1, s_1) and (S_2, s_2) are isomorphic by Theorem 2.16. Now assume (S_1, s_1) isomorphic to (S_2, s_2), then the composition of f_1 with this isomorphism makes X_1 a minimal desingularization of S_2 and thus isomorphic to X_2. □

Now we want to reduce the question from the base change to the completion back to finite thickenings of the exceptional locus: First we note that the $X_1 \otimes \mathcal{O}_{S_1,s_1}/m_{s_1}^{n+1}$ together with the inclusions form a direct system. The conditions on the morphism are fulfilled by the associativity of the fibre product. Now we have:

Theorem 2.25. *Two normal, two-dimensional singularities (S_i, s_i) with minimal desingularizations $f_i : X_i \to S_i$ are isomorphic if and only if we have an isomorphism of direct systems*

$$X_1 \otimes \mathcal{O}_{S_1,s_1}/m_{s_1}^{n+1} \cong X_2 \otimes \mathcal{O}_{S_2,s_2}/m_{s_2}^{n+1}, \ \ n \geq 0.$$

The same is true for the minimal good desingularization.

Proof. First assume (S_1, s_1) isomorphic to (S_2, s_2), that is $\mathcal{O}_{S_1,s_1} \cong \mathcal{O}_{S_2,s_2}$. Then we have also $\mathcal{O}_{S_1,s_1}/m_{s_1}^{n+1} \cong \mathcal{O}_{S_2,s_2}/m_{s_2}^{n+1}$ for every n. Now by Corollary 2.24 we know $X_1 \cong X_2$, and thus we get $X_1 \otimes \mathcal{O}_{S_1,s_1}/m_{s_1}^{n+1} \cong X_2 \otimes \mathcal{O}_{S_2,s_2}/m_{s_2}^{n+1}$ for all n. The compatibility needed for a direct system follows directly from the associativity of the fibre-product ([Gro60], 3.3.9.1).

Now assume $X_1 \otimes \mathcal{O}_{S_1,s_1}/m_{s_1}^{n+1} \cong X_2 \otimes \mathcal{O}_{S_2,s_2}/m_{s_2}^{n+1}$ for all n, and this is an isomorphism of direct systems. Then we have

$$H^0(X_1 \otimes \mathcal{O}_{S_1,s_1}/m_{s_1}^{n+1}, \mathcal{O}_{X_1 \otimes \mathcal{O}_{S_1,s_1}/m_{s_1}^{n+1}}) \cong H^0(X_1 \otimes \mathcal{O}_{S_2,s_2}/m_{s_2}^{n+1}, \mathcal{O}_{X_2 \otimes \mathcal{O}_{S_2,s_2}/m_{s_2}^{n+1}}),$$

and by the functoriality of taking global sections this is an isomorphism of inverse systems.

Now be Lemma 2.15 we have $H^0(X_i, \mathcal{O}_{X_i}) \cong \mathcal{O}_{S_i,s_i}$ and we use [Gro61], 4.1.7, the theorem on formal functions, to get:

$$\begin{aligned} \widehat{\mathcal{O}}_{S_1,s_1} &\cong (H^0(X_1, \mathcal{O}_{X_1}))^\wedge \\ &\cong \varprojlim H^0(X_1 \otimes \mathcal{O}_{S_1,s_1}/m_{s_1}^{n+1}, \mathcal{O}_{X_1 \otimes \mathcal{O}_{S_1,s_1}/m_{s_1}^{n+1}}) \\ &\cong \varprojlim H^0(X_2 \otimes \mathcal{O}_{S_2,s_2}/m_{s_2}^{n+1}, \mathcal{O}_{x_2 \otimes \mathcal{O}_{S_2,s_2}/m_{s_2}^{n+1}}) \\ &\cong (H^0(X_2, \mathcal{O}_{X_2}))^\wedge \cong \widehat{\mathcal{O}}_{S_2,s_2} \end{aligned} \tag{2.2}$$

But the $\mathcal{O}_{S_i,s_i}$ are complete, so we get $\mathcal{O}_{S_1,s_1} = \widehat{\mathcal{O}}_{S_1,s_1} \cong \widehat{\mathcal{O}}_{S_2,s_2} = \mathcal{O}_{S_2,s_2}$. □

The limits in (2.2) can also be calculated with other infinitesimal neighbourhoods of the exceptional divisor. We want to formulate this as a corollary, but before that, we need the following Theorem. Recall that for a scheme X and a quasi-coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ the scheme $\operatorname{Spec}(\mathcal{O}_X/\mathcal{I})$ has $\operatorname{supp}(\mathcal{O}_X/\mathcal{I})$ as underlying topological space, and $\mathcal{O}_X/\mathcal{I}$ as structure sheaf. Now we get:

Theorem 2.26. *Let (S_i, s_i) be two normal, two-dimensional singularities with minimal good desingularizations $f_i : X_i \to S_i$ and $E_{i,l} = \operatorname{Spec}(\mathcal{O}_{X_i}/\mathcal{I}_{i,l})$ the integral components of the exceptional divisor. Further assume that for some $n_1, ..., n_n$ we have ideal sheaves $\mathcal{J}_i$ with $\sum_{l=1}^{n} n_l E_{i,l} \subset \operatorname{Spec}(\mathcal{O}_{X_i}/\mathcal{J}_i)$ and $\sum_{l=1}^{n} E_{i,l} = \operatorname{Spec}(\mathcal{O}_{X_i}/\sqrt{\mathcal{J}_i})$ and that we have an isomorphism $\varphi : \operatorname{Spec}(\mathcal{O}_{X_1}/\mathcal{J}_1) \to \operatorname{Spec}(\mathcal{O}_{X_2}/\mathcal{J}_2)$. Then φ induces*

$$Z_1 = \sum_{l=1}^{n} n_l E_{1,l} \cong \sum_{l=1}^{n} n_l E_{2,l} = Z_2$$

Proof. Let $(U_j)_{j \in \mathbb{N}}$ be a covering of $\operatorname{Spec}(\mathcal{O}_{X_1}/\mathcal{J}_1)$ with open affine subsets, then $V_j = \varphi(U_j)$ is an open affine covering of $\operatorname{Spec}(\mathcal{O}_{X_1}/\mathcal{J}_1)$. We know that we may assume $Z_1 \cap U_j \cong \operatorname{Spec}(A_j/(f_j^l g_j^m))$ with f_j, g_j irreducible and $Z_2 \cap V_j \cong \operatorname{Spec}(\widetilde{A}_j/(\widetilde{f}^l \widetilde{g}^m))$.

Now by the functoriality of the reduction ([Gro60], 5.1.5) φ_{red} is an isomorphism between $\operatorname{Spec}(\mathcal{O}_{X_1}/\sqrt{\mathcal{J}_1})$ and $\operatorname{Spec}(\mathcal{O}_{X_1}/\sqrt{\mathcal{J}_1})$ so we may assume that we have units λ_f, λ_g such that $\varphi|_{U_j}^{\#}$ maps f_j to $\lambda_f \widetilde{f}_j$ and g_j to $\lambda_g \widetilde{g}_j$. But this is just another way to say that $\varphi|_{U_j}$ induces a local isomorphism $\varphi|_{Z_1 \cap U_j} : Z_1 \cap U_j \to Z_2 \cap V_j$. Now, because the $\varphi|_{Z_1 \cap U_j}$ are all just restrictions of φ they agree on $U_{i,j}$, and thus glue to a global isomorphism $\psi : Z_1 \to Z_2$. □

In the proof we never used that the $\mathcal{O}_{S_i,s_i}$ are complete, so we immediately get the following corollary:

Corollary 2.27. *The previous theorem holds also if we replace one or both of the S_i by an algebraization.*

Now we can formulate the corollary of the proof of Theorem 2.25:

Corollary 2.28. *Let (S_i, s_i) be two normal, two-dimensional singularities with minimal good desingularizations $f_i : X_i \to S_i$ and let $E_{i,l}$ be the integral components of the exceptional divisors. Further let $(n_{1,j}, ..., n_{n,j})_{j\in\mathbb{N}}$ be a sequence with $n_{l,j+1} \geq n_{l,j}$ and $\lim_{j\to\infty} n_{l,j} = \infty$ for all l. Then (S_1, s_1) is isomorphic to (S_2, s_2) if and only if we have an isomorphism of direct systems*

$$Z_{1,j} = \sum_{l=1}^{n} n_{l,j} E_{1,l} \cong \sum_{l=1}^{n} n_{l,j} E_{2,l} = Z_{2,j}$$

of schemes.

Proof. First suppose we have an isomorphism of direct systems. Let $\mathcal{I}_{i,j}$ be the ideal sheaf defining $Z_{i,j}$ in X_i, then the isomorphism of direct systems induces an isomorphism of inverse systems

$$\mathcal{O}_{Z_{1,j}} = \mathcal{O}_{X_1}/\mathcal{I}_{1,j} \cong \mathcal{O}_{X_2}/\mathcal{I}_{2,j} = \mathcal{O}_{Z_{2,j}},$$

which gives us

$$\varprojlim_{\mathbb{N}} H^0(Z_{1,j}, \mathcal{O}_{X_1}/\mathcal{I}_{1,j}) \cong \varprojlim_{\mathbb{N}} H^0(Z_{2,j}, \mathcal{O}_{X_2}/\mathcal{I}_{2,j}). \tag{2.3}$$

Now by (2.2) in the proof of Theorem 2.25 it suffices to prove that we have

$$\varprojlim_{\mathbb{N}} H^0(Z_{i,j}, \mathcal{O}_{X_i}/\mathcal{I}_{i,j}) \cong \varprojlim_{\mathbb{N}} H^0(X_i \otimes \mathcal{O}_{S_i,s_i}/m_{s_i}^{n+1})$$

and by symmetry, we may assume $i = 1$.

Now let $\mathcal{J}_{1,j}$ be the ideal sheaf defining $X_1 \otimes \mathcal{O}_{S_1,s_1}/m_{s_1}^{j+1}$, then we have $\sqrt{\mathcal{I}_{i,j}} = \sqrt{\mathcal{J}_{i,j}}$, and because X_i is noetherian we get α_j and γ_j for each j such that we have $\mathcal{J}_{1,\alpha_j} \subset \mathcal{I}_{1,j} \subset \mathcal{J}_{i,\gamma_j}$. But then the induced maps

$$H^0(X_1 \otimes \mathcal{O}_{S_1,s_1}/m_{s_1}^{\alpha_j+1}) \longrightarrow H^0(Z_{1,j}, \mathcal{O}_{X_1}/\mathcal{I}_{1,j}) \longrightarrow H^0(X_1 \otimes \mathcal{O}_{S_1,s_1}/m_{s_1}^{\gamma_j+1})$$

satisfy the condition (2.1) of Corollary 2.4, and the condition on the $(n_{1,j}, ..., n_{n,j})$ gives us that the sets $\{\alpha_j\}$ and $\{\gamma_j\}$ are right directed and cofinal in $\mathbb{N}$. So we get the needed isomorphism from Corollary 2.4.

For the othjer direction, if (S_1, s_1) and (S_2, s_2) are isomorphic, than we know from Theorem 2.25 that we have an isomorphism of direct systems

$$(X_1 \otimes \mathcal{O}_{S_1,s_1}/m_{s_1}^{n+1}) \cong (X_2 \otimes \mathcal{O}_{S_2,s_2}/m_{s_2}^{n+1}).$$

Now because the X_i are noetherian for all j we find an n such that we have $Z_{i,j} \subset X_i \otimes \mathcal{O}_{S_i,s_i}/m_{s_i}^{n+1}$. So by Theorem 2.26 the isomorphism between the systems $(X_i \otimes \mathcal{O}_{S_i,s_i}/m_{s_i}^{n+1})$ induces an isomorphism between the $(Z_{i,j})$. □

Now the previous theorem reduces the question whether two normal, two-dimensional singularities are isomorphic to the question whether two inverse systems of divisors are isomorphic. But the schemes involved in this systems can be calculated on an arbitrary algebraization:

Lemma 2.29. *Let* (S,s) *be a normal, two-dimensional singularity,* $\mathrm{Spec}(A)$ *an algebraization of* S *and* X *a desingularization of* $\mathrm{Spec}(A)$*. Let* m_s *be the maximal ideal of* $\mathcal{O}_{S,s}$ *and* m_a *the one of* A*. If we set* $X' = X \otimes \mathcal{O}_{S,s}$ *we have*

$$X' \otimes \mathcal{O}_{S,s}/m_s^{n+1} \cong X \otimes A/m_a^{n+1}$$

for all $n \geq 0$*.*

Proof. By Lemma 2.5 (2) we have

$$\mathcal{O}_{S,s}/m_s^{n+1} \cong A/m_a^{n+1}$$

so the claim follows directly from the associativity of the fibre product ([Gro60], 3.3.9.1). □

Now the conditions for being a minimal (good) desingularization can already be checked on this infinitesimal thickenings, and by the proof of Lemma 4.2 of [Bǎd01] being regular is stable under the needed base change, so we get:

Lemma 2.30. *Let* (S,s) *be a normal, two-dimensional singularity and* $\mathrm{Spec}(A)$ *be an arbitrary algebraization of* S*. Let* $f : X \to \mathrm{Spec}(A)$ *be a desingularization. Then* $X \otimes \mathcal{O}_{S,s}$ *is a minimal desingularization of* S *if and only if* X *is a minimal desingularization of* $\mathrm{Spec}(A)$*.*

The same is true for the minimal good desingularization.

In particular, if we set $S_2 = A$ in Theorem 2.26 we get:

Corollary 2.31. *Let* (S,s) *be a normal, two-dimensional singularity and* $\mathrm{Spec}(A)$ *be an arbitrary algebraization of* S*. Let* $f : X \to \mathrm{Spec}(A)$ *be the minimal desingularization of* $\mathrm{Spec}(A)$ *and* $f' : X' \to S$ *the one of* S*. Further let* E_l *and* E'_l *be the integral components of the exceptional divisors. Then for all* $(n_1, \cdots, n_n)$ *we have isomorphisms of schemes*

$$\sum_{l=1}^{n} n_l E_l \cong \sum_{l=1}^{n} n_l E'_l.$$

The same is true for the minimal good desingularization.

If we combine Lemma 2.30 with Theorem 2.25 we get the following corollary:

Corollary 2.32. *Let* (S_i, s_i) *be two normal, two-dimensional singularity and* $\mathrm{Spec}(A_i)$ *be an arbitrary algebraization of* S_i *with minimal desingularization* $f : X_i \to \mathrm{Spec}(A_i)$*, then* (S_1, s_1) *is isomorphic to* (S_2, s_2) *if and only if for all* $n \geq 0$ *we have*

$$X_1 \otimes A_1/m_{a_1}^{n+1} \cong X_2 \otimes A_2/m_{a_2}^{n+1}$$

and those isomorphisms are compatible with the natural morphisms

$$X_i \otimes A_i/m_{a_i}^{n+1} \longrightarrow X_i \otimes A_i/m_{a_i}^{n+2}.$$

The same is true for the minimal good desingularization.

Note that the last condition just says that we have an isomorphism of direct systems. The nice consequence of this corollary is that if we work with the $X \otimes \mathcal{O}_{S,s}/m_s^{n+1}$, we may always assume them to be embedded into some regular, two-dimensional scheme of finite type over k, which thus is smooth. Finally we get an analogue of Corollary 2.28 for an algebraization:

Corollary 2.33. *Let (S_i, s_i) be two normal, two-dimensional singularities and* $\mathrm{Spec}(A_i)$ *an arbitrary algebraization of S_i. Further let $f_i : X_i \to \mathrm{Spec}(A_i)$ be minimal good desingularizations and $E_{i,l}$ the integral components of the exceptional divisor. Further let $(n_{1,j}, ..., n_{n,j})_{j \in \mathbb{N}}$ be a sequence with $n_{l,j+1} \geq n_{l,j}$ and $\lim_{j \to \infty} n_{l,j} = \infty$ for all l. Then (S_1, s_1) is isomorphic to (S_2, s_2) if and only if we have an isomorphism of direct systems*

$$Z_{1,j} = \sum_{l=1}^{n} n_{l,j} E_{1,l} \cong \sum_{l=1}^{n} n_{l,j} E_{2,l} = Z_{2,j}$$

of schemes.

2.3 Taut singularities

From the previous section we know that if we want to understand whether two normal, two-dimensional singularities are isomorphic, we have to understand finite thickenings of the exceptional divisor of their minimal good desingularization. Now we want to reduce this question further. For this we look at the exceptional divisor E. This is a divisor on X whose irreducible components have regular reduction and intersect transversally. Using this, we can assign some combinatorial data to the exceptional divisor. We want to ask the following question: Does this combinatorial data describe the singularity up to equivalence?

As an addition to the well-known invariants like the genus we need:

Definition 2.34. Let $E = \sum_{l=1}^{n} n_l E_l$ be a closed, one-dimensional subscheme of a regular, two-dimensional scheme, such that the E is projective over k. Let $(E_i \cdot E_j)$ be the symmetric matrix with $E_i \cdot E_j$ as ij-th entry. We call $(E_i \cdot E_j)$ the *intersection matrix of E.*

Suppose now $E = f^{-1}(s)_{\mathrm{red}}$ where f is a desingularization of a normal, two-dimensional singularity. We want to collect some facts on $(E_i \cdot E_j)$ we need later.

From [Mum61], Page 230 we get:

Lemma 2.35. *$(E_i \cdot E_j)$ is negative definite.*

Now assume additionally that E_i and E_j intersect transversally for $i \neq j$, so by Lemma V 1.3 of [Har77] we have

Lemma 2.36. *$0 \leq \#(E_i \cap E_j) = E_i \cdot E_j$ for $i \neq j$.*

Then Lemma 4.10 of [Lau71] gives us:

Lemma 2.37. *There exist positive integers $n_1, \dots, n_n$ such that for all i:*

$$E_i \cdot \sum_{l=1}^{n} n_l E_l < 0.$$

By the negative definiteness we also have:

Remark 2.38. We have $E_i^2 < 0$ for all i.

If we assume additionally that all E_l are regular, we can encode all the combinatorial data of E into one object:

Definition 2.39. Let $E = \sum_{l=1}^{n} n_l E_l$ be a closed, one-dimensional subscheme of a regular, two-dimensional scheme, such that E is projective over k and the E_l are regular. The *dual graph* Γ_E of E is the following graph:

- For each E_l we add a vertex v_l.
- We add $E_l \cdot E_i$ edges $e_{l,i}^j$ between v_l and v_i.
- Each vertex v_l is decorated by three weights: the arithmetic genus $[p_a(E_l)]$, the multiplicity (n_l) and the self-intersection E_l^2. If $p_a(E_l) = 0$ or $n_l = 1$ we omit the $[0]$ or (1).

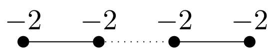
We say that two dual graphs are isomorphic if we have a bijection φ_v between the sets of vertices respecting the decorations and a bijection φ_e between the sets of edges such that $e_{l,i}^j$ is mapped to an edge between $\varphi_v(v_l)$ and $\varphi_v(v_i)$.

The exceptional divisor, or at least its reduction, fulfils the assumptions of the previous definition, so we define:

Definition 2.40. Let (S, s) be a normal, two-dimensional singularity and $f : X \to S$ its minimal good desingularization, and Γ the dual graph for $f^{-1}(s)_{\text{red}}$. The we call Γ also the *dual graph for* (S, s), or we say (S, s) is a Γ-*singularity*.

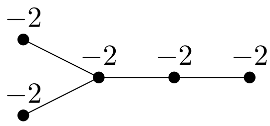
Examples Now we want to give the dual graphs for the examples above:

1. The dual graph of $(S_1, 0)$ is:

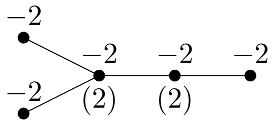


   where we have n vertices.

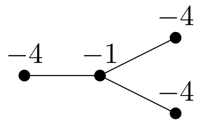
2. The dual graph of $(S_2, 0)$ is:



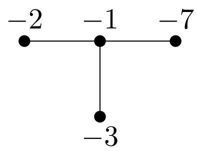
Now with Theorem 2.64, which will cite later, we can actually calculate $Z = f^{-1}(0)$ for this singularity, and we get as dual graph for Z:



3. For the singularity $(S_3, 0)$ the minimal and the minimal good desingularization differ, and the dual graph is by definition the graph to the minimal good desingularization, so we get:



4. For $(S_4, 0)$ again we have to remember to take the dual graph to the minimal good desingularization:



With this we have all combinatorial invariants we can assign to the exceptional divisor, which leads to the following definition:

Definition 2.41. Let $E = \sum_{l=1}^{n} n_l E_l$ on X and $E' = \sum_{l=1}^{n'} n'_l E'_l$ on X' be two closed, one-dimensional subscheme of regular, two-dimensional schemes, such that E and E' are projective over k and the E_l, E'_l are regular. We say that E and E' are *combinatorially equivalent* if their dual graphs are isomorphic.

We say that E is *defined by its dual graph* if every E' combinatorially equivalent to E is already isomorphic to E.

Now for a given E as in the previous definition, we look at the set $S(E)$ of all tuple (E', X') where X' is a regular, two-dimensional scheme and $E' \subset X'$ is combinatorially equivalent to E. Note that we need the ambient scheme X' only to define the self-intersection numbers E'_l. So for the question whether two combinatorially equivalent schemes E and E' are isomorphic the ambient scheme does not matter. So we say that $(E', X'), (E'', X'') \in S(E)$ are equivalent if and only if E' is isomorphic to E'' as k-schemes. This gives an equivalence relation on $S(E)$.

Definition 2.42. We define $\mathrm{CEQ}(E)$ as $S(E)$ divided by the equivalence relation defined above.

Remark 2.43. E is defined by its dual graph if and only if $\mathrm{CEQ}(E) = \{[E]\}$.

Now the definition of tautness is:

Definition 2.44. Let (S, s) be a normal, two-dimensional singularity with minimal good desingularization $f : X \to S$, we say that (S, s) is *taut* if (S, s) is isomorphic to any other normal, two-dimensional singularity with isomorphic dual graph.

Our main tool to proof tautness of some normal, two-dimensional singularity is the following Lemma, which is a direct consequence of the discussion before and Corollary 2.28:

Lemma 2.45. *Let (S, s) be a normal, two-dimensional singularity with minimal good desingularization $f : X \to S$ and let $E_1, \dots, E_n$ be the integral components of $f^{-1}(s)$. Let $(n_{1,j}, ..., n_{n,j})_{j\in\mathbb{N}}$ be a sequence with $n_{l,j+1} \geq n_{l,j}$ and $\lim_{j\to\infty} n_{l,j} = \infty$ for all l. We set $Z_j = \sum_{l=1}^{n} n_{l,j} E_l$. Then (S, s) is taut if for every j the Z_j is defined by its dual graph.*

The reverse of this lemma is more delicate. We have to deal with two questions: Given $\widetilde{E}$ and jZ' combinatorial equivalent, but not isomorphic, then by Theorem 2.26 we have a whole system of schemes, combinatorial equivalent, but not isomorphic. We know (by definition) that $\widetilde{E}$ is embedded in a regular, two-dimensional scheme $\widetilde{X}$. The first question is, can we contract $\widetilde{E} \subset \widetilde{X}$ to get a normal singularity?

To answer this question we need the following definition:

Definition 2.46. Let X be a regular scheme and $E \subset X$ a closed subset. A *contraction* Y of E is a proper morphism $f : X \to Y$ with $f(E) = y$ and $f_*(\mathcal{O}_X) = \mathcal{O}_Y$ which is an isomorphism away from E.

Note the condition $f_*(\mathcal{O}_X) = \mathcal{O}_Y$ which is equivalent to the normality of Y. This condition is not important if we want to know whether a contraction exists, because if we have a proper, birational morphism $\overline{f} : X \to \overline{Y}$ with $\overline{f}(E) = y$ which is an isomorphism away from E, we may take Y as the normalization of $\overline{Y}$ and f as the induced morphism $f : X \to Y$.

2.4 Algebraic spaces

Our goal is to find a reverse of Lemma 2.45: Suppose we have a normal, two-dimensional singularity (S, s) with minimal good desingularization X and exceptional divisor Z. Suppose further we find a regular, two-dimensional scheme X' with a divisor Z' on it such that jZ' is combinatorially equivalent to jZ for all j. Now suppose that at least for one j the scheme jZ' is not isomorphic to jZ. Then we can not find an isomorphism of inverse systems for Theorem 2.25. So if we have a contraction of Z' then the resulting normal, two-dimensional singularity can not be isomorphic to (S, s).

Unfortunately for an arbitrary X' this contraction may not exist as a scheme but as an algebraic space. The goal of this section is to give the needed definitions and statements for this, and finally prove that such a contraction exists as a scheme if we change X' "away from Z'".

One motivation for the introduction of algebraic spaces is the existence of categorical quotients: If we have a category $\mathcal{C}$ with fibre products, two objects $U, R \in \mathrm{Ob}(\mathcal{C})$ and two morphism $a, b : R \to U$ in $\mathcal{C}$ we get a diagram $R \overset{a}{\underset{b}{\rightrightarrows}} U$. We write just $R \rightrightarrows U$ if a, b are the composition of a map $R \to U \times U$ with the canonical projections $\pi_i : U \times U \to U$. A diagram $R \to U \times U$ is called a *categorical equivalence relation* if

$$\mathrm{Hom}_{\mathcal{C}}(Z, R) \subset \mathrm{Hom}_{\mathcal{C}}(Z, U) \times \mathrm{Hom}_{\mathcal{C}}(Z, U)$$

is an equivalence relation for every $Z \in \mathrm{Ob}(\mathcal{C})$.

Let $R \overset{a}{\underset{b}{\rightrightarrows}} U$ be a diagram and $f : U \to X$ a map in $\mathcal{C}$. We say that f is the *coequalizer* of $R \overset{a}{\underset{b}{\rightrightarrows}} U$ if $f \circ a = f \circ b$ and for every other $f' : U \to X'$ with $f' \circ a = f' \circ b$ there exists an unique morphism g in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{array}{ccccc} R & \overset{a}{\underset{b}{\rightrightarrows}} & U & \xrightarrow{f} & X \\ & & & \searrow^{f'} & \downarrow g \\ & & & & X' \end{array}$$

Now a map $f : U \to X$ in $\mathcal{C}$ is called the *categorical quotient* of the categorical equivalence relation $R \rightrightarrows U$, if $U \to X$ is the coequalizer of $R \rightrightarrows U$.

Now let $\mathcal{C}$ be the category of schemes. Then one may ask: Does a coequalizer exist for every categorical equivalence relation? The answer is "no" but it is not easy to give a counterexample, because a posteriori one gets that for a huge classes of categorical equivalence relations we have a coequalizer which is a scheme.

Now we "make the problem into a definition" by taking the following definition of algebraic spaces from [Art71]:

Definition 2.47. An *algebraic space* X consists of an affine scheme U and a closed subscheme R of $U \times U$ such that

1. R is a categorical equivalence relation.
2. The projection maps $\pi_i : R \to U$ are étale.

The underlying set of points $|X|$ of an algebraic space is the set $|U|/|R|$.

To get a category we also need morphisms. But for this work we only need morphism from affine schemes into algebraic spaces in detail, so we define only this and refer to [Art71] for the general definition and for the definition of the fibre product of algebraic spaces.

Definition 2.48. Let X be an algebraic space as above and let V be an affine scheme. A map $V \to X$ is a closed subscheme $W \subset U \times V$ such that:

1. The projection $W \to V$ is étale and surjective,
2. The two closed subschemes $R \times_U W$ and $W \times_V W$ of $U \times U \times V$ are equal.

Now by Corollary (6.12) of [Art70] we get:

Theorem 2.49. *Let X be a regular, two-dimensional scheme and $E \subset X$ a connected, closed, one dimensional subset with $(E_i \cdot E_j)$ negative-definite, then there exists a contraction $f : X \to Y$ with Y an algebraic space.*

Now we have the contraction as an algebraic space, and we want to show that we can modify X such that the contraction is already a scheme.

Corollary 2.50. *Let $Z = \sum_{l=1}^{n} n_l E_l$ be a closed, one-dimensional subscheme of a regular, two-dimensional scheme X, such that E is projective over k and the E_l are integral. Assume further that Z_{red} satisfies the conditions of the exceptional divisor of a minimal good desingularization. Then there exists a normal, two-dimensional singularity (S', s') with minimal good desingularization $f' : X' \to S'$ and an embedding $\iota : Z \to X'$ with $f'(\iota(Z)) = s'$.*

Proof. Let $f : X \to S$ be the contraction of Z with S an algebraic space and $s = f(Z)$. Then by Theorem II 6.4 of [Knu71] we have an affine scheme U and an étale map $U \to S$ such that the embedding $s \to S$ factors $s \to U \to S$. We may assume U to be normal. We take the fibre product of algebraic spaces $X' = X \times_S \operatorname{Spec}(\widehat{\mathcal{O}}_{U,s})$. Now $S' = \operatorname{Spec}(\widehat{\mathcal{O}}_{U,s})$ is a scheme, and by Proposition II 1.7 of [Knu71] we know that the fibre product of two schemes over an algebraic space is a scheme, so X' is a scheme. Let s' be the closed point of S'. Then we know that (S', s') is a normal, two-dimensional singularity and X' is regular. Because the reduction of the exceptional fibre of $f' : X' \to S'$ is Z_{red}, we know that f' is the minimal good desingularization of S'.

It remains to prove the existence of ι. First we remark that by the same argumentation as in the proof of Corollary 2.28 we get $Z \subset X \otimes \operatorname{Spec}(\widehat{\mathcal{O}}_{U,s}/m^{i+1})$ for an i large enough. But by definition we have $\widehat{\mathcal{O}}_{U,s}/m^{i+1} = \mathcal{O}_{S',s'}/m_s^{i+1}$, and so the associativity of the fibre product gives us

$$Z \subset X \otimes \operatorname{Spec}(\widehat{\mathcal{O}}_{U,s}/m^{i+1}) \cong X' \otimes \operatorname{Spec}(\mathcal{O}_{S',s'}/m_s^{i+1})$$

and this gives the wanted $\iota : Z \to X'$. □

Now we get the reverse of Lemma 2.45:

Lemma 2.51. *Let (S, s) be a normal, two-dimensional singularity with minimal good desingularization $f : X \to S$ and let E_i be the n integral components of $f^{-1}(s)$. Let $(n_{1,j}, ..., n_{n,j})_{j \in \mathbb{N}}$ be a sequence with $n_{l,j+1} \geq n_{l,j}$ and $\lim_{j \to \infty} n_{l,j} = \infty$ for all l. We set $Z_j = \sum_{l=1}^{n} n_{l,j} E_l$. If (S, s) is taut, then for all j the Z_j is defined by its dual graph.*

Proof. Suppose that for one j_0 the Z_{j_0} is not defined by its dual graph, that is we have a scheme Z'_{j_0} such that Z_{j_0} is combinatorially equivalent to Z'_{j_0} but not isomorphic. By the definition of combinatorial equivalent we know that Z'_{j_0} is embedded in some regular, two-dimensional scheme X'. But then by Corollary 2.50 we can get a singularity (S', s') such that $Z'_{j_0} = \sum_{l=1}^{n} n_{l,j} E'_l$, where E'_l are the integral components of the exceptional divisor of the minimal good desingularization of (S', s'). In particular the dual graphs of (S, s) and (S', s') are isomorphic.

If we now define $Z'_j = \sum_{l=1}^{n} n_{l,j} E'_l$, then we have $Z'_{j_0} \ncong Z_{j_0}$, and so the direct systems (Z_j) and (Z'_j) can not be isomorphic, and with Corollary 2.28 this means that (S, s) and (S', s') are not isomorphic, and thus (S, s) is not taut. □

If we combine this Lemma with Lemma 2.45 and Corollary 2.31, we get the following Theorem:

Theorem 2.52. *Let (S, s) be a normal, two-dimensional singularity. Let f be the minimal good desingularization of S or of any algebraization of S and let E_i be the n integral components of its exceptional divisor. Let $(n_{1,j}, ..., n_{n,j})_{j \in \mathbb{N}}$ be a sequence with $n_{l,j+1} \geq n_{l,j}$ and $\lim_{j \to \infty} n_{l,j} = \infty$ for all l. We set $Z_j = \sum_{l=1}^{n} n_{l,j} E_l$. Then (S, s) is taut if, and only if for all j the Z_j is defined by its dual graph.*

2.5 Cycles supported by the exceptional locus

Let X be a regular, two-dimensional scheme and B a closed, one-dimensional subscheme of X proper over k with integral components E_l. Then we call $\sum_{l=1}^{n} n_l E_l$ a cycle supported by B. By the last theorem we know that the question of tautness reduces to the question whether cycles supported by the exceptional loci of the minimal good desingularizations are isomorphic as schemes. Now we want to introduce two special types of such cycles. First, we need the fundamental cycle of Artin ([Art66], Page 131):

Lemma 2.53. *Let (S, s) be a normal, two-dimensional singularity with desingularization $f : X \to S$, and $E = f^{-1}(s)_{\text{red}}$. There exists a smallest divisor Z on X with* $\operatorname{supp}(Z) = \operatorname{supp}(E)$ *and $Z \cdot E_i \leq 0$ for all i.*

Now we want to give this cycle a name:

Definition 2.54. The cycle Z of Lemma 2.53 is called the *fundamental cycle* of (S, s).

We can calculate Z with a quite simple algorithm from $(E_i \cdot E_j)$: We define $Z_0 = E$ and construct $Z_{\nu+1}$ from Z_ν as follows: If $Z_\nu \cdot E_i \leq 0$ for all i we stop. Else we set $Z_{\nu+1} = Z_\nu E_i$ for one i with $Z_\nu \cdot E_i > 0$

Now we need some facts to define the other cycle:

Lemma 2.55. *Let $f : X \to S$ be the minimal good desingularization of a normal, two-dimensional singularity or its algebraization, and E_i the integral components of the exceptional locus. If $\widetilde{Z} = \sum_{l=1}^{n} r_l E_l$ satisfies $\widetilde{Z} \cdot E_i \leq -1$ for all i, then $-\widetilde{Z}$, or to be precise $\mathcal{O}_X(-\widetilde{Z})$, is ample.*

Proof. We have $\mathcal{O}_X(-\widetilde{Z}) \cdot E_i = -\widetilde{Z} \cdot E_i > 0$ for all i, so the lemma follows by the proof of [Lip69], Theorem 12.1 (iii) because ample holds without the additional hypothesis $H^1(X, \mathcal{O}_X) = 0$. □

To simplify the reference we give those divisors a name:

Definition 2.56. Let (S, s) be a normal, two-dimensional singularity. An *anti-ample cycle* for (S, s) is a divisor $\widetilde{Z} = \sum_{l=1}^{n} r_l E_l$ on the minimal good desingularization of some algebraization of (S, s) satisfying the previous lemma.

Now Lemma 2.37 shows that we always have an anti-ample cycle for (S, s). But unlike the fundamental cycle it is not unique. Actually there is also a smallest anti-ample cycle, but later we need the freedom to choose specific anti-ample cycles. For example for positive p, we often need an anti-ample cycle with all coefficients prim to p. For this we need a function from $\mathbb{Z}$ to $\{0, 1\}$ which maps n to 0 exactly if $p \neq 0$ and $p|n$. This is the following “dual gcd” or “binary gcd”:

Definition 2.57. Let a, b be integers. We define

$$\delta_{\text{gcd}}(a, b) = \begin{cases} 1 & \text{if } ab = 0; \\ 1 & \text{if } \gcd(a, b) = 1; \\ 0 & \text{else.} \end{cases}$$

In all application later we have $b = p$.

We get the existence of such anti-ample cycles from the following Lemma:

Lemma 2.58. *Let (S, s) be a normal, two-dimensional singularity, $p > 0$, then we always have an anti-ample cycle $\widetilde{Z} = \sum_{l=1}^{n} n_l E_l$ for (S, s) such that $\gcd(n_l, p) = 1$ for all l.*

Proof. Let $\widetilde{Z}'$ any anti-ample cycle for (S, s), that is $\widetilde{Z}' \cdot E_i \leq -1$ for all i. We define $t = \max_i \{E_i \cdot \sum_{\substack{l=1 \\ l \neq i}}^{n} E_l\}$. Now $(t+1)\widetilde{Z}'$ has $(t+1)\widetilde{Z}' \cdot E_i \leq -(t+1)$. We write $(t+1)\widetilde{Z}' = \sum_{l=1}^{n} n'_l E_l$, and define n_l by $n_l = n'_l + 1$ if $p|n'_l$ and $n_l = n'_l$ else. By Lemma 2.36 we have $E_i \cdot E_l \geq 0$ for $i \neq l$, and thus we get for all i with $n_i = n'_i + 1$:

$$E_i \cdot \sum_{l=1}^{n} n_l E_l \leq E_i \cdot \sum_{l=1}^{n} (n'_l + 1) E_l = E_i \cdot \sum_{l=1}^{n} n'_l E_l + E_i^2 + E_i \cdot \sum_{\substack{l=1 \\ l \neq i}}^{n} E_l \leq -t - 1 + E_i^2 + t < -1$$

Where the last inequality follows from $E_i^2 < 0$, see Remark 2.38. For the i with $n_i = n_i'$ an analogue calculation shows

$$E_i \cdot \sum_{l=1}^{n} n_l E_l \leq -t - 1 + t \leq -1$$

So $\sum_{l=1}^{n} n_l E_l$ is the desired cycle. □

Now we want to address the following question: Suppose we have n smooth, one-dimensional schemes, we glue them in such a way that $E = \sum_{l=1}^{n} n_l E_l$ is connected and has only transversal intersections. Also we choose E_i^2 such that $(E_i \cdot E_j)$ is negative definite. Is there a normal, two-dimensional singularity (S, s) with desingularization $f : X \to S$ such that E is a cycle on $f^{-1}(s)$? The answer is "yes", but before we can prove this, we need some notation.

