Possible Worlds Semantics for Indicative and Counterfactual Conditionals? A Formal-Philosophical Inquiry into Chellas-Segerberg Semantics

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Abstract

Conditional logic is a sub-discipline of philosophical logic. It aims to provide an alternative account of conditionals in contrast to the traditional material implication analysis. The present thesis focuses on a specific possible worlds semantics for conditional logics, the Chellas-Segerberg (CS) semantics (Chellas, 1975; Segerberg, 1989), which has not been widely investigated, save by Nejdl (1992) and Delgrande (1987, 1988).

The main thesis of this dissertation is that CS-semantics is an adequate framework for both (i) indicative and (ii) counterfactual conditionals. To argue for (i) and (ii) we, first, discuss the general need of a conditional logic approach, which goes beyond a material implication analysis. We address the difference between indicative and counterfactual conditionals. We focus, then, on two arguments brought forward by Bennett (2003) against accounts of indicative conditionals in terms of truth and falsehood, which arguably include possible worlds semantics such as CS-semantics: (a) D. Lewis' (1976) triviality result and (b) Bennett's (2003) Gibbardian stand-off argument, which goes back to Gibbard (1980).

We, furthermore, investigate a lattice of conditional logics based on the basic proof theoretic system for CS-semantics plus 29 further axioms. This framework allows us to describe – as we shall show – a range of conditional logic systems, such as the indicative conditional logic system by Kraus, Lehmann, and Magidor (1990) and Lehmann and Magidor (1992) and the counterfactual system of D. Lewis (1973/2001). For our formal investigation we distinguish between Chellas frames (Chellas, 1975) on the one hand and Segerberg frames (Segerberg, 1989) on the other hand: While Chellas frames are generalizations of Kripke frames, Segerberg frames rather correspond to what is often called 'general frames'. We give, then, correspondence proofs for the lattice of systems on the basis of Chellas frames and discuss the notion of trivial frame correspondence. We, then, provide a completeness result for the lattice of conditional logics for standard Segerberg frames. This type of Segerberg frames is solely based on structural conditions and is – unlike the notion of (simple) Segerberg frame completeness – not trivial in the sense that any conditional logic is complete w.r.t. some class of frames.

We finally, provide an objective and a subjective interpretation of CS-semantics by drawing on the notion of alethic modality and the Ramsey-test, respectively. We, then, argue that our objective and subjective account of CS-semantics can serve as basis for indicative and counterfactual conditionals, respectively.

Contents

Preface

In this thesis we investigate possible worlds semantics for both indicative and counterfactual conditional logics. We will do this both from a formal and a philosophical perspective. In particular, we aim to show that a specific type of possible worlds semantics, namely Chellas-Segerberg (CS) semantics (see Chellas, 1975; Segerberg, 1989), is a philosophically plausible and technically viable semantics for conditionals. To show this, we discuss a range of topics.

First, we argue for the general need of a conditional logic project. An alternative treatment of conditionals – in contrast to the material implication approach in propositional logic (p.c.) or first-order logic (f.o.l.) – was already suggested by C. I. Lewis (1912; cf. Hughes & Cresswell, 1996/2003, p. 194f) and are also advocated in relevance logic approaches (e.g. Weingartner & Schurz, 1986, p. 10f). Although there is strong agreement among experts that the conditional logic project – which provides an analysis of conditionals, which goes beyond the material implication analysis – allows for a better understanding of the formal and philosophical underpinnings of scientific and everyday conditionals, this is less obvious for non-experts. So, we were often asked by non-experts what the point of a conditional logic is, since the material implication would allow for an adequate account of conditionals in natural language. In Chapter 1 we, thus, describe some arguments, which aim to show that the material implication analysis of conditionals in p.c. and f.o.l. does not suffice and that, hence, a conditional logic approach is needed, which goes beyond p.c. or f.o.l.

Second, we describe some interdisciplinary ramifications of the conditional logic approach (see Chapter 2), such as linguistics, philosophy of conditionals, psychology of reasoning and non-monotonic reasoning. This is done for two reasons: (i) We aim to show that a conditional logic approach can be fruitfully applied to a range of disciplines. (ii) We describe the interrelation between conditional logics and default logics, which are both discussed in the non-monotonic reasoning literature. Although both approaches deal with non-monotonicity, they have clearly distinct motivations and represent distinct formal approaches. We contrast both approaches to allow for a sound discussion of conditional default logics (Adams, 1975; Schurz, 1997b) and pure conditional logics in the non-monotonic literature (e.g. Kraus et al., 1990; Lehmann & Magidor, 1992). Note that this topic has been largely ignored in the philosophical literature on conditionals, such as by Bennett (2003).

Third, in Chapter 3 we (i) describe important probabilistic semantics (e.g. Adams, 1965, 1966, 1977, 1975; Schurz, 1997b; Adams, 1986; Schurz, 1998, 2005) and possible worlds semantics (e.g. D. Lewis, 1973/2001; Stalnaker, 1968; Stalnaker & Thomason, 1970) for conditional logics, (ii) discuss the difference between conditional logics for indicative and counterfactual conditionals and (iii) provide on that basis a defense of possible worlds semantics for indicative conditionals against criticism by Bennett (2003). We will, for that purpose focus on two arguments by Bennett (2003) against accounts of indicative conditionals in terms of truth and falsehood (short: truth-value accounts): (a) D. Lewis' (1976) triviality result and (b) Bennett's Gibbardian stand-off argument (which is based on Gibbard, 1980, p 231f). We will in particular investigate whether (a) and (b) are decisive against possible worlds semantics such as CS-semantics, which draw on the notion of truth and falsehood in an essential way.

Finally, we argue for a positive account of indicative and counterfactual conditionals in terms of a specific possible worlds semantics, namely CS-semantics. This semantics was first investigated by Chellas (1975) and Segerberg (1989) and has, except for Nejdl (1992) and Delgrande (1987, 1988), largely been ignored in the conditional logic literature. We go beyond the existing literature insofar as we argue that CS-semantics can serve as basis for both indicative and counterfactual conditionals. For this purpose we proceed in two steps: First, we represent and discuss 29 principles from the conditional logic literature and provide soundness, completeness and correspondences results for a lattice of conditional logic systems resulting from these principles and the basic CS-system (see Chapters 4–5). While soundness and correspondence proofs are based on Chellas models as described in Chellas (1975), the completeness result relies on the notion of Segerberg models (see Segerberg, 1989). This is due to the fact that CS-semantics deviates from Kripke semantics (Hughes & Cresswell, 1996/2003, p. 38) and its multi-modal extensions (see Blackburn, de Rijke, & Venema, 2001, p. 20; Schurz, 1997a, p. 166f; Gabbay, Kurucz, Wolter, & Zakharyaschev, 2003, p. 21) in essential ways. We extend Chellas' and Segerberg's results insofar as we, first, define the notion of non-trivial frame conditions. Second, we provide a translation from conditional logic formulas α to trivial frame conditions C_α and prove a general frame correspondence result for formulas α and their translations C_α. Third, we discuss singleton frames in CS-semantics in order to provide a better understanding of types of trivial Chellas and Segerberg models. Fourth, we identify non-trivial frame conditions for 21 further principles than given by Chellas and Segerberg and provide soundness, canonicity and correspondence proofs for all 29 principles. Note that Segerberg (1989, p. 163) does not give proofs for his canonicity results explicitly, but instead appeals to the reader's intuitions for the 9 principles he investigated. However, in order to provide a strong basis for CS-semantics, we think it is worthwhile to draw out these proofs in some detail, the more since this type of semantics is by far more complex than standard Kripke semantics.

Finally, in Chapter 7 we provide a positive argument for the plausibility and usefulness of CS-semantics for both indicative and counterfactual conditionals. For that purpose we give both (a) an objective interpretation of CS-semantics in line with alethic interpretations of Kripke-semantics and (b) a subjective interpretation by means of a modified Ramsey-test criterion. Most plausibly (a) and (b) serve as basis for the interpretation of CS-semantics in terms of indicative conditional logics and counterfactual logics, respectively. We, furthermore, show that a range of indicative and counterfactual conditional logics, such as **C**, **CL**, **P** (Kraus et al., 1990), **R** (Lehmann & Magidor, 1992), **V**, **VC** (D. Lewis, 1973/2001), **P***(Adams, 1965, 1966, 1977) and **S** (Stalnaker, 1968; Stalnaker & Thomason, 1970) can be described by means of CS-semantics. In addition, we discuss the monotonic system **CM**, **M** (Kraus et al., 1990) and the system **MC**, for which the conditional formulas collapse with the material implication, and provide representation results in terms of frame conditions for systems **M** and **MC**. Note that we use for that purpose a full conditional logic language (as described in Section 4.2.1), which allows for the formulation of bridge principles and nestings of conditionals (see Section 4.2.1). This is not the case for the systems described by Kraus et al. (1990), Lehmann and Magidor (1992), Adams (1965, 1966, 1977, 1975, 1986) and Schurz (1997b, 1998, 2005). We conclude this investigation by a list of further issues that might be worth being pursued and summarize the advantages of CS-semantics over existing semantics for indicative and counterfactual conditional logics (see Chapter 8).

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Part I

Foundational Issues

Chapter 1

An Argument for a Conditional Logic

Traditionally, conditionals and conditional structures are analyzed in terms of propositional calculus (p.c.) or first-order logic (f.o.l.) by means of the material implication ('$\rightarrow$'). Conditionals of the form 'if p then q' are represented by formulas, such as '$p \rightarrow q$'. For example, sentences like 'if the weather is nice, Thomas goes for a walk' are represented as '$p \rightarrow q$', where 'p' and 'q' stand for 'the weather is nice' and 'Thomas goes for a walk', respectively. Suppose we want to find out whether the sentence 'Thomas goes for a walk' follows from the sentences 'the weather is nice' and 'if the weather is nice Thomas goes for a walk'. The standard procedure for this problem is the following: Represent the sentences in p.c. or f.o.l. and accept the natural language inference as admissible if the formal translation is an admissible inference in p.c. or f.o.l. Since the above sentences can be adequately represented by formulas of the form q, p and $p \rightarrow q$, respectively, and the formulas p and $p \rightarrow q$ imply q in p.c., we accept the above inference as being admissible in English.

1.1 Counterexamples to a Material Implication Analysis of Conditionals

If we take this commonplace approach and use the material implication to analyze natural language conditionals, the logical properties of the material implication are imposed on conditionals in natural language. There are, however, cases, in which this is clearly counter-intuitive, such as the following (examples E1-E4 are essentially taken from Adams, 1965, p. 166):

E1 John will arrive on the 10 o'clock plane. Therefore, if John will not arrive on the 10 o'clock plane, he will arrive on the 11 o'clock plane.

E2 John will arrive on the 10 o'clock plane. Therefore, if John misses his plane in New York, he will arrive on the 10 o'clock plane.

E3 If Brown wins the election, Smith will retire to private life. If Smith dies before the election, Brown will win it. Therefore, if Smith dies before the election, then he will retire to private life.

E4 If Brown wins the election, Smith will retire to private life. Therefore, if Smith dies before the election and Brown wins it, Smith will retire to private live.

E5 If Andrea wins the lottery, she will donate 500.000 $ to UNICEF. Therefore, if Andrea does not donate 500.000 $ to UNICEF, then she will not win the lottery.

Inferences of type E1 and E2 are often called 'fallacies of material implication' (Adams, 1975, p. 4 and p. 11; cf. also Weingartner & Schurz, 1986, p. 10f). Strategies to avoid making inferences E1 and E2 were already investigated by C. I. Lewis (1912, p. 529). Despite their counter-intutiveness, inferences E1–E5 are valid according to the standard account of conditionals in p.c. In order to see that more clearly, let us formalize examples such as E1-E5 in terms of inferences (i.e. 'if α then β') rather than in terms of formulas (i.e. '$\alpha \rightarrow \beta$'). The expressions α and β stand, then, for arbitrary formulas and expressions like 'if ... then ...' for an inference relation between sets of formulas Γ and a formula α (formally: $\Gamma \vdash \alpha$; cf. Section 4.2.2). For the sake of simplicity we abbreviate inferences with a finite number of premises, such as '$\{\alpha_1, \alpha_2\} \vdash \beta$' by expression, such as 'if

α_1 and α_2, then β'. We, hence, regard 'therefore' as a conclusion indicator rather than a conditional connective. Let us now translate E1–E5 into formal inferences, where conditionals are represented by the material implication ('$\rightarrow$'):

S1	If p then $\neg p \rightarrow q$	(Ex Falso Quodlibet)
S2	If q then $p \rightarrow q$	(Verum Ex Quodlibet)
S3	If $p \rightarrow q$ and $q \rightarrow r$ then $p \rightarrow r$	(Transitivity)
S4	If $p \rightarrow r$ then $p \wedge q \rightarrow r$	(Monotonicity)
S5	If $p \rightarrow q$ then $\neg q \rightarrow \neg p$	(Contraposition)

The formal inferences S1–S5 correspond to E1–E5, respectively. In inferences S1–S5 we use in addition to the material implication '$\rightarrow$' the connectives '$\neg$' ("Negation") and '$\wedge$' ("Conjunction"), which can be read 'not' and 'and', respectively. We shall, henceforth, also employ the connective $\vee$ ("Disjunction", read 'or'). We, furthermore, assume that the connective '$\neg$' binds strongest, while both '$\wedge$' and '$\vee$' bind stronger than '$\rightarrow$' (for a more detailed description see Section 4.2.1).

As observed above, inference schemas S1–S5 are all p.c.-valid. Since S1–S5 are formalizations of inferences E1–E5 respectively, the above criterion gives us that we should regard E1–E5 as valid inferences. Intuitively, however, E1–E5 seem hardly acceptable.

In order to provide the basis for a general discussion, which does not presuppose a material implication analysis of conditionals, we represent E1–E5 in a more neutral way. For that purpose we employ the two-place conditional connective '$\Box\!\!\rightarrow$'. This conditional connective can – but need not – be specified to have logical properties of the material implication. This symbol is used in D. Lewis (1973/2001) for representing counterfactual conditionals. We, however, use the symbol for various types of conditionals, such as counterfactuals and indicative conditionals. In this way we can attribute logical properties to different types of conditionals without presupposing that the properties of the material implication hold for these conditionals. Let us, accordingly, reformulate S1–5 by using the conditional connective '$\Box\!\!\rightarrow$':[1]

[1]Here $\alpha, \beta, \ldots$ stand for arbitrary formulas, while $p, q, \ldots$ are restricted to atomic formulas.

