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Subgroup Into-Conjugacy Separability Property for Groups

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Zusammenfassung

Subgroup conjugacy separability ist eine Eigenschaft einer Gruppe, die logische Fortsetzung der folgenden Reihe von gut bekannten Gruppeneigenschaften der Gruppen ist: residually finiteness (RF), conjugacy separability (CS), and subgroup separability (LERF).

Definition 1: Eine Gruppe G heißt *subgroup conjugacy separable* (SCS), wenn es für je zwei nicht konjugierte endlich erzeugte Untergruppen H_1 und H_2 in G, einen Homomorphismus ϕ von G auf eine endliche Gruppe $\overline{G}$ gibt, so dass $\phi(H_1)$ nicht zu $\phi(H_2)$ konjugiert ist.

Diese Eigenschaft wurde erst im preprint [6] von Bogopolski und Grunewald definiert. Es gibt viele Artikel über RF, CS und LERF Gruppen, aber wir kennen nur zwei Artikel über SCS Gruppen. In [15] haben Grunewald und Segal bewiesen, dass alle fast polyzyklische Gruppen SCS sind (Siehe auch [34, Kapitel 4, Satz 7]). In [6] haben Bogopolski und Grunewald bewiesen, dass freie Gruppen und einige fast freie Gruppen SCS sind.

Eine Gruppe G heißt *hereditary subgroup conjugacy separable* (HSCS), wenn jede endliche Index Untergruppe H in G SCS ist.

In Proposition 1.6.8 geben wir eine hinreichende Bedingung für eine SCS Gruppe um HSCS zu sein.

Außerdem haben wir viele Ergebnisse über eine andere Eigenschaft, die mit der SCS Eigenschaft verwandt ist, bekommen.

Definition 2: Eine Gruppe G heißt *subgroup into-conjugacy separable*

(SICS), wenn für je zwei endlich erzeugte Untergruppen H_1, H_2 in G gilt: wenn H_2 nicht zu einer Untergruppe von H_1 konjugiert ist, dann existiert einen Homomorphismus ϕ von G auf eine endliche Gruppe $\overline{G}$, so dass $\phi(H_2)$ nicht zu einer Untergruppe von $\phi(H_1)$ konjugiert ist.

Das Studium der SICS Gruppen wurde durch Bogopolski und Grunewald in [6] angefangen. Insbesondere haben sie bewiesen, dass freie Gruppen SICS sind, und jede fast freie Gruppe, die SICS ist, auch SCS ist.

Die Hauptfrage, die wir in dieser Arbeit beantwortet haben, lautet:

Ist die Klasse der SICS Gruppen unter den verschiedenen Arten von Produkten geschlossen?

Die Antwort der analogen Frage für die Klassen der RF, CS, und LERF Gruppen kann in der folgenden Tabelle zusammengefaßt werden.

A and B	RF	CS	LERF
$A * B$	ja	ja	ja
$A \times B$	ja	ja	nicht immer
$A \rtimes B$	nicht immer	nicht immer	nicht immer
$A \wr B$	nicht immer	nicht immer	nicht immer

Tab. 1. Die Klassen der RF, CS, und LERF Gruppen mit den verschiedenen Arten von Produkten.

Um diese Frage für die Klasse der SICS Gruppen zu beantworten, haben wir die folgenden Ergebnisse bewiesen.

Hauptsatz (O. Bogopolski, A. Elsawy). *Das freie Produkt zweier Gruppen, die SICS und LERF sind, ist wieder SICS (und LERF nach [7] und [32]).*

Wir haben auch weitere Ergebnisse für verschiedene Arten von Produkten bewiesen. Wir haben bewiesen, dass das direkte Produkt $F_m \times F_n$ für alle $m, n \geq 2$ nicht SICS ist, wobei F_m die freie Gruppe vom Rang m ist.

Auf der anderen Seite haben wir ein Beispiel einer nicht SICS Gruppe gefunden, die ein semidirektes Produkt zweier SICS Gruppen ist.

In der Tat haben wir gezeigt, dass das semidirekt Produkt $F_2 \rtimes F_1$ nicht SICS ist, wobei $F_2 = \langle x, y \mid \rangle, F_1 = \langle a \mid \rangle$, und die Operation von a auf F_2 durch $axa^{-1} = y^{-1}x$, $aya^{-1} = y$ definiert ist.

Wir haben ebenfalls bewiesen, dass die Klasse der SICS Gruppen unter dem Kranzprodukt nicht geschlossen ist.

Zudem haben wir ein Beispiel einer nicht SICS Gruppe gefunden, die das amalgamierte Produkt einer SICS Gruppe mit einer SCS Gruppe über einer zyklischen Untergruppe ist.

So können wir die Tabelle 1 erweitern.

A and B	RF	CS	LERF	SICS
$A * B$	ja	ja	ja	ja mit Beding.
$A \times B$	ja	ja	nicht immer	nicht immer
$A \rtimes B$	nicht immer	nicht immer	nicht immer	nicht immer
$A \wr B$	nicht immer	nicht immer	nicht immer	nicht immer

Tab. 2. Die Klassen der RF, CS, LERF und SICS Gruppen mit den verschiedenen Arten von Produkten.

Contents

Notation

We have tried to use only standard notation and list below only a few usages that might cause difficulty.

$\emptyset$ is the empty set.

$\mathbb{Z}$ denotes the set of all integers.

$\mathbb{N}$ is the set of positive integers.

$\mathbb{Q}$ denotes the set of all rational numbers.

$|X|$ denotes the cardinal of the set X.

$X \setminus Y$ is set difference.

$X \subseteq Y$ denotes inclusion, proper or not; $X \subsetneqq Y$ denotes strict inclusion.

$G = \langle X \mid R \rangle$ denote the presentation corresponding to a group G with generators $x \in X$ and relators $r \in R$.

$G = \langle X \mid \, \rangle$ denotes the free group with basis X.

$|w|$, for $w \in \langle X \mid \, \rangle$, is the length of w as a reduced word relative to the basis X.

F_n is the free group of rank $n \in \mathbb{N}$.

$G = \langle X \rangle$ denotes the group generated by the set X.

$H \leqslant G$ means that H is a subgroup of G.

$H < G$ means that H is a proper subgroup of G.

$N \trianglelefteq G$ means that N is a normal subgroup of G.

$|G : H|$ is the index of H in G.

$[a, b] = aba^{-1}b^{-1}$, $x^g = gxg^{-1}$, and $H^g = gHg^{-1}$.

$C_G(H)$ is the centralizer of the subgroup H in G.

$N_G(H)$ is the normalizer of the subgroup H in G.

$A \times B$ is the direct product.

$A \rtimes B$ is the semidirect product.

$A * B$ is the free product.

$|w|$, for $w \in A * B$, is the length of w as an alternative product of elements from A and B.

$A *_X B$ denotes the free product of A and B with $X = A \cap B$ amalgamated.

$A \wr B$ is the wreath product.

Introduction

The subgroup conjugacy separability property is a logical extension of the following series of well-known residual properties of groups: residual finiteness (RF), conjugacy separability (CS), and subgroup separability (LERF).

Definition 0.0.1. A group G is called *subgroup conjugacy separable* (abbreviated to SCS) if for every two finitely generated subgroups H_1 and H_2 in G such that H_2 is not conjugate to H_1, there exists a homomorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(H_1)$ is not conjugate to $\phi(H_2)$ in $\overline{G}$.

This definition was first introduced by Bogopolski and Grunewald in [6]. There are many papers on residually finite, conjugacy separable, and LERF groups. However, to our knowledge there are ony two papers on SCS groups. In [15] , Grunewald and Segal proved that all virtually polycyclic groups are SCS (see also [34, Ch. 4, Thm. 7]). Bogopolski and Grunewald [6] proved that free groups and some virtually free groups are SCS.

A group G is called *hereditarily subgroup conjugacy separable* if every finite index subgroup $H \leqslant G$ is subgroup conjugacy separable.

In Proposition 1.6.8, we give a sufficient condition for a subgroup conjugacy separable group to be hereditarily subgroup conjugacy separable.

On the other hand, we have studied another property called SICS. The definition of SICS groups can be obtained by replacement the words "conjugate to" in Definition 0.0.1 by the words "conjugate into".

Definition 0.0.2. A group G is called *subgroup into conjugacy separable* (or simply SICS) if for every two finitely generated subgroups H_1 and H_2 in G so that H_2 is not conjugate into H_1 in G, there exists a homomorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(H_2)$ is not conjugate into $\phi(H_1)$ in $\overline{G}$; Bogopolski and Grunewald [6].

(Here we say that H_2 is *conjugate into* H_1 if there exists $g \in G$ such that $H_2^g \leqslant H_1$.)

The study of SICS groups was initiated by Bogopolski and Grunewald in [6]. They have proved that free groups are SICS, and every SICS virtually free group is SCS.

The present thesis is primarily an attempt to extend our knowledge of the SICS groups. Namely, the main question we have considered is the following

Question: Is the class of SICS groups closed under different kinds of products?

In Chapters 2 and 3, we give an answer to this question with respect to free, direct, semidirect, and wreath products.

The answer to the analogous question for the classes of RF, CS, and LERF groups can be summarized in the following table:

A and B	RF	CS	LERF
$A * B$	yes	yes	yes
$A \times B$	yes	yes	not nec.
$A \rtimes B$	not nec.	not nec.	not nec.
$A \wr B$	not nec.	not nec.	not nec.

Tab. 0.0.1. *The classes RF, CS, and LERF with different kinds of products.*

In order to answer this question for the class of SICS groups, we have proved the following

Main theorem (O. Bogopolski, A. Elsawy). *The free product of two groups which are simultaneously SICS and LERF is again SICS (and LERF by [7] and [32]).*

On the other hand, we proved that the direct product $F_m \times F_n$ of two free groups is not SICS for all $m, n \geq 2$. However, free groups are LERF [17] and SICS [6]. We also observed that this example implies that the right-angled Artin group G corresponding to the complete bipartite graph $K_{n,m}$ is not SICS for all $n, m \geq 2$.

We also gave an example of a non-SICS group which is a semidirect product of two SICS groups. Namely, we proved that the semidirect product $F_2 \rtimes F_1$ is not SICS, where $F_2 = \langle x, y \,|\, \rangle$, $F_1 = \langle a \,|\, \rangle$, and the action of a on F_2 is given by $axa^{-1} = y^{-1}x$, $aya^{-1} = y$.

Moreover, we have noticed that the class of SICS groups is not closed under the wreath product as expected.

Finally, we gave an example of a non-SICS group which is the free product of an SICS group and an SCS group with amalgamated cyclic subgroup.

Therefore, in the following table we can see the comparison between the classes RF, CS, LERF, and SICS with respect to the different kinds of products mentioned above.

A and B	RF	CS	LERF	SICS
$A * B$	yes	yes	yes	yes with conditions
$A \times B$	yes	yes	not nec.	not nec.
$A \rtimes B$	not nec.	not nec.	not nec.	not nec.
$A \wr B$	not nec.	not nec.	not nec.	not nec.

Tab. 0.0.2. *The classes RF, CS, LERF, and SICS with different kinds of products.*

Chapter 1

Preliminaries

Following P. Hall as outlined by Gruenberg [14], we say that a group G is *residually* $\mathscr{P}$ where $\mathscr{P}$ is any group property if for every non-trivial element $g \in G$ there exists a quotient group $\overline{G}$ satisfying $\mathscr{P}$ so that the corresponding element $\overline{g}$ of g in $\overline{G}$ is non-trivial.

As in [14], it is obvious to see that any group which satisfies $\mathscr{P}$ is residually $\mathscr{P}$, and it follows further that residually "residually $\mathscr{P}$" is always the same property as residually $\mathscr{P}$. We also can easily notice that if $\mathscr{P}$ implies another property $\mathscr{P}'$, then also residually $\mathscr{P}$ implies residually $\mathscr{P}'$.

This last fact shows, in particular, that a residually nilpotent group is residually soluble. Furthermore, residual "of order p^k", where p is a prime number and k is a positive integer, implies both residual nilpotency and residual finiteness.

Residual properties attracted a lot of mathematicians. For example, Magnus [23] proved that every free group is residually nilpotent. Mal'cev [25], Marshall Hall [16], and Takahasi [38] proved that every free group is residually "of order p^k" for any given prime number p and some positive integer k.

In this thesis we focus on the finiteness property. In particular, in this chapter we give an overview of the class of residually finite groups, some of its properties, and the relation between some of its interesting proper classes (for more details see [24, Sec. 6.5]).

If we consider finiteness as the property $\mathscr{P}$, then we get the following definition: A group G is called *residually finite* if for every non-trivial element $g \in G$, there exists a finite quotient group $\overline{G}$ such that $\overline{g}$ in $\overline{G}$ is non-trivial. Equivalently, by setting $g = xy^{-1}$, a group G is residually finite if for every two different elements $x, y \in G$, there exists a homomorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(x) \neq \phi(y)$ in $\overline{G}$.

In the following section many equivalent definitions and examples of residually finite groups will be introduced.

1.1 Residually finite groups

Definition 1.1.1. A group G is called *residually finite* (abbreviated to RF) if one of the following equivalent conditions holds:

(a) for every two elements $x \neq y$ in G, there exists a homomorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(x) \neq \phi(y)$ in $\overline{G}$;

(b) for every non-trivial element $g \in G$ there is a finite index (normal) subgroup D of G which does not contain g;

(c) the intersection of all finite index normal subgroups in G is the trivial subgroup of G;

(d) the intersection of all finite index subgroups in G is the trivial subgroup of G.

For completeness, we show that these conditions are indeed equivalent.

(a)⇒(b): Let g be a non-trivial element in G, then there exists a homomorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(g)$ in $\overline{G}$ is non-trivial. Denote $\ker(\phi)$ by D, then D is a finite index (normal) subgroup of G such that $g \notin D$.

(b)⇒(c): Let H be the intersection of all finite index normal subgroups of G, then H is the trivial subgroup of G.

Indeed, if g is a non-trivial element in H, then g belongs to every finite index normal subgroup of G. On the other hand, according to (b), there exists a finite index subgroup D of G such that $g \notin D$. Since $N := \cap_{x \in G} D^x$ is a normal subgroup in G and $|G : N| \leq (|G : D|)! < \infty$ [5, p. 8, Poincaré Theorem], the subgroup N is a finite index normal subgroup in G such that $g \notin N \leqslant D$, which leads to a contradiction.

(c)⇒(d): The implication is obvious, since the intersection of all finite index subgroups in G is a subgroup of the intersection of all finite index normal subgroups in G.

(d)⇒(a): Let $x, y \in G$ such that $x \neq y$, then $g := xy^{-1}$ is a non-trivial element in G. It follows from (d) that there exists a finite index subgroup D such that $g \notin D$.

The subgroup $N := \cap_{z \in G} D^z$ is a finite index normal subgroup in G such that $g \notin N \leqslant D$, then $xN \neq yN$. Let $\overline{G} = G/N$ and ϕ be the natural epimorphism from G onto $\overline{G}$, then $\phi(x) \neq \phi(y)$ in $\overline{G}$. □

The class of residually finite groups contains all finite groups, free groups [21], polycyclic groups, in particular, finitely generated nilpotent groups [19], and polycyclic-by-finite groups [26].

Moreover, by the work of Meskin [29], the Baumslag-Solitar group

$$BS(n,m) = \langle a, b \mid a^{-1} b^n a = b^m \rangle, \text{ where } n, m \in \mathbb{Z},$$

is residually finite if and only if at least one of the following three cases holds: $|n| = 1, |m| = 1$, or $|n| = |m|$.

For any finitely presented residually finite group G the following theorems have been proven:

I. G has solvable word problem (see [22, p. 195, Thm. 4.6]).

That is, there exists an algorithm that takes as input two words w_1, w_2 in the generators of G and decides after finitely many steps whether $w_1 = w_2$ in G or not.

II. G is Hopfian, that is, every epimorphism from G onto itself is an automorphism (see for example [5, p. 120, Thm. 29.9]).

However, Hopfian groups are not always residually finite. For example, the Baumslag-Solitar group $BS(2,4)$ is Hopfian but not residually finite. According to Baumslag and Solitar [3], $BS(n,m)$ is Hopfian if and only if at least one of the following three cases holds: $n \mid m$, $m \mid n$, or n and m have precisely the same prime divisors.

III. The automorphism group of G is residually finite (see [24, p. 414]).

IV. Every subgroup of G is residually finite, and every group K which contains G as a subgroup of finite index is residually finite, Scott [33].

However, this may not be true for all quotient groups of G. For instance, the free group F_2 with two free generators is residually finite but its quotient group $BS(2,3)$ is not.

Another example of a non-residually finite group is given by G. Higman in [18]. He proved that the group

$$G = \langle a_1, a_2, \ldots, a_n \mid a_i^{-1} a_{i+1} a_i = a_{i+1}^2,\ a_n^{-1} a_1 a_n = a_1^2,\ i = 1, 2 \ldots, n-1,\ n \geq 4 \rangle$$

has no normal subgroup of finite index, and therefore, the only finite quotient group of G is the trivial one.

In the rest of this chapter we will discuss in detail the class of residually finite groups to investigate its subclasses.

In Sections 1.2 and 1.3 we consider important subclasses of the class of residually finite groups. Definitions of these subclasses LERF, π_C, and ERF can be obtained by replacement the word “elements” in Definition 1.1.1 (a) by “finitely generated subgroups”, “cyclic subgroups”, or “subgroups”, respectively.

1.2 LERF, π_C, and ERF groups

Definition 1.2.1. I. A group G is called *locally extended residually finite* (abbreviated to LERF) if one of the following equivalent conditions is true:

(a) for every two finitely generated subgroups $H_1 \neq H_2$ in G, there exists a homomorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(H_1) \neq \phi(H_2)$;

(b) for every finitely generated subgroup H of G and every $g \in G \setminus H$, there exists a homomorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(g) \notin \phi(H)$;

(c) for every finitely generated subgroup H of G and every $g \in G \setminus H$, a finite index subgroup D of G exists and satisfies $H \leqslant D$ and $g \notin D$.

II. A group G is called a *π_C group*, if a similar condition to one of the equivalent conditions (a), (b), or (c) holds whenever H_1, H_2, and H are cyclic subgroups of G.

III. A group G is called *extended residually finite* (ERF) if a similar condition to one of the equivalent conditions (a), (b), or (c) holds whenever H_1, H_2, and H are subgroups of G.

For completeness, we give a proof that these conditions are indeed equivalent. We consider only the case of LERF groups; the same argument holds for π_C and ERF groups.

(a)⇒(b): Let H be a finitely generated subgroup of G and let $g \in G \setminus H$. Let H' be the subgroup of G defined by $H' = \langle H, g \rangle$, then $H \neq H'$.

Applying (a), we conclude that there exists a homomorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(H) \neq \phi(H')$ in $\overline{G}$.

Since $\phi(H') = \langle \phi(H), \phi(g) \rangle$ and $\phi(H) \neq \phi(H')$, we have $\phi(g) \notin \phi(H)$.

(b)⇒(c): Let H be a finitely generated subgroup of G and let $g \in G \setminus H$.

According to (b), there exists a homomorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(g) \notin \phi(H)$ in $\overline{G}$.

Therefore, the finite index normal subgroup N :=ker(ϕ) of G satisfies $gN \notin \{hN : h \in H\} = HN$. Let $D = HN$, then D is a finite index subgroup of

G such that $H \leqslant D$ and $g \notin D$. Indeed, if $g \in D$, then $gN \subseteq DN = D$, which is a contradiction.

(c)⇒(a): Let H_1, H_2 be two different finitely generated subgroups of G, then, without loss of generality, there exists $g \in H_2$ such that $g \notin H_1$.

Using (c), we can find a finite index subgroup D of G such that $H_1 \leqslant D$ and $g \notin D$. Since $N := \cap_{x \in G} D^x$ is a finite index normal subgroup in G, the group $\overline{G} := G/N$ is finite. Let ϕ be the natural epimorphism from G onto $\overline{G}$.

We claim that $\phi(g) \notin \phi(H_1)$, which obviously implies $\phi(H_1) \neq \phi(H_2)$ as desired. Indeed, if $\phi(g) \in \phi(H_1)$, then $gN \subseteq H_1 N$, hence $g \in H_1 N \subseteq D \cdot D = D$, a contradiction. □

Clearly, any ERF, LERF, or π_C group is residually finite. It follows that any finitely presented ERF, LERF, or π_C group has solvable word problem. In addition, one of the special properties of LERF groups is that they have solvable generalized word problem.

Recall that a group G has solvable generalized word problem, if there is an algorithm that takes as input a finite subset X of G and an element $g \in G$ and decides after finitely many steps whether $g \in \langle X \rangle$ or not.

There are many theorems about LERF groups. We cite here some of them:

- Every subgroup of a LERF group is LERF and every finite extension of a LERF group is LERF, Scott [33].
- Polycyclic groups are LERF, Mal'cev [26].
- Free groups are LERF, M. Hall [17].
- The free product of two LERF groups is LERF, Romanovskii [32] and Burns [7].
- The free product of two LERF groups amalgamated along a finite subgroup is LERF, Allenby and Gregorac [2].

- The free product of two free groups amalgamated along a cyclic subgroup is LERF, Brunner, Burns, and Solitar [4].
- The free product of two nilpotent groups amalgamated along a cyclic subgroup is LERF, Tang [39].
- The free product of a LERF group and a free group F amalgamated along a maximal cyclic subgroup in F is LERF, Gitik [11].
- The free product of two LERF groups amalgamated along a cyclic subgroup need not be LERF, Gitik and Rips [12].
- The direct product of two LERF groups need not be LERF, Allenby and Gregorac [2].

1.3 Relations between the classes ERF, LERF, π_C, and RF

Relations between the classes ERF, LERF, π_C, and RF are described by Allenby and Gregorac, in [2], as the following:

$$\text{ERF} \subsetneqq \text{LERF} \subsetneqq \pi_C \subsetneqq \text{RF}.$$

It is clear that, the inclusions follow directly from the definitions, however, the definitions are not enough to show the non-equality.

Therefore, we consider the following examples to show that these inclusions are strict.

LERF but not ERF group: Consider the free group $F_2(a,b)$. M. Hall [17] proved that free groups are LERF. We give a proof that F_2 is not ERF.

Consider the subgroup $H = \langle b^i a b^{-i} \mid i \in \mathbb{Z} \setminus \{0\} \rangle$ of $F_2(a,b)$, then it is obvious that $a \notin H$. We claim that a belongs to every finite index subgroup D of F_2 which contains H, therefore F_2 is not ERF.

Since D has finite index in F_2, there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $b^n \in D$. Hence $a = b^{-n}(b^n a b^{-n})b^n \in D$ (note that $b^n a b^{-n} \in H \leqslant D$).

π_C **but not LERF group:** Consider the direct product $F_2 \times F_2$. In [2], Allenby and Gregorac proved that $F_2 \times F_2$ is not LERF.

However, $F_2 \times F_2$ is π_C, because free groups are π_C, and the direct product of two π_C groups is π_C, which is proved by Stebe [37, Thm. 1, 4].

RF but not π_C group: Consider the Baumslag-Solitar group

$$BS(1,m) = \langle a, b \,|\, a^{-1}ba = b^m \rangle \text{ for all } m > 1.$$

Although this group is residually finite (see for example [29]), it is easy to show that $BS(1,m)$ is not π_C.

Indeed, it is clear that $b \notin \langle b^m \rangle$. Let ϕ be an epimorphism from $BS(1,m)$ onto a finite group of order s, then

$$\phi(b) = \phi(a)^{-s}\phi(b)\phi(a)^s = \phi(a^{-s}ba^s) = \phi(b^{m^s}) \in \phi(\langle b^m \rangle).$$

Since s is arbitrary, we claim that $\phi(b) \in \phi(\langle b^m \rangle)$ for every epimorphism ϕ from $BS(1,m)$ onto a finite group. Thus $BS(1,m)$ is not π_C. □

Now, an interesting question arises naturally and need to be answered.

Question: Are the classes of ERF, LERF, π_C, and RF groups closed under different kinds of products?

In the following, we introduce the complete answer to this question concerning the wreath product, the direct product, the semidirect product, and the free product.

The wreath product: By wreath product we mean the restricted wreath product in which the direct product is used.

The wreath product of two ERF, LERF, π_c, or RF groups is not necessary to be RF, and therefore need not be ERF, LERF, and π_C.

For instance, consider the group $W = S_3 \wr \mathbb{Z}$, where S_3 is the symmetric group of degree 3. It is well-known that the groups S_3 and $\mathbb{Z}$ are ERF, LERF, π_C, and RF. Now we show that W is not RF.

Let $\mathscr{P}$ be any property such that whenever a group has $\mathscr{P}$ then all its subgroups also have $\mathscr{P}$. Gruenberg [14, Thm. 3.1] proved that if the wreath product $W = A \wr B$ is residually $\mathscr{P}$, then either B is $\mathscr{P}$ or A is abelian.

Obviously, S_3 is not abelian and $\mathbb{Z}$ is infinite. In addition, the subgroup of any finite group is finite. It follows immediately that, the group W is not RF, and therefore W is not ERF, LERF, or π_C.

The direct product: According to Allenby and Gregorac [2], it has been proven by Mal'cev [26], that the direct product of two ERF groups is ERF.

On the other hand, the direct product of two LERF groups may not be LERF. For instance, the direct product $F_2 \times F_2$ is not LERF, by the work of Allenby and Gregorac [2], although F_2 is LERF.

It follows that, $F_n \times F_m$ is not LERF for all n and m greater than 1, because every subgroup of LERF group is LERF, Scott [33].

They also proved, in [2], that $F_1 \times F_2$ is LERF.

Moreover, for the class of π_C groups, it is proved by Stebe [37] that the direct product of two π_C groups is π_C. For the class of RF groups, it is obvious that the direct product of two RF groups is RF.

The semidirect product: In [2], Allenby and Gregorac proved that the semidirect product $G = A \rtimes B$ of two groups A and B satisfies the following assertions:

- If A and B are ERF, then G is ERF.
- If A is ERF and B is LERF, then G is LERF.
- If A and B are π_C, then G is π_C.

If A and B are LERF groups, then the semidirect product $A \rtimes B$ need not be LERF. For instance, Burns, Karrass, and Solitar [8] proved that $F_2 \rtimes F_1$ is not LERF for some particular action of F_1 on F_2, although F_2 and F_1 are LERF (note that F_2 is not ERF).

Similarly, if A and B are RF groups, then the semidirect product $A \rtimes B$ may not be RF. For instance, consider the group $W = S_3 \wr \mathbb{Z}$, where S_3 is the symmetric group of degree 3.

As we have seen in the discussion of the wreath product, W is not residually finite. However, W can be written as $W = (\prod_{i\in\mathbb{Z}} S_3) \rtimes \mathbb{Z}$, where $\mathbb{Z}$ and $\prod_{i\in\mathbb{Z}} S_3$ are residually finite. Note that Gruenberg [14] proved that the direct product of RF groups is RF.

The free product: The free product of two ERF groups may not be ERF. For instance, the infinite cyclic group $\mathbb{Z}$ is ERF, however the free product $\mathbb{Z} * \mathbb{Z} = F_2$ is not ERF, as we have proved in the beginning of Section 1.3.

On the other hand, it is proved by Burns [7] and Romanovskii [32] that the free product of two LERF groups is LERF, and also it is proved by Stebe [37] that the free product of two π_C groups is π_C.

Finally, we can summarize almost all the results we have discussed in the following table.

A and B	ERF	LERF	π_C	RF
$A \wr B$	not nec.	not nec.	not nec.	not nec.
$A \times B$	yes	not nec.	yes	yes
$A \rtimes B$	yes	not nec.	yes	not nec.
$A * B$	not nec.	yes	yes	yes

Tab. 1.3.1. *The classes ERF, LERF, π_C, and RF with different kinds of products.*

Another modification of the condition on the elements of a residually finite group is to replace the equality between the elements in Definition 1.1.1 (a) by the conjugacy.

In this way, we obtain another proper class of the class of residually finite groups, which we consider in the next section.

1.4 Conjugacy separable groups

Definition 1.4.1. A group G is called *conjugacy separable* (CS) if for every two non-conjugate elements $x, y \in G$, there exists a homomorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(x)$ is not conjugate to $\phi(y)$ in $\overline{G}$.

Clearly, every CS group is RF. Indeed, let G be a CS group with the identity element e and let g be a non-trivial element in G. According to Definition 1.1.1 (b), to show that G is RF, we need to find a finite index subgroup D of G such that D does not contain g.

Since g is not conjugate to e and G is CS, there exists a homomorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(g)$ is not conjugate to $\phi(e)$ in $\overline{G}$. So $\phi(g)$ is non-trivial in $\overline{G}$. Thus $D :=\ker(\phi)$ is a finite index subgroup of G such that $g \notin D$.