Definition 2.59. Let Γ be a dual graph as in Definition 2.39 such that Γ is connected and the associated intersection matrix $(E_i \cdot E_j)$ is negative definite. A *realisation* of Γ is a smooth, two-dimensional scheme X with an closed, one-dimensional subscheme E, such that Γ is the dual graph of E.

Now we get that such a realisation always exist, even if we additionally fix the E_i:

Theorem 2.60. *For any connected dual graph Γ with negative definite $(E_i \cdot E_j)$, and any n smooth, one-dimensional schemes E_l with $p_a(E_l)$ as in Γ, we have a realisation X with $E = \sum_{l=1}^{n} n_l E_l$.*

Proof. First we note that it suffices to prove the theorem for one chosen n-tuple $(\widetilde{n}_1, \ldots, \widetilde{n}_n)$ of natural numbers, which may differ from the n_l of Γ. Now we glue the E_l to a scheme $E = \sum_{l=1} E_l$ such that all intersections are transversal, and E_i and E_j intersect exactly as often as the matrix $(E_i \cdot E_j)$ requires.

The main difficulty now is not to find a X into which E embeds, but to find a X such that we have E_i^2 equal to the ii-th entry of $(E_i \cdot E_j)$ for all i. To find this we use the following fact: Suppose we have a closed, one-dimensional subscheme $Z' = \sum_{l=1} n_l' E_l'$ of a smooth, two-dimensional scheme X' such that Z' is the fibre of a map from X' to a smooth, one-dimensional scheme. Then by (4.1) of [Win74] we have for all i:

$$0 = Z' \cdot E_i' = n_i'(E_i')^2 + E_i' \cdot \sum_{\substack{l=1 \\ l \neq i}}^{n'} n_l' E_l' \tag{2.4}$$

and thus the $(E_i')^2$ are controlled by the n_l' and the $E_i' \cdot E_l'$.

Our strategy is now: First we choose $(\widetilde{n}_1, \ldots, \widetilde{n}_n)$ such that

$$0 \geq -r_i = \widetilde{n}_i (E_i \cdot E_j)_{ii} + \sum_{\substack{l=1 \\ l \neq i}}^{n} \widetilde{n}_l (E_i \cdot E_j)_{il} \tag{2.5}$$

that is as in Lemma 2.58, respectively Lemma 2.53 if $p = 0$.

Now we construct a new E' from E as follows: At every E_i we choose r_i points which are smooth in E and we glue additional smooth, one-dimensional schemes $E_{i,j}$ transversally to them such that $E_{i,j}$ only intersects with E_i. If we now find an embedding of E' as a fibre, then for each $E_i \subset E$ the equation (2.4) gives:

$$0 = E' \cdot E_i = \widetilde{n}_i E_i^2 + \sum_{\substack{l=1 \\ l \neq i}}^{n} \widetilde{n}_l E_i \cdot E_l + \sum_{j=1}^{r_i} E_i \cdot E_{i,j} + \sum_{l=1}^{n} \sum_{j=1}^{r_l} E_i \cdot E_{l,j}.$$

But we have by construction: $E_i \cdot E_l = (E_i \cdot E_j)_{il}$, the last term vanishes, and the third term is r_i, so we get $\widetilde{n}_i E_i^2 = \widetilde{n}_i (E_i \cdot E_j)_{ii}$ as wanted, with (2.5).

So it remains only to prove that we can embed E' as a fibre. But this is now just Proposition 4.2 of [Win74] because all assumptions are fulfilled by construction. □

Together with Corollary 2.50 we get:

Corollary 2.61. *For any connected dual graph Γ, with negative definite $(E_i \cdot E_j)$ and any n smooth, one-dimensional schemes E_l with $p_a(E_l)$ as in Γ, we have normal, two-dimensional singularity (S, s) with desingularization $f : X \to S$ such that Γ is the dual graph of $\sum_{l=1}^{n} n_l E_l$ on the exceptional locus.*

The theorem gives us also a necessary condition on E for (S, s) to be taut. By Theorem 2.52 we know that the cycle $\sum_{l=1}^{n} E_l$ on the exceptional locus of the minimal good desingularization must be defined by its dual graph. But for example suppose we have $n = 1$ and $p_a(E_1) = 1$, that is an elliptic curve. To such a scheme we have the j-invariant. Then by [Sil92], Proposition III 1.4 we know that the j gives an one-to-one mapping between isomorphism classes of elliptic curves over k and the k-rational points of the scheme $\mathbb{A}^1_k$. In particular, there are two non-isomorphic elliptic curves. But the j-invariant is not encoded in the dual graph. So with Theorem 2.60 we can embed both curves with a given negative self-intersection into smooth surfaces. But then this curves are combinatorially equivalent, but not isomorphic. This implies that if we contract these curves, the resulting singularities are not isomorphic, and thus not taut.

This example generalises in the following way:

Lemma 2.62. *Let (S, s) be a normal, two-dimensional singularity, $f : X \to S$ its minimal good desingularization, and E_i the integral components of the exceptional locus. If S is taut, then we have necessary $p_a(E_i) = 0$ and each E_i intersects with at most 3 others.*

Proof. First suppose by contradiction that we have an i with $p_a(E_i) > 0$. The case $p_a(E_i) = 1$ is just a direct generalisation of the last paragraph: We choose an elliptic curve E_i' non-isomorphic to E_i and do the same argumentation with $E = \sum_{l=1}^{n} E_l$ and $E' = E_i' + \sum_{\substack{l=1 \\ l \neq i}}^{n} E_l$.

Also the general case for $g = p_a(E_i) > 1$ follows analogously: By [DM69] we have an irreducible quasi-projective scheme $\mathfrak{M}_g$ of dimension $3g - 3$ whose k-rational points are in one-to-one correspondence with the isomorphism classes of smooth, one-dimensional schemes over k, so we can again find a non-isomorphic E_i' with $p_a(E_i') = p_a(E_i)$.

So we have necessarily $p_a(E_i) = 0$ for all i. Because k is algebraically closed, this is equivalent to $E_i \cong \mathbb{P}_k^1$.

Now assume we have an E_i which intersects with 4 others. Now we use that $\mathrm{Aut}(\mathbb{P}_k^1)$ is isomorphic to $\mathrm{PGL}(1, k)$. The idea behind this isomorphism is that we know an automorphism of $\mathbb{P}_k^1$ if we now the image of three distinct points. So we may assume that 3 of the 4 other components intersecting E_i intersect at 0, 1 and ∞. Now we take E and E', such that the 4-th component intersects E_i at different points. Then we can again embed E and E', and their dual graphs are isomorphic, but E is not isomorphic to E', and so S is not taut. □

2.6 Rational singularities

Finally we want to define the well known class of rational singularities. One reason for this is that for some rational singularities the question of tautness is already known in arbitrary characteristic we discuss that in Section 5. The definition is:

Definition 2.63. A two-dimensional singularity (S, s) is called *rational* if there exists a desingularization $f : X \to S$, such that $R^1 f_* \mathcal{O}_X = 0$.

Rational singularities have many nice properties. We will recall those we need in the following theorems. We have by [Art66], Theorem 3 and 4:

Theorem 2.64. *Let (S, s) be a normal, two-dimensional singularity with desingularization $f : X \to S$ and fundamental cycle Z, then:*

1. *We have $p_a(Z) \geq 0$, and (S, s) is rational if and only if $p_a(Z) = 0$.*
2. *If (S, s) is rational, then for all $l \geq 1$ we have $X \otimes \mathcal{O}_{S,s}/m_s^l \cong lZ$.*

Further we have by [Băd04], 3.32.3:

Theorem 2.65. *Let (S, s) be a rational, normal, two-dimensional singularity. Then the minimal desingularization is the minimal good desingularization.*

Now let Z' be a positive cycle supported by E. From the proof of [Art66], Theorem 3 we get

$$p_a(Z' + E_i) = p_a(Z') + p_a(E_i) + Z' \cdot E_i - 1. \tag{2.6}$$

Now take the construction of the fundamental cycle from above. By construction we have $Z_\nu \cdot E_i \geq 1$ and thus

$$p_a(Z_{\nu+1}) \geq Z_\nu$$

and equality if and only if $Z_\nu \cdot E_i = 1$. So for (S, s) to be rational we have necessarily $p_a(E) = 0$ and using (2.6) again we get $p_a(E_i) = 0$ for all E_i.

This leads to the following corollary:

Corollary 2.66. *Let (S, s) be a normal, two-dimensional singularity. The question whether (S, s) is rational can be decided using only the dual graph of (S, s).*

Proof. As we discussed above, (S, s) is rational if and only if all $p_a(E_i) = 0$ and we can find a sequence (Z_ν) constructing Z with $Z_\nu \cdot E_i = 1$ for $Z_{\nu+1} = Z_\nu + E_i$ and every ν. But all this can be decided by just using data we find in the dual graph. □

Finally [Art62], Theorem 2.3 shows us that for rational singularities, we may prove Lemma 2.51 without using algebraic spaces:

Theorem 2.67. *Let X be a smooth surface and $E \subset X$ a connected, closed, one-dimensional subset with $(E_i \cdot E_j)$ negative definite, Z the fundamental cycle. If $p_a(Z) = 0$, then the algebraic space Y of Theorem 2.49 is a scheme, and (Y, y) is a rational singularity.*

If we combine this with (ii) of Theorem 2.64 and use Theorem 2.25 we get the following corollary:

Corollary 2.68. *Let (S, s) be a rational singularity with minimal desingularization $f : X \to S$ and fundamental cycle Z. Then (S, s) is taut if, and only if, for every j the cycle jZ is defined by its dual graph.*

3 Extending isomorphisms

Our aim is to classify taut singularities. That is, we want to give some criteria, such that, given a normal, two-dimensional singularity (S, s) with some dual graph satisfying this criteria, every other singularity with the same dual graph is isomorphic to S.

Lets suppose that $(\widetilde{S}, \widetilde{s})$ is a normal, two-dimensional singularities with isomorphic dual graph. For $(n_{1,j}, ..., n_{n,j})_{j\in\mathbb{N}}$ let Z_j and $\widetilde{Z}_j$ the cycles as in Theorem 2.52 for (S, s) respectively $(\widetilde{S}, \widetilde{s})$. From Theorem 2.52 we know that $(S, s) \cong (\widetilde{S}, \widetilde{s})$ if and only if all Z_j and $\widetilde{Z}_j$ are isomorphic. The goal of this section is to show, that we have $(S, s) \cong (\widetilde{S}, \widetilde{s})$ if and only if $Z_{j_0} \cong \widetilde{Z}_{j_0}$ for one sufficiently large j_0.

The main tool for this is the following obstruction-theory: Suppose we know $Z_{j_0} \cong \widetilde{Z}_{j_0}$. What is the obstruction against extending this isomorphism to one between $Z_{j_0} + E_l$ and $\widetilde{Z}_{j_0} + \widetilde{E}_l$? To get the obstruction we first show that we can extend this isomorphism locally. This is trivial in the analytic category because one may choose every open set isomorphic to the zero-set of y^{n_l} or $x^{n_j}y^{n_l}$ there. But in the algebraic world we have to work for this.

After we know that we can extend the isomorphism locally we can transfer results of Grauert ([Gra62]), Laufer ([Lau71]) and Tjurina ([Tju68]) into the algebraic category. From this we get the obstruction-theory, which tells us when the local extensions of isomorphism glue to a global one.

If we have now a $(n_{1,j}, \ldots, n_{n,j}) \geq (n_{1,j_0}, \ldots, n_{n,j_0})$, then by adding one E_{l_i} respectively $\widetilde{E}_{l_i}$ at a time, we can get a new n-tuple $(n_{1,j'}, \ldots, n_{n,j'}) \geq (n_{1,j}, \ldots, n_{n,j})$. We show that one can choose the $n_{l,j'}$ and l_i in a way that the obstruction space is trivial in every step. Thus $Z_{j_0} \cong \widetilde{Z}_{j_0}$ implies $Z_{j'} \cong \widetilde{Z}_{j'}$ which then implies $Z_j \cong \widetilde{Z}_j$.

Setting for this section By Theorem 2.52 we may replace S with an algebraization of it. Then we know that the exceptional divisor is a local complete intersection in a regular, two-dimensional k-scheme of finite type, or, because k is algebraically closed, in a smooth, two-dimensional k-scheme. So we may reformulate the question above as follows:

Given a smooth, two-dimensional k-scheme X and an effective divisor as one of a good desingularization $B = \sum_{l=1}^{n} n_l B_l$ with $B_l = \operatorname{Spec}(\mathcal{O}_X/\mathcal{I}_l)$ regular, and $B = \operatorname{Spec}(\mathcal{O}_X/\mathcal{I}_B)$ with $\mathcal{I}_B = \prod_{l=1}^{n} \mathcal{I}_l^{n_l}$. Suppose further we have a second smooth, two-dimensional k-scheme $\widetilde{X}$ with an effective divisor $\widetilde{B} = \sum_{l=1}^{n} n_l \widetilde{B}_l$ and we know for a choice of $0 < n'_l \leq n_l$ that $C = \sum_{l=1}^{n} n'_l B_l$ is as scheme isomorphic to $\widetilde{C} = \sum_{l=1}^{n} n'_l \widetilde{B}_l$. Is then B isomorphic to $\widetilde{B}$?

3.1 Preliminaries

Before we can work on the answer to the previous question, we need to introduce Čech $\check{H}^1$ for sheaves of non-abelian groups. Also to simplify the proofs we recall/discuss some properties of the push-forward for sheaves along closed immersion. Finally for the local question we have to cite some results of Illusie on extensions of algebras.

3.1.1 H^1 for sheaves of non-abelian groups

Let X be a topological space and $\mathcal{F}$ a sheaf of (not necessary abelian) groups on X. If $\mathcal{F}$ is abelian, we have cohomology groups $H^i(X,\mathcal{F})$, which are defined as right-derived of the global section functor. If $\mathcal{F}$ is non-abelian, we have no such description, but in the abelian case under some additional conditions, we know that we have $H^i(X,\mathcal{F}) = \check{H}^i(X,\mathcal{F})$, where the second term is the Čech-cohomology. Now one can go through the construction of $\check{H}^i$ and reformulate it for non-abelian sheaves. Then one gets $\check{H}^0 = H^0$ as usual as the global sections, and a pointed set $\check{H}^1(X,\mathcal{F})$. For $i > 1$ one may construct some objects $H^i(X,\mathcal{F})$, but this is complicated, and we do not need them, so we only tread the case $\check{H}^1$ for non-abelian sheaves.

We want to recall the construction of $\check{H}^i(X,\mathcal{F})$ for $\mathcal{F}$ abelian. For this we follow partly [Liu02], Section 5.2.1 respectively [Ser55], §3.

Let X be a topological space, $\mathcal{U} = \{(U_i)\}_{i\in I}$ be an open covering of X with I a totally ordered set. For $i_0,...,i_n \in I$ we set

$$U_{i_0...i_n} = U_{i_0} \cap ... \cap U_{i_n}.$$

Then we define for every $n \geq 0$ the *n-cochains* (of $\mathcal{U}$ in $\mathcal{F}$) as

$$C^n(\mathcal{U},\mathcal{F}) = \prod_{\substack{(i_0,...,i_n)\in I^{n+1} \\ i_0<...<i_n}} \mathcal{F}(U_{i_0...i_n}).$$

Now we define a differential d_n from $C^n(\mathcal{U},\mathcal{F})$ to $C^{n+1}(\mathcal{U},\mathcal{F})$ as

$$(d_nf)_{i_0...i_{n+1}} = \sum_{l=0}^{n+1}(-1)^l f_{i_0...\widehat{i_l}...i_{n+1}}|_{U_{i_0...i_{n+1}}}$$

where, as usual, '$\widehat{i_l}$' means that we remove the index i_l. Also we omit the restriction to $U_{i_0...i_{n+1}}$. A direct calculation shows that we have $d_{n+1}d_n = 0$ so we get a complex $C^\bullet(\mathcal{U},\mathcal{F})$ and we define

$$\check{H}^n(\mathcal{U},\mathcal{F}) = H^n(C^\bullet(\mathcal{U},\mathcal{F})).$$

Or in other words: If we define the *n-cocycle* as

$$Z^n(\mathcal{U},\mathcal{F}) = \ker(d_n)$$

and the *n-coboundaries* as

$$B^n(\mathcal{U},\mathcal{F}) = \operatorname{im}(d_{n-1}),$$

then we have

$$\check{H}^n(\mathcal{U},\mathcal{F}) = Z^n(\mathcal{U},\mathcal{F})/B^n(\mathcal{U},\mathcal{F}).$$

Now we want to do the same for $\mathcal{F}$ non-abelian. To distinguish between the abelian and the non-abelian case we now write the group-law multiplicative. The definition of the $C^n(\mathcal{U},\mathcal{F})$ is independent of the commutativity of the $\mathcal{F}(U_{i_0\dots i_n})$, so we take the same definition. We also may define the d_n, but because taking the inverse is a morphism if and only if the group is abelian, these will be just set-maps, not morphisms of groups. But if we look at the "kernel" of d_0 and d_1 as set-maps, for d_0, we have $s_i s_j^{-1} = 1$ which is equivalent to the relation

$$s_i = s_j$$

so we simply define $Z^0(\mathcal{U},\mathcal{F})$ as the subgroup if elements in $C^0(\mathcal{U},\mathcal{F})$ fulfilling this relation, and because this is just the glueing condition, we get

$$\check{H}^0(\mathcal{U},\mathcal{F}) = Z^0(\mathcal{U},\mathcal{F}) = H^0(X,\mathcal{F}).$$

For $Z^1(\mathcal{U},\mathcal{F})$ the condition coming from d_1 is $f_{i_1i_2}f_{i_0i_2}^{-1}f_{i_0i_1} = 1$ which via $f_{i_0i_2}^{-1} = f_{i_1i_2}^{-1}f_{i_0i_1}^{-1}$ gives

$$f_{i_0i_1}f_{i_1i_2} = f_{i_0i_2}. \tag{3.1}$$

Now we define $Z^1(\mathcal{U},\mathcal{F})$ as the subset of elements in $C^1(\mathcal{U},\mathcal{F})$ fulfilling this relation. Finally we have to translate the quotient by $B^1(\mathcal{U},\mathcal{F})$. The relation in the abelian case may be written as

$$f_{i_0i_1} = f_{i_0} + f'_{i_0i_1} - f_{i_1}$$

which translates to the non-abelian cases as

$$f_{i_0i_1} = f_{i_0}f'_{i_0i_1}f_{i_1}^{-1} \tag{3.2}$$

and we define $\check{H}^1(\mathcal{U},\mathcal{F})$ as $Z^1(\mathcal{U},\mathcal{F})$ divided by this equivalence relation. Then $\check{H}^1(\mathcal{U},\mathcal{F})$ is not a group, but we have a distinguished element $*$ given by $1 \in U_{i_0i_1}$, so $\check{H}^1(\mathcal{U},\mathcal{F})$ is a pointed set.

Before we start with the usual stuff of passing to a refinement and to the inductive limit we want to discuss our definition of n-cochains. One alternate definition of n-cochains is

$$C'^n(\mathcal{U},\mathcal{F}) = \prod_{(i_0,\dots,i_n)\in I^{n+1}} \mathcal{F}(U_{i_0\dots i_n}),$$

and every following definition as above, just with $C^n(\mathcal{U},\mathcal{F})$ replaced by $C'^n(\mathcal{U},\mathcal{F})$.

For $\mathcal{F}$ abelian it is well known that starting with this definition leads to canonical isomorphic $\check{H}^i(\mathcal{U},\mathcal{F})$, see for example [Liu02], Corollary 5.2.4. But for $\mathcal{F}$ non-abelian, we have not found any reference for this. Now suppose that $\mathcal{F}$ is non-abelian, then clearly $\check{H}^0(\mathcal{U},\mathcal{F}) \cong H^0(X,\mathcal{F})$ with both definitions. For $Z^1(\mathcal{U},\mathcal{F})$ we have a look at the relation (3.1): If take $i_0 = i_1 = i_2$ we get $f_{i_0i_0} = 1$ and from this with $i_0 = i_2$ we get $f_{i_0i_1} = f_{i_1i_0}^{-1}$, so $Z^1(\mathcal{U},\mathcal{F})$ and $Z'^1(\mathcal{U},\mathcal{F})$ are isomorphic sets. This isomorphism can be chosen such that it is compatible with the relation (3.2), so the resulting quotients are isomorphic. So $\check{H}^i(\mathcal{U},\mathcal{F})$ does not depend on the choice between this two definitions of a n-cochain.

The next steps are standard again and we need only the results, so we just quote them: Let $\mathcal{U} = \{V_j\}_{j\in J}$ a refinement of $\mathcal{U}$, that is we have a map $\sigma : J \to J$ and $V_j \subset U_{\sigma(j)}$ for all j. Then by [Gro55], (5.1.6) in the non-abelian and [Liu02], Lemma 5.2.8 in the abelian case, we have a map $\check{H}^1(\mathcal{U},\mathcal{F}) \to \check{H}^1(\mathcal{U},\mathcal{F})$. This maps make the set $\{\check{H}^1(\mathcal{U},\mathcal{F})\}_{\mathcal{U}}$ a direct system, and we define

$$\check{H}^1(X,\mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U},\mathcal{F}).$$

Note that $\check{H}^1$ depends on the chosen topology on X.

Now we need two more facts. First we want to state that, under some conditions, the Čech-cohomology groups are the same as the cohomology groups if we define $H^n(X,\mathcal{F})$ as the right-derived functor of Γ:

Theorem 3.1. *Let X be a noetherian, separated scheme and $\mathcal{F}$ be a quasi-coherent sheaf on X, then for all n, we have*

$$\check{H}^1(X,\mathcal{F}) \cong H^1(X,\mathcal{F}).$$

where $\check{H}^1$ was calculated using the Zariski topology.

Proof. Let $\mathcal{U}$ be an open affine covering of X. By [Liu02], Theorem 5.2.19 we have $\check{H}^1(X,\mathcal{F}) \cong \check{H}^1(\mathcal{U},\mathcal{F})$ and by [Har77], Theorem III 4.5 we have $\check{H}^1(\mathcal{U},\mathcal{F}) \cong H^1(X,\mathcal{F})$. □

Remark 3.2. This holds even more general for abelian sheaves on topological spaces.

Finally we need the long exact sequence in cohomology sets. For this we first need two definitions:

Definition 3.3. A map $\lambda : A \to B$ of pointed sets is a map of sets which maps the distinguished point of A to the distinguished point of B. A sequence $A \xrightarrow{\lambda} B \xrightarrow{\tau} C$ of pointed sets is exact if $\tau^{-1}(*) = \lambda(A)$, where $*$ is the distinguished point of C.

Now we get the long sequence from [Lau71], Theorem 6.5:

Theorem 3.4. *Let X be a paracompact Hausdorff space, and $1 \to \mathcal{F}' \xrightarrow{\lambda} \mathcal{F} \xrightarrow{\tau} \mathcal{F}'' \to 1$ be an exact sequence of sheaves of groups over X, then*

$$1 \longrightarrow H^0(X,\mathcal{F}') \longrightarrow H^0(X,\mathcal{F}) \longrightarrow H^0(X,\mathcal{F}'')$$
$$\xrightarrow{\delta} \check{H}^1(X,\mathcal{F}') \longrightarrow \check{H}^1(X,\mathcal{F}) \longrightarrow \check{H}^1(X,\mathcal{F}'')$$

is an exact sequence of pointed sets. For $t \in H^0(X,\mathcal{F}'')$ the image $\delta(t)$ is defined as follows: $t = \tau(s_{i_0})$ with $(s_{i_0}) \in C^0(\mathcal{U},\mathcal{F})$ for some open cover $\mathcal{U}$ of X. Then $(\delta(t))_{i_0i_1} = \tau^{-1}(s_{i_0}^{-1}s_{i_1})$ determines the equivalence class $\delta(t) \in \check{H}^1(X,\mathcal{F}')$.

Torsors There is also a more geometric interpretation of $\check{H}^1(X, \mathcal{F})$ which is sometimes helpful. We want to give a short view on this interpretation, citing [GW10], Section (11.5).

Let X be a topological space, and let $\mathcal{G}$ be a sheaf of groups on X. If $\mathcal{T}$ is a sheaf on X, then $\mathcal{T}$ is a $\mathcal{G}$-sheaf if we have morphism of sheaves $\mathcal{G} \times \mathcal{T} \to \mathcal{T}$, which is on every open $U \subset X$ a left action of $\mathcal{G}(U)$ on $\mathcal{T}(U)$. A morphism between two $\mathcal{G}$-sheaves $\mathcal{T}$ and $\mathcal{T}'$ is a morphism of sheaves such that on every open $U \subset X$ the map $\mathcal{T}(U) \to \mathcal{T}'(U)$ is $\mathcal{G}(U)$-equivariant.

We say that $\mathcal{G}(U)$ acts *simply transitively* on $\mathcal{T}(U)$ if for all $t_1, t_2 \in \mathcal{T}(U)$ there exists a unique $f \in \mathcal{G}(U)$ with $ft_1 = t_2$.

Now we get the notion of a $\mathcal{G}$-torsor:

Definition 3.5. A $\mathcal{G}$-sheaf $\mathcal{T}$ is a $\mathcal{G}$-*torsor* if it satisfies:

1. The group $\mathcal{G}(U)$ acts simply transitively on $\mathcal{T}(U)$ for every open $U \subset X$.

2. There exists an open covering $\mathcal{U}$ of X such that $\mathcal{T}(U) \neq \emptyset$ for all $U \in \mathcal{U}$.

For every $\mathcal{G}$ we have always at least one $\mathcal{G}$-torsor because $\mathcal{G}$ acts on itself by left multiplication. We call this torsor the *trivial torsor*.

Definition 3.6. Let $\mathcal{T}$ be a $\mathcal{G}$-torsor and $\mathcal{U}$ an open covering of X. We say that $\mathcal{U}$ *trivializes* T if for every $U \in \mathcal{U}$ the restricted torsor $\mathcal{T}|_U$ is isomorphic to the trivial $\mathcal{G}|_U$-torsor. Or equivalent, if every $\Gamma(U, \mathcal{T}) \neq 0$.

Now from [GW10], Proposition 11.12 we get the following description of $\check{H}^1(X, \mathcal{G})$ and $\check{H}^1(\mathcal{U}, \mathcal{G})$:

Theorem 3.7. *Let X be a topological space, $\mathcal{G}$ a sheaf of groups on X and $\mathcal{U}$ an open covering of X. Then $\check{H}^1(X, \mathcal{G})$ is isomorphic to the set of $\mathcal{G}$-torsors, and $\check{H}^1(\mathcal{U}, \mathcal{G})$ is isomorphic to the set of $\mathcal{G}$-torsors which are trivialized by $\mathcal{U}$.*

From this we get the Leray acyclicity theorem for non-abelian sheaves:

Theorem 3.8. *Let X be a topological space, and let $\mathcal{G}$ be a sheaf of groups on X. If I is a totally ordered set and $\mathcal{U} = \{(U_i)\}_{i \in I}$ is an open covering of X such that $\check{H}^1(U_i, \mathcal{G}|_{U_i}) = 0$ for all $i \in I$, then we have*

$$\check{H}^1(X, \mathcal{G}) \cong \check{H}^1(\mathcal{U}, \mathcal{G}).$$

Proof. First we note that by Proposition 5.1.1 of [Gro55] the natural map from $\check{H}^1(\mathcal{U}, \mathcal{G})$ to $\check{H}^1(X, \mathcal{G})$ is injective. So the theorem follows if we can show that it is also surjective, that is by the previous theorem, that every $\mathcal{G}$-torsor $\mathcal{T}$ is trivialized by $\mathcal{U}$. For this let $\mathcal{T}$ be any $\mathcal{G}$-torsor. Then $\mathcal{T}|_{U_i}$ is a $\mathcal{G}|_{U_i}$-torsor, and thus the trivial $\mathcal{G}|_{U_i}$-torsor by the assumption $\check{H}^1(U_i, \mathcal{G}|_{U_i}) = 0$ and again the previous theorem. □

3.1.2 Closed immersions

In the next section, we often need the push-forward of a quasi-coherent sheaf on a closed subscheme along the inclusion. So we want to collect some facts for this now. Let X be a scheme and Y a closed subscheme, we denote by ι the inclusion. Let $\mathcal{F}$ be a quasi-coherent sheaf on X and $\mathcal{G}$ one on Y.

Then by Proposition 3.2.4 and 3.3.9 of [Liu02] ι is separated and quasi-compact, so by Proposition 5.1.14 (Ibid.) $\iota_*(\mathcal{G})$ is quasi-coherent. Now assume that $\mathcal{F}_x = 0$ for all $x \in X \backslash Y$. Then by Corollaire 9.3.5 of [Gro60] we have $\mathcal{F} = \iota_*(\iota^*(\mathcal{F}))$. Also by looking at the stalks we get that the canonical map $\iota^*(\iota_*(\mathcal{G})) \to \mathcal{G}$ (Ibid. 0 (4.4.3.3)) is an isomorphism. So ι_* gives us a bijection between the quasi-coherent sheaves on X with support contained in Y and the quasi-coherent sheaves on Y.

Furthermore, for a second quasi-coherent sheaf $\mathcal{G}'$ on Y we have

$$\iota_*(\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, \mathcal{G}')) \cong \mathrm{Hom}_{\mathcal{O}_X}(\iota_*(\mathcal{G}), \iota_*(\mathcal{G}'))$$

by 0 4.2.5 of [Gro60].

Also, ι is an affine morphism, that is we can find an affine covering $\{U_i\}$ of X such that $\iota^{-1}(U_i)$ is affine. By Corollaire 1.3.3 of [Gro61] this means that $H^i(X, \iota_*(\mathcal{G}))$ is isomorphic to $H^i(Y, \mathcal{G})$ for all $i \geq 0$.

These facts justify the standard convention that one writes $\mathcal{G}$ for $\iota_*(\mathcal{G})$. We follow this convention. Only if we think it helps the understanding we write explicitly $\iota_*(\mathcal{G})$.

If we only assume $\mathcal{G}$ to be a sheaf of (not necessarily abelian) groups, then for $i = 0, 1$ we have still

$$\check{H}^i(X, \iota_*(\mathcal{G})) \cong \check{H}^i(Y, \mathcal{G}). \tag{3.3}$$

This follows for $i = 0$ directly from the definition. For $i = 1$ one uses that every open covering of Y comes from an open covering of X, and by the definition of $\iota_*(\mathcal{G})$ the calculation of the left side only depends on the opens in X with non-empty intersection with Y.

3.1.3 Extensions of algebras

Let $j : Y \to S$ be a morphism of schemes and $\mathcal{L}$ an $\mathcal{O}_Y$-module. We want to classify the schemes $\widetilde{Y}/S$ so that Y is a closed sub-scheme of $\widetilde{Y}$ with ideal sheaf $\mathcal{L}$ such that we have $\mathcal{L}^2 = 0$ in $\mathcal{O}_{\widetilde{Y}}$. To do this, we first want to construct a group

$$\mathrm{Exal}_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{L})$$

classifying sequences

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{O}_{\widetilde{Y}} \longrightarrow \mathcal{O}_Y \longrightarrow 0 \tag{3.4}$$

of $j^{-1}(\mathcal{O}_S)$-algebras. For this we recall the construction in [Ill71], III 1.1: First we consider the category

$$\underline{\mathrm{Exal}}_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{L})$$

with objects sequences of the form (3.4) and $j^{-1}(\mathcal{O}_S)$-algebra-morphism of this sequences of the form

$$\begin{array}{ccccc} \mathcal{L} & \longrightarrow & \mathcal{O}_{\widetilde{Y}} & \longrightarrow & \mathcal{O}_Y \\ \downarrow \text{id} & & \downarrow & & \downarrow \text{id} \\ \mathcal{L} & \longrightarrow & \mathcal{O}_{\widetilde{Y}'} & \longrightarrow & \mathcal{O}_Y \end{array} . \tag{3.5}$$

We denote by $\mathrm{Exal}_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{L})$ the set of objects of $\underline{\mathrm{Exal}}_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{L})$ modulo isomorphism. Then by [Ill71], III 1.1.5 this set carries a group structure and this is the group we want.

Now we need the derived category of $\mathcal{O}_Y$-modules $D(Y)$ as a tool. All we need to know about this category is that it arises from $Ch(Y)$, the category of chain complexes of $\mathcal{O}_Y$-modules, in the following way: First one goes to the quotient category $K(Y)$, where the morphisms are equivalence-classes of morphisms between chain complexes modulo chain homotopy. Then one gets $D(Y)$ from $K(Y)$ by inverting quasi-isomorphisms.

The main tool to understand the group $\underline{\mathrm{Exal}}_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{L})$ is Illusies "Théorème Fondamental" ([Ill71], III 1.2.3). For this theorem we need the cotangent-complex of Y over S, which we call $L^\bullet_{Y/S}$. The cotangent-complex is an object in $D(Y)$. We do not need the formal definition, for which we refer to [Ill71], II 1.2.3. We only need $L^\bullet_{Y/S}$ in the following two special cases, which are Proposition III 3.1.2 (ii) and Corollaire III 3.2.7 of [Ill71]:

Lemma 3.9. *1. If Y is a smooth S-scheme, then we have*

$$L^\bullet_{Y/S} \overset{q.\ i.}{\cong} \Omega^1_{Y/S}$$

with $\Omega^1_{Y/S}$ in degree 0.

2. If we have a smooth S-scheme X and a regular embedding $\iota : Y \to X$ with kernel I. Then we have

$$L^\bullet_{Y/S} \overset{q.\ i.}{\cong} (I/I^2 \xrightarrow{d\otimes 1} \iota^*(\Omega^1_{X/S}))$$

with $\iota^*(\Omega^1_{X/S})$ in degree 0.

To understand $L^\bullet_{Y/S}$ further, we need the *Hyperext* functor $\mathbb{E}\mathrm{xt}^i_{\mathcal{O}_Y}$, which is the right derived functor of $\mathrm{Hom}_{D(Y)}(\ \ ,\ \ )$. By (2.2) on Page 50 of [Huy06] we have

$$\mathbb{E}\mathrm{xt}^i_{\mathcal{O}_Y}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathrm{Hom}_{D(Y)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[i]).$$

Then the Théorème Fondamental gives us an isomorphism

$$\mathrm{Exal}_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{L}) \cong \mathbb{E}\mathrm{xt}^1_{D(Y)}(L^\bullet_{Y/S}, \mathcal{L}).$$

Using the quasi-isomorphisms above we get

$$\mathrm{Exal}_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{L}) \cong \mathbb{E}\mathrm{xt}^1_{D(Y)}(\Omega^1_{Y/S}, \mathcal{L})$$

if Y is smooth, respectively

$$\mathrm{Exal}_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{L}) \cong \mathbb{E}\mathrm{xt}^1_{D(Y)}(I/I^2 \xrightarrow{d\otimes 1} \iota^*(\Omega^1_{X/S}), \mathcal{L})$$

if Y is regularly embedded. Now we get:

Lemma 3.10. *If Y is smooth, then we have* $\mathrm{Exal}_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{L}) = 0$.

Proof. We have $\mathrm{Exal}_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{L}) = \mathbb{E}\mathrm{xt}^1_{D(Y)}(\Omega^1_{Y/S}, \mathcal{L}) = \mathrm{Hom}^1_{D(Y)}(\Omega^1_{Y/S}, \mathcal{L}[1])$ where $\Omega^1_{Y/S}$ and $\mathcal{L}$ are complexes concentrated in degree 0, so $\mathcal{L}[1]$ is concentrated in degree -1. So by the definition of morphism in $Ch(Y)$ we have $\mathrm{Hom}_{C(Y)}(\Omega^1_{Y/S}, \mathcal{L}[1]) = 0$ and both complexes are not quasi-isomorphic, so we have also $\mathrm{Hom}_{D(Y)}(\Omega^1_{Y/S}, \mathcal{L}[1]) = 0$. □

If Y is regularly embedded into some smooth scheme, we can use the spectral-sequence for Hyperext. By Example 2.70 of [Huy06] we have a spectral-sequence

$$\mathbb{E}\mathrm{xt}^p_{D(Y)}(\mathcal{H}^{-q}(\mathcal{F}^\bullet), \mathcal{G}^\bullet) \Rightarrow \mathbb{E}\mathrm{xt}^{p+q}_{D(Y)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$$

from the construction of $\mathbb{E}\mathrm{xt}_{\mathcal{O}_Y}$ as right derived functor. In our situation $\mathcal{F}^\bullet = L^\bullet_{Y/S}$ is concentrated in degree -1 and 0, so we are in the well known "two rows" case of a spectral sequence. This gives us a long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_Y}(\Omega^1_{Y/S}, \mathcal{L}) \longrightarrow \mathbb{E}\mathrm{xt}^1_{D(Y)}(L^\bullet_{Y/S}, \mathcal{L}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_Y}(\ker(d\otimes 1), \mathcal{L}) \\ \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_Y}(\Omega^1_{Y/S}, \mathcal{L}) \longrightarrow \dots \end{aligned} \tag{3.6}$$

where the outer terms are just ordinary Ext-groups.