S1′	If α then $\neg\alpha \mathrel{\Box\!\!\rightarrow} \beta$	(Ex Falso Quodlibet, EFQ)
S2′	If β then $\alpha \mathrel{\Box\!\!\rightarrow} \beta$	(Verum Ex Quodlibet, VEQ)
S3′	If $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ and $\beta \mathrel{\Box\!\!\rightarrow} \gamma$ then $\alpha \mathrel{\Box\!\!\rightarrow} \gamma$	(Transitivity)
S4′	If $\alpha \mathrel{\Box\!\!\rightarrow} \gamma$ then $\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$	(Monotonicity)
S5′	If $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ then $\neg\beta \mathrel{\Box\!\!\rightarrow} \neg\alpha$	(Contraposition)

C. I. Lewis (1912) already investigates – as we mentioned above – inferences S1 and S2. His main motivation was to find a logic of conditionals, which is more in accord with ordinary inferences than the standard p.c. account in terms of the material implication C. I. Lewis (1912, p. 522). For that purpose, D. Lewis focused on inferences S1 and S2 rather than S3-S5 (cf. C. I. Lewis, 1912, p. 528f; Hughes & Cresswell, 1996/2003, p. 194f). S1 and S2 differ from inferences of type S3–S5 insofar, as S1 and S2 but not S3–S5 represent bridge principles, viz. principles, which specify a fixed logical relationship between conditional and non-conditional formulas (formulas without conditional operator; cf. Section 4.2.1). C. I. Lewis (1912, p. 528f), however, rejects this property of the conditional: (a) A false proposition does not "imply" everything (principle S1, Ex Falso Quodlibet) and (b) a true proposition is not "implied" by arbitrary propositions (principle S2, Verum Ex Quodlibet). D. Lewis argues that in the case of (a) factual falsehood and in case of (b) factual truth do not guarantee that the respective conditional is true.

A second line of research, which aims to make inferences S1 and S2 invalid, is relevance logic (e.g. Weingartner & Schurz, 1986; Schurz, 1991). The basic idea in the approaches of Weingartner and Schurz (1986) and Schurz (1991) is to avoid parts of the conclusion (or the premises), which seem to be irrelevant for the inference in question. The validity of S1 does not depend on the logical form of the sub-formula α. In a similar vain, S2's validity does not draw on the sub-formula β. In this approach an inference is irrelevant either (i) if sub-formulas in the conclusion exists, which is not present in one of the premises (Weingartner & Schurz, 1986, Definition 2, p. 7) or when (ii) a sub-formula in an inference is replaceable by its negation salva validate (Weingartner & Schurz,

Although the former expressions are formula schemata rather than formulas, for the sake of brevity we shall often refer to them as formulas.

1986, Definition 3, p. 9). Alternatively one can characterize irrelevant inferences by identifying sub-formulas in the conclusion, which can be replaced by arbitrary formulas salva validate (Schurz, 1991, Definition 21 and 22, pp. 409–411). Note, however, that the main aim of the relevance logic approach is not to provide a more intuitive analysis of conditionals than a material implication analysis, but to identify inferences, which produce irrelevant inferences. The focus of these relevance logics is not on material implications, but relevance logic investigations involve a range of other logical connectives (e.g. Weingartner & Schurz, 1986; Schurz, 1991).

The starting point of the conditional logic approach employed here is the following observation: If we apply the above procedure for checking whether a natural language inference is valid and accept a material implication analysis of conditionals, E1–E5 are rendered valid, despite being counter-intuitive. In line with philosophers, such as Adams (1965, 1975), Stalnaker (1968), D. Lewis (1971, 1973/2001) and C. I. Lewis (1912) we hence aim to develop conditional logical systems, which do not make S1′–S5′ valid and allow, hence, for an interpretation of conditionals, which is more in line with the logical properties intuitively attributed to natural language conditionals. We call such an approach ‘conditional logic project approach’. It is the view that we need to extend p.c. or f.o.l. to allow for a (philosophically) adequate representation of conditionals. These extensions of p.c. and f.o.l. are, then, referred to as ‘conditional logics’. Note that conditional logics need not be designed to allow for an analysis of all types of conditionals in natural language. D. Lewis (1973/2001), for example, proposes conditional logic systems, which are intended to allow for an analysis of counterfactual conditionals (D. Lewis, 1973/2001, p. 1). In our terminology he still counts as a proponent of a conditional logic project. In contrast, an opponent of a conditional logic project argues that p.c. and f.o.l. suffice and no conditional logic is needed for a philosophically adequate analysis of any type of conditionals. In its most traditional form, an opponent of a conditional logic project adheres to a material implication analysis of conditionals.[2]

[2]Some opponents of a conditional logic project admit that not all types of conditionals are analyzable in terms of a material implication, such as counterfactual conditionals. Given that these opponents presuppose a material implication analysis of conditionals, their view implies that counterfactuals are not adequately representable in a logical system. We discuss such an approach

An opponent of a conditional logic project might be completely unimpressed by our above argument. She might argue – in the tradition of Tarski – that these examples merely appear to be paradoxical, although they are not. It is due to the ambiguity and elusiveness of natural language that these examples appear to be problematic. Once the conditionals are properly analyzed, their paradoxical nature disappears. Since $\neg p \to q$ is p.c.-equivalent to $p \vee q$, we might paraphrase the conclusion of E1 as 'John arrives at the 10 o'clock plane or he will arrive on the 11 o'clock plane'. It seems much less paradoxical to infer the latter conclusion from 'John arrives on the 10 o'clock plane'.[3]

This analysis of the conditional logic opponent, however, begs the very question. If we presuppose that the truth conditions of conditionals (of the form 'if α then β') agree with the truth conditions of disjuncts of a certain sort (namely, disjuncts of the form 'not α or β'), it is no wonder that E1-E5 only appear to be paradoxical, since we reduce the truth conditions of conditionals to truth conditions of disjuncts. It is, however, this reduction of conditionals to disjunctions, which proponents of a conditional logic reject, since this reduction implies the properties S1′–S5′ based on p.c.

The conditional logic opponent might still be unimpressed given the above argumentation. She has, however, to concede that any material implication analysis of natural language conditionals implies that S1′–S5′ hold for the conditionals analyzed in this way. Since the conditional logic proponent argues for an extension of p.c. or f.o.l. in terms of a conditional logic (i.e. by specifying logical axioms for a conditional operator $\Box\!\!\rightarrow$), she is not forced to make such a move.

1.2 Stronger Counterexamples to a Material Implication Analysis

There are two important classes of conditionals, for which an analysis in terms of the material implication is particularly counter-intuitive: (a) counterfactual condi-

in Section 1.2.1 in some detail and argue that such a move is not advisable.

[3]The relevance logician might object and argue that the inference from p to $p \vee q$ is indeed paradoxical, since the conclusion $p \vee q$ contains a formula q, which is irrelevant with respect to (w.r.t.) the premise p. We, however, concede this point to the conditional logic opponent.

tionals (short: counterfactuals) and (b) normic conditionals. Although (a) and (b) play – as we shall argue – an important part in the sciences, humanities and in everyday reasoning, they do not lend themselves to a material implication analysis. Due to our argument, the conditional logic opponent is forced either to (A) refrain from an analysis of both types of conditionals or else (B) give up an analysis in terms of the material implication. Since counterfactuals and normic conditionals play, as we argue, a key role in the sciences, the humanities (including philosophy) and in everyday reasoning, (A) is not advisable. (B) is, however, obviously not an option for a conditional logic opponent (see previous section). We will first focus on counterexamples of type (a) and, then, discuss counterexamples of type (b).

1.2.1 Counterfactual Conditionals

Counterfactuals, such as the one described before allow for causal inferences (cf. D. Lewis, 1973, p. 557). Questions as ‘What would be the case if ...?’ lie at the heart of philosophy and the scientific method in general and employ counterfactual conditional structures. In everyday reasoning we also employ questions of this type rather frequently. For example, a driver might ask himself after a car accident: ‘What would have happened if I reacted differently?’ Further examples from the sciences, the humanities and everyday reasoning are easily constructed.

In order to argue against a material implication analysis of counterfactuals, let us consider the following examples E1′-E5′ (example E3′ is borrowed from D. Lewis, 1973/2001, p. 33):

E1′ The Titanic sank. Therefore, if the Titanic had not sunk, Kennedy would not have been assassinated.

E2′ The Titanic sank. Therefore, if the Titanic would not have been built, then it would have sunk.

E3′ If J. Edgar Hoover had been born a Russian, he would have been a Communist. If J. Edgar Hoover had been a Communist, he would have been a traitor. Therefore, if he had been born a Russian, he would have been a traitor.

E4′ If Hitler had not been born, there would have been no World War II. Therefore, if Hitler had not been born and a similarly vile dictator emerged in Germany in the 1930s, there would have been no World War II.

E5′ If Kennedy had not been assassinated by Oswald, Oswald would still have been arrested. Therefore, if Oswald had not been arrested, Kennedy would have been assassinated by Oswald.

Note that examples E1′–E5′ have the logical forms S1′–S5′, respectively. Hence, if we use the material implication to analyze examples E1′–E5′, the inferences E1′–E5′ would turn out to be logically valid. Let us discuss S1′–S5′ with respect to (w.r.t.) counterfactuals in some more detail: According to S1′ any counterfactual with a false antecedent would be true, no matter what the consequent is. Analogously, by S2′ any counterfactual with a true consequent would hold, whatever the antecedent.[4] It is, however, a generally accepted feature of counterfactuals that they are not vacuously true despite the antecedent being false or the consequent being true: If the antecedent is false, it also depends on the consequent whether we regard the counterfactual being true. Moreover, if the consequent of a counterfactual is true, not in all cases the counterfactual is regarded true. Our acceptance of the counterfactual also depends on the antecedent.

Examples E3′- E5′ also present counterexamples for a material implication analysis. Note that in E3′–E5′ no interpretation of counterfactuals of the form $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ in terms of $\neg\alpha \vee \beta$ seems viable. Let us now focus on normic conditionals.

1.2.2 Normic Conditionals

The Quantified Case

Before turning to examples of normic conditionals, let us first describe a standard analysis of quantified sentences for first-order logic. This will prove useful, since many types of normic conditionals are naturally formulated in what can be considered a quantified form. Consider, for instance, the following example:

[4] From a linguistic perspective the discussion is complicated by the tense of the clauses and the tendency that true consequents are indicated by modifiers such as ‘still’ (cf. E5′). We will, however, abstract from these difficulties here.

E6 All fishes are cold-blooded.[5]

Conditionals, such as E6 are traditionally analyzed in the following way: For all x holds, if x is a fish, then x is cold-blooded. The very same analysis is not restricted to the above sentence, but also extends to sentences, such as ‘fishes are cold-blooded’. Both types of conditionals are, hence, treated as universally quantified conditionals (short: uq-conditionals).

In the sciences, humanities, and everyday reasoning, there are also conditionals of the following sort:

E7 99.9% of fishes are cold-blooded
E8 Most fishes are cold-blooded.
E9 Fishes are probably cold-blooded.
E10 Fishes are normally cold-blooded.

E7–E10 also qualify as quantified conditionals. E7 indicates an exact numeric value of frequency/probability, while E8–E10 do not. Furthermore, E7–E9 suggest a frequentistic/probabilistic interpretation (henceforth called ‘probabilistic conditionals’), whereas E9 does not immediately do so. For E10 at least two interpretations can be distinguished: ‘normally’ might either be understood in the sense of prototypical normality or statistical normality (Schurz, 2001b, p. 478). In a statistical normality interpretation, E10 states that the probability of x being cold-blooded, given that x is a fish, is high. Hence, E10 is in this interpretation roughly synonymous with E8 or E9. However, in a prototypical normality interpretation E10 rather states that a prototypical fish is cold-blooded. E7-E9 and E10 are examples for probabilistic conditionals and normic conditionals, respectively (cf. Schurz, 2008, p. 89f; Schurz, 2001b, p. 476f). Both sorts of conditionals, however, are by no means independent. Given appropriate constraints (described below), it is reasonable to assume that normic conditionals, as understood here, “imply” statistical conditionals of type E8, but not vice versa (Schurz, 2008, p. 90).

[5]Note that in fact most, but not all fishes are cold-blooded. For example, some species of sharks and tunas are exceptional, since they can raise their temperature significantly above ambient temperature (Helfman, Collette, Facey, & Bowen, 2009, p. 96f).

Two questions arise here: What do we mean by 'imply' and what reasons are there to assume that such an implicative relation holds? The above use of the term 'imply' refers to a conceptional analysis rather logical implication w.r.t. a logical system. This means roughly that due to our use of language the following holds: If conditionals of type E10 are true, then conditionals of type E8 are true. The converse, however, needs not be the case. Hence, probabilistic conditionals are a specific type of normic conditionals. Schurz (2001b) provides a justification for this relationship in life sciences, social sciences and humanities: For an evolutionary system to be successful, the normal case has to be in the long run the statistically normal case (Schurz, 2008, p. 91). Statistically normal cases are, however, specified by conditionals of type E8.

Schurz's argumentation shows, that it is rational to explicate the relation between normic conditionals and statistical conditionals in the above terms. It, however, does not show that conditionals of type E8 and E10 are used in natural language in that sense. This is an empirical question and can hardly be expected to be addressed by conceptual analysis.

Conditionals of type E7-E10 are not uq-conditionals in a strict sense, but qualify only as quasi-universally quantified conditionals (short: quq-conditionals). All four examples are close in meaning to E6, but they allow for exceptions. Hence, there might be exceptions in the sense that there are individuals, which are fishes, but not cold-blooded. This, however, implies that exceptions are not counterexamples to quq-conditionals. Hence, they do not falsify those conditionals strictly. Nevertheless, exceptions can accumulate in such a way that they become the rule rather than the exception. So, if the majority of fishes are not cold-blooded, then E8 and E9 cannot hold. On the basis of Schurz's argument, this line of reasoning also applies to conditionals of type E10. Since an analysis of conditionals in terms of f.o.l. (cf. example E6) does not allow for any exceptions/counterexamples, conditionals such as E7–E10 are prima facie excluded from a traditional f.o.l. analysis.[6]

[6]It might be argued that we can use quantifiers to count elements in the domain and by these means are able to represent a frequentistic interpretation of quq-conditionals. Please note that this approach does not work for cases E8–E10. This is due to the fact that we are in general not able to identify a non-arbitrary threshold of "counterexamples", above which one rejects E8–E10. Moreover, for infinite domains even arbitrary thresholds are not viable.