On the other hand, every RF group is not necessary to be CS. For instance, in [36], Stebe proved that the group $\mathrm{SL}(n, \mathbb{Z})$ of $n \times n$ integer matrices with determinant 1 is not CS for all $n \geq 3$, although $\mathrm{SL}(n, \mathbb{Z})$ is RF.

To show that $\mathrm{SL}(n, \mathbb{Z})$ is RF, let $(a_{ij}) = A \neq B = (b_{ij}) \in \mathrm{SL}(n, \mathbb{Z})$, then there exist at least one i and one j such that $a_{ij} \neq b_{ij}$. Let $m > max\{|a_{ij}|, |b_{ij}|\}$, then the map $\phi : \mathrm{SL}(n, \mathbb{Z}) \to \mathrm{SL}(n, \mathbb{Z}_m)$ is a homomorphism onto a finite group such that $\phi(A) \neq \phi(B)$.

Since CS groups are RF, all finitely presented CS groups have solvable word problem. Moreover, in [28], McKinsey proved that if G is finitely generated recursively presented group and G is conjugacy separable then the conjugacy problem for G is solvable.

Recall that the conjugacy problem for a group G is solvable if there exists an algorithm that takes as input two words w_1, w_2 in the generators of G and decides after finitely many steps whether w_1 is conjugate to w_2 in G or not.

According to Remeslennikov [31], the direct product of two CS groups is CS. In the same paper he proved that free groups are CS, and the free product of two CS groups is CS.

Furthermore, he gave a necessary and sufficient condition for the wreath product of two CS groups to be CS. More precisely, he proved that if A and B are two CS groups, then the wreath product $W = A \wr B$ is CS if and only if either B is finite or A is abelian and π_C.

According to his condition the wreath product of two CS groups may not be CS. For instance, although S_3 and $\mathbb{Z}$ are CS, the group $W = S_3 \wr \mathbb{Z}$ is not CS because S_3 is not abelian and $\mathbb{Z}$ is not finite. This was expected because W is not RF and every CS group is RF.

The same example can be also used as a proof that the semidirect product of two CS groups need not be CS. Indeed, it is proved by Remeslennikov [31] that the direct product of CS groups is CS, in particular, $\prod_{i\in\mathbb{Z}} S_3$ is CS, and we know that $\mathbb{Z}$ is CS. However, $W = (\prod_{i\in\mathbb{Z}} S_3) \rtimes \mathbb{Z}$ is not CS.

So we can add the CS class to Table 1.3.1 to get the following table:

A and B	ERF	LERF	π_C	CS	RF
$A \wr B$	not nec.	not nec.	not nec.	not nec.	not nec.
$A \times B$	yes	not nec.	yes	yes	yes
$A \rtimes B$	yes	not nec.	yes	not nec.	not nec.
$A * B$	not nec.	yes	yes	yes	yes

Tab. 1.4.1. *The classes ERF, LERF, π_C, CS, and RF with different kinds of products.*

One of the properties which distinguish the class of CS groups from the other classes EFR, LERF, π_C, and RF is that a subgroup of a CS group need not be CS.

In [27], Martino and Minasyan have produced an example of finitely presented conjugacy separable groups that contain non-conjugacy separable subgroups of finite index.

In addition, Chagas and Zalesskii [10] have constructed another example with the same property. Furthermore, they gave a sufficient condition for a conjugacy separable group to preserve this property when passing to subgroups of finite index (see Section 1.6).

On the other hand, some special subgroups of a CS group are CS. For instance, it is proved by Minasyan [30] that every retract subgroup of a CS group is CS.

Another special property of CS groups is that the class of conjugacy separable groups is not closed under finite extensions. In [13], Goryaga gave an example of a non-CS group which is a finite extension of a CS group.

The relation between CS and LERF groups is not completely known. However, we can easily notice that every CS group is not necessary to be LERF. For instance, the group $F_2 \times F_2$ is CS [31] but not LERF [2].

In Section 1.5 we consider another subclasses, SCS and SICS, of the class of residually finite groups. The definition of the subclass SCS can be obtained by replacement the word "elements" in Definition 1.4.1 (a) by "finitely generated subgroups". The definition of the subclass SICS can be obtained by replacement the words "conjugate to" in the definition of SCS groups by the words "conjugate into". The definition of the into-conjugacy can be found in Definition 1.5.2.

1.5 SCS and SICS groups

The following definition was first introduced in [6] by Bogopolski and Grunewald.

Definition 1.5.1. A group G is called *subgroup conjugacy separable* (SCS) if for every two non-conjugate finitely generated subgroups $H_1, H_2 \leqslant G$, there exists a homomorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(H_1)$ is not conjugate to $\phi(H_2)$ in $\overline{G}$.

The study of SCS groups was initiated by Grunewald and Segal in [15], although the class of SCS groups were not defined yet. They have proved that all virtually polycyclic groups are SCS (see also [34, Ch. 4, Thm. 7]).

In [6], Bogopolski and Grunewald proved that free groups and some finite extensions of free groups are SCS.

Obviously, every SCS group is RF. Indeed, let G be an SCS group and g be a non-trivial element in G, then $\langle g \rangle$ is not conjugate to the subgroup $\langle e \rangle$, where e is the identity element in G. Since G is SCS, there exists a homomorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(\langle g\rangle)$ is not conjugate to $\phi(\langle e\rangle)$. Thus $\phi(g)$ is non-trivial in $\overline{G}$, and therefore G is RF.

Since SCS groups are RF, finitely presented SCS groups have solvable word problem. Moreover, as it was observed by Bogopolski and Grunewald [6], finitely presented SCS groups have solvable generalized conjugacy problem.

Recall that a group G has solvable generalized conjugacy problem, if there is an algorithm that takes as input two finite subsets X and Y of G and decides after finitely many steps whether the subgroups $\langle X\rangle$ and $\langle Y\rangle$ are conjugate in G or not.

Definition 1.5.2. Let A and B be two subgroups of a group G. We say that A is *conjugate into* B, if there is an element $g \in G$ such that A^g is a subgroup of B.

Definition 1.5.3. (Bogopolski and Grunewald [6]) A group G is called *subgroup into-conjugacy separable* (SICS) if for every two finitely generated subgroups $H_1, H_2 \leqslant G$ such that H_2 is not conjugate into H_1, one of the following equivalent conditions holds:

(a) there exists a homomorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(H_2)$ is not conjugate into $\phi(H_1)$ in $\overline{G}$.

(b) there exists a finite index subgroup D of G such that H_1 is contained in D and H_2 is not conjugate into D.

For completeness, we show that these conditions are indeed equivalent.

(a) ⇒ (b): Let $N := \ker(\phi)$, where ϕ is the homomorphism defined in (a). Since $\phi(H_2)$ is not conjugate into $\phi(H_1)$ in $\overline{G}$, the subgroup H_2N is not conjugate into H_1N in G.

Set $D := H_1N$, then clearly $H_1 \leqslant D$, and H_2 is not conjugate into D. Indeed, if there exists $g \in G$ such that $H_2^g \leqslant D$, then

$$(H_2N)^g = H_2^g N \leqslant DN = D = H_1N,$$

which contradicts that H_2N is not conjugate into H_1N in G.

(a) $\Leftarrow$ (b): Let D be the finite index subgroup of G which is given by (b). Since $N := \cap_{g\in G} D^g$ is a finite index normal subgroup of G, consider the natural homomorphism ϕ from G onto the finite group G/N.

To prove that ϕ satisfies (a), assume to the contrary that $\phi(H_2) = H_2N$ is conjugate into $\phi(H_1) = H_1N$. It follows directly that $(H_2N)^g = H_2^gN \leqslant H_1N$ for some $g \in G$. Then $H_2^gN \leqslant D$, and therefore, $H_2^g \leqslant D$, which contradicts that H_2 is not conjugate into D. Hence $\phi(H_2)$ is not conjugate into $\phi(H_1)$. □

Analogous to SCS groups, one can easily show that SICS groups are RF. However, RF groups may not be SICS. For instance, in Chapter 3, we will prove that the residually finite group $F_2 \times F_2$ is not SICS.

If we denote the conjugacy by $\sim$ and the conjugacy into by $\rightsquigarrow$, all the properties which we introduced can be summarized in the following table.

	$\neq$	$\nsim$	$\not\rightsquigarrow$
elements	RF	CS	—
subgroups	ERF	—	—
cyclic subgroups	π_C	—	—
finitely generated subgroups	LERF	SCS	SICS

Tab. 1.5.1. *The classes ERF, LERF, π_C,CS, SCS, SICS and RF.*

The unique result we knew, before writing this thesis, about SICS groups is that free groups are SICS, and every SICS virtually free group is SCS, Bogopolski and Grunewald [6]. This was one of the motivations for the present thesis.

We know, by the examples in [27] and [10], that the subgroups of a CS group need not be CS. However, we do not know whether the same is true for SCS or SICS groups or not. On the other hand, following the observation given by Minasyan [30] that every retract of a CS group is CS, we claim that the same is true for SCS and SICS groups.

Proposition 1.5.4. *If H is a retract of an SICS (SCS) group G, then H is SICS (SCS).*

Proof. Let G be an SICS group, and ϕ be a retraction from G onto H, then ϕ is an epimorphism and ϕ restricted to $H \leqslant G$ is the identity map. Let H_1 and H_2 be two finitely generated subgroups of H such that H_2 is not conjugate into H_1 in H. We will show that H_2 is not conjugate into H_1 by any element of $G \setminus H$.

Suppose the contrary, i.e. there exists $g \in G \setminus H$ such that $gH_2g^{-1} \leqslant H_1$. Then, we have

$$\phi(g)H_2\phi(g)^{-1} = \phi(g)\phi(H_2)\phi(g^{-1}) = \phi(gH_2g^{-1}) \leqslant \phi(H_1) = H_1.$$

Since $\phi(g)$ belongs to H, and H_2 is not conjugate into H_1 in H, we get a contradiction. Hence H_2 is not conjugate into H_1 in G.

Since G is SICS, there exists a homomorphism ϕ_1 from G onto a finite group $\overline{G}$ such that $\phi_1(H_2)$ is not conjugate into $\phi_1(H_1)$ in $\overline{G}$.

Let $\overline{H} := \phi_1(H)$ and ϕ_2 from H onto $\overline{H}$ be the restriction of ϕ_1 to H. By construction, it follows that $\phi_2(H_2)$ is not conjugate into $\phi_2(H_1)$ in $\overline{H}$. Therefore H is SICS.

This proof can be easily adapted to obtain a proof for the case of SCS groups: it suffices to replace the word ”into“ by ”to” and the symbol “$\leqslant$” by ”=“. □

The relations between the classes of CS, SCS, and SICS groups are not completely known. It is clear that they intersect, for instance, finite and free groups are CS, SCS, and SICS. The complement of these classes also intersect. For example, every non-residually finite group is not CS, SCS, or SICS.

On the other hand, we have proved the following results:

- *CS groups need not be SICS.* Indeed, in Chapter 3 Section 3.2 we will prove that $F_2 \times F_2$ is not an SICS group. However, free groups are CS [37] and the direct product of two CS groups is CS [31]. Therefore, the group $F_2 \times F_2$ is CS but not SICS.

- *SCS groups need not be SICS.* Indeed, it is proved by Grunewald and Segal [15] that all virtually polycyclic groups are SCS (see also [34, Ch. 4, Thm. 7]). However, in Chapter 3 Section 3.3 we will see that polycyclic groups need not be SICS.

- *An HNN extension of an SICS group may not be SICS.* Indeed, in Chapter 3 Section 3.3 we will show that $G = \langle x, y, a \,|\, axa^{-1} = y^{-1}x,\ aya^{-1} = y\rangle$ is not SICS. Clearly, G is an HNN extension of F_2 determined by the automorphism $\phi : F_2 \to F_2$ given by $\phi(x) = y^{-1}x$ and $\phi(y) = y$.

Proposition 1.5.5. *Let G be a torsion free SCS group, then G is CS if for every g_1 and g_2 in G the following is valid:*

g_1 is not conjugate to g_2 implies that g_1 is not conjugate to g_2^{-1}.

Proof. Let g_1, g_2 be two elements in G such that g_1 is not conjugate to g_2. To prove that G is CS, we will find a homomorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(g_1)$ is not conjugate to $\phi(g_2)$.

First, we show that $\langle g_1\rangle$ is not conjugate to $\langle g_2\rangle$. Without loss of generality, we may assume that g_1 and g_2 are non-trivial. Assume to the contrary that $\langle g_1\rangle$ is conjugate to $\langle g_2\rangle$, then there exists an element $x \in G$ such that $\langle g_1\rangle^x = \langle g_2\rangle$ or equivalently $\langle g_1\rangle = \langle g_2\rangle^{x^{-1}}$. Therefore, there exist $n \neq \pm 1$ and $m \neq \pm 1$ such that $xg_1^n x^{-1} = g_2$ and $g_1 = x^{-1}g_2^m x$. It follows that

$$g_1 = x^{-1}g_2^m x = (x^{-1}g_2x)^m = (x^{-1}xg_1^n x^{-1}x)^m = g_1^{nm}.$$

Since G is torsion free and $g_1 \neq 1$, we get a contradiction. Thus $\langle g_1\rangle$ is not conjugate to $\langle g_2\rangle$.

Second, since G is SCS, there exists a homomorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(\langle g_1\rangle)$ is not conjugate to $\phi(\langle g_2\rangle)$. Since ϕ is a homomorphism, it follows that $\phi(g_1)$ is not conjugate to $\phi(g_2)$. □

Proposition 1.5.6. *If G is an SICS group and for every finitely generated subgroup $H \leqslant G$ there is no non-trivial element $g \in G$ such that $H^g < H$, then G is SCS.*

Proof. Let H_1 and H_2 be two finitely generated subgroups of G such that H_1 is not conjugate to H_2. Our aim is to find a homomorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(H_1)$ is not conjugate to $\phi(H_2)$.

First, we claim that H_2 is not conjugate into H_1 or H_1 is not conjugate into H_2. Indeed, assume to the contrary that there exist g_1 and g_2 in G such that $H_2^{g_1} < H_1$ and $H_1^{g_2} < H_2$, then $H_2^{g_1 g_2} = (H_2^{g_1})^{g_2} < H_1^{g_2} < H_2$ contradicts the assumption.

Thus, without loss of generality, we may assume that H_2 is not conjugate into H_1. Since G is SICS, there exists a homomorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(H_2)$ is not conjugate into $\phi(H_1)$. Therefore, $\phi(H_2)$ is not conjugate to $\phi(H_1)$. □

The relations between the classes of LERF, SCS, and SICS goups also are not completely known. However, for abelian groups we can easily show that they are equivalent.

Let G be an abelian group.

SCS $\Leftrightarrow$ LERF: Since two subgroups of G are conjugate if and only if they are equal, G is LERF if and only if G is SCS.

SICS $\Rightarrow$ LERF: Let H be a finitely generated subgroup of G, and let $g \notin H$, then $\langle g \rangle \nleq H$. Since G is SICS, there exists an epimorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(\langle g \rangle) \nleq \phi(H)$, and then $\phi(g) \notin \phi(H)$.

SICS $\Leftarrow$ LERF: Let H_1, H_2 be two finitely generated subgroups of G such that H_2 is not conjugate into H_1, equivalently $H_2 \nleq H_1$. Then there exists at least one element $h \in H_2$ such that $h \notin H_1$. Since G is LERF, there exists an epimorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(h) \notin \phi(H_1)$, and therefore $\phi(H_2) \nleq \phi(H_1)$, equivalently $\phi(H_2)$ is not conjugate into $\phi(H_1)$. □

Corollary 1.5.7. *Finitely generated abelian groups are SCS and SICS.*

Proof. Any finitely generated abelian group is polycyclic, and therefore it is LERF, by [26], so it is SCS and SICS. □

Remark 1.5.8. There exist abelian groups which are neither SCS nor SICS. For instance, the group $\mathbb{Q}$ is of this type.

To this moment we do not know whether the class of SCS groups is closed under different kinds of products or not. For SICS groups we will prove, in Chapters 2 and 3, the following statements:

- The free product of two simultaneously SICS and LERF groups is SICS.
- The direct product of two SICS groups need not be SICS.
- The semidirect product of two SICS groups need not be SICS.
- The wreath product of two SICS (SCS) groups need not be SICS (SCS).
- The free product of SICS (SCS) group with SCS group amalgamated over a cyclic subgroup need not be SICS.

If we denote the word "yes" by "+", "yes with conditions" by "⊕", "not necessary" by "–", and "not known" by "?", then we can extend Table 1.4.1 and obtain the following table:

A and B	ERF	LERF	π_C	CS	SCS	SICS	RF
$A \times B$	+	–	+	+	?	–	+
$A \rtimes B$	+	–	+	–	?	–	–
$A \wr B$	–	–	–	–	–	–	–
$A * B$	–	+	+	+	?	⊕	+

Tab. 1.5.2. *The classes ERF, LERF, π_C,CS, SCS, SICS and RF with different kinds of products.*

1.6 Topological interpretation

Definition 1.6.1. Let G be a group, the *profinite topology* $\mathcal{PT}(G)$ is the topology whose basic open sets are all cosets to finite index normal subgroups in G.

Every finite index subgroup $D \leqslant G$ contains the finite index normal subgroup $N := \cap_{g \in G} D^g$. Since $D = \cup_{g \in D} gN$ and $G \setminus D = \cup_{g \in G \setminus D} gN$, it follows that every finite index subgroup $D \leqslant G$ is both closed and open in $\mathcal{PT}(G)$.

Moreover, since $hD = \cup_{g \in D} hgN$ and $G \setminus hD = h(G \setminus D) = \cup_{g \in G \setminus D} hgN$ for every h in G, it follows that every left coset of finite index subgroup $D \leqslant G$ is both closed and open in $\mathcal{PT}(G)$. The same holds for write cosets of finite index subgroups in G.

Proposition 1.6.2. *The profinite topology $\mathcal{PT}(G)$ of a group G is Hausdorff if and only if the intersection of all finite index normal subgroups in G is trivial.*

Proof. Suppose that $\mathcal{PT}(G)$ is Hausdorff. Let g be a non-trivial element in G such that g belongs to all finite index normal subgroups in G, then g and the identity element e in G belong to the same open sets in $\mathcal{PT}(G)$, which contradicts the assumption.

Now, assume that the intersection of all finite index normal subgroups in G is trivial. To prove that $\mathcal{PT}(G)$ is Hausdorff, we take two different elements $x, y \in G$ and show that there exist two open sets $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{PT}(G)$ such that $x \in \mathcal{A}$, $y \in \mathcal{B}$, and $\mathcal{A} \cap \mathcal{B} = \emptyset$.

Since the intersection of all finite index normal subgroups in G is trivial, there exists a finite index normal subgroup N in G such that $xy^{-1} \notin N$. Then we take $\mathcal{A} = xN$ and $\mathcal{B} = yN$. □

Below we give other definitions of RF, LERF, CS, and SCS groups using profinite topology on groups, and we will show that these definitions are equivalent to those which are given in Sections 1.1–1.5.

Definition 1.6.3. The group G is called *residually finite (or RF)*, if the profinite topology $\mathcal{PT}(G)$ is Hausdorff.

This definition is equivalent to Definition 1.4.1 by Proposition 1.6.2.

Definition 1.6.4. A subset H of a group G is called *separable* if H is closed in the profinite topology $\mathcal{PT}(G)$.

Definition 1.6.5. The group G is called *subgroup separable (or LERF)*, if every finitely generated subgroup in G is separable.

Note that if we replace the words "finitely generated subgroup" by "cyclic groups" (or "groups"), we obtain another definition of the π_C groups (or ERF groups).

Now we prove that Definition 1.6.5 and Definition 1.2.1 are indeed equivalent.

Definition 1.2.1 ⇒ Definition 1.6.5: Let H be a finitely generated subgroup in G, we will prove that H is closed, or equivalently, $G\backslash H$ is open in $\mathcal{PT}(G)$.

Let $g \in G\backslash H$. By Definition 1.2.1 (c), there exists a finite index subgroup D in G such that $H \leqslant D$ and $g \notin D$. Since $N := \cap_{x\in G} D^x$ is a finite index normal subgroup in G, the coset gN is an open neighbourhood of g in G.

Since $g \notin D$, the coset gN is contained in $G \setminus D \subseteq G \setminus H$, which implies that $G \setminus H$ is open.

Definition 1.6.5 ⇒ Definition 1.2.1: Let H be a finitely generated subgroup in G and $g \notin H$. Since H is separable or equivalently $G \setminus H$ is open, there exists a finite index normal subgroup N in G such that $gN \subseteq G \setminus H$.

Let ϕ be the natural homomorphism from G onto G/N, then $\phi(g) \notin \phi(H)$. Otherwise gN would equal to hN for some $h \in H$, then $h \in gN \cap H$, which is a contradiction. □

Definition 1.6.6. The group G is called *conjugacy separable (or CS)*, if for every element $g \in G$, its conjugacy class $g^G := \{xgx^{-1} \mid x \in G\} \subseteq G$ is separable.

In the following we show that Definition 1.6.6 is equivalent to Definition 1.4.1.

Definition 1.6.6 ⇒ Definition 1.4.1: Let g_1, g_2 be two non-conjugate elements in G, then $g_2 \notin g_1^G$. Since g_1^G is closed in $\mathcal{PT}(G)$, the complement of g_1^G in G is open. Then there exists a finite index normal subgroup N in G such that g_2N is contained in $(G \setminus g_1^G)$.

An equivalent assertion is that $g_2N \neq gN$ for all $g \in g_1^G$. Therefore, the natural homomorphism ϕ from G onto the finite group G/N is the required homomorphism. That is, $\phi(g_1)$ is not conjugate to $\phi(g_2)$.

Definition 1.4.1 ⇒ Definition 1.6.6: We want to prove that g^G is closed for every element $g \in G$. Let $g' \in G \setminus g^G$, then g' is not conjugate to g in G.

According to Definition 1.4.1, there exists a homomorphism ϕ from G onto a finite quotient G/N such that $g'N$ is not conjugate to gN in G/N, and therefore $g'N \neq xN$ for all $x \in g^G$. Thus g' has an open neighbourhood $g'N \subseteq G \setminus g^G$, which implies that $G \setminus g^G$ is open. □

The completion $\widehat{G}$ of G with respect to the profinite topology $\mathcal{PT}(G)$ is called the *profinite completion* of G. It can be expressed as an inverse limit of all finite quotients of G, we write $\widehat{G} = \varprojlim_N G/N$.

Moreover, there exists a natural homomorphism i from G to $\widehat{G}$ that sends g to (gN). We can easily see that i is a monomorphism when G is residually finite, since the kernel of i is the intersection of all finite index normal subgroups in G.

We also can look at the CS groups from another topological point of view. The group G is CS, if G is residually finite and for every two elements $x, y \in G$ such that $y = x^\gamma$ for some $\gamma \in \widehat{G}$, there exists $g \in G$ such that $y = x^g$.

If S is a subset of G, we denote, in this section, by $\overline{S}$ its closure in $\mathcal{PT}(G)$. In [10], Chagas and Zalesskii gave a sufficient condition for a conjugacy separable group to be hereditarily conjugacy separable. A group G is called *hereditarily conjugacy separable* if every finite index subgroup $H \leqslant G$ is conjugacy separable.

Namely, they have proved that if G is a conjugacy separable group and

$$\overline{C_G(g)} = C_{\widehat{G}}(g) \text{ for every element } g \in G,$$

then G is hereditarily conjugacy separable.

In the same sense, we can topologically consider the SCS groups as follows.

Definition 1.6.7. The group G is called *subgroup conjugacy separable (or SCS)*, if G is residually finite and for every two finitely generated subgroups $H_1, H_2 \leqslant G$ such that $H_2 = H_1^{\gamma}$ for some $\gamma \in \widehat{G}$, there exists $g \in G$ such that $H_2 = H_1^g$.

Following Chagas and Zalesskii, we give a sufficient condition for a subgroup conjugacy separable group to be hereditarily subgroup conjugacy separable.

Analogous to CS groups, a group G is *hereditarily subgroup conjugacy separable* if every finite index subgroup $H \leqslant G$ is subgroup conjugacy separable.

Proposition 1.6.8. *Let G be a subgroup conjugacy separable group and for every finitely generated subgroup $H \leqslant G$,*

$$\overline{N_G(H)} = N_{\widehat{G}}(H).$$

Then G is hereditarily subgroup conjugacy separable.

Proof. Let H be a finite index subgroup of G. Let H_1, H_2 be two finitely generated subgroups of H such that $H_2 = H_1^{\gamma}$ for some $\gamma \in \widehat{H}$. Since G is subgroup conjugacy separable, there exists $g \in G$ such that $H_2 = H_1^g$.

Then $\delta := \gamma g^{-1} \in N_{\widehat{G}}(H_1)$. It follows that $g = \delta^{-1}\gamma \in N_{\widehat{G}}(H_1)\widehat{H} \cap G$. Since H is of finite index in G, the set $N_G(H_1)H$ is closed in the profinite topology, that is $\overline{N_G(H_1)H} = N_G(H_1)H$.

By hypothesis $N_{\widehat{G}}(H_1) = \overline{N_G(H_1)} \subseteq G$, so

$$N_{\widehat{G}}(H_1)\widehat{H} \cap G = \overline{N_G(H_1)}(\widehat{H} \cap G) = \overline{N_G(H_1)}H = \overline{N_G(H_1)H} = N_G(H_1)H.$$

Note that $\overline{N_G(H_1)}H = \overline{N_G(H_1)H}$, since H has finite index in G and so H is closed. Therefore, $g = ch$ for some $c \in N_G(H_1)$, $h \in H$.

Hence $H_2 = H_1^g = H_1^h$, that is H_2 is conjugate to H_1 in H. □

It would be useful to reformulate the SICS property in terms of profinite topology on groups. However, we do not expect that this is possible.

In the following proposition we show that the subgroup into conjugacy separability for a group G implies the closeness, in the profinite topology, of the union of all subgroups conjugated to H, where H is a finitely generated subgroup of G.

Proposition 1.6.9. *If G is an SICS group and H is a finitely generated subgroup of G, then the set ${}^GH := \{g^{-1}hg \mid g \in G,\ h \in H\}$ is separable.*

Proof. We will prove that ${}^GH$ is closed for every finitely generated subgroup $H \leqslant G$. Let $g \in (G \setminus {}^GH)$, then the cyclic subgroup $\langle g \rangle$ is not conjugate into H.

Since G is SICS, there exists a homomorphism ϕ from G onto a finite quotient G/N such that $\langle g \rangle N$ is not conjugate into HN in G/N. It follows that gN is not conjugate to hN for all $h \in H$, or equivalently $gN \neq xN$ for all $x \in {}^GH$.

So gN is an open neighbourhood of g such that $gN \subseteq (G \setminus {}^GH)$, and this implies that $(G \setminus {}^GH)$ is open. □

The following basic topological definitions and theorems will be used in Chapter 2 in order to prove that the free product of two simultaneously LERF and SICS groups is SICS.

Theorem 1.6.10. *([35] page 147) For any group G there is a topological space $\mathcal{X}$ with $\pi_1(\mathcal{X}, x_0) \cong G$, where $\pi_1(\mathcal{X}, x_0)$ is the fundamental group of $\mathcal{X}$ based at x_0.*

Definition 1.6.11. Let $\widetilde{\mathcal{X}}, \mathcal{X}$ be topological spaces. A continuous map p from $\widetilde{\mathcal{X}}$ onto $\mathcal{X}$ is called a *covering map* if p is surjective and every point $x \in \mathcal{X}$ has an open neighbourhood $\mathcal{U}$ such that $p^{-1}(\mathcal{U})$ is a disjoint union of open subsets of $\widetilde{\mathcal{X}}$ each of which is mapped homeomorphically onto $\mathcal{U}$ by p.

The space $\widetilde{\mathcal{X}}$ is called the *covering space* and $\mathcal{X}$ is called the *base space* of the covering map p.

A topological space $\mathcal{X}$ is called *locally path connected* if every point has a local base consisting of path connected neighbourhoods. A topological space $\mathcal{X}$ is called *semilocally simply connected* if every point $x \in \mathcal{X}$ has a neighbourhood $\mathcal{U}$ such that every closed path at x in $\mathcal{U}$ is homotopic in $\mathcal{X}$ to the constant path at x.

Theorem 1.6.12. *([20] page 174) Suppose that $\mathcal{X}$ is a connected, locally path connected, and semilocally simply connected topological space. If H is a subgroup of $\pi_1(\mathcal{X}, x_0)$, then there exists a covering $\phi : (\mathcal{X}_H, x_H) \to (\mathcal{X}, x_0)$ such that $H = \phi_*(\pi_1(\mathcal{X}_H, x_H))$, where ϕ_* is the induced monomorphism from $\pi_1(\mathcal{X}_H, x_H)$ to $\pi_1(\mathcal{X}, x_0)$.*

Definition 1.6.13. Let $\widetilde{\mathcal{X}}, \mathcal{X}$ and $\mathcal{Y}$ be topological spaces. If $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ is a covering and $f : \mathcal{Y} \to \mathcal{X}$ is a continuous map then a *lift* of f is a continuous map $\widetilde{f} : \mathcal{Y} \to \widetilde{\mathcal{X}}$ such that $p \circ \widetilde{f} = f$.