3.2 Local extensions of isomorphism

Now we want to show, that we can always extend an isomorphism locally. Remember, this is trivial in the analytic category.

Theorem 3.11. *Let B, $\widetilde{B}$, C and $\widetilde{C}$ be as described in the setting for this section. Suppose that $\varphi : C \to \widetilde{C}$ is the isomorphism. Then for every $x \in B$ there exists an open $x \in U_x \subset B$ and an isomorphism $\psi : U_x \to \varphi(U_x)$ such that $\psi|_{C\cap U_x} = \varphi|_{U_x}$.*

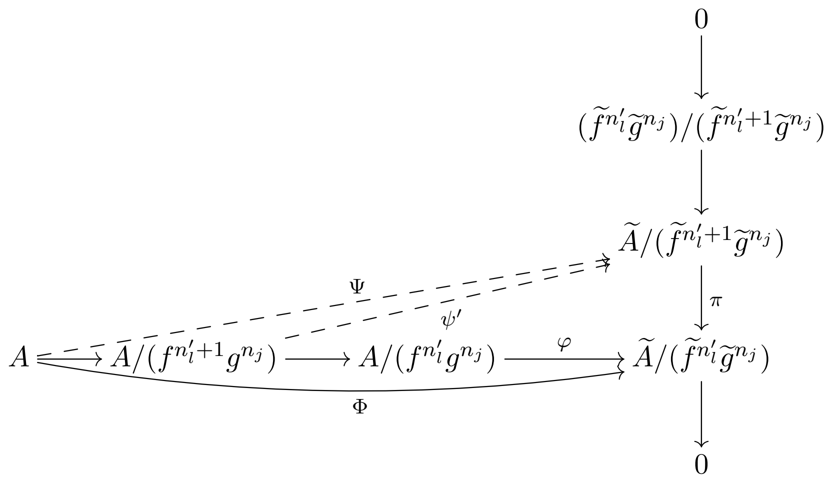
We split the proof of this theorem into two cases. The setting is always the following: Suppose that we have $x \in B_l$, then we can suppose that $U_x = \mathrm{Spec}(A) \cap B$ where $\mathrm{Spec}(A)$ is an affine subset of X, so A is a regular k-algebra of finite type, $B_l \cap U = \mathrm{Spec}(A/f)$ and if we have an other component B_j with $x \in B_j$, then $B_j \cap U = \mathrm{Spec}(A/g)$. So we get $U_x = \mathrm{Spec}(A/(f^{n_l} g^{n_j}))$ where we set $g = 1$ if we did not have a B_j. On $\varphi(U_x)$ we may suppose the same with $\widetilde{f}, \widetilde{g} \in \widetilde{A}$. Then we have, by abuse of notation, $\varphi : A/(f^{n'_l} g^{n'_j}) \to \widetilde{A}/(\widetilde{f}^{n'_l} \widetilde{g}^{n'_j})$. Further, if $n'_l > 1$ or $g \neq 1$, then we may choose $\widetilde{f}$ and $\widetilde{g}$ in such a way that the φ maps the residue class of f to the residue class of $\widetilde{f}$, and the one of g to the one of $\widetilde{g}$.

Now it suffices to show the theorem for the case $n_l = n'_l + 1$ and $n_j = n'_j$.

First we do the cases that $\mathrm{Spec}(A/(fg))$ is not smooth:

Lemma 3.12. *Theorem 3.11 is true for $g \neq 1$.*

Proof. We look at the following commutative diagram of A-modules:



(3.7)

Since A is regular and of finite type, and $k = \overline{k}$, Spec(A) is smooth, we get Ψ from Φ by Definition 2.1. First we want to show that we can choose Ψ such that it maps f and g to the residue class of $\widetilde{f}$ and $\widetilde{g}$. The ideal $(\widetilde{f}^{n_l'}\widetilde{g}^{n_j})/(\widetilde{f}^{n_l'+1}\widetilde{g}^{n_j})$ is an $\widetilde{A}/(\widetilde{f}^{n_l'}\widetilde{g}^{n_j})$-module, and so using Φ also an A-module. Let now ∂ be any k-derivation from A to $(\widetilde{f}^{n_l'}\widetilde{g}^{n_j})/(\widetilde{f}^{n_l'+1}\widetilde{g}^{n_j})$, that is a k-linear map fulfilling the Leibniz rule $\partial(ab) = \Phi(a)\partial(b) + \Phi(b)\partial(a)$ for all $a, b \in A$. Then for $\Psi' = \Psi + \partial$ we have $\Psi'(ab) = \Psi'(a)\Psi'(b)$ by a straight-forward calculation using the Leibniz rule and $\Psi \cdot \partial = \Phi \cdot \partial$ in $(\widetilde{f}^{n_l'}\widetilde{g}^{n_j})/(\widetilde{f}^{n_l'+1}\widetilde{g}^{n_j})$.

So Ψ' is also a lifting of Φ. Now $\Psi(f)-\widetilde{f}$ and $\Psi(g)-\widetilde{g}$ in are in the kernel of π and thus in $(\widetilde{f}^{n_l'}\widetilde{g}^{n_j})/(\widetilde{f}^{n_l'+1}\widetilde{g}^{n_j})$, so we can choose Ψ in the described way if we find a derivation ∂ such that $\partial(f) = -(\Psi(f) - \widetilde{f})$ and $\partial(g) = -(\Psi(g) - \widetilde{g})$. Now we use the standard identification between derivations and elements of $\mathrm{Hom}_A(\Omega^1_{A/k}, (\widetilde{f}^{n_l'}\widetilde{g}^{n_j})/(\widetilde{f}^{n_l'+1}\widetilde{g}^{n_j}))$. If y is is the singular point of Spec$(A/(fg))$, then by [Liu02], Proposition 9.1.8 $\Omega^1_{A_x/k}$ is generated by $d_x f$ and $d_x g$, but $\Omega^1_{A/k}$ is finitely generated and quasi-coherent, so maybe after shrinking U_x we may assume that df and dg generate $\Omega^1_{A/k}$. This shows the existence of a derivation ∂ with $\partial(f) = -(\Psi(f) - \widetilde{f})$ and $\partial(g) = -(\Psi(g) - \widetilde{g})$ and thus we can assume $\Psi(f) = \widetilde{f}$ and $\Psi(g) = \widetilde{g}$ in $\widetilde{A}/(\widetilde{f}^{n_l'+1}\widetilde{g}^{n_j})$.

Now we have $\Psi(f^{n_l'+1}g^{n_j}) = 0$ and we get ψ' by the universal property of the kernel. If we do the same for φ^{-1} and $\widetilde{A}$ we get $\widetilde{\psi} : \widetilde{A}/(\widetilde{f}^{n_l'+1}\widetilde{g}^{n_j}) \to A/(f^{n_l'+1}g^{n_j})$.

From the commutativity of Diagram (3.7) we get for all $a \in A/(f^{n_l'+1}g^{n_j})$ and all $b \in \widetilde{A}/(\widetilde{f}^{n_l'+1}\widetilde{g}^{n_j})$:

$$\widetilde{\psi} \circ \psi'(a) = a + f^{n_l'}g^{n_j}\partial(a) \text{ and } \psi' \circ \widetilde{\psi}(b) = b + \widetilde{f}^{n_l'}\widetilde{g}^{n_j}\widetilde{\partial}(b)$$

Additionally we have

$$\widetilde{\psi}(b) + f^{n_l'}g^{n_j}\partial(\widetilde{\psi}(b)) = \widetilde{\psi} \circ \psi' \circ \widetilde{\psi}(b) = \widetilde{\psi}(b + \widetilde{f}^{n_l'}\widetilde{g}^{n_j}\widetilde{\partial}(b)) = \widetilde{\psi}(b) + \widetilde{\psi}(\widetilde{f}^{n_l'}\widetilde{g}^{n_j})\widetilde{\psi}(\widetilde{\partial}(b))$$

that is $f^{n'_l}g^{n_j}\partial(\widetilde{\psi}(b)) = \widetilde{\psi}(\widetilde{f}^{n'_l}\widetilde{g}^{n_j})\widetilde{\psi}(\widetilde{\partial}(b))$, and by the choice of Ψ:

$$\widetilde{\psi}(\psi'(f^{n'_l}g^{n_j})) = f^{n'_l}g^{n_j} \text{ and } \psi'(\widetilde{\psi}(\widetilde{f}^{n'_l}\widetilde{g}^{n_j})) = \widetilde{f}^{n'_l}\widetilde{g}^{n_j}$$

Now we set for $a \in A/(f^{n'_l+1}g^{n_j})$:

$$\xi(a) = a - f^{n'_l}g^{n_j}\partial(a) \text{ and } \psi = \psi' \circ \xi$$

then we get:

$$\begin{aligned}
\widetilde{\psi}(\psi(a)) &= \widetilde{\psi}(\psi'(a - f^{n'_l}g^{n_j}\partial(a))) \\
&= a + f^{n'_l}g^{n_j}\partial(a) - \widetilde{\psi}(\psi'(f^{n'_l}g^{n_j}))(\partial(a) + f^{n'_l}g^{n_j}\partial(\partial(a))) \\
&= a + f^{n'_l}g^{n_j}\Big(\partial(a) - \partial(a) - f^{n'_l}g^{n_j}\partial(\partial(a))\Big) \\
&= a
\end{aligned}$$

and using $f^{n'_l}g^{n_j}\partial(\widetilde{\psi}(b)) = \widetilde{\psi}(\widetilde{f}^{n'_l}\widetilde{g}^{n_j})\widetilde{\psi}(\widetilde{\partial}(b))$ we get

$$\begin{aligned}
\psi(\widetilde{\psi}(b)) &= \psi'(\widetilde{\psi}(b) - f^{n'_l}g^{n_j}\partial(\widetilde{\psi}(b)) = \psi'(\widetilde{\psi}(b)) - \psi'(f^{n'_l}g^{n_j}\partial(\widetilde{\psi}(b))) \\
&= b + \widetilde{f}^{n'_l}\widetilde{g}^{n_j}\Big(\widetilde{\partial}(b) - \widetilde{\partial}(b) - \widetilde{f}^{n'_l}\widetilde{g}^{n_j}\widetilde{\partial}(\widetilde{\partial}(b))\Big) \\
&= b
\end{aligned}$$

So ψ is the isomorphism we need. □

If $\mathrm{Spec}(A/(fg))$ is smooth we have:

Lemma 3.13. *Theorem 3.11 is true if $g = 1$.*

Proof. The argumentation of the previous proof holds also for $g = 1$ if $n'_l \geq 2$, we only have to replace dg with some dg' such that $\Omega^1_{A/k}$ are generated by df and dg' at one place. So we have only to do the case $n'_l = 1$. In this case we have

$$0 \longrightarrow f/f^2 \longrightarrow A/(f^2) \longrightarrow A/(f) \longrightarrow 0 \ \in \mathrm{Exal}_k(A/(f), f/f^2)$$

Now f/f^2 respectively $\widetilde{f}/\widetilde{f}^2$ are free A/f $(\widetilde{A}/(\widetilde{f}))$ module of rank one. So the isomorphism $\varphi : A/f \to \widetilde{A}/(\widetilde{f})$ gives us

$$0 \longrightarrow f/f^2 \longrightarrow \widetilde{A}/(\widetilde{f}^2) \longrightarrow A/(f) \longrightarrow 0 \ \in \mathrm{Exal}_k(A/(f), f/f^2)$$

but $\mathrm{Spec}(A/(f))$ is smooth, so by Lemma 3.10 we have $\mathrm{Exal}_k(A/(f), f/f^2) = 0$ and thus we have an isomorphism $\psi : A/(f^2) \to \widetilde{A}/(\widetilde{f}^2)$, and $\psi|_{\mathrm{Spec}\,A/(f))} = \varphi$. □

This shows that in our situation local thickening of isomorphism is always possible. We want to emphasize that the extendibility of isomorphisms really depends on the fact that we only look at such extensions of algebras which embed into a smooth scheme. If we take for example $Y = \operatorname{Spec}(k[x,y]/(y^2))$ embedded into $X = \operatorname{Spec}(k[x,y])$, and calculate $\operatorname{Exal}_k(\mathcal{O}_Y, (y^2)/(y^3))$ using the sequence (3.6), then we have the following result: If $\operatorname{Char}(k) \neq 2$ then $\operatorname{Ext}^1_k(\Omega^1_{Y/k}, (y^2)/(y^3))) = k[x]$, and also $d \otimes 1 : (y^2)/y^4) \to \Omega^1_{X/k}|_Y$ has a non-trivial kernel (generated by y^3) so the third term in (3.6) is also non-zero. For $\operatorname{Char}(k) = 2$ the sequence is quite different: In this situation we have $\Omega^1_{Y/k} \cong \Omega^1_{X/k}|_Y$ and $d \otimes 1$ is the zero-mapping, so the first and fourth term of (3.6) vanish, and $\operatorname{Exal}_k(\mathcal{O}_Y, (y^2)/(y^3)) \cong \operatorname{Hom}_{\mathcal{O}_Y}((y^2)/(y^4), (y^2)/(y^3)) \neq 0$.

So we see that $\operatorname{Exal}_k(\mathcal{O}_Y, (y^2)/(y^3))$ is not trivial, but all extensions which can be embedded into a smooth, two-dimensional scheme are isomorphic as schemes.

3.3 Reducing the obstruction to cohomology

As discussed in the beginning of this section, we are in the following situation: B and C are closed subschemes of a regular, two-dimensional scheme X, given by the ideal sheaves $\mathcal{I}_B = \prod_{l=1}^{n} \mathcal{I}_l^{n_l}$ and $\mathcal{I}_C = \prod_{l=1}^{n} \mathcal{I}_l^{n'_l}$ with $0 < n'_l \leq n_l$. That is, we have an exact sequence

$$0 \longrightarrow \mathcal{I}_C/\mathcal{I}_B \longrightarrow \mathcal{O}_X/\mathcal{I}_B \longrightarrow \mathcal{O}_X/\mathcal{I}_C \longrightarrow 0$$

We denote the inclusion $C \hookrightarrow B$ by $\iota_{C,B}$ and the projection $\mathcal{O}_B \to \mathcal{O}_C$ by $\pi_{C,B}$.

Now we want to construct a sheaf classifying automorphism α of B which are the identity on C, that is the sections of this sheaf are not automorphisms of B, but of the $\mathcal{O}_X$-algebra $\mathcal{O}_X/\mathcal{I}_B$. So by the well known contravariant correspondence between automorphisms of B and automorphisms of $\mathcal{O}_X/\mathcal{I}_B$, the sections of this sheaf are the opposite group to the group of automorphisms of B. The identity condition restricted to C then translates to the commutativity of the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_C/\mathcal{I}_B & \longrightarrow & \mathcal{O}_X/\mathcal{I}_B & \longrightarrow & \mathcal{O}_X/\mathcal{I}_C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha|_{\mathcal{I}_C/\mathcal{I}_B}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathcal{I}_C/\mathcal{I}_B & \longrightarrow & \mathcal{O}_X/\mathcal{I}_B & \longrightarrow & \mathcal{O}_X/\mathcal{I}_C & \longrightarrow & 0 \end{array} \tag{3.8}$$

From this commutativity we get, that α maps $\mathcal{I}_C/\mathcal{I}_B$ necessarily to $\mathcal{I}_C/\mathcal{I}_B$ and using the snake-lemma we get that the restriction $\alpha|_{\mathcal{I}_C/\mathcal{I}_B}$ must be already surjective.

Now we define the pre-sheaf $\mathcal{A}ut_C(B)$ whose sections for an open $U \subset B$ are defined as the set of all isomorphisms $\alpha : \Gamma(U, \mathcal{O}_B|_U) \to \Gamma(U, \mathcal{O}_B|_U)$ such that α is the identity on the set U and for all $x \in U$ we have $\alpha_x((\mathcal{I}_C/\mathcal{I}_B)_x) = (\mathcal{I}_C/\mathcal{I}_B)_x$ and α induces the identity on $\mathcal{O}_{C,x}$. Then $\Gamma(U, \mathcal{A}ut_C(B))$ together with the composition is a group. By the discussion above the automorphism making the diagram above commutative are exactly the global sections of $\mathcal{A}ut_C(B)$. Also the pre-sheaf $\mathcal{A}ut_C(B)$ is a sheaf.

Now the proof of Theorem 6.6. of [Lau71] applies without change in our situation, so we get:

Theorem 3.14. *Let $C \subset B$ and $\widetilde{C} \subset \widetilde{B}$ be two schemes with $\psi : C \to \widetilde{C}$ an isomorphism and assume that we can extend ψ locally. Then the local extensions determine a class $o \in \check{H}^1(B, \mathcal{A}ut_C(B))$, and $o = *$ if and only if we can glue the local extensions to a global isomorphism $\Psi : B \to \widetilde{B}$.*

The other direction is also true: If two schemes become isomorphic after thickening, the are isomorphic. This is just a special case of Theorem 2.26.

Now, under some additional conditions, the pointed set $\check{H}^1(B, \mathcal{A}ut_C(B))$ is actually computable, and is in most cases even a group.

The sheaf $\mathcal{A}ut_C(B)$ has a subsheaf $\mathcal{A}ut_{C,\mathcal{I}_C/\mathcal{I}_B}(B)$ of normal subgroups given by

$$\Gamma(U, \mathrm{Aut}_{C,\mathcal{I}_C/\mathcal{I}_B}) = \{\alpha \in \Gamma(U, \mathrm{Aut}_C) \mid \varphi_x \text{ is the identity on } (\mathcal{I}_C/\mathcal{I}_B)_x \forall x \in U\}$$

and if we denote by Q the quotient sheaf we get an exact sequence of sheaves of groups:

$$1 \longrightarrow \mathcal{A}ut_{C,\mathcal{I}_C/\mathcal{I}_B}(B) \longrightarrow \mathcal{A}ut_C(B) \longrightarrow Q \to 1 \tag{3.9}$$

Now as first condition, we assume $\mathcal{I}_C^2 \subset \mathcal{I}_B$, that is for every open $U \in X$ we have $(\mathcal{I}_C(U))^2 \subset \mathcal{I}_B(U)$ in $\mathcal{O}_X(U)$. Then like [Lau71], Proposition 6.4, we can construct a morphism λ from the sheaf of groups $\mathcal{A}ut_{C,\mathcal{I}_C/\mathcal{I}_B}(B)$ to a coherent $\mathcal{O}_B$-module, which turns out to be an isomorphism.

Lemma 3.15. *If $\mathcal{I}_C^2 \subset \mathcal{I}_B$, then we have a morphism*

$$\lambda : \mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{C/k}, \mathcal{I}_C/\mathcal{I}_B) \longrightarrow \mathcal{A}ut_{C,\mathcal{I}_C/\mathcal{I}_B}(B).$$

Proof. We define λ for every open $U \subset B$. For every $\xi \in \mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{C/k}, \mathcal{I}_C/\mathcal{I}_B)(U)$ we define

$$\begin{aligned} \lambda_U(\xi) : \mathcal{O}_B(U) &\longrightarrow \mathcal{O}_B(U) \\ f &\longmapsto f + \xi(d(f)) \end{aligned}$$

where d is the composition of the projection $\pi_{C,B} : \mathcal{O}_B \to \mathcal{O}_C$ with the differential d_C on C. For reasons of readability we write $\pi_{C,B}$ also for $\pi_{C,B}|_U$ if $U \subset C$ is open. Now we show that λ_U maps to $\mathcal{A}ut_{C,\mathcal{I}_C/\mathcal{I}_B}(B)(U)$. Then, because this construction commutes with restrictions, the λ_U glue to a morphism λ of sheaves.

First we have to show that for every $\xi \in \mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{C/k}, \mathcal{I}_C/\mathcal{I}_B)(U)$, we have

$$\lambda_U(\xi) \in \mathcal{H}om_{\mathcal{O}_B}(\mathcal{O}_B, \mathcal{O}_B)(U).$$

For this take $f, g \in \mathcal{O}_B(U)$, we have clearly $\lambda_U(\xi)(f+g) = \lambda_U(\xi)(f) + \lambda_U(\xi)(g)$, so it remains to show that $\lambda_U(\xi)(f \cdot g) = \lambda_U(\xi)(f) \cdot \lambda_U(\xi)(g)$.

We calculate both sides:

$$\begin{aligned} \lambda_U(\xi)(f \cdot g) &= fg + \xi\big(\pi_{C,B}(f) \cdot d(g) + \pi_{C,B}(g) \cdot d(f)\big) \\ &= fg + \pi_{C,B}(f)\xi(d(g)) + \pi_{C,B}(g)\xi(d(f)) \end{aligned}$$

and

$$\begin{aligned}\lambda_U(\xi)(f)\cdot\lambda_U(\xi)(g) &= (f+\xi(d(f)))\cdot(g+\xi(d(g)))\\ &= fg+f\xi(d(g))+g\xi(d(f))+\xi(d(g))\xi(d(f))\end{aligned}$$

But the last summand is zero in $\mathcal{O}_B(U)=\mathcal{O}_X/\mathcal{I}_B(U)$, because

$$\xi(d(g))\xi(d(f))\in\mathcal{I}_C^2(U)\subset\mathcal{I}_B(U)$$

and for the same reason we have $\pi_{C,B}(f)\cdot\xi(d(g))=f\cdot\xi(d(g))$, and we have, as wanted, $\lambda_U(\xi)\in\mathcal{H}om_{\mathcal{O}_B}(\mathcal{O}_B,\mathcal{O}_B)(U)$.

Now we want to prove that λ_U is a morphism of monoids from the additive group $\mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{C/k},\mathcal{I}_C/\mathcal{I}_B)(U)$ to the monoid $\mathcal{H}om_{\mathcal{O}_B}(\mathcal{O}_B,\mathcal{O}_B)(U)$ with the composition. For this take $\xi_1,\xi_2\in\mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{C/k},\mathcal{I}_C/\mathcal{I}_B)(U)$, then we have:

$$\begin{aligned}\lambda_U(\xi_1)(\lambda_U(\xi_2)(f)) &= \lambda_U(\xi_1)(f+\xi_1(d(f))) = f+\xi_1(d(f))+\xi_2(d(f+\xi_1(d(f))))\\ &= f+\xi_1(d(f))+\xi_2(d(f))+\xi_2(d(\xi_1(d(f)))\\ &= f+\xi_1(d(f))+\xi_2(d(f))=\lambda_U(\xi_1+\xi_2)(f)\end{aligned}$$

The last term of the second row is zero because for every $\xi\in\mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{C/k},\mathcal{I}_C/\mathcal{I}_B)(U)$ we have $d\circ\xi=d_C\circ\pi_{C,B}\circ\xi=0$.

It remains to show that for all $\xi\in\mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{C/k},\mathcal{I}_C/\mathcal{I}_B)(U)$ we have

$$\lambda_U(\xi)\in\mathcal{A}ut_{C,\mathcal{I}_C/\mathcal{I}_B}(B)(U).$$

But the calculation above shows: $\lambda_U(\xi)$ is an isomorphism with inverse $\lambda_U(-\xi)$, so we have $\lambda_U(\xi)\in\mathcal{A}ut(B)(U)$.

Now we have to show that $\lambda_U(\xi)$ is the identity on the set U and on $\mathcal{O}_{C,x}$ for all $x\in U$. This is true because $\lambda_U(\xi)$ is the identity on $C\cap U$, which follows from

$$\pi_{C,B}(\lambda_U(\xi)(f))=\pi_{C,B}(f)+\pi_{C,B}(\xi(d(f)))=\pi_{C,B}(f).$$

Finally we have to prove that $\lambda_U(\xi)$ induces the identity on $(\mathcal{I}_C/\mathcal{I}_B)_x$. For this take $h\in(\mathcal{I}_C/\mathcal{I}_B)_x\subset\mathcal{O}_{B,x}$, then we have:

$$(\lambda_U(\xi))_x(h)=h+\xi_x(d_x(h))=h+\xi(d_x(\pi_{C,B,x}(h)))=h$$

because $\pi_{C,B,x}(h)=0$. This finishes the proof. □

Now we have the claimed isomorphism:

Theorem 3.16. *If $\mathcal{I}_C^2\subset\mathcal{I}_B$, then λ is an isomorphism.*

Proof. We show this by showing it for an arbitrary open $U \subset B$. We use the same notations as in the previous proof. We have to show that λ_U is injective and surjective. We start with the first one: Take $\xi \in \mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{C/k}, \mathcal{I}_C/\mathcal{I}_B)(U)$ with $\lambda_U(\xi) = \mathrm{id}_U$, then we have for all $f \in \mathcal{O}_B(U)$:

$$f = \mathrm{id}_U(f) = \lambda_U(\xi)(f) = f + \xi(d(f)),$$

which means $\xi(d(f)) = 0$. But because $\pi_{C,B}$ is surjective and $d = d_C \circ \pi_{C,B}$ we know that

$$\{d(f) \mid f \in \mathcal{O}_B(U)\} = \{d_C(g) \mid g \in \mathcal{O}_C(U)\}$$

and this is a set of generators for $\Omega^1_{C/k}(U)$, so we have $\xi = 0$.

For surjectivity: For a given $\alpha \in \mathcal{A}ut_{C,\mathcal{I}_C/\mathcal{I}_B}(B)(U)$ define

$$\begin{aligned} \widetilde{\xi} : \mathcal{O}_B(U) &\longrightarrow \mathcal{I}_C/\mathcal{I}_B(U) \\ f &\longmapsto \alpha(f) - f \end{aligned}$$

Then $\widetilde{\xi}$ maps really to $\mathcal{I}_C/\mathcal{I}_B(U) = \ker(\pi_{C,B})$ because from the commutativity of (3.8) we get $\pi_{C,B} \circ \alpha = \pi_{C,B}$ and with that:

$$\pi_{C,B}(\widetilde{\xi}(f)) = \pi_{C,B}(\alpha(f) - f) = \pi_{C,B}(\alpha(f)) - \pi_{C,B}(f) = 0$$

If we show that $\widetilde{\xi}$ is a derivation, that is it satisfies the Leibniz rule it induces a $\xi' \in \mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{B/k}, \mathcal{I}_C/\mathcal{I}_B)(U)$ with $\widetilde{\xi} = \xi' \circ d_B$. For this, let $f, g \in \mathcal{O}_B(U)$ arbitrary, then we have:

$$\begin{aligned} \widetilde{\xi}(f \cdot g) &= \alpha(fg) - fg = \alpha(f)\alpha(g) - f\alpha(g) + f\alpha(g) - fg \\ &= \alpha(g)\big(\alpha(f) - f\big) + f\big(\alpha(g) - g\big) \\ &= \alpha(g)\widetilde{\xi}(f) + f\widetilde{\xi}(g) \\ &= g\widetilde{\xi}(f) + f\widetilde{\xi}(g) \end{aligned}$$

The last equality follows because we have $\widetilde{\xi}(g) = \alpha(g) - g \in \mathcal{I}_C/\mathcal{I}_B(U)$, and with $\mathcal{I}_C^2 \subset \mathcal{I}_B$ we get:

$$0 = \alpha(g)\widetilde{\xi}(f) - g\widetilde{\xi}(f) \in \mathcal{I}_C/\mathcal{I}_B(U)$$

Now we look at the map

$$\Psi : \mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{C/k}, \mathcal{I}_C/\mathcal{I}_B)(U) \longrightarrow \mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{B/k}, \mathcal{I}_C/\mathcal{I}_B)(U)$$

which is given in the following way: For every $\psi \in \mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{C/k}, \mathcal{I}_C/\mathcal{I}_B)(U)$ we compose the derivation $\psi \circ d_C$ with $\pi_{C,B}$. This gives a derivation on $\mathcal{O}_B(U)$, and thus by the universal property of $\Omega^1_{B/k}(U)$ an element in $\mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{B/k}, \mathcal{I}_C/\mathcal{I}_B)(U)$. In particular all elements in the image of Ψ are zero on $\mathcal{I}_C/\mathcal{I}_B(U)$. If on the other hand, we have a

$\psi' \in \mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{B/k}, \mathcal{I}_C/\mathcal{I}_B)(U)$ such that ψ' is zero on $\mathcal{I}_C/\mathcal{I}_B(U)$, then composing ψ' with (the set map) $\pi^{-1}_{C,B}$ gives an element of $\mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{C/k}, \mathcal{I}_C/\mathcal{I}_B)(U)$. So the image of Ψ are exactly the elements which are zero on $\mathcal{I}_C/\mathcal{I}_B(U)$.

So it remains to prove that ξ is zero on $\mathcal{I}_C/\mathcal{I}_B(U)$, but this follows directly from the definition, because α induces the identity on $\mathcal{I}_C/\mathcal{I}_B(U)$. So we have a $\xi \in \mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{C/k}, \mathcal{I}_C/\mathcal{I}_B)(U)$ with $\xi' \circ d_B = \xi \circ d_C \circ \pi_{C,B}$, and we have for $f \in \mathcal{O}_B(U)$:

$$\lambda_U(\xi)(f) = f + \xi(d_C(\pi_{C,B}(f))) = f + \xi'(d_B(f)) = f + \widetilde{\xi}(f) = f + \alpha(f) - f = \alpha(f)$$

and this finishes the proof. □

Now $\mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{C/k}, \mathcal{I}_C/\mathcal{I}_B)$ is a quasi-coherent sheaf on B, so for cohomology Theorem 3.1 gives us:

Corollary 3.17. *If $\mathcal{I}_C^2 \subset \mathcal{I}_B$, then for $i = 0, 1$ the isomorphism λ induces*

$$\check{H}^i(B, \mathcal{A}ut_{C,\mathcal{I}_C/\mathcal{I}_B}(B)) \cong H^i(B, \mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{C/k}, \mathcal{I}_C/\mathcal{I}_B)).$$

Now we want to restrict C even more, and with this restriction we have are able to calculate the obstruction space. This is the step-by-step way we mentioned at the beginning of this section.

So from now on, we always assume the following condition:

Definition 3.18. If we say that we *assume (S)*, then we assume that we have exactly one l_0 with $n_{l_0} = n'_{l_0} + 1$ and for all other l we have $n_l = n'_l$.

First we note, that (S) implies $\mathcal{I}_C^2 \subset \mathcal{I}_B$, so Theorem 3.16 and Corollary 3.17 hold. Moreover we have $\mathcal{I}_B = \mathcal{I}_{B_{l_0}}\mathcal{I}_C$ so because of $\mathcal{O}_{B_{l_0}} = \mathcal{O}_X/\mathcal{I}_{B_{l_0}}$ the ideal sheaf $\mathcal{I}_C/\mathcal{I}_B$ of C in B is a quasi-coherent sheaf on B_{l_0} and we have $\mathcal{I}_C/\mathcal{I}_B = {\iota_{B_{l_0},B}}_*(\mathcal{I}_C/\mathcal{I}_B)$ Now we get a sequence of theorems leading to a way to calculate the obstruction space:

Lemma 3.19. *If we assume (S), then the sheaves $\mathcal{A}ut_C(B)$, $\mathcal{A}ut_{C,\mathcal{I}_C/\mathcal{I}_B}(B)$ and Q have trivial stalks outside from B_{l_0}.*

Proof. This follows because the condition defining $\mathcal{A}ut_C(B)$ and $\mathcal{A}ut_{C,\mathcal{I}_C/\mathcal{I}_B}(B)$ are on the stalks, and because $(\mathcal{I}_C/\mathcal{I}_B)_x$ is trivial outside B_{l_0} we have $\mathcal{O}_{B,x} \cong \mathcal{O}_{C,x}$ there. □

Lemma 3.20. *If we assume (S) and additionally $n_{l_0} > 2$, then we have $Q = 1$.*

Proof. A sheaf is trivial if all stalks are trivial, so we take a $x \in B$, and look at Q_x.

By Lemma 3.19 Q_x is trivial outside B_{l_0}, so we may assume $x \in B_{l_0}$. By the discussion after Definition 2.22 we get, that at every point we have $\mathcal{O}_{B,x} \cong \mathcal{O}_{X,x}/(f_{l_0}^{n_{l_0}} f_l^{n_l})$, with $f_{l_0} \in \mathcal{I}_{l_0}$ and $f_l \in \mathcal{I}_l$ if $x \in E_{l_0} \cap E_l$ and $f_l = 1$ else. It follows, that we have $(\mathcal{I}_C/\mathcal{I}_B)_x = (f_{l_0}^{n_{l_0}-1} f_l^{n_l})/(f_{l_0}^{n_{l_0}} f_l^{n_l})$

Take now $\psi \in \mathcal{A}ut_C(B)_x$. Because $\psi|_{\mathcal{O}_{C,x}} = \mathrm{id}_{\mathcal{O}_{C,x}}$ we have for $g \in \mathcal{O}_{B,x}$: $\psi(g) = g + \varphi(g)$ with some morphism $\varphi : \mathcal{O}_{B,x} \to (\mathcal{I}_C/\mathcal{I}_B)_x$. In particular we get

$$\begin{aligned}\psi(f_{l_0}^{n_{l_0}-1}) &= (f_{l_0} + \varphi(f_{l_0}))^{n_{l_0}-1} \\ &= f_{l_0}^{n_{l_0}-1} + \sum_{i=1}^{n_{l_0}-2} \binom{n_{l_0}-1}{i} f_{l_0}^i \varphi(f_{l_0})^{n_{l_0}-1-i} + \varphi(f_{l_0})^{n_{l_0}-1} \\ &= f_{l_0}^{n_{l_0}-1}\end{aligned}$$

The summands in the middle vanish because they are in $(\mathcal{I}_{l_0}\mathcal{I}_C)_x = (\mathcal{I}_B)_x$ and the last one because $n_{l_0} - 1 \geq 2$ and so $\varphi(f_{l_0})^{n_{l_0}-1} \in (\mathcal{I}_C^2)_x \subset (\mathcal{I}_B)_x$.

Take now $g f_{l_0}^{n_{l_0}-1} f_l^{n_l} \in (\mathcal{I}_C/\mathcal{I}_B)_x$, we have:

$$\begin{aligned}\psi(f) &= \psi(g f_{l_0}^{n_{l_0}-1} f_l^{n_l}) = \psi(g)\psi(f_{l_0})^{n_{l_0}-1}\psi(f_l^{n_l}) \\ &= (g + \varphi(g))(f_{l_0}^{n_{l_0}-1}(f_l^{n_l} + \varphi(f_l^{n_l}))) \\ &= (g + \varphi(g))(f_{l_0}^{n_{l_0}-1} f_l^{n_l} + f_{l_0}^{n_{l_0}-1}\varphi(f_l^{n_l})) \\ &= g f_{l_0}^{n_{l_0}-1} f_l^{n_l} + f_{l_0}^{n_{l_0}-1} f_l^{n_l}\varphi(g) = g f_{l_0}^{n_{l_0}-1} f_l^{n_l}\end{aligned}$$

Here the last summands in the last two rows vanish again because they are in $(\mathcal{I}_{l_0}\mathcal{I}_C)_x = (\mathcal{I}_B)_x$.

So every $\psi \in \mathcal{A}ut_C(B)_x$ is the identity on $(\mathcal{I}_C/\mathcal{I}_B)_x$ and we have $Q_x = 1$. □

Again this has an impact on the cohomology:

Corollary 3.21. *If we assume (S) and additionally $n_{l_0} > 2$, then we have for $i = 0, 1$:*

$$\check{H}^i(B, \mathcal{A}ut_C(B)) = H^i(B, \mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{C/k}, \mathcal{I}_C/\mathcal{I}_B))$$

Proof. From the sequence (3.9) and Lemma 3.20 we get $\mathcal{A}ut_C(B) \cong \mathcal{A}ut_{C,\mathcal{I}_C/\mathcal{I}_B}(B)$, so we get

$$\check{H}^1(B, \mathcal{A}ut_C(B)) = \check{H}^1(B, \mathcal{A}ut_{C,\mathcal{I}_C/\mathcal{I}_B}(B)) = H^1(B, \mathcal{H}om_{\mathcal{O}_C}(\Omega^1_{C/k}, \mathcal{I}_C/\mathcal{I}_B))$$

from Corollary 3.17. □

The key tool for the calculation of $H^1(B, \mathcal{H}om_{\mathcal{O}_C}(\Omega^1_{C/k}, \mathcal{I}_C/\mathcal{I}_B))$ is the following reduction to B_{l_0}:

Lemma 3.22. *If we assume (S), then*

$$\mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{C/k}, \mathcal{I}_C/\mathcal{I}_B) \cong \mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\Omega^1_{C/k}|_{B_{l_0}}, \mathcal{I}_C/\mathcal{I}_B).$$

Proof. By Section 3.1.2 we have ${\iota_{B_{l_0},C}}_*(\iota^*_{B_{l_0},C}(\Omega^1_{C/k})) = \Omega^1_{C/k}$, so we have

$$\iota_{C,B_*}(\Omega^1_{C/k}) = \iota_{C,B_*}({\iota_{B_{l_0},C}}_*(\iota^*_{B_{l_0},C}(\Omega^1_{C/k}))) = {\iota_{B_{l_0},B}}_*(\iota^*_{B_{l_0},C}(\Omega^1_{C/k}))$$

by the functoriality of push forward. But then we get

$$\mathcal{H}om_{\mathcal{O}_B}(\iota_{C,B_*}(\Omega^1_{C/k}), {\iota_{B_{l_0},B}}_*(\mathcal{I}_C/\mathcal{I}_B)) \cong {\iota_{B_{l_0},B}}_*(\mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\Omega^1_{C/k}|_{B_{l_0}}, \mathcal{I}_C/\mathcal{I}_B))$$

again from Section 3.1.2. □

For the cohomology this implies together with Section 3.1.2:

Corollary 3.23. *If we assume (S), then we have for $i \geq 0$:*

$$H^i(B, \mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{C/k}, \mathcal{I}_C/\mathcal{I}_B)) = H^i(B_{l_0}, \mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\Omega^1_{C/k}|_{B_{l_0}}, \mathcal{I}_C/\mathcal{I}_B)).$$

From now on, we have to differ between $n_{l_0} = 2$ and $n_{l_0} > 2$. We do the first case first:

Lemma 3.24. *If we assume (S) and additionally $n_{l_0} = 2$, then:*

$$\mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\Omega^1_{C/k}|_{B_{l_0}}, \mathcal{I}_C/\mathcal{I}_B) \cong \mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\Omega^1_{B_{l_0}/k}, \mathcal{I}_C/\mathcal{I}_B)$$

Proof. Let $I = (\mathcal{I}_{B_{l_0}}/\mathcal{I}_C)/(\mathcal{I}_{B_{l_0}}/\mathcal{I}_C)^2$. Then we have the canonical exact sequence

$$I \xrightarrow{d_{B_{l_0}}} \Omega^1_{C/k}|_{B_{l_0}} \longrightarrow \Omega^1_{B_{l_0}/k} \longrightarrow 0$$

Now we apply $\mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\quad, \mathcal{I}_C/\mathcal{I}_B)$ and get:

$$0 \to \mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\Omega^1_{B_{l_0}/k}, \mathcal{I}_C/\mathcal{I}_B) \to \mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\Omega^1_{C/k}|_{B_{l_0}}, \mathcal{I}_C/\mathcal{I}_B) \xrightarrow{\circ d_{B_{l_0}}} \mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(I, \mathcal{I}_C/\mathcal{I}_B)$$

So all we have to show is $\circ d_{B_{l_0}} = 0$, which we prove on the level of stalks.