The Qualification Problem

Note that the foregoing argumentation shows only that quq-conditionals cannot be analyzed as uq-conditionals in the traditional way. The conditional logic opponent can, however, argue that, despite this fact, we can represent quq-conditionals in f.o.l. Although this assumption might seem plausible at first glance, any person who aims to achieve this goal will find it extremely hard to do so. In fact we shall argue that it is in general not possible to account for quq-conditionals in f.o.l. This type of argument is well known in the non-monotonic reasoning literature and is discussed under the label "qualification problem" (cf. McCarthy, 1980, p. 27; Horty, 2001, p. 340f). It is used in the non-monotonic reasoning literature as a motivation for a deviation from f.o.l. The present argumentation, however, provides a more stringent argument in favor of the insolvability of the qualification problem in f.o.l.

Note that the qualification problem is closely related to the discussion of so-called ceteris paribus laws (cf. Schurz, 2002b; Earman, Roberts, & Smith, 2002; Cartwright, 2002). One of the main issues in this controversy is the question whether ceteris paribus laws, such as 'ceteris paribus, planets have elliptical orbits' (cf. Schurz, 2002b, p. 352), can be adequately described by uq-conditionals. Such an analysis, however, presupposes that there are is only a finite number of exceptions (Earman et al., 2002, p. 284; Schurz, 2002b, p. 359). However, a more promising alternative for a reconstruction of ceteris paribus laws is suggested by Schurz (2002b): Schurz (2002b, p. 364) suggests to account for ceteris paribus laws in terms of normic conditionals (see Section 1.2.2), at least for the non-physical sciences (cf. 370f). As the qualification problem taps the interrelation between quq-conditionals and uq-conditionals, we will, hence, also indirectly draw on the issue of ceteris paribus laws.

Let us now focus on the qualification problem. A conditional logic project opponent might argue that we can in general transform quq-conditionals into uq-conditionals and that, hence, we can treat quq-conditionals in f.o.l. To do so we have to specify the antecedent in such a way that it excludes all (potential) exceptions.[7] Consider, for example, the following quq-conditional:

[7] Note that such an approach is most plausible for normic conditionals. To make the opponent's

E11 Mammals are normally viviparous.

'Viviparous' means that the respective species' females do not lay eggs, but give live birth.[8] Note that mammal species normally give live birth (Macdonald et al., 2007a). A subclass of mammals, namely the prototheria, are defined as those mammals, which do not give live birth (Macdonald et al., 2007b). Platypoda (platypuses) and tachyglossa (spiny anteaters) are the only orders of the subclass prototheria (McKenna & Bell, 1997, p. 35). Ignoring that there exists a possibly infinite number of heterogeneous exceptions why the a mammal specimen might not lay eggs (cf. Schurz, 2002b, p. 359 and p. 370), we can change E11 based on these facts to the following strict uq-conditional:

E12 All mammals, except for spiny anteaters and platypus, are viviparous.

If we restrict ourselves to living species, we might accept E12 exactly if we accept E11. Note, however, that our acceptance of E12 essentially hinges on our state of knowledge. Given the present state of science, it is plausible to accept E12 if we accept E11 and vice versa. Hence, the opponent's argument is rather weak, since she refers in this version only to acceptability conditions relativized to a certain state of knowledge. The opponent might, hence, argue that she means something stronger, namely that given our present scientific knowledge E12 and E11 are equivalent. This, however, does not allow us in general to express E11 alternatively in terms of E12. This is due to the fact that it can very well be that new evidence and theories emerge, such that we reject E12, but not E11, or vice versa.

One possible way out of this predicament is to use the subclass of *prototheria*: All species of this subclass are defined to be mammals, which are non-viviparous (Macdonald et al., 2007b). This allows us to reformulate E12 in a knowledge independent way:

E13 All mammals, except for prototheria, are viviparous.

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case as strong as possible, we restrict the present discussion to this type of conditionals.

[8]Note that viviparousness is a disposition of individual specimen rather than species. In order to make the case as strong as possible for the conditional project opponent, we, however, abstract from the following two difficulties: (i) how to account for dispositions and (ii) how to relate dispositions of specimens to dispositions of species.

Although example E13 is now knowledge-independent, it cannot be equivalent to E11 for the following reasons: E13 is analytically true, since prototheria are defined to encompass all mammal species, which are non-viviparous. However, unlike E13, example E11 has no inherent feature, which guarantees its truth. Hence, E11 is not analytically true.

Our argument shows that there are only two ways for the conditional logic opponent to reduce quq-conditionals to uq-conditionals: (a) either use particular features, which conjointly guarantee that a quq-conditional is equivalent to a uq-conditional given some state of knowledge or (b) use defining features to make sure that the respective uq-conditional holds, regardless of one's state of knowledge. We saw that (b) does in general not work. We have, however, not yet ruled out that (a) is a weak, but still viable approach. We, hence, aim to show that (a) does in general not suffice. Ignoring the knowledge-dependence of approach (a), we argue that we cannot construct a uq-conditional, which is equivalent to E11, but does not represent approach (b). We thereby already abstract from the fact that a possibly infinite number of heterogeneous exceptions exists why a mammal specimen may not lay eggs (cf. Schurz, 2002b, p. 359 and p. 370).

To keep our argument simple, presuppose that a specimen's genetic code determines, which species it belongs to.[9] The genetic code, however, determines a specimen's metabolism only on the basis of environmental conditions. Environmental influences (i.e. medication, toxic substances) can even alter a specimen's expression of genes, without affecting the specimen's genes. In particular, it is not precluded that there an environmental condition A, which inhibit the organism's metabolic processes that lead to the organism's viviparousness, but do not affect its genetic code.[10] The environmental condition A might, for example, be the presence of a toxic substance. It is furthermore, not precluded that there is a second environmental condition B, which inhibits A's inhibition of the organism's viviparousness and which again does not affect the organism's genetic code. This might, for example, be a chemical substance, which neutralizes the toxic substance's effect on the organism. If both conditions A and B apply, the organ-

[9]This assumption is controversial among biologists and philosophers of biology (cf. Ereshefsky, 2007).

[10]These processes might be extremely unlikely in a natural environment. This, however, does not affect the present argument.

ism retains its viviparousness. However, it is not precluded that there is a third environmental condition C, which inhibits B's inhibition of A's inhibition of the organism's viviparousness. Again we presuppose that environmental condition B does not affect the organism's genetic code. Environmental condition C might, for example, be the presence of a neutralizer of the neutralizer of the toxic substance on the organism's viviparousness. If conditions A, B and C hold, then C inhibits B's inhibition of A. Thus, A can inhibit the organism's viviparousness (B does not interfere due to C). We can continue this example with environmental conditions D, E, F ... A fortiori there is no reason why we might not produce such a chain of inhibitions of arbitrary length. Note, however, that the total environmental condition, in which A inhibits the organism's viviparousness is different from the situation in which conditions A, B and C are present. In the latter situation only the inhibition of B by C allows A to inhibit the organism's viviparousness. No such conditions are present in a situation, in which only A is present. Hence, our argument shows that we might create an arbitrary number of non-equivalent conditions that inhibit an organism's viviparousness.

However, to make sure that a quq-conditional is equivalent to E11, we have to exclude all environmental influences, which inhibit the organism's viviparousness. Since we can construct an arbitrary number of such environmental conditions, we essentially get an infinite number of environmental conditions, which have this effect, namely inhibitory conditions simpliciter, inhibitions of inhibitions of inhibitions and so on. Since formulas only allow for a finite number of sub-formulas, we cannot possibly list all those exceptions in the antecedent. It might be argued that we could instead succeed by specifying that there are no environmental conditions, which inhibit the organism's viviparousness. This step is, however, not viable, since this would make the respective uq-conditional again analytic: Environmental conditions are defined as conditions, which do not solely pertain to the organism's genetic code. Hence, such a move would result in approach (b).In addition, both strategies (a) and (b) cannot account for a possibly infinite number of heterogeneous exceptions why the a mammal specimen might not lay eggs. The latter observation provides further support that quq-conditionals cannot in general be reduced to uq-conditionals.

The Propositional Case

Note that there are conditionals, which can be reconstructed without reference to any type of quantification, but which also qualify as probabilistic or normic conditionals. They are propositional conditionals, such as the following:

E14 If specimen 214 is a fish, then it is probably cold-blooded.

E15 If specimen 214 is a fish, then it is normally cold-blooded.

E16 If Peter pulls the red lever now, the engine will probably start.

E17 If it rains now, the car will normally not start.

E14 and E16 are probabilistic conditionals and E15 and E17 normic conditionals. The modifiers ‘probably’ and ‘normally’ are interpreted to pertain to the whole conditional structure, not only to the consequent. E14 and E15 can be regarded as specific instances of the quq-conditionals E9 and E10, respectively. Moreover, it can be argued – as in the quantificational case – that probabilistic conditionals are a subtype of normic conditionals. We, hence, restrict ourselves to the discussion to normic conditionals. One can easily check that our argumentation also applies to probabilistic conditionals.

Analogous to the quq-cases, the conditionals E14-E17 do not preclude counterexamples. In E7-E10 counterexamples are instances of quq-conditionals (fishes that are not cold-blooded), in E14 and E15 counterexamples are state of affairs, in which the antecedent is true, but the consequent is false (specimen 214 being a fish and not being cold-blooded). This means that the set of formulas $\{\alpha \mathrel{\Box\!\!\rightarrow} \beta, \alpha, \neg\beta\}$ is consistent in the propositional case, where the normic conditional is represented as $\alpha \mathrel{\Box\!\!\rightarrow} \beta$. Note that this condition is equivalent with $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ not implying $\alpha \rightarrow \beta$.[11] Hence, the following inference cannot hold for normic conditionals:

S6′ if $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ then $(\alpha \rightarrow \beta)$ (MP).

If we analyze conditionals in terms of the material implication, MP holds trivially. Given that normic conditionals do not satisfy S6′, this suggests that a material implication analysis is inappropriate for normic conditionals. In addition, the

[11] The quq-case is complicated by semantic considerations (see Section 3.4). In this thesis we, however, focus on the propositional case only.

inferences S1′–S5′, which are valid for the material implication, are also counter-intuitive for normic conditionals:

E1″ It rains today. Therefore, if it does not rain today, then Fireball will normally win the race tomorrow.

E2″ It rains today. Therefore, if I Fireball wins then race tomorrow, it will normally rain today.

E3″ Michaela studies into the night today, she normally gets much work done. If Michaela gets much work done today, she normally does not study into the night. Therefore, if Michaela studies into the night today, then she normally does not study into the night.

E4″ If Tina turns the ignition key now, the car normally starts. Therefore, if Tina turns the ignition key now and the motor does not get any fuel, the car normally starts.

E5″ If Ludwig does not ascend Mt. Everest, then he normally (still) does not win the Nobel Prize for Physics. Therefore, if Ludwig wins the Nobel Prize for Physics, then he normally ascends Mt. Everest.

Examples E1″-E5″ correspond to inferences S1′-S5′, respectively. Our examples are, however, in need of some explanation: In E3″ and E4″ it seems more natural to use the expression ‘to be expected’ rather than ‘normally’. Note, however, that expectancy can be viewed in this context as synonymous to normality, since it allows for an interpretation in terms of both prototypical and statistical normality (see Section 1.2.2): We can accept the consequent either since it is (proto)typically expected on the basis of the antecedent or since the consequent is (merely) probable given the antecedent. We, however, use here the expression ‘normally’ in order to allow for a uniform treatment of normic conditionals.

Note that inference S5′ is problematic for a normic conditional $\alpha \mathrel{\Box\!\!\rightarrow} \beta$, if $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ is the case, but α and β are also normally true. It is, however, hard to find formulas α and β, which conform to the above restrictions and are contingent, but plausible candidates for being true unconditionally. To provide a counterexample for S5′, we use elementary propositions, which are for any person (including Ludwig) non-normal/exceptional (‘Ludwig ascends Mt. Everest’, ‘Ludwig wins the Nobel Prize for Physics’) and negate them. Moreover, the first conditional ‘if

Ludwig does not ascend Mt. Everest, then he normally does not win the Nobel Prize for Physics' should be regarded true, since for any person (including Ludwig) it is not normal to win the Nobel Prize for Physics regardless if he ascended Mt. Everest or not.

The Role of Normic Conditionals

Normic conditionals are important means for argumentation in the sciences, the humanities and everyday reasoning. Probabilistic reasoning lies at the heart of empirical sciences and has shown to be a useful framework for describing everyday reasoning. Normic conditionals play an important role in rational argumentation in the humanities (including philosophy). The type of argumentation, which is most typically endorsed in humanities, eventually draws on our intuitions and appeals to individuals' judgments of plausibility. The argumentation is considered as substantial support for the conclusion. It is, however, not conclusive in the sense that the conclusion argued for stays plausible, no matter which conditions are added to the argumentation. In other words, the presuppositions of the argument might not ensure in all cases that the conclusion holds. The same holds, for probabilistic reasoning: The probability, for instance, of an arbitrary mammal being viviparous is high (non-viviparous mammals are an exception). Despite this fact it does not follow that the probability of platypus being viviparous is high. One can see that normic conditionals are employed in the sciences, the humanities and in everyday reasoning, if one considers sets of statements, such as the following (cf. Delgrande, 1987, p. 109):

(A) If specimen 213 is a mammal, it is normally viviparous.
If specimen 213 is a platypus, it is normally not viviparous.
Specimen 213 is a platypus and a mammal.

(B) Mammals are viviparous.
Mammals, which are platypus, are not viviparous.
There are mammals, which are platypus.

In the sciences, the humanities and in everyday reasoning sets, such as (A) and (B), are considered consistent. If we, however, analyze both sets of statements in terms of the material implication, these sets are inconsistent. Since S6′ rather than S1′–S5′ is responsible for the inconsistency of (A) and (B), this suggests that the conditionals in (A) and (B) are interpreted as normic conditionals. Analogous examples from the sciences, the humanities and everyday reasoning can easily be constructed. In a conditional logic, however, this type of set can be rendered consistent.