Theorem 1.6.14. *([20] page 147) Let $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a covering. Given a path $f : I \to \mathcal{X}$ and $\widetilde{x_0} \in \widetilde{\mathcal{X}}$ with $p(\widetilde{x_0}) = f(0)$ there exists a unique lift $\widetilde{f} : I \to \widetilde{\mathcal{X}}$ such that $\widetilde{f}(0) = \widetilde{x_0}$.*

Chapter 2

SICS for Free Product of Groups

2.1 Introduction

In order to deepen our knowledge of the class of SICS groups we need to discover its properties. Since the class of SICS groups shines recently (2010 [6]), many problems still open (for example see Appendix A).

In this thesis we are interested to know whether this class is closed under different kinds of products or not. In particular, in this Chapter we prove that the free product of two finitely generated groups which are simultaneously LERF and SICS is SICS, Theorem 2.2.1.

Many problems in group theory are difficult to solve by using only algebraic methods. Presentations of groups in terms of generators and relations give the possibility to identify these groups with fundamental groups of some path connected topological spaces.

Applying some geometrical and topological theorems on these topological spaces we can prove some algebraic properties of the original groups. And this is exactly what we will do in this chapter to prove Theorem 2.2.1.

Before we formulate and prove Theorem 2.2.1, we introduce a preparatory, in which we explain the idea of the proof and prove the needed results which we use in the proof.

Let A and B be two simultaneously LERF and SICS groups. To prove that the free product $G = A * B$ is SICS, all what we need to do is to verify Definition 1.5.3.

In other words, we want to prove that for every two finitely generated subgroups H_1, H_2 in G such that H_2 is not conjugate into H_1, the following holds:

there exists a homomorphism ϕ from G onto a finite group $\overline{G}$ such that $\phi(H_2)$ is not conjugate into $\phi(H_1)$ in $\overline{G}$.

Or equivalently, according to Definition 1.5.3 (b), there exists a finite index subgroup D of G such that $H_1 \leqslant D$ and H_2 is not conjugate into D.

Starting from this point, using Theorems 1.6.10 and 1.6.12, we will translate our problem from the algebraic language to the topological language.

It follows from Theorem 1.6.10 that there exists a topological space, which will be denoted by $\Gamma_A \overset{e}{-} \Gamma_B$, such that G is isomorphic to the fundamental group of it.

Since H_1, H_2, and D are subgroups of G, it follows from Theorem 1.6.12 that there exist covering spaces of $\Gamma_A \overset{e}{-} \Gamma_B$ corresponding to H_1, H_2, and D.

Since we want to prove that D exists, we will prove that it could be created for every finitely generated subgroups H_1 and H_2 in G such that H_2 is not conjugated into H_1.

Using a special geometrical and topological algorithm, we will construct a covering space $(\widetilde{Z}, \tilde{z})$ and a covering map $\phi : (\widetilde{Z}, \tilde{z}) \to (\Gamma_A \overset{e}{-} \Gamma_B, v)$ such that the subgroup $\phi_*(\pi_1(\widetilde{Z}, \tilde{z}))$ of G plays the role of D, where ϕ_* is the induced monomorphism from $\pi_1(\widetilde{Z}, \tilde{z})$ to $\pi_1(\Gamma_A \overset{e}{-} \Gamma_B, v)$, $\tilde{z} \in \widetilde{Z}$, and $v \in \Gamma_A \overset{e}{-} \Gamma_B$.

In other words, $\phi_*(\pi_1(\widetilde{Z}, \tilde{z}))$ would be a finite index subgroup of G such that H_1 is contained in $\phi_*(\pi_1(\widetilde{Z}, \tilde{z}))$, and H_2 is not conjugate into $\phi_*(\pi_1(\widetilde{Z}, \tilde{z}))$.

Now we explain how we represent $G = A * B$ and its subgroups geometrically as fundamental groups of topological spaces.

Let Γ be a directed graph, the set of its vertices will be denoted by $V(\Gamma)$ and the set of its edges will be denoted by $E(\Gamma)$. The initial and the terminal vertices of an edge $e \in E(\Gamma)$ will be denoted by $\iota(e)$ and $\tau(e)$ respectively. By $\overline{e}$ we denote the inverse of an edge $e \in E(\Gamma)$.

A *path* of length $n \geq 0$ in a graph Γ is the sequence $p = v_0 e_1 v_1 \cdots e_n v_n$ such that $v_i \in V(\Gamma)$, $e_j \in E(\Gamma)$, $\iota(e_j) = v_{j-1}$, and $\tau(e_j) = v_j$ for all $i \in \{0, 1, \ldots, n\}$ and $j \in \{1, 2, \ldots, n\}$. For simplicity we write $p = e_1 e_2 \cdots e_n$. A *reduced path* is a path $p = e_1 e_2 \cdots e_n$ such that $e_{i+1} \neq \overline{e_i}$ for all $i \in \{1, 2, \ldots, n-1\}$.

A reduced path $p = e_1 e_2 \cdots e_n$ such that $\tau(e_n) = \iota(e_1)$, $e_1 \neq \overline{e_n}$, $n \geq 1$, will be called a *cycle* of length n. The cycle of length 1 is called a *loop*. The *girth* of a graph is the minimum length of the cycles contained in the graph, and if the graph does not contain any cycles, its girth is defined to be infinity.

Let $G = \langle X \,|\, R \rangle$ be a group, let $X^{\pm} = \{x, x^{-1} : x \in X\}$. A labeling of a graph Γ by the set X is a function $Lab : E(\Gamma) \to X^{\pm}$ such that

(1) $Lab(\overline{e}) = (Lab(e))^{-1}$ for every $e \in E(\Gamma)$,

(2) if $Lab(e_1) = Lab(e_2)$ and $\iota(e_1) = \iota(e_2)$, then $e_1 = e_2$.

A graph with a labeling function is called a labeled graph. Denote the set of all words in $X^{\pm}$ by $W(X)$. The label of a path $p = e_1 e_2 \cdots e_n$ is

$$Lab(p) := Lab(e_1) Lab(e_2) \cdots Lab(e_n) \in W(X).$$

Let $G = \langle X \,|\, R \rangle$ be a group. To represent G geometrically, let Γ_G be the two dimensional CW-complex consisting of the following cells:

(1) One vertex v.

(2) $|X|$ oriented labeled loops e_x based at v such that for every $x \in X$ there is a unique loop labeled by x.

(3) $|R|$ two-dimensional cells (discs) such that for every $r \in R$ there is a unique disc whose boundary represents the word r and is identified with the path in Γ_G labeled by r (see Figure 2.1.1).

We identify G with $\pi_1(\Gamma_G, v)$ as follows.

(1) Identify 1_G with the homotopy class of the path of length 0 at v.

(2) For every $x \in X$, identify x with the homotopy class of the loop e_x.

(3) For every path p in Γ_G, identify the word $Lab(p)$ in $W(X)$ with the corresponding element in G.

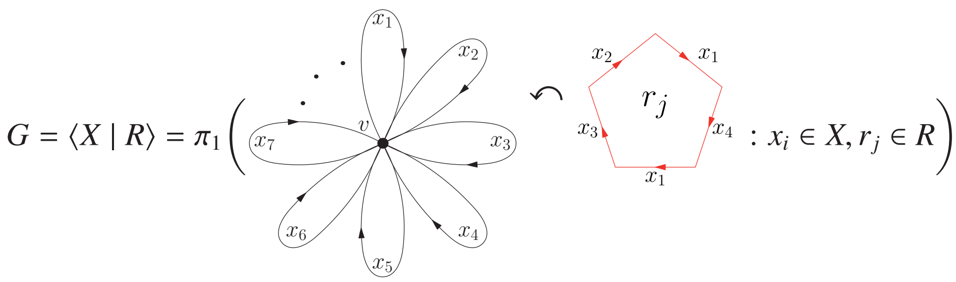
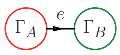


Fig. 2.1.1. *An Example of the geometrical representation of G.*

Let A, B be two groups, we denote by  the CW-complex consisting of Γ_A, Γ_B, and an edge e with $\iota(e) = v_A, \tau(e) = v_B$ (see Figure 2.1.2). Here Γ_A and Γ_B are the corresponding CW-complexes to A and B respectively which are constructed exactly as in the previous paragraph, v_A is the vertex in Γ_A, and v_B is the vertex in Γ_B.

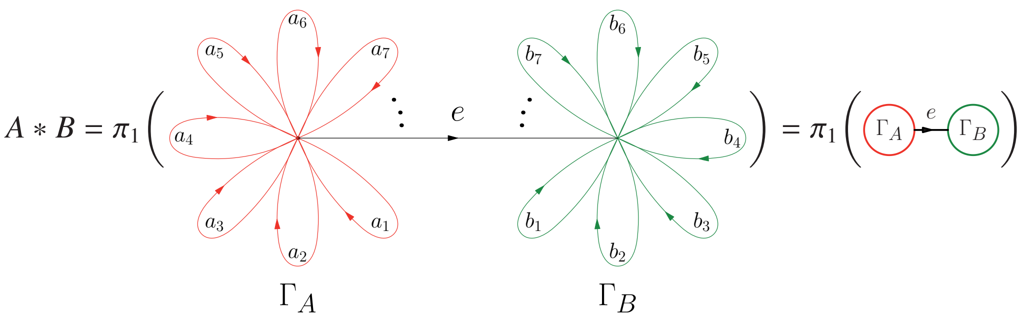


Fig. 2.1.2. *The geometrical representation of $A * B$.*

Note that $(\Gamma_A)\overset{e}{\longleftarrow}(\Gamma_B)$ is homotopy equivalent to Γ_{A*B}.

By Theorem 1.6.12, to every subgroup $H \leqslant G$ corresponds a covering map $\varphi : (C_H, v_H) \to (\Gamma_G, v)$, where C_H is a covering space of Γ_G such that im$(\varphi_*)= H$, where φ_* is the induced map $\varphi_* : \pi_1(C_H, v_H) \to \pi_1(\Gamma_G, v)$. We lift the labeling of Γ_G to C_H. So an edge e of C_H is labeled by x if its image $\varphi(e)$ is labeled by x.

Let $\varphi : (C_H, v_H) \to ((\Gamma_A)\overset{e}{\longleftarrow}(\Gamma_B), v)$ be a covering map corresponding to a subgroup H of $A * B$. We can assume that C_H consists of coverings of Γ_A and Γ_B joined by copies of the edge e (see Figure 2.2.1). The coverings of Γ_A and Γ_B will be called $A-$ and $B-$*components* of C_H respectively.

The following lemma will be used in the construction of the covering space $\widetilde{Z}$ of the space $(\Gamma_A)\overset{e}{\longleftarrow}(\Gamma_B)$.

Lemma 2.1.1. *Let Γ be a finite connected graph different from a vertex and let m be a natural number. Then there exists a finite connected graph $\widetilde{\Gamma}$ that covers Γ and its girth is larger than m.*

Proof. If the girth of Γ is greater than m, then we take $\widetilde{\Gamma} = \Gamma$. Assume that the girth of Γ is less than or equal to m and let F be the fundamental group of Γ with respect to some point v, then F is free with some basis set.

In order to choose a basis set of F, we choose a maximal tree T in Γ, then the set of edges of Γ which are outside T corresponds to a basis set X of F. The length function on F with respect to X will be called X-*length*.

Since F is residually finite, F has a finite index normal subgroup N such that N has no elements of X-length up to m.

Then N does not contain conjugates to non-trivial elements of X-length up to m, because N is normal. Let $(\widetilde{\Gamma}, \tilde{v})$ be the based covering space of (Γ, v) corresponding to N. We claim that $(\widetilde{\Gamma}, \tilde{v})$ has girth larger than m.

Assume to the contrary that there exists a cycle $\tilde{c}$ in $(\widetilde{\Gamma}, \tilde{v})$ of length smaller than or equal to m. Choose a path $\tilde{l}$ from $\tilde{v}$ to the cycle $\tilde{c}$. Then the homotopy class $[\tilde{l}\tilde{c}\bar{\tilde{l}}]$ belongs to $\pi_1(\widetilde{\Gamma}, \tilde{v})$.

Now we consider the projection $lc\bar{l}$ of the path $\tilde{l}\tilde{c}\bar{\tilde{l}}$ in Γ. Let l_1 be a path in the tree T connecting the initial and terminal points of l, then $[lc\bar{l}] \in N$ and because of normality of N we have $[l_1 c\overline{l_1}] = [l_1\bar{l}][lc\bar{l}][l_1\bar{l}] \in N$.

Since l_1 is a path in T and the number of edges in c is smaller then or equal to m, the X−length of $[l_1 c\overline{l_1}]$ is smaller than or equal to m. Thus we have found an element in N with length smaller than or equal to m, a contradiction. □

Now we are ready to formulate and prove the main theorem.

2.2 The main theorem

Theorem 2.2.1. *(O. Bogopolski and A. N. Elsawy) Let A, B be finitely generated groups which are simultaneously LERF and SICS groups, then the free product $A * B$ is LERF and SICS.*

Proof. Burns [7] and Romanovskii [32] proved that $A * B$ is LERF. It remains to prove that $A * B$ is SICS. Let H_1, H_2 be two finitely generated subgroups of $A * B$ such that H_2 is not conjugate into H_1.

According to Definition 1.5.3 (b), it suffices to construct a finite index subgroup D of $A * B$ such that D contains H_1 and does not contain a conjugate of H_2.

If H_1 has finite index in $A * B$, then we take $D = H_1$. Assume that H_1 has infinite index and let (C_{H_1}, z) be a covering space of the base space $\big((\Gamma_A)\overset{e}{-}(\Gamma_B), v\big)$ corresponding to the subgroup H_1 of the group $A * B$, for some $z \in C_{H_1}$ and $v \in (\Gamma_A)\overset{e}{-}(\Gamma_B)$.

We may assume that C_{H_1} consists of coverings of Γ_A and coverings of Γ_B joined by copies of the edge e. Let A_i be the finitely generated subgroups of A corresponding to the coverings Γ_{A_i} of Γ_A in C_{H_1}. That is, $A_i \cong \pi_1(\Gamma_{A_i}, z_i)$ for some z_i in Γ_{A_i}. Similarly, let B_j be the finitely generated subgroups of B corresponding to the coverings Γ_{B_j} of Γ_B in C_{H_1}.

The coverings Γ_{A_i} and Γ_{B_j} will be called $A-$ and $B-$*components* of C_{H_1} respectively and the copies of e will be called $e-$*edges*.

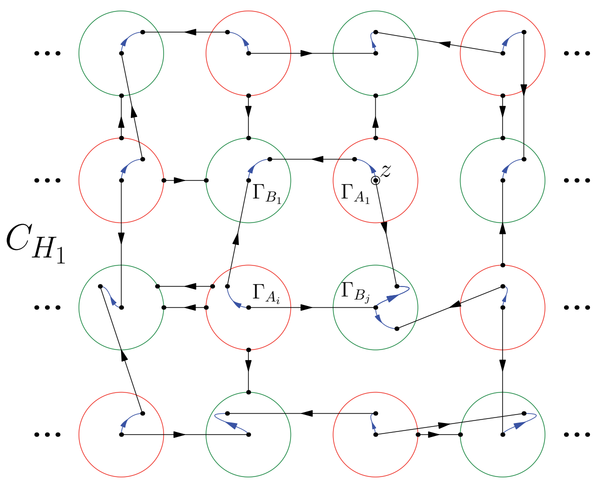


Fig. 2.2.1. *The covering C_{H_1} corresponding to the subgroup H_1*

Denote by $p_{x_1}, p_{x_2}, \ldots, p_{x_m}$ the cycles in C_{H_1} based at z which correspond to some generating set $\{x_1, x_2, \ldots, x_m\}$ of H_1. That is, first we fix a set of generators of A, a set of generators of B, and a set $\{x_1, x_2, \ldots, x_m\}$ of generators of H_1. Then, we choose some reduced words $W_{x_1}, W_{x_2}, \ldots, W_{x_m}$ in the generators of A and B such that $W_{x_i} = x_i$ in $A * B$ for all $i \in \{1, \ldots, m\}$. Finally, we denote by p_{x_i} the cycles in C_{H_1} such that p_{x_i} are based at z and $Lab(p_{x_i}) = W_{x_i}$ for all $i \in \{1, \ldots, m\}$.

Now we start a sequence of steps to obtain a covering space $(\widetilde{Z}, \tilde{z})$ of the base space $((\Gamma_A \overset{e}{-} \Gamma_B), v)$ through a covering map ϕ such that the subgroup $\phi_*(\pi_1(\widetilde{Z}, \tilde{z}))$ of $A * B$ plays the role of D.

Before we start our procedure we will eliminate the useless components of C_{H_1} and leave only the components which are important for H_1.

Let $Core(C_{H_1})$ be a subcomplex of C_{H_1} such that:

(1) An $A-$component or $B-$component belongs to $Core(C_{H_1})$ if at least one of the cycles $p_{x_1}, p_{x_2}, \ldots, p_{x_m}$ passes through it.

(2) An $e-$edge e' belongs to $Core(C_{H_1})$ if e' belongs to at least one of the cycles $p_{x_1}, p_{x_2}, \ldots, p_{x_m}$.

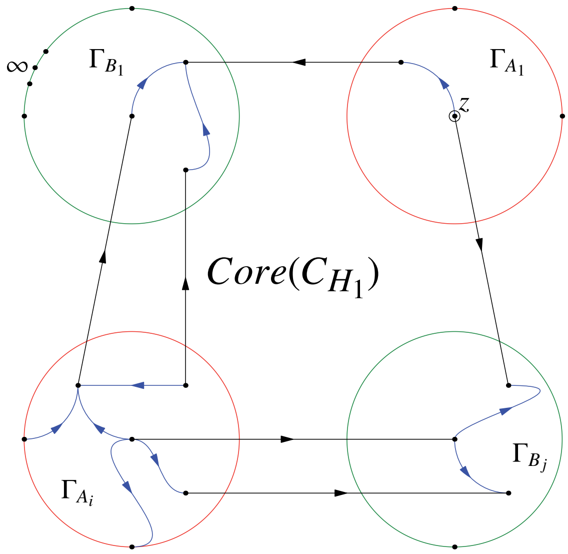


Fig. 2.2.2. *The subcomplex* $(Core(C_{H_1}), z)$*. An* $A-$ *or* $B-$*components in* $Core(C_{H_1})$ *can have infinitely many outer vertices.*

A vertex v in $Core(C_{H_1})$ will be called *inner* vertex if $v = \iota(e')$ or $v = \tau(e')$ for some $e-$edge e' in $Core(C_{H_1})$, and called *outer* vertex if it is not inner.

Obviously, the subcomplex $Core(C_{H_1})$ satisfies $\pi_1(Core(C_{H_1}), z) = H_1$. Moreover, it contains finitely many $A-$ and $B-$components and finitely many inner vertices (but it may have infinitely many outer vertices).

Now, using the complex $(Core(C_{H_1}), z)$, we will construct a new finite complex $(\widetilde{Z}, \tilde{z})$ in two main steps.

Since the covering space $(\widetilde{Z}, \tilde{z})$ should be finite, the $A-$ and $B-$components in $(Core(C_{H_1}), z)$ which have infinitely many outer vertices represent an obstacle to construct $(\widetilde{Z}, \tilde{z})$.

So in the first step, using a special algorithm, we replace all the $A-$ and $B-$components which have infinitely many outer vertices by some finite coverings of Γ_A and Γ_B, and get a finite complex denoted by $(\overline{Z}, \overline{z})$. We also will show that the properties LERF and SICS of A and B ensure the existence of such finite coverings.

The complex $(\overline{Z}, \overline{z})$ is finite, however it not necessarily covers the space $(\Gamma_A \overset{e}{-} \Gamma_B, v)$. In the second step, using a geometrical algorithm, we glue some special finite coverings of Γ_A and Γ_B to the complex $(\overline{Z}, \overline{z})$ and use Lemma 2.1.1 to obtain a finite covering $(\widetilde{Z}, \tilde{z})$ of $(\Gamma_A \overset{e}{-} \Gamma_B, v)$.

Now, we explain in detail the construction process of $(\widetilde{Z}, \tilde{z})$.

The construction of $(\widetilde{Z}, \tilde{z})$.

Step 1. The construction of the finite complex $(\overline{Z}, \overline{z})$.

We will construct the complex $(\overline{Z}, \overline{z})$ by using the global structure of $(Core(C_{H_1}), z)$. Locally we will replace all the $A-$ and $B-$components which have infinitely many outer vertices by some finite coverings of Γ_A and Γ_B. For that we will use Constructions 1-6 (see below).

If Γ_{A_i} or Γ_{B_j} in $(Core(C_{H_1}), z)$ has finitely many outer vertices, then we will not make any replacement.

In the following constructions we consider only Γ_{A_i}; the constructions for Γ_{B_j} are similar. Without loss of generality, let z belong to Γ_{A_0}, where z is the global basepoint of $Core(C_{H_1})$.

Construction 1. Suppose that Γ_{A_i} has at least two inner vertices. Then for every two inner vertices u, v in Γ_{A_i} choose two paths p_u, p_v in Γ_{A_i} such that p_u, p_v begin at z_i and end at u, v respectively (see Figure 2.2.3).

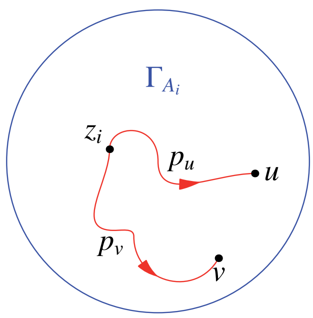


Fig. 2.2.3. *The paths* p_u, p_v *in* Γ_{A_i}.

Denote $Lab(p_u)$ by a_u and $Lab(p_v)$ by a_v. Then $a_u a_v^{-1} \notin A_i \leqslant A$ whenever $u \neq v$. As A_i is finitely generated and A is LERF, there exists a finite index subgroup $A_{i,u,v} \leqslant A$ such that $A_i \leqslant A_{i,u,v}$ and $a_u a_v^{-1} \notin A_{i,u,v}$.

Let $\Gamma_{A_i} \xrightarrow{\psi_{i,u,v}} \Gamma_{A_{i,u,v}} \to \Gamma_A$ be covering maps which correspond to the chains $A_i \leqslant A_{i,u,v} \leqslant A$. Clearly, $\Gamma_{A_{i,u,v}}$ is finite.

Then $\pi_1(\Gamma_{A_{i,u,v}}, \psi_{i,u,v}(z_i)) \cong A_{i,u,v}$ and $\psi_{i,u,v}(u) \neq \psi_{i,u,v}(v)$ because $a_u a_v^{-1}$ does not belong to $A_{i,u,v}$.

So, we can conclude that for every two inner vertices u, v in Γ_{A_i}, we can find a finite covering space $\Gamma_{A_{i,u,v}}$ of Γ_{A_i} in which the images of u and v are different.

Now, let $H_2 = \langle y_1, y_2, \ldots, y_n \rangle$ and $y_s = a_{s1}b_{s1}\ldots a_{st}b_{st}$, where $t \in \mathbb{N}$, $a_{si} \in A$, and $b_{si} \in B$ for all $s = 1, \ldots, n$ and $i = 1, \ldots, t$ (note that t depends on y_s).

$$\text{Let} \quad A_{H_2} = \bigcup_{s=1}^{n} \{a_{s1}, \ldots, a_{st}\} \setminus \{1_A\}, \quad A^*_{H_2} = \{a^{\pm 1}, (aa')^{\pm 1} : a, a' \in A_{H_2}\},$$
$$B_{H_2} = \bigcup_{s=1}^{n} \{b_{s1}, \ldots, b_{st}\} \setminus \{1_B\}, \quad B^*_{H_2} = \{b^{\pm 1}, (bb')^{\pm 1} : b, b' \in B_{H_2}\}.$$

Observe that these four sets are finite.

Construction 2. Suppose that Γ_{A_i} has a path p_{uw} which begins at an inner vertex u and ends at an outer vertex w and satisfies $Lab(p_{uw}) \in A^*_{H_2}$. Then we choose a path p_w in Γ_{A_i} which begins at the basepoint z_i of Γ_{A_i} and ends at w. Also, for every inner vertex v in Γ_{A_i} we choose a path p_v in Γ_{A_i} which begins at z_i and ends at v (see Figure 2.2.4).

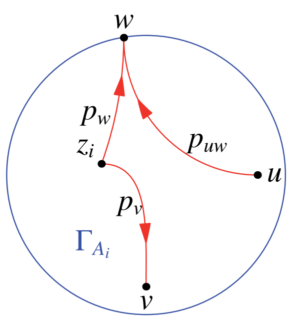


Fig. 2.2.4. *The paths p_w, p_v, and p_{uw} in Γ_{A_i}.*

Note that in Γ_{A_i} there are finitely many outer vertices like w, because Γ_{A_i} has finitely many inner vertices and the set $A^*_{H_2}$ is finite, and also note that v can be equal to u.

Denote $Lab(p_v)$ by a_v and $Lab(p_w)$ by a_w. It follows that $a_v a_w^{-1} \notin A_i \leqslant A$, since $v \neq w$. As A_i is finitely generated and A is LERF, there exists a finite index subgroup $A_{i,u,v,w} \leqslant A$ such that $A_i \leqslant A_{i,u,v,w}$ and $a_v a_w^{-1} \notin A_{i,u,v,w}$.

Let $\Gamma_{A_i} \xrightarrow{\psi_{i,u,v,w}} \Gamma_{A_{i,u,v,w}} \to \Gamma_A$ be covering maps corresponding to the chains $A_i \leqslant A_{i,u,v,w} \leqslant A$. Then $\Gamma_{A_{i,u,v,w}}$ is finite and $\pi_1(\Gamma_{A_{i,u,v,w}}, \psi_{i,u,v,w}(z_i)) \cong A_{i,u,v,w}$. Moreover, $\psi_{i,u,v,w}(v) \neq \psi_{i,u,v,w}(w)$ because $a_v a_w^{-1} \notin A_{i,u,v,w}$.

Therefore, we can conclude that, if in Γ_{A_i} there is a path p_{uw} which begins at an inner vertex u and ends at an outer vertex w and satisfies $Lab(p_{uw}) \in A^*_{H_2}$, then for every inner vertex $v \in \Gamma_{A_i}$ we can find a finite covering space $\Gamma_{A_{i,u,v,w}}$ of Γ_{A_i} in which the images of v and w are different.

Construction 3. Suppose that Γ_{A_i} has two paths $p_{u_1w_1}, p_{w_2u_2}$ such that $p_{u_1w_1}$ begins at an inner vertex u_1 and ends at an outer vertex w_1, $p_{w_2u_2}$ begins at a different outer vertex w_2, i.e., $w_1 \neq w_2$ and ends at an inner vertex u_2, and $Lab(p_{u_1w_1}), Lab(p_{w_2u_2}) \in A^*_{H_2}$. Then for every two such vertices w_1 and w_2 we choose two paths p_{w_1}, p_{w_2} in Γ_{A_i} which begin at z_i and end at w_1, w_2 respectively (see Figure 2.2.5).

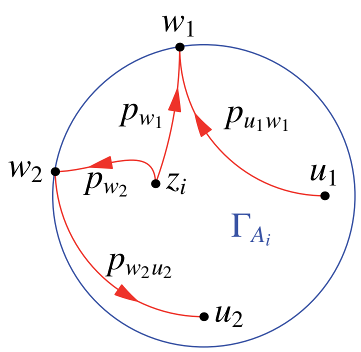


Fig. 2.2.5. *The paths p_{w_1}, p_{w_2}, $p_{u_1w_1}$ and $p_{u_2w_2}$ in Γ_{A_i}.*

Note that in Γ_{A_i} there are finitely many outer vertices like w_1, w_2. Also note that u_1 and u_2 could be equal but w_1 and w_2 are different.

Denote $Lab(p_{w_1})$ by a_{w_1} and $Lab(p_{w_2})$ by a_{w_2}. Then $a_{w_1} a_{w_2}^{-1} \notin A_i \leqslant A$, because $w_1 \neq w_2$. As A_i is finitely generated and A is LERF, there exists a subgroup $A_{i,u_1,u_2,w_1,w_2} \leqslant A$ of finite index such that $A_i \leqslant A_{i,u_1,u_2,w_1,w_2}$ and $a_{w_1}a_{w_2}^{-1} \notin A_{i,u_1,u_2,w_1,w_2}$.

Let $\Gamma_{A_i} \xrightarrow{\psi_{i,u_1,u_2,w_1,w_2}} \Gamma_{A_{i,u_1,u_2,w_1,w_2}} \to \Gamma_A$ be covering maps corresponding to the chains $A_i \leqslant A_{i,u_1,u_2,w_1,w_2} \leqslant A$.