For this take a $x \in B_{l_0}$, $f \in I_x$ and $\varphi \in \mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\Omega^1_{C/k}|_{B_{l_0}}, \mathcal{I}_C/\mathcal{I}_B)_x$ arbitrary. We want to show $\varphi(d_{B_{l_0},x}(f)) = 0$.

Now take $g \in \prod_{\substack{i=1 \\ i \neq l_0}}^n (\mathcal{I}_i)_x$ such that g is no zero-divisor in $(\mathcal{O}_{B_{l_0}})_x$. Because f is the residue class of some element of $(\mathcal{I}_{B_{l_0}})_x$ and $n'_{l_0} = 1$, we have $fg \in (\mathcal{I}_C)_x$, so $fg = 0$ in I_x, which means:

$$0 = d_{B_{l_0},x}(fg) = f d_{B_{l_0},x}(g) + g d_{B_{l_0},x}(f)$$

This implies for φ:

$$g\varphi(d_x(f)) = \varphi(g d_{B_{l_0},x}(f)) = -\varphi(f d_x(g)) = -f\varphi(d_{B_{l_0},x}(g)) = 0$$

because the last term is in $(\mathcal{I}_{B_{l_0}}\mathcal{I}_C)_x = (\mathcal{I}_B)_x$. This shows $\varphi(df) = 0$, because by choice g is no zero-divisor. □

If we denote by $^\vee$ the dual sheaf $\mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\quad, \mathcal{O}_{B_{l_0}})$, then we get the following corollary for the cohomology:

Corollary 3.25. *If we assume (S) and additionally $n_{l_0} = 2$, then we have a long exact sequence:*

$$0 \longrightarrow H^0(B_{l_0}, (\Omega^1_{B_{l_0}/k})^\vee \otimes \mathcal{I}_C/\mathcal{I}_B) \longrightarrow H^0(B, \mathcal{A}ut_C(B)) \longrightarrow H^0(B, Q)$$
$$\longrightarrow H^1(B_{l_0}, (\Omega^1_{B_{l_0}/k})^\vee \otimes \mathcal{I}_C/\mathcal{I}_B) \longrightarrow \check{H}^1(B, \mathcal{A}ut_C(B)) \longrightarrow \check{H}^1(B, Q)$$

Proof. By Corollary 3.17 and 3.23 and the previous Lemma we have

$$\begin{aligned}\check{H}^i(B, \mathcal{A}ut_{C,\mathcal{I}_C/\mathcal{I}_B}(B)) &= H^i(B, \mathcal{H}om_{\mathcal{O}_B}(\Omega^1_{C/k}, \mathcal{I}_C/\mathcal{I}_B)) \\ &= H^i(B_{l_0}, \mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\Omega^1_{C/k}|_{B_{l_0}}, \mathcal{I}_C/\mathcal{I}_B)) \\ &= H^i(B_{l_0}, \mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\Omega^1_{B_{l_0}/k}, \mathcal{I}_C/\mathcal{I}_B)) \\ &= H^i(B_{l_0}, \mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\Omega^1_{B_{l_0}/k}, \mathcal{O}_{B_{l_0}}) \otimes \mathcal{I}_C/\mathcal{I}_B)\end{aligned}$$

Where the last equality follows because [Gro60], 0 5.4.2 allows us to put the tensor product with $\mathcal{I}_C/\mathcal{I}_B$ in the second argument. So the claimed sequence is just the long exact sequence of (3.9) using Theorem 3.4. □

Now we do the second case, $n_{l_0} > 2$:

Lemma 3.26. *If we assume (S) and additionally $n_{l_0} > 2$, then we have*

$$\Omega^1_{C/k}|_{B_{l_0}} \cong \Omega^1_{X/k}|_{B_{l_0}}.$$

Proof. Again we look at the canonical exact sequence

$$\mathcal{I}_C/\mathcal{I}_C^2 \overset{d_C \otimes 1}{\longrightarrow} \Omega^1_{X/k}|_C \longrightarrow \Omega^1_{C/k} \longrightarrow 0$$

Now the pullback-functor $\iota^*_{B_{l_0},C}$ is right-exact, so we get a sequence:

$$\mathcal{I}_C/\mathcal{I}_C^2|_{B_{l_0}} \overset{d_C \otimes 1 \otimes 1}{\longrightarrow} \Omega^1_{X/k}|_{B_{l_0}} \longrightarrow \Omega^1_{C/k}|_{B_{l_0}} \longrightarrow 0$$

It remains to show $d_C \otimes 1 \otimes 1 = 0$. This can again be done by passing to the stalks, and again, the only interesting points are the $x \in B_{l_0}$.

As above, if $f \in (\mathcal{I}_C/\mathcal{I}_C^2)_x$, then $f = g f_{l_0}^{n_{l_0}-1} f_i^{n_i}$, now with $g \in \mathcal{O}_{C,x}$. We get

$$d_{C,x}(f) = d_{C,x}(g f_{l_0}^{n_{l_0}-1} f_i^{n_i}) = f_{l_0} d_{C,x}(g f_{l_0}^{n_{l_0}-2} f_i^{n_i}) + g f_{l_0}^{n_{l_0}-2} f_i^{n_i} d_{C,x}(f_{l_0}) \in (\mathcal{I}_{l_0}\Omega^1_{X/k}|_C)_x$$

but this implies $d_C \otimes 1 \otimes 1 = 0$. □

If we combine the previous Lemma with Corollary 3.21 and 3.23 we get immediately:

Corollary 3.27. *If we assume (S) and additionally $n_{l_0} > 2$, then*

$$\check{H}^1(B, \mathcal{A}ut_C(B)) \cong H^1(B_{l_0}, \mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\Omega^1_{X/k}|_{B_{l_0}}, \mathcal{I}_C/\mathcal{I}_B)).$$

As next step, we want to show that $\mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\Omega^1_{X/k}|_{B_{l_0}}, \mathcal{I}_C/\mathcal{I}_B)$ is the extension of two invertible sheaves on B_{l_0}. For this we need the *normal bundle* of B_{l_0} in X: Because B_{l_0} and X are regular and of finite type over the algebraically closed field k, they are smooth, and so by [Har77], Theorem II 8.17 the canonical sequence is exact on the left:

$$0 \longrightarrow \mathcal{I}_{B_{l_0}}/\mathcal{I}^2_{B_{l_0}} \longrightarrow \Omega^1_{X/k}|_{B_{l_0}} \longrightarrow \Omega^1_{B_{l_0}/k} \longrightarrow 0$$

also by Theorem II 8.15 from there we know that $\Omega^1_{B_{l_0}/k}$ is locally free of rank 1, and so the sequence above stays exact if we apply $\mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\ \ , \mathcal{O}_{B_{l_0}})$:

$$\begin{aligned} 0 \longrightarrow \mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\Omega^1_{B_{l_0}/k}, \mathcal{O}_{B_{l_0}}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/k}|_{B_{l_0}}, \mathcal{O}_{B_{l_0}}) \\ \longrightarrow \mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\mathcal{I}_{B_{l_0}}/\mathcal{I}^2_{B_{l_0}}, \mathcal{O}_{B_{l_0}}) \longrightarrow 0 \end{aligned} \tag{3.10}$$

and we define

$$\mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\mathcal{I}_{B_{l_0}}/\mathcal{I}^2_{B_{l_0}}, \mathcal{O}_{B_{l_0}}) = \mathcal{N}_{B_{l_0}/X}$$

So we get:

Corollary 3.28. *If we assume (S) and additionally $n_{l_0} > 2$, we have a long exact sequence*

$$\begin{aligned} 0 &\longrightarrow H^0(B_{l_0}, \Omega^1_{B_{l_0}/k}{}^{\vee} \otimes \mathcal{I}_C/\mathcal{I}_B) \longrightarrow H^0(B_{l_0}, \mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\Omega^1_{X/k}|_{B_{l_0}}, \mathcal{I}_C/\mathcal{I}_B)) \\ &\longrightarrow H^0(B_{l_0}, \mathcal{N}_{B_{l_0}/X} \otimes \mathcal{I}_C/\mathcal{I}_B) \to H^1(B_{l_0}, \Omega^1_{B_{l_0}/k}{}^{\vee} \otimes \mathcal{I}_C/\mathcal{I}_B) \\ &\longrightarrow \check{H}^1(B, \mathcal{A}ut_C(B)) \longrightarrow H^1(B_{l_0}, \mathcal{N}_{B_{l_0}/X} \otimes \mathcal{I}_C/\mathcal{I}_B) \longrightarrow 0 \end{aligned}$$

Proof. All sheaves in (3.10) are locally free over B_{l_0} and so is $\mathcal{I}_C/\mathcal{I}_B$, so tensoring gives us an exact sequence

$$\begin{aligned} 0 &\longrightarrow \Omega^1_{B_{l_0}/k}{}^{\vee} \otimes_{\mathcal{O}_{B_{l_0}}} \mathcal{I}_C/\mathcal{I}_B \longrightarrow \mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\Omega^1_{X/k}|_{B_{l_0}}, \mathcal{O}_{B_{l_0}}) \otimes_{\mathcal{O}_{B_{l_0}}} \mathcal{I}_C/\mathcal{I}_B \\ &\longrightarrow \mathcal{N}_{B_{l_0}/X} \otimes_{\mathcal{O}_{B_{l_0}}} \mathcal{I}_C/\mathcal{I}_B \longrightarrow 0 \end{aligned}$$

Again [Gro60], 0 5.4.2 allows us to put the tensor product with $\mathcal{I}_C/\mathcal{I}_B$ in the second argument, and the theorem follows with Corollary 3.27 because $\dim(B_{l_0}) = 1$ implies $H^2(B_{l_0}, \mathcal{H}om_{\mathcal{O}_{B_{l_0}}}(\Omega^1_{B_{l_0}/k}, \mathcal{I}_C/\mathcal{I}_B)) = 0$ □

Summarising this section we see that the result of [Lau71], Page 110 stays valid in our setting:

Corollary 3.29. *If we assume (S), then $\check{H}^1(B, \mathcal{A}ut_C(B))$ vanishes, if the following cohomology vanishes:*

- *If $n_{l_0} = 2$: $H^1(B_{l_0}, (\Omega^1_{B_{l_0}/k})^\vee \otimes_{\mathcal{O}_{B_{l_0}}} \mathcal{I}_C/\mathcal{I}_B)$ and $\check{H}^1(B, Q)$*
- *if $n_{l_0} > 2$: $H^1(B_{l_0}, (\Omega^1_{B_{l_0}/k})^\vee \otimes_{\mathcal{O}_{B_{l_0}}} \mathcal{I}_C/\mathcal{I}_B)$ and $H^1(B_{l_0}, \mathcal{N}_{B_{l_0}/X} \otimes_{\mathcal{O}_{B_{l_0}}} \mathcal{I}_C/\mathcal{I}_B)$*

3.4 Reducing the obstruction to combinatorial data

Now, $(\Omega^1_{B_{l_0}/k})^\vee$, $\mathcal{N}_{B_{l_0}/X}$ and $\mathcal{I}_C/\mathcal{I}_B$ are invertible sheaves on B_{l_0}, so we can calculate their degrees. As we will see, this helps us to control the vanishing of some of the cohomology groups in Corollary 3.29.

By [Liu02], 7.3.31 we have

$$\deg_{B_{l_0}}((\Omega^1_{B_{l_0}/k})^\vee) = -\deg_{B_{l_0}}(\Omega^1_{B_{l_0}/k}) = 2 - 2p_a(B_{l_0}) \tag{3.11}$$

Now we want to calculate $\deg(\mathcal{I}_C/\mathcal{I}_B)$ and $\deg_{B_{l_0}}(\mathcal{N}_{B_{l_0}/X})$. For this we tensor the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-B_{l_0}) \longrightarrow \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_{B_{l_0}} \longrightarrow 0$$

with $\mathcal{O}_X(-\sum_{l=1}^{n} n'_l B_l)$. Together with

$$\mathcal{I}_C = \mathcal{O}_X(-\sum_{l=1}^{n} n'_l B_l) \text{ and } \mathcal{I}_B = \mathcal{O}_X(-B_{l_0} - \sum_{l=1}^{n} n'_l B_l),$$

and the three short exact sequences

$$0 \to \mathcal{I}_C \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{I}_C \longrightarrow 0,$$
$$0 \to \mathcal{I}_B \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y/\mathcal{I}_B \longrightarrow 0$$
$$0 \to \mathcal{I}_C/\mathcal{I}_B \longrightarrow \mathcal{O}_X/\mathcal{I}_B \longrightarrow \mathcal{O}_X/\mathcal{I}_C \longrightarrow 0$$

we get:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & \mathcal{I}_C/\mathcal{I}_B & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-B_{l_0} - \sum_{l=1}^{n} n'_l B_l) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X/\mathcal{I}_B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-\sum_{l=1}^{n} n'_l B_l) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X/\mathcal{I}_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}_X/\mathcal{I}_{B_{l_0}}(-\sum_{l=1}^{n} n'_l B_l) & & 0 & & 0 & &
\end{array}$$

Now the snake lemma gives us $\mathcal{I}_C/\mathcal{I}_B \cong \mathcal{O}_X/\mathcal{I}_{B_{l_0}}(-\sum_{l=1}^{n} n_i' B_l)$. Thus we get:

$$\deg_{B_{l_0}}(\mathcal{I}_C/\mathcal{I}_B) = \deg_{B_{l_0}}(\mathcal{O}_{B_{l_0}}(-\sum_{l=1}^{n} n_l' B_l)) = B_{l_0} \cdot (-\sum_{l=1}^{n} n_l' B_l) = -B_{l_0} \cdot \sum_{i=1}^{n} n_l' B_l,$$

As a special case we get

$$\mathcal{N}_{B_{l_0}/X} = (\mathcal{I}_{l_0}/\mathcal{I}_{l_0}^2)^{\vee} = \mathcal{O}_{B_{l_0}}(-B_{l_0})^{\vee} = \mathcal{O}_{B_{l_0}}(B_{l_0})$$

and with that we have by definition

$$\deg_{B_{l_0}}(\mathcal{N}_{B_{l_0}/X}) = B_{l_0} \cdot B_{l_0}.$$

Adding all results together, we get:

$$\begin{aligned} \deg_{B_{l_0}}((\Omega^1_{B_{l_0}/k})^{\vee} \otimes_{\mathcal{O}_{B_{l_0}}} \mathcal{I}_C/\mathcal{I}_B) &= 2 - 2p_a(B_{l_0}) - B_{l_0} \cdot \sum_{l=1}^{n} n_l' B_l \\ \deg_{B_{l_0}}(\mathcal{N}_{B_{l_0}/X} \otimes_{\mathcal{O}_{B_{l_0}}} \mathcal{I}_C/\mathcal{I}_B) &= B_{l_0} \cdot B_{l_0} - B_{l_0} \cdot \sum_{l=1}^{n} n_l' B_l \end{aligned} \tag{3.12}$$

Now we want to relate the degree of an invertible sheaf with the vanishing of global sections:

Lemma 3.30. *Let Y be an integral, one-dimensional, proper k-scheme and $\mathcal{L} \in \mathrm{Pic}(Y)$ with $\deg(\mathcal{L}) < 0$. Then we have $h^0(Y, \mathcal{L}) = 0$.*

Proof. Suppose, by contradiction, that we have a non-zero $\sigma \in H^0(Y, \mathcal{L})$. From this we get a short exact sequence

$$0 \to \mathcal{O}_Y \overset{\cdot\sigma}{\to} \mathcal{L} \to \mathcal{F} \to 0$$

The exactness is nearly clear, only the injectivity of $\cdot\sigma$ needs an argument: Let $y \in U \subset Y$, such that $\mathcal{L}|_U \cong \mathcal{O}_X|_U$, we look at $\mathcal{O}_X(U) \overset{\cdot\sigma|_U}{\to} \mathcal{O}_X(U)$. By [Liu02], Proposition 2.4.18 σ non-zero implies $\sigma|_U \in \mathcal{O}_U$ non-zero. But by Proposition 2.4.17 from there we know that $\mathcal{O}_X(U)$ is integral, so the multiplication with $\sigma|_U$ is injective.

Now $\mathcal{F}$ is a skyscraper-sheaf, which implies $h^1(Y, \mathcal{F}) = 0$. Then by additivity of χ we get:

$$0 > \deg(\mathcal{L}) = \chi(\mathcal{L}) - \chi(\mathcal{O}_Y) = \chi(\mathcal{F}) = h^0(Y, \mathcal{F}) \geq 0$$

a contradiction. □

Now, for a locally free sheaf $\mathcal{L}$ on a smooth, one-dimensional scheme Y over k we get from Serre-duality:

$$H^1(Y, \mathcal{L}) = H^0(Y, \mathcal{L}^{\vee} \otimes \Omega^1_{Y/k})^{\vee}$$

If we combine this with Section 3.2, Theorem 3.14, Lemma 3.29 and (3.12) plus (3.11) and also use $\deg(\mathcal{L}^{\vee}) = -\deg(\mathcal{L})$, we get:

Corollary 3.31. *If we assume (S), then:*

$$H^1(B_{l_0}, (\Omega^1_{B_{l_0}/k})^\vee \otimes_{\mathcal{O}_{B_{l_0}}} \mathcal{I}_C/\mathcal{I}_B) = 0 \text{ if } 2(2p_a(B_{l_0}) - 2) + B_{l_0} \cdot \sum_{l=1}^{n} n'_l B_l < 0$$

and if we additionally have $n'_{l_0} \geq 2$, then

$$H^1(B_{l_0}, \mathcal{N}_{B_{l_0}/Y} \otimes_{\mathcal{O}_{B_{l_0}}} \mathcal{I}_C/\mathcal{I}_B) = 0 \text{ if } 2p_a(B_{l_0}) - 2 - B_{l_0} \cdot B_{l_0} + B_{l_0} \cdot \sum_{l=1}^{n} n'_l B_l < 0$$

If B_{l_0} is isomorphic to $\mathbb{P}^1_k$ we get a better result:

Corollary 3.32. *If we assume (S), and if $B_{l_0} \cong \mathbb{P}^1_k$, then:*

$$H^1(B_{l_0}, (\Omega^1_{B_{l_0}/k})^\vee \otimes_{\mathcal{O}_{B_{l_0}}} \mathcal{I}_C/\mathcal{I}_B) = 0 \text{ if } -2 + B_{l_0} \cdot \sum_{l=1}^{n} n'_l B_l \leq 1$$

and if we additionally have $n'_{l_0} \geq 2$, then

$$H^1(B_{l_0}, \mathcal{N}_{B_{l_0}/Y} \otimes_{\mathcal{O}_{B_{l_0}}} \mathcal{I}_C/\mathcal{I}_B) = 0 \text{ if } -B_{l_0} \cdot B_{l_0} + B_{l_0} \cdot \sum_{l=1}^{n} n'_l B_l \leq 1$$

Proof. For $\mathcal{L} \in \text{Pic}(\mathbb{P}^1_k) \cong \mathbb{Z}$ we have $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1_k}(\deg(\mathcal{L}))$, and thus Lemma 5.3.1 of [Liu02] and an explicit calculation based on Section 5.1 from there gives us:

$$h^1(\mathbb{P}^1_k, \mathcal{L}) = \begin{cases} 0 & \deg(\mathcal{L}) \geq 0 \\ -\deg(\mathcal{L}) - 1 & \deg(\mathcal{L}) < 0 \end{cases} \tag{3.13}$$

So $h^1(\mathbb{P}^1_k, \mathcal{L}) = 0$ if $\deg(\mathcal{L}) \geq -1$, and the corollary is a direct consequence of (3.12). □

Now Corollary 3.31 gives us a new proof of Theorem 6.8 of [Lau71]:

Theorem 3.33. *If we assume (S), $n'_{l_0} \geq 2$ and if the conditions of Corollary 3.31 or Corollary 3.32 are fulfilled, then the map* $\text{CEQ}(C) \longrightarrow \text{CEQ}(B)$ *mapping $[C']$ to the equivalence class of any extension of C' is a bijection.*

Proof. From the conditions of Corollary 3.31 or Corollary 3.32 we know that the cohomology groups in Corollary 3.29 vanish. This implies $\check{H}^1(B, \mathcal{A}ut_C(B)) = 0$ which then with Theorems 3.14 and 3.11 implies that the map is well defined and surjective. But by Theorem 2.26 the map is also injective. □

3.5 Applications for singularities

Now we want to come back to the situation where B is supported by the exceptional divisor of the minimal good desingularization of an algebraization of a normal, two-dimensional singularity. For this fix one anti-ample cycle $\widetilde{Z} = \sum_{l=1}^{n} r_l E_l$ for (S, s) (Definition 2.56).

Now we construct $\widetilde{Z}$ stepwise, for this let $r = \sum_{l=1}^{n} r_l$, we choose $j_0, ..., j_{r-1} \in \{1, ..., n\}$ as follows:

- $\widetilde{Z}_1 = E_{j_0}$
- $\widetilde{Z}_r = \widetilde{Z}$
- For all $i \in \{1, ..., r-1\}$ we have $\widetilde{Z}_{i+1} = \widetilde{Z}_i + E_{j_i}$

By construction, if we set $B = \widetilde{Z}_{i+1}$ and $C = \widetilde{Z}_i$ condition (S) is true, and we can use our calculations above.

We define

$$\tau = \max_{i \in \{1,...,r-1\}} (E_{j_i} \cdot \widetilde{Z}_i)$$

and

$$\lambda = \max_{j \in \{1,...,n\}} \{0, 2(2p_a(E_j) - 2), 2p_a(E_j) - 2 - E_j^2\}$$

and as a consequence we get the result we wanted. This means, we have proven Theorem 6.9 of [Lau71]:

Theorem 3.34. *Let $\widetilde{Z}$, τ and λ as above. If we have $\nu \geq \max\{\lambda + \tau + 1, 1\}$ and if at least one n_l is equal to 1 additionally $\nu \geq 2$, than Theorem 3.33 gives us a bijection*

$$\mathrm{CEQ}(\nu\widetilde{Z}) \longrightarrow \mathrm{CEQ}((\nu + 1)\widetilde{Z})$$

Proof. Let the $\widetilde{Z}_i$ be as chosen above. We define $Z_i = \nu\widetilde{Z} + \widetilde{Z}_i = \sum_{l=1}^{n} s_{l,i} E_l$, then $Z_0 = \nu\widetilde{Z}$ and $Z_r = (\nu+1)\widetilde{Z}$, so all we have to show is that the natural map $\mathrm{CEQ}(Z_i) \to \mathrm{CEQ}(Z_{i+1})$ is bijective for all $i \in \{0, \cdots, r-1\}$. But by the second condition on ν we have $n'_{l_0} \geq 2$ in Theorem 3.33, so the map is bijective using this theorem if we have:

$$\begin{aligned} 0 &> 2(2p_a(E_{j_i}) - 2) + E_{j_i} \cdot \sum_{l=1}^{n} s_{l,i} E_l \\ 0 &> 2p_a(E_{j_i}) - 2 - E_{j_i} \cdot E_{j_i} + E_{j_i} \cdot \sum_{l=1}^{n} s_{l,i} E_l \end{aligned} \tag{3.14}$$

But by construction, we have (using $E_{j_i} \cdot \widetilde{Z} \leq -1$ and $-\lambda - \tau - 1 \geq -\nu$):

$$\begin{aligned} E_{j_i} \cdot \sum_{l=1}^{n} s_{l,i} E_l &= E_{j_i} \cdot (\nu \widetilde{Z} + \widetilde{Z}_i) = \nu E_{j_i} \cdot \widetilde{Z} + E_{j_i} \cdot \widetilde{Z}_i \\ &\leq -\nu + \tau \leq -\lambda - \tau - 1 + \tau \\ &\leq -\lambda - 1 \end{aligned} \tag{3.15}$$

So because of $\lambda \geq 2(2p_a(E_{j_i}) - 2)$ and $\lambda \geq 2p_a(E_{j_i}) - 2 - E_{j_i} \cdot E_{j_i}$, also by construction, (3.14) is true. □

As a corollary we get the same result for an arbitrary $B \geq \nu \widetilde{Z}$:

Corollary 3.35. *If $B \geq \nu \widetilde{Z}$, then Theorem 3.33 gives us a bijection*

$$\mathrm{CEQ}(\nu \widetilde{Z}) \longrightarrow \mathrm{CEQ}(B)$$

Proof. There exist a $l \in \mathbb{N}$ with $B \leq (\nu + l)\widetilde{Z}$. Now Theorem 2.26 tells us that the maps

$$\mathrm{CEQ}(\nu \widetilde{Z}) \longrightarrow \mathrm{CEQ}(B) \longrightarrow \mathrm{CEQ}((\nu + l)\widetilde{Z}),$$

are injective, but by the previous theorem the composition is also bijective, so the first map is already bijective. □

Now our ν still depends on the choice of the j_i, but there are only finitely many choices, so we have a minimal τ, which we call τ_{min}. Then we define:

Definition 3.36. Let (S, s) be a normal, two-dimensional singularity and $\widetilde{Z}$ an anti-ample cycle for (S, s). The *significant multiplicity of $\widetilde{Z}$* is the smallest integer ν such that $\nu \geq \lambda + \tau_{min} + 1$, $\delta_{\mathrm{gcd}}(\nu, p) = 1$ and if at least one n_l in $\widetilde{Z} = \sum_{l=1}^{n} n_l E_l$ is equal to 1, then also $\nu \geq 2$.

Note that the condition $\delta_{\mathrm{gcd}}(\nu, p) = 1$ is not necessary for the theorems of this section, but later it simplifies the formulations.

Remark 3.37. By definition the ν only depends on the dual graph of $\widetilde{Z}$.

Now we can simply take one order $j_0, \ldots, j_{r-1}$ such that τ is minimal, and immediately get the following corollary of Theorem 3.34 respectively Corollary 3.35:

Corollary 3.38. *If ν is the significant multiplicity of $\widetilde{Z}$ and $\mathrm{CEQ}(\nu \widetilde{Z}) = \{[\nu \widetilde{Z}]\}$, then for all $B = \sum_{l=1}^{l} n_l E_l$ we have $\mathrm{CEQ}(B) = \{[B]\}$*

The translation back to singularities is:

Corollary 3.39. *Let (S,s) be a normal, two-dimensional singularity, $\widetilde{Z}$ an anti-ample cycle for (S,s) and $\widetilde{\nu}$ its significant multiplicity. Then S is taut if and only if*

$$\mathrm{CEQ}(j\widetilde{Z}) = \{[j\widetilde{Z}]\}$$

for one $j \geq \widetilde{\nu}$.

Proof. Let ν be as in Theorem 3.34. By Corollary 3.35 we have $\mathrm{CEQ}(j\widetilde{Z}) = \{[j\widetilde{Z}]\}$ for one $j \geq \widetilde{\nu} \geq \nu$ if and only if $\mathrm{CEQ}(\nu\widetilde{Z}) = \{[\nu\widetilde{Z}]\}$. So the Corollary is an immediate consequence of Theorem 2.52 and Corollary 3.38. □

4 The plumbing scheme and its applications

The last corollary of the previous section reduces the question whether a singularity (S, s) is taut to the question whether $\mathrm{CEQ}(\nu\widetilde{Z})$ is trivial. For $n = 1$ and $n = 2$ this question can also easily be answered with the techniques of the previous section. But already for $n = 3$ some of the obstruction groups given in Corollary 3.29 are not trivial. Also we have neglected the role of $\check{H}^1(B, Q)$ and we have no satisfactory answer to this in arbitrary characteristics, so we bypass this question.

The idea behind this section is the following: Let $f : X \to S$ the minimal good desingularization and $E = f^{-1}(s)_{\mathrm{red}}$. Then by Lemma 2.62 for S to be taut, we have necessarily that all integral components E_l of E are isomorphic to $\mathbb{P}^1_k$ and every E_l intersects with at most three others. Let Γ be the dual graph for some cycle $Z = \sum_{l=1}^{n} n_l E_l$ on E. Now we want to construct a special realisation of Γ, that is another scheme P, combinatorially equivalent to Z. This P is the plumbing scheme for Z or, because it only depends on Γ, for the dual graph.

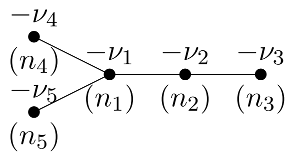
For $p = 0$ and in the complex-analytic category Laufer has shown that $\mathrm{CEQ}(Z)$ is trivial if and only if $H^1(P, \mathcal{H}om_{\mathcal{O}_P}(\Omega^1_{P/k}, \mathcal{O}_P)) = 0$ for this special P. (Theorem 3.9 of [Lau73a]). This criterion cannot be transferred to $p > 0$ without changes, because we have an easy example of a taut singularity having $H^1(P, \mathcal{H}om_{\mathcal{O}_P}(\Omega^1_{P/k}, \mathcal{O}_P)) \neq 0$ for some extension of $f^{-1}(s)$. We give this example in Section 4.5. But at least we can prove that $H^1(P, \mathcal{H}om_{\mathcal{O}_P}(\Omega^1_{P/k}, \mathcal{O}_P)) = 0$ implies $\mathrm{CEQ}(Z) = \{[Z]\}$. We use this in the next section to show that Laufer's tautness criterion implies that tautness for $p = 0$ implies tautness for almost all $p > 0$.

Now we first give an example of P, then the general construction of the plumbing, and finally we discuss Laufer's criterion in the algebraic setting and for $p > 0$.

4.1 Example and notations

4.1.1 Example

As an example, we want to construct P for the following dual graph Γ:

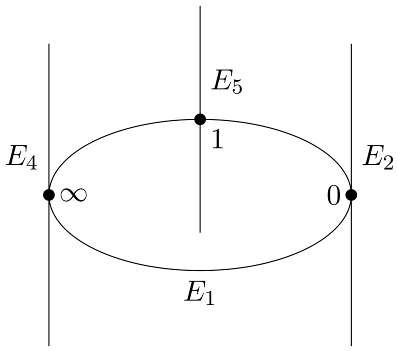


That is, we have $E_1 = \mathbb{P}^1_k$ which intersects with three others. We put this intersection at 0, ∞ and 1. For $E_2 = \mathbb{P}^1_k$ we put the two intersections at 0 and ∞, the three other E_l have the intersection always at 0. Now we give five open (but not affine) pieces W_l of P and then we specify how we glue them. The piece W_l always consists of one E_l and we have affine parts of E_j in W_l if and only if E_j intersects E_l.

The first piece covers W_1 and consists of the two affine charts $\mathrm{Spec}(R_{1,0})$ and $\mathrm{Spec}(R_{1,1})$ glued along $\mathrm{Spec}(R_{1,01})$:

$$\begin{aligned} R_{1,0} &= k[x_{1,0}, y_{1,0}]/(x_{1,0}^{n_2}(x_{1,0}-1)^{n_5}y_{1,0}^{n_1}) \\ R_{1,1} &= k[x_{1,1}, y_{1,1}]/(x_{1,1}^{n_4}(x_{1,1}-1)^{n_5}y_{1,1}^{n_1}) \\ R_{1,01} &= k[x_{1,0}, y_{1,0}, x_{1,1}, y_{1,1}]/(x_{1,0}x_{1,1}-1, y_{1,0}-x_{1,1}^{\nu_1}y_{1,1}, (x_{1,0}-1)^{n_4}y_{1,0}^{n_1}) \end{aligned} \tag{4.1}$$

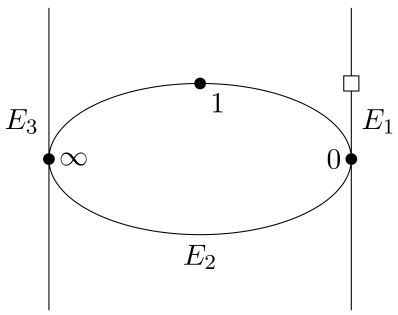
This piece is a $\mathbb{P}^1_k$ with three $\mathbb{A}^1_k$ attached at 0, ∞ and 1. Later we glue W_1 along this three affine arms to the E_l. If we draw the $\mathbb{P}^1_k$ as a circle, and the $\mathbb{A}^1_k$ as straight lines, we get the following picture:



For W_2 we glue the following affine charts:

$$\begin{aligned} R_{2,0} &= k[x_{2,0}, y_{2,0}, s_{2,0}^{-1}]/(s_{2,0}^{-1}(y_{2,0}-1)-1, x_{2,0}^{n_1}y_{2,0}^{n_2}) \\ R_{2,1} &= k[x_{2,1}, y_{2,1}]/(x_{2,1}^{n_3}y_{2,1}^{n_2}) \\ R_{2,01} &= k[x_{2,0}, y_{2,0}, x_{2,1}, y_{2,1}, s_{2,0}^{-1}]/(s_{2,0}^{-1}(y_{2,0}-1)-1, x_{2,0}x_{2,1}-1, y_{2,0}-x_{2,1}^{\nu_2}y_{2,1}, y_{2,0}^{n_2}) \end{aligned} \tag{4.2}$$

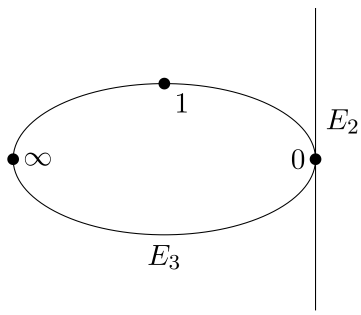
That is, W_2 is a $\mathbb{P}^1_k$ with two affine arms, where the arm in the $R_{2,1}$-chart will later be glued with E_3 in W_3 and the one in the $R_{2,0}$-chart will later be glued with E_1 in W_1. We have to remove the point 1 in the $R_{2,0}$-chart because later we glue E_5 to E_1 at this point. In this case the picture is



Now W_3 is given by the following charts:

$$\begin{aligned} R_{3,0} &= k[x_{3,0}, y_{3,0}]/(x_{3,0}^{n_2} y_{3,0}^{n_3}) \\ R_{3,1} &= k[x_{3,1}, y_{3,1}]/(y_{3,1}^{n_3}) \\ R_{3,01} &= k[x_{3,0}, y_{3,0}, x_{3,1}, y_{3,1}]/(x_{3,0}x_{3,1} - 1, y_{3,0} - x_{3,1}^{\nu_3} y_{3,1}, y_{3,0}^{n_3}) \end{aligned} \tag{4.3}$$

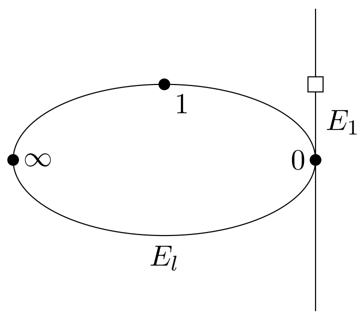
This is a $\mathbb{P}_k^1$ one $\mathbb{A}_k^1$ attached at 0.



Finally the last two pieces, W_4 and W_5, are given by gluing the following affine schemes:

$$\begin{aligned} R_{l,0} &= k[x_{l,0}, y_{l,0}, s_{l,0}^{-1}]/(s_{l,0}^{-1}(y_{l,0} - 1) - 1, x_{l,0}^{n_1} y_{l,0}^{n_l}) \\ R_{l,1} &= k[x_{l,1}, y_{l,1}]/(y_{l,1}^{n_l}) \\ R_{l,01} &= k[x_{l,0}, y_{l,0}, x_{l,1}, y_{l,1}]/(s_{l,0}^{-1}(y_{l,0} - 1) - 1, x_{l,0}x_{l,1} - 1, y_{l,0} - x_{l,1}^{\nu_l} y_{l,1}, y_{l,0}^{n_l}) \end{aligned} \tag{4.4}$$

Again we have to remove the point 1 in the affine arm corresponding to E_1.