1.3 Conversational Implicatures – A Possible Way Out?

Despite our arguments, the conditional logic opponent has still a potential way out of this predicament. She can appeal to the pragmatic analysis of natural language by Grice (Grice, 1989, see also Bennett, 2003, pp. 22–26). Grice held the view that an analysis of conditionals in terms of the material implication is appropriate. According to that view, inferences of type S1′-S5′ appear to be odd, although they are in fact not. This appearance is due to the rules that govern natural language use, called ‘maxims of conversational implicature’.

Maxims of conversational implicature include the following (Bennett, 2003, p. 22f): be appropriately informative, be truthful, be relevant, be orderly and brief. Language is used in such a way that the language user conveys information “without outright asserting it” (Bennett, 2003, p. 22). An example for a Gricean implicature is the following (cf. Bennett, 2003, p. 23): If Jane Doe says ‘God exists’, then she conveys that she believes that God exists. Despite this fact, however the proposition *God exists* does not entail that Jane Doe believes that God exists. Asserting a proposition without believing it, however, results in a violation of the maxim of truthfulness. So, if Jane Doe asserts *God exists* without believing it, she violates the maxim of truthfulness and her assertion is be considered awkward or at least in need of further explanation.

The conditional logic opponent can, then, defend an analysis of natural language conditionals in terms of the material conditional by Gricean maxims along the following lines: (a) A material implication analysis is appropriate from a normative stance, and (b) we can explain why inferences of type S1′–S5′ strike us as counter-intuitive by reference to Gricean maxims. It is important to note that according to this view the conclusion of S1′–S5′ must be true, give the premises are true. However, asserting the conclusion rather than the premise(s) is not appropriate given conversational maxims.

Bennett's (2003, p. 24f) discussion shows that inferences of type S1′ and S2′ might be considered awkward for the following reason: In both cases the maxim of brevity and the maxim of informativeness suggest that one should assert the premise rather than the conclusion: First, the premise is shorter (maxim of informativeness). Second, the premise implies the conclusion, but not vice versa. Thus, the premise is more informative (maxim of informativeness). Hence, applied to E1 and E2, Gricean maxims suggest that one should assert the premises of E1 and E2 rather than the conclusions.[12]

As an alternative to above analysis, we can employ a relevance approach, as described by Schurz (1991). In Schurz (1991, Definition 21 and 22, pp. 409–411) the p.c.-valid inference S is considered irrelevant if one can uniformly substitute a propositional variable (or an axiom schema letter) in the conclusion of S by an arbitrary formula without making S invalid (cf. Section 1.1). The rationale underlying this approach is that an inference is irrelevant if it allows us to infer a formula from premises, which contain an irrelevant "element", viz. which does not depend on any sub-formula in the premises. Note that according to Schurz's (1991) irrelevance criterion, both inference schemas S1′ and S2′ turn out to be irrelevant, since S1′ and S2′ both contain a sub-formula, which can be replaced salva validate (cf. Schurz, 1991, p. 410). In the case of S1′ the irrelevant element is the formula schema β and in the case S2′ it is the formula schema α.

Note that we can interpret Schurz's (1991) approach also in terms of Gricean maxims. This is due to the fact that the exclusion of irrelevant inferences follows a non-redundancy preference, insofar as inferences with irrelevant conclusions elements are explicitly excluded. Non-redundancy preferences, however, are tapped by Grice's (1975, p. 46) maxim "relation". The main difference between Schurz's (1991) (ir)relevance criterion and the Gricean approach described above is that in

[12] In our opinion the analysis of the counterfactual and normic counterexamples E1′–E5′ and E1″-E5″, respectively, in terms of conversational implicature is less plausible than for counterexamples E1-E5. To make the case for a Gricean analysis as strong as possible, we, hence, restrict ourselves to examples E1-E5.

Schurz's (1991) account, irrelevance inferences are not just "not assertable", but regarded as normatively deficient (see criterion (a) above).

Let us continue with the discussion of the Gricean approach described above. Observe that not all inferences of type S1′–S5′ can be explained by appeal to the two criteria brevity and informativeness equally well. While an explanation of S4′ seems viable in terms of the maxims of brevity and informativeness, an application of both maxims to S3′ and S5′ yields a much less clearer result. Let us, first, focus on S4′: The premise in E4, for example, is shorter than the conclusion (maxim of brevity) and the premise implies the conclusion, but not vice versa (maxim of informativeness). Hence, both maxims suggest that one should endorse the premise rather than the conclusion. In case of S3′, however, both maxims conflict with each other: For example, in E3 the premises imply the conclusion, but not vice versa. Hence, the maxim of informativeness suggests that one should assert the premises of E3 rather than its conclusion. The maxim of brevity, however, gives us reasons to assert the conclusion rather than the premises, since the conclusion is shorter than the conjunction of both premises. In case of S5′, there is even less support: In E5, for example, the conclusion and the premise imply each other. So, the maxim of informativeness is indecisive. Moreover, the conclusion contains an unnegated antecedent and consequent, while the premise employs negated versions of both sub-clauses (as consequent and antecedent, respectively). Hence, the conclusion is shorter than the premise. Thus, the maxim of brevity suggests that one should assert the conclusion of E5 rather than its premise. Note that the failure to account for S5′ inferences, such as E5, is more severe than it might seem first. This is due to the fact that, given reasonable restrictions, S5′ implies S3′ and S4′, but not vice versa (Kraus et al., 1990, p. 180f). Observe that Schurz's (1991) approach does not give us the same results as the Gricean account: Only S4′ turns out to be irrelevant in this Schurz's account, while in Schurz's (1991) approach S3′ and S4′ are regarded relevant inferences.

We shall now discuss *what* a Gricean approach, as outlined by Bennett (2003, p. 24f), can explain at maximum. For inferences S1′–S5′, this approach gives us only that, provided one is certain of the premise(s) (and the conclusion) of S1′-S5′, one should assert the respective premise(s) rather than its conclusion. Note that an interpretation in terms of Gricean maxims (as above) does not allow for a general account of factor (b): A Gricean approach of this sort tells us – provided one is in a particular type of communicative situation and one is sure of the premise(s) (and the conclusion) – that one should assert the premise(s) rather than the conclusion. Hence, this Gricean account does not hold unrestrictedly, but only w.r.t. certain types of communicative situations.

There is, however, a range of communicative situations, which do not conform with these restrictions and for which, hence, the maxims of brevity and informativeness do not apply. One type of situation, which is particularly important in this context are logical reasoning situations: Given some type of reasoning problem, we might be forced to puzzle out whether propositions allow for inferences of type S1′–S5′. To infer propositions from other propositions one needs not be sure that these propositions hold. Moreover, in such a context it seems natural that one might communicate inferences. Descriptions of (logical) inferences, however, often violate the maxims of brevity and informativeness. Hence, the above approach does not in general work for these type of situations. Furthermore, in reasoning situations the conclusions of inferences S1′–S5′ are – unlike in the situation Grice describes – not just awkward to assert or inappropriate, if one asserts the premises, but factually false. The instances of S1′–S5′ discussed in the previous sections show that the conclusions are not warranted given one adds the premises to one's (hypothetical) stock of beliefs. The present problem is aggravated by the fact that almost any type of (communicative) situation involves some sort of reasoning element and can, hence, (at least partially) be regarded as a reasoning situation.

However, even if we accept that a Gricean approach (as described above) might account for point (b), we still need to explain point (a). We, however, are not aware of such an argument in the literature. Moreover, there are in fact reasons why, from a normative perspective, a material implication approach seems inappropriate: If we use such an approach, we cannot, for instance, allow for exceptions as described in Section 1.2.2. Note that Schurz's (1991) relevance account agrees with a conditional logic approach insofar as it does not presuppose that an analysis of conditionals in terms of the material implication is appropriate from a normative perspective. Since we focus in this thesis on conditionals rather than relevant inferences, we will not discuss Schurz's (1991) approach here any further.

1.4 No Material Implication Analysis

We argued that counterfactuals and normic conditionals play an essential role in the sciences, the humanities and everyday reasoning. Our discussion in Section 1.2, however, showed that both types of conditionals do not lend themselves to a material implication analysis, but have logical properties, which differ from the material implication. In addition, we saw that a Gricean analysis does not allow for a general way out of this problem (see Section 1.3).

One might still argue that a material implication approach is still appropriate for conditionals, which are neither counterfactuals nor normic conditionals. Such a position is, however, also highly problematic. In order to argue for such a position one has to have clear linguistic criteria to distinguish (a) indicative conditionals from counterfactual conditionals and (b) non-normic indicative conditionals from normic indicative conditionals. In Section 2.1.4 we argue that no purely linguistic criterion for (a) exists. We, however, think that no clear linguistic criterion for (b) exists either. We can, for instance, hardly expect modifiers, such as 'normally', to do the whole job. Let us, for that purpose, consider the following modification of E17:

E17′ If it rains now, the car will not start.

Despite its form, E17′ can alternatively be interpreted as a normic conditional, in the sense that it does not preclude that it rains now, but that the car will start. If we, however, interpret E17′ in terms of a material implication, this cannot be the case. Moreover, due to the vagueness and ambiguity of language use it is not plausible to argue that the linguistic form of a conditional determines whether it is to be understood either as a normic or a non-normic indicative conditional. Hence, it seems best to use a more general approach in terms of a conditional logic. This approach does not presuppose that, for example, properties S1′-S5′ hold and allows for a much less problematic analysis of conditionals on a much more general basis.

Chapter 2

Interdisciplinary Dimensions

The present chapter focuses on two main points. First, we discuss the interdisciplinary ramifications of the conditional logic approach (see Chapter 1). By means of our discussion we aim to show that the conditional logic project can be fruitfully applied in a range of disciplines, such as philosophy, linguistics, psychology and computers science. Our discussion of this issue is intended to emphasize a point made in Chapter 1, namely that a conditional logic approach is needed, which goes beyond p.c. and f.o.l.

Second, we use an idea from the non-monotonic logic literature (discussed in the computer science and artificial intelligence literature) to describe a main motivation of default logic approaches and contrast them with conditional logic approaches. Our discussion also helps to clarify the aims of the conditional logic project and, in addition, serves as basis for our survey of conditional logics in Chapter 3.

2.1 The Conditional Logic Project and Related Projects

2.1.1 Overview

The study of conditional logics started in the philosophical literature[1], but conditional logics were later investigated in a range of disciplines such as linguistics (e.g. Stalnaker, 1975; Kratzer, 1977, 1981; Portner, 2009), computer science[2] and psychology (e.g. Evans & Over, 2004). In all those disciplines several distinct but related projects emerged. These projects can be described as follows:

(1) the linguistics of conditionals project
(2) the philosophy of conditionals project
(3) the psychology of reasoning project
(4) the non-monotonic reasoning project

Projects (1)-(4) are all related to *the conditional logic project*, but pursue distinct goals. We, first, describe the conditional logic project and, then, discuss projects (1)-(4) and their relation to the conditional logic project.

2.1.2 The Conditional Logic Project

The aim of *the conditional logic project* is to specify logical systems, which can describe conditionals more adequately than p.c. or f.o.l. (This motivation is discussed in detail in Chapter 1.) By means of conditional logics the scope of logical analyses is, then, broadened and logical tools are developed, which allow for a more fine-grained language-based analysis of concepts and their application. Examples are the accounts of causality (e.g. D. Lewis, 1973) and the analysis of dispositions (e.g. D. Lewis, 1997).

2.1.3 The Linguistics of Conditionals Project

The linguistics of conditionals project aims to provide an adequate empirical description of conditionals in natural language. This can either be done primarily on

[1] See, for example, Adams (1965, 1966), Stalnaker (1968), Stalnaker and Thomason (1970) and D. Lewis (1971, 1973/2001).

[2] See, for example, Ginsberg (1986), Delgrande (1987, 1988, 1998) and Nejdl (1992).

a syntactic/grammatical level (e.g. Haegeman, 2003), on a semantic (e.g. Portner, 2009) or pragmatic level (e.g. Declerck & Reed, 2001). Its aim is, however, always descriptive in the sense that a description of actual linguistic practice is aimed for.

The linguistics literature does not seem to have a large impact on the conditional logic literature. The conditional logic literature, however, influences linguistics to some extent. Philosophers such as Adams, D. Lewis and Stalnaker, who investigated conditional logics, also published in linguistic journals (e.g. Adams, 1970; D. Lewis, 1970; Stalnaker, 1975). The conditional logic literature's influence is not limited to that. Portner (2009), for example, gives a semantic account of modalities and conditionals in natural language, based on Angelika Kratzer's work (i.e. Kratzer, 1977, 1981), which had itself a strong impact on linguistics. This approach is based on D. Lewis' systems of spheres semantics (D. Lewis, 1973/2001) and standard modal logic (e.g. Hughes & Cresswell, 1996/2003). Portner (2009) is, however, not interested in the logical and meta-logical properties of logical systems, but rather uses the formal framework to model the meaning of natural language conditionals and modal expressions (Portner, 2009, p. 49f). For that purpose he takes the context, in which such expressions occur into account. He uses it as an additional parameter for determining the meaning of linguistic entities. In his approach the context specifies the speaker, the addressee, time of utterance and place of utterance (Portner, 2009, p. 49).

In the Portner approach modal expressions in natural language such as 'must' and 'should' are viewed as indexicals, whose actual meaning depends also on the context. For example, 'must' can be interpreted in a deontic sense ('dog owners must keep their animals indoors'), but also epistemically ('it must be raining outside', Portner, 2009, p. 30). Portner (2009, pp. 50–52) argues that there is an additional contextual parameter called 'conversational background'. This parameter is used to determine the meaning of sentences with modifiers, such as 'in the view of what I know' and 'in the view of the rules of the secret committee' and is used to determine the type of modality employed (Portner, 2009, pp. 50–52). Portner, then, uses both parameters, *conversational background* and *context*, to describe a semantics for natural language conditionals (p. 81f). Both parameters are used to determine an ordering of possible worlds, which describes a system of spheres in a D. Lewis' type semantics and gives truth conditions for the conditional.