Then $\Gamma_{A_{i,u_1,u_2,w_1,w_2}}$ is finite and $\pi_1(\Gamma_{A_{i,u_1,u_2,w_1,w_2}}, \psi_{i,u_1,u_2,w_1,w_2}(z_i)) \cong A_{i,u_1,u_2,w_1,w_2}$. Moreover, $\psi_{i,u_1,u_2,w_1,w_2}(w_1) \neq \psi_{i,u_1,u_2,w_1,w_2}(w_2)$ because $a_{w_1}a_{w_2}^{-1} \notin A_{i,u_1,u_2,w_1,w_2}$.

So we can conclude that, if u_1, u_2 are inner vertices, w_1, w_2 are two outer vertices in Γ_{A_i}, and there exist two paths $p_{u_1w_1}, p_{w_2u_2}$ such that $Lab(p_{u_1w_1})$, $Lab(p_{w_2u_2}) \in A_{H_2}^*$, then we can find a finite covering space $\Gamma_{A_{i,u_1,u_2,w_1,w_2}}$ of Γ_{A_i} in which the images of w_1 and w_2 are different.

Construction 4. Suppose that $H_2 = \langle y_1, \ldots, y_n \rangle$ is conjugate into A in $A * B$. Then $H_2^g \leqslant A$ for some g in $A * B$. Let $k_s := gy_sg^{-1}$ for all $s = 1, 2, \ldots, n$, then $K := \langle k_1, \ldots, k_n \rangle = H_2^g \leqslant A$.

We claim that K is not conjugate into A_i in A for all A_i, that is $K^a \nleqslant A_i$ for all $a \in A$ and all A_i.

Assume to the contrary that $K^a \leqslant A_i$ for some $a \in A$. In Γ_{A_i} there exists a path which begins at z_i and has lable a. Let u be the terminal vertex of the path p such that $\iota(p) = z_i$ and $Lab(p) = a$, then $K \leqslant \pi_1(\Gamma_{A_i}, u)$.

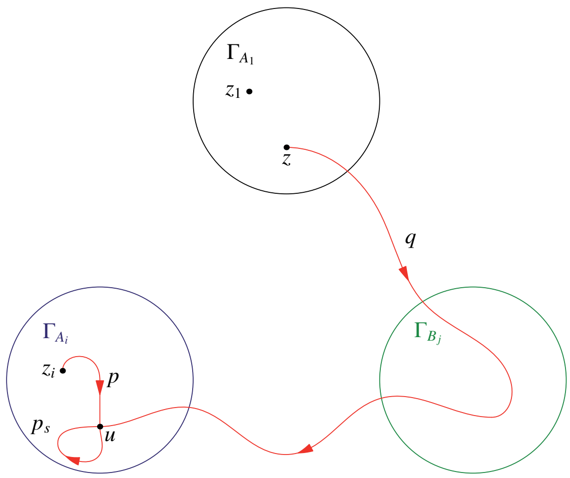


Fig. 2.2.6. *The paths p, p_s, and q in $(Core(C_{H_1}), z)$.*

Let p_s be the cycles in Γ_{A_i} based at u which correspond to k_s, for all $s \in \{1, \ldots, n\}$ (see Figure 2.2.6).

Let q be a path in $Core(C_{H_1})$ from the global basepoint z to u, and let g' be the label of q.

The path $qp_s\overline{q}$ is closed in $(Core(C_{H_1}), z)$ for all $s = 1, 2, \ldots, n$. Then $g'k_sg'^{-1} \in \pi_1(Core(C_{H_1}), z)$ for all $s = 1, 2, \ldots, n$.

In other words $g'gy_sg^{-1}g'^{-1} \in H_1$ for all $s = 1, 2, \ldots, n$, and therefore $H_2^{g'g} \leqslant H_1$ leads to a contradiction, and the claim is proved.

Since A_i is finitely generated and A is SICS, there exists a finite index subgroup $A_i' \leqslant A$ such that $A_i \leqslant A_i'$ and $K^a \nleqslant A_i'$ for all $a \in A$.

Let $\Gamma_{A_i} \xrightarrow{\psi_i'} \Gamma_{A_i'} \to \Gamma_A$ be covering maps which are corresponding to the chains $A_i \leqslant A_i' \leqslant A$. Then $\pi_1(\Gamma_{A_i'}, \psi_i'(z_i)) \cong A_i'$ and $K^a \nleqslant A_i'$ for all $a \in A$.

So we conclude that, if $K := H_2^g \leqslant A$ for some $g \in A * B$, then, for every A_i, there exists a finite index subgroup $A_i' \leqslant A$ such that $A_i \leqslant A_i'$ $K^a \nleqslant A_i'$ for all $a \in A$. In particular, if $H_2 \leqslant A$, then there exists a finite index subgroup $A_i' \leqslant A$ such that $H_2^a \nleqslant A_i'$ for all $a \in A$.

Construction 5. Suppose that Γ_{A_i} has exactly one inner vertex, say u, and for every outer vertex w in Γ_{A_i} there is no paths p_{uw} in Γ_{A_i} which begin at u, end at w, and satisfy $Lab(p_{uw}) \in A^*_{H_2}$. Let $A_i'' := A$ and $\Gamma_{A_i''} := \Gamma_A$.

Construction 6. Let A_i^* be the finite index subgroup of A obtained by intersection of all the subgroups $A_{i,u,v}, A_{i,u,v,w}, A_{i,u_1,u_2,w_1,w_2}, A_i'$ and A_i'', which we obtained in *Constructions 1-6*, respectively.

Note that A_i^* is an intersection of finitely many finite index subgroups, and using the fact that $|G : H \cap K| \leqslant |G : H||G : K|$, where H, K are subgroups in a group G, we deduce that A_i^* is a finite index subgroup in A.

Let $\Gamma_{A_i} \xrightarrow{\psi_i} \Gamma_{A_i^*} \to \Gamma_A$ be the covering maps which correspond to the chains $A_i \leqslant A_i^* \leqslant A$.

Analogously, we define finite index subgroups B_j^* of B and covering maps $\Gamma_{B_j} \xrightarrow{\phi_j} \Gamma_{B_j^*} \to \Gamma_B$ which correspond to the chains $B_j \leqslant B_j^* \leqslant B$.

Let $(\overline{Z}, \overline{z})$ be the complex which is obtained by replacing each Γ_{A_i} and Γ_{B_j} in $Core(C_{H_1})$ with infinitely many vertices by the new complexes $\Gamma_{A_i^*}$ and $\Gamma_{B_j^*}$ respectively. This gives a natural map $\Phi : (Core(C_{H_1}), z) \to (\overline{Z}, \overline{z})$ (see Figure 2.2.7).

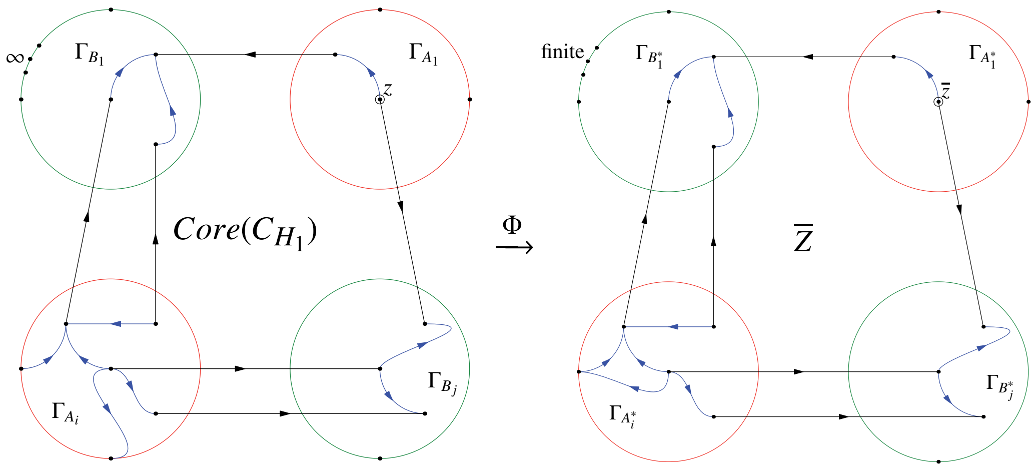


Fig. 2.2.7. *From the infinite complex* $(Core(C_{H_1}), z)$ *to the finite complex* $(\overline{Z}, \overline{z})$.

A vertex $\overline{u} \in \overline{Z}$ will be called an *inner vertex in* $\overline{Z}$ if $\overline{u}$ is the image of an inner vertex in $(Core(C_{H_1}), z)$. If $\overline{u} \in \overline{Z}$ is not an inner vertex, then $\overline{u}$ will be called an *outer vertex in* $\overline{Z}$. It follows from the definition that a preimage of an outer vertex in $\overline{Z}$ is an outer vertex in $(Core(C_{H_1}), z)$.

Remark 2.2.2. It follows from the conclusion in Construction 1 that for any inner vertex v in $\overline{Z}$ the set $\Phi^{-1}(v)$ contains exactly one inner vertex and possibly several outer vertices in $Core(C_{H_1})$.

Remark 2.2.3. It follows from the conclusion in Construction 2 that the image of an outer vertex $w \in Core(C_{H_1})$ is an outer vertex in $\overline{Z}$, if the following holds: There is a path p in the $A-$ or $B-$component of $Core(C_{H_1})$, say Γ_{A_i}, such that $w \in \Gamma_{A_i}$, $\iota(p)$ is an inner vertex, $\tau(p) = w$, and $Lab(p) \in A^*_{H_2}$.

Remark 2.2.4. It follows from the conclusions in Construction 2 and 3 that the images of two different outer vertices $w_1, w_2 \in Core(C_{H_1})$ are two different outer vertices in $\overline{Z}$, if the following holds:

There are two paths p_1, p_2 in the $A-$ or $B-$component of $Core(C_{H_1})$, say Γ_{A_i}, such that $w_1, w_2 \in \Gamma_{A_i}$, the vertices $\iota(p_1), \tau(p_2)$ are inner vertices, $\tau(p_1) = w_1, \iota(p_2) = w_2$, and $Lab(p_1), Lab(p_2) \in A^*_{H_2}$.

Although the complex $(\overline{Z}, \overline{z})$ is finite, it is possibly not a covering of the complex $(\Gamma_A \overset{e}{\longleftarrow} \Gamma_B, v)$. In order to get a finite covering of $(\Gamma_A \overset{e}{\longleftarrow} \Gamma_B, v)$, we glue some special components to $(\overline{Z}, \overline{z})$ and use Lemma 2.1.1, as we will explain in the second step, to obtain a finite covering $(\widetilde{Z}, \tilde{z})$ of $(\Gamma_A \overset{e}{\longleftarrow} \Gamma_B, v)$.

Step 2. The construction of the finite covering $(\widetilde{Z}, \widetilde{z})$**.**

To construct the finite complex $(\widetilde{Z}, \widetilde{z})$, we apply the following procedure:

Step 2.1. If $(\overline{Z}, \overline{z})$ has no outer vertices, then $(\overline{Z}, \overline{z})$ covers $(\Gamma_A \overset{e}{\longleftarrow} \Gamma_B, v)$. Take $(\widetilde{Z}, \widetilde{z}) = (\overline{Z}, \overline{z})$.

Step 2.2. If $(\overline{Z}, \overline{z})$ has at least one outer vertex, then $(\overline{Z}, \overline{z})$ does not cover $(\Gamma_A \overset{e}{\longleftarrow} \Gamma_B, v)$. Therefore, to construct the finite complex $(\widetilde{Z}, \widetilde{z})$, we apply the following three steps:

Step 2.2.1. In the present step, we construct two special finite covering spaces U and V of Γ_A and Γ_B respectively.

Since A and B are RF (which follows from either LERF or SICS for A and B), there exist two finite index normal subgroups $K_1 \trianglelefteq A, K_2 \trianglelefteq B$ such that $a \notin K_1$ for all $a \in A^*_{H_2}$ and $b \notin K_2$ for all $b \in B^*_{H_2}$.

Indeed, for every $a \in A^*_{H_2}$ there exists a finite index normal subgroup $N_a \trianglelefteq A$ such that $a \notin N_a$, since A is RF. Since $A^*_{H_2}$ is finite, $K_1 := \cap_{a \in A^*_{H_2}} N_a$ is a finite index normal subgroup in A such that $a \notin K_1$ for all $a \in A^*_{H_2}$. Similarly, $K_2 := \cap_{b \in B^*_{H_2}} N_b$ is a finite index normal subgroup in B such that $b \notin K_2$ for all $b \in B^*_{H_2}$.

Let U and V be covering spaces of Γ_A and Γ_B corresponding to K_1 and K_2 respectively. To every vertex in U we glue an outgoing $e-$edge, and denote the resulting complex by $\mathrm{Ext}(U)$. Similarly, to every vertex in V

we glue an incoming e–edge, and denote the resulting complex by Ext(V). The idea behind this is that the e–edge in $(\Gamma_A \xrightarrow{e} \Gamma_B, v)$ goes from Γ_A to Γ_B.

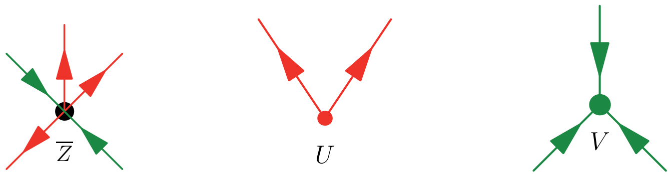


Fig. 2.2.8. *U, V, and $\overline{Z}$ as vertices.*

Moreover, we glue to every outer vertex in every A–component in $\overline{Z}$ an outgoing e–edge, and to every outer vertex in every B–component in $\overline{Z}$ an incoming e–edge. Denote the resulting complex by Ext($\overline{Z}$).

***Step* 2.2.2.** Consider U, V, and $\overline{Z}$ as vertices and construct a tree T with root $\overline{Z}$ inductively as follows.

Let $M = \max\{|y_1|, |y_2|, \ldots, |y_n|\}$. We will construct a chain of trees $T_0 \subseteq T_1 \subseteq \cdots \subseteq T_{M+1} = T$. Let $T_0 = \text{Ext}(\overline{Z})$.

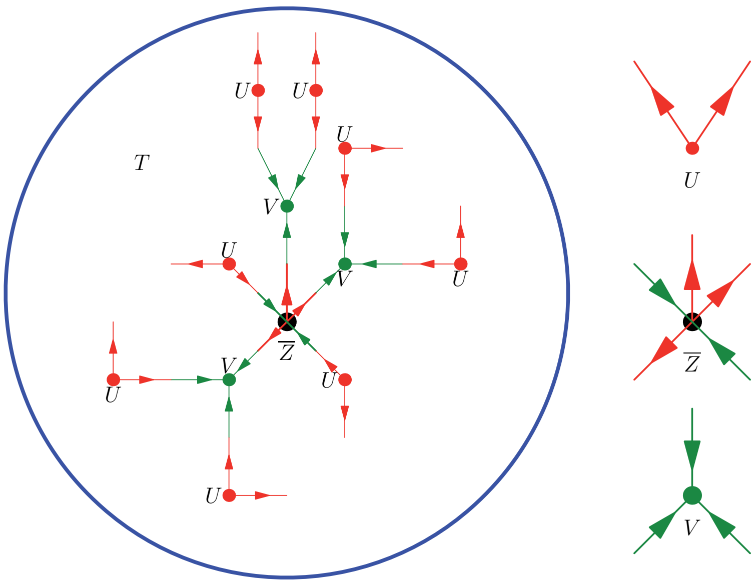


Fig. 2.2.9. *The tree T.*

Suppose that we have constructed T_i for some i less than M. To construct T_{i+1}, we glue to each outgoing e–edge of T_i a copy of Ext(V) through an incoming e−edge of Ext(V). Similarly, we glue to each incoming e−edge of T_i a copy of Ext(U) through an outgoing e−edge of Ext(U). The glued two e−edges will be considered as one e−edge.

Inductively, we construct T_M. Let T_{M+1} be the result of gluing to each incoming e−edge of T_M a copy of Ext(U) through an outgoing e−edge of U. On the other hand, we leave the outgoing e−edges of T_M as they are. So we obtain a tree $T = T_{M+1}$ which has only outgoing free edges (see Figure 2.2.9).

***Step* 2.2.3.** Now we consider the tree T itself as a vertex with $m \geq 0$ outgoing e−edges. If $m = 0$, we set $\widetilde{Z} := T$.

Now we consider the case $m > 0$. Suppose that Ext(V) has n incoming e−edges (clearly $n > 0$), then m copies of Ext(V) have nm incoming edges and n copies of T have nm outgoing edges.

We glue m copies of Ext(V) to n copies of T as follows

(1) every copy of Ext(V) is connected with all the copies of T by exactly one edge;

(2) different copies of Ext(V) are not connected directly by an edge;

(3) different copies of T are not connected directly by an edge.

So we obtain a graph Γ with girth equals 4 (see Figure 2.2.10).

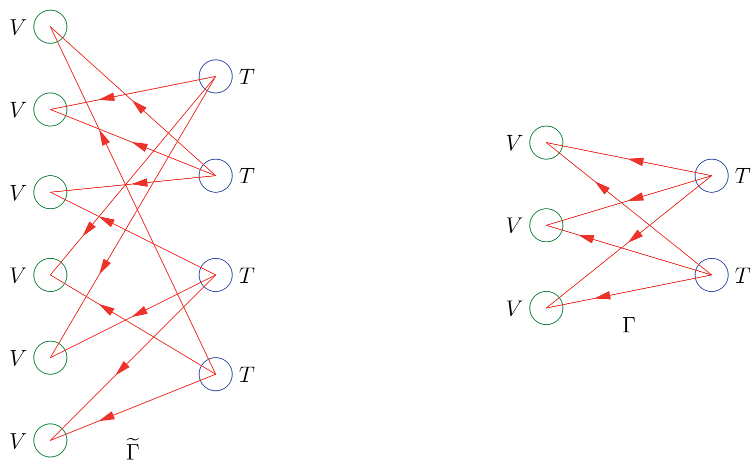


Fig. 2.2.10. *A finite connected graph $\widetilde{\Gamma}$ which covers Γ and has girth $> M$.*

By Lemma 2.1.1, there exists a finite connected graph $\widetilde{\Gamma}$ such that $\widetilde{\Gamma}$ covers Γ and the girth of $\widetilde{\Gamma}$ exceeds $M = \max\{|y_1|, |y_2|, \ldots, |y_n|\}$. Set $\widetilde{Z} := \widetilde{\Gamma}$.

Finally consider the graph $\widetilde{Z}$ as a based CW-complex $(\widetilde{Z}, \widetilde{z})$, where $\widetilde{z}$ is the image of the basepoint z of $Core(C_{H_1})$ in $\widetilde{Z}$ (see Figure 2.2.11). Therefore, $(\widetilde{Z}, \widetilde{z})$ is a finite CW-complex which covers $((\Gamma_A) \overset{e}{\longrightarrow} (\Gamma_B), v)$.

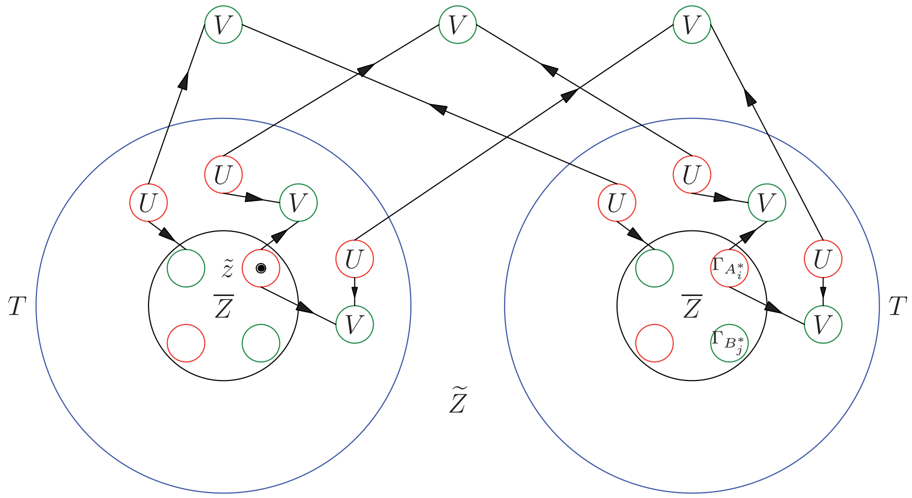


Fig. 2.2.11. *The finite complex $(\widetilde{Z}, \widetilde{z})$ corresponding to the subgroup D.*

By construction, $\widetilde{Z}$ contains several disjoint copies of $Core(C_{H_1})$. Therefore all the generators $x_1, \ldots, x_m$ of H_1 are represented by cycles based at $\widetilde{z}$ in $(\widetilde{Z}, \widetilde{z})$. This implies the following claim:

Claim. Let $\phi : (\widetilde{Z}, \widetilde{z}) \to ((\Gamma_A) \overset{e}{\longrightarrow} (\Gamma_B), v)$ be a covering map, and $D = \phi_*(\pi_1(\widetilde{Z}, \widetilde{z})) \leqslant A * B$. Then D is a finite index subgroup in $A * B$ and contains H_1.

The proof that H_2 is not conjugate into D.

To gain a complete proof of the theorem, it remains to prove that H_2 is not conjugate into D. So the rest of the proof will be devoted to show that all the subgroups in D are not conjugate to H_2.

Assume to the contrary that there exists $g \in A * B$ such that ${H_2}^g$ is a subgroup of $D \cong \pi_1(\widetilde{Z}, \tilde{z})$. Let $\tilde{d}$ be the terminal vertex of the path in $\widetilde{Z}$ which starts at $\tilde{z}$ and has label g. Then every element y in the generator set $\{y_1, y_2, \ldots, y_n\}$ of H_2 can be represented by a closed path based on $\tilde{d}$ in $(\widetilde{Z}, \tilde{d})$. In the following, we show that all possible positions of $\tilde{d}$ in $\widetilde{Z}$ lead to contradictions.

All the possible positions of $\tilde{d}$ can be summarized in two cases.

Case 1. $\tilde{d}$ belongs to one of the $A-$ or $B-$components in a copy of $\overline{Z}$.

Case 2. $\tilde{d}$ belongs to one of the copies of U or V.

Now we discuss these two cases in detail.

Consider Case 1. Suppose that $\tilde{d}$ belongs to one of the $A-$ or $B-$components in a copy of $\overline{Z}$; we denote this copy by $\overline{\mathscr{Z}}$, and by $\mathscr{T}$ the copy of T which contains $\overline{\mathscr{Z}}$. Let $p_{y_1}, p_{y_2}, \ldots, p_{y_n}$ be the cycles in $(\widetilde{Z}, \tilde{d})$ corresponding to the basis $y_1, y_2, \ldots, y_n$ of H_2. Then either at least one of the cycles $p_{y_1}, p_{y_2}, \ldots, p_{y_n}$ passes through a vertex outside $\overline{\mathscr{Z}}$ or all of them are completely inside $\overline{\mathscr{Z}}$.

Case 1.1. Suppose that at least one of the paths $p_{y_1}, p_{y_2}, \ldots, p_{y_n}$, say p_{y_s}, passes through a vertex outside $\overline{\mathscr{Z}}$. Since the normal subgroups $\pi_1(U) \trianglelefteq A$ and $\pi_1(V) \trianglelefteq B$ do not contain any element from $A^*_{H_2}$ and $B^*_{H_2}$ respectively, the path p_{y_s} has no closed subpaths in any copy of U or V. In addition, $\mathscr{T}$ (considered as a graph) does not contain any cycle, since $\mathscr{T}$ is a tree.

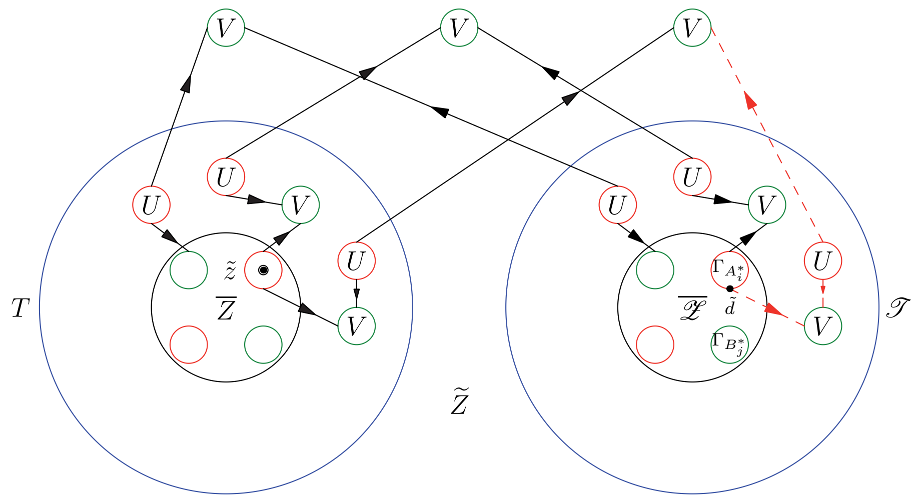


Fig. 2.2.12. *The path p_{y_s} has a subpath p such that p has length $> M$.*

Then the case $\widetilde{Z} = \mathcal{T}$ is impossible. Now we consider the case in which $\widetilde{Z}$ is larger than $\mathcal{T}$. In this case the path p_{y_s} has a subpath p which starts at $\tilde{d}$ and ends at a leaf of $\mathcal{T}$ (see Figure 2.2.12). Then the length of p is greater than M. However, by the definition of the constant M, the path $p_{y_s} \in \{p_{y_1}, p_{y_2}, \ldots, p_{y_n}\}$ has length at most M, which is a contradiction.

Case 1.2. Suppose that all the paths $p_{y_1}, p_{y_2}, \ldots, p_{y_n}$ are completely inside $\overline{\mathscr{Z}}$, then we claim that one of the following holds:

(1) H_2 is conjugate into H_1 in $A * B$, i.e., $H_2^g \leqslant H_1$ for some $g \in A * B$.

(2) $H_2 \leqslant A$ and H_2 is conjugate into A_i^* in A, i.e., $H_2^a \leqslant A_i^*$ for some $a \in A$.

(3) $H_2 \leqslant B$ and H_2 is conjugate into B_j^* in B, i.e., $H_2^b \leqslant B_j^*$ for some $b \in B$.

Recall that definitions of A_i^* and B_j^* can be found in Construction 6. Each one of these statements leads to a contradiction. Indeed, (1) contradicts the assumption. In addition, according to Construction 4, H_2 is not conjugate into A_i' in A, and H_2 is not conjugate into B_j' in B. Consequently, H_2 is not conjugate into A_i^* in A, and H_2 is not conjugate into B_j^* in B, since $A_i^* \leqslant A_i'$ and $B_j^* \leqslant B_j'$, according to Construction 6.

In order to prove our claim, it suffices to show that one of the following is true:

(1) There exist closed lifts of $p_{y_1}, \ldots, p_{y_n}$ in $Core(C_{H_1})$ with labels $y_1, \ldots, y_n$ such that all these lifts are based at the same vertex. (Here we consider lifts with respect to the natural map $\Phi : Core(C_{H_1}) \to \overline{\mathscr{Z}}$.

(2) The cycles $p_{y_1}, p_{y_2}, \ldots, p_{y_n}$ are completely inside $\Gamma_{A_i^*}$ for some i, which contains $\tilde{d}$ and all of them are based at the same vertex.

(3) The cycles $p_{y_1}, p_{y_2}, \ldots, p_{y_n}$ are completely inside $\Gamma_{B_j^*}$ for some j, which contains $\tilde{d}$ and all of them are based at the same vertex.

Without loss of generality we may assume that $\tilde{d}$ belongs to an $A-$component of $\overline{\mathscr{Z}}$, say $\Gamma_{A_1^*}$. Therefore, $\tilde{d}$ can be either an inner or an outer vertex in $\Gamma_{A_1^*}$.

Case 1.2.1. Suppose that $\tilde{d}$ is an inner vertex in $\Gamma_{A_1^*}$.

Then, by Remark 2.2.2, the set $\psi_1^{-1}(\tilde{d})$ has exactly one inner vertex in Γ_{A_1} of $Core(C_{H_1})$, denote it by d. Note that $\psi_1^{-1}(\tilde{d})$ possibly contains several outer vertices.

Let y be an arbitrary element of the set $\{y_1, y_2, \ldots, y_n\}$, then y satisfies one of the following three cases: y belongs to A, y belongs to B, or y does not belong to A and does not belong to B.

Case 1.2.1.1. Suppose that $y \in A$, then p_y is a closed path in $\Gamma_{A_1^*}$ (see Figure 2.2.13).

Let p'_y be the unique lift of p_y in Γ_{A_1} such that $\iota(p'_y) = d$ and $Lab(p'_y) = y$. Denote $\tau(p'_y)$ by w. Since $w \in \psi_1^{-1}(\tilde{d})$, it follows that $w = d$ or w is an outer vertex in Γ_{A_1}.