We specify the gluing of this five pieces by giving the affine schemes $W_1 \cap W_l$, $l = 2, 4, 5$ and $W_2 \cap W_3$ as $\mathrm{Spec}(R_{il})$ with the following R_{il}:

$$\begin{aligned} R_{12} &= k[x_{1,0}, y_{1,0}, x_{2,0}, y_{2,0}, s_{2,0}^{-1}]/(s_{2,0}^{-1}(y_{2,0} - 1) - 1, x_{1,0} - y_{2,0}, y_{1,0} - x_{2,0}, x_{1,0}^{n_2} y_{1,0}^{n_1}) \\ R_{23} &= k[x_{2,1}, y_{2,1}, x_{3,0}, y_{3,0}]/(x_{2,1} - y_{3,0}, y_{2,1} - x_{3,0}, x_{2,1}^{n_3} y_{2,1}^{n_2}) \\ R_{14} &= k[x_{1,1}, y_{1,1}, x_{4,0}, y_{4,0}, s_{4,0}^{-1}]/(s_{4,0}^{-1}(y_{4,0} - 1) - 1, x_{1,1} - y_{4,0}, y_{1,1} - x_{4,0}, x_{1,1}^{n_4} y_{1,1}^{n_1}) \\ R_{15} &= k[x_{1,0}, y_{1,0}, x_{5,0}, y_{5,0}, s_{5,0}^{-1}]/(s_{5,0}^{-1}(y_{5,0} - 1) - 1, (x_{1,0} - 1) - y_{5,0}, y_{1,0} - x_{5,0}, y_{5,0}^{n_5} y_{1,0}^{n_1}) \end{aligned}$$

The resulting scheme is the *plumbing scheme* for Γ. The scheme P is embedded into a smooth, two-dimensional scheme, because if we omit the last equation in every affine chart, we get regular, two-dimensional, affine schemes.

4.1.2 Notations

Before we start with the general construction of the plumbing, we have to fix some notations for the rest of this section. For this section, by Z we always denote a divisor on a smooth, two-dimensional scheme X which satisfies the properties of an exceptional divisor of a good desingularization of an algebraization of a normal, two-dimensional singularity. We denote by E_l the n integral components of Z. By ν_l we denote the negative of the self-intersection-number of E_l, that is $v_l = -E_l \cdot E_l$. Then we necessarily have $v_l \geq 1$. Finally by t_l we denote the number of E_j, $j \neq l$ with $E_l \cap E_j \neq \emptyset$.

We also make some assumptions on Z, first we assume that $Z = \sum_{l=0}^{n} n_l E_l$ with $n_l > 0$. Also we assume that $p_a(E_l) = 0$ and $t_l < 4$ for all l. This is no important restriction, because if the last two conditions are not fulfilled, then we know already from Lemma 2.62 that the corresponding singularity can not be taut.

4.2 The plumbing scheme

Now we want to give the general construction for the plumbing scheme for some Z. The example in Section 4.1.1 already contains all the ideas we need. First we want to prove that, for any Z, we can decompose Z into open but not affine pieces isomorphic to schemes given by gluing affine charts like (4.1) to (4.4). To give the general form of this pieces we need some more notation. For E_l and $1 \leq i \leq t_l$ let E_{j_i} be the t_l components with $E_l \cap E_{j_i} \neq \emptyset$. As we have seen in the example, for $t_{j_i} = 3$ we need to invert some elements of the form $s_{l,0} = y_{l,0} - 1$ or $s_{l,1} = y_{l,1} - 1$. These s are more or less a technical problem. We need to invert these elements so that every W_l only contains points of Z which lie on E_l or on one E_{j_i}. For all practical calculations we need later, the s can be ignored, because inverting this elements is just a localization, and those commute for example with taking Kähler differentials.

First case: If $t_l = 1$, then W_l has the following affine charts:

$$\begin{aligned} R_{l,0} &= k[x_{l,0}, y_{l,0}, s_{l,0}^{-1}]/(s_{l,0}^{-1} s_{l,0} - 1, x_{l,0}^{n_{j_1}} y_{l,0}^{n_l}) \\ R_{l,1} &= k[x_{l,1}, y_{l,1}]/(y_{l,1}^{n_l}) \\ R_{l,01} &= k[x_{l,0}, y_{l,0}, x_{l,1}, y_{l,1}]/(s_{l,0}^{-1} s_{l,0} - 1, x_{l,0} x_{l,1} - 1, y_{l,0} - x_{l,1}^{\nu_l} y_{l,1}, y_{l,0}^{n_l}) \end{aligned} \tag{4.5}$$

Second case: If $t_l = 2$, then with W_l has the following affine charts:

$$\begin{aligned} R_{l,0} &= k[x_{l,0}, y_{l,0}, s_{l,0}^{-1}]/(s_{l,0}^{-1}s_{l,0} - 1, x_{l,0}^{n_{j_1}} y_{l,0}^{n_l}) \\ R_{l,1} &= k[x_{l,1}, y_{l,1}, s_{l,1}^{-1}]/(s_{l,1}^{-1}s_{l,1} - 1, x_{l,1}^{n_{j_2}} y_{l,1}^{n_l}) \\ R_{l,01} &= k[x_{l,0}, y_{l,0}, x_{l,1}, y_{l,1}, \underset{m=0,1}{s_{l,m}^{-1}}]/(s_{l,m}^{-1}s_{l,m} - 1, x_{l,0}x_{l,1} - 1, y_{l,0} - x_{l,1}^{\nu_l}y_{l,1}, y_{l,0}^{n_l}) \end{aligned} \tag{4.6}$$

Third case: If $t_l = 3$, W_l has the following affine charts:

$$\begin{aligned} R_{l,0} &= k[x_{l,0}, y_{l,0}, s_{l,0}^{-1}]/(s_{l,0}^{-1}s_{l,0} - 1, x_{l,0}^{n_{j_1}}(x_{l,0} - 1)^{n_{j_3}} y_{l,0}^{n_l}) \\ R_{l,1} &= k[x_{l,1}, y_{l,1}, s_{l,1}^{-1}]/(s_{l,1}^{-1}s_{l,1} - 1, x_{l,1}^{n_{j_2}}(x_{l,1} - 1)^{n_{j_3}} y_{l,1}^{n_l}) \\ R_{l,01} &= k[x_{l,0}, y_{l,0}, x_{l,1}, y_{l,1}, \underset{m=0,1}{s_{l,m}^{-1}}]/(s_{l,m}^{-1}s_{l,m} - 1, x_{l,0}x_{l,1} - 1, y_{l,0} - x_{l,1}^{\nu_l}y_{l,1}, (x_{l,0} - 1)^{n_{j_3}} y_{l,0}^{n_l}) \end{aligned} \tag{4.7}$$

Now let V_l be an open neighbourhood of E_l in Z. We want to show that we can choose V_l isomorphic to one W_l:

Lemma 4.1. *For every E_l we have an open $V_l \subset Z$ isomorphic to W_l given by the appropriate one of the affine charts* (4.5), (4.7) *or* (4.7).

Proof. We have $E_l \cong \mathbb{P}_k^1$ and we know that, for every 3 points on $\mathbb{P}_k^1$, we can map these three points on 0, ∞ and 1. So we can choose V_l in such a way that $(V_l)_{\text{red}}$ is "a $\mathbb{P}_k^1$ with t_l affine arms", or to be precise, $(V_l)_{\text{red}}$ is isomorphic to $(W_l)_{\text{red}}$,

Now we want to extend the isomorphism between $(W_l)_{\text{red}}$ and $(V_l)_{\text{red}}$ to one between W_l and V_l. We do this as in Section 3.3. That is, we thicken either the $\mathbb{P}_k^1$-part or one of the affine arms from the n-th to the $(n+1)$-th infinitesimal neighbourhood and show that we can extend the isomorphism.

First we observe that extending at the affine parts is always possible because we can always extend locally on each affine arm via Theorem 3.11, and this glues because the extensions are trivial on the $\mathbb{P}_k^1$-part because there is simply nothing to extend. So we first extend at the $\mathbb{P}_k^1$-part as much as needed, and then simply extend at the affine parts. The only difficult step for the $\mathbb{P}_k^1$-part is the first one. For this we calculate the $\check{H}^1(W_l, \mathrm{Aut}_{(W_l)_{\text{red}}}(W_l))$ by hand. We do this in Section 6.2.5 and get that two such schemes must be isomorphic if and only if the ν_l are equal.

Now we want to use Corollary 3.32 to show that all cohomology groups in Corollary 3.29 vanish. For this we use the degrees calculated in (3.12) and like in (6.2) we get the vanishing using $\nu_l \geq 1$ and $t_l \leq 3$. This shows that we can indeed choose W_l and V_l isomorphic. □

Now we can construct the special scheme P.

Definition 4.2. Take an open covering W_l of Z as in Lemma 4.1. The *Plumbing scheme* for this Z is the scheme that we get if we glue the W_l in the following way: For all

$E_l \cap E_j = \{x_{lj}\}$ let i_l and i_j be such that $V_l \cap V_j = \mathrm{Spec}(R_{l,i_l}) \cap \mathrm{Spec}(R_{j,i_j})$. Then we glue W_l and W_j along $\mathrm{Spec}(\widetilde{R}_{lj})$ where

$$\begin{aligned}\widetilde{R}_{lj} = k[\widetilde{x}_{l,i_l}, y_{l,i_l}, \widetilde{x}_{j,i_j}, y_{j,i_j}, s_{l,i_l}^{-1}, s_{j,i_j}^{-1}]/(\ & s_{l,i_l}^{-1}s_{l,i_l} - 1, s_{j,i_j}^{-1}s_{j,i_j} - 1,\\ & \widetilde{x}_{j,i_j} - y_{l,i_l}, y_{j,i_j} - \widetilde{x}_{l,i_l}, \widetilde{x}_{l,i_l}^{n_j} y_{l,i_l}^{n_l})\end{aligned}$$

with $\widetilde{x}_{l,i_l} = x_{l,i_l} - 1$ if $j_4 = j$ in W_l and $\widetilde{x}_{l,i_l} = x_{l,i_l}$ else, and analogously for $\widetilde{x}_{j,i_j}$ depending on $j_4 = l$ in W_j.

P only depends on the dual graph of Z, so we also say that P is the plumbing scheme for this dual graph.

Remark 4.3. P is by construction embedded into a regular, two-dimensional scheme, so P is combinatorially equivalent to Z.

From this we can get the following global description of Z:

Lemma 4.4. *Let $V_l \cong W_l$ be open neighbourhoods of E_l in Z as in Lemma 4.1. For $E_l \cap E_j = \{x_{lj}\}$ let i_l, i_j, $\widetilde{x}_{l,i_l}$ and $\widetilde{x}_{j,i_j}$ as in Definition 4.2. Then we can find relations*

$$\widetilde{x}_{j,i_j} = y_{l,i_l}(a_{y,l,j} + \widetilde{x}_{l,i_l} y_{l,i_l} p_{y,l,j}) \text{ and } y_{j,i_j} = \widetilde{x}_{l,i_l}(a_{x,l,j} + \widetilde{x}_{l,i_l} y_{l,i_l} p_{x,l,j})$$

with

$$a_{x,l,j}, a_{y,l,j} \in k^\times \text{ and } p_{x,l,j}, p_{y,l,j} \in k[\widetilde{x}_{l,i_l}, y_{l,i_l}, s_{l,i_l}^{-1}, s_{j,i_j}^{-1}],$$

such that $V_l \cap V_j = \mathrm{Spec}(R_{l,i_l}) \cap \mathrm{Spec}(R_{j,i_j}) = \mathrm{Spec}(R_{lj})$ *with*

$$\begin{aligned}R_{lj} = k[\widetilde{x}_{l,i_l}, y_{l,i_l}, \widetilde{x}_{j,i_j}, y_{j,i_j}, s_{l,i_l}^{-1}, s_{j,i_j}^{-1}]/(\ & s_{l,i_l}^{-1}s_{l,i_l} - 1, s_{j,i_j}^{-1}s_{j,i_j} - 1,\\ & \widetilde{x}_{j,i_j} - y_{l,i_l}(a_{y,l,j} + \widetilde{x}_{l,i_l} y_{l,i_l} p_{y,l,j}),\\ & y_{j,i_j} - \widetilde{x}_{l,i_l}(a_{x,l,j} + \widetilde{x}_{l,i_l} y_{l,i_l} p_{x,l,j}),\\ & \widetilde{x}_{l,i_l}^{n_j} y_{l,i_l}^{n_l})\end{aligned}$$

and Z is isomorphic to the glueing of the W_l along the $\mathrm{Spec}(R_{lj})$

Proof. The idea behind this is that if two schemes are built of isomorphic open charts, then both of them can be obtained by glueing via an automorphism of the double intersections U_{lj} of the open charts. Now P and Z have a common set of open charts, the W_l.

For all l, j with $E_l \cap E_j \neq \emptyset$ we do the following: Let $\varphi_l : W_l \to V_l$ and $\varphi_j : W_j \to V_j$ be the isomorphisms, then $\varphi_j^{-1} \circ \varphi_l$ induces an automorphism $\varphi_{l,j}$ of $W_l \cap W_j$. Now let $g_{l,j}$ be composition of the canonical map $W_l \cap W_j \to W_l$ with φ_{lj}. If we do this for all l, j, then by [Liu02], Lemma 2.3.33 the scheme Z is isomorphic to the scheme that we get if we glue the W_l via the $g_{l,j}$.

By Definition 4.2 we know that $W_l \cap W_j = \operatorname{Spec}(\widetilde{R}_{lj})$ and we have

$$\widetilde{R}_{lj} = k[\widetilde{x}_{l,i_l}, y_{l,i_l}, s_{l,i_l}^{-1}, s_{j,i_j}^{-1}]/(s_{l,i_l}^{-1}s_{l,i_l} - 1, s_{j,i_j}^{-1}s_{j,i_j} - 1, \widetilde{x}_{l,i_l}^{n_j} y_{l,i_l}^{n_l}).$$

So it remains to prove that $\varphi_{l,j}$ maps $\widetilde{x}_{l,i_l}$ to $\widetilde{x}_{l,i_l}(a_{x,l,j} + x_{l,i_l}y_{l,i_l}p_{x,l,j})$ and y_{l,i_l} to $y_{l,i_l}(a_{y,l,j} + \widetilde{x}_{l,i_l}y_{l,i_l}p_{y,l,j})$. We show this without loss of generality for $\widetilde{x}_{l,i_l}$.

By construction $\varphi_{l,j}$ is the identity on the underling topological space, so the only possibility is that $\widetilde{x}_{l,i_l}$ is multiplied by some unit in $\widetilde{R}_{lj}$. But the units in $\widetilde{R}_{lj}$ are of the form $a_{x,l,j} + \widetilde{x}_{l,i_l}y_{l,i_l}p_{x,l,j}$ with $a_{x,l,j} \in k[\widetilde{x}_{l,i_l}, y_{l,i_l}, s_{l,i_l}^{-1}, s_{j,i_j}^{-1}]/(s_{l,i_l}^{-1}s_{l,i_l} - 1, s_{j,i_j}^{-1}s_{j,i_j} - 1)^{\times}$ and $p_{x,l,j} \in k[\widetilde{x}_{l,i_l}, y_{l,i_l}, s_{l,i_l}^{-1}, s_{j,i_j}^{-1}]$.

First suppose that $a_{x,l,j} = s_{l,i_l}^m$ for $m > 0$, then, because of $s_{l,i_l} = y_{l,i_l} - 1$, we get that $\widetilde{x}_{l,i_l}$ maps to $\widetilde{x}_{l,i_l}s_{l,i_l}^m = (-1)^m\widetilde{x}_{l,i_l} + \widetilde{x}_{l,i_l}y_{l,i_l}p_s$ for some polynomial p_s. But then we can simply add p_s to $p_{x,l,j}$. Now suppose $a_{x,l,j} = s_{l,i_l}^{-m}$ for $m > 0$, then we have

$$\widetilde{x}_{l,i_l} = \widetilde{x}_{l,i_l}s_{l,i_l}^{-m}s_{l,i_l}^{m} = \widetilde{x}_{l,i_l}s_{l,i_l}^{-m}((-1)^m + y_{l,i_l}p_s)$$

or

$$\widetilde{x}_{l,i_l}s_{l,i_l}^{-m} = (-1)^m(\widetilde{x}_{l,i_l} - \widetilde{x}_{l,i_l}y_{l,i_l}s_{l,i_l}^{-m}p_s)$$

and again we can add $s_{l,i_l}^{-m}p_s$ to $p_{x,l,j}$.

Suppose finally that $a_{x,l,j} = s_{j,i_j}^m = (\widetilde{x}_{l,i_l} - 1)^m$, with $m \neq 0$. Then the unit $s_{j,i_j} = \widetilde{x}_{l,i_l} - 1$ maps to $\widetilde{x}_{l,i_l}s_{j,i_j}^m - 1$ which is no unit, a contradiction. Because the multiplicative group in which $a_{x,l,j}$ lies is generated by $k^{\times}$, s_{l,i_l} and s_{j,i_j}^{-1} this shows that we indeed have $a_{x,l,j} \in k^{\times}$. □

4.3 Calculating $H^1(P, \Theta_P)$

4.3.1 The Mayer–Vietoris sequence

In the next section, we need the following variant of the well-known Mayer–Vietoris sequence. All notations are defined in Section 3.1.1.

Theorem 4.5. *Let X be a separated scheme, and $\mathcal{F}$ a sheaf on X and I a totally ordered set. Further let $\mathcal{U} = \{(U_i)\}_{i\in I}$ be an open covering of X. There is an exact sequence*

$$0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow \bigoplus_{i\in I} H^0(U_i, \mathcal{F}|_{U_j}) \longrightarrow Z^1(\mathcal{U}, \mathcal{F}) \to H^1(X, \mathcal{F}) \longrightarrow \bigoplus_{i\in I} H^1(U_i, \mathcal{F}|_{U_j})$$

Proof. From the definition of Čech cohomology, we get an exact sequence

$$0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow \bigoplus_{i\in I} H^0(U_i, \mathcal{F}|_{U_i}) \longrightarrow Z^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\lambda} \check{H}^1(\mathcal{U}, \mathcal{F}) \longrightarrow 0 \qquad (4.8)$$

This sequence gives us the first three terms of our sequence. Then, again by Proposition 5.1.1 of [Gro55], we know that the natural map $\tau : \check{H}^1(\mathcal{U}, \mathcal{F}) \to H^1(X, \mathcal{F})$ is injective. Now we interpret $H^1(X, \mathcal{F}) = \check{H}^1(X, \mathcal{F})$ as the group of $\mathcal{F}$-torsors as in Theorem 3.7.

If we have a $\mathcal{F}$-torsor, then by restricting to U_i we get a $\mathcal{F}|_{U_i}$-torsor. If we take the direct sum over all these restrictions, we get a map $H^1(X,\mathcal{F}) \to \bigoplus_{i\in I} H^1(U_i,\mathcal{F}|_{U_i})$, and the kernel of this map are exactly the torsors trivialized by $\mathcal{U}$. But by Theorem 3.7 again, those are given by $\check{H}^1(\mathcal{U},\mathcal{F})$. Summarizing we get that

$$0 \longrightarrow \check{H}^1(\mathcal{U},\mathcal{F}) \xrightarrow{\tau} H^1(X,\mathcal{F}) \longrightarrow \bigoplus_{j\in I_n} H^1(U_j,\mathcal{F}|_{U_j}) \tag{4.9}$$

is exact and thus also the concatenation of (4.8) and (4.9) via $\tau \circ \lambda$, which is the Mayer–Vietoris sequence we wanted. □

4.3.2 Reduction to the rank of a matrix

Now we want to calculate $H^1(P,\mathcal{H}om_{\mathcal{O}_P}(\Omega^1_{P/k},\mathcal{O}_P))$. In this section we show that, at least if all n_l are prime to p, we may calculate $H^1(P,\mathcal{H}om_{\mathcal{O}_P}(\Omega^1_{P/k},\mathcal{O}_P))$ by calculating the rank of a matrix M_P over k. For this we first show that $H^1(P,\mathcal{H}om_{\mathcal{O}_P}(\Omega^1_{P/k},\mathcal{O}_P))$ is isomorphic to the quotient of two — a priori infinite dimensional — k-vector spaces. Then we reduce this quotient to the quotient of two finite dimensional k-vector spaces, and finally we construct the matrix M_P.

To simplify the notation we use the following standard definition:

Definition 4.6. For a k-scheme X we set $\Theta_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/k},\mathcal{O}_X)$.

Reduction to a quotient We want to use the Mayer–Vietoris sequence to reduce the calculation of $H^1(P,\Theta_P)$ to a quotient. For this we set $I = \{1,...,n\}$ and $\mathcal{U} = \{W_l\}_{l\in I}$. Then Theorem 4.5 provides us with an exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0(P,\Theta_P) \longrightarrow \bigoplus_{l=1}^{n} H^0(W_l,\Theta_P|_{W_l}) \longrightarrow Z^1(\mathcal{U},\Theta_P) \\ &\longrightarrow H^1(P,\Theta_P) \longrightarrow \bigoplus_{l=1}^{n} H^1(W_l,\Theta_P|_{W_l}) \end{aligned} \tag{4.10}$$

This sequence is in this form still not much help, but under some conditions on the n_l and p we can show that the last term vanishes, and taking direct limit makes the third term nicer. Using [Liu02], Proposition 6.1.24 (c) we observe:

$$\Theta_P|_{W_l} = \mathcal{H}om_{\mathcal{O}_P|_{W_l}}(\Omega^1_{P/k}|_{W_l},\mathcal{O}_P|_{W_l}) \cong \mathcal{H}om_{\mathcal{O}_{W_l}}(\Omega^1_{W_l/k},\mathcal{O}_{W_l}) = \Theta_{W_l}$$

and use the explicit calculations done in Section 6.1 to see that $H^1(W_l,\Theta_P|_{W_l}) = 0$ if and only if $\delta_{\mathrm{gcd}}(n_l,p) = 1$. So the last term of (4.10) vanishes if and only if all $\delta_{\mathrm{gcd}}(n_l,p) = 1$.

Now we want to make the third term of (4.10) nicer. Let P_l be the image of E_l under the isomorphism from V_l to W_l. We take a decreasing system of open coverings $\mathcal{U}^j = \{U_l^j\}$, $j \in \mathbb{N}$ such that for every l we have $P_l \subset U_l^j \subset W_l$ and $P_l = \bigcap_{j\in\mathbb{N}} U_l^j$ (as topological spaces). Now, as discussed in Section 3.1.1, $Z^1(\mathcal{U}, \Theta_P)$ is defined via the exact sequence

$$0 \longrightarrow Z^1(\mathcal{U}^j, \Theta_P) \longrightarrow \bigoplus_{\substack{(l_0,l_1)\in I^2 \\ l_0<l_1}} H^0(U^j_{l_0 l_1}, \Theta_P) \longrightarrow \bigoplus_{\substack{(l_0,l_1,l_2)\in I^3 \\ l_0<l_1<l_2}} H^0(U^j_{l_0 l_1 l_2}, \Theta_P).$$

Because of $P_{l_0} \cap P_{l_1} \cap P_{l_2} = \emptyset$ we have

$$\varinjlim_{j\in\mathbb{N}} H^0(U^j_{l_0} \cap U^j_{l_1} \cap U^j_{l_2}, \Theta_P) = 0$$

and in the case $P_{l_0} \cap P_{l_1} = \emptyset$ we have also

$$\varinjlim_{j\in\mathbb{N}} H^0(U^j_{l_0} \cap U^j_{l_1}, \Theta_P) = 0.$$

In the remaining case $x_{l_0,l_1} \in P_{l_0} \cap P_{l_1}$ we have

$$\varinjlim_{j\in\mathbb{N}} H^0(U^j_{l_0} \cap U^j_{l_1}, \Theta_P) = \Theta_{P,x_{l_0,l_1}}.$$

Now taking direct limits preserves exactness, so we get

$$\varinjlim_{j\in\mathbb{N}} Z^1(\mathcal{U}^j, \Theta_P) = \bigoplus_{\substack{(l_0,l_1)\in I^2 \\ x_{l_0,l_1}\in P_{l_0}\cap P_{l_1}}} \Theta_{P,x_{l_0,l_1}} \tag{4.11}$$

Finally we define

$$\Theta_{P,P_l} = \varinjlim_{j\in\mathbb{N}} H^0(U^j_l, \Theta_P). \tag{4.12}$$

and use the Mayer–Vietoris argument (4.10) for every $\mathcal{U}^j$ and take the direct limit, so we get an exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(P, \Theta_P) \longrightarrow \bigoplus_{l=1}^{n} \Theta_{P,P_l} \xrightarrow{\rho_P} \bigoplus_{\substack{(l_0,l_1)\in I^2 \\ x_{l_0,l_1}\in P_{l_0}\cap P_{l_1}}} \Theta_{P,x_{l_0,l_1}} \longrightarrow H^1(P, \Theta_P) \\ \longrightarrow \bigoplus_{\substack{l\in I \\ \delta_{\text{gcd}}(n_l,p)=0}} \varinjlim_{j\in\mathbb{N}} H^1(U^j_l, \Theta_P|_{U^j_l}) \end{aligned} \tag{4.13}$$

in particular, we get the reduction we wanted:

Lemma 4.7. *If $\delta_{\text{gcd}}(n_l, p) = 1$ for all l, then*

$$H^1(P, \Theta_P) \cong \bigoplus_{\substack{(l_0,l_1)\in I^2 \\ x_{l_0,l_1}\in P_{l_0}\cap P_{l_1}}} \Theta_{P,x_{l_0,l_1}} \Big/ \rho_P\Big(\bigoplus_{l=1}^{n} \Theta_{P,P_l}\Big) \tag{4.14}$$

Remark 4.8. If we do this also for $H^1(Z,\Theta_Z)$ the terms we get in (4.11) and (4.12) are isomorphic to those of P. So the only term in (4.13) which differs is the map ρ_P which is replaced by a map ρ_Z and the difference depends on the differences in the glueing of Z and P.

In particular one might reformulate Lemma 4.7 for Z.

Reduction to finite dimensional vector spaces Now we want to reduce the calculation of the quotient in (4.14) to a quotient of finite dimensional vector spaces. For this we look at the elements of $\Theta_{P,x_{l,j}}$ for $P_l \cap P_{lj} \neq \emptyset$ and show that all but finitely many of them are always in the image of ρ_P.

For $P_l \cap P_{lj} \neq \emptyset$ every element of $\Theta_{P,x_{l,j}}$ is of the form

$$\sum_{s=\delta_{\mathrm{gcd}}(n_j,p)} \sum_{t=0} \alpha_{st} x_{l,i}^s y_{l,i}^t \frac{\partial}{\partial x_{l,i}} + \sum_{u=0} \sum_{v=\delta_{\mathrm{gcd}}(n_l,p)} \beta_{uv} x_{l,i}^u y_{l,i}^v \frac{\partial}{\partial y_{l,i}} \tag{4.15}$$

with $\alpha_{st}, \beta_{uv} \in k$ and i equals 0 or 1, depending on the chart of W_l in which we find $x_{l,j}$. To simplify the notation, for the next two paragraphs we assume without any loss of generality $i = 0$ and $j = j_1$.

The following two lemmata reduce the elements of (4.15) which are relevant for the calculation of $H^1(P,\Theta_P)$ to only finitely many:

Lemma 4.9. *For all $a \geq n_l$, $b \geq 0$ there are elements $f, g \in \Theta_{P,P_l}$ with*

$$\rho_P(f) = y_{l,0}^a x_{l,0}^{\delta_{\mathrm{gcd}}(n_{j_1},p)+b} \frac{\partial}{\partial x_{l,0}}$$

and

$$\rho_P(g) = y_{l,0}^a x_{l,0}^b \frac{\partial}{\partial y_{l,0}}$$

in $\Theta_{P,x_{l,j}}$ and $\rho_P(f) = \rho_P(g) = 0$ at every other $\Theta_{P,x_{i_1,i_2}}$.

Proof. For $t_l \leq 2$ this follows easily from the local calculations in Section 6.1, because for example $y_{l,0}^a x_{l,0}^b \frac{\partial}{\partial y_{l,0}}$ is in $\Theta_{R_l,0}$ and trivial in $\Theta_{R_l,01}$ ($y_{l,0}^a = 0$ there) so it glues to an element of Θ_{P,P_l}. For $t_l = 3$ we need an extra argument. Formally on $\mathcal{O}_{P,x_{l,j}}$ we have $(x_{l,0}-1)^{-1} = -\sum_{n=0}^{\infty} x_{l,0}^n$, and so we get

$$(x_{l,0}-1)^{-n_{j_3}} y_{l,0}^{n_l} = -y_{l,0}^{n_l} (\sum_{n=0}^{\infty} x_{l,0}^n)^{n_{j_3}} = -y_{l,0}^{n_l} (\sum_{n=0}^{n_{j_1}} x_{l,0}^n)^{n_{j_3}}$$

there. So with

$$f = -x_{l,0}^b (x_{l,0}-1)^{n_{j_3}} (\sum_{n=0}^{n_{j_1}} x_{l,0}^n)^{n_{j_3}} y_{l,0}^a \frac{\partial}{\partial x_{l,0}}$$

we get $\rho_P(f) = x_{l,0}^b y_{l,0}^a \frac{\partial}{\partial x_{l,0}}$. Finally we get g with $\rho_P(g) = x_{l,0}^b y_{l,0}^a \frac{\partial}{\partial y_{l,0}}$ simply by replacing $\frac{\partial}{\partial x_{l,0}}$ with $\frac{\partial}{\partial y_{l,0}}$ in f. ☐

Lemma 4.10. *For all $a \geq 0$, $b \geq n_j$ there are elements $f, g \in \Theta_{P,P_j}$ with*

$$\rho_P(f) = y_{l,0}^a x_{l,0}^b \frac{\partial}{\partial x_{l,0}}$$

and

$$\rho_P(g) = y_{l,0}^{\delta_{\mathrm{gcd}}(n_l,p)+a} x_{l,0}^b \frac{\partial}{\partial y_{l,0}}$$

in $\Theta_{P,x_{l,j}}$ and $\rho_P(f) = \rho_P(g) = 0$ at every other $\Theta_{P,x_{i_1,i_2}}$.

Proof. The argument is the same as before, but here we additionally use the gluing for the plumbing from Section 6.3.1. □

This shows: For the calculation of $H^1(P, \Theta_P)$, we only have to know whether for all l the following finitely many elements of $\Theta_{P,x_{l,j}}$ are in the image of ρ_p:

$$\sum_{s=\delta_{\mathrm{gcd}}(n_j,p)}^{n_j-1} \sum_{t=0}^{n_l-1} \alpha_{st} x_{l,0}^s y_{l,0}^t \frac{\partial}{\partial x_{l,0}} + \sum_{u=0}^{n_j-1} \sum_{v=\delta_{\mathrm{gcd}}(n_l,p)}^{n_l-1} \beta_{uv} x_{l,0}^u y_{l,0}^v \frac{\partial}{\partial y_{l,0}} \tag{4.16}$$

The matrix M_P Now we have a closer look at the remaining elements of Θ_{P,P_l}. These are only finitely many, but depending on the value of t_l we get different lists. For better readability we assume $\delta_{\mathrm{gcd}}(n_l, p) = 1$ for all l. If $\delta_{\mathrm{gcd}}(n_l, p) = 0$ for some l, then the lists remain finite, but we get some extra terms. For the calculations we use the given covering for the W_l. So the elements of Θ_{P,P_l} are just the global sections coming from the calculations in Section 6.1.

Depending on t_l the elements of Θ_{P,P_l} are contained in the following lists: In all three cases the $\frac{\partial}{\partial y_{l,0}}$ are with $0 \leq a \leq \nu_l(b-1)$ and $0 < b$ given by:

$$x_{l,0}^a y_{l,0}^b \frac{\partial}{\partial y_{l,0}} = x_{l,1}^{\nu_l(b-1)-a} y_{l,1}^b \frac{\partial}{\partial y_{l,1}} \tag{4.17}$$

For $\frac{\partial}{\partial x_{l,0}}$ we have look at t_l. For $t_l = 1, 2$ we have with $0 < a \leq (\nu_l b + 1)$ and $0 \leq b$:

$$x_{l,0}^a y_{l,0}^b \frac{\partial}{\partial x_{l,0}} = -x_{l,1}^{\nu_l b-a+2} y_{l,1}^b \frac{\partial}{\partial x_{l,1}} + \nu_l x_{l,1}^{\nu_l b-a+1} y_{l,1}^{b+1} \frac{\partial}{\partial y_{l,1}} \tag{4.18}$$

For $t_l = 1$ we have additionally for $0 \leq b$:

$$y_{l,1}^b \frac{\partial}{\partial x_{l,1}} = -x_{l,0}^{\nu_l b+2} y_{l,0}^b \frac{\partial}{\partial x_{l,0}} + \nu_l x_{l,0}^{\nu_l b+1} y_{l,0}^{b+1} \frac{\partial}{\partial y_{l,0}} \tag{4.19}$$

Finally, for $t_l = 3$ we have for $0 < a \leq \nu_l b$ and $0 < b$:

$$x_{l,0}^a y_{l,0}^b (x_{l,0} - 1) \frac{\partial}{\partial x_{l,0}} = x_{l,1}^{\nu_l b-a+1} y_{l,1}^b (x_{l,1} - 1) \frac{\partial}{\partial x_{l,1}} - \nu_l x_{l,1}^{\nu_l b-a} y_{l,1}^{b+1} (x_{l,1} - 1) \frac{\partial}{\partial y_{l,1}} \tag{4.20}$$

From this and Lemma 4.7 we immediately get the following Theorem:

Theorem 4.11. *If $\delta_{\text{gcd}}(n_l, p) = 1$ for all l, then $H^1(P, \Theta_P) = 0$ if and only if the image of* (4.17)*,* (4.19)*,* (4.18) *or* (4.20) *under ρ_P generates all elements of the form* (4.16)*.*

A nice consequence of this theorem is that it provides a way to actually calculate $\dim_k(H^1(P, \Theta_P))$. For this we construct a $r_P \times c_P$ matrix M_P over k in the following way: For every point $x_{l,j}$ and every element of (4.16) we add one row to M_P. Then for every P_l and every Element of (4.17), (4.19), (4.18) or (4.20) we add a column to M_P. The entries in M_P are simply the coefficients of the element associated to the column as an expansion in the element associated to the row. We give an explicit example of M_P in Section 5.2.2. Note that, by construction, the entries of M_P are integers. Also by the construction of M_P we get the following corollary of Theorem 4.11:

Corollary 4.12. *If $\delta_{\text{gcd}}(n_l, p) = 1$ for all l, then* $\dim(H^1(P, \Theta_P)) = r_P - \operatorname{rank}(M_P)$

Remark 4.13. Theorem 4.11 and Corollary 4.12 work analogously for $H^1(Z, \Theta_Z)$, but M_Z is in practice much harder to write down explicitly than M_P.

As a consequence of the corollary we get the following comparison theorem between $p = 0$ and $p > 0$:

Theorem 4.14. *Let P_0 be a plumbing scheme over $\mathbb{C}$, and for all $p > 0$ with $\delta_{\text{gcd}}(n_l, p) = 1$ for all l let P_p be the plumbing scheme for the same dual graph over an algebraically closed field of characteristic p. Then we have*

$$\dim(H^1(P_0, \Theta_{P_0})) \leq \dim(H^1(P_p, \Theta_{P_p}))$$

and equality for all but finitely many p.

Proof. By Corollary 4.12 we have $\dim(H^1(P_p, \Theta_{P_p})) = r_{P_p} - \operatorname{rank}(M_{P_p})$ for $p = 0$ and $p > 0$. By construction r_{P_p} only depends on the dual graph, so it does not change for different p. Also we get M_{P_p} if we take all entries of M_{P_0} modulo p. Now $\operatorname{rank}(M_{P_0}) = m$ is equivalent to the existence of one non-vanishing $m \times m$ minor, and all $(m+1) \times (m+1)$ minors vanish. But the minors of M_{P_p} are just the minors of M_{P_0} modulo p, so the rank can only decrease, thus the $\dim(H^1(P_p, \Theta_{P_p}))$ can only increase.

Finally the rank decreases if and only if $p > 0$ divides all $m \times m$ minors of M_{P_0}, so it decreases for exactly the prime factors of the gcd of all non vanishing $m \times m$ minors of M_{P_0}. □

Consequences for $H^1(Z, \Theta_Z)$ Our goal is to show, that $H^1(P, \Theta_P) = 0$ implies that every Z combinatorially equivalent to P is already isomorphic to P. We prove this later, but now we are able to prove that $H^1(P, \Theta_P) = 0$ already implies $H^1(Z, \Theta_Z) = 0$ for all Z combinatorial equivalent to P, which of course is a necessary condition for Z to be isomorphic to P:

Theorem 4.15. *If $\delta_{\text{gcd}}(n_l, p) = 1$ for all l, and we have $H^1(P, \Theta_P) = 0$, then we have $H^1(Z, \Theta_Z) = 0$ for all Z combinatorial equivalent to P.*

Proof. By Lemma 4.7 we have to prove that the surjectivity of ρ_P on every $\Theta_{P,x_{l,j}}$ implies the surjectivity of ρ_Z on every $\Theta_{Z,x_{l,j}}$. By Remark 4.8 we know that the only difference between ρ_P and ρ_Z is the gluing. To make this precise: By the calculations of Section 6.3.1 and 6.3.2 we know that $\Theta_{P,x_{l,j}} \cong \Theta_{Z,x_{l,j}}$, and they are as $k[\widetilde{x}_{l,i_l}, y_{l,i_l}]/(\widetilde{x}_{l,i_l}^{n_j} y_{l,i_l}^{n_l})$-module generated by $\widetilde{x}_{l,i_l}\frac{\partial}{\partial \widetilde{x}_{l,i_l}}$ and $y_{l,i_l}\frac{\partial}{\partial y_{l,i_l}}$. Now, for all $f = \widetilde{x}_{l,i_l}^a y_{l,i_l}^b \frac{\partial}{\partial \widetilde{x}_{l,i_l}} \in \Theta_{P,P_l}$ we have $\rho_P(f) = \rho_Z(f) = \widetilde{x}_{l,i_l}^a y_{l,i_l}^b \frac{\partial}{\partial \widetilde{x}_{l,i_l}}$, and the same with $\frac{\partial}{\partial y_{l,i_l}}$. In particular, Lemma 4.9 stays true with ρ_Z instead of ρ_P.