2.1.4 The Philosophy of Conditionals Project

The aim of *the philosophy of conditionals project* is to provide a plausible and rationally justified theory of conditionals. For that purpose often ordinary language is employed as a guide. Despite this fact, however, its primary aim is normative in nature. Note that this approach is, at its core, restricted to an informal discussion of these issues. The philosophical discussion and the interpretation of conditional logics and philosophical discussion can, then, be used to provide a more adequate analysis of philosophical problems from areas such as philosophy of language, epistemology and philosophy of science (e.g. Schurz, 2008, p. 90f).

Both the philosophy of conditionals project and the conditional logic project have a strong mutual influence on each other. On the one hand, conditional logics are taken as formal descriptions of normative theories of conditionals in the philosophy of conditionals literature (e.g. Bennett, 2003, p. 163–168). On the other hand, in the conditional logic project often the philosophical underpinnings of the formalisms are discussed.

A range of philosophers such as Adams (1975), Bennett (2003), Edgington (2007) and Gibbard (1980) share the view that (a) indicative and counterfactual conditionals require a completely different theoretical treatment and that (b) indicative conditionals should be analyzed in terms of a subjective probabilistic semantics such as Adams (1975). For the analysis of counterfactuals Adams, Bennett, Edgington and Gibbard disagree. For example, Adams (1975, Chapter 4) and Edgington (2007, p. 202) argue for an analysis of counterfactuals in terms of a probabilistic semantics, while Bennett (2003, p. 291) prefers a temporal analysis based on D. Lewis (1979). Note that not all authors employ a treatment of conditionals in terms of subjective probabilities. For instance, Schurz (1997b, 2001b) advocates a treatment of conditionals in terms of objective frequency-based probabilities (cf. Pearl, 1988; D. Lewis, 1980; Bacchus, 1990).

For the remainder of this section we will focus on the distinction between indicative and counterfactual conditionals. Due to point (a) it is of pivotal importance for the conditional logic approaches of Adams (1965, 1966, 1975), Bennett

(2003), Edgington (2007) and Gibbard (1980) to have a clear criterion for distinction between indicative and counterfactual conditionals. Bennett (2003, p. 8), for example, holds the view that there exist countless pairs of indicative and counterfactual conditionals, such as the following (Adams, 1970, p. 90):

E18 If Oswald hadn't shot Kennedy in Dallas, then no one else would have.
E19 If Oswald didn't shoot Kennedy in Dallas, then no one else did.

Given our historical knowledge about the assassination of Kennedy, conditional E18 appears to be probably true, whereas E19 is clearly not. Bennett (2003), then, argues that, since E18 and E19 use the "same" antecedent and consequent (p. 8), both differ in their logical properties (p. 8).

As we observed before, analyses of conditionals by Adams (1975), Bennett (2003), Edgington (2007) and Gibbard (1980) presuppose a clear criterion for the distinction between indicative and counterfactual conditionals. At best this criterion should, in addition, be independent from their analysis of conditionals. If one assumes the account of Adams (1975), Bennett (2003), Edgington (2007) or Gibbard (1980) and identifies a conditional on the basis of this criterion as indicative, a subjective probabilistic semantics is in order. If it is a counterfactual conditional, an alternative semantics has to be applied (cf. Bennett, 2003, p. 9).

There are two basic approaches to describe the difference between indicative and counterfactual conditionals, such as E18 and E19: (A) by linguistic mood (indicative vs. subjunctive, respectively) or (B) by the fact that the counterfactuals' antecedents are "counter to the facts". According to criterion (A), the main difference between E18 and E19 is the fact that E18 is in the indicative mood, while E19 is in the subjunctive mood. Criterion (B) identifies suppositions about the truth-value of the antecedent as the essential difference between E18 and E19. In E18 it is presupposed that the antecedent is "counter to facts" viz. false, while in E19 the antecedent is neither presupposed to be true nor presupposed to be false. It is essential to specify what we mean by 'presuppose' here. In his book *Counterfactuals* D. Lewis (1973/2001) argues that this presupposition may be due to "conversational implicature, without any effect on truth conditions" (p. 3). He, then, provides truth conditions for counterfactuals, according to which the antecedent of a counterfactual is not required to be false (p. 26). In this context a more fine-grained distinction by Declerck and Reed (2001) is useful. In order to provide a general terminology for the categorization of conditionals, Declerck and Reed (2001) distinguish between two types of counterfactual conditionals: (B1) "genuine" counterfactuals and (B2) tentative counterfactuals. Counterfactuals of type (B1) are counterfactuals, for which the antecedent is false, while counterfactuals of type (B2) include counterfactuals whose antecedent is merely improbable (p. 13f).

Criterion (A) appears to be a well-established criterion in linguistics for the distinction between indicative and counterfactual conditionals. This is, however, not the case. It is not an established fact that the subjunctive mood describes 'would'-conditionals of type E18 in the English language reliably (Bennett, 2003, p. 11). Moreover, in French and Spanish no subjunctive mood, but a conditional tense is used instead to formulate counterfactuals (Bennett, 2003, p. 11).

So, should one choose criterion (A), (B1) or else (B2) as a criterion for distinguishing between indicative and counterfactual conditionals? Bennett (2003, p. 12) favors criterion (A), but implicates that the difference between criteria (A) and (B) is a rather weak point important only for labeling two sorts of conditionals (indicative vs. counterfactual conditionals). We disagree, since there are substantial reasons for philosophers who subscribe to point (a) and (b) above (as for example Bennett), to accept criterion (A) but not criteria (B1) and (B2). To describe Bennett's own motivation for accepting criterion (A), we have to outline some basic assumptions of subjective approaches to indicative conditionals.

Point (a) gives us that subjective probabilistic semantics for conditionals, as described by Adams (1965, 1966, 1975; cf. Section 3.5), is in order. In this subjective probabilistic framework, all formulas, to which probabilities are assigned, represent propositions that can either be true or else false. A probability assigned to a formula α is, then, interpreted as the probability that α is true (cf. Bennett, 2003, p. 47f). Truth and falsity in this framework are interpreted in an objective way, in contrast to truth and falsity according to an agents' set of beliefs.

Bennett (2003, p. 12) argues for criterion (A) by objecting against criterion (B1). (Bennett does not discuss criterion (B2)). According to Bennett (2003, p. 12) criterion (B1) is implausible for the following reasons: (i) Our classification of counterfactual conditionals would not so much depend on the fact whether the antecedent is objectively false but rather on the fact whether the speaker believes so. Furthermore, (ii) the other type of conditionals, indicative conditionals, would not – in contrast to counterfactual conditionals – in any specific sense be "factual", "pro-factual" or the like. Moreover, (iii) the label 'counterfactual' would only depend on pragmatic assumptions regarding conditionals and should, therefore not serve as basis for the categorization of indicative and counterfactual conditional.

We think that (ii) and (iii) are rather weak points for the following reasons: First, indicative conditionals are factual in the sense that they do not presuppose either (B1) or (B2). Second, it is not so clear that (B1) and (B2) are only made on a pragmatic level as opposed to a semantic level. Even if we presuppose that (B1) and (B2) are only made on a pragmatic level, (B1) and (B2) might still be essential for the application of counterfactual conditionals to real world reasoning and, thus, be an appropriate criterion for distinguishing between indicative and counterfactual conditionals.

Let us now focus on point (i). Point (i) seems, given a subjective probabilistic framework as described above, particularly plausible. This is due to the fact that this approach only draws on the notions of objective truth and subjective probabilities (see above). In general, however, it is far from clear that one must accepts such a subjective probabilistic framework. One might alternatively use a subjective notion of truth and interpret both conditional and non-conditional formulas in terms of truth and falsehood according to an agent's beliefs. In that interpretation a counterfactual is, then, "contrary to the facts the agent believes".

A person who accepts points (a) and (b) above, might, in addition, reject criterion (B2) for the following reasons: Since (B2) states that the antecedent of a counterfactual as opposed to indicative conditionals is regarded as unlikely, it seems plausible to describe this probabilistic component also in terms in the subjective probabilistic framework already used in the probabilistic semantics (point b). By point (a), we are not allowed to use the same characterization of indicative and counterfactual conditionals. So, if one accepts criterion (B2) and allows for a subjective probabilistic semantics for counterfactual conditionals, it seems again plausible (i) that there are no counterfactuals with subjectively probable antecedents and (ii) that there are no indicative conditionals with subjectively improbable antecedents. While (i) might be regarded plausible on general grounds, it is much less clear that (ii) is adequate from a position in line with points (a) and (b), such as Adams (1975), Bennett (2003), Edgington (2007) and Gibbard (1980). As we saw above these authors argue that one should distinguish between indicative and counterfactual conditionals, such as E18 and E19, respectively, since both have different assertability conditions. To accept (B2) and, hence, (ii), would, however, deny exactly that. This is due to the fact that conditionals with a low subjective antecedent probability, such as E18 must according to (ii) also be regarded as counterfactual conditionals. Hence, both E18 and E19 would be regarded counterfactual conditionals and, thus, should be subjected to the same philosophical analysis. In sum, there are substantial reasons for persons who subscribe to position (a) to prefer an analysis of conditionals in terms of criterion (A) rather than (B1) or else (B2).

In this thesis we will, however, spell out the difference between indicative and counterfactual conditionals in terms of criteria (B1) and (B2) (see Section 3.3). This seems natural, since we explicitly allow for the notion of truth in indicative and counterfactual conditionals (see Sections 3.3). We also feel that it is essential for a philosophically adequate conditional logic to discuss its normative adequacy. We can, however, hardly see how criterion (A) – which refers solely to syntactical features of natural language – can lead to a fruitful discussion with respect to (w.r.t.) the normative adequacy of conditional logics for both types of conditionals. Criteria (B1) and (B2), however, make – from a normative perspective – more substantial claims concerning the difference between both types of conditionals.

2.1.5 The Psychology of Reasoning Project

A further project is *the psychology of reasoning project*. The aim of this project is to provide an empirically adequate account of human reasoning. One of the most active and interesting areas in this project is reasoning with conditionals (c.f. Evans & Over, 2004). This project is descriptive in the sense that it aims to provide an empirically adequate description of human beings' actual reasoning. It is, however, not so much concerned with linguistic representations of conditionals.

One motivation for the study of conditional logics is that these logics are also intended to be descriptively more accurate in accounting for human beings' reasoning with conditionals than p.c. or f.o.l. Most often the rationale is to asks one's own intuitions about human beings' way of reasoning with conditionals (see Chapter 1). It is, then, argued that these intuitions largely agree with human beings' actual reasoning (also called 'common sense reasoning'). However, to provide support for this claim we cannot restrict ourselves to rational analyses, but have to investigate empirically whether these claims really hold (cf. Pelletier, Elio, & Hanson, 2008). One aim of the psychology of reasoning project is to do exactly that.

Traditionally, psychologists took p.c., f.o.l. and syllogistic as their normative standard. The performance of participants on conditional reasoning tasks was, then, judged against the predictions made by a material implication analysis of the task. A still dominant approach originating from this tradition is the mental model theory of Johnson-Laird and Byrne (2002). According to that approach, human beings use semantic means to perform logical reasoning with conditionals, very much like reasoning with truth-tables. In the simplest case they only represent those lines of the truth-table, in which the conditional is true. Moreover, human beings do not start with a fully fleshed-out "mental model" encompassing all lines, in which the conditional is true, but rather start with the line, in which the antecedent and the consequent are true. Effort put into the task and other factors might help participants to flesh out a complete "mental model" and produce the normatively correct solution.

More recently, the normative standard in this literature seems to change. Evans, Handley, and Over (2003) and Oberauer and Wilhelm (2003), for example, report results which show that subjects interpret conditionals in terms of conditional probabilities (in line with a probabilistic conditional logic semantics) rather than the material implication. For reasoning with counterfactual conditionals, Evans and Over (2004, Chapter 7) specifically discuss D. Lewis' system of spheres semantics. Furthermore, counterfactual reasoning is also investigated from a developmental perspective (e.g. Beck, Robinson, Carroll, & Apperly, 2006; Rafetseder, Cristi-Vargas, & Perner, 2010; Rafetseder & Perner, 2010). Schurz (2007) provides also evidence from a default logic perspective: He investigated reasoning with normic conditionals and regular indicative conditionals (cf. Section 1.2.2) in the context of conflicting information such as described by example (A) in

Section 1.2.2. His results show that human reasoning is line with probabilistic default reasoning and conditional logic approaches rather than a material implication analysis.

Pfeifer and Kleiter (in press), moreover, give an overview over a range of their probabilistic reasoning experiments, in which specific axioms of system **P** (Adams, 1965, 1966 or alternatively Kraus et al., 1990 and Lehmann & Magidor, 1992) were investigated and contrasted them with inferences S3′–S5′ (Transitivity, Monotonicity and Contraposition, respectively) from Chapter 1. Their results show that subjects' reasoning concurs largely with the inferences predicted by the conditional logic semantics.

2.1.6 The Non-Monotonic Reasoning Project

Starting in the late 70s in computer science and in particular in the artificial intelligence community a number of logical systems were developed, which deviate from p.c. and f.o.l. in an essential and more radical way than, for example, modal logic (e.g. Hughes & Cresswell, 1996/2003). Central to this deviation is the notion of a consequence relation.[3] A consequence relation describes whether a given formula α is logically implied by a given set of formulas Γ (for short: $\Gamma \vDash \alpha$). In p.c., f.o.l. and modal logic (e.g. Hughes & Cresswell, 1996/2003) this consequence relation is monotonic. That means that if $\Gamma \vDash \beta$, then it holds for any α that $\Gamma \cup \{\alpha\} \vDash \beta$. So, no matter what one adds to Γ, the formulas implied by Γ are still consequences of $\Gamma \cup \{\alpha\}$. This principle, however, does not hold generally for the systems investigated in the computer science literature. Since this seems to be the characteristic feature of this approach, these systems are often called 'non-monotonic logics'.

A central motivation for the study of this type of formalism is the observation that the notion of defaults (a type of non-monotonic rule) is required in order to provide an adequate account of general intelligence (McCarthy & Hayes, 1969; McCarthy, 1980, 1989). Since this motivation is closely related to the notion of defaults and non-monotonic rules discussed in this literature, we will investigate

[3] One can, alternatively, describe the following by means of a proof-theoretic derivability relation.

this issue in more detail in Section 2.2.