Fig. 2.2.13. *$y \in A$, where $y \in \{y_1, y_2, \ldots, y_n\}$.*

Suppose that w is an outer vertex (see Figure 2.2.13). Since $\iota(p'_y)$ is an inner vertex and $Lab(p'_y) \in A^*_{H_2}$, it follows from Remark 2.2.3 that $\psi_1(w)$ is an outer vertex in Γ_{A_1}. Which contradicts the fact that $w \in \psi_1^{-1}(\tilde{d})$.

Hence $w = d$, and p'_y is a closed lift of p_y in $Core(C_{H_1})$ based at d.

Case 1.2.1.2. Suppose that $y \in B$, then p_y passes only through $\tilde{d}$ in $\Gamma_{A_1^*}$ and some vertices in $\Gamma_{B_j^*}$ for some j, say $\Gamma_{B_1^*}$ (see Figure 2.2.14).

Let $\tilde{u}$ be a vertex in $\Gamma_{B_1^*}$ such that $\tilde{u} = \tau(\tilde{e})$, where $\tilde{e}$ is the e–edge which begins at $\tilde{d}$. Then $p_y = \tilde{e}\widetilde{p_y}\overline{\tilde{e}}$, for some closed path $\widetilde{p_y}$ in $\Gamma_{B_1^*}$ based at $\tilde{u}$.

Let u be the unique inner vertex in the set $\phi_1^{-1}(\tilde{u})$, and p'_y be the unique lift of $\widetilde{p_y}$ in Γ_{B_1} such that $\iota(p'_y) = u$ and $Lab(p'_y) = y$. As in Case 1.2.1, the path p'_y is a closed path in Γ_{B_1} based at u.

Since u is the unique inner vertex in $\phi_1^{-1}(\tilde{u})$, and the same for $d \in \psi_1^{-1}(\tilde{d})$, there is exactly one e–edge e' from d to u. Then the path $e'p'_y\overline{e'}$ is a closed lift of p_y in $Core(C_{H_1})$ based at d.

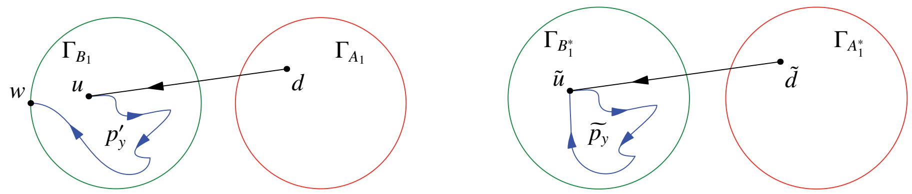


Fig. 2.2.14. *$y \in B$, where $y \in \{y_1, y_2, \ldots, y_n\}$.*

Case 1.2.1.3. Suppose that $y \notin A$ and $y \notin B$. Then y can be written as $y = a_1b_1a_2b_2 \cdots a_tb_ta_{t+1}$, such that $t \in \mathbb{N}$, $a_i \in A$ for all $i \in \{1, \ldots, t+1\}$, $b_j \in B$ for all $j \in \{1, \ldots, t\}$, $a_1 \neq 1$ or $a_{t+1} \neq 1$ for $t = 1$, $a_i \neq 1$ for all $i \in \{2, \ldots, t\}$ and $t \geq 2$, and $b_j \neq 1$ for all $j \in \{1, \ldots, t\}$ and $t \geq 1$ (see Figure 2.2.15).

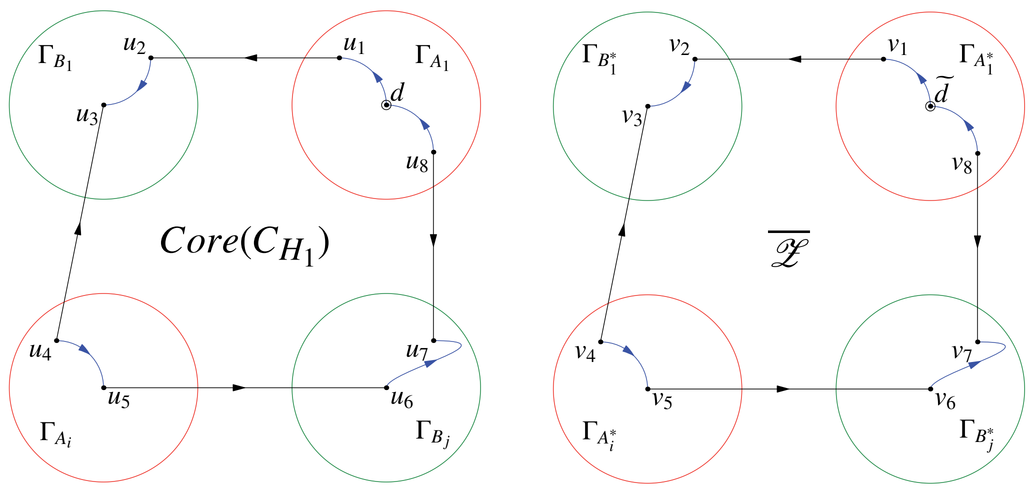


Fig. 2.2.15. *$y \notin A$ and $y \notin B$, where $y \in \{y_1, y_2, \ldots, y_n\}$.*

Let $p_y = p_{a_1}e_1p_{b_1}e_2 \cdots p_{a_t}e_{2t-1}p_{b_t}e_{2t}p_{a_{t+1}}$ be the cycle in $(\overline{\mathscr{X}}, \tilde{d})$ corresponding to y, where p_{a_i} and p_{b_j} are the paths in $\Gamma_{A_i^*}$ and $\Gamma_{B_j^*}$ corresponding to a_i and b_j respectively, and e_k are e–edges.

Obviously $A_1^* = A_{t+1}^*$ and $\tilde{d} = \iota(p_{a_1}) = \tau(p_{a_{t+1}})$. Also it is clear that, the vertices $\iota(p_{a_i})$, $\tau(p_{a_i})$, $\iota(p_{b_j})$, and $\tau(p_{b_i})$ are inner vertices for all $i \in \{1, \ldots, t+1\}$ and $j \in \{1, \ldots, t\}$.

Let p'_{a_1} be the unique lift of p_{a_1} in Γ_{A_1} such that $\iota(p'_{a_1}) = d$ and $Lab(p'_{a_1}) = a_1$. Denote $\tau(p'_{a_1})$ by u_1 and $\tau(p_{a_1})$ by v_1. Note that v_1 and u_1 can be equal to $\tilde{d}$ and d, respectively. First we show that u is an inner vertex in Γ_{A_1}.

If $a_1 = 1$, then p_{a_1} is the path of length 0 at $\tilde{d}$ and p'_{a_1} is the path of length 0 at d. So in this case $v_1 = \tilde{d}$ and $u_1 = d$, and therefore u_1 is an inner vertex in Γ_{A_1}.

Now suppose that $a_1 \neq 1$. Since $u_1 \in \psi_1^{-1}(v_1)$ is the terminal vertex of the path p'_{a_1}, where $\iota(p'_{a_1}) = d$ is an inner vertex and $Lab(p'_{a_1}) \in A^*_{H_2}$, the vertex u_1 can not be outer by Remark 2.2.3.

Thus, the vertex $u_1 = \tau(p'_{a_1})$ is an inner vertex in Γ_{A_1} in all the cases of a_1. Let e'_1 be the unique lift of the edge e_1 in $Core(C_{H_1})$ such that $\iota(e'_1) = u_1$. Let $u_2 = \tau(e'_1)$. Clearly, u_2 is an inner vertex in Γ_{B_1}.

Let p'_{b_1} be the unique lift of p_{b_1} in Γ_{B_1} such that $\iota(p'_{b_1}) = u_2$ and $Lab(p'_{b_1}) = b_1$. Denote $\tau(p'_{b_1})$ by u_3 and $\tau(p_{b_1})$ by v_3.

As above we can show that u_3 is an inner vertex in Γ_{B_1}.

We lift consecutively $e_2, p_{a_2}, e_3, p_{b_2}, \ldots, e_{2t}, p_{a_{t+1}}$ and get the path $p'_y = p'_{a_1}e'_1p'_{b_1}e'_2 \cdots p'_{a_t}e'_{2t-1}p'_{b_t}e'_{2t}p'_{a_{t+1}}$ in $Core(C_{H_1})$ with $Lab(p'_y) = y$. We show that p'_y is closed. Indeed, the inner vertices $\iota(p'_{a_1}), \tau(p'_{a_{t+1}})$ belong to $\psi_1^{-1}(\tilde{d})$, so they must coincide by Remark 2.2.2.

Therefore, from Case 1.2.1 we can conclude that, if $\tilde{d}$ is an inner vertex in $\Gamma_{A_1^*}$ and all the paths $p_{y_1}, \ldots, p_{y_n}$ are completely inside $\overline{Z}$, then for every y in $\{y_1, \ldots, y_n\}$ there exists a closed path p'_y in $Core(C_{H_1})$ based at d such that $Lab(p'_y) = y$. Which implies that H_2 is conjugate into H_1.

Case 1.2.2. Suppose that $\tilde{d}$ is an outer vertex in $\Gamma_{A_1^*}$. Then, by the definition of the outer vertices in $\overline{Z}$, the set $\psi_1^{-1}(\tilde{d})$ has no inner vertices. The generators $y_1, y_2, \ldots, y_n$ of H_2 satisfy one of the following cases:

(1) $y \in A$ for all $y \in \{y_1, y_2, \ldots, y_n\}$.

(2) $y \notin A$ for all $y \in \{y_1, y_2, \ldots, y_n\}$

(3) $y \in A$ for some $y \in \{y_1, y_2, \ldots, y_n\}$, and at least one $y' \in \{y_1, y_2, \ldots, y_n\}$ does not belong to A.

Case 1.2.2.1. Suppose that $y \in A$ for all $y \in \{y_1, \ldots, y_n\}$, equivalently $H_2 \leqslant A$. Then $p_{y_1}, \ldots, p_{y_n}$ lie completely in $\Gamma_{A_1^*}$ (see Figure 2.2.16).

Choose a path p_a in $\Gamma_{A_1^*}$ such that $\tau(p_a) = \tilde{d}$ and $\iota(p_a) = \overline{z_1}$, where $\overline{z_1}$ is the basepoint of $\Gamma_{A_1^*}$. Denote $Lab(p_a)$ by $a \in A$.

Then, for every $y \in \{y_1, y_2, \ldots, y_n\}$, the path $p_a p_y \overline{p_a}$ is closed at $\overline{z_1}$. So, $y^a \in \pi_1(\Gamma_{A_1^*}, \overline{z_1}) \cong A_1^*$, which implies that $H_2^a \leqslant A_1^*$. However, according to the conclusion in Construction 4, H_2 is not conjugate into A_i' in A. Consequently, H_2 is not conjugate into A_i^* in A, since $A_i^* \leqslant A_i'$, according to Construction 6.

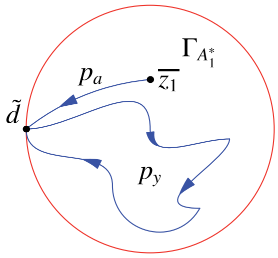


Fig. 2.2.16. *$y \in A$, for all $y \in \{y_1, y_2, \ldots, y_n\}$.*

Case 1.2.2.2. Suppose that $y \notin A$ for all $y \in \{y_1, y_2, \ldots, y_n\}$. Then y can be written as $y = a_1 b_1 a_2 b_2 \cdots a_t b_t a_{t+1}$, such that $t \in \mathbb{N}$, $a_i \in A$, $a_i \neq 1$, $b_j \in B$, and $b_j \neq 1$ for all $i \in \{1, 2, \ldots, t+1\}$ and $j \in \{1, 2, \ldots, t\}$ (see Figure 2.2.17).

Let $p_y = p_{a_1} e_1 p_{b_1} e_2 \cdots p_{a_t} e_{2t-1} p_{b_t} e_{2t} p_{a_{t+1}}$ be the cycle based at $\tilde{d}$ in $(\widetilde{Z}, \tilde{d})$ corresponding to y, where p_{a_i} and p_{b_j} are the paths in $\Gamma_{A_i^*}$ and $\Gamma_{B_j^*}$ corresponding to a_i and b_j respectively, and e_k are e–edges. Obviously $A_1^* = A_{t+1}^*$ and $\tilde{d} = \iota(p_{a_1}) = \tau(p_{a_{t+1}})$. Furthermore, the vertices $\tau(p_{a_1})$, $\iota(p_{a_{t+1}})$, $\iota(p_{a_i})$, $\tau(p_{a_i})$, $\iota(p_{b_j})$, and $\tau(p_{b_i})$ are inner for all $i \in \{2, \ldots, t\}$ and $j \in \{1, 2, \ldots, t\}$.

Denote $\tau(p_{a_1})$ by v_1. Since v_1 is an inner vertex in $\Gamma_{A_1^*}$, the set $\psi_1^{-1}(v_1)$ contains exactly one inner vertex, by Remark 2.2.2. Denote the unique inner vertex in the set $\psi_1^{-1}(v_1)$ by u_1. Let e_1' be the unique lift of the edge e_1 in $Core(C_{H_1})$ such that $\iota(e_1') = u_1$. Let $u_2 = \tau(e_1')$ and $v_2 = \tau(e_1)$. Clearly, u_2 is an inner vertex in Γ_{B_1}.

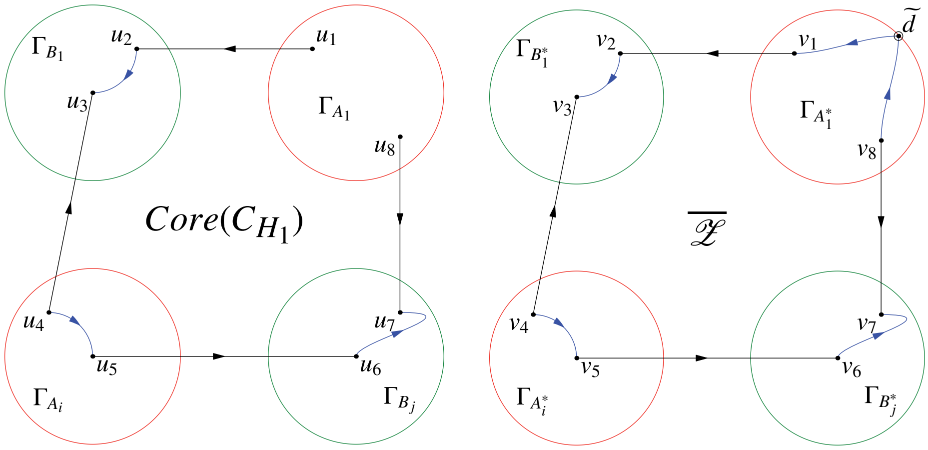


Fig. 2.2.17. *The path* $p' = p'_{b_1}e'_2 \cdots p'_{a_t}e'_{2t-1}p'_{b_t}$ *in* $Core(C_{H_1})$.

Let p'_{b_1} be the unique lift of p_{b_1} in Γ_{B_1} such that $\iota(p'_{b_1}) = u_2$ and $Lab(p'_{b_1}) = b_1$. Denote $\tau(p'_{b_1})$ by u_3 and $\tau(p_{b_1})$ by v_3.

Since $u_3 \in \psi_1^{-1}(v_3)$ is the terminal vertex of the path p'_{b_1}, where $\iota(p'_{b_1}) = u_2$ is an inner vertex and $Lab(p'_{b_1}) \in B^*_{H_2}$, the vertex u_3 can not be outer by Remark 2.2.3. Thus u_3 is an inner vertex in Γ_{B_1}.

We lift consecutively the paths $e_2, p_{a_2}, e_3, p_{b_2}, \ldots, e_{2t}$ and get the path $p' = e'_1p'_{b_1}e'_2 \cdots p'_{a_t}e'_{2t-1}p'_{b_t}e'_{2t}$ in $Core(C_{H_1})$ with $Lab(p') = b_1a_2b_2 \cdots a_tb_t$, $\iota(p') = u_2$, and $\tau(p') = u_{4t-1}$.

Let p'_{a_1} be the unique lift of p_{a_1} in Γ_{A_1} such that $\tau(p'_{a_1}) = u_1$, $Lab(p'_{a_1}) = a_1$. Denote $\iota(p'_{a_1})$ by d_1. Similarly, let $p'_{a_{t+1}}$ be the unique lift of $p_{a_{t+1}}$ in $\Gamma_{A_{t+1}} = \Gamma_{A_1}$ such that $\iota(p'_{a_{t+1}}) = u_{4t-1}$, $Lab(p'_{a_{t+1}}) = a_{t+1}$. Denote $\tau(p'_{a_{t+1}})$ by d_2 (see Figure 2.2.18).

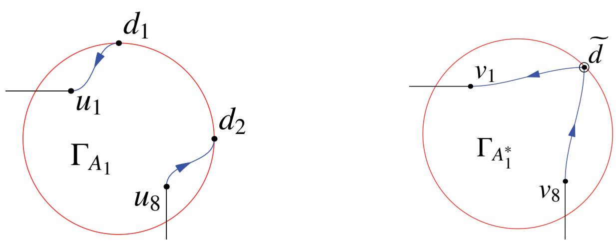


Fig. 2.2.18. *The outer vertices* $d_1, d_2 \in \psi_1^{-1}(\tilde{d})$.

Since $d_1, d_2 \in \psi_1^{-1}(\tilde{d})$, it follows from the definition of outer vertices in $\overline{Z}$ that d_1 and d_2 are outer vertices in Γ_{A_1}. Since $\iota(p'_{a_{t+1}})$ and $\tau(p'_{a_1})$ are inner vertices, $\iota(p'_{a_1}) = d_1$, $\tau(p'_{a_{t+1}}) = d_2$, and $Lab(p'_{a_1}), Lab(p'_{a_{t+1}}) \in A^*_{H_2}$, the outer vertices d_1, d_2 must coincide by Remark 2.2.4.

Denote $d_1 = d_2$ by d. Then the path $p'_y = p'_{a_1} p' p'_{a_{t+1}}$ in $Core(C_{H_1})$ is a closed lift of p_y based at d with $Lab(p'_y) = y$.

Now we show that all the paths p'_y are based at the same outer vertex d for all $y \in \{y_1, y_2, \ldots, y_n\}$. Assume to the contrary that p'_{y_1} is based at $d_1 \in \psi_1^{-1}(\tilde{d})$ and p'_{y_2} is based at $d_2 \in \psi_1^{-1}(\tilde{d})$ such that $d_1 \neq d_2$.

As in the previous paragraph, we can write p'_{y_1} and p'_{y_2} as $p'_{a_1} p'_1 p'_{a_{t+1}}$ and $p'_{a'_1} p'_2 p'_{a'_{t'+1}}$ respectively, where $\iota(p'_{a_1}) = \tau(p'_{a_{t+1}}) = d_1$ and $\iota(p'_{a'_1}) = \tau(p'_{a'_{t'+1}}) = d_2$.

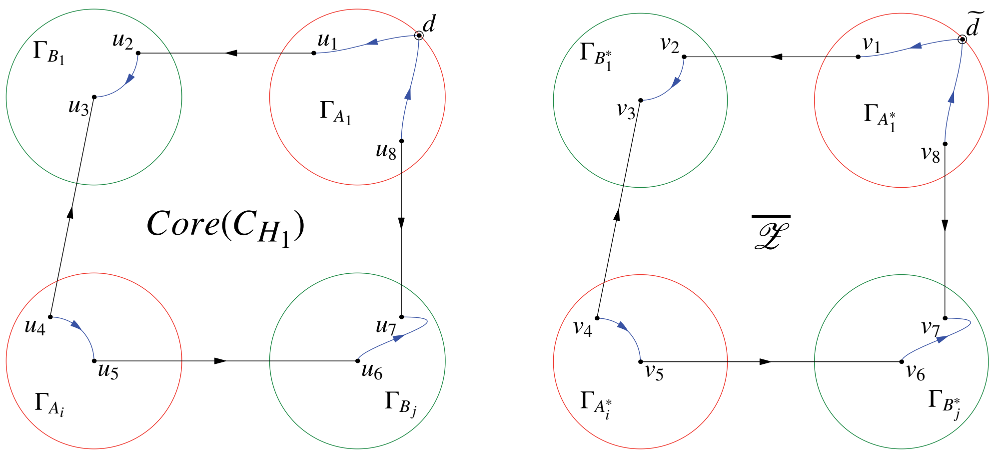


Fig. 2.2.19. *$y \notin A$ for all $y \in \{y_1, y_2, \ldots, y_n\}$.*

Since $\iota(p'_{a'_{t'+1}})$ and $\tau(p'_{a_1})$ are inner vertices, $\iota(p'_{a_1}) = d_1$, $\tau(p'_{a'_{t'+1}}) = d_2$, and $Lab(p'_{a_1}), Lab(p'_{a'_{t'+1}}) \in A^*_{H_2}$, the outer vertices d_1, d_2 must coincide by Remark 2.2.4.

Therefore, for every $y \in \{y_1, \ldots, y_n\}$ there exists a closed path p'_y in $Core(C_{H_1})$ based at d with $Lab(p'_y) = y$. It follows that H_2 is conjugate into H_1, which is a contradiction (see Figure 2.2.19).

Case 1.2.2.3. Suppose that $y \in A$ for some $y \in \{y_1, \ldots, y_n\}$, and $y' \notin A$ for at least one $y' \in \{y_1, \ldots, y_n\}$. Then for every element $y \in \{y_1, \ldots, y_n\}$ we can finde a closed path p'_y in $Core(C_{H_1})$ such that $Lab(p'_y) = y$ and all of these paths are based at the same vertex.

As in Case 1.2.2.2, if $y' \notin A$ for some $y' \in \{y_1, \ldots, y_n\}$, then there exists a closed path $p'_{y'} = p'_{a_1} p' p'_{a_{t+1}}$ in $Core(C_{H_1})$ such that $Lab(p'_{y'}) = y'$. Furthermore, all such paths are based at the same outer vertex, say $d \in \psi_1^{-1}(\tilde{d})$ (see Figure 2.2.20).

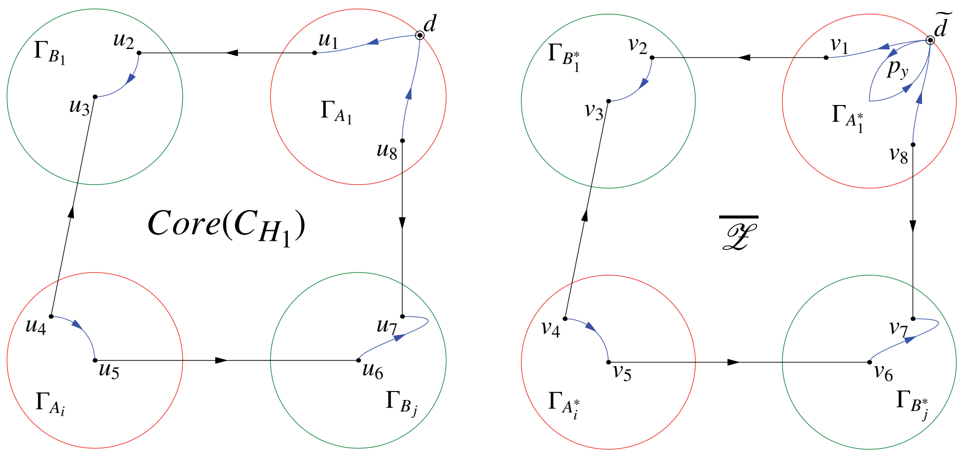


Fig. 2.2.20. *The path* $p'_{y'} = p'_{a_1} p' p'_{a_{t+1}}$ *in* $Core(C_{H_1})$.

Now we prove that, if $y \in A$ for some $y \in \{y_1, y_2, \ldots, y_n\}$, then there exists a closed path p'_y in $Core(C_{H_1})$ based at d with $Lab(p'_y) = y$.

Since $y \in A$, the cycle p_y in $(\widetilde{Z}, \tilde{d})$ with $Lab(p_y) = y$ lies completely inside $\Gamma_{A_1^*}$. Let p'_y be the the unique lift of p_y in Γ_{A_1} such that $\iota(p'_y) = d$ and $Lab(p'_y) = y$. Denote $\tau(p'_y)$ by w. Since $w \in \psi_1^{-1}(\tilde{d})$, it follows from the definition of outer vertices in $\overline{Z}$ that w is an outer vertex in Γ_{A_1} (see Figure 2.2.21).

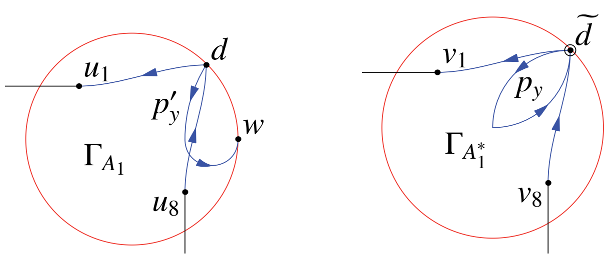


Fig. 2.2.21. *The outer vertices* $d, w \in \psi_1^{-1}(\tilde{d})$.

Since $\tau(p'_{a_1})$ and $\iota(p'_{a_{t+1}}p'_y)$ are inner vertices, $\iota(p'_{a_1}) = d$, $\tau(p'_{a_{t+1}}p'_y) = w$, and $Lab(p'_{a_1}), Lab(p'_{a_{t+1}}p'_y) \in A^*_{H_2}$, the outer vertices w and d must coincide by Remark 2.2.4.

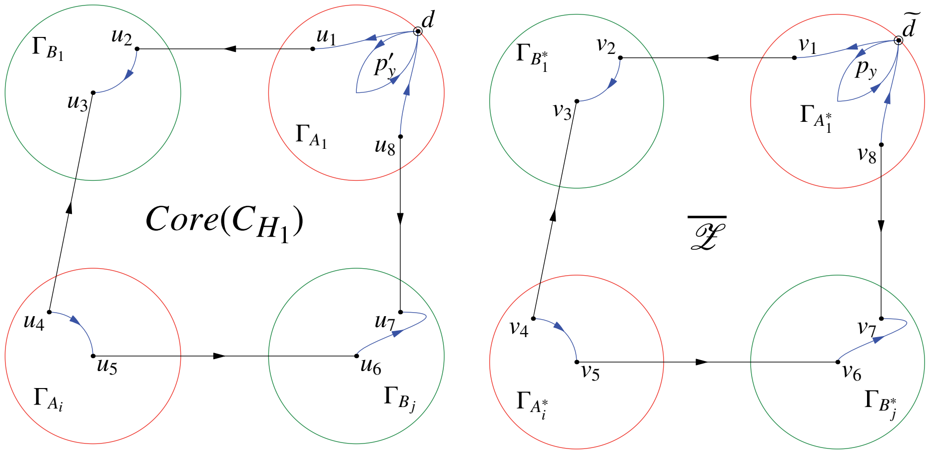


Fig. 2.2.22. *H_2 is conjugate into H_1.*

Therefore, for every $y \in \{y_1, \ldots, y_n\}$ there exists a closed path p'_y in $Core(C_{H_1})$ based at d with $Lab(p'_y) = y$. It follows that H_2 is conjugate into H_1, which is a contradiction (see Figure 2.2.22).

So we can conclude that Case 1, in which $\tilde{d}$ belongs to one of the $A-$ or $B-$components in a copy of $\overline{Z}$ in $\widetilde{Z}$, leads to a contradiction.

Consider Case 2. Suppose that $\tilde{d}$ belongs to one of the copies of U or V in $(\widetilde{Z}, \tilde{d})$. Let $p_{y_1}, p_{y_2}, \ldots, p_{y_n}$ be the cycles in $(\widetilde{Z}, \tilde{d})$ based at $\tilde{d}$ such that $Lab(p_y) = y$ for all $y \in \{y_1, y_2, \ldots, y_n\}$.

Since $\pi_1(U)$ and $\pi_1(V)$ does not contain any element from $A^*_{H_2}$ and $B^*_{H_2}$ respectively, all the cycles $p_{y_1}, p_{y_2}, \ldots, p_{y_n}$ have no closed subpaths in any copy of U or V in $\widetilde{Z}$. Then the cycles $p_{y_1}, p_{y_2}, \ldots, p_{y_n}$ satisfy one of the following two cases: At least one of them passes through several copies of T, or every one of them passes through only one of the copies of T.

Case 2.1. Suppose that p_y lies in several copies of T for some y in $\{y_1, y_2, \ldots, y_n\}$. Then p_y is a cycle of length at most M in the graph $\widetilde{\Gamma}$ which has girth greater than M, a contradiction.