Next we want to look at the image of Θ_{P,P_j} in $\Theta_{P,x_{l,j}}$. Suppose we have a $f \in \Theta_{P,x_{l,j}}$ with $\rho_P(f) = y_{l,i_l}^a \widetilde{x}_{l,i_l}^b \frac{\partial}{\partial \widetilde{x}_{l,i_l}}$ which we also may write as $\widetilde{x}_{j,i_j}^a y_{j,i_j}^b \frac{\partial}{\partial y_{j,i_j}}$. But then by (6.5) and the charts of Lemma 4.4 we get:

$$\rho_Z(f) = \widetilde{x}_{j,i_j}^a y_{j,i_j}^b \frac{\partial}{\partial y_{j,i_j}} = a_{y,l,j}^{a+1} y_{l,i_l}^a a_{x,l,j}^b \widetilde{x}_{l,i_l}^b \frac{\partial}{\partial \widetilde{x}_{l,i_l}} + y_{l,i_l}^{a+1} \widetilde{x}_{l,i_l}^b R_f \tag{4.21}$$

with some R_f. Analogously, if we have some $g \in \Theta_{P,x_{l,j}}$ with

$$\rho_P(g) = y_{l,i_l}^a \widetilde{x}_{l,i_l}^b \frac{\partial}{\partial y_{l,i_l}} = x_{j,i_j}^a y_{j,i_j}^b \frac{\partial}{\partial \widetilde{x}_{j,i_j}}$$

then we have

$$\begin{aligned} \rho_Z(f) &= \widetilde{x}_{j,i_j}^a y_{j,i_j}^b \frac{\partial}{\partial \widetilde{x}_{j,i_j}} \\ &= a_{y,l,j}^a y_{l,i_l}^a a_{x,l,j}^b \widetilde{x}_{l,i_l}^b (a_{x,l,j}^2 \widetilde{x}_{l,i_l}^2 p_{y,j,l} \frac{\partial}{\partial \widetilde{x}_{l,i_l}} + a_{y,l,j} \frac{\partial}{\partial y_{l,i_l}}) + y_{l,i_l}^{a+1} \widetilde{x}_{l,i_l}^b R_g \end{aligned} \tag{4.22}$$

Now we want to prove that we have Lemma 4.10 for Z. For this, let $b \geq n_j$. Because we have Lemma 4.9 for Z, we only have to care for $a < n_l$. For $a = n_l - 1$ the terms $y_{l,i_l}^{a+1} \widetilde{x}_{l,i_l}^b R_f$ and $y_{l,i_l}^{a+1} \widetilde{x}_{l,i_l}^b R_g$ vanish. But $a_{y,l,j}$ and $a_{x,l,j}$ are units in k, so (4.21) shows us that $y_{l,i_l}^a \widetilde{x}_{l,i_l}^b \frac{\partial}{\partial \widetilde{x}_{l,i_l}}$ is in the image of ρ_Z, and with this (4.22) shows that also $y_{l,i_l}^a \widetilde{x}_{l,i_l}^b \frac{\partial}{\partial y_{l,i_l}}$ is in the image of ρ_Z. So by doing inverse induction on a we see that we have Lemma 4.10 for Z.

It remains to show that the surjectivity of ρ_P implies, that for $a < n_l$ and $b < n_l$ also $y_{l,i_l}^a \widetilde{x}_{l,i_l}^b \frac{\partial}{\partial \widetilde{x}_{l,i_l}}$ and $y_{l,i_l}^a \widetilde{x}_{l,i_l}^b \frac{\partial}{\partial y_{l,i_l}}$ are in the image of ρ_Z. But with (4.21) and (4.22) this follows analogously to the argumentation before. We only have to to a double inverse induction on $a + b$: We start with $a = n_l - 1$ and $b = n_j - 1$. In each step we reduce a until $a = 0$ and then we reduce b by one and start again with $a = n_l - 1$. □

Remark 4.16. The inverse of this theorem does not hold. There is a counterexample with $H^1(P, \Theta_P) = \mathbb{C}$ but $H^1(Z, \Theta_Z) = 0$ of Laufer in [Lau73a], §4 (end of page 93).

4.4 Deformations of curves

To prove that $H^1(P, \Theta_P) = 0$ implies that every Z combinatorially equivalent to P is already isomorphic to P, we need one more technique. Laufer's analytic proof of this fact uses the deformation theory of Kodaira–Spencer, and as described in [Ser06] on page 79f, this cannot be directly translated in the language of the modern deformation theory. For example, Laufer needs a versal family not only in the formal setting, but as an "honest" deformation over a manifold. But some observations can be transferred to the algebraic world and turn out to be rather useful.

So in this section, we want to translate as much of Laufer's results on deformations of (exceptional) curves to the algebraic world as possible and needed.

First we have to cite some definitions, which we take mainly from [Ser06], Sections 1.2.1 and 2.4.1.

Let X be a k-scheme. A *deformation η of X over (S, s)* is a cartesian diagram

$$\eta\colon \quad \begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \operatorname{Spec}(k) & \xrightarrow{\ s\ } & S \end{array}$$

where π is flat and surjective, S is connected and s is a k-rational point of S. If

$$\xi\colon \quad \begin{array}{ccc} X & \longrightarrow & \mathcal{Y} \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec}(k) & \longrightarrow & S \end{array}$$

is another deformation of X, then an *isomorphism of η with ξ* is a S-isomorphism $\Phi : \mathcal{X} \to \mathcal{Y}$ inducing the identity on X. Let S, S' be two connected k-scheme S and let s and s' be k-rational points of S and S', and $\varphi : S' \to S$ be a morphism mapping s' to s. Then for every deformation η of X over (S, s), we get a deformation

$$\eta_{S'}\colon \quad \begin{array}{ccc} X & \longrightarrow & \mathcal{X} \times_S S' \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \operatorname{Spec}(k) & \xrightarrow{\ s'\ } & S' \end{array}$$

over (S', s'). This is a deformation because flatness and surjectivity are stable under base change. Also for (S, s) as above we have the *trivial deformation*

$$\begin{array}{ccc} X & \longrightarrow & X \times S \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec}(k) & \xrightarrow{\ s\ } & S \end{array}$$

of X. Finally we say that a deformation η of X is *locally trivial* if for every point $x \in X$ we find an open neighbourhood $U_x \subset X$ such that the induced deformation of U_x

$$\eta|_{U_x}\colon \quad \begin{array}{ccc} U_x & \longrightarrow & \mathcal{X}|_{U_x} \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \operatorname{Spec}(k) & \xrightarrow{\ s\ } & S \end{array}$$

is isomorphic to the trivial deformation of U_x.

For a k-scheme X we define the following functor from the category of connected schemes together with a k-rational point to sets:

$$\mathrm{Def}'_X(S, s) = \{\text{locally trivial deformations of } X \text{ over } (S, s)\}/\text{isomorphism}$$

Note that being locally trivial is a rather big restriction. For example, if X is a smooth, proper, one-dimensional scheme over k, then we know that another smooth, proper, one-dimensional scheme X' over k is isomorphic to X if and only if we have open $U \subset X$ and $V \subset X'$ such that U and V are isomorphic. In particular, if $\eta \in \mathrm{Def}'_X(S, s)$, then for every $s' \in S$ with smooth fibre $\pi^{-1}(s')$, this fibre is isomorphic to X. On the other hand, this restriction makes the functor better understandable, in particular if we assume the schemes S to be spectra of artinian rings. Also this restriction is no problem for us, because Lemma 4.1 shows that if we have a deformation of Z, and if all fibres are combinatorially equivalent to Z, then they are locally isomorphic.

The following theorem is the main help in the next section:

Theorem 4.17. *Let $[P] \in \mathrm{CEQ}(Z)$ be the plumbing scheme for Z. Then there exists an integral affine scheme Y, a k-rational point $y' \in Y$ and $\eta \in \mathrm{Def}'_Z(Y, y)$ with $[\pi^{-1}(y')] = [P] \in \mathrm{CEQ}(Z)$.*

Proof. The existence of such a deformation is a consequence of Lemma 4.4. From this lemma we know that for Z the glueing along every $W_{lj} \neq \emptyset$ is done via $x_{j,i_j} = y_{l,i_l}(a_{y,l,j} + x_{l,i_l} y_{l,i_l} p_{y,l,j})$ and $y_{j,i_j} = x_{l,i_l}(a_{x,l,j} + x_{l,i_l} y_{l,i_l} p_{x,l,j})$.

Let $A = k[u_{x,l,j}, u_{y,l,j}, u_{x,l,j}^{-1}, u_{y,l,j}^{-1}, t_x, t_y]$ (with lj running over all lj such that $W_{lj} \neq \emptyset$), and $Y = \mathrm{Spec}(A)$. We define $\mathcal{X}$ as follows: We glue the $W_l \times Y$ along the $W_{lj} \times Y$ via $x_{j,i_j} = y_{l,i_l}(u_{y,l,j} + x_{l,i_l} y_{l,i_l} p_{y,l,j} t_y)$ and $y_{j,i_j} = x_{l,i_l}(u_{x,l,j} + x_{l,i_l} y_{l,i_l} p_{x,l,j} t_x)$ which defines an automorphism, because the right factors are of the form "invertible + nilpotent".

Let now π be the projection. By construction of $\mathcal{X}$ we have $P \cong \pi^{-1}(1, 1, \ldots, 1, 1, 0, 0)$ and $Z \cong \pi^{-1}(a_{y,1,2}, a_{x,1,2}, \ldots, a_{x,l,n}, a_{y,l,n}, 1, 1)$.

Now π is locally trivial by construction, and thus also flat, because flatness is a local condition, and the trivial deformation is flat. □

4.5 $H^1(P, \Theta_P) = 0$ implies P isomorphic to Z

Now we are able to prove that $H^1(P, \Theta_P) = \{0\}$ implies $\mathrm{CEQ}(Z) = \{[Z]\}$.

Theorem 4.18. *Let Z as described in Section 4.1.2, with $\delta_{\mathrm{gcd}}(n_l, p) = 1$ for all l, and let P be the plumbing scheme for Z. If $H^1(P, \Theta_P) = 0$, then Z is isomorphic to P.*

Proof. From Theorem 4.17 we get a locally trivial deformation

$$\eta\colon\quad \begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(k) & \longrightarrow & Y \end{array}$$

of Z into P. Now the base of this deformation is an integral affine scheme, so, via localisation, we may assume that we have $Y = \operatorname{Spec}(R)$, where R is an integral semi-local ring with exactly two maximal ideals m_1 and m_2. Let y_i be the point given by m_i, and let $X_i = \pi^{-1}(y_i)$. Suppose that we have $X_1 \cong Z$ and $X_2 \cong P$.

Localizing further we get two local rings (R_1, m_1) and (R_2, m_2) both with residue field k and a common quotient field K. Then we have $X_i \cong \mathcal{X} \times_{\operatorname{Spec}(R)} \operatorname{Spec}(R_i/m_i)$. Let $\widehat{R}_i$ be the completion of R_i, and K_i the quotient field of $\widehat{R}_i$. By the universal property of the quotient field we get maps $K \to K_i$, and there exists a field $\widetilde{K}$ containing K_1 and K_2. In other words, we get a commutative diagram:

$$
\begin{array}{ccccccc}
R_1 & \longrightarrow & K & & K & \longleftarrow & R_2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\widehat{R}_1 & \longrightarrow & K_1 & & K_2 & \longleftarrow & \widehat{R}_2 \\
 & & & \searrow \quad \swarrow & & & \\
 & & & \widetilde{K} & & &
\end{array}
$$

From this we get the dual diagram for the spectra, and using this and standard properties of the fibre-product from [Gro60], §3.3, we get:

$$
\begin{aligned}
& \mathcal{X} \times_{\operatorname{Spec}(R_1)} \operatorname{Spec}(\widehat{R}_1) \times_{\operatorname{Spec}(\widehat{R}_1)} \operatorname{Spec}(\widetilde{K}) \\
\cong\ & \mathcal{X} \times_{\operatorname{Spec}(R_1)} K \times_{\operatorname{Spec}(K)} \operatorname{Spec}(K_1) \times_{\operatorname{Spec}(K_1)} \operatorname{Spec}(\widetilde{K}) \\
\cong\ & \mathcal{X} \times_{\operatorname{Spec}(R_2)} K \times_{\operatorname{Spec}(K)} \operatorname{Spec}(K_1) \times_{\operatorname{Spec}(K_1)} \operatorname{Spec}(\widetilde{K}) \\
\cong\ & \mathcal{X} \times_{\operatorname{Spec}(R_2)} \operatorname{Spec}(\widehat{R}_2) \times_{\operatorname{Spec}(\widehat{R}_2)} \operatorname{Spec}(\widetilde{K})
\end{aligned}
\tag{4.23}
$$

Our next goal is to show that we have

$$\mathcal{X} \times_{\operatorname{Spec}(R_i)} \operatorname{Spec}(\widehat{R}_i) \cong X_i \times_{\operatorname{Spec}(k)} \operatorname{Spec}(\widehat{R}_i),$$

because using (4.23) this gives us

$$Z \times_{\operatorname{Spec}(k)} \operatorname{Spec}(\widetilde{K}) \cong P \times_{\operatorname{Spec}(k)} \operatorname{Spec}(\widetilde{K}).$$

For this we look at the functors Def'_P and Def'_Z and restrict them to spectra of local artinian k-algebras. By Theorem 2.4.1 of [Ser06] the tangent-space of this functors are $H^1(P, \Theta_P)$ and $H^1(Z, \Theta_Z)$ respectively and thus trivial; the first one by the assumption, the second one by Theorem 4.15. So by the same Theorem of [Ser06], they have a semi-universal element. Now Proposition 2.2.8 of [Ser06] tells us $\operatorname{Def}'_P = \operatorname{Hom}(k, \quad) = \operatorname{Def}'_Z$. From this we get $\operatorname{Def}'_Z(\widehat{R}_1) = \operatorname{Hom}(k, \widehat{R}_1)$ and $\operatorname{Def}'_P(\widehat{R}_2) = \operatorname{Hom}(k, \widehat{R}_2)$, or in other words, for every n we have:

$$\mathcal{X} \times_{\operatorname{Spec}(R_i)} \operatorname{Spec}(\widehat{R}_i/m_i^{n+1}) \cong X_i \times_{\operatorname{Spec}(k)} \operatorname{Spec}(\widehat{R}_i/m_i^{n+1})$$

That is, if we use the theory of formal schemes as a black box we have an isomorphism

$$\widehat{\mathcal{X}}|_{X_i} \cong \widehat{X_i}|_{X_i},$$

which by [Gro61], 5.4.1 gives us

$$\mathcal{X} \times_{\mathrm{Spec}(R_i)} \mathrm{Spec}(\widehat{R}_i) \cong X_i \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\widehat{R}_i),$$

as wanted.

So we have not yet that P and Z are isomorphic, but we know that they are isomorphic after base change to some field extension of k. Now we want to get the isomorphism between P and Z from this isomorphism. For this we take a look at the isomorphism functor

$$\begin{aligned} \mathcal{I}som_k(Z, P) : \{\text{Schemes}/k\} &\to \{\text{groups}\} \\ S &\mapsto \mathrm{Iso}_k(Z \times_k S, P \times_k S) \end{aligned}$$

We have shown

$$\mathcal{I}som_k(Z, P)(\mathrm{Spec}(\widetilde{K})) \neq \emptyset,$$

and the theorem follows if we can show

$$\mathcal{I}som_k(Z, P)(\mathrm{Spec}(k)) \neq \emptyset.$$

Fortunately, because Z and P are proper, one-dimensional schemes over a field and thus projective, by [Gro95], Section 4c together with Théorème 3.1 and Page 265 the functor $\mathcal{I}som_k(Z, P)$ is represented by a scheme I locally of finite type over k. That is, we have a natural isomorphism between the functor $\mathcal{I}som_k(Z, P)$ and the functor $\mathrm{Hom}(\ , I)$. In particular, we have bijections from $\mathcal{I}som_k(Z, P)(\mathrm{Spec}(\widetilde{K}))$ and $\mathcal{I}som_k(Z, P)(\mathrm{Spec}(k))$ to the $\widetilde{K}$-rational respectively k-rational points of I.

So we know $I(\widetilde{K}) \neq \emptyset$, thus I is not the empty scheme. But I is locally of finite type over k, so by Hilbert's Nullstellensatz I has a $\overline{k}$-rational point, but k is algebraically closed, so we have a k-rational point. □

Because the plumbing P only depends on the dual graph of Z we immediately get the following corollary:

Corollary 4.19. *Let Z and P be as in the theorem. If $H^1(P, \Theta_P) = 0$, then every B' combinatorially equivalent to Z is already isomorphic to Z. That is, Z is defined by its dual graph.*

For singularities this has the following consequence:

Theorem 4.20. *Let (S, s) be a normal, two-dimensional singularity, $\widetilde{Z} = \sum_{l=1}^{n} n_l E_l$ an anti-ample divisor for (S, s) with all $\delta_{\mathrm{gcd}}(n_l, p) = 1$ for all l. Further let ν be the significant multiplicity for $\widetilde{Z}$. If P is the plumbing scheme for $\nu\widetilde{Z}$, then (S, s) is taut if $H^1(P, \Theta_P) = 0$.*

Proof. By Corollary 4.19 we have $\mathrm{CEQ}(P) = \mathrm{CEQ}(\nu\widetilde{Z}) = \{[\nu\widetilde{Z}]\}$, so (S,s) is taut by Corollary 3.39. □

Finally we are able to prove the next comparison theorem between $p = 0$ and $p > 0$:

Theorem 4.21. *Let Γ be the dual graph of some plumbing scheme P_0 over $\mathbb{C}$, and for all $p > 0$ with $\delta_{\mathrm{gcd}}(n_l, p) = 1$ for all l let P_p be the plumbing scheme for Γ over an algebraically closed field of characteristic p. Then $\mathrm{CEQ}(P_0) = \{[P_0]\}$ implies $\mathrm{CEQ}(P_p) = \{[P_p]\}$ for all but finitely many p.*

Proof. By [Lau73a], Theorem 3.9 from $\mathrm{CEQ}(P_0) = \{[P_0]\}$ we get $H^1(P_0, \Theta_{P_0}) = 0$, which by Theorem 4.14 implies $H^1(P_p, \Theta_{P_p}) = 0$ for all but finitely many $p > 0$. So we get $\mathrm{CEQ}(P_p) = \{[P_p]\}$ for the good p with Corollary 4.19. □

4.6 Open questions I

Before we come to the application of these theorems to the question of tautness, which we will do in the next section, we want to discuss the first step in the last proof. This is actually the only place in this work where we need to use a Theorem of Laufer, which we are not able to modify and reprove for arbitrary characteristics. Theorem 3.9 of [Lau73a] is a stronger version of our Corollary 4.19, which also has the inverse implication. That is, it says $\mathrm{CEQ}(Z) = \{[Z]\}$ if and only if $H^1(P, \Theta_P) = 0$.

Now we want to look at the following example for $p > 0$: We take $E = E_1 = \mathbb{P}^1_k$ and $Z = pE_1$ and $\nu_1 > 1$. With an explicit calculation, done in Section 6.2.1, we get $\mathrm{CEQ}(2E_1) = \{[2E_1]\}$, and in Corollary 3.38 we have $\widetilde{Z} = E$ and $\nu = 2$, so this implies $\mathrm{CEQ}(jE_1) = \{[jE_1]\}$ for all j. In particular, we have $P \cong Z$.

But again a calculation in local coordinates, done in Section 6.1.2, shows that we have $h^1(Z, \Theta_Z) = h^1(P, \Theta_P) = \nu_1 - 1$ for $p|i$.

So Theorem 3.9 of [Lau73a] cannot be true for $p > 0$ without modifications. But if we demand the n_l to be prime to $p > 0$, then with Corollary 4.19 we have the “if” statement of Laufer's Theorem 3.9, and we think that this is also the modification needed for the “only if” direction, but we are not able to prove this.

In our very simple example “n_l prime to p” is a working modification. Later in Section 5 we get additional evidence for this, because we can show that a rational double point is taut if and only if $H^1(P, \Theta_P) = 0$ for all Z with all $\delta_{\mathrm{gcd}}(n_l, p) = 1$: We have the explicit list of all taut and non-taut rational double points of Artin ([Art66]) and for all taut rational double points Section 5.2.1 shows $H^1(P, \Theta_P) = 0$ for any large enough anti-ample cycle Z supported by the exceptional locus with all $\delta_{\mathrm{gcd}}(n_l, p) = 1$. On the other hand, for all non-taut rational double points Section 5.2.2 shows $H^1(P, \Theta_P) > 0$ for all cycle Z supported by the exceptional locus with all $\delta_{\mathrm{gcd}}(n_l, p) = 1$. This leads to the following conjecture:

Conjecture 4.22. Let Z be as described in Section 4.1.2, with $\delta_{\mathrm{gcd}}(n_l, p) = 1$ for all l, and let P be the plumbing scheme for Z. Then we have $\mathrm{CEQ}(Z) = \{[Z]\}$ if and only if $H^1(P, \Theta_P) = 0$.

5 Taut and non taut singularities

5.1 Taut over $\mathbb{C}$ implies taut for nearly all p

Now we want to transfer the last theorem of the previous section to the question of tautness of normal, two-dimensional singularities. We get the following theorem:

Theorem 5.1. *Let (S_0, s_0) be a normal, two-dimensional singularity over $\mathbb{C}$ with dual graph Γ. For all primes p let (S_p, s_p) be a Γ-singularity over an algebraically closed field of characteristic p with dual graph Γ. If (S_0, s_0) is taut, then (S_p, s_p) is taut for all but finitely many p.*

Proof. First by Lemma 2.62 we know that (S_p, s_p) for $p = 0$ and $p > 0$ is not taut if Γ is not of the form which we assume for the plumbing. So from now on, we may assume that we have a plumbing scheme for Γ. Let P_p for $p = 0$ and $p > 0$ this plumbing scheme. Then, by construction, P_p is embedded into a regular, two-dimensional scheme, and by Corollary 2.50 we get a Γ-singularity (S_p, s_p) for every $p > 0$.

Now let $\widetilde{Z}_0$ be an anti-ample divisor for (S_0, s_0) and $\widetilde{Z}_p$ one for (S_p, s_p). Denote by ν_0 and ν_p their significant multiplicity. By Corollary 3.39 the tautness of (S_0, s_0) implies $\mathrm{CEQ}(\nu_0 \widetilde{Z}_0) = \{[\nu_0 \widetilde{Z}_0]\}$.

$\widetilde{Z}_0$ and $\widetilde{Z}_p$ are defined by combinatorial data, so we can assume that all coefficients of $\widetilde{Z}_0$ and $\widetilde{Z}_p$ are equal. By the construction of ν we have $\nu_p = \nu_0$ and $\delta_{\mathrm{gcd}}(n_l, p) = 1$ for all l for all but finitely many p.

Let P_0 be the plumbing scheme for $\nu_0 \widetilde{Z}_0$ and P_p the ones for $\nu_p \widetilde{Z}_p$. We have $\mathrm{CEQ}(\nu_0 \widetilde{Z}_0) = \{[\nu_0 \widetilde{Z}_0]\} = \mathrm{CEQ}(P_0)$, so we are in the situation of Theorem 4.21, that is we have $\mathrm{CEQ}(P_p) = \{[P_p]\}$ for all but finitely many p. So for all good p we have $\mathrm{CEQ}(P_p) = \mathrm{CEQ}(\nu_p \widetilde{Z}_p)$, so (S_p, s_p) is taut by Corollary 3.39. □

Note that for a given Γ we can compute the good and the bad p for this Γ. With "good" we mean that for this p the tautness of (S_0, s_0) implies the tautness of (S_p, s_p). The two places in the proof where we had to exclude some primes can be healed. The first place is very simple: For all p with $\nu_p = \nu_0 + 1$ we simply do the proof again, with ν_0 replaced by $\nu_0 + 1$. The second place needs a little more thinking, but with Lemma 2.58 we see that we can always choose the coefficients of $\widetilde{Z}_0$ prim to every fixed p. So going through the proof finitely many times shows that a p is good if it is not one of the finitely many primes excluded by Theorem 4.21. That is p is good if and only if we have equality in Theorem 4.14. So theoretically we are able to calculate all good p for a given singularity, but in practice the matrix M_{P_0} is huge.

If we can calculate all good p, we also get all bad p. As discussed after Theorem 4.21, if we had an inverse of Corollary 4.19, we could also show that the bad p have (S_p, s_p) not taut. With this we would get "Then (S_0, s_0) is taut if and only if (S_p, s_p) is taut for all but finitely many p". But we cannot prove this, so all we can prove is that for the bad p, we have $H^1(P_p, \Theta_{P_p}) > 0$. This is Theorem 4.14.

For a special class of normal, two-dimensional singularities, the rational double points, one knows exactly which are taut for which p, and we can show that they are taut exactly for the good p. We will do this in the next section.

5.2 Rational double points

Now we want to discuss the question of tautness for a special class of singularities. We say that a normal, two-dimensional singularity (S, s) is a *rational double point* if it is rational and we have have $Z^2 = -2$, where Z is the fundamental cycle of (S, s). The last condition is by [Art66], Corollary 6 equivalent to the fact that $\mathcal{O}_{S,s}$ has multiplicity 2, which explains the name.

Now being a rational double point gives rather strong restrictions on the combinatorics of Z, and it is a result of Artin in [Art66] that the rational double points are exactly the normal, two-dimensional singularities with dual graph isomorphic to one of the Dynkin-diagrams A_n, D_n, E_6, E_7 or E_8 (See [Băd01] Theorem 3.32 for a detailed proof).

In [Art77] Artin calculated a complete list of all equations of rational double points in every characteristic. This list shows that all rational double points are taut, except D_n for $p = 2$, E_6 and E_7 for $p = 2, 3$ and E_8 for $p = 2, 3, 5$. We can reprove the tautness results of Artin with our methods, and we also can show certain non-vanishing results for the non-taut cases.

5.2.1 Taut rational double points

For this section (S, s) is always a normal, two-dimensional singularity, $\widetilde{Z} = \sum_{l=1} n_l E_l$ an anti-ample cycle for (S, s) with all $\delta_{\text{gcd}}(n_l, p) = 1$, as in Lemma 2.58, and ν its significant multiplicity. Further let X be the smooth, two-dimensional scheme with $\widetilde{Z} \subset X$ from the definition of $\widetilde{Z}$. That is, $f : X \to \text{Spec}(A)$ is a minimal good desingularization of an algebraization of (S, s) with $\widetilde{Z}$ supported on the exceptional fibre of f.

Let $E = \sum_{l=1} E_l$. We define the sheaf $\Theta_X(-\log(E))$ on X via

$$0 \longrightarrow \Theta_X(-\log(E)) \longrightarrow \Theta_X \longrightarrow \bigoplus_{l=1}^{n} \mathcal{N}_{E_l/X} \longrightarrow 0 \tag{5.1}$$

By [Wah76], Proposition 2.2 this is a locally free $\mathcal{O}_X$-module of rank 2. Now we need the notion of *cohomology with supports*: Let Y be a scheme and C a closed subscheme, and $\mathcal{F}$ a sheaf on X, then we define

$$H^0_C(\mathcal{F}) = \{s \in H^0(\mathcal{F}) \mid s_p = 0 \;\forall_{p \in Y \setminus C}\}$$

This defines a left exact functor form the category of abelian sheaves on Y to the category of groups, and we denote with $H^i_C(\mathcal{F})$ the left derived functor.

By Theorem 5.19 of [Wah75] we have $H^1_E(\Theta_X(-\log(E))(E)) = 0$ for the taut rational double points. Our goal is to show that $H^1_E(\Theta_X(-\log(E))(E)) = 0$ implies $H^1(j\widetilde{Z}, \Theta_{j\widetilde{Z}}) = 0$ for some j, and that we get the tautness from this.

First we want to show that the usual cohomology of both sheaves agrees:

Lemma 5.2. *For j sufficiently large with $\delta_{\mathrm{gcd}}(j,p) = 1$ we have*

$$H^1(j\widetilde{Z}, \Theta_{j\widetilde{Z}}) = H^1(X, \Theta_X(-\log(E))).$$

Proof. For $p = 0$ we get from (1.6) of [BW74] the sequence

$$0 \to \Theta_{j\widetilde{Z}} \longrightarrow \Theta_X \otimes \mathcal{O}_{j\widetilde{Z}} \longrightarrow \bigoplus_{l=1}^{n} \mathcal{N}_{E_l/X} \longrightarrow 0 \tag{5.2}$$

The formal calculations proving the exactness of this sequence do not change for $p > 0$ if all coefficients of $j\widetilde{Z}$ are prime to p. So because we have chosen all coefficients of $\widetilde{Z}$ with $\delta_{\mathrm{gcd}}(n_l, p) = 1$ we have the sequence (5.2) for $\delta_{\mathrm{gcd}}(j,p) = 1$.

If we look at the constructions of (5.1) ([Wah76], Page 333) and (5.2) ([BW74], Page 71) we get map $\Theta_X(-\log(E)) \to \Theta_{j\widetilde{Z}}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
 & & & & \Theta_X \otimes \mathcal{O}_X(-j\widetilde{Z}) & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Theta_X(-\log(E)) & \longrightarrow & \Theta_X & \longrightarrow & \bigoplus_{l=1}^{n} \mathcal{N}_{E_l/X} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \mathrm{id} & & \\
0 & \longrightarrow & \Theta_{j\widetilde{Z}} & \longrightarrow & \Theta_X \otimes \mathcal{O}_{j\widetilde{Z}} & \longrightarrow & \bigoplus_{j=1}^{n} \mathcal{N}_{E_l/X} & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

The column in the middle arises from the standard sequence

$$0 \longrightarrow \mathcal{O}_X(-j\widetilde{Z}) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{j\widetilde{Z}} \longrightarrow 0$$

after tensoring with Θ_X which is locally free, because X is smooth. Now the snake lemma gives us an exact sequence

$$0 \longrightarrow \Theta_X \otimes \mathcal{O}_X(-j\widetilde{Z}) \longrightarrow \Theta_X(-\log(E)) \to \Theta_{j\widetilde{Z}} \to 0$$

Taking cohomology, we get using Section 3.1.2 for the third term:

$$\begin{aligned}
&H^1(X, \Theta_X \otimes \mathcal{O}_X(-j\widetilde{Z})) \longrightarrow H^1(X, \Theta_X(-\log(E))) \longrightarrow H^1(j\widetilde{Z}, \Theta_{j\widetilde{Z}}) \\
\to\ &H^2(X, \Theta_X \otimes \mathcal{O}_X(-j\widetilde{Z}))
\end{aligned}$$

Now $\mathcal{O}_X(-\widetilde{Z})$ is ample by Lemma 2.55, so the first and the last term vanish for j sufficiently large by [Liu02], Proposition 5.3.6. And thus we have

$$H^1(j\widetilde{Z}, \Theta_{j\widetilde{Z}}) = H^1(X, \Theta_X(-\log(E)))$$

for all j sufficiently large with $\delta_{\text{gcd}}(j,p) = 1$. □

Let ω_X be the dualizing sheaf of X, that is, because X is smooth of dimension 2, we have

$$\omega_X = \bigwedge^2 \Omega^1_{X/k}$$

Then we get the following isomorphism of k-vector spaces:

Lemma 5.3. *We have*

$$H^1_E(\Theta_X(-\log(E))(E) \otimes \omega_X^{\otimes 2}) \cong H^1(X, \Theta_X(-\log(E)))^\vee.$$

Proof. From Theorem 4.9 of [Băd01] we get

$$H^1_E(\Theta_X(-\log(E))(E) \otimes \omega_X^{\otimes 2}) \cong H^1(X, \left(\Theta_X(-\log(E))(E) \otimes \omega_X^{\otimes 2}\right)^\vee \otimes \omega_X)$$

and because taking the dual commutes with tensor products the last term is just

$$H^1(X, \Theta_X(-\log(E))(E))^\vee \otimes \omega_X^\vee),$$

and the lemma follows if we show

$$(\Theta_X(-\log(E))(E))^\vee \cong \Theta_X(-\log(E)) \otimes \omega_X.$$

Now, for a locally free sheaf $\mathcal{F}$ of rank 2 by [Wah85], Page 276 we have

$$\mathcal{F}^\vee = (\mathcal{F}^\vee)^\vee \otimes \bigwedge^2 \mathcal{F}^\vee.$$

So we get with (3.5) and (1.2) of [Wah85]:

$$\Theta_X(-\log(E))^\vee = \Theta_X(-\log(E)) \otimes \bigwedge^2 \Omega^1_{X/k}(\log(E)) = \Theta_X(-\log(E)) \otimes \omega_X \otimes \mathcal{O}_X(E)$$

and thus:

$$(\Theta_X(-\log(E))(E))^\vee = \Theta_X(-\log(E))^\vee \otimes \mathcal{O}_X(-E) = \Theta_X(-\log(E)) \otimes \omega_X$$

which is the isomorphism we need. □

Summarizing the previous lemmata we get:

Theorem 5.4. *Let (S, s) be a normal, two-dimensional singularity such that for one anti-ample cycle $\widetilde{Z}$ with significant multiple ν, the multiple $\nu\widetilde{Z}$ is isomorphic to its plumbing scheme. Then (S, s) is taut if*

$$H^1_E(\Theta_X(-\log(E))(E) \otimes \omega_X^{\otimes 2}) = 0$$

Proof. With Lemma 5.3 and 5.2 we get

$$0 = H^1_E(\Theta_X(-\log(E))(E) \otimes \omega_X^{\otimes 2}) = H^1(j\widetilde{Z}, \Theta_{j\widetilde{Z}})$$

and the last term is by assumption isomorphic to $H^1(P, \Theta_P)$, so the tautness follows with Theorem 4.20. □

For rational double points the situation becomes more simple, because we may assume $\omega_X \cong \mathcal{O}_X$. To show this we need some more definitions: Let $\widetilde{S}$ be the algebraization of (S, s) such that $f : X \to \widetilde{S}$ is the minimal good resolution of $\widetilde{S}$. By Theorem 4 of [Art66] we may choose $\widetilde{S}$ such that it embeds into $\mathbb{A}^m_k$ with $m \geq 2$. Then we define

$$\omega_{\widetilde{S}} = \mathcal{E}xt^{m-2}_{\mathbb{A}^3_k}(\mathcal{O}_{\widetilde{S}}, \bigwedge^m \Omega^1_{\mathbb{A}^m/k})$$

We say that $\widetilde{S}$ is *Gorenstein* if $\omega_{\widetilde{S}}$ is invertible. We note that our definition is just a special case of a more general definition. In particular by Theorem 18.3 of [Mat89] the question whether $\widetilde{S}$ is Gorenstein only depends on S, so we may also say that (S, s) is a Gorenstein singularity.

Now for a Gorenstein singularity we can say something more on ω_X:

Lemma 5.5. *If the normal, two-dimensional singularity (S, s) is Gorenstein, then maybe after shrinking of S we have*

$$\omega_X = \begin{cases} \mathcal{O}_X & \textit{if } (S,s) \textit{ is rational;} \\ \mathcal{O}_X(-D) \ : \ D > 0, \ \mathrm{Supp}(D) = E & \textit{else.} \end{cases}$$

Proof. Because (S, s) is Gorenstein we have $\omega_{\widetilde{S}}|U \cong \mathcal{O}_U$ for some open $s \in U \subset \widetilde{S}$. So maybe after shrinking we may assume $U = \widetilde{S}$ and thus $\omega_{\widetilde{S}} \cong \mathcal{O}_{\widetilde{S}}$. But then we get $f^*(\omega_{\widetilde{S}}) \cong \mathcal{O}_X$, and the lemma is just a reformulation of Theorem 4.17 of [Băd01]. □

Now by a result of Artin we know that a normal, two-dimensional singularity is rational and Gorenstein if and only if it is a rational double point, so Lemma 5.5 and Theorem 5.4 give the following corollary:

Corollary 5.6. *Let (S, s) be a rational double point such that for one anti-ample cycle $\widetilde{Z}$ with significant multiple ν, the multiple $\nu\widetilde{Z}$ is isomorphic to its plumbing scheme. Then (S, s) is taut if*

$$H^1_E(\Theta_X(-\log(E))(E)) = 0$$

Now the sheaf $\Theta_X(-\log(E))(E)$ is the sheaf $S(E)$ of [Wah75], so we get the tautness part of Artin's classification:

Corollary 5.7. *A rational double point is taut if its dual graph is isomorphic to one of the following Dynkin-diagrams:*

$$\begin{cases} A_n & p = 2; \\ A_n, D_n & p = 3; \\ A_n, D_n, E_6, E_7 & p = 5; \\ A_n, D_n, E_6, E_7, E_8 & p = 0 \text{ or } p \geq 7. \end{cases} \tag{5.3}$$

Proof. From all rational double points with dual graph isomorphic to one of the Dynkin-diagrams in the corollary let (S, s) be the one with $\nu\widetilde{Z}$ isomorphic to its plumbing scheme. Then by Theorem D (for $p = 0$) and Theorem 5.19 (for $p > 0$ and the Dynkin-diagrams as in (5.3)) of [Wah75] we have $H^1_E(\Theta_X(-\log(E))(E)) = 0$. Then the tautness follows with Corollary 5.6. □

The main point of this proof is hidden in Theorem D and 5.19 of [Wah75]. This theorems calculate $H^1_E(\Theta_X(-\log(E))(E) = 0$ only using the combinatorial data of $\nu\widetilde{Z}$, so for a given dual graph, we know the vanishing actually for all singularities with this dual graph, in particular also for the one whose anti-ample cycle is isomorphic to its plumbing scheme.