Interestingly the impact of the non-monotonic reasoning literature on the conditional logic literature is very limited. We are not aware of any paper from the latter area, which refers explicitly to the non-monotonic reasoning literature. The conditional logic approach, however, is well-perceived in the non-monotonic reasoning literature (e.g. Ginsberg, 1986; Delgrande, 1987, 1988; Kraus et al., 1990; Nejdl, 1992; Lehmann & Magidor, 1992; Delgrande, 1998; Schurz, 1998).

2.2 General Intelligence, Defaults, Non-Monotonic Rules and Conditionals

In this section we introduce an important distinction, namely between conditional logics and default logics. Before we do this, it seems, however, sensible to provide a motivation for the study of default logics and non-monotonic logics first.

2.2.1 A Motivation for the Study of Non-Monotonic Logics

McCarthy and colleagues (McCarthy & Hayes, 1969; McCarthy, 1980, 1989) describe a motivation for non-monotonic reasoning formalisms. McCarthy and colleagues aim to lay the groundwork for a general formal account of intelligence. They are, however, not interested in mechanisms, which solve problems on the basis of pre-established formal representations of problems. Instead, they focus on how representations of the world must be, such that "the solution of the problems follows from the facts expressed in the representation" (McCarthy & Hayes, 1969, p. 466). Their approach is, hence, akin to the logical programming paradigm (i.e. the programming language PROLOG). In this paradigm a program is not told what to do, but is instead told what is true and asked to draw conclusions from those specifications (Clocksin & Mellish, 2003, p. 255).

For a discussion of adequate representations of the world, let us focus on a standard problem from computer science, namely the cannibals and missionaries problem:

> "Three missionaries and three cannibals come to a river. A row-

> boat that seats two is available. If the cannibals ever outnumber the missionaries on either bank of the river, the missionaries will be eaten. How shall they cross the river[?]" (McCarthy, 1980, p. 29)

For an adequate representation of this problem, we have to specify that there exist three cannibals (say c_1, c_2 and c_3), three missionaries (m_1, m_2 and m_3), a boat (b), etc. Let us call the resulting set of factual formulas Γ. However, in order for the agent to draw any (relevant) conclusion from the problem situation, we are forced to introduce in addition certain closed-world conditions, namely that there are no more cannibals, missionaries, boats than specified, that the boat is the only means to cross the river etc. (c.f. McCarthy, 1980, p. 30). We can specify these constraints in f.o.l., namely by adding formalizations of the closed world condition (such as 'the boat is the only means to cross the river') to Γ. Let us call the set of closed world assumptions Δ. Such an approach, however, has one great disadvantage. If we change the problem situation slightly – for example by including a bridge, on which both the cannibals and missionaries can cross the river – the representation in f.o.l. becomes inconsistent. This is due to the fact that given this change still all closed world conditions specified earlier hold (i.e. that the boat is the only means to cross the river.), due to the monotonicity principle. Given this rigidity, such a representation seems hardly adequate for a general account of intelligence, since for a general account of intelligence one is interested in finding non-arbitrary representations of the world. The above specifications in Δ, however, seem rather arbitrary, since we have to introduce for each such situation different sets of closed world assumptions, although the problem situation might only differ slightly.

Let us take a closer look at the above representation of the problem situation. The set of factual formulas Γ specified above is not problematic. Rather the addition of the set of closed world assumptions Δ created this inflexibility. To avoid this inflexibility, we are, hence, not allowed to include the closed world assumptions unconditionally, but only on the basis of certain preconditions. But on which preconditions? How should we account for this closed world conditions then?

In the non-monotonic literature the notion of default is used to specify those closed world conditions. Defaults, informally stated, are rules of the form "in absence of information to the contrary, assume …"(Reiter, 1980, p. 81). Note that this rule allows for exceptions: If we have information to the contrary, we are not allowed to draw the conclusion. However, for a consequence relation to be monotonic, no such exceptions are allowed to hold. To see this more clearly let us first specify formally, what these defaults look like.

2.2.2 Reiter Defaults

The type of defaults discussed in the preceding section can be described in terms of Reiter's formalism (Reiter, 1980, p. 88f). Defaults in Reiter's approach have the following form:[4]

$$\frac{\alpha : \mathbf{M}\beta_1, \ldots, \mathbf{M}\beta_n}{\gamma} \quad \text{(for } n \in \mathbb{N}\text{)}$$

Here, $\mathbb{N}$ is the set of positive integers. The formulas α and γ are the prerequisite of the default and its consequent, respectively (Reiter, 1980, p. 88). The above Reiter default is to be read the following way: if α is the case and $\beta_1, \ldots \beta_n$ are possible, then one is allowed to conclude γ.

In Reiter (1980), defaults are interpreted in the context of extensions of default theories. A default theory Θ is an ordered pair $\langle \Delta', \Gamma \rangle$, where Δ' is a set of defaults (as specified above) and Γ a set of f.o.l.-formulas. An extension of such a default theory extends the set Γ in such a way that it is deductively closed under f.o.l. and closed under the set of defaults Δ'. Note here that the expressions $\mathbf{M}\beta_1, \ldots, \mathbf{M}\beta_n$ (for $n \in \mathbb{N}$) in the Reiter default described above essentially boil down to $\neg\beta_1, \ldots,$ $\neg\beta_n$ are not in the extension under consideration (c.f. Reiter, 1980, p. 89).

To specify extensions formally, let us denote the set of f.o.l.-theorems of a set of formulas Γ by $\mathrm{Th}(\Gamma)$. Then, an extension E of a default theory $\Theta = \langle \Delta', \Gamma \rangle$ is a set of f.o.l.-formulas, which satisfies the following conditions: (a) $\Gamma \subseteq E$, (b) $E = \mathrm{Th}(E)$, and (c) if $\alpha : \mathbf{M}\beta_1, \ldots, \mathbf{M}\beta_n / \gamma$ is a default in Δ'[5], then the following holds: if $\alpha \in E$ and $\neg\beta_1, \ldots, \neg\beta_n \notin E$, then $\gamma \in E$.

[4] We restrict ourselves to closed defaults. This type of default contains exclusively closed f.o.l.-formulas, namely formulas, which are not allowed to contain free individual variables (c.f. Reiter, 1980, p. 88).

[5] Here '/' stands for a conclusion indicator as used in the Reiter default described above.

To describe the defaults discussed in the previous section and to apply them to normic conditionals (cf. Section 1.2.2), a less general framework suffices, namely normal defaults. Reiter (1980, p. 94f) specifies normal defaults the following way:

(D1) $$\frac{\alpha : \mathbf{M}\beta}{\beta}$$

This type of default has only one possibility requirement and this requirement refers, in addition, to the consequent of the default. Schurz (1994) shows that normal defaults of type (D1) can alternatively be read as normic conditionals of the form $\alpha \Box\!\!\rightarrow \beta$. For that purpose Schurz (1994) gives an interpretation of modified default theories $\Theta = \langle \Delta', \Gamma \rangle$, in which Δ' contains only normal defaults, in terms of probabilistic conditional logic semantics of Adams (1975). In Schurz's framework extensions E of default theories Θ correspond to sets of default assumptions, which are generated by E (see Schurz, 1994, esp. Theorems 4 and 5, p. 255f): Whenever a f.o.l.-formula α is added to an extension of a default theory by means of a normal default $\beta \Box\!\!\rightarrow \alpha$, all other facts β in Γ must be regarded irrelevant insofar as they do not conjointly with Δ' and other f.o.l. formulas in Γ imply $\neg\alpha$. A set of default for an extension E for a default theory Θ, which corresponds to a set of default assumptions encodes, then, the irrelevance assumptions, which are made in the course of the construction of extension E.[6]

In the next section we will we reformulate the possibility requirements first in terms of non-derivability and consistency conditions. Although from a technical perspective this is quite trivial, these reformulations allows us to see why normal Reiter defaults suffice for the specification of defaults discussed in the previous section. This procedure also allows to compare the default approach enacted in the non-monotonic reasoning literature more easily with conditional logic approaches.

[6] Schurz (1997b, p. 545f), then, extends this type of analysis to Poole (1988) extensions. We will, however, not focus further on this issue here.

2.2.3 Reformulations

In Reiter's (1980) formalism a possibility requirement ($\mathbf{M}\alpha$) of a default can be reformulated the following way:

C1 $\neg\alpha \notin \mathrm{Th}(\Gamma)$

Note that C1 makes the conditions for the application of defaults encoded in the definition of default theories explicit. Both Reiter's original formulation and C1 are purely syntactic and, hence, a part of a system's proof theory rather than of its model theory. This condition, however, has an equivalent formulation in terms of consistency:

C2 $\Gamma \cup \{\alpha\}$ is consistent

To see this, consider the following: If $\neg\alpha \in \mathrm{Th}(\Gamma)$, then the addition of α would make the set $\Gamma \cup \{\alpha\}$ inconsistent. Furthermore, if the set $\Gamma \cup \{\alpha\}$ is inconsistent, then it is the case that $\neg\alpha \in \mathrm{Th}(\Gamma)$. As can easily be seen, C2 is equivalent to the following condition:

C3 $\Gamma \nvdash \neg\alpha$

Here, $\Gamma \nvdash \alpha$ denotes that α is not a theorem of Γ (while $\Gamma \vdash \alpha$ states that α is a theorem of Γ).

2.2.4 Consistency, Non-Derivability and Default Logics

We can now describe the defaults discussed in the previous section in terms of (normal) Reiter defaults. We specified defaults informally as rules of the form: "in absence of information to the contrary, assume ..."(Reiter, 1980, p. 81). In the above example, we might construct a default such as 'in absence of information to the contrary, assume that the boat is the only means to cross the river'. Defaults of this type can be represented the following way: conclude α, if $\neg\alpha$ is not the case. This might sound awkward at first glance, although it is not as we will show shortly.

For a description of the cannibals and missionaries problem we start with a set of formulas of facts Γ and apply conditional closed world conditions on them. We add, for example,'the boat is the only means to cross the river', if the set Γ does not imply that the boat is not the only means to cross the river. More formally, this precondition corresponds to C1 and results in the following default rule: we add α to Γ , if $\neg\alpha$ is not a theorem of Γ. In terms of Reiter defaults this default is a normal default. It is, however, a degenerated one, since this default does not have an additional prerequisite.

Note in this context that we do not consider all logically possible normal defaults as candidates for closed worlds assumptions, but only normal Reiter defaults, which appear acceptable as normic conditionals in the given context. The default rule discussed in the previous two paragraphs is such an acceptable default assumption. An absurd, but logically possible default assumption would be the following: 'In absence of information to the contrary, every cannibal marries a missionary'. Observe that the latter default assumption would in the present context not hold up as an acceptable normic conditional.

Let us now focus on reformulations of the normal default rules. The present default rule can – due to the equivalence of C1 and C2 – also be reformulated the following way: we add α to Γ, if $\Gamma \cup \{\alpha\}$ is consistent. This formulation shows, why such defaults are more appropriate for the representation of closed world conditions than the simple addition of unconditional closed world assumptions. They allow to add a closed world assumption, such as 'the boat is the only means to cross the river', only if the resulting set is not inconsistent. This procedure, hence, allows to preserve the consistency of a set Γ and might be regarded as a consistency-based approach.

Let us reformulate this type of default in terms of C3. Then, we are allowed to add α to Γ given that $\Gamma \nvdash \neg\alpha$ holds. Non-derivability conditions are, however, not a part of any f.o.l. systems. F.o.l. is strictly based on rules, which only refer to derivability conditions. A canonical example of a f.o.l.-rule is the rule Modus Ponens. According to that rule we are allowed to conclude $\beta \in \Gamma$ for a formula β and a set of formulas Γ, if there is a formula $\alpha \in \Gamma$, such that $\alpha \rightarrow \beta \in \Gamma$ holds also. In the case of Modus Ponens the addition of formulas can also be described in terms of a f.o.l.-derivability relation $\vdash$ the following way: if $\Gamma \vdash \alpha \rightarrow \beta$ and $\Gamma \vdash \alpha$, then $\Gamma \vdash \beta$. The expression $\vdash$ is to be understood here in terms of f.o.l.-derivability.

Since defaults are designed to go beyond f.o.l., a characterization of these inferences in terms of the f.o.l.-derivability relation $\vdash$ is not appropriate. Hence, in the non-monotonic literature often the alternative symbol $\mid\!\sim$ is used instead. We might characterize the above default, thus, the following way: $\Gamma \mid\!\sim \alpha$, if $\Gamma \not\mid\!\sim \neg\alpha$. According to this analysis, Reiter defaults can, hence, be described the following way:

$$\frac{\Gamma \mid\!\sim \alpha\ , \Gamma \not\mid\!\sim \neg\beta_1, \ldots, \Gamma \not\mid\!\sim \neg\beta_n}{\Gamma \mid\!\sim \gamma} \quad \text{(for } n \in \mathbb{N}\text{)}$$

Note that $\mid\!\sim$ is to be understood always relative to a default theory $\Theta = \langle \Delta', \Gamma \rangle$. Moreover, unlike in the previous section the premises of the rule employ a non-monotonic inference relation $\mid\!\sim$ (derivability and non-derivability conditions). The main reason for this change is that $\mid\!\sim$ is used in such a framework to replace extensions. The non-monotonic inference relation is employed to specify the set of formulas, which follows from the set Γ of a default theory $\Theta = \langle \Delta', \Gamma \rangle$ relative to the defaults in Δ'. This set is, then, an extension in the above sense. Moreover, we can also describe normal Reiter defaults the following way:

$$\frac{\Gamma \mid\!\sim \alpha\ , \Gamma \not\mid\!\sim \neg\beta}{\Gamma \mid\!\sim \beta}$$

The inference relation $\mid\!\sim$ is, unlike f.o.l.-derivability relation $\vdash$, non-monotonic. (An inference relation R is monotonic exactly if from $\Gamma R \alpha$ follows $\Gamma \cup \{\beta\} R \alpha$ for any β.) The non-monotonicity property results from the reliance of default logics on non-derivability conditions in non-monotonic rules.[7] Default rules are, hence, also called 'non-monotonic rules' (Schurz, 2008, p. 55f; Schurz, 2001a, p. 373; cf. also Section 4.2.2). While monotonic rules only refer to derivability conditions, non-monotonic rules also refer to at least one (non-trivial) non-derivability

[7]Note that not all default rules might make the inference relation non-monotonic. The following normal default rule does not require inference relations to be non-monotonic: $\Gamma \mid\!\sim \alpha$, $\Gamma \not\mid\!\sim \top / \Gamma \mid\!\sim \gamma$. As $\top$ is a theorem of any set of formulas whatsoever, this type of default rule is inapplicable.

condition. Since it is characteristic for default logic approaches to include non-monotonic rules, we will regard any logical system, which includes (non-trivial) non-monotonic rules as a default logic.