Case 2.2. Suppose that every cycle in $\{p_{y_1}, p_{y_2}, \dots, p_{y_n}\}$ passes through only one of the copies of T. Then the cycles $p_{y_1}, p_{y_2}, \dots, p_{y_n}$ satisfy one of two cases: There exist at least two cycles p_y and $p_{y'}$ such that p_y passes through a copy of T and $p_{y'}$ passes through another copy of T or all the cycles $p_{y_1}, p_{y_2}, \dots, p_{y_n}$ pass through the same copy of T.

Note that all the cycles $p_{y_1}, p_{y_2}, \dots, p_{y_n}$ based at the same vertex $\tilde{d}$.

Case 2.2.1. Suppose that $p_{y'}$, $p_{y''} \in \{p_{y_1}, p_{y_2}, \dots, p_{y_n}\}$ pass through different copies of T, denote these copies by T' and T'' respectively. Then the vertex $\tilde{d}$ belongs to T', T'' or to a copy of V outside T' and T''.

If $\tilde{d}$ belongs to T', then $p_{y''}$ passes through T' and T''. Similarly, if $\tilde{d}$ belongs to T'', then $p_{y'}$ passes through T' and T''. Which contradicts the assumption in Case 2.2.

Now suppose that $\tilde{d}$ belongs to a copy of V, denote it by V_0, outside T' and T''. Then each of p_y and $p_{y'}$ has a subpath p such that p starts at V_0 and ends at a copy of $\overline{Z}$. Otherwise, $p_{y'}$ and $p_{y''}$ pass through more than one copy of T, since T has no cycles.

Therefore, according to the construction of T the length of p is greater than M, which contradicts the definition of M.

Case 2.2.2. Suppose that all the cycles $p_{y_1}, p_{y_2}, \dots, p_{y_n}$ pass through the same copy of T, denote it by $\mathscr{T}_0$. Then $\tilde{d}$ belongs to a copy of V outside $\mathscr{T}_0$ or to a copy of U or V in $\mathscr{T}_0$.

Case 2.2.2.1. Suppose that $\tilde{d}$ belongs to a copy of V, denote it by V_1, outside $\mathscr{T}_0$. Denote the copy of $\overline{Z}$ in $\mathscr{T}_0$ by $\overline{\mathscr{Z}}_0$ Then every cycle p_y in $\{p_{y_1}, p_{y_2}, \dots, p_{y_n}\}$ has a subpath p such that p starts at V_1 and ends at $\overline{\mathscr{Z}}_0$ in $\mathscr{T}_0$, because $\mathscr{T}_0$ has no cycles and $\mathscr{T}_0$ is joined with V_1 by only one edge. The length of p is greater than M, which contradicts the definition of M.

Case 2.2.2.2. Suppose that $\tilde{d}$ belongs to a copy of U or V in $\mathscr{T}_0$ such that the length of the path which starts at this copy of U or V and ends at $\overline{\mathscr{Z}}_0$ is greater than $\frac{M}{2}$. Then we get a contradiction with the definition of M.

Suppose that $\tilde{d}$ belongs to one of the copies of U or V in $\mathscr{T}_0$ such that the length of the path which starts at this copy of U or V and ends at $\overline{\mathscr{Z}}_0$ is less than or equal $\frac{M}{2}$. We show that this assumption also leads to a contradiction.

Without loss of generality, we may assume that $\tilde{d}$ belongs to one of the copies of U, denote it by U_0. Obviously, $\mathscr{T}_0$ has no cycles and also recall that all the cycles $p_y \in \{p_{y_1}, \ldots, p_{y_n}\}$ pass only through $\mathscr{T}_0$. Therefore, every $p_y \in \{p_{y_1}, \ldots, p_{y_n}\}$ has a closed subpath in $\overline{\mathscr{Z}}_0$.

Since U_0 and $\overline{\mathscr{Z}}_0$ are joind by exactly one path, all the cycles p_y in $\{p_{y_1}, \ldots, p_{y_n}\}$ have the form $p_1 p_y^* \overline{p_1}$, where p_y^* are cycles in $\overline{\mathscr{Z}}_0$ based at the same outer vertex, say $\widetilde{d_0}$, and p_1 is the path which begins at $\tilde{d}$ in U_0 and ends at $\widetilde{d_0}$ in $\overline{\mathscr{Z}}_0$ (see Figure 2.2.23).

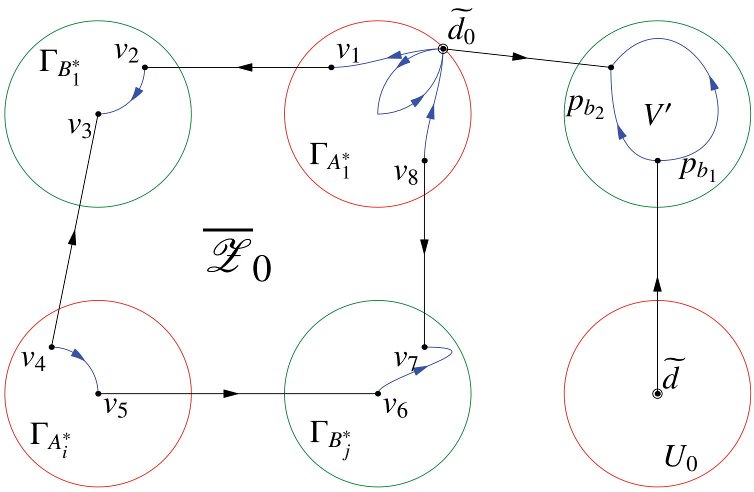


Fig. 2.2.23. *The paths p_1 and p_2.*

We show that, the path p_1 is a common subpath of all the paths p_y. Indeed, assume to the contrary that $p_y = p_1 p_y^* \overline{p_1}$ and $p_{y'} = p_2 p_{y'}^* \overline{p_2}$ such that $p_1 \neq p_2$, $\iota(p_1) = \iota(p_2) = \tilde{d}$, and $\tau(p_1) = \tau(p_2) = \widetilde{d_0}$. Since $\mathscr{T}_0$ is a tree, the paths p_1 and p_2 pass through the same copies of U and V.

Since $p_1 \neq p_2$, there exists at least one copy of U or V such that the subpaths of p_1 and p_2 in this copy are different, denote this copy by V'. Let

p_{b_1} be the subpath of p_1 in V' and p_{b_2} be the subpath of p_2 in V' such that $\iota(p_{b_1}) = \iota(p_{b_2})$, $\tau(p_{b_1}) = \tau(p_{b_2})$, and $p_{b_1} \neq p_{b_2}$. Therefore, the path $p_{b_1}\overline{p_{b_2}}$ is a closed path in V' with $Lab(p_{b_1}\overline{p_{b_2}}) \in B^*_{H_2}$, which contradicts the structure of V' (see Figure 2.2.23).

Denote $Lab(p_1)$ by g_1, then every $y \in \{y_1, y_2, \ldots, y_n\}$ is equal to $g_1 y^* g_1^{-1}$, where y^* is the label of a cycle based at $\widetilde{d}_0$ in $(\overline{\mathscr{X}}_0, \widetilde{d}_0)$.

Let $K = H_2^{g_1^{-1}} = \langle y_1^*, y_2^*, \ldots, y_n^* \rangle$, then $K \leqslant \pi_1(\overline{\mathscr{X}}_0, \widetilde{d}_0)$ which is the same as *Case 1.2.2*, where $\widetilde{d}_0$ in $\overline{\mathscr{X}}_0$ plays the role of the outer vertex $\widetilde{d}$ in $\overline{\mathscr{X}}$ and $\{y_1^*, y_2^*, \ldots, y_n^*\}$ play the role of $\{y_1, y_2, \ldots, y_n\}$.

Therefore, the generators $y_1^*, y_2^*, \ldots, y_n^*$ satisfy one of three cases:

(1) $y \in A$ for all $y \in \{y_1^*, y_2^*, \ldots, y_n^*\}$.

(2) $y \notin A$ for all $y \in \{y_1^*, y_2^*, \ldots, y_n^*\}$.

(3) $y \in A$ for some $y \in \{y_1^*, y_2^*, \ldots, y_n^*\}$, and at least one $y' \in \{y_1^*, y_2^*, \ldots, y_n^*\}$ does not belong to A.

Denote by $p_{y_1^*}, \ldots, p_{y_n^*}$ the cycles in $(\overline{\mathscr{X}}_0, \widetilde{d}_0)$ corresponding to $y_1^*, \ldots, y_n^*$ respectively. Note that all of them are based at the same vertex $\widetilde{d}_0$.

Case 2.2.2.2.1. Suppose that $y \in A$ for all $y \in \{y_1^*, y_2^*, \ldots, y_n^*\}$, equivalently $H_2^{g_1^{-1}} \leqslant A$. Then the cycles p_y lie completely in $\Gamma_{A_1^*}$. Choose a path p_a in $\Gamma_{A_1^*}$ such that $\tau(p_a) = \tilde{d}_0$ and $\iota(p_a) = \overline{z_1}$, where $\overline{z_1}$ is the basepoint of $\Gamma_{A_1^*}$. Denote $Lab(p_a)$ by $a \in A$.

Therefore, the paths $p_a p_y \overline{p_a}$ are closed in $(\Gamma_{A_1^*}, \overline{z_1})$ for all $y \in \{y_1^*, y_2^*, \ldots, y_n^*\}$. It follows further that $y^a \in \pi_1(\Gamma_{A_1^*}, \overline{z_1}) \cong A_1^*$ for all $y \in \{y_1^*, y_2^*, \ldots, y_n^*\}$. Hence $K^a \leqslant A_1^*$, which contradicts the construction of $\overline{\mathscr{X}}_0$, according to the conclusion in Construction 4.

Case 2.2.2.2.2. and ***Case 2.2.2.2.3.*** Consider the following two cases: $y \notin A$ for all y in $\{y_1^*, y_2^*, \ldots, y_n^*\}$, or $y \in A$ for some $y \in \{y_1^*, y_2^*, \ldots, y_n^*\}$ and $y' \notin A$ for at least one $y' \in \{y_1^*, y_2^*, \ldots, y_n^*\}$.

Following exactly the same procedure in Case 1.2.2.2 and Case 1.2.2.3, we can find a closed path p_y in $Core(C_{H_1})$ with $Lab(p_y) = y$ for all y in

$\{y_1^*, y_2^*, \ldots, y_n^*\}$. Furthermore, all the paths $p_{y_1^*}, p_{y_2^*}, \ldots, p_{y_n^*}$ are based at the same outer vertex, say d_0, such that $d_0 \in \psi_1^{-1}(\widetilde{d_0})$.

Choose a path p_g in $Core(C_{H_1})$ such that $\tau(p_g) = d_0$ and $\iota(p_g) = z$, where z is the basepoint of $Core(C_{H_1})$. Denote $Lab(p_g)$ by g, where $g \in A * B$. Therefore, the paths $p_g p_y \overline{p_g}$ are closed in $(Core(C_{H_1}), z)$ for all $y \in \{y_1^*, y_2^*, \ldots, y_n^*\}$. It follows further that, $y^g \in \pi_1(Core(C_{H_1}), z) \cong H_1$.

Hence $K^g = (H_2^{g_1^{-1}})^g = H_2^{g_1^{-1}g} \leqslant H_1$, which implies that H_2 is conjugate into H_1.

Finally we conclude that $D \cong \pi_1(\widetilde{Z}, \tilde{z})$ is a finite index subgroup of $A * B$, which contains H_1 and does not contain any conjugate of H_2, and therefore $A * B$ is SICS. □

Note that, by induction we obtain the following corollary:

Corollary 2.2.5. *Let A_i be a LERF and SICS group for all $i = 1, 2 \ldots, n$, then the group $A_1 * A_2 * \cdots * A_n$ is SICS.*

Chapter 3

SICS for Direct, Semidirect, Wreath, and Amalgamated Product of Groups

3.1 Introduction

In this chapter we will not use any geometrical arguments. We will use only group theory to answer the question: Is the class of SICS groups closed under the direct, semidirect, or wreath product or not?

The answer to this question is "no". We give examples of groups each one of them is a product of two simultaneously LERF and SICS groups and we prove that the product groups are not SICS.

In addition, we give an example of a non-SICS group which is a free product of an SICS group and an SCS group with an amalgamated cyclic subgroup.

Direct product: Allenby and Gregorac [2] gave an example of a non-LERF group which is a direct product of two LERF groups.

Namely, they have proved that $F_2 \times F_2$ is not LERF, though F_2 is LERF. It follows further that $F_m \times F_n$ is not LERF for all $n, m \geq 2$, since every subgroup of a LERF group is LERF (Scott [33]).

In Section 3.2, we prove that the group $F_m \times F_n$ is not SICS for all $n, m \geq 2$, though free groups are SICS (Bogopolski and Grunewald [6]).

Semidirect product: Burns, Karrass, and Solitar [8] gave an example of a non-LERF group which is a semidirect product of two LERF groups. Namely, they have proved that $F_2 \rtimes_\phi F_1$, for some action ϕ, is not LERF.

In Section 3.3, we prove that the same group is not SICS.

Amalgemated product: Gitik and Rips [12] gave an example of a non-LERF group which is a free product of two LERF groups with an amalgamated cyclic subgroup.

Allenby and Doniz [1] have modified their example to get a simpler example of a non-LERF group which is a free product of two LERF groups with an amalgamated cyclic subgroup.

In Section 3.4, we prove that the group given in [12] is not SICS, however it is a free product of an SICS group and an SCS group with an amalgamated cyclic subgroup.

Wreath product: The wreath product of two SCS or SICS groups is not necessary to be SCS or SICS.

In [9], Campbell proved that $W = A_5 \wr \mathbb{Z}$ is not residually finite, where A_5 is the alternating group of degree 5. Clearly, A_5 and $\mathbb{Z}$ are simultaneously SCS and SICS, because finite groups and free groups are SCS and SICS [6]. However, W is neither SCS nor SICS because it is not residually finite.

3.2 A direct product of SICS groups which is not SICS

In the following theorem we prove that $F_m \times F_n$ is not SICS, for all $m, n \geq 2$. We will adapt the proof which is given by Allenby and Gregorac [2], to obtain a proof of our result.

Theorem 3.2.1. *Let* $F = \langle x_1, x_2, \ldots, x_{n+k} \mid \rangle$ *and* $G = \langle a_1, a_2, \ldots, a_n \mid \rangle$, *then* $F \times G$ *is not SICS, for all* $k \geq 0$, $n \geq 2$.

Proof. Suppose that $H = \langle (x_1, a_1), (x_2, a_2), \ldots, (x_n, a_n), (1, c) \rangle \leqslant F \times G$, where $c = a_2^{-1} a_1^2 a_2 a_1^{-3}$.

Let $G' = \{1_F\} \times G$, $F' = F \times \{1_G\}$, and $K = \langle (1, c) \rangle \leqslant F \times G$, then we claim that $K^{G'} = H \cap G'$, where $K^{G'}$ is the normal closure of K in G'.

Clearly, $K^{G'} \leqslant G'$. To prove that $K^{G'} \leqslant H$, let $g \in G$ and $f \in F$ such that f is obtained from g by replacing every a_i in g by x_i (for example, if $g = a_1^2 a_3^{-1}$, then $f = x_1^2 x_3^{-1}$).

So $(1, g)(1, c)(1, g)^{-1} = (1, gcg^{-1}) = (f, g)(1, c)(f, g)^{-1}$ which is an element in H. Therefore, $K^{G'} \leqslant H \cap G'$.

Conversely, if $(1, d) \in H \cap G'$, then $(1, d) \in K^{G'}$. Indeed, since $(1, d) \in H$, we can write

$(1, d) = \prod_{i=1}^{r} (x_1, a_1)^{l_i} \cdots (x_n, a_n)^{m_i} (1, c)^{n_i}$ for some $l_i, m_i, n_i \in \mathbb{Z}$, and $r \in \mathbb{N}$.

So

$$(1, d) = (\prod_{i=1}^{r} x_1^{l_i} \cdots x_n^{m_i}, \prod_{i=1}^{r} a_1^{l_i} \cdots a_n^{m_i} c^{n_i}).$$

Set $g_i = a_1^{l_i} \cdots a_n^{m_i} \in G$. Then $(1, d) = (1, \prod_{i=1}^{r} g_i c^{n_i}) =$

$$(1, g_1 c^{n_1} g_1^{-1} (g_1 g_2) c^{n_2} (g_1 g_2)^{-1} (g_1 g_2 g_3) c^{n_3} \cdots (\prod_{i=1}^{r-1} g_i) c^{n_{r-1}} (\prod_{i=1}^{r-1} g_i)^{-1} (\prod_{i=1}^{r} g_i) c^{n_r}).$$

Since $\prod_{i=1}^{r} x_1^{l_i} \cdots x_n^{m_i} = 1$, we have $(\prod_{i=1}^{r} g_i) = 1$. Thus $K^{G'} = H \cap G'$.

Now we have $\widetilde{G} := G'/(H \cap G') \cong \langle a_1, a_2 \mid a_2^{-1} a_1^2 a_2 = a_1^3 \rangle * \langle a_3, a_4, \ldots, a_n \mid \rangle$ is a non-residually finite group. Indeed, if $\widetilde{G}$ is residually finite, then all its subgroups are residually finite. However, according to [3], the group $\langle a_1, a_2 \mid a_2^{-1} a_1^2 a_2 = a_1^3 \rangle$ is not Hopfian, and therefore it is not residually finite.

Therefore, there exists a non-trivial element $\tilde{t} \in \widetilde{G}$ such that $\tilde{t}$ belongs to every finite index normal subgroup $\widetilde{N}$ of $\widetilde{G} = G'/(H \cap G')$. Let $\tilde{t} = t(H \cap G')$ for some $t \in G'$, then

> t belongs to every finite index normal subgroup $N \trianglelefteq G'$ which contains $H \cap G'$. $(*)$

To show that $F \times G$ is not an SICS group, we will prove that $\langle t \rangle$ is not conjugate into H, although $\phi(\langle t \rangle)$ is contained in $\phi(H)$ for every epimorphism ϕ from $F \times G$ onto a finite group.

Since $\tilde{t}$ is a non-trivial element in $\widetilde{G} = G'/(H \cap G')$, it follows immediately that $t \notin H$. We claim that $gtg^{-1} \notin H$ for all $g \in F \times G$, and therefore, $\langle t \rangle$ is not conjugate into H.

To prove our claim, assume to the contrary that $gtg^{-1} \in H$. Since $t \in G'$ and G' is a normal subgroup in $F \times G$, we have $gtg^{-1} \in G'$ for all $g \in F \times G$.

Therefore, $gtg^{-1} \in H \cap G' = K^{G'}$, which implies that

$$gtg^{-1} = (1, \textstyle\prod_{i=1}^{r} g_i c^{n_i} g_i^{-1}) \text{ for some } g_i \in G \text{ and } r \in \mathbb{N}.$$

Let $g = (W'(x_1, \ldots, x_{n+k}), W(a_1, \ldots, a_n))$, then

$$t = (1, (W(a_1, \ldots, a_n))^{-1} \textstyle\prod_{i=1}^{r} g_i c^{n_i} g_i^{-1} (W(a_1, \ldots, a_n)))$$
$$= (1, \textstyle\prod_{i=1}^{r} ((W(a_1, \ldots, a_n))^{-1} g_i) c^{n_i} ((W(a_1, \ldots, a_n))^{-1} g_i)^{-1}) \in K^{G'} = H \cap G'.$$

In particular, t belongs to H, which is a contradiction. So we have proved that $\langle t \rangle$ is not conjugate into H.

Finally, let M be a finite index normal subgroup in $F \times G$, then $M \cap G'$ is a finite index normal subgroup in G'.

Let $N := (H \cap G')(M \cap G') = K^{G'}(M \cap G')$, then N is a finite index normal subgroup in G' and contains $H \cap G'$. Thus $t \in N \leqslant HM$, by $(*)$. Let ϕ be an epimorphism from $F \times G$ onto $(F \times G)/M$, then $\phi(t) = tM \subseteq HM = \phi(H)$.

So $\langle t \rangle$ is not conjugate into H, but $\phi(\langle t \rangle)$ is contained in $\phi(H)$ for every homomorphism ϕ from $F \times G$ onto a finite group. □

Remark 3.2.2. Clearly, $F_1 \times F_1$ is SCS and SICS. However, we do not know whether $F_1 \times F_m$ is SCS and/or SICS or not, for all $m \geq 2$.

Definition 3.2.3. Let Γ be a finite graph. The *graph group* or the *right-angled Artin group* $G(\Gamma)$ (or G for simplicity) is given by the presentation with a generator a_i for every vertex v_i of Γ and a defining relation $[a_i, a_j] = 1$ for each edge between vertices v_i and v_j in Γ.

Remark 3.2.4. The right-angled Artin group G corresponding to the complete graph K_n on n vertices is the free abelian group of rank n, so it is SCS and SICS for all $n \geq 1$.

Example: The right-angled Artin group G corresponding to a square is not SICS. Indeed, let S be a square with vertices v_a, v_b, v_c, and v_d, then the right-angled Artin group G corresponding to S can be represented as follows.

$G = \langle a, b, c, d \,|\, [a, b] = [b, c] = [c, d] = [d, a] = 1\rangle = \langle a, c\rangle \times \langle b, d\rangle$, which is isomorphic to $F_2 \times F_2$. Hence, by Theorem 3.2.1, G is not SICS.

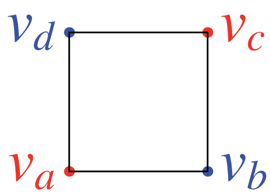


Fig. 3.2.1. $S = (V = \{v_a, v_b, v_c, v_d\}, E = \{(v_a, v_b), (v_b, v_c), (v_c, v_d), (v_d, v_a)\})$

Definition 3.2.5. A *bipartite graph* $\Gamma = (U, V, E)$ is a graph whose vertices can be divided into two disjoint sets U and V such that every vertex in U is connected by an edge to one in V, every vertex in V is connected by an edge to one in U, and no vertices in the same set are adjacent.

Definition 3.2.6. The *complete bipartite graph* on n and m vertices, denoted by $K_{n,m}$, is the bipartite graph $\Gamma = (V, U, E)$, where $|V| = n$, $|U| = m$, and E connects every vertex in V with all vertices in U.

Let $V = \{v_{a_1}, v_{a_2}, \ldots, v_{a_n}\}$ and $U = \{v_{b_1}, v_{b_2}, \ldots, v_{b_m}\}$. The the right-angled Artin group G corresponding to $K_{n,m}$ can be represented as follows.

$$G = \langle a_1, \ldots, a_n, b_1, \ldots, b_m \,|\, [a_i, b_j] = 1, i = 1, \ldots, n, j = 1, \ldots, m\rangle$$

$$= \langle a_1, \ldots, a_n\rangle \times \langle b_1, \ldots, b_m\rangle,$$

which is isomorphic to $F_n \times F_m$.

Therefore, using Theorem 3.2.1, we obtain the following remark:

Remark 3.2.7. The right-angled Artin group G corresponding to the complete bipartite graph $K_{n,m}$ is not SICS for all $n, m \geq 2$.

3.3 A semidirect product of SICS groups which is not SICS

It is well known that, the direct product $A \times B$ of two groups A and B is the semidirect product $A \rtimes B$ with the trivial action of B on A.

Therefore, by Theorem 3.2.1, we conclude that the class of SICS groups is not closed under the semidirect product. In this section we consider a semidirect product with a nontrivial action, and we prove that it is not SICS.

In the following theorem we prove that $F_2 \rtimes F_1$ is not SICS. Assume that $F_1 = \langle a \,|\, \rangle$ and $F_2 = \langle x, y \,|\, \rangle$. If we consider the action of a on F_2 as

$$axa^{-1} = y^{-1}x, \; aya^{-1} = y,$$

then we can write $F_2 \rtimes F_1$ as follows.

$F_2 \rtimes F_1 = \langle x, y, a \,|\, axa^{-1} = y^{-1}x, \; aya^{-1} = y \rangle$. Since $y = xax^{-1}a^{-1}$, it follows

$$F_2 \rtimes F_1 = \langle x, a \,|\, axax^{-1}a^{-1}a^{-1} = xax^{-1}a^{-1} \rangle$$
$$= \langle x, a \,|\, axax^{-1}a^{-1} = xax^{-1} \rangle = \langle x, a \,|\, [a, xax^{-1}] = 1 \rangle.$$

Theorem 3.3.1. *Let $G = \langle x, a \,|\, [a, xax^{-1}] = 1 \rangle$, then G is not SICS.*

Proof. Let $A = \langle a_i \,|\, a_i = x^i a x^{-i}, \; [a_i, a_{i+1}] = 1, \; i \in \mathbb{Z} \rangle$, then $G = A \rtimes \langle x \rangle$, where the action of x on A is given by $x a_i x^{-1} = a_{i+1}$.

Let $H = \langle x, \; a_0 a_1^{-2} \rangle$, $H_1 = H \cap A$, and $H_2 = \langle [a_0, a_2] \rangle$.

To prove that G is not SICS, we prove the following:

(1) H_2 is not conjugate into H.

(2) H_2 is conjugate into every finite index subgroup L in G, which contains H.

The proof of **(1)** follows from (i), (ii) and (iii)

(i) $H_1 = \langle a_i a_{i+1}^{-2} : i \in \mathbb{Z}\rangle$.

It is clear that $\langle a_i a_{i+1}^{-2} : i \in \mathbb{Z}\rangle \leqslant H_1$, since

$$a_i a_{i+1}^{-2} = (x^i a_0 x^{-i})(x^i a_1 x^{-i})^{-2} = x^i a_0 x^{-i} x^i a_1^{-2} x^{-i} = x^i a_0 a_1^{-2} x^{-i}.$$

Conversely, let $h \in H \cap A$. Since $h \in H$, there exist $r \in \mathbb{N}$ and $i_1, \ldots, i_r$, $n_1, \ldots, n_r \in \mathbb{Z}$, such that

$$\begin{aligned} h &= x^{i_1}(a_0 a_1^{-2})^{n_1} x^{i_2}(a_0 a_1^{-2})^{n_2} x^{i_3} \cdots x^{i_r}(a_0 a_1^{-2})^{n_r} \\ &= x^{i_1} a_0^{n_1} x^{-i_1} x^{i_1} a_1^{-2n_1} x^{-i_1} x^{i_1+i_2} \cdots x^{i_1+\cdots+i_r} a_0^{n_r} x^{-i_1-\cdots-i_r} x^{i_1+\cdots+i_r} a_1^{-2n_r} x^{-i_1-\cdots-i_r} x^{i_1+\cdots+i_r} \\ &= a_{i_1}^{n_1} a_{i_1+1}^{-2n_1} \cdots a_{i_1+\cdots+i_r}^{n_r} a_{i_1+\cdots+i_r+1}^{-2n_r} x^{i_1+\cdots+i_r} = (a_{i_1} a_{i_1+1}^{-2})^{n_1} \cdots (a_{i_1+\cdots+i_r} a_{i_1+\cdots+i_r+1}^{-2})^{n_r} x^{i_1+\cdots+i_r}. \end{aligned}$$

Since $h \in A$, we get $x^{i_1+\cdots+i_r} = 1$. So $h \in \langle a_i a_{i+1}^{-2} : i \in \mathbb{Z}\rangle$.

(ii) $[a_0, a_2]^g \notin H_1$ for all $g \in G$.

Assume the contrary, that there exists $g = a' x^l \in G$ for some $l \in \mathbb{Z}$ and $a' \in A$ such that $[a_0, a_2]^g \in H_1$. Thus $[a_l, a_{l+2}]^{a'} \in H_1$, indeed

$$\begin{aligned} [a_l, a_{l+2}]^{a'} &= a' x^l a_0 x^{-l} x^l a_2 x^{-l} x^l a_0^{-1} x^{-l} x^l a_2^{-1} x^{-l} a'^{-1} = a' x^l a_0 a_2 a_0^{-1} a_2^{-1} x^{-l} a'^{-1} \\ &= [a_0, a_2]^g \in H_1. \end{aligned}$$

Therefore, there exists $h \in H_1$ such that $[a_l, a_{l+2}]^{a'} = h$.

Now we prove that, $[a_l, a_{l+2}]^{a'} = h$ leads to a contradiction. To clearify this contradiction we use the following notation:

For every $b \in A$, a *representation* W_b of b is the sequence $(c_1, c_2, \ldots, c_k)$ such that $b = c_1 c_2 \ldots c_k$ in A, and $c_1, c_2, \ldots, c_k \in \{a_i^{\pm 1} : i \in \mathbb{Z}\}$. The length of W_b is k.

A representation W_b of an element $b \in A$ will be called *reduced* if it has minimum length. Note that an element $b \in A$ may have many reduced representations.

We claim that the sequence $(a_l, a_{l+2}, a_l^{-1}, a_{l+2}^{-1})$, or one of its other three cyclic permutations $(a_{l+2}, a_l^{-1}, a_{l+2}^{-1}, a_l)$, $(a_l^{-1}, a_{l+2}^{-1}, a_l, a_{l+2})$, $(a_{l+2}^{-1}, a_l, a_{l+2}, a_l^{-1})$, is a partial sequence of a reduced representation W of the element $[a_l, a_{l+2}]^{a'}$ in A.