Finally for Gorenstein but not rational singularities we get the following corollary of Lemma 5.5 and Theorem 5.4:

Corollary 5.8. *Let (S, s) be a Gorenstein but not rational normal, two-dimensional singularity, such that for one anti-ample cycle $\widetilde{Z}$ with significant multiple ν, the multiple $\nu\widetilde{Z}$ is isomorphic to its plumbing scheme. Then (S, s) is taut if*

$$H^1_E(\Theta_X(-\log(E))(E - 2D)) = 0.$$

5.2.2 Non taut rational double points

Now we want to look at the rational double points which are not taut. As first example we want to look at D_4 for $p = 2$. For $2E$ we present the matrix $M_{P_{2E}}$ at the following page. To fit this matrix on the page, we had to write $a\frac{\partial}{\partial b}$ as $\frac{a}{\frac{\partial}{\partial b}}$ at the labels and replace $(x_{1,0} - 1)$ by $y_{4.0}$ for the last three columns.

	$y_{2,0}\frac{\partial}{\partial y_{2,0}}$	$y_{3,0}\frac{\partial}{\partial y_{3,0}}$	$y_{4,0}\frac{\partial}{\partial y_{4,0}}$	$\frac{\partial}{\partial x_{2,1}}$	$\frac{\partial}{\partial x_{3,1}}$	$\frac{\partial}{\partial x_{4,1}}$	$x_{2,0}\frac{\partial}{\partial x_{2,0}}$	$x_{3,0}\frac{\partial}{\partial x_{3,0}}$	$x_{4,0}\frac{\partial}{\partial x_{4,0}}$	$y_{2,0}x_{2,0}\frac{\partial}{\partial x_{2,0}}$	$y_{3,0}x_{3,0}\frac{\partial}{\partial x_{3,0}}$	$y_{4,0}x_{4,0}\frac{\partial}{\partial x_{4,0}}$	$y_{1,0}x_{1,0}y_{4,0}\frac{\partial}{\partial x_{1,0}}$	$y_{1,0}x_{1,0}^2y_{4,0}\frac{\partial}{\partial x_{1,0}}$	$y_{1,0}\frac{\partial}{\partial y_{1,0}}$
$y_{2,0}\frac{\partial}{\partial y_{2,0}}$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$y_{3,0}\frac{\partial}{\partial y_{3,0}}$	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$y_{4,0}\frac{\partial}{\partial y_{4,0}}$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$x_{2,0}y_{2,0}\frac{\partial}{\partial y_{2,0}}$	0	0	0	2	0	0	0	0	0	0	0	0	-1	0	0
$x_{3,0}y_{3,0}\frac{\partial}{\partial y_{3,0}}$	0	0	0	0	2	0	0	0	0	0	0	0	0	-1	0
$x_{4,0}y_{4,0}\frac{\partial}{\partial y_{4,0}}$	0	0	0	0	0	2	0	0	0	0	0	0	1	1	0
$x_{2,0}\frac{\partial}{\partial x_{2,0}}$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1
$x_{3,0}\frac{\partial}{\partial x_{3,0}}$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
$x_{4,0}\frac{\partial}{\partial x_{4,0}}$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1
$x_{2,0}y_{2,0}\frac{\partial}{\partial x_{2,0}}$	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
$x_{3,0}y_{3,0}\frac{\partial}{\partial x_{3,0}}$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
$x_{4,0}y_{4,0}\frac{\partial}{\partial x_{4,0}}$	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0

Now we clearly see that the first 12×12 minor is 8, and so the $\mathrm{rank}(M_{P_{2E}}) = 12$ for $p \neq 2$ and if we omit the three columns with 2 in it, we get another 12×12 minor, but this is 0 for every p. So we get $\mathrm{rank}(M_{P_{2E}}) = 11 < 12$ for $p = 2$. Now 2 divides 2, so we cannot calculate $H^1(\Theta_{P_{2E}})$ this way, but if we take any positive divisor Z supported on E with $Z > 2E$ and all coefficients odd, then maybe after a reordering we have

$$M_{P_Z} = \begin{pmatrix} M_{P_{2E}} & 0 \\ A & B \end{pmatrix} \tag{5.4}$$

with matrices A, B, and thus $\mathrm{rank}(M_{P_Z}) < r_{P_Z}$ which now implies $H^1(\Theta_{P_Z}) \neq 0$.

Furthermore the same is true for D_n, $n > 4$: If we construct M_{P_Z} with an appropriate ordering for the rows/columns, we find $M_{P_{2E}}$ from D_4 (minus one "2 column") at the upper left corner, and again we get $H^1(\Theta_{P_Z}) \neq 0$. More general this is true whenever the dual graph of E has a star, that is E has one component which intersects with 3 others.

The next question now is: What is $H^1(\Theta_{P_Z})$ for E_6 and $p = 3$? The rank of $M_{P_{2E}}$ does not differ between $p = 3$ and $p = 0$, and the next choice, $M_{P_{3E}}$, is already a 60×69 matrix. So we cannot calculate this by hand. Now the construction of the matrix M_P is very explicit, and can easy be done using a computer. The only problem is the needed memory to store M_P. So we construct M_P for some cycle supported by E and compute $\mathrm{rank}(M_{P_p})$ for this cycle with some computer algebra system.

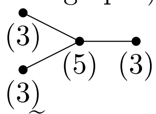
The natural candidate for this calculation would be $\nu\widetilde{Z}$ for a chosen anti-ample cycle $\widetilde{Z} = \sum_{l=1}^{n} n_l E_l$ with significant multiplicity ν. But to simplify the construction of M_P we want to stick to some cycle of the form jE. This is no problem because if we choose j bigger then $\nu \cdot \max\{n_l\}$, then with Corollary 3.35 we know that we have $\mathrm{CEQ}(jE) \cong \mathrm{CEQ}(\nu\widetilde{Z})$. To make sure that p does not divide j, we take the next prime bigger than $\nu \cdot \max\{n_l\}$ as j (and $j > 7$, the biggest p we are interested in).

We want to discuss the significant multiplicity ν first. By definition ν depends on λ and τ_{min} defined previous to Theorem 3.34. The calculation of λ depending on the Γ is easy, in particular we have $\lambda = 0$ for all rational double points, because $p_a(E_l) = 0$ and $E_l^2 = -2$ for all l.

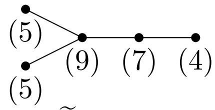
The calculation of τ_{min} is not so easy. Of course with the help of a computer we might just try every possibility for the j_i. But if $\widetilde{Z} = \sum_{l=1}^{n} n_l E_l$, then the number of possible combinations is the faculty of the sum of the n_l divided by the product of the faculties of the n_l. For D_4 this is not such a big number, but already for D_5 this number is so huge that the program did not terminate after a reasonable time.

But at least for the rational double points we found a nice way to compute a good upper bound for τ_{min}. We take $j_1 = 1$ and then we construct j_i inductively as follows: Let $\widetilde{Z}_{i-1} = \sum_{l=1}^{n} s_{l,i-1} E_l$. Let $\widetilde{j}$ be the smallest integer between 1 and n such that $s_{\widetilde{j},i-1} < n_{\widetilde{j}}$ and $E_{\widetilde{j}} \cdot (\widetilde{Z}_{i-1} + E_{\widetilde{j}})$ is maximal among these $\widetilde{j}$. Then we set $j_i = \widetilde{j}$. If we now calculate τ for this j_i and our $\widetilde{Z}$ chosen (see below) with the help of a computer, we get always $\tau = 1$. So because all n_l are greater then 1, we simply take $\nu = 2$.

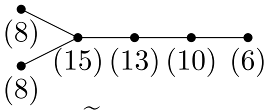
Now the anti-ample cycles we used are (for reasons of readability we omit the -2 in the dual graphs):



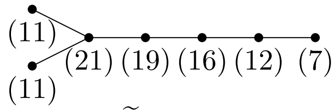
$\widetilde{Z}$ for D_4



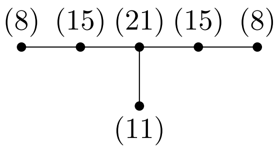
$\widetilde{Z}$ for D_5



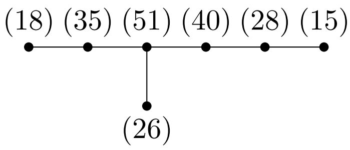
$\widetilde{Z}$ for D_6



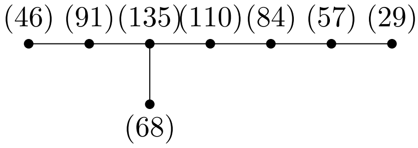
$\widetilde{Z}$ for D_7



$\widetilde{Z}$ for E_6



$\widetilde{Z}$ for E_7



$\widetilde{Z}$ for E_7

Now we generated text files containing the entries of M_P processable by various computer algebra systems. Because the size of M_P grows relatively fast, we hat do experiment with different ways of generating this files and also with different computer algebra systems. We mainly tested Maple([map]) and Sage([sag]), and hat to write the generator in Perl and C++.

The main problem is the growth of the matrix. If pt is the number of intersection points x_{lj} then we have $r_P = 2 \cdot pt \cdot (j^2 - j)$, and even if this just grows quadratically, for E_8 and $j = 203$ we have already $r_P = 1024380$. On the other hand, the matrix M_P is a sparse matrix with only less then $\frac{1}{1000}$ of its entries non-zero. It is crucial to use this fact. For example for E_7 and E_8 and $j = 53$ the resulting input file for maple is 2.2 GB respectively 3.1 GB large, and it takes nearly a day to generate them with perl. We tried to run Maple with this files, but, even with sufficiently large memory, after one week Maple was still "reading" the matrix.

The calculation in the table then where done using Sage, because Sage implements an algorithm for exactly our problem ([DV02]). If we additionally pass the matrix the right way to Sage, the file size reduces drastically (48MB for E_7 and $j = 103$). Also the calculations need much less memory and can therefore be done with 16 GB of ram. The only exception is E_8 and $j = 203$. Here perl was simple to slow to generate the text files in reasonable time, so we hat to switch to C++. Also we hat to reduce mod p already during the generation to get files of a few hundred MB.

Finally, after solving all the technical difficulties we get the following table:

Γ	max $\{n_l\}$	j	$r_P \times c_P$	rank M_P			
				$p=2$	3	5	7
D_4	5	11	660×735	659	660	660	660
D_5	9	19	2736×2944	2735	2736	2736	2736
D_6	15	31	9300×9827	9298	9300	9300	9300
D_7	21	43	21672×22662	21670	21672	21672	21672
E_6	21	43	18060×19049	18059	18059	18060	18060
E_7	51	103	126072×131532	126069	126071	126072	126072
E_8	135	271	1024380×1116997	1024376	1024378	1024379	1024380

5.3 Open questions II

If we take the last table and calculate from it the dimension of $h^1(P, \Theta_P)$ we get:

Γ	$h^1(P, \Theta_P)$			
	$p = 2$	3	5	7
D_4	1	0	0	0
D_5	1	0	0	0
D_6	2	0	0	0
D_7	2	0	0	0
E_6	1	1	0	0
E_7	3	1	0	0
E_8	4	2	1	0

If one compares this table with Artin's list one notices: For all non-taut rational double points $h^1(P, \Theta_P) + 1$ is exactly the number of isomorphism classes of singularities. This suggests that Theorem 3.1 of [Lau73b] may be still true for $p > 0$ if we restrict the n_l as before. So we can continue the discussion from the end of Section 4.5 and propose a stronger version of Conjecture 4.22:

Conjecture 5.9. Let (S_0, s_0) be a taut, normal, two-dimensional singularity over $\mathbb{C}$ with dual graph Γ. For a prime p let (S_p, s_p) be a Γ-singularity over an algebraically closed field k of characteristic p with dual graph Γ. Let $\widetilde{Z}_p$ be an anti-ample divisor for S_p with $\delta_{\mathrm{gcd}}(n_l, p) = 1$ for all l. Further let ν be its significant multiplicity and P_p the plumbing scheme for $\widetilde{Z}$. Then we have exactly $1+\dim(H^1(P_p, \Theta_{P_p}))$ isomorphism classes of Γ-singularities over k.

In particular we could reformulate Theorem 5.1 as "(S_0, s_0) is taut if and only if (S_p, s_p) is taut for all but finitely many p".

Additionally it would be interesting to know whether Corollary 5.8 can be used to show tautness of non-rational Gorenstein, normal, two-dimensional singularities. If one knows a way to calculate D, then there should be a way to use [Wah75] again.

For normal, two-dimensional singularities which are rational but not Gorenstein, all we know is the following recent result of Lee and Nakayama: In [LN12] they show $H^1(\Theta_X(-\log(E))) = 0$ all Hirzebruch-Jung singularities and for every p, which with Lemma 5.2 and Theorem 4.20 would imply the tautness, but Lee and Nakayama prove the tautness using other methods and they deduce the vanishing using the tautness.

The last interesting observation for which we have no explanation is that a rational double point is taut for $p > 0$ if and only if its fundamental cycle has all multiplicities prime to p. In our calculations we never need this fact, but maybe this can be explained with the calculations for Theorem 5.19 of [Wah75].

6 Local calculations

In this section we do all the local calculations needed in the previous ones. For reasons of readability we skip the elements s_{i,i_i}^{-1}. This is no real restriction, because for each of them we would get an additional generator ds_{i,i_i}^{-1} of $\Omega^1_{R_{i,i_i}/k}$, and an additional relation $s_{i,i_i}ds_{i,i_i}^{-1} + s_{i,i_i}^{-1}ds_{i,i_i}$ and $ds_{i,i_i} = dy_{i,i_i}$. Also for $\Theta_{R_{i,i_i}}$ we get an additional generator $\frac{\partial}{\partial s_{i,i_i}^{-1}}$, but using the new relation in $\Omega^1_{R_{i,i_i}/k}$, we get $\frac{\partial}{\partial s_{i,i_i}^{-1}} = (s_{i,i_i}^{-1})^2 \frac{\partial}{\partial y_{i,i_i}}$.

6.1 Θ_{P,P_l}

First we want to calculate $\Theta_{P,P_l} = \varinjlim_{j \in \mathbb{N}} H^0(U_l^j, \Theta_P)$.

For the calculations we use the fact that for $R = k[x_1, \cdots, x_n]/(f_1, \ldots, f_l)$ we can calculate

$$\Omega^1_{R/k} = \langle dx_1, \cdots, dx_n \rangle_R / \langle d(f_1), \cdots, d(f_l) \rangle_R$$

and then for Θ_R we use the exact sequence

$$0 \to \Theta_R \to \mathrm{Hom}_R(\Omega^1_{k[x_1,\cdots,x_n]/k} \otimes R, R) \to \mathrm{Hom}_R((f_1, \cdots, f_l)/(f_1, \cdots, f_l)^2, R)$$

coming from the standard exact sequence

$$(f_1, \ldots, f_l)/(f_1, \cdots, f_l)^2 \to \Omega^1_{k[x_1,\cdots,x_n]/k} \otimes R \to \Omega^1_{R/k} \to 0$$

Because every step behaves nicely under direct limits, we can omit the limit.

6.1.1 $t_l = 1$

From the charts (4.5) we get for $\Omega^1_{W_l/k}$:

$$\begin{aligned}
\Omega^1_{R_{l,0}/k} &= \langle dx_{l,0}, dy_{l,0} \rangle_{R_{l,0}} / \langle n_{j_1} x_{l,0}^{n_{j_1}-1} y_{l,0}^{n_l} dx_{l,0} + n_l x_{l,0}^{n_{j_1}} y_{l,0}^{n_l-1} dy_{l,0} \rangle_{R_{l,0}} \\
\Omega^1_{R_{l,1}/k} &= \langle dx_{l,1}, dy_{l,1} \rangle_{R_{l,1}} / \langle n_l y_{l,1}^{n_l-1} dy_{l,1} \rangle_{R_{l,1}} \\
\Omega^1_{R_{l,01}/k} &= \langle dx_{l,0}, dy_{l,0}, dx_{l,1}, dy_{l,1} \rangle_{R_{l,01}} / Rl_{\Omega^1_{R_{l,01}/k}}
\end{aligned}$$

with

$$Rl_{\Omega^1_{R_{l,01}/k}} = \langle x_{l,0}dx_{l,1} + x_{l,1}dx_{l,0}, dy_{l,0} - \nu_l x_{l,1}^{\nu_l-1} y_{l,1} dx_{l,1} - x_{l,1}^{\nu_l} dy_{l,1}, n_l y_{l,0}^{n_l-1} dy_{l,0} \rangle_{R_{l,01}}$$

Thus we get for Θ_{W_l}:

$$\begin{aligned}
\Theta_{R_{l,0}} &= \langle x_{l,0}^{\delta_{\gcd(n_{j_1},p)}} \frac{\partial}{\partial x_{l,0}}, y_{l,0}^{\delta_{\gcd(n_l,p)}} \frac{\partial}{\partial y_{l,0}} \rangle_{R_{l,0}} \\
\Theta_{R_{l,1}} &= \langle \frac{\partial}{\partial x_{l,1}}, y_{l,1}^{\delta_{\gcd(n_l,p)}} \frac{\partial}{\partial y_{l,0}} \rangle_{R_{l,1}} \\
\Theta_{R_{l,01}} &= \langle \frac{\partial}{\partial x_{l,0}}, y_{l,0}^{\delta_{\gcd(n_l,p)}} \frac{\partial}{\partial y_{l,0}}, \frac{\partial}{\partial x_{l,1}}, y_{l,1}^{\delta_{\gcd(n_l,p)}} \frac{\partial}{\partial y_{l,1}} \rangle_{R_{l,01}} / Rl_{\Theta_{R_{l,01}}}
\end{aligned}$$

with

$$Rl_{\Theta_{R_{l,01}}} = \langle \frac{\partial}{\partial x_{l,1}} + x_{l,0}^2 \frac{\partial}{\partial x_{l,0}} - \nu_l x_{l,0} y_{l,0} \frac{\partial}{\partial y_{l,0}}, \\ y_{l,1}^{\delta_{\text{gcd}}(n_l,p)} \frac{\partial}{\partial y_{l,1}} - y_{l,0}^{\delta_{\text{gcd}}(n_l,p)} x_{l,1}^{(1-\delta_{\text{gcd}}(n_l,p))\nu_l} \frac{\partial}{\partial y_{l,0}} \rangle_{R_{l,01}}$$

The calculations for $Rl_{\Theta_{R_{l,01}}}$ are:

$$\frac{\partial}{\partial x_{l,1}}(dx_{l,0}) = -\frac{\partial}{\partial x_{l,1}}(x_{l,0}^2 dx_{l,1}) = -x_{l,0}^2$$

$$\frac{\partial}{\partial x_{l,1}}(dy_{l,0}) = \frac{\partial}{\partial x_{l,1}}(\nu_l x_{l,1}^{\nu_l-1} y_{l,1} dx_{l,1} + x_{l,1}^{\nu_l} dy_{l,1}) = \nu_l x_{l,1}^{\nu_l-1} y_{l,1} = \nu_l x_{l,0} y_{l,0}$$

$$y_{l,1}^{\delta_{\text{gcd}}(n_l,p)} \frac{\partial}{\partial y_{l,1}}(dx_{l,0}) = -y_{l,1}^{\delta_{\text{gcd}}(n_l,p)} \frac{\partial}{\partial y_{l,1}}(x_{l,0}^2 dx_{l,1}) = 0$$

$$y_{l,1}^{\delta_{\text{gcd}}(n_l,p)} \frac{\partial}{\partial y_{l,1}}(dy_{l,0}) = y_{l,1}^{\delta_{\text{gcd}}(n_l,p)} \frac{\partial}{\partial y_{l,1}}(\nu_l x_{l,1}^{\nu_l-1} y_{l,1} dx_{l,1} + x_{l,1}^{\nu_l} dy_{l,1}) = y_{l,1}^{\delta_{\text{gcd}}(n_l,p)} x_{l,1}^{\nu_l} \\ = y_{l,0}^{\delta_{\text{gcd}}(n_l,p)} x_{l,1}^{(1-\delta_{\text{gcd}}(n_l,p))\nu_l}$$

Now we want to calculate $H^1(W_l, \Theta_{W_l})$: Because we have covert W_l with two affine charts, this is very simple with Čech-cohomology: We only have to calculate which elements of $H^0(\Theta_{R_{l,01}})$ are restrictions of elements of $H^0(\Theta_{R_{l,0}})$ and $H^0(\Theta_{R_{l,1}})$.

For $\delta_{\text{gcd}}(n_l,p) = 1$, we want to show that we have $H^1(W_l, \Theta_{W_l}) = 0$, that is that every element in $H^0(\Theta_{R_{l,01}})$ is a restriction. Lets look at $x_{l,0}^b y_{l,0}^a \frac{\partial}{\partial y_{l,0}}$ first. Because of $\delta_{\text{gcd}}(n_l,p) = 1$ we have $a > 0$. For $b \geq 0$ it is the restriction of the same element of $H^0(\Theta_{R_{l,0}})$. For $b < 0$ we have

$$x_{l,1}^{-b} y_{l,0}^a \frac{\partial}{\partial y_{l,0}} = x_{l,1}^{-b+(a-1)\nu_l} y_{l,1}^a \frac{\partial}{\partial y_{l,1}} \tag{6.1}$$

so these are restrictions of elements of $H^0(\Theta_{R_{l,1}})$. Now look at $x_{l,0}^b y_{l,0}^a \frac{\partial}{\partial x_{l,0}}$. Here we have $a \geq 0$. For $b > 0$ this is again the restriction of the same element of $H^0(\Theta_{R_{l,0}})$. For $b \leq 0$ we have

$$x_{l,1}^{-b} y_{l,0}^a \frac{\partial}{\partial x_{l,0}} = x_{l,1}^{-b+\nu_l a+2} y_{l,1}^a x_{l,0}^2 \frac{\partial}{\partial x_{l,0}} = x_{l,1}^{-b+\nu_l a} y_{l,1}^a (-\frac{\partial}{\partial x_{l,1}} + \nu_l x_{l,0} y_{l,0} \frac{\partial}{\partial y_{l,0}}) \\ = x_{l,1}^{-b+\nu_l a+1} y_{l,1}^a (-x_{l,1} \frac{\partial}{\partial x_{l,1}} + \nu_l y_{l,1} \frac{\partial}{\partial y_{l,1}})$$

so these are restrictions of elements of $H^0(\Theta_{R_{l,1}})$, and thus we have $H^1(W_l, \Theta_{W_l}) = 0$.

Now we do the case $\delta_{\text{gcd}}(n_l, p) = 0$: For $\frac{\partial}{\partial x_{l,0}}$ nothing changes. For $\frac{\partial}{\partial y_{l,0}}$ we have (6.1) again, and we immediately see that $x_{l,0}^b \frac{\partial}{\partial y_{l,0}}$ ($a = 0$ and $0 < -b < \nu_l$) is not a restriction, and thus we have $h^1(W_l, \Theta W_l) = \nu_l - 1$.

6.1.2 $t_l = 0$

This is a special calculation we only need for the counterexample at the end of Section 4.5. Our charts for $W_l = E_l$ are simply:

$$\begin{aligned}
\Omega^1_{R_{l,0}/k} &= \langle dx_{l,0}, dy_{l,0}\rangle_{R_{l,0}} / \langle n_l y_{l,0}^{n_l-1} dy_{l,0}\rangle_{R_{l,0}} \\
\Omega^1_{R_{l,1}/k} &= \langle dx_{l,1}, dy_{l,1}\rangle_{R_{l,1}} / \langle n_l y_{l,1}^{n_l-1} dy_{l,1}\rangle_{R_{l,1}} \\
\Omega^1_{R_{l,01}/k} &= \langle dx_{l,0}, dy_{l,0}, dx_{l,1}, dy_{l,1}\rangle_{R_{l,01}} / Rl_{\Omega^1_{R_{l,01}/k}}
\end{aligned}$$

with

$$Rl_{\Omega^1_{R_{l,01}/k}} = \langle x_{l,0} dx_{l,1} + x_{l,1} dx_{l,0}, dy_{l,0} - \nu_l x_{l,1}^{\nu_l - 1} y_{l,1} dx_{l,1} - x_{l,1}^{\nu_l} dy_{l,1}, n_l y_{l,0}^{n_l-1} dy_{l,0}\rangle_{R_{l,01}}$$

Thus we get for Θ_{W_l}:

$$\begin{aligned}
\Theta_{R_{l,0}} &= \langle \frac{\partial}{\partial x_{l,0}}, y_{l,0}^{\delta_{\text{gcd}}(n_l,p)} \frac{\partial}{\partial y_{l,0}} \rangle_{R_{l,0}} \\
\Theta_{R_{l,1}} &= \langle \frac{\partial}{\partial x_{l,1}}, y_{l,1}^{\delta_{\text{gcd}}(n_l,p)} \frac{\partial}{\partial y_{l,0}} \rangle_{R_{l,1}} \\
\Theta_{R_{l,01}} &= \langle \frac{\partial}{\partial x_{l,0}}, y_{l,0}^{\delta_{\text{gcd}}(n_l,p)} \frac{\partial}{\partial y_{l,0}}, \frac{\partial}{\partial x_{l,1}}, y_{l,1}^{\delta_{\text{gcd}}(n_l,p)} \frac{\partial}{\partial y_{l,1}} \rangle_{R_{l,01}} / Rl_{\Theta_{R_{l,01}}}
\end{aligned}$$

with

$$\begin{aligned}
Rl_{\Theta_{R_{l,01}}} = \langle\ & \frac{\partial}{\partial x_{l,1}} + x_{l,0}^2 \frac{\partial}{\partial x_{l,0}} - \nu_l x_{l,0} y_{l,0} \frac{\partial}{\partial y_{l,0}}, \\
& y_{l,1}^{\delta_{\text{gcd}}(n_l,p)} \frac{\partial}{\partial y_{l,1}} - y_{l,0}^{\delta_{\text{gcd}}(n_l,p)} x_{l,1}^{(1-\delta_{\text{gcd}}(n_l,p))\nu_l} \frac{\partial}{\partial y_{l,0}} \rangle_{R_{l,01}}
\end{aligned}$$

The calculations for $Rl_{\Theta_{R_{l,01}}}$ are exactly as in the case $t_l = 1$. Also the calculations for $H^1(W_l, \Theta_{W_l})$ are exactly the same. In particular, we get $H^1(W_l, \Theta_{W_l}) = 0$ for $\delta_{\text{gcd}}(n_l, p) = 1$ and $h^1(W_l, \Theta W_l) = \nu_l - 1$ for $\delta_{\text{gcd}}(n_l, p) = 0$.

6.1.3 $t_l = 2$

From the charts (4.6) we get for $\Omega^1_{W_l/k}$:

$$\begin{aligned}
\Omega^1_{R_{l,0}/k} &= \langle dx_{l,0}, dy_{l,0}\rangle_{R_{l,0}}/\langle n_{j_1} x_{l,0}^{n_{j_1}-1} y_{l,0}^{n_l} dx_{l,0} + n_l x_{l,0}^{n_{j_1}} y_{l,0}^{n_l-1} dy_{l,0}\rangle_{R_{l,0}} \\
\Omega^1_{R_{l,1}/k} &= \langle dx_{l,1}, dy_{l,1}\rangle_{R_{l,1}}/\langle n_l y_{l,1}^{n_l-1} x_{l,1}^{n_{j_2}} dy_{l,1} + n_{j_2} x_{l,1}^{n_{j_2}-1} y_{l,1}^{n_l} dx_{l,1}\rangle_{R_{l,1}} \\
\Omega^1_{R_{l,01}/k} &= \langle dx_{l,0}, dy_{l,0}, dx_{l,1}, dy_{l,1}\rangle_{R_{l,01}}/Rl_{\Omega^1_{R_{l,01}/k}}
\end{aligned}$$

with

$$Rl_{\Omega^1_{R_{l,01}/k}} = \langle x_{l,0}dx_{l,1} + x_{l,1}dx_{l,0}, dy_{l,0} - \nu_l x_{l,1}^{\nu_l-1} y_{l,1} dx_{l,1} - x_{l,1}^{\nu_l} dy_{l,1}, n_l y_{l,0}^{n_l-1} dy_{l,0}\rangle_{R_{l,01}}$$

Thus we get for Θ_{W_l}:

$$\begin{aligned}
\Theta_{R_{l,0}} &= \langle x_{l,0}^{\delta_{\mathrm{gcd}}(n_{j_1},p)} \frac{\partial}{\partial x_{l,0}}, y_{l,0}^{\delta_{\mathrm{gcd}}(n_l,p)} \frac{\partial}{\partial y_{l,0}}\rangle_{R_{l,0}} \\
\Theta_{R_{l,1}} &= \langle x_{l,1}^{\delta_{\mathrm{gcd}}(n_{j_2},p)} \frac{\partial}{\partial x_{l,1}}, y_{l,1}^{\delta_{\mathrm{gcd}}(n_l,p)} \frac{\partial}{\partial y_{l,0}}\rangle_{R_{l,1}} \\
\Theta_{R_{l,01}} &= \langle \frac{\partial}{\partial x_{l,0}}, y_{l,0}^{\delta_{\mathrm{gcd}}(n_l,p)} \frac{\partial}{\partial y_{l,0}}, \frac{\partial}{\partial x_{l,1}}, y_{l,1}^{\delta_{\mathrm{gcd}}(n_l,p)} \frac{\partial}{\partial y_{l,1}}\rangle_{R_{l,01}}/Rl_{\Theta_{R_{l,01}}}
\end{aligned}$$

with

$$\begin{aligned}
Rl_{\Theta_{R_{l,01}}} = \langle\ & \frac{\partial}{\partial x_{l,1}} + x_{l,0}^2 \frac{\partial}{\partial x_{l,0}} - \nu_l x_{l,0} y_{l,0} \frac{\partial}{\partial y_{l,0}}, \\
& y_{l,1}^{\delta_{\mathrm{gcd}}(n_l,p)} \frac{\partial}{\partial y_{l,1}} - y_{l,0}^{\delta_{\mathrm{gcd}}(n_l,p)} x_{l,1}^{(1-\delta_{\mathrm{gcd}}(n_l,p))\nu_l} \frac{\partial}{\partial y_{l,0}}\rangle_{R_{l,01}}
\end{aligned}$$

The calculations for $Rl_{\Theta_{R_{l,01}}}$ are exactly as in the case $t_l = 1$. Also the calculations for $H^1(W_l, \Theta_{W_l})$ are exactly the same.

6.1.4 $t_l = 3$

From the charts (4.7) we get for $\Omega^1_{W_l/k}$:

$$\begin{aligned}
\Omega^1_{R_{l,0}/k} &= \langle dx_{l,0}, dy_{l,0}\rangle_{R_{l,0}}/Rl_{\Omega^1_{R_{l,0}/k}} \\
\Omega^1_{R_{l,1}/k} &= \langle dx_{l,1}, dy_{l,1}\rangle_{R_{l,1}}/Rl_{\Omega^1_{R_{l,1}/k}} \\
\Omega^1_{R_{l,01}/k} &= \langle dx_{l,0}, dy_{l,0}, dx_{l,1}, dy_{l,1}\rangle_{R_{l,01}}/Rl_{\Omega^1_{R_{l,01}/k}}
\end{aligned}$$

with

$$\begin{aligned}
Rl_{\Omega^1_{R_{l,0}/k}} &= \langle\ x_{l,0}^{n_{j_1}-1}(x_{l,0}-1)^{n_{j_3}-1}y_{l,0}^{n_l-1}((n_{j_1}+n_{j_3})x_{l,0}-n_{j_1})y_{l,0}dx_{l,0} \\
&\quad +n_l x_{l,0}(x_{l,0}-1)dy_{l,0}\rangle_{R_{l,0}} \\
Rl_{\Omega^1_{R_{l,1}/k}} &= \langle\ x_{l,1}^{n_{j_2}-1}(x_{l,1}-1)^{n_{j_3}-1}y_{l,1}^{n_l-1}((n_{j_2}+n_{j_3})x_{l,0}-n_{j_1})y_{l,0}dx_{l,0} \\
&\quad +n_l x_{l,0}(x_{l,0}-1)dy_{l,0}\rangle_{R_{l,1}} \\
Rl_{\Omega^1_{R_{l,01}/k}} &= \langle\ x_{l,0}dx_{l,1}+x_{l,1}dx_{l,0}, dy_{l,0}-\nu_l x_{l,1}^{\nu_l-1}y_{l,1}dx_{l,1}-x_{l,1}^{\nu_l}dy_{l,1}, \\
&\quad n_{j_3}(x_{l,0}-1)^{n_{j_3}-1}y_{l,0}^{n_l}dx_{l,0}+n_l x_{l,0}^{n_{j_3}}y_{l,0}^{n_l-1}dy_{l,0}\rangle_{R_{l,01}}
\end{aligned}$$

Thus we get for Θ_{W_l}:

$$\begin{aligned}
\Theta_{R_{l,0}} &= \langle x_{l,0}^{\delta_{\gcd}(n_{j_1},p)}(x_{l,0}-1)^{\delta_{\gcd}(n_{j_3},p)}\frac{\partial}{\partial x_{l,0}}, y_{l,0}^{\delta_{\gcd}(n_l,p)}\frac{\partial}{\partial y_{l,0}}\rangle_{R_{l,0}} \\
\Theta_{R_{l,1}} &= \langle x_{l,1}^{\delta_{\gcd}(n_{j_2},p)}(x_{l,1}-1)^{\delta_{\gcd}(n_{j_3},p)}\frac{\partial}{\partial x_{l,1}}, y_{l,1}^{\delta_{\gcd}(n_l,p)}\frac{\partial}{\partial y_{l,0}}\rangle_{R_{l,1}}
\end{aligned}$$

and

$$\begin{aligned}
\Theta_{R_{l,01}} = \langle\ &(x_{l,0}-1)^{\delta_{\gcd}(n_{j_3},p)}\frac{\partial}{\partial x_{l,0}}, y_{l,0}^{\delta_{\gcd}(n_l,p)}\frac{\partial}{\partial y_{l,0}}, (x_{l,1}-1)^{\delta_{\gcd}(n_{j_3},p)}\frac{\partial}{\partial x_{l,1}}, \\
&y_{l,1}^{\delta_{\gcd}(n_{j_1},p)}\frac{\partial}{\partial y_{l,1}}\rangle_{R_{l,01}}/Rl_{\Theta_{R_{l,01}}}
\end{aligned}$$

with

$$\begin{aligned}
Rl_{\Theta_{R_{l,01}}} = \langle\ &(x_{l,1}-1)^{\delta_{\gcd}(n_{j_3},p)}\frac{\partial}{\partial x_{l,1}} - (x_{l,0}-1)^{\delta_{\gcd}(n_{j_3},p)}x_{l,0}^{2-\delta_{\gcd}(n_{j_3},p)}\frac{\partial}{\partial x_{l,0}} \\
&+\nu_l(x_{l,0}-1)^{\delta_{\gcd}(n_{j_3},p)}x_{l,0}^{1-\delta_{\gcd}(n_{j_3},p)}y_{l,0}\frac{\partial}{\partial y_{l,0}}, \\
&y_{l,1}^{\delta_{\gcd}(n_l,p)}\frac{\partial}{\partial y_{l,1}} - y_{l,0}^{\delta_{\gcd}(n_l,p)}x_{l,1}^{(1-\delta_{\gcd}(n_l,p))\nu_l}\frac{\partial}{\partial y_{l,0}}\rangle_{R_{l,01}}
\end{aligned}$$

The calculations for $Rl_{\Theta_{R_{l,01}}}$ are:

$$\begin{aligned}
(x_{l,1}-1)^{\delta_{\mathrm{gcd}}(n_{j_3},p)}\frac{\partial}{\partial x_{l,1}}(dx_{l,0}) &= -(x_{l,1}-1)^{\delta_{\mathrm{gcd}}(n_{j_3},p)}\frac{\partial}{\partial x_{l,1}}(x_{l,0}^2 dx_{l,1}) \\
&= -(x_{l,1}-1)^{\delta_{\mathrm{gcd}}(n_{j_3},p)}x_{l,0}^2 \\
&= (x_{l,0}-1)^{\delta_{\mathrm{gcd}}(n_{j_3},p)}x_{l,0}^{2-\delta_{\mathrm{gcd}}(n_{j_3},p)} \\
(x_{l,1}-1)^{\delta_{\mathrm{gcd}}(n_{j_3},p)}\frac{\partial}{\partial x_{l,1}}(dy_{l,0}) &= (x_{l,1}-1)^{\delta_{\mathrm{gcd}}(n_{j_3},p)}\frac{\partial}{\partial x_{l,1}}(\nu_l x_{l,1}^{\nu_l-1}y_{l,1}dx_{l,1}+x_{l,1}^{\nu_l}dy_{l,1}) \\
&= (x_{l,1}-1)^{\delta_{\mathrm{gcd}}(n_{j_3},p)}\nu_l x_{l,1}^{\nu_l-1}y_{l,1} \\
&= (x_{l,1}-1)^{\delta_{\mathrm{gcd}}(n_{j_3},p)}\nu_l x_{l,0}y_{l,0} \\
&= -\nu_l(x_{l,0}-1)^{\delta_{\mathrm{gcd}}(n_{j_3},p)}x_{l,0}^{1-\delta_{\mathrm{gcd}}(n_{j_3},p)}y_{l,0}
\end{aligned}$$

The calculations for $y_{l,1}^{\delta_{\mathrm{gcd}}(n_l,p)}\frac{\partial}{\partial y_{l,1}}$ are again exactly as in the case $t_l = 1$.