Before we discuss the non-monotonicity of non-trivial default rules, let us compare non-monotonic rules with inductive inferences. An inductive inference is, for example, the following (cf. Section 1.2.2): All up to now observed fishes are cold-blooded. Hence, (probably) all fishes are cold-blooded (Schurz, 2008, p. 47). This inference is, according to f.o.l., logically not valid. For any finite number of fishes, which are cold-blooded, there might still be a larger number of fishes, which are not cold-blooded. Hence, the observation of a finite number of fishes being cold-blooded does not guarantee that all fishes or even the majority of fishes are cold-blooded. Nevertheless, we would accept the inference in many practical situations (where the number of observed fishes is large enough). Note, however, that this inference is non-monotonic. Let Γ' be the observation that n fishes ($n \in \mathbb{N}$) are cold-blooded. Suppose we are allowed to draw the inference $\Gamma' \mathrel{|\!\sim} \alpha$ for a sufficiently high n, where the formula α represents the state of affairs that all (most) fishes are cold-blooded. This inference, however, does not hold under all circumstances. Let β represent the (unlikely) observation that m fishes ($m \in \mathbb{N}$) with $m > n$ are not cold-blooded. Then, it does not hold that $\Gamma' \cup \{\beta\} \mathrel{|\!\sim} \alpha$.

Both non-monotonic rules and inductive inferences share that the truth of the formulas in Γ' does not guarantee the truth of the conclusion α, whatever the circumstances. In the case of non-monotonic rules, circumstances are possible, which qualify as exceptions, as indicated by non-derivability conditions (see below). In the case of inductive inferences there might still be convincing counter-evidence that blocks the inductive inference. Carnap went as far as to postulate two different types of logics: deductive logics and inductive logics. Although this terminology is controversial (Schurz, 2008, p. 47), we can identify logical systems, which draw only on monotonic rules as deductive logics and default logics as inductive logics (cf. Schurz, 2008, pp. 54–56).

Given this analysis, it seems hardly surprising that a general account of intelligence is in need of a default logic approach. For intelligent behavior, agents have to be able to draw both deductive and inductive inferences. Default logics, however, allow – contrary to f.o.l. – for the representation of inductive inferences, as required for a non-rigid account of closed world conditions (of problem situations). In the framework of McCarthy and colleagues the agents are, then, able to draw inferences (e.g. that the boat is the only means to cross the river) from the absence of other facts (e.g. the boat is the only means to cross the river that is *explicitly mentioned* in the problem description). Moreover, as non-derivability conditions can also be formulated in terms of consistency requirements, default logics represent a consistency-based approach, which relies on consistency requirements instead.

Prima facie these closed world conditions might seem very natural and almost self-evident. From a formal perspective they are by far non-trivial and require a major deviation from f.o.l. Note that the exact specification of those closed world conditions is no easy task. Moreover, closed world conditions, as discussed here seem to be closely connected to Gricean conversational implicatures (see Section 1.3). For these an exact specification is also far from obvious.

2.2.5 Non-Derivability and Axiomatization of Default Theories

We might ask ourselves why our reformulation of Reiter's (1980) approach in terms of extensions refers to the notion of derivability (non-derivability) of formulas from sets of formulas (i.e. $\Gamma \vdash \alpha$) instead of derivability (non-derivability) of formulas simpliciter (i.e. $\vdash \beta$). Here $\vdash \beta$ abbreviates derivability of β from the empty set. Let us, however, first describe derivability and non-derivability relations from a formal perspective. Derivability of a formula α from a formula set Γ in f.o.l. (short: $\Gamma \vdash \alpha$) is always reducible to derivability of a finite subset Γ' of Γ. Hence, it holds that $\Gamma \vdash \alpha$ exactly if $\Gamma' \vdash \alpha$ applies, which holds exactly if $\vdash \bigwedge \Gamma' \to \alpha$ is the case.[8] The expression $\bigwedge \Delta$ represents the conjunction of all elements in Δ. In a similar vain we can describe non-derivability. A formula α is non-derivable from Γ (short: $\Gamma \nvdash \alpha$) exactly if there is no finite subset Γ' of Γ, such that $\Gamma' \vdash \alpha$. This condition holds exactly if there is no finite subset Γ' of Γ, such that $\vdash \bigwedge \Gamma' \to \alpha$. Let us now take a look at Reiter's (1980) default logic approach.

In Reiter's (1980) account derivability and non-derivability conditions of a

[8]These equivalences rest on a somewhat stronger monotonicity property of $\vdash$ in f.o.l. than discussed before, namely $\Gamma' \vdash \alpha$, if $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Gamma'$.

default theory $\Theta = \langle \Gamma, \Delta' \rangle$ essentially depend on the set of formulas Γ and supersets of Γ. Extensions of such a default theory are construed by adding formulas to Γ according to the default rules in Δ'. Each application of such a rule "extends" Γ and results finally in so-called extensions of the default theory Θ. The sets referenced by defaults, hence, may change steadily.

If we construct derivability (non-derivability) conditions in terms of formulas alone (i.e. $\vdash \alpha$), we only check whether a given formula α is derivable (non-derivable) simpliciter, without reference to a set Γ'. In the case of Reiter defaults we are, however, neither interested in derivability conditions simpliciter nor in consistency-checks of formulas. Instead, we rather need to know whether a formula α follows from a particular set of formulas Γ' or is consistent with it. This is due to the following facts: In Reiter defaults, sequences of applications of default rules are permitted. In the course of these applications, the parameter Γ' might be referenced repeatedly and, thus, be changed. Hence, in these cases we cannot replace the (non-)derivability condition of a formula α from a formula set Γ' in a rule by a (non-)derivability of a single formula β for some formula β. Notice that we can only replace $\Gamma' \vdash \alpha$ by $\vdash \bigwedge \Gamma \to \alpha$, where $\bigwedge X$ denotes the conjunction of all formulas in a formula set X, if we presuppose that Γ' is finite. In the case Γ' is not finite, we only have that $\Gamma' \vdash \alpha$ holds iff there is a finite subset Γ'_f of Γ' such that $\Gamma'_f \vdash \alpha$ is the case. In order to eliminate $\Gamma' \vdash \alpha$ in a rule we, however, need to be able to reference to Γ and not just to Γ'_f. Since we do not presuppose that the set Γ' is finite, the latter strategy will not work. However, only in the case of derivability of single formulas (i.e. $\nvdash \beta$) derivability conditions are tantamount to consistency-checking conditions of single formulas (i.e. β).

In other cases reliance on derivability (non-derivability) conditions of formulas from sets (i.e. $\Gamma \vdash \alpha$) is inessential. For example, in the system of Adams (1975; see also Schurz, 1997b, 1998) there exists a non-monotonic rule, which draws on a consistency condition. Let us represent conditionals, as in Chapter 1, in terms of formulas of the form $\alpha \mathrel{\Box\!\!\to} \beta$, where α and β represent the antecedent formula and the consequent formula, respectively. The non-monotonic rule in Adams (1975, Rule R7, p. 61) gives us, then, that any set of formulas, which contains both formulas $\alpha \mathrel{\Box\!\!\to} \beta$ and $\alpha \mathrel{\Box\!\!\to} \neg\beta$, is inconsistent, provided α is consistent. In natural language this seems plausible, since it is, for example, natural to regard both conditionals 'if Anna goes shopping she is happy' and 'if Anna goes shopping she is not happy' as contradicting each other.

To account for Adams' (1975) consistency principle, we have to require that the antecedent of such a conditional formula is consistent. Otherwise it would contradict a very intuitive and central axiom of conditional logic called 'Refl', namely $\alpha \;\Box\!\!\rightarrow\; \alpha$ (see Sections 3.5.3). Since this is the only precondition for Adams' rule, we can specify it the following way:[9]

RCNC if $\nvdash_{p.c.} \neg\alpha$ then $\neg((\alpha \;\Box\!\!\rightarrow \beta) \wedge (\alpha \;\Box\!\!\rightarrow \neg\beta))$

'RCNC' stands for 'Restricted Conditional Non-Contradiction'. Note that $\neg\alpha$ being non-derivable is tantamount to α being consistent (see Section 2.2.3). RCNC is, since it refers to a non-trivial non-derivability condition, a non-monotonic rule. Thus, as a variant of RCNC is used in Adams' (1975) logic, Adams' (1975) system should be regarded as a genuine default logic. Observe that Adams' (1975) conditional logic is a default logic in a weaker sense than, for example, Reiter's (1980) default logic and presumably also than system **Z** of Pearl (1990) and Goldszmidt and Pearl (1996), since the former refers only to non-derivability of single formulas in p.c. Note that Adams' alternative conditional logic systems (Adams, 1965, 1966, 1986) do not endorse any kind of non-monotonic rule (cf. Schurz, 1998, p. 84; see also Section 3.5.3). Adams' own formulation of RCNC differs from our exposition here, since it does not allow for boolean combinations of conditional formulas (see Section 3.4.3). In a rich enough language, however, both variants are equivalent (cf. Section 3.5.3).

2.2.6 Problems of Present Default Logic Accounts

So far we focused on default logics in line with Reiter (1980) in proof-theoretic terms. For $\mid\!\sim$ we often used the somewhat more neutral term 'inference relation'. In the non-monotonic literature, however, the relation $\mid\!\sim$ is more often described as a model-theoretic consequence relation rather than a proof-theoretic derivability relation (e.g. Makinson, 1994). Authors in the non-monotonic literature, however, are not very specific as to why they use a model-theoretic notion rather than

[9] For a formal definition of logical connectives etc. see Chapter 4.2.1.

a proof-theoretic one. One motivation, however, might be that often no syntactic procedure exists, which formalizes the set of non-derivable formulas. We will inquire this issue in some more detail in this section.

Typical examples for recent default logics are Pearl (1990), Goldszmidt and Pearl (1996), Levesque (1990) and Giordano, Gliozzi, and Olivetti (2005). In the approaches of Levesque (1990) and Giordano et al. (2005) non-monotonic rules are used, which require that either (a) a formula is falsifiable (axiom NvsB, Levesque, 1990, p. 286) or (b) that a formula is not an element of a deductively closed set (B*4, Giordano et al., 2005, p. 12). It should be clear from the discussion in foregoing sections that both systems cannot be represented only in terms of monotonic rules, but need to employ in addition non-monotonic rules.[10]

Although Levesque (1990) and Giordano et al. (2005) employ these preconditions, they do neither axiomatize nor formalize them. They regard them rather as a part of the meta-language, which governs the application of the axioms.[11] This procedure is, however, problematic. It is far from evident, how these non-derivability conditions should be axiomatized. This problem has a strong impact on the nature of proofs in that system. Levesque (1990, p. 287), for example, provides a proof in his propositional system, in which he refers to his principle NvsB. Instead of explicitly requiring in the proof that the formula under consideration is falsifiable, he informally restricts the set of formulas, for which the proof ought to apply (p. 287), so that the precondition is satisfied.

The main problem of these approaches is that there is no standard procedure to account for non-derivability proofs. Hence, if we have to refer in a default logic to non-derivability conditions, it remains unclear whether there exists a syntactic, mechanical procedure to infer the non-derivability of a formula in a finite number of steps. The existence of such a procedure can, however, be regarded as the minimum requirement for an adequate proof-theoretic characterization of a logical system. Otherwise the distinction between the model theory and the proof theory (and, hence, soundness and completeness proofs) would not make much sense.

[10]Levesque (1990, p. 286) refers to the falsifiability of formulas in an "object language" proof. Since falsifiability is a semantic concept, it is preferable to employ the concept of consistency instead. We shall enact such a syntactic approach here. We will, however, postpone a general discussion of this issue to the end of this section.

[11]Note that Reiter (1980) neither gives a formalization of the operator **M** (see Section 2.2.2).

How can one axiomatize non-monotonic rules then? It should be clear that the default logic of Adams (1975) is finitely axiomatizable. Such a result is to be expected due to the fact that this system exclusively draws on the non-derivability of single formulas in p.c. But how to axiomatize the default logic of Adams (1975)? One strategy is to isolate all theorems, which require reliance on non-derivability conditions and add them to the set of axioms. In a variation of Adams' (1975) system, we might include any formula $\neg((\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta))$, in which α is an atomic proposition. Note, however, that this procedure does not give us all substitution instances of RCNC. Moreover, an exact delineation of the theorems to be added is not an easy task. In the somewhat less complex default logic $\mathbf{C}^{\mathbf{def}}$of Carnap this procedure is a viable method (cf. Hendry & Prokriefka, 1985; cf. Schurz, 2001a, p. 39).[12] For other default logic systems such an approach is not available. Moreover, it is not clear that other default logics, which do not only draw non-derivability in p.c. are finitely axiomatizable. This is, however, required for a finite, mechanical notion of proof (cf. Enderton, 2001, p. 109).

More natural than the first approach is a simultaneous axiomatization of the sets of derivable and non-derivable formulas. Schurz (2001a, 1996) provides a full axiomatization of theorems and non-theorems for Carnap's system $\mathbf{C}^{\mathbf{def}}$ and a set of default logics discussed in the non-monotonic literature (e.g. Nute, 1992), respectively. This approach is, however, exceptional in the sense that the issue addressed and its solution are largely ignored in the non-monotonic reasoning literature. The axiomatization of system $\mathbf{C}^{\mathbf{def}}$, on the other hand, is also not broadly discussed.

A disadvantage of the latter approach is that it is not always applicable. Note that the set of theorems and non-theorems encompasses the whole set of formulas. In order to be able to axiomatize sets of formulas – as required by our notion of syntactic proof – these sets have to be decidable, viz. there has to exist a syntactic, mechanic and finite procedure to check whether an arbitrary formula is a member of either set. This, however, implies that both sets (theorems and non-theorems) have to be decidable. This is only the case, as Kleene's theorem tells us (Enderton, 2001, Theorem 17F; c.f. Schurz, 2004, pp. 42–44), if the whole set of formulas

[12]We use here the expression $\mathbf{C}^{\mathbf{def}}$ rather than $\mathbf{C}$ (Schurz, 2001a, p. 365) to avoid ambiguity, since we discuss also Kraus et al.'s (1990, p. 176) system $\mathbf{C}$ in Section 7.2.1.

is decidable. Since f.o.l. is undecidable (Church's Theorem, c.f. Enderton, 2001, §3.5), it remains far from clear how f.o.l. defaults logics, such as Levesque's (1990) and Reiter's (1980) systems, should be completely axiomatized. For a specific set of logics, this might, however, be viable: In case a logic is a complete f.o.l. theory, it is decidable and, hence, allows for a direct axiomatization.