On the other hand, we will prove that the sequence $(a_l, a_{l+2}, a_l^{-1}, a_{l+2}^{-1})$, and all its other three cyclic permutations, are not partial sequence of every reduced representation of any element $h \in H_1 = H \cap A$, which contradicts that $[a_l, a_{l+2}]^{a'} = h$.

First we prove our claim. Let $W_{a'} = (c_1, c_2, \ldots, c_k)$ be a reduced representation of a', it follows that $W = (c_1, \ldots, c_k, a_l, a_{l+2}, a_l^{-1}, a_{l+2}^{-1}, c_k^{-1}, \ldots, c_1^{-1})$ is a representation of $[a_l, a_{l+2}]^{a'}$ in A. Using the relations of A we want to reduce W and see how it could be. So we consider the following cases of W:

Case 1. Let $a_{l+1}^{\pm 1}, a_l^{\pm 1}, a_{l+2}^{\pm 1} \notin \{c_1, c_2, \ldots, c_k\}$, then W is a reduced representation of $[a_l, a_{l+2}]^{a'}$ in A with partial sequence $(a_l, a_{l+2}, a_l^{-1}, a_{l+2}^{-1})$.

Case 2. Let $a_{l+1}^{\pm 1}, a_l^{\pm 1}$, or $a_{l+2}^{\pm 1} \in \{c_1, c_2, \ldots, c_k\}$, and no one of them, under all possible permutations using the relations of A, is equal to c_k, then $(a_l, a_{l+2}, a_l^{-1}, a_{l+2}^{-1})$ is a partial sequence of the reduced representation W of $[a_l, a_{l+2}]^{a'}$ in A.

Case 3. Let $a_{l+1}^{\pm 1}, a_l^{\pm 1}$, or $a_{l+2}^{\pm 1} \in \{c_1, c_2, \ldots, c_k\}$, and consider the following cases:

Case 3.1. If $a_{l+1}^{\pm 1}$, after some permutations using the relations of A, is equal to c_k, then the reduction of W will reduce the length of W without any change of the sequence $(a_l, a_{l+2}, a_l^{-1}, a_{l+2}^{-1})$. Note that $a_{l+1}^{\pm 1}$ commutes with $a_l^{\pm 1}$ and $a_{l+2}^{\pm 1}$.

Case 3.2. If $a_l^{\pm 1}$ or $a_{l+2}^{\pm 1}$, after some permutations using the relations of A, is equal to c_k, then for every step of reduction, the length of W will be reduced and an element of the sequence $(a_l, a_{l+2}, a_l^{-1}, a_{l+2}^{-1})$ will be canceled from one side and its inverse will appear in the other side.

So we conclude that, for every $a' \in A$ there exists a reduced representation W' of $[a_l, a_{l+2}]^{a'}$ in A such that the sequence $(a_l, a_{l+2}, a_l^{-1}, a_{l+2}^{-1})$, or one of its other three cyclic permutations, is a partial sequence of W'.

Conversely, we prove that the sequence $(a_l, a_{l+2}, a_l^{-1}, a_{l+2}^{-1})$, and all its other three cyclic permutations, are not partial sequences of every reduced representation of any element $h \in H_1$.

First, we consider the sequences $(a_l, a_{l+2}, a_l^{-1}, a_{l+2}^{-1})$ and $(a_l^{-1}, a_{l+2}^{-1}, a_l, a_{l+2})$.

Let $h \in H_1$, and assume to the contrary that one of the sequences $(a_l, a_{l+2}, a_l^{-1}, a_{l+2}^{-1})$ and $(a_l^{-1}, a_{l+2}^{-1}, a_l, a_{l+2})$ is a partial sequence of a reduced representation W_h of h.

Since $h \in H_1 = \langle a_i a_{i+1}^{-2} : i \in \mathbb{Z}\rangle$, we have $h = \prod_{j=1}^{k}(\prod_{i=t_j}^{r_j}(a_i a_{i+1}^{-2})^{n_{i,j}})$, for some $r_j, t_j, n_{i,j} \in \mathbb{Z}$, $t_j \leq r_j$, and $k \in \mathbb{N}$.

Let $s_1 = a_{l-1}a_l^{-2}$, $s_2 = a_l a_{l+1}^{-2}$, $s_3 = a_{l+1}a_{l+2}^{-2}$, and $s_4 = a_{l+2}a_{l+3}^{-2}$, then the element $a_l^{\pm 1}$ can appear in the reduced representation W_h of h if and only if the product $s_1^{n_1}s_2^{n_2}$ or $s_2^{n_2}s_1^{n_1}$, where $n_1, n_2 \in \mathbb{Z}$, is a partial product of the product $\prod_{j=1}^{k}(\prod_{i=t_j}^{r_j}(a_i a_{i+1}^{-2})^{n_{i,j}})$.

Similarly, the element $a_{l+2}^{\pm 1}$ can appear in the reduced representation W_h of h if and only if the product $s_3^{n_3}s_4^{n_4}$ or $s_4^{n_4}s_3^{n_3}$, where $n_3, n_4 \in \mathbb{Z}$, is a partial product of the product $\prod_{j=1}^{k}(\prod_{i=t_j}^{r_j}(a_i a_{i+1}^{-2})^{n_{i,j}})$.

Therefore, the sequence $(a_l, a_{l+2}, a_l^{-1}, a_{l+2}^{-1})$ or $(a_l^{-1}, a_{l+2}^{-1}, a_l, a_{l+2})$ is a partial sequence of W_h if and only if there exist $h_1, h_2, u \in H_1$ such that $h = h_1 u h_2$, where u is one of the following 16 types:

$$
\begin{aligned}
u_1 &= (s_1^{n_1}s_2^{n_2})(s_3^{n_3}s_4^{n_4})(s_1^{n_5}s_2^{n_6})(s_3^{n_7}s_4^{n_8}), & u_2 &= (s_1^{n_1}s_2^{n_2})(s_3^{n_3}s_4^{n_4})(s_1^{n_5}s_2^{n_6})(s_4^{n_7}s_3^{n_8}),\\
u_3 &= (s_1^{n_1}s_2^{n_2})(s_3^{n_3}s_4^{n_4})(s_2^{n_5}s_1^{n_6})(s_3^{n_7}s_4^{n_8}), & u_4 &= (s_1^{n_1}s_2^{n_2})(s_3^{n_3}s_4^{n_4})(s_2^{n_5}s_1^{n_6})(s_4^{n_7}s_3^{n_8}),\\
u_5 &= (s_1^{n_1}s_2^{n_2})(s_4^{n_3}s_3^{n_4})(s_1^{n_5}s_2^{n_6})(s_3^{n_7}s_4^{n_8}), & u_6 &= (s_1^{n_1}s_2^{n_2})(s_4^{n_3}s_3^{n_4})(s_1^{n_5}s_2^{n_6})(s_4^{n_7}s_3^{n_8}),\\
u_7 &= (s_1^{n_1}s_2^{n_2})(s_4^{n_3}s_3^{n_4})(s_2^{n_5}s_1^{n_6})(s_3^{n_7}s_4^{n_8}), & u_8 &= (s_1^{n_1}s_2^{n_2})(s_4^{n_3}s_3^{n_4})(s_2^{n_5}s_1^{n_6})(s_4^{n_7}s_3^{n_8}),\\
u_9 &= (s_2^{n_1}s_1^{n_2})(s_3^{n_3}s_4^{n_4})(s_1^{n_5}s_2^{n_6})(s_3^{n_7}s_4^{n_8}), & u_{10} &= (s_2^{n_1}s_1^{n_2})(s_3^{n_3}s_4^{n_4})(s_1^{n_5}s_2^{n_6})(s_4^{n_7}s_3^{n_8}),\\
u_{11} &= (s_2^{n_1}s_1^{n_2})(s_3^{n_3}s_4^{n_4})(s_2^{n_5}s_1^{n_6})(s_3^{n_7}s_4^{n_8}), & u_{12} &= (s_2^{n_1}s_1^{n_2})(s_3^{n_3}s_4^{n_4})(s_2^{n_5}s_1^{n_6})(s_4^{n_7}s_3^{n_8}),\\
u_{13} &= (s_2^{n_1}s_1^{n_2})(s_4^{n_3}s_3^{n_4})(s_1^{n_5}s_2^{n_6})(s_3^{n_7}s_4^{n_8}), & u_{14} &= (s_2^{n_1}s_1^{n_2})(s_4^{n_3}s_3^{n_4})(s_1^{n_5}s_2^{n_6})(s_4^{n_7}s_3^{n_8}),\\
u_{15} &= (s_2^{n_1}s_1^{n_2})(s_4^{n_3}s_3^{n_4})(s_2^{n_5}s_1^{n_6})(s_3^{n_7}s_4^{n_8}), & u_{16} &= (s_2^{n_1}s_1^{n_2})(s_4^{n_3}s_3^{n_4})(s_2^{n_5}s_1^{n_6})(s_4^{n_7}s_3^{n_8}),
\end{aligned}
$$

for some $n_i \in \mathbb{Z}$, $i \in \{1, \ldots, 8\}$.

Now we prove that all these 16 types lead to contradictions.

$$
\begin{aligned}
u_1 &= a_{l-1}^{n_1} a_l^{n_2-2n_1} a_{l+1}^{n_3-2n_2} a_{l+2}^{n_4-2n_3} a_{l+3}^{-2n_4} a_{l-1}^{n_5} a_l^{n_6-2n_5} a_{l+1}^{n_7-2n_6} a_{l+2}^{n_8-2n_7} a_{l+3}^{-2n_8} \\
&= a_{l-1}^{n_1} a_{l+1}^{n_3-2n_2} a_l^{n_2-2n_1} a_{l+2}^{n_4-2n_3} a_{l+3}^{-2n_4} a_{l-1}^{n_5} a_l^{n_6-2n_5} a_{l+2}^{n_8-2n_7} a_{l+1}^{n_7-2n_6} a_{l+3}^{-2n_8}
\end{aligned}
$$

Since a_{l+3} and a_{l-1} do not commute with a_l and a_{l+2} respectively, the sequence $(a_l, a_{l+2}, a_l^{-1}, a_{l+2}^{-1})$ or $(a_l^{-1}, a_{l+2}^{-1}, a_l, a_{l+2})$ appears in u_1 if and only if $n_4 = n_5 = 0$.

It follows that

$$
u_1 = a_{l-1}^{n_1} a_{l+1}^{n_3-2n_2} (a_l^{n_2-2n_1} a_{l+2}^{-2n_3} a_l^{n_6} a_{l+2}^{n_8-2n_7}) a_{l+1}^{n_7-2n_6} a_{l+3}^{-2n_8}.
$$

Hence $-2n_3 = \pm 1$ which contradicts that $n_3 \in \mathbb{Z}$.

Similarly for all other 15 types as follows.

$$
\begin{aligned}
u_2 &= a_{l-1}^{n_1} a_l^{n_2-2n_1} a_{l+1}^{n_3-2n_2} a_{l+2}^{n_4-2n_3} a_{l+3}^{-2n_4} a_{l-1}^{n_5} a_l^{n_6-2n_5} a_{l+1}^{-2n_6} a_{l+2}^{n_7} a_{l+3}^{-2n_7} a_{l+1}^{n_8} a_{l+2}^{-2n_8} \\
&= a_{l-1}^{n_1} a_{l+1}^{n_3-2n_2} a_l^{n_2-2n_1} a_{l+2}^{n_4-2n_3} a_{l+3}^{-2n_4} a_{l-1}^{n_5} a_l^{n_6-2n_5} a_{l+2}^{n_7-2n_8} a_{l+1}^{-2n_6} a_{l+3}^{-2n_7} a_{l+1}^{n_8}.
\end{aligned}
$$

Then $n_4 = n_5 = 0$ and $u_2 = a_{l-1}^{n_1} a_{l+1}^{n_3-2n_2} (a_l^{n_2-2n_1} a_{l+2}^{-2n_3} a_l^{n_6} a_{l+2}^{n_7-2n_8}) a_{l+1}^{-2n_6} a_{l+3}^{-2n_7} a_{l+1}^{n_8}$. Hence $-2n_3 = \pm 1$, which contradicts that $n_3 \in \mathbb{Z}$.

$$
\begin{aligned}
u_3 &= a_{l-1}^{n_1} a_l^{n_2-2n_1} a_{l+1}^{n_3-2n_2} a_{l+2}^{n_4-2n_3} a_{l+3}^{-2n_4} a_l^{n_5} a_{l+1}^{-2n_5} a_{l-1}^{n_6} a_l^{-2n_6} a_{l+1}^{n_7} a_{l+2}^{n_8-2n_7} a_{l+3}^{-2n_8} \\
&= a_{l-1}^{n_1} a_{l+1}^{n_3-2n_2} a_l^{n_2-2n_1} a_{l+2}^{n_4-2n_3} a_{l+3}^{-2n_4} a_{l+1}^{-2n_5} a_{l-1}^{n_6} a_{l+1}^{n_7} a_l^{n_5-2n_6} a_{l+2}^{n_8-2n_7} a_{l+3}^{-2n_8}.
\end{aligned}
$$

Then $n_4 = n_6 = 0$ and $u_3 = a_{l-1}^{n_1} a_{l+1}^{n_3-2n_2} (a_l^{n_2-2n_1} a_{l+2}^{-2n_3} a_l^{n_5} a_{l+2}^{n_8-2n_7}) a_{l+1}^{n_7-2n_5} a_{l+3}^{-2n_8}$. Hence $-2n_3 = \pm 1$, which contradicts that $n_3 \in \mathbb{Z}$.

$$
\begin{aligned}
u_4 &= a_{l-1}^{n_1} a_l^{n_2-2n_1} a_{l+1}^{n_3-2n_2} a_{l+2}^{n_4-2n_3} a_{l+3}^{-2n_4} a_l^{n_5} a_{l+1}^{-2n_5} a_{l-1}^{n_6} a_l^{-2n_6} a_{l+2}^{n_7} a_{l+3}^{-2n_7} a_{l+1}^{n_8} a_{l+2}^{-2n_8} \\
&= a_{l-1}^{n_1} a_{l+1}^{n_3-2n_2} a_l^{n_2-2n_1} a_{l+2}^{n_4-2n_3} a_{l+3}^{-2n_4} a_{l+1}^{-2n_5} a_{l-1}^{n_6} a_l^{n_5-2n_6} a_{l+2}^{n_7-2n_8} a_{l+3}^{-2n_7} a_{l+1}^{n_8}.
\end{aligned}
$$

Then $n_4 = n_6 = 0$ and $u_4 = a_{l-1}^{n_1} a_{l+1}^{n_3-2n_2} (a_l^{n_2-2n_1} a_{l+2}^{-2n_3} a_l^{n_5} a_{l+2}^{n_7-2n_8}) a_{l+1}^{-2n_5} a_{l+3}^{-2n_7} a_{l+1}^{n_8}$. Hence $-2n_3 = \pm 1$, which contradicts that $n_3 \in \mathbb{Z}$.

$$
\begin{aligned}
u_5 &= a_{l-1}^{n_1} a_l^{n_2-2n_1} a_{l+1}^{-2n_2} a_{l+2}^{n_3} a_{l+3}^{-2n_3} a_{l+1}^{n_4} a_{l+2}^{-2n_4} a_{l-1}^{n_5} a_l^{n_6-2n_5} a_{l+1}^{n_7-2n_6} a_{l+2}^{n_8-2n_7} a_{l+3}^{-2n_8} \\
&= a_{l-1}^{n_1} a_{l+1}^{-2n_2} a_l^{n_2-2n_1} a_{l+2}^{n_3-2n_4} a_{l+3}^{-2n_3} a_{l+1}^{n_4} a_{l-1}^{n_5} a_l^{n_6-2n_5} a_{l+2}^{n_8-2n_7} a_{l+1}^{n_7-2n_6} a_{l+3}^{-2n_8}.
\end{aligned}
$$

Then $n_3 = n_5 = 0$ and $u_5 = a_{l-1}^{n_1} a_{l+1}^{n_4-2n_2} (a_l^{n_2-2n_1} a_{l+2}^{-2n_4} a_l^{n_6} a_{l+2}^{n_8-2n_7}) a_{l+1}^{n_7-2n_6} a_{l+3}^{-2n_8}$. Hence $-2n_4 = \pm 1$, which contradicts that $n_4 \in \mathbb{Z}$.

$$
u_6 = a_{l-1}^{n_1} a_l^{n_2-2n_1} a_{l+1}^{-2n_2} a_{l+2}^{n_3} a_{l+3}^{-2n_3} a_{l+1}^{n_4} a_{l+2}^{-2n_4} a_{l-1}^{n_5} a_l^{n_6-2n_5} a_{l+1}^{-2n_6} a_{l+2}^{n_7} a_{l+3}^{-2n_7} a_{l+1}^{n_8} a_{l+2}^{-2n_8}
$$

$$= a_{l-1}^{n_1} a_{l+1}^{-2n_2} a_l^{n_2-2n_1} a_{l+2}^{n_3-2n_4} a_{l+3}^{-2n_3} a_{l+1}^{n_4} a_{l-1}^{n_5} a_l^{n_6-2n_5} a_{l+2}^{n_7-2n_8} a_{l+1}^{-2n_6} a_{l+3}^{-2n_7} a_{l+1}^{n_8}.$$

Then $n_3 = n_5 = 0$ and $u_6 = a_{l-1}^{n_1} a_{l+1}^{n_4-2n_2} (a_l^{n_2-2n_1} a_{l+2}^{-2n_4} a_l^{n_6} a_{l+2}^{n_7-2n_8}) a_{l+1}^{-2n_6} a_{l+3}^{-2n_7} a_{l+1}^{n_8}$. Hence $-2n_4 = \pm 1$, which contradicts that $n_4 \in \mathbb{Z}$.

$$\begin{aligned} u_7 &= a_{l-1}^{n_1} a_l^{n_2-2n_1} a_{l+1}^{-2n_2} a_{l+2}^{n_3} a_{l+3}^{-2n_3} a_{l+1}^{n_4} a_{l+2}^{-2n_4} a_l^{n_5} a_{l+1}^{-2n_5} a_{l-1}^{n_6} a_l^{-2n_6} a_{l+1}^{n_7} a_{l+2}^{n_8-2n_7} a_{l+3}^{-2n_8} \\ &= a_{l-1}^{n_1} a_{l+1}^{-2n_2} a_l^{n_2-2n_1} a_{l+2}^{n_3-2n_4} a_{l+3}^{-2n_3} a_{l+1}^{n_4-2n_5} a_{l-1}^{n_6} a_l^{n_5-2n_6} a_{l+2}^{n_8-2n_7} a_{l+1}^{n_7} a_{l+3}^{-2n_8}. \end{aligned}$$

Then $n_3 = n_6 = 0$ and $u_7 = a_{l-1}^{n_1} a_{l+1}^{n_4-2n_2} (a_l^{n_2-2n_1} a_{l+2}^{-2n_4} a_l^{n_5} a_{l+2}^{n_8-2n_7}) a_{l+1}^{n_7-2n_5} a_{l+3}^{-2n_8}$. Hence $-2n_4 = \pm 1$, which contradicts that $n_4 \in \mathbb{Z}$.

$$\begin{aligned} u_8 &= a_{l-1}^{n_1} a_l^{n_2-2n_1} a_{l+1}^{-2n_2} a_{l+2}^{n_3} a_{l+3}^{-2n_3} a_{l+1}^{n_4} a_{l+2}^{-2n_4} a_l^{n_5} a_{l+1}^{-2n_5} a_{l-1}^{n_6} a_l^{-2n_6} a_{l+2}^{n_7} a_{l+3}^{-2n_7} a_{l+1}^{n_8} a_{l+2}^{-2n_8} \\ &= a_{l-1}^{n_1} a_{l+1}^{-2n_2} a_l^{n_2-2n_1} a_{l+2}^{n_3-2n_4} a_{l+3}^{-2n_3} a_{l+1}^{n_4-2n_5} a_{l-1}^{n_6} a_l^{n_5-2n_6} a_{l+2}^{n_7-2n_8} a_{l+3}^{-2n_7} a_{l+1}^{n_8}. \end{aligned}$$

Then $n_3 = n_6 = 0$ and $u_8 = a_{l-1}^{n_1} a_{l+1}^{n_4-2n_2} (a_l^{n_2-2n_1} a_{l+2}^{-2n_4} a_l^{n_5} a_{l+2}^{n_7-2n_8}) a_{l+1}^{-2n_5} a_{l+3}^{-2n_7} a_{l+1}^{n_8}$. Hence $-2n_4 = \pm 1$, which contradicts that $n_4 \in \mathbb{Z}$.

$$\begin{aligned} u_9 &= a_l^{n_1} a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_l^{-2n_2} a_{l+1}^{n_3} a_{l+2}^{n_4-2n_3} a_{l+3}^{-2n_4} a_{l-1}^{n_5} a_l^{n_6-2n_5} a_{l+1}^{n_7-2n_6} a_{l+2}^{n_8-2n_7} a_{l+3}^{-2n_8} \\ &= a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_{l+1}^{n_3} a_l^{n_1-2n_2} a_{l+2}^{n_4-2n_3} a_{l+3}^{-2n_4} a_{l-1}^{n_5} a_l^{n_6-2n_5} a_{l+2}^{n_8-2n_7} a_{l+1}^{n_7-2n_6} a_{l+3}^{-2n_8}. \end{aligned}$$

Then $n_4 = n_5 = 0$ and $u_9 = a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_{l+1}^{n_3} (a_l^{n_1-2n_2} a_{l+2}^{-2n_3} a_l^{n_6} a_{l+2}^{n_8-2n_7}) a_{l+1}^{n_7-2n_6} a_{l+3}^{-2n_8}$. Hence $-2n_3 = \pm 1$, which contradicts that $n_3 \in \mathbb{Z}$.

$$\begin{aligned} u_{10} &= a_l^{n_1} a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_l^{-2n_2} a_{l+1}^{n_3} a_{l+2}^{n_4-2n_3} a_{l+3}^{-2n_4} a_{l-1}^{n_5} a_l^{n_6-2n_5} a_{l+1}^{-2n_6} a_{l+2}^{n_7} a_{l+3}^{-2n_7} a_{l+1}^{n_8} a_{l+2}^{-2n_8} \\ &= a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_{l+1}^{n_3} a_l^{n_1-2n_2} a_{l+2}^{n_4-2n_3} a_{l+3}^{-2n_4} a_{l-1}^{n_5} a_l^{n_6-2n_5} a_{l+2}^{n_7-2n_8} a_{l+1}^{-2n_6} a_{l+3}^{-2n_7} a_{l+1}^{n_8}. \end{aligned}$$

Then $n_4 = n_5 = 0$ and $u_{10} = a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_{l+1}^{n_3} (a_l^{n_1-2n_2} a_{l+2}^{-2n_3} a_l^{n_6} a_{l+2}^{n_7-2n_8}) a_{l+1}^{-2n_6} a_{l+3}^{-2n_7} a_{l+1}^{n_8}$. Therefore, $-2n_3 = \pm 1$, which contradicts that $n_3 \in \mathbb{Z}$.

$$\begin{aligned} u_{11} &= a_l^{n_1} a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_l^{-2n_2} a_{l+1}^{n_3} a_{l+2}^{n_4-2n_3} a_{l+3}^{-2n_4} a_l^{n_5} a_{l+1}^{-2n_5} a_{l-1}^{n_6} a_l^{-2n_6} a_{l+1}^{n_7} a_{l+2}^{n_8-2n_7} a_{l+3}^{-2n_8} \\ &= a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_{l+1}^{n_3} a_l^{n_1-2n_2} a_{l+2}^{n_4-2n_3} a_{l+3}^{-2n_4} a_{l+1}^{-2n_5} a_{l-1}^{n_6} a_{l+1}^{n_7} a_l^{n_5-2n_6} a_{l+2}^{n_8-2n_7} a_{l+3}^{-2n_8}. \end{aligned}$$

Then $n_4 = n_6 = 0$ and $u_{11} = a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_{l+1}^{n_3} (a_l^{n_1-2n_2} a_{l+2}^{-2n_3} a_l^{n_5} a_{l+2}^{n_8-2n_7}) a_{l+1}^{n_7-2n_5} a_{l+3}^{-2n_8}$. Therefore, $-2n_3 = \pm 1$, which contradicts that $n_3 \in \mathbb{Z}$.

$$\begin{aligned} u_{12} &= a_l^{n_1} a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_l^{-2n_2} a_{l+1}^{n_3} a_{l+2}^{n_4-2n_3} a_{l+3}^{-2n_4} a_l^{n_5} a_{l+1}^{-2n_5} a_{l-1}^{n_6} a_l^{-2n_6} a_{l+2}^{n_7} a_{l+3}^{-2n_7} a_{l+1}^{n_8} a_{l+2}^{-2n_8} \\ &= a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_{l+1}^{n_3} a_l^{n_1-2n_2} a_{l+2}^{n_4-2n_3} a_{l+3}^{-2n_4} a_{l+1}^{-2n_5} a_{l-1}^{n_6} a_l^{n_5-2n_6} a_{l+2}^{n_7-2n_8} a_{l+3}^{-2n_7} a_{l+1}^{n_8}. \end{aligned}$$

Then $n_4 = n_6 = 0$ and $u_{12} = a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_{l+1}^{n_3} (a_l^{n_1-2n_2} a_{l+2}^{-2n_3} a_l^{n_5} a_{l+2}^{n_7-2n_8}) a_{l+1}^{-2n_5} a_{l+3}^{-2n_7} a_{l+1}^{n_8}$. Therefore, $-2n_3 = \pm 1$, which contradicts that $n_3 \in \mathbb{Z}$.

$$\begin{aligned} u_{13} &= a_l^{n_1} a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_l^{-2n_2} a_{l+2}^{n_3} a_{l+3}^{-2n_3} a_{l+1}^{n_4} a_{l+2}^{-2n_4} a_{l-1}^{n_5} a_l^{n_6-2n_5} a_{l+1}^{n_7-2n_6} a_{l+2}^{n_8-2n_7} a_{l+3}^{-2n_8} \\ &= a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_l^{n_1-2n_2} a_{l+2}^{n_3-2n_4} a_{l+3}^{-2n_3} a_{l+1}^{n_4} a_{l-1}^{n_5} a_l^{n_6-2n_5} a_{l+2}^{n_8-2n_7} a_{l+1}^{n_7-2n_6} a_{l+3}^{-2n_8}. \end{aligned}$$

Then $n_3 = n_5 = 0$ and $u_{13} = a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_{l+1}^{n_4} (a_l^{n_1-2n_2} a_{l+2}^{-2n_4} a_l^{n_6} a_{l+2}^{n_8-2n_7}) a_{l+1}^{n_7-2n_6} a_{l+3}^{-2n_8}$. Therefore, $-2n_4 = \pm 1$, which contradicts that $n_4 \in \mathbb{Z}$.

$$\begin{aligned} u_{14} &= a_l^{n_1} a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_l^{-2n_2} a_{l+2}^{n_3} a_{l+3}^{-2n_3} a_{l+1}^{n_4} a_{l+2}^{-2n_4} a_{l-1}^{n_5} a_l^{n_6-2n_5} a_{l+1}^{-2n_6} a_{l+2}^{n_7} a_{l+3}^{-2n_7} a_{l+1}^{n_8} a_{l+2}^{-2n_8} \\ &= a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_l^{n_1-2n_2} a_{l+2}^{n_3-2n_4} a_{l+3}^{-2n_3} a_{l+1}^{n_4} a_{l-1}^{n_5} a_l^{n_6-2n_5} a_{l+2}^{n_7-2n_8} a_{l+1}^{-2n_6} a_{l+3}^{-2n_7} a_{l+1}^{n_8}. \end{aligned}$$

Then $n_3 = n_5 = 0$ and $u_{14} = a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_{l+1}^{n_4} (a_l^{n_1-2n_2} a_{l+2}^{-2n_4} a_l^{n_6} a_{l+2}^{n_7-2n_8}) a_{l+1}^{-2n_6} a_{l+3}^{-2n_7} a_{l+1}^{n_8}$. Therefore, $-2n_4 = \pm 1$, which contradicts that $n_4 \in \mathbb{Z}$.

$$\begin{aligned} u_{15} &= a_l^{n_1} a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_l^{-2n_2} a_{l+2}^{n_3} a_{l+3}^{-2n_3} a_{l+1}^{n_4} a_{l+2}^{-2n_4} a_l^{n_5} a_{l+1}^{-2n_5} a_{l-1}^{n_6} a_l^{-2n_6} a_{l+1}^{n_7} a_{l+2}^{n_8-2n_7} a_{l+3}^{-2n_8} \\ &= a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_l^{n_1-2n_2} a_{l+2}^{n_3-2n_4} a_{l+3}^{-2n_3} a_{l+1}^{n_4-2n_5} a_{l-1}^{n_6} a_l^{n_5-2n_6} a_{l+2}^{n_8-2n_7} a_{l+1}^{n_7} a_{l+3}^{-2n_8}. \end{aligned}$$

Then $n_3 = n_6 = 0$ and $u_{15} = a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_{l+1}^{n_4} (a_l^{n_1-2n_2} a_{l+2}^{-2n_4} a_l^{n_5} a_{l+2}^{n_8-2n_7}) a_{l+1}^{n_7-2n_5} a_{l+3}^{-2n_8}$. Therefore, $-2n_4 = \pm 1$, which contradicts that $n_4 \in \mathbb{Z}$.