Calculating $H^1(W_l, \Theta_{W_l})$:
For $\delta_{\mathrm{gcd}}(n_l,p) = 1$, we want to show again that every element in $H^0(\Theta_{R_{l,01}})$ is a restriction of elements of $H^0(\Theta_{R_{l,0}})$ and $H^0(\Theta_{R_{l,1}})$. For $x_{l,0}^b y_{l,0}^a \frac{\partial}{\partial y_{l,0}}$ this is again as in the case $t_l = 1$. Also $x_{l,0}^b y_{l,0}^a (x_{l,0}-1)^{\delta_{\mathrm{gcd}}(n_{j_3},p)}\frac{\partial}{\partial x_{l,0}}$ is for $b \geq 1$ and $a \geq 0$ again a restriction. For $b \leq 0$ we have:

$$\begin{aligned}
& x_{l,1}^{-b}y_{l,0}^a(x_{l,0}-1)^{\delta_{\mathrm{gcd}}(n_{j_3},p)}\frac{\partial}{\partial x_{l,0}} \\
&= x_{l,1}^{-b+\nu_l a+2-\delta_{\mathrm{gcd}}(n_{j_3},p)}y_{l,1}^a\left((x_{l,0}-1)^{\delta_{\mathrm{gcd}}(n_{j_3},p)}x_{l,0}^{2-\delta_{\mathrm{gcd}}(n_{j_3},p)}\frac{\partial}{\partial x_{l,0}}\right) \\
&= x_{l,1}^{-b+\nu_l a+2-\delta_{\mathrm{gcd}}(n_{j_3},p)}y_{l,1}^a\left(-(x_{l,1}-1)^{\delta_{\mathrm{gcd}}(n_{j_3},p)}\frac{\partial}{\partial x_{l,1}}\right. \\
&\quad \left.+\nu_l(x_{l,0}-1)^{\delta_{\mathrm{gcd}}(n_{j_3},p)}x_{l,0}^{1-\delta_{\mathrm{gcd}}(n_{j_3},p)}y_{l,0}\frac{\partial}{\partial y_{l,0}}\right) \\
&= x_{l,1}^{-b+\nu_l a}y_{l,1}^a\left(-x_{l,1}^{2-\delta_{\mathrm{gcd}}(n_{j_3},p)}(x_{l,1}-1)^{\delta_{\mathrm{gcd}}(n_{j_3},p)}\frac{\partial}{\partial x_{l,1}}\right. \\
&\quad \left.-\nu_l(x_{l,1}-1)^{\delta_{\mathrm{gcd}}(n_{j_3},p)}y_{l,1}\frac{\partial}{\partial y_{l,1}}\right)
\end{aligned}$$

So this is also a restriction and we get $H^1(W_l, \Theta_{W_l}) = 0$ again.

With the same arguments as in the previous cases we get $h^1(W_l, \Theta_{W_l}) = \nu_l - 1$ for $\delta_{\mathrm{gcd}}(n_l,p) = 0$.

6.2 $\check{H}^1(W_l, \mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l))$

In this section we calculate the first cohomology of the (non-abelian) sheaf $\mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l)$, showing that we always can lift the isomorphism between the reductions of W_l and W_l to the scheme with the $\mathbb{P}^1_k$ in W_l not reduced. Again for $\mathcal{U}$ consisting of two affine charts the calculation of $\check{H}^1(\mathcal{U}, \mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l))$ is not very complicated, because we only have to calculate which elements of $\mathrm{Aut}_{R_{l,01}/N_{l,01}}(R_{l,01})$ are restrictions of elements of $\mathrm{Aut}_{R_{l,0}/N_{l,0}}(R_{l,0})$ and $\mathrm{Aut}_{R_{l,1}/N_{l,1}}(R_{l,1})$, where $N_{l,i}$ is the nilradical.

Now $\mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l)$ is not quasi-coherent, so we do not have Serre vanishing, so we cannot use Theorem 3.8 here. But in the calculation of $\check{H}^1(\mathcal{U}, \mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l))$ we see that in this special case the sequence (3.9) splits, and we can use this to calculate $\check{H}^1(W_l, \mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l))$.

How do we calculate the elements ψ of $\mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l)(U)$? Suppose $R = k[x, y, s^{-1}]/I$ with $I = (y^2), (xy^2), (x(x-1)y^2)$. Because ψ must be the identity on the reduction, $\psi(x) = x + a \cdot p_x$, where a is a generator of nilradical, that is $a = y, xy, xy(x-1)$. Because each of these three terms vanishes in R if we multiply it with y, we have $a \cdot s = -a$ and so we have $p_x \in k[x]$. By explicit calculation one sees that the inverse map is given by $x \mapsto x - ap_x$.

For the y a priori we may multiply y simply by any $b \in R^\times$. But calculating when this morphism has an inverse shows that we have $b = \lambda x^j$ with $\lambda \in k^\times$ for $a = y$ in R, and $b = \lambda$ else. For $a = xy, xy(x-1)$ we have $b = 1$ already because the morphism is the identity on the reduction. This implies $b = 1 + ap$, but $ya = 0$.

6.2.1 $t_l = 0$

This is a calculation we need for the counterexample at the end of Section 4.5. Here we did not need the s_{l,i_l}, so $\mathcal{U} = \{\mathrm{Spec}(R_{l,0}), \mathrm{Spec}(R_{l,1})\}$ with:

$$\begin{aligned} R_{l,0} &= k[x_{l,0}, y_{l,0}]/(y_{l,0}^2) \\ R_{l,01} &= k[x_{l,0}, y_{l,0}, x_{l,1}, y_{l,1}]/(x_{l,0}x_{l,1} - 1, y_{l,0} - x_{l,1}^{\nu_l} y_{l,1}, y_{l,0}^2) \\ R_{l,1} &= k[x_{l,1}, y_{l,1}]/(y_{l,1}^2) \end{aligned}$$

Then by the discussion above, the elements of $\mathrm{Aut}_{R/N}(R)$ are given by:

$$\begin{aligned} \psi_{l,0}(x_{l,0}) &= x_{l,0} + y_{l,0}\varphi_{x,0}(x_{l,0}) & \varphi_{x,0}(x_{l,0}) &\in k[x_{l,0}] \\ \psi_{l,0}(y_{l,0}) &= \lambda_0 y_{l,0} & \lambda_0 &\in k^\times \\ \psi_{l,1}(x_{l,1}) &= x_{l,1} + y_{l,1}\varphi_{x,1}(x_{l,1}) & \varphi_{x,1}(x_{l,1}) &\in k[x_{l,1}] \\ \psi_{l,1}(y_{l,1}) &= y_{l,1}\lambda_1 & \lambda_1 &\in k^\times \\ \psi_{l,10}(x_{l,0}) &= x_{l,0} + y_{l,0}\varphi_{x,01}(x_{l,0}, x_{l,1}) & \varphi_{x,01}(x_{l,0}, x_{l,1}) &\in k[x_{l,0}, x_{l,1}] \\ \psi_{l,10}(y_{l,0}) &= y_{l,0}\lambda_{01} x_{l,0}^j & \lambda_{01} &\in k^\times,\ j \in \mathbb{Z} \end{aligned}$$

For the restriction to $R_{l,01}$ we calculate

$$\begin{array}{l|l} \psi_{l,0}|_{l,01}(y_{l,0}) = y_{l,0}\lambda_0 & \psi_{l,0}|_{l,01}(x_{l,0}) = x_{l,0} + y_{l,0}\varphi_{x,0}(x_{l,0}) \\ \psi_{l,1}|_{l,01}(y_{l,0}) = y_{l,0}\lambda_1 & \psi_{l,1}|_{l,01}(x_{l,0}) = (\psi_{l,1}|_{l,01}(x_{l,1}))^{-1} = x_{l,0} - y_{l,0}x_{l,0}^{\nu_l+2}\varphi_{x,1}(x_{l,1}) \end{array}$$

With this and $y_{l,0} \cdot (\psi_{l,1}|_{l,01})^{-1} = y_{l,0} \cdot \mathrm{id}$ we get:

$$\begin{aligned} (\psi_{l,1}|_{l,01})^{-1}(\psi_{l,0}|_{l,01}(x_{l,0})) &= (\psi_{l,1}|_{l,01})^{-1}(x_{l,0} + y_{l,0}\varphi_{x,0}(x_{l,0})) \\ &= x_{l,0} + y_{l,0}x_{l,0}^{\nu_l+2}\varphi_{x,1}(x_{l,1}) + \lambda_1^{-1}y_{l,0}(\psi_{l,1}|_{l,01})^{-1}(\varphi_{x,0}(x_{l,0})) \\ &= x_{l,0} + y_{l,0}(x_{l,0}^{\nu_l+2}\varphi_{x,1}(x_{l,1}) + \lambda_1^{-1}\varphi_{x,0}(x_{l,0})) \end{aligned}$$

$$(\psi_{l,1}|_{l,01})^{-1}(\psi_{l,0}|_{l,01}(y_{l,0})) = (\psi_{l,1}|_{l,01})^{-1}(y_{l,0}\lambda_0) = \lambda_0\lambda_1^{-1}y_{l,0}$$

So we see that $\check{H}^1(\mathcal{U}, \mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l))$ is generated by the automorphism of $R_{l,01}$ which maps $x_{l,0}$ to $x_{l,0}$ and $y_{l,0}$ to $y_{l,0}x_{l,0}$, so $\check{H}^1(\mathcal{U}, \mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l)) \cong \mathbb{Z}$, and for any extension of E_l the image in $\mathbb{Z}$ is simply the self-intersection number $-v_l$.

6.2.2 $t_l = 1$

Here we need only $s_{l,0}^{-1}$, so in this case our covering is $\mathcal{U} = \{\mathrm{Spec}(R_{l,0}), \mathrm{Spec}(R_{l,1})\}$ with:

$$\begin{aligned} R_{l,0} &= k[x_{l,0}, y_{l,0}, s_{l,0}^{-1}]/(s_{l,0}^{-1}s_{l,0} - 1, x_{l,0}y_{l,0}^2) \\ R_{l,01} &= k[x_{l,0}, y_{l,0}, x_{l,1}, y_{l,1}]/(s_{l,0}^{-1}s_{l,0} - 1, x_{l,0}x_{l,1} - 1, y_{l,0} - x_{l,1}^{\nu_l}y_{l,1}, y_{l,0}^2) \\ R_{l,1} &= k[x_{l,1}, y_{l,1}]/(y_{l,1}^2) \end{aligned}$$

Again by the discussion above, the elements of $\mathrm{Aut}_{R/N}(R))$ are given by:

$$\begin{aligned} \psi_{l,0}(x_{l,0}) &= x_{l,0} + x_{l,0}y_{l,0}\varphi_{x,0}(x_{l,0}) && \varphi_{x,0}(x_{l,0}) \in k[x_{l,0}] \\ \psi_{l,0}(y_{l,0}) &= y_{l,0} && \\ \psi_{l,1}(x_{l,1}) &= x_{l,1} + y_{l,1}\varphi_{x,1}(x_{l,1}) && \varphi_{x,1}(x_{l,1}) \in k[x_{l,1}] \\ \psi_{l,1}(y_{l,1}) &= y_{l,1}\lambda_1 && \lambda_1 \in k^\times \\ \psi_{l10}(x_{l,0}) &= x_{l,0} + y_{l,0}\varphi_{x,01}(x_{l,0}, x_{l,1}) && \varphi_{x,01}(x_{l,0}, x_{l,1}) \in k[x_{l,0}, x_{l,1}] \\ \psi_{l10}(y_{l,0}) &= y_{l,0}\lambda_{01}x_{l,0}^j && \lambda_{01} \in k^\times,\ j \in \mathbb{Z} \end{aligned}$$

For the restriction to $R_{l,01}$ we calculate

$$\begin{array}{l|l} \psi_{l,0}|_{l,01}(y_{l,0}) = y_{l,0} & \psi_{l,0}|_{l,01}(x_{l,0}) = x_{l,0} + x_{l,0}y_{l,0}\varphi_{x,0}(x_{l,0}) \\ \psi_{l,1}|_{l,01}(y_{l,0}) = y_{l,0}\lambda_1 & \psi_{l,1}|_{l,01}(x_{l,0}) = (\psi_{l,1}|_{l,01}(x_{l,1}))^{-1} = x_{l,0} - y_{l,0}x_{l,0}^{\nu_l+2}\varphi_{x,1}(x_{l,1}) \end{array}$$

With this and $y_{l,0}\cdot(\psi_{l,1}|_{l,01})^{-1}=y_{l,0}\cdot\mathrm{id}$ we get:

$$\begin{aligned}(\psi_{l,1}|_{l,01})^{-1}(\psi_{l,0}|_{l,01}(x_{l,0})) &= (\psi_{l,1}|_{l,01})^{-1}(x_{l,0}+x_{l,0}y_{l,0}\varphi_{x,0}(x_{l,0}))\\ &= x_{l,0}+y_{l,0}x_{l,0}^{\nu_l+2}\varphi_{x,1}(x_{l,1})+\lambda_1^{-1}y_{l,0}(\psi_{l,1}|_{l,01})^{-1}(x_{l,0}\varphi_{x,0}(x_{l,0}))\\ &= x_{l,0}+y_{l,0}(x_{l,0}^{\nu_l+2}\varphi_{x,1}(x_{l,1})+\lambda_1^{-1}x_{l,0}\varphi_{x,0}(x_{l,0}))\end{aligned}$$

$$(\psi_{l,1}|_{l,01})^{-1}(\psi_{l,0}|_{l,01}(y_{l,0})) = (\psi_{l,1}|_{l,01})^{-1}(y_{l,0})=\lambda_1^{-1}y_{l,0}$$

So we see that $\check{H}^1(\mathcal{U},\mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l))$ is generated by the automorphism of $R_{l,01}$ which maps $x_{l,0}$ to $x_{l,0}$ and $y_{l,0}$ to $y_{l,0}x_{l,0}$, so $\check{H}^1(\mathcal{U},\mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l))\cong\mathbb{Z}$, and for any scheme the image in $\mathbb{Z}$ is simply the difference of the self-intersection number $-v_l$.

6.2.3 $t_l=2$

In this case our covering is $\mathcal{U}=\{\mathrm{Spec}(R_{l,0}),\mathrm{Spec}(R_{l,1})\}$ with:

$$\begin{aligned}R_{l,0} &= k[x_{l,0},y_{l,0},s_{l,0}^{-1}]/(s_{l,0}^{-1}s_{l,0}-1,x_{l,0}y_{l,0}^2)\\ R_{l,01} &= k[x_{l,0},y_{l,0},x_{l,1},y_{l,1}]/(s_{l,0}^{-1}s_{l,0}-1,x_{l,0}x_{l,1}-1,y_{l,0}-x_{l,1}^{\nu_l}y_{l,1},y_{l,0}^2)\\ R_{l,1} &= k[x_{l,1},y_{l,1},s_{l,1}^{-1}]/(s_{l,1}^{-1}s_{l,1}-1,x_{l,1}y_{l,1}^2)\end{aligned}$$

Again by the discussion above, the elements of $\mathrm{Aut}_{R/N}(R))$ are given by:

$$\begin{aligned}\psi_{l,0}(x_{l,0}) &= x_{l,0}+x_{l,0}y_{l,0}\varphi_{x,0}(x_{l,0}) && \varphi_{x,0}(x_{l,0})\in k[x_{l,0}]\\ \psi_{l,0}(y_{l,0}) &= y_{l,0}\\ \psi_{l,1}(x_{l,1}) &= x_{l,1}+y_{l,1}\varphi_{x,1}(x_{l,1}) && \varphi_{x,1}(x_{l,1})\in k[x_{l,1}]\\ \psi_{l,1}(y_{l,1}) &= y_{l,1}\\ \psi_{l10}(x_{l,0}) &= x_{l,0}+y_{l,0}\varphi_{x,01}(x_{l,0},x_{l,1}) && \varphi_{x,01}(x_{l,0},x_{l,1})\in k[x_{l,0},x_{l,1}]\\ \psi_{l10}(y_{l,0}) &= y_{l,0}\lambda_{01}x_{l,0}^j && \lambda_{01}\in k^\times,\ j\in\mathbb{Z}\end{aligned}$$

For the restriction to $R_{l,01}$ we calculate:

$$\begin{array}{l|l}\psi_{l,0}|_{l,01}(y_{l,0})=y_{l,0} & \psi_{l,0}|_{l,01}(x_{l,0})=x_{l,0}+x_{l,0}y_{l,0}\varphi_{x,0}(x_{l,0})\\ \psi_{l,1}|_{l,01}(y_{l,0})=y_{l,0} & \psi_{l,1}|_{l,01}(x_{l,0})=x_{l,0}-y_{l,0}x_{l,0}^{\nu_l+1}\varphi_{x,1}(x_{l,1})\end{array}$$

With this and $y_{l,0} \cdot (\psi_{l,1}|_{l,01})^{-1} = y_{l,0} \cdot \mathrm{id}$ we get:

$$\begin{aligned}(\psi_{l,1}|_{l,01})^{-1}(\psi_{l,0}|_{l,01}(x_{l,0})) &= (\psi_{l,1}|_{l,01})^{-1}(x_{l,0} + x_{l,0}y_{l,0}\varphi_{x,0}(x_{l,0})) \\ &= x_{l,0} + y_{l,0}x_{l,0}^{\nu_l+1}\varphi_{x,1}(x_{l,1}) + y_{l,0}(\psi_{l,1}|_{l,01})^{-1}(x_{l,0}\varphi_{x,0}(x_{l,0})) \\ &= x_{l,0} + y_{l,0}(x_{l,0}^{\nu_l+1}\varphi_{x,1}(x_{l,1}) + x_{l,0}\varphi_{x,0}(x_{l,0}))\end{aligned}$$

$$(\psi_{l,1}|_{l,01})^{-1}(\psi_{l,0}|_{l,01}(y_{l,0})) = (\psi_{l,1}|_{l,01})^{-1}(y_{l,0}) = y_{l,0}$$

So we see that $\check{H}^1(\mathcal{U}, \mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l))$ is generated by the automorphism of $R_{l,01}$ which maps $x_{l,0}$ to $x_{l,0}$ and $y_{l,0}$ to $y_{l,0}x_{l,0}$, and the ones mapping $y_{l,0}$ to $y_{l,0}\lambda$. So $\check{H}^1(\mathcal{U}, \mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l)) \cong \mathbb{Z} \times k^\times$, and again for any scheme the image in $\mathbb{Z}$ is the difference of the self-intersection number $-v_l$.

Also we see that if $\psi_{l,0}$ and $\psi_{l,1}$ came from a lifting of an isomorphism, then we have necessarily $\psi_{l,i}(y_{l,0}) = 0$, so the $k^\times$ part of $\check{H}^1(\mathcal{U}, \mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l))$ is never hit as an obstruction on the global lifting.

6.2.4 $t_l = 3$

In this case our covering is $\mathcal{U} = \{\mathrm{Spec}(R_{l,0}), \mathrm{Spec}(R_{l,1})\}$ with:

$$R_{l,0} = k[x_{l,0}, y_{l,0}, s_{l,0}^{-1}]/(s_{l,0}^{-1}s_{l,0} - 1, x_{l,0}(x_{l,0} - 1)y_{l,0}^2)$$

$$R_{l,01} = k[x_{l,0}, y_{l,0}, x_{l,1}, y_{l,1}]/(s_{l,0}^{-1}s_{l,0} - 1, x_{l,0}x_{l,1} - 1, y_{l,0} - x_{l,1}^{\nu_l}y_{l,1}, (x_{l,0} - 1)y_{l,0}^2)$$

$$R_{l,1} = k[x_{l,1}, y_{l,1}, s_{l,1}^{-1}]/(x_{l,1}(s_{l,1}^{-1}s_{l,1} - 1, x_{l,1} - 1)y_{l,1}^2)$$

Again by the discussion above, the elements of $\mathrm{Aut}_{R/N}(R))$ are given by:

$$\begin{aligned}\psi_{l,0}(x_{l,0}) &= x_{l,0} + x_{l,0}(x_{l,0} - 1)y_{l,0}\varphi_{x,0}(x_{l,0}) & \varphi_{x,0}(x_{l,0}) &\in k[x_{l,0}] \\ \psi_{l,0}(y_{l,0}) &= y_{l,0} \\ \psi_{l,1}(x_{l,1}) &= x_{l,1} + y_{l,1}x_{l,1}(x_{l,1} - 1)\varphi_{x,1}(x_{l,1}) & \varphi_{x,1}(x_{l,1}) &\in k[x_{l,1}] \\ \psi_{l,1}(y_{l,1}) &= y_{l,1} \\ \psi_{l10}(x_{l,0}) &= x_{l,0} + y_{l,0}(x_{l,0} - 1)\varphi_{x,01}(x_{l,0}, x_{l,1}) & \varphi_{x,01}(x_{l,0}, x_{l,1}) &\in k[x_{l,0}, x_{l,1}] \\ \psi_{l10}(y_{l,0}) &= y_{l,0}x_{l,0}^j & j &\in \mathbb{Z}\end{aligned}$$

Now we could calculate $\check{H}^1(\mathcal{U}, \mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l))$ but later on we see that this is not isomorphic to $\check{H}^1(W_l, \mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l))$ so we skip this. We only need the local description for the results of the next section. Also, as in the previous cases these charts help us to interpret $\check{H}^1(W_l, \mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l))$.

6.2.5 The final calculation

From the local charts in the previous sections we see that the Sequence (3.9) splits, that is we have

$$\mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l) \cong \mathrm{Aut}_{(W_l)_{\mathrm{red}},N}(W_l) \oplus Q$$

and by Corollary 3.25 we have $H^1(W_l, \mathrm{Aut}_{(W_l)_{\mathrm{red}},N}(W_l)) = H^1(B_{l_0}, (\Omega^1_{B_{l_0}/k})^\vee \otimes \mathcal{I}_C/\mathcal{I}_B)$, where B_{l_0} is the image of the inclusion $\iota : \mathbb{P}^1_k \to W_l$. Now we have $v_l \geq 0$ and $t_l \leq 3$ so we get

$$\deg_{B_{l_0}}((\Omega^1_{B_{l_0}/k})^\vee \otimes_{\mathcal{O}_{B_{l_0}}} \mathcal{I}_C/\mathcal{I}_B) = 2 - 2g_{l_0} - B_{l_0} \cdot \sum_{l=1}^{n} n'_l B_l = 2 + v_l - t_l \geq 0 \quad (6.2)$$

and thus $H^1(B_{l_0}, (\Omega^1_{B_{l_0}/k})^\vee \otimes \mathcal{I}_C/\mathcal{I}_B) = 0$ by (3.13).

So we have $\check{H}^1(W_l, \mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l)) = \check{H}^1(W_l, Q)$. From the local charts we see that we have an exact sequence

$$1 \longrightarrow Q \longrightarrow \iota_*(\mathcal{O}^\times_{\mathbb{P}^1_k}) \longrightarrow \mathcal{F} \longrightarrow 1$$

where $\mathcal{F}$ is a skyscraper sheaf concentrated on the singular points of $(W_l)_{\mathrm{red}}$ and the stalks there are $k^\times$. So in cohomology, we get the long exact sequence

$$\begin{aligned} 1 \longrightarrow H^0(W_l, Q) \longrightarrow H^0(W_l, \iota_*(\mathcal{O}^\times_{\mathbb{P}^1_k})) \longrightarrow H^0(W_l, \mathcal{F}) \\ \longrightarrow \check{H}^1(W_l, Q) \longrightarrow \check{H}^1(W_l, \iota_*(\mathcal{O}^\times_{\mathbb{P}^1_k})) \longrightarrow 1 \end{aligned}$$

and with (3.3) we get

$$H^0(W_l, \iota_*(\mathcal{O}^\times_{\mathbb{P}^1_k})) = H^0(\mathbb{P}^1_k, \mathcal{O}^\times_{\mathbb{P}^1_k}) = k^\times$$

and

$$\check{H}^1(W_l, \iota_*(\mathcal{O}^\times_{\mathbb{P}^1_k})) = \check{H}^1(\mathbb{P}^1_k, \mathcal{O}^\times_{\mathbb{P}^1_k}) \cong \mathrm{Pic}(\mathbb{P}^1_k) \cong \mathbb{Z}.$$

Now we get: For $t_l = 0$ the sheaf $\mathcal{F}$ is trivial, so we have

$$\check{H}^1(W_l, \mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l)) = \check{H}^1(W_l, Q) = \check{H}^1(W_l, \iota_*(\mathcal{O}^\times_{\mathbb{P}^1_k})) \cong \mathbb{Z}$$

here. For $t_l > 0$ we see in the local charts that $H^0(W_l, Q) = 1$, so the map $H^0(W_l, \iota_*(\mathcal{O}^\times_{\mathbb{P}^1_k})) \to H^0(W_l, \mathcal{F})$ is injective. Also we have $H^0(W_l, \mathcal{F}) = (k^\times)^{t_l}$. Thus we get

$$\check{H}^1(W_l, \mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l)) = \check{H}^1(W_l, Q) \cong \mathbb{Z} \times (k^\times)^{t_l - 1}$$

and as discussed after the calculation of $\check{H}^1(\mathcal{U}, \mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l))$ for $t_l = 2$, the $(k^\times)^{t_l-1}$ part is never hit by an obstruction. Also, if we have V_l and W_l such that there reductions are isomorphic, and we lift this isomorphism locally with Theorem 3.11, then the element generated by this local lifts in $\check{H}^1(W_l, \mathrm{Aut}_{(W_l)_{\mathrm{red}}}(W_l))$ maps to the difference of the self-intersection number of the $\mathbb{P}^1_k$ in W_l and V_l. In particular V_l and W_l are isomorphic if these numbers are equal.

6.3 $\Theta_{W_l \cap W_j}$

6.3.1 Plumbing

For the plumbing $W_l \cap W_j = \operatorname{Spec}(R_{lj})$ with

$$k[\widetilde{x}_{l,i_l}, y_{l,i_l}, \widetilde{x}_{j,i_j}, y_{j,i_j}]/(\widetilde{x}_{j,i_j} - y_{l,i_l}, y_{j,i_j} - \widetilde{x}_{l,i_l}, \widetilde{x}_{l,i_l}^{n_j} y_{l,i_l}^{n_l}).$$

From this we get

$$\Omega^1_{R_{lj}/k} = \langle d\widetilde{x}_{l,i_l}, dy_{l,i_l}, d\widetilde{x}_{j,i_j}, dy_{j,i_j}\rangle_{R_{lj}} / \langle\, d\widetilde{x}_{l,i_l} - dy_{j,i_j}, dy_{l,i_l} - d\widetilde{x}_{j,i_j}, \\ n_j \widetilde{x}_{l,i_l}^{n_j-1} y_{l,i_l}^{n_l} d\widetilde{x}_{l,i_l} + n_l \widetilde{x}_{l,i_l}^{n_j} y_{l,i_l}^{n_l-1} dy_{l,i_l}\rangle_{R_{lj}}$$

and

$$\begin{aligned}\Theta_{R_{lj}} = \langle\ & \widetilde{x}_{l,i_l}^{\delta_{\mathrm{gcd}}(n_j,p)} \frac{\partial}{\partial \widetilde{x}_{l,i_l}}, y_{l,i_l}^{\delta_{\mathrm{gcd}}(n_l,p)} \frac{\partial}{\partial y_{l,i_l}}, \widetilde{x}_{j,i_j}^{\delta_{\mathrm{gcd}}(n_l,p)} \frac{\partial}{\partial \widetilde{x}_{j,i_j}}, y_{j,i_j}^{\delta_{\mathrm{gcd}}(n_j,p)} \frac{\partial}{\partial y_{j,i_j}}\rangle_{R_{lj}} \\ & /\langle \widetilde{x}_{l,i_l}^{\delta_{\mathrm{gcd}}(n_j,p)} \frac{\partial}{\partial \widetilde{x}_{l,i_l}} - y_{j,i_j}^{\delta_{\mathrm{gcd}}(n_j,p)} \frac{\partial}{\partial y_{j,i_j}}, \\ & \quad y_{l,i_l}^{\delta_{\mathrm{gcd}}(n_l,p)} \frac{\partial}{\partial y_{l,i_l}} - \widetilde{x}_{j,i_j}^{\delta_{\mathrm{gcd}}(n_l,p)} \frac{\partial}{\partial \widetilde{x}_{j,i_j}}\rangle_{R_{lj}}\end{aligned}$$

The calculations are:

$$\begin{aligned}\widetilde{x}_{j,i_j}^{\delta_{\mathrm{gcd}}(n_l,p)} \frac{\partial}{\partial \widetilde{x}_{j,i_j}}(d\widetilde{x}_{l,i_l}) &= \widetilde{x}_{j,i_j}^{\delta_{\mathrm{gcd}}(n_l,p)} \frac{\partial}{\partial \widetilde{x}_{j,i_j}}(dy_{j,i_j}) = 0 \\ \widetilde{x}_{j,i_j}^{\delta_{\mathrm{gcd}}(n_l,p)} \frac{\partial}{\partial \widetilde{x}_{j,i_j}}(dy_{l,i_l}) &= \widetilde{x}_{j,i_j}^{\delta_{\mathrm{gcd}}(n_l,p)} \frac{\partial}{\partial \widetilde{x}_{j,i_j}}(d\widetilde{x}_{j,i_j}) \\ &= \widetilde{x}_{j,i_j}^{\delta_{\mathrm{gcd}}(n_l,p)} = y_{l,i_l}^{\delta_{\mathrm{gcd}}(n_l,p)} \\ y_{j,i_j}^{\delta_{\mathrm{gcd}}(n_j,p)} \frac{\partial}{\partial y_{j,i_j}}(d\widetilde{x}_{l,i_l}) &= y_{j,i_j}^{\delta_{\mathrm{gcd}}(n_j,p)} \frac{\partial}{\partial y_{j,i_j}}(dy_{j,i_j}) = y_{j,i_j}^{\delta_{\mathrm{gcd}}(n_j,p)} = \widetilde{x}_{l,i_l}^{\delta_{\mathrm{gcd}}(n_j,p)} \\ y_{j,i_j}^{\delta_{\mathrm{gcd}}(n_j,p)} \frac{\partial}{\partial y_{j,i_j}}(dy_{l,i_l}) &= y_{j,i_j}^{\delta_{\mathrm{gcd}}(n_j,p)} \frac{\partial}{\partial y_{j,i_j}}(d\widetilde{x}_{j,i_j}) = 0\end{aligned} \tag{6.3}$$

6.3.2 *B*

In this case we know from Lemma 4.4:

$$R_{lj} = k[\widetilde{x}_{l,i_l}, y_{l,i_l}, \widetilde{x}_{j,i_j}, y_{j,i_j}]/(\widetilde{x}_{j,i_j} - y_{l,i_l}(a_{y,l,j} + \widetilde{x}_{l,i_l}y_{l,i_l}p_{y,l,j}),$$
$$y_{j,i_j} - \widetilde{x}_{l,i_l}(a_{x,l,j} + \widetilde{x}_{l,i_l}y_{l,i_l}p_{x,l,j}), \widetilde{x}^{n_j}_{l,i_l}y^{n_l}_{l,i_l}).$$

But on the other hand, we know that $a_{y,l,j} + \widetilde{x}_{l,i_l}y_{l,i_l}p_{y,l,j}, a_{x,l,j} + \widetilde{x}_{l,i_l}y_{l,i_l}p_{x,l,j} \in R^{\times}_{lj}$, and if $a_{y,j,l} + \widetilde{x}_{l,i_l}y_{l,i_l}p_{y,j,l}, a_{x,j,l} + \widetilde{x}_{l,i_l}y_{l,i_l}p_{x,j,l}$ are there inverse, then $R_{jl} \cong R_{lj}$ can be written as

$$R_{jl} = k[\widetilde{x}_{l,i_l}, y_{l,i_l}, \widetilde{x}_{j,i_j}, y_{j,i_j}]/(\widetilde{x}_{l,i_l} - y_{j,i_j}(a_{y,j,l} + \widetilde{x}_{j,i_j}y_{j,i_j}p_{y,j,l}),$$
$$y_{l,i_l} - \widetilde{x}_{j,i_j}(a_{x,j,l} + \widetilde{x}_{j,i_j}y_{j,i_j}p_{x,j,l}), \widetilde{x}^{n_j}_{l,i_l}y^{n_l}_{l,i_l}).$$

For our calculations before, the latter form is more handy, so we use it to get:

$$\begin{aligned}\Omega^1_{R_{lj}/k} = {} & \langle d\widetilde{x}_{l,i_l}, dy_{l,i_l}, d\widetilde{x}_{j,i_j}, dy_{j,i_j}\rangle_{R_{lj}} \\ & /\langle d\widetilde{x}_{l,i_l} - a_{y,j,l}dy_{j,i_j} - y^2_{j,i_j}p_{y,j,l}d\widetilde{x}_{j,i_j} - 2\widetilde{x}_{j,i_j}p_{y,j,l}y_{j,i_j}dy_{j,i_j} - \widetilde{x}_{j,i_j}y^2_{j,i_j}d(p_{y,j,l}), \\ & dy_{l,i_l} - a_{x,j,l}d\widetilde{x}_{j,i_j} - 2\widetilde{x}_{j,i_j}p_{x,j,l}y_{j,i_j}d\widetilde{x}_{j,i_j} - p_{x,j,l}x'^2_{j,i_j}dy_{j,i_j} - \widetilde{x}^2_{j,i_j}y_{j,i_j}d(p_{x,j,l}), \\ & n_j\widetilde{x}^{n_j-1}_{l,i_l}y^{n_l}_{l,i_l}d\widetilde{x}_{l,i_l} + n_l\widetilde{x}^{n_j}_{l,i_l}y^{n_l-1}_{l,i_l}dy_{l,i_l}\rangle_{R_{lj}}\end{aligned} \tag{6.4}$$

Which leads to:

$$\Theta_{R_{lj}} = \langle \widetilde{x}^{\delta_{\gcd}(n_j,p)}_{l,i_l}\frac{\partial}{\partial\widetilde{x}_{l,i_l}}, y^{\delta_{\gcd}(n_l,p)}_{l,i_l}\frac{\partial}{\partial y_{l,i_l}}, \widetilde{x}^{\delta_{\gcd}(n_l,p)}_{j,i_j}\frac{\partial}{\partial\widetilde{x}_{j,i_j}}, y^{\delta_{\gcd}(n_j,p)}_{j,i_j}\frac{\partial}{\partial y_{j,i_j}}\rangle_{R_{lj}}/(6.5)$$

Where (6.5) are the following relations, which we get by using the relations of (6.4).

$$\begin{aligned}\widetilde{x}^{\delta_{\gcd}(n_l,p)}_{j,i_j}\frac{\partial}{\partial\widetilde{x}_{j,i_j}}(d\widetilde{x}_{l,i_l}) &= \widetilde{x}^{\delta_{\gcd}(n_l,p)}_{j,i_j}(y^2_{j,i_j}p_{y,j,l} + \widetilde{x}_{j,i_j}y^2_{j,i_j}\frac{\partial}{\partial\widetilde{x}_{j,i_j}}(d(p_{y,j,l}))) \\ \widetilde{x}^{\delta_{\gcd}(n_l,p)}_{j,i_j}\frac{\partial}{\partial\widetilde{x}_{j,i_j}}(dy_{l,i_l}) &= \widetilde{x}^{\delta_{\gcd}(n_l,p)}_{j,i_j}(a_{x,j,l} + 2\widetilde{x}_{j,i_j}y_{j,i_j}p_{x,j,l} + \widetilde{x}^2_{j,i_j}y_{j,i_j}\frac{\partial}{\partial\widetilde{x}_{j,i_j}}(d(p_{x,j,l})) \\ y^{\delta_{\gcd}(n_j,p)}_{j,i_j}\frac{\partial}{\partial y_{j,i_j}}(d\widetilde{x}_{l,i_l}) &= y^{\delta_{\gcd}(n_j,p)}_{j,i_j}(a_{y,j,l} + 2\widetilde{x}_{j,i_j}y_{j,i_j}p_{y,j,l} + \widetilde{x}_{j,i_j}y^2_{j,i_j}\frac{\partial}{\partial y_{j,i_j}}(d(p_{y,j,l}))) \\ y^{\delta_{\gcd}(n_j,p)}_{j,i_j}\frac{\partial}{\partial y_{j,i_j}}(dy_{l,i_l}) &= y^{\delta_{\gcd}(n_j,p)}_{j,i_j}(\widetilde{x}^2_{j,i_j}p_{x,j,l} + \widetilde{x}^2_{j,i_j}y_{j,i_j}\frac{\partial}{\partial y_{j,i_j}}(d(p_{x,j,l}))\end{aligned} \tag{6.5}$$

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