Finally, note that in default logics valid formulas are in general not closed under substitution (cf. Schurz, 2004, pp. 35–38), viz. the rule of substitution is not validity-preserving. Hence, in such default logics, the inference relation is *not* logical in the sense that it is not structural, viz. it is not structure-invariant. This is unlike systems which enact only monotonic rules, such as p.c. and f.o.l. In order to see this, let us take a look at Adams' (1975) RCNC rule. Although $\neg((p \rightarrow q) \wedge (p \rightarrow \neg q))$ is a theorem in such a system, $\neg((p \wedge \neg p \rightarrow q) \wedge (p \wedge \neg p \rightarrow \neg q))$ is not, since $p \wedge \neg p$ is inconsistent. There exists, however, a more restricted rule of substitution (isomorphic substitutions) which preserves validity of formulas in default-logics (cf. cf. Schurz, 2001a, p. 370; Schurz, 2004, p. 38).

2.2.7 Non-Monotonic Logics, Conditional Logics and Default Logics

In Chapter 1 we argued that, for an adequate representation of natural language conditionals, a conditional logic approach is needed which goes beyond p.c. and f.o.l. In this chapter we saw that p.c. and f.o.l. neither suffice for a general account of intelligence nor for a general representation of inductive inferences. For the latter purposes, however, a major deviation in terms of default logics is needed. Both, default logic formalisms and conditional logic formalisms, are in a sense non-monotonic, and are, hence, often subsumed under the label 'non-monotonic reasoning'. Despite this fact, however, there are strong differences between both approaches.

In conditional logics, such as Stalnaker and Thomason (1970) and D. Lewis (1973/2001), a conditional connective, such as $\Box\!\!\rightarrow$, is used to represent conditional structures from natural language (see Section 1.1). This conditional connective is a two-place (modal) operator which connects two formulas, namely an antecedent formula α and a consequent formula β in terms of $\alpha \Box\!\!\rightarrow \beta$. The formal expression $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ is an object language formula and can in standard conditional logic approaches, such as Stalnaker and Thomason (1970) and D. Lewis (1973/2001), be negated, iterated, nested and so on. Hence, in these accounts expressions, such as $\neg(\alpha \mathrel{\Box\!\!\rightarrow} \beta)$, $\alpha \mathrel{\Box\!\!\rightarrow} (\beta \mathrel{\Box\!\!\rightarrow} \gamma)$ and $\alpha \mathrel{\Box\!\!\rightarrow} (\delta \wedge (\beta \mathrel{\Box\!\!\rightarrow} \gamma))$ are also regarded as formulas (see also Chapter 4.2.1).

In default logics the inference relation $\mathrel{|\!\sim}$ is due to non-monotonic rules non-monotonic. Note that $\mathrel{|\!\sim}$ is not a part of the object language, but a meta-language expression which describes the formal relation between sets of object language formulas (i.e. Γ) and object language formulas (i.e. α). Hence, $\Gamma \mathrel{|\!\sim} \alpha$ is not an object language formula. In addition, nestings and iterations of $\mathrel{|\!\sim}$ neither result in object language nor in meta-language expressions.

On the one hand a non-monotonic conditional operator can be introduced without making the inference relation non-monotonic. This is actually the case for most conditional logics, such as Stalnaker and Thomason (1970) D. Lewis (1973/2001), Adams (1966), Segerberg (1989) and Delgrande (1987). On the other hand, we can have default logics which do not have a conditional operator and whose non-monotonic inference relation does not lend itself into a plausible interpretation in terms of conditional assertions (see below). Examples for such default logics are Reiter's (1980) and Levesque's (1990) systems. These systems do not possess a conditional operator or the like. Moreover, it is quite implausible to interpret the non-monotonic inference relation in either of these systems as a type of conditional assertion. There are, however, logics which are both conditional logics and default logics. Adam's (1975) system is a particularly prominent example for such a system and also system **Z** by Pearl (1990) and Goldszmidt and Pearl (1996). Also the system of Giordano et al. (2005) can be regarded as such a hybrid logic.

Given this discussion, the results from the default logic literature seem to bear only superficially on a conditional logic approach. This, however, is not the case. First, often intuitively plausible principles, such as Adams' rule RCNC (see above) are postulated for a conditional logic approach, without making explicit which formal consequences such an approach has (see Section 2.2.5). Bennett (2003, p. 84), for example, employs a stronger version of this principle when arguing against an objective interpretation of conditionals. He, however, seems unaware of the consequences of his approach (see Section 3.7). In addition, there is a number of studies which associate conditional logics with belief revision (e.g. Lindström & Rabinowicz, 1995; Giordano et al., 2005). This is not an easy task, since AGM belief revision (Alchourrón, Gärdenfors, & Makinson, 1985) – the dominant approach in belief revision – conflicts with a Ramsey-test interpretation of conditionals (Gärdenfors, 1986). Note, however, that the belief revision approach is consistency-based, and, hence, in our terminology a default logic account. In the literature this is, however, often not taken seriously.

Second, in the non-monotonic reasoning literature many formalisms are discussed which are formulated in default logic terminology. These formalisms, however, can sometimes easily be translated into a conditional logic terminology and bear directly on the conditional logic literature (i.e. Kraus et al., 1990; Lehmann & Magidor, 1992). To distinguish between both, we have to be able to distinguish between mere default logic terminology and essential default logic assumptions. The discussion of the preceding sections enables us to do that.

Kraus et al. (1990) and Lehmann and Magidor (1992) use a non-monotonic inference relation $\mid\!\sim$ for the discussion of non-monotonic systems. They, however, do not specify $\mid\!\sim$ in terms of a relation between sets of formulas (i.e. Γ) and formulas (i.e. α), but rather as a relation between formulas (i.e. $\alpha \mid\!\sim \beta$). A so-called conditional assertion $\alpha \mid\!\sim \beta$ is, then, interpreted the following way: if α, normally β (Kraus et al., 1990, p. 173).

Kraus et al. (1990) and Lehmann and Magidor (1992), hence, do not interpret $\mid\!\sim$ in terms of a formal, logical inference relation, but in terms of a material inference relation. A material inference relation differs from a formal, logical version insofar, as it is not closed under isomorphic substitution, viz. it is not closed under the substitution of atomic propositional letters by (other) atomic propositional letters (cf. Schurz, 2001a, p. 370; Schurz, 2004, p. 38). Hence, in the framework of Kraus et al. (1990) and Lehmann and Magidor (1992), the fact that $p_1 \mid\!\sim q_1$ holds does not guarantee that its isomorphic substitution $p_2 \mid\!\sim q_2$ holds, too.

These considerations, hence, suggest that conditional assertions (i.e. $\alpha \mid\!\sim \beta$) in the framework of Kraus et al. (1990) and Lehmann and Magidor (1992) correspond to conditional formulas (i.e. $\alpha \mathrel{\Box\!\!\rightarrow} \beta$) rather than to non-monotonic inference relations in the sense of Reiter (1980). This argument is strengthened by two facts. First, we can identify a second-order inference relation for conditional assertions of type $\alpha \mathrel{|\!\sim} \beta$. This second-order inference relation is, however, monotonic (e.g. for system **P**, Lehmann & Magidor, 1992, p. 9). Hence, the Kraus et al. (1990) and Lehmann and Magidor (1992) systems should be regarded as conditional logics with a monotonic (second-order) inference relation.

Second, the systems of Kraus et al. (1990) and Lehmann and Magidor (1992) do not employ non-monotonic rules with non-derivability conditions in the sense of Reiter (1980). Although Kraus et al. (1990)[13] and Lehmann and Magidor (1992, p. 5) allow for negations of conditional assertions in the meta-language (i.e. $\alpha \not\mathrel{|\!\sim} \beta$), Kraus et al. (1990) do not use this notion in their specification of rules. Lehmann and Magidor (1992, p. 18f), however, discuss, for example, the following rule:

RM $$\frac{\alpha \mathrel{|\!\sim} \gamma \,,\, \alpha \not\mathrel{|\!\sim} \neg\beta}{\alpha \wedge \beta \mathrel{|\!\sim} \gamma}$$

'RM' abbreviates here 'rational monotonicity'.[14] The rule RM appears to be non-monotonic, since it relies on a non-derivability condition (or its semantic equivalent). Hence, system **R** of Lehmann and Magidor (1992) would qualify as a default logic. Observe, however, that the expression $\alpha \not\mathrel{|\!\sim} \neg\beta$ is somewhat ambiguous. A semantic interpretation of RM in terms of models makes the interpretation of $\not\mathrel{|\!\sim}$ more perspicuous: If $\alpha \mathrel{|\!\sim} \gamma$ holds in a system **R** model and $\alpha \mathrel{|\!\sim} \neg\beta$ is *not* the case in that model, then also $\alpha \wedge \beta \mathrel{|\!\sim} \gamma$ holds in that model. Hence, $\alpha \not\mathrel{|\!\sim} \neg\beta$ should not be interpreted in terms of non-derivability in the sense of default logics discussed earlier. The latter interpretation in terms of "non-derivability" conditions would require that the set $\{\alpha, \beta\}$ is consistent in a logical system such as p.c. (see Section 2.2.3). This, however, would not make too much sense in the present context.[15] Given this analysis, it is much more plausible to interpret the "non-derivability"

[13] Kraus et al. (1990) do not explicitly indicate whether $\alpha \not\mathrel{|\!\sim} \beta$ is a part of their meta-language. However, their use of $\not\mathrel{|\!\sim}$ suggests that they do (e.g. p. 184 and p. 188).

[14] We use here a slightly different, but equivalent version of this principle (c.f. Lehmann & Magidor, 1992, p. 18).

[15] In addition, the semantics of Kraus et al. (1990) and Lehmann and Magidor (1992) behaves like a conditional logic rather than a typical default logic, such as Levesque (1990) or Carnap's system $\mathbf{C^{def}}$ (Schurz, 2001a).

condition of RM – translated in our conditional logic language – as $\neg(\alpha \mathrel{\Box\!\!\rightarrow} \beta)$ being the case. Hence, both the Kraus et al. (1990) and Lehmann and Magidor (1992) systems should be understood as conditional logics rather than a genuine default logics.

Chapter 3

A Defense of Possible Worlds Semantics for Indicative Conditionals

The present chapter serves at least three purposes. First, we provide an overview over formal-philosophical approaches to conditionals. We start with Ramsey-test interpretations and describe some important standard conditional logics with possible world semantics, such as D. Lewis, 1973/2001 and Stalnaker (1968; Stalnaker & Thomason, 1970; see Section 3.2). Then, we discuss the difference between indicative and counterfactual conditionals w.r.t. formal semantics for both types of conditionals (see Section 3.3). We, in addition, focus on foundational issues for probabilistic conditional systems (see Section 3.4) and describe in Section 3.5 probabilistic standard systems, such as Adams (1965, 1966, 1977), Adams, (1986; see also Schurz, 1998) and Adams (Adams, 1975; see also Schurz, 1997b).

Second, we aim to defend an account of indicative conditionals on the basis of possible worlds semantics, such as Chellas-Segerberg (CS) semantics (Chellas, 1975; Segerberg, 1989; Chapters 4–5). Our defense is targeted against the criticism of some philosophers, such as Adams (1975), Bennett (2003), Edgington (2007) and Gibbard (1980) against truth-value approaches. Adams (1975), Bennett (2003), Edgington (2007) and Gibbard (1980) argue in particular that conditionals do not in general have truth-values (NTV: "No Truth-Value") (Bennett, 2003, p. 94; see also Section 2.1.4). Since possible worlds semantics, such as CS-semantics and the Stalnaker (1968) semantics for indicative conditionals, are based on the notions of truth and falsehood, these semantics seem also prone to the criticism of NTV proponents (cf. Adams, 1975, p. 7; cf. Section 2.1.4). In our defense of possible worlds semantics, we will address two arguments against truth-value arguments put forth by Bennett (2003): (a) D. Lewis' (1976) triviality result and (b) Bennett's (2003, pp. 83–93) Gibbardian stand-off argument, which goes back to Gibbard (1980, p. 231f). We will describe our strategy of our defense of possible worlds semantics against (a) and (b) in more detail in the next section.

Third, the present chapter serves also as basis for our positive account of possible worlds semantics in terms of CS-semantics in Chapter 4–7. We shall in particular draw for our interpretation of CS-semantics in Section 7.1 on our discussion of indicative vs. counterfactual conditional logics (see Section 3.3) and the Ramsey-test (see Section 3.2). For that purpose we discuss in Section 3.2 different interpretations of the Ramsey-test, including Bennett's (2003) probabilistic interpretation and Stalnaker's (1968) original account. We, furthermore, describe the conditional logic system of Stalnaker (1968; Stalnaker & Thomason, 1970) and discuss some important differences between Stalnaker's semantics, which is interpreted by Stalnaker (1968) in terms of the Ramsey-test, and CS-semantics. We also distinguish between possible worlds semantics, which can directly be interpreted in terms of the Ramsey-test such as CS-semantics, and ordering semantics, such as D. Lewis (1973/2001), Burgess (1981), Kraus et al. (1990) and Lehmann and Magidor (1992). In the latter approach orderings of possible worlds serve as basis for the semantics rather than the Ramsey-test.

3.1 Our Defense of Possible Worlds Semantics

To defend possible worlds semantics for indicative conditionals, we discuss two arguments against truth-value accounts put forth by Bennett (2003): (a) D. Lewis' (1976) triviality result and (b) Bennett's (2003, pp. 83–93) Gibbardian stand-off argument. Bennett (2003) draws for (b) on pairs of conditionals with conflicting consequences, which were already investigated by Gibbard (1980, p. 231f).

Bennett discusses two further argument against truth-value accounts, (c) one by Edgington (Bennett, 2003, p. 102) and (d) one by Bradley (Bennett, 2003, p. 105