Finally, $$\begin{aligned} u_{16} &= a_l^{n_1} a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_l^{-2n_2} a_{l+2}^{n_3} a_{l+3}^{-2n_3} a_{l+1}^{n_4} a_{l+2}^{-2n_4} a_l^{n_5} a_{l+1}^{-2n_5} a_{l-1}^{n_6} a_l^{-2n_6} a_{l+2}^{n_7} a_{l+3}^{-2n_7} a_{l+1}^{n_8} a_{l+2}^{-2n_8} \\ &= a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_l^{n_1-2n_2} a_{l+2}^{n_3-2n_4} a_{l+3}^{-2n_3} a_{l+1}^{n_4-2n_5} a_{l-1}^{n_6} a_l^{n_5-2n_6} a_{l+2}^{n_7-2n_8} a_{l+3}^{-2n_7} a_{l+1}^{n_8}. \end{aligned}$$

Then $n_3 = n_6 = 0$ and $u_{16} = a_{l+1}^{-2n_1} a_{l-1}^{n_2} a_{l+1}^{n_4} (a_l^{n_1-2n_2} a_{l+2}^{-2n_4} a_l^{n_5} a_{l+2}^{n_7-2n_8}) a_{l+1}^{-2n_5} a_{l+3}^{-2n_7} a_{l+1}^{n_8}$. Therefore, $-2n_4 = \pm 1$, which contradicts that $n_4 \in \mathbb{Z}$.

So we conclude that, no one of $(a_i, a_{i+2}, a_i^{-1}, a_{i+2}^{-1})$ and $(a_i^{-1}, a_{i+2}^{-1}, a_i, a_{i+2})$ is a partial sequence of a reduced representation W_h of h.

Equivalently, we consider the other two sequences $(a_{i+2}, a_i^{-1}, a_{i+2}^{-1}, a_i)$ and $(a_{i+2}^{-1}, a_i, a_{i+2}, a_i^{-1})$. Similarly, either $(a_{i+2}, a_i^{-1}, a_{i+2}^{-1}, a_i)$ or $(a_{i+2}^{-1}, a_i, a_{i+2}, a_i^{-1})$ is a partial sequence of W_h if and only if there exist $h_1, h_2, u \in H_1$ such that $h = h_1 u h_2$, where u is one of 16 possibility. Each one of them leads to a contradiction.

For example, if $u = s_3^{n_1} s_4^{n_2} s_1^{n_3} s_2^{n_4} s_3^{n_5} s_4^{n_6} s_1^{n_7} s_2^{n_8}$, for some $n_i \in \mathbb{Z}$, $i \in \{1, \ldots, 8\}$, then $u = a_{l+1}^{n_1} a_{l+2}^{n_2-2n_1} a_{l+3}^{-2n_2} a_{l-1}^{n_3} a_l^{n_4-2n_3} a_{l+1}^{n_5-2n_4} a_{l+2}^{n_6-2n_5} a_{l+3}^{-2n_6} a_{l-1}^{n_7} a_l^{n_8-2n_7} a_{l+1}^{-2n_8}$

$$= a_{l+1}^{n_1} a_{l+3}^{-2n_2} a_{l+2}^{n_2-2n_1} a_l^{n_4-2n_3} a_{l-1}^{n_3} a_{l+1}^{n_5-2n_4} a_{l+3}^{-2n_6} a_{l+2}^{n_6-2n_5} a_l^{n_8-2n_7} a_{l-1}^{n_7} a_{l+1}^{-2n_8}.$$

Then $n_3 = n_6 = 0$ and $u = a_{l+1}^{n_1} a_{l+3}^{-2n_2} a_{l+1}^{n_5-2n_4} (a_{l+2}^{n_2-2n_1} a_l^{n_4} a_{l+2}^{-2n_5} a_l^{n_8-2n_7}) a_{l-1}^{n_7} a_{l+1}^{-2n_8}$. Therefore, $-2n_5 = \pm 1$, which contradicts that $n_5 \in \mathbb{Z}$.

It follows that, no one of $(a_{l+2}, a_l^{-1}, a_{l+2}^{-1}, a_l)$ and $(a_{l+2}^{-1}, a_l, a_{l+2}, a_l^{-1})$ is a partial sequence of a reduced representation W_h of h.

Thus, $[a_l, a_{l+2}]^{a'}$ does not belong to H_1 for all $a' \in A$, which implies that $[a_0, a_2]^g$ does not belong to H_1 for all $g \in G$. So that the proof of (ii) is complete.

(iii) $[a_0, a_2]^g \notin H$ for all $g \in G$.

Assume to the contrary that there exists an element $h \in H$ such that $h = [a_0, a_2]^g$ for some $g \in G$. Since $H = H_1 \rtimes \langle x \rangle$, there exist $h_1 \in H_1$ and $n \in \mathbb{Z}$ such that $h = x^n h_1$. Therefore, $h = x^n h_1 = [a_0, a_2]^g$, which implies that $n = 0$ because the sum of the powers of x in $[a_0, a_2]^g$ is zero. So $[a_0, a_2]^g \in H_1$, which contradicts (ii).

Thus, the proof of **(1)** is complete.

Now we prove that $[a_0, a_2]$ belongs to every finite index subgroup L in G which contains H, and therefore **(2)** is proved.

Let $C = \cap_{g \in G} L^g$, then C is a finite index normal subgroup in G. Clearly, C is normal in G.

In addition, C has finite index in G because $|G : C| \leqslant (|G : L|)! < \infty$ [5, p. 8, Poincaré Theorem]. It follows further that $C \cap A$ is normal in G and has finite index in A, because C and A are normal in G, and $AC/C \cong A/(C \cap A)$.

Let $N = (C \cap A)H_1$. Since $C, H_1 \leqslant L$, it follows that $N \leqslant L$. Moreover, $N \leqslant A$, since $H_1 = H \cap A$. We prove further that $N \trianglelefteq A$.

Since $C \cap A \trianglelefteq A$ and $H_1 = \langle a_i a_{i+1}^{-2} \rangle$, it remains to prove that $a_j^{\epsilon} a_i a_{i+1}^{-2} a_j^{-\epsilon} \in N$ for all $i, j \in \mathbb{Z}$ and $\epsilon = \pm 1$.

Consider the following two cases of i and j:

Case 1. Let $j \leqslant i$. Since a_i commutes with a_{i+1} for all i, we get

$a_j a_i a_{i+1}^{-2} a_j^{-1} =$

$$(a_j a_{j+1}^{-2})(a_{j+1} a_{j+2}^{-2})^2 \cdots (a_i a_{i+1}^{-2})^{2^n} a_i a_{i+1}^{-2} (a_i a_{i+1}^{-2})^{-2^n} \cdots (a_{j+1} a_{j+2}^{-2})^{-2} (a_j a_{j+1}^{-2})^{-1},$$

where $n = i - j$. Then $a_j a_i a_{i+1}^{-2} a_j^{-1} \in H_1 \leqslant N$.

Similarly $a_j^{-1} a_i a_{i+1}^{-2} a_j \in N$, since $a_j^{-1} a_i a_{i+1}^{-2} a_j =$

$$(a_j a_{j+1}^{-2})^{-1} (a_{j+1} a_{j+2}^{-2})^{-2} \cdots (a_i a_{i+1}^{-2})^{-2^n} a_i a_{i+1}^{-2} (a_i a_{i+1}^{-2})^{2^n} \cdots (a_{j+1} a_{j+2}^{-2})^2 (a_j a_{j+1}^{-2}).$$

Case 2. Let $j > i$. Consider the set $\{a_i \mid i \in \mathbb{Z}\}$. Since $C \cap A$ has finite index in A, there exist two integers $n, m \in \mathbb{Z}$ such that $n \neq m$ and $a_m a_n^{-1} \in C \cap A$.

Without loss of generality we can assume that $n - m < 0$, then

$$a_0 a_{n-m}^{-1} = x^{-m}(x^m a_0 x^{-m})(x^n a_0^{-1} x^{-n}) x^m = x^{-m} a_m a_n^{-1} x^m \in C \cap A,$$

because $C \cap A$ is normal in G.

Let $n - m = t$, then $x^t a_0 a_t^{-1} x^{-t} = a_t a_{2t}^{-1} \in C \cap A$, and therefore $a_0 a_{2t}^{-1} \in C \cap A$. So inductively $a_0 a_{dt}^{-1} \in C \cap A$ for all $d > 0$. Choose d such that $dt \leqslant i - j$ and set $k = dt + j$, then $k \leqslant i$ such that $a_j a_k^{-1} = x^j a_0 a_{dt}^{-1} x^{-j} \in C \cap A$.

Therefore, $a_j a_i a_{i+1}^{-2} a_j^{-1} = (a_j a_k^{-1})(a_k a_i a_{i+1}^{-2} a_k^{-1})(a_k a_j^{-1}) \in N$. Indeed, since $k \leqslant i$, similar to Case 1, it follows that $(a_k a_i a_{i+1}^{-2} a_k^{-1}) \in H_1$.

Similarly, to prove that $a_j^{-1} a_i a_{i+1}^{-2} a_j \in N$, we take $a_m^{-1} a_n \in C \cap A$, which implies that

$$a_0^{-1} a_t = a_0^{-1} a_{n-m} = x^{-m}(x^m a_0^{-1} x^{-m})(x^n a_0 x^{-n}) x^m = x^{-m} a_m^{-1} a_n x^m \in C \cap A.$$

Since $x^t a_0^{-1} a_t x^{-t} = a_t^{-1} a_{2t} \in C \cap A$, we have $a_0^{-1} a_{2t} \in C \cap A$, and it follows inductively that $a_0^{-1} a_{dt} \in C \cap A$ for all $d > 0$.

So we can find $k' \leqslant i$ such that $a_j^{-1} a_{k'} \in C \cap A$.

Thus $a_j^{-1} a_i a_{i+1}^{-2} a_j = (a_j^{-1} a_{k'})(a_{k'}^{-1} a_i a_{i+1}^{-2} a_{k'})(a_{k'}^{-1} a_j) \in N$. So the proof that $N \trianglelefteq A$ is complete.

Since $a_i a_{i+1}^{-2} \in H_1 \leqslant N$ for all $i \in \mathbb{Z}$, it follows that $a_i N = a_{i+1}^2 N$ for all $i \in \mathbb{Z}$. Using that $N \trianglelefteq A$, it follows that $a_0 N = a_1^2 N = a_2^4 N$.

Therefore, $a_0 a_2 N = a_2^4 a_2 N = a_2 a_2^4 N = a_2 a_0 N$. Thus $[a_0, a_2] \in N \leqslant L$. □

Remark 3.3.2. Let $D = \langle x, y\rangle$ be the free abelian group of rank 2, and $\langle a\rangle$ be the infinite cyclic group, then the group $D \rtimes \langle a\rangle$, where the action of a on D is given by $axa^{-1} = y^{-1}x$, $aya^{-1} = y$, is polycyclic.

Since polycyclic groups are SCS, by [15], therefore, $D \rtimes \langle a\rangle$ is SCS. However, similar to Theorem 3.3.1, it is not an SICS group. Therefore, an SCS group need not be SICS.

Remark 3.3.3. According to D. Segal, oral communication, there exists a nilpotent group G which is SCS, but not SICS (his example is very complicated and based on a solution by Matyasevich of the 10th Hilbert problem).

Remark 3.3.4. The group $G = \langle x, y, a \,|\, axa^{-1} = y^{-1}x,\ aya^{-1} = y\rangle$ can be considered as an HNN extension of F_2 with the automorphism $\phi : F_2 \to F_2$ which is given by $\phi(x) = y^{-1}x$ and $\phi(y) = y$.

Therefore, by Theorem 3.3.1, the class of SICS groups is not closed under the HNN extension.

3.4 An amalgamated product of groups which is not SICS

We do not know an example of an amalgamated product of two SICS groups which is not SICS. It is still an open question whethere the class of SICS groups is closed under the amalgamated product or not.

In the following theorem we adapt the proof given by Gitik and Rips [12] to give an example of a non-SICS group which is the free product of an SICS group and an SCS group with amalgamated cyclic subgroup.

Theorem 3.4.1. *Let D be a free abelian group with the basis $d_1, d_2, \ldots, d_8$. Let $A = D \rtimes \langle a\rangle$, where* $a^{-1}d_1a = d_1d_2$, $a^{-1}d_2a = d_2$, $a^{-1}d_3a = d_4$, $a^{-1}d_4a = d_3$,
$a^{-1}d_5a = d_6$, $a^{-1}d_6a = d_5$, $a^{-1}d_7a = d_7$, $a^{-1}d_8a = d_8$.

Let B be any group containing an element b of infinite order. If B contains an element c such that $cb = bc$ and $c \notin \langle b \rangle$, then $G = A *_{a=b} B$ is not SICS, though A is SCS.

Remark: We do not know whether G is SCS, or not.

Proof. Since A is a polycyclic group, by [15], it follows that A is SCS.

Let $c_1 = cd_1$, $c_2 = d_3d_7cd_3$, $c_3 = d_4cd_4$, $c_4 = d_5d_8cd_5$, and $c_5 = d_6cd_6$. Let $H_1 = \langle c_1, c_2, \ldots, c_5, a^3d_2, ad_7d_8 \rangle$ and $H_2 = \langle a \rangle \leqslant G$. To prove that G is not SICS, we prove that:

(1) H_2 is not conjugate into H_1.

(2) For every subgroup $H \leqslant G$ with $H_1 \leqslant H$, either $H_2 \leqslant H$ or $H \cap H_2 = \{1\}$.

Since $H \cap H_2 = \{1\}$ does not occur for finite index subgroups H in G, the proof of **(2)** implies that $H_2 \leqslant H$ for every finite index subgroup H in G with $H_1 \leqslant H$.

The proof of **(1)** follows from (i), (ii), (iii).

(i) $H_2 \nleqslant H_1$, that is, $a \notin H_1$.

Every element $x \in H_1$ can be written as

$$x = \prod_{i=1}^{t} c_1^{l_{1i}} c_2^{l_{2i}} \cdots c_5^{l_{5i}} (a^3d_2)^{l_{6i}} (ad_7d_8)^{l_{7i}}$$

where $l_{1i}, l_{2i}, \ldots, l_{7i} \in \mathbb{Z}$, $t \in \mathbb{N}$.

If $l_{6i} = l_{7i} = 0$ for all $i = 1, \ldots, t$, then $x \neq a$. If $l_{1i} = l_{2i} = \cdots = l_{5i} = 0$ for all $i = 1, \ldots, t$, then $x = (a^3d_2)^k(ad_7d_8)^l = a^{3k+l}d_2^k d_7^l d_8^l$ for some $k, l \in \mathbb{Z}$ and thus $x \neq a$ for all $k, l \in \mathbb{Z}$.

So it is enough to prove that
$y = u(a^3d_2)^k(ad_7d_8)^l v = ua^{3k+l}d_2^k d_7^l d_8^l v \in \langle a \rangle$ if and only if $y = 1$, where
$k, l \in \mathbb{Z}$, $u \in L = \{1, d_1, d_3, d_3^{-1}d_7^{-1}, d_4, d_4^{-1}, d_5, d_5^{-1}d_8^{-1}, d_6, d_6^{-1}\}$, and
$v \in R = \{1, d_1^{-1}, d_3^{-1}, d_3d_7, d_4, d_4^{-1}, d_5^{-1}, d_5d_8, d_6, d_6^{-1}\}$.

Note that a, d_2, d_7, d_8 commute and $d_1 a d_1^{-1} = a d_2$.

It is easy to check that all these cases lead to $y \in \langle a \rangle \Leftrightarrow y = 1$.

For instance, let $u = d_1$ and $v = 1$, then

$$y = d_1 a^{3k+l} d_2^k d_7^l d_8^l = a^{3k+l} d_1 d_2^{4k+l} d_7^l d_8^l \notin \langle a \rangle$$

for all $k, l \in \mathbb{Z}$.

Let $u = d_1$ and $v = d_1^{-1}$, then

$$y = d_1 a^{3k+l} d_1^{-1} d_2^k d_7^l d_8^l = a^{3k+l} d_2^{4k+l} d_7^l d_8^l \in \langle a \rangle \Leftrightarrow y = 1.$$

The complete list of a similar y could be found in [12].

(ii) $H_2^g \nleq \langle c_1, c_2, \ldots, c_5 \rangle$ for all $g \in G$. Indeed, every element $x \in H_2^g$ can be written as $x = g a^t g^{-1}$ for some $t \in \mathbb{Z}$. The cyclic length of x is 1, although the cyclic length of every non-trivial element $y \in \langle c_1, c_2, \ldots, c_5 \rangle$ is greater than 1.

(iii) $gag^{-1} \neq (a^3 d_2)^k (a d_7 d_8)^l$ for all $g \in G$ and $l, k \in \mathbb{Z}$.

Case 1: Let $g \in B$. If $gag^{-1} \in B \setminus \langle b \rangle$, then $gag^{-1} \notin A$, and therefore $gag^{-1} \neq (a^3 d_2)^k (a d_7 d_8)^l$. If $gag^{-1} \in \langle b \rangle$, then $gag^{-1} = a^j$ for some $j \in \mathbb{Z} \setminus \{0\}$.

Since $(a^3 d_2)^k (a d_7 d_8)^l \in \langle a \rangle \Leftrightarrow (a^3 d_2)^k (a d_7 d_8)^l = 1$, and $gag^{-1} \neq 1$, it follows that $gag^{-1} \neq (a^3 d_2)^k (a d_7 d_8)^l$.

Case 2: Let $g \in A$. If $gag^{-1} \in \langle a \rangle$, then $gag^{-1} \neq (a^3 d_2)^k (a d_7 d_8)^l$, because $(a^3 d_2)^k (a d_7 d_8)^l \in \langle a \rangle \Leftrightarrow (a^3 d_2)^k (a d_7 d_8)^l = 1$.

If $gag^{-1} \in A \setminus \langle a \rangle$, then $g = d a^s$ for some $s \in \mathbb{Z}$ and $d \neq 1$ in D, and so that $gag^{-1} = dad^{-1} = aa^{-1}dad^{-1}$.

If $aa^{-1}dad^{-1} = gag^{-1} = (a^3 d_2)^k (a d_7 d_8)^l = a^{3k+l} d_2^k d_7^l d_8^l$ for some $l, k \in \mathbb{Z}$, then $3k + l = 1$ and $a^{-1}dad^{-1} = d_2^k d_7^l d_8^l$.

Let $d = d_1^{t_1} d_2^{t_2} \cdots d_8^{t_8}$ then $a^{-1}dad^{-1} = d_2^{t_1} d_3^{t_4 - t_3} d_4^{t_3 - t_4} d_5^{t_6 - t_5} d_6^{t_5 - t_6}$ for some t_i in $\mathbb{Z}$, $i \in \{1, \ldots, 8\}$. Therefore $a^{-1}dad^{-1} = d_2^k d_7^l d_8^l$ implies that $l = 0, 3k = 1$, a contradiction.

Thus $gag^{-1} \neq (a^3 d_2)^k (a d_7 d_8)^l$ for all $g \in G$ and $l, k \in \mathbb{Z}$.

Case 3: Let $g \notin A$ and $g \notin B$ such that $gag^{-1} = (a^3d_2)^k(ad_7d_8)^l$ for some $l, k \in \mathbb{Z}$. Choose a representation $W_g = \prod_{i=1}^{t} g_i$ of $g \in A *_{a=b} B$ with minimal length satisfies that $gag^{-1} = (a^3d_2)^k(ad_7d_8)^l$, where $g_i \in A \cup B \setminus \langle a \rangle$ for all $i = 1, \dots, t,\ t \geq 2$ and W_g is an alternative product of elements from A and B.

Without loss of generality we can assume that $g_t \in B$. If $g_tag_t^{-1} \in B \setminus \langle b \rangle$ then the length of gag^{-1} is greater than 1 and thus $gag^{-1} \neq (a^3d_2)^k(ad_7d_8)^l$ for all $l, k \in \mathbb{Z}$.

Let $g_tag_t^{-1} \in \langle b \rangle$, then $g_tag_t^{-1} = a^j$ for some $j \in \mathbb{Z} \setminus \{0\}$, and therefore $g_{t-1}a^jg_{t-1}^{-1} \in A$. Then $g_{t-1}a^jg_{t-1}^{-1} \in \langle a \rangle$ or $\in A \setminus \langle a \rangle$. If $g_{t-1}a^jg_{t-1}^{-1} \in \langle a \rangle$, we claim that $g_{t-1}a^jg_{t-1}^{-1} = a^j$ and then $g_tag_t^{-1} = g_{t-1}g_tag_t^{-1}g_{t-1}^{-1}$, which contradicts the minimality of the length of W_g.

Now we prove our claim. Since $g_{t-1} \in A \setminus \langle a \rangle$, we assume that $g_{t-1} = da^s$ for some $s \in \mathbb{Z}$ and $d \neq 1$ in D, then $g_{t-1}a^jg_{t-1}^{-1} = da^jd^{-1} = a^ja^{-j}da^jd^{-1}$. Since $a^{-j}da^jd^{-1} \in D$ and $a^ja^{-j}da^jd^{-1} \in \langle a \rangle$, we get $a^{-j}da^jd^{-1} = 1$, and therefore $g_{t-1}a^jg_{t-1}^{-1} = a^j$.

If $g_{t-1}a^jg_{t-1}^{-1} \in A \setminus \langle a \rangle$ and $t > 2$, then the length of gag^{-1} is greater than 1 and thus $gag^{-1} \neq (a^3d_2)^k(ad_7d_8)^l$ for all $l, k \in \mathbb{Z}$.

If $g_{t-1}a^jg_{t-1}^{-1} \in A \setminus \langle a \rangle$ and $t = 2$, then $gag^{-1} = g_{t-1}a^jg_{t-1}^{-1}$, where $g_{t-1} \in A \setminus \langle a \rangle$. Let $g_{t-1} = da^s$ for some $s \in \mathbb{Z}$ and $d \neq 1$ in D, then $gag^{-1} = da^jd^{-1} = a^ja^{-j}da^jd^{-1}$.

If $gag^{-1} = (a^3d_2)^k(ad_7d_8)^l$, then $a^ja^{-j}da^jd^{-1} = a^{3k+l}d_2^kd_7^ld_8^l$ for some $l, k \in \mathbb{Z}$, and therefore $3k + l = j$ and $a^{-j}da^jd^{-1} = d_2^kd_7^ld_8^l$. Let $d = d_1^{t_1}d_2^{t_2} \cdots d_8^{t_8}$ for some $t_i \in \mathbb{Z},\ i \in \{1, \dots, 8\}$, then $a^{-j}da^jd^{-1} = d_2^{jt_1}d_3^{t_4-t_3}d_4^{t_3-t_4}d_5^{t_6-t_5}d_6^{t_5-t_6}$, where j is odd, or $a^{-j}da^jd^{-1} = d_2^{jt_1}$, where j is even.

Therefore $a^{-j}da^jd^{-1} = d_2^kd_7^ld_8^l$ implies that $l = 0, k = jt_1$, and then $3t_1 = 1$, a contradiction. Thus $gag^{-1} \neq (a^3d_2)^k(ad_7d_8)^l$ for all $g \in G$ and $l, k \in \mathbb{Z}$.

$gag^{-1} = (\prod_{i=1}^{t-2} a_ib_i)(a_{t-1}a_taa_t^{-1}a_{t-1}^{-1})(\prod_{i=1}^{t-2} a_ib_i)^{-1}$.

Now we prove **(2)**, let $H \leqslant G$ such that $H_1 \leqslant H$, and $H \cap H_2 \neq \{1\}$, we want to prove that $H_2 \leqslant H$. Since $H \cap H_2 \neq \{1\}$, assume that $a^n \in H$ for some

$0 \neq n \in \mathbb{N}$.

If $n = 2m$, then

$$(cd_1)a^{2n}(a^3d_2)^{-m}(cd_1)^{-1} = c(d_1a^md_2^{-m}d_1^{-1})c^{-1} = c(d_1a^md_1^{-1})d_2^{-m}c^{-1} = ca^mc^{-1} = a^m.$$

If n is odd then $a^{-n}(d_7d_3cd_3)a^n(d_4cd_4)^{-1} = d_7(d_4cd_4)(d_4cd_4)^{-1} = d_7 \in H$, and $a^{-n}(d_8d_5cd_5)a^n(d_6cd_6)^{-1} = d_8(d_6cd_6)(d_6cd_6)^{-1} = d_8 \in H$.

Thus $a \in H$, and therefore $H_2 \leqslant H$. □

Remark 3.4.2. If B contains an element c_0 such that $bc_0 = c_0b^{-1}$, take

$c_1 = c_0d_1$, $c_2 = d_3d_7c_0d_3c_0^{-1}d_3$, $c_3 = d_4c_0d_4c_0^{-1}d_4$,

$c_4 = d_5d_8c_0d_5c_0^{-1}d_5$, $c_5 = d_6c_0d_6c_0^{-1}d_6$.

Let $H_1 = \langle c_1, c_2, \ldots, c_5, a^3d_2, ad_7d_8\rangle$, $H_2 = \langle a\rangle \leqslant G$. We similarly show that $G = A *_{a=b} B$ is not SICS.

Corollary 3.4.3. Let B be an SICS group and $b \in B$ be of infinite order. If for any SCS group A and $a \in A$ of infinite order $A *_{a=b} B$ is SICS then $N_B(\langle b\rangle) = \langle b\rangle$.

Appendix A

In this appendix we introduce some open problems which we want to solve in the next night.

1. For each pair (A, B) of classes from the set (CS, LERF, SCS, and SICS) decide whether A lies in B or not.
2. Is the direct, semidirect, or free product of two SCS groups again SCS?
3. We know that for $n \geq 2$ the group $BS(1, n)$ is RF but not LERF (π_C). Is $BS(n, n)$ LERF (π_C)?
4. Are $BS(1, n)$ and $BS(n, n)$ SCS and/or SICS?
5. We know that limit groups are LERF. Are they SCS and/or SICS?
6. Is $F_1 \times F_m$ SICS?
7. Is $F_n \times F_m$ SCS?
8. Is $F_n *_{\mathbb{Z}} F_m$ SCS and/or SICS?
9. If A and B are two nilpotent groups, is $A *_{\mathbb{Z}} B$ SCS and/or SICS?
10. If A and B are two SCS and/or SICS groups, and X is a finite subgroup of A and B, is $A *_X B$ SCS and/or SICS?
11. If A and B are two SCS and/or SICS groups, and X is a malnormal subgroup of A and B, is $A *_X B$ SCS and/or SICS?

12. If G is a finite extension of a group H, and H is SCS and/or SICS, is G SCS and/or SICS?

13. Is every subgroup of an SCS and/or SICS group again SCS and/or SICS?

14. If A is an Artin group, is A LERF, SCS and/or SICS?

15. What are the necessary and sufficient conditions for the right-angled Artin group to be SCS and/or SICS?

16. Let A be an SCS and/or SICS group. What are the necessary and sufficient conditions for an HNN extension of A to be SCS and/or SICS?

17. Let A and B be two LERF, SCS, and/or SICS, under which conditions $A \wr B$ is LERF, SCS, and/or SICS? In particular, we know that Lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ is RF and CS, is it LERF, SCS, and/or SICS?

18. **Definition:** (a) A group G is called *subgroup free separable* (SFS) if for every two finitely generated subgroups $H_1 \neq H_2$, there exists a homomorphism ϕ from G to a free group F such that $\phi(H_1) \neq \phi(H_2)$.

    (b) A group G is called *subgroup conjugacy free separable* (SCFS) if for every two finitely generated subgroups H_1 and H_2 such that H_2 is not conjugate to H_1 in G , there exists a homomorphism ϕ from G to a free group F such that $\phi(H_1)$ is not conjugate to $\phi(H_2)$ in F.

    (c) A group G is called *subgroup into-conjugacy free separable* (SICFS) if for every two finitely generated subgroups H_1 and H_2 such that H_2 is not conjugate into H_1 in G , there exists a homomorphism ϕ from G to a free group F such that $\phi(H_1)$ is not conjugate into $\phi(H_2)$ in F.

    It is easy to see that every SFS is LERF, every SCFS is SCS, and every SICFS is SICS, since every free group is LERF, SCS, and SICS. Which interesting classes of groups are SFS, SCFS, and/or SICFS?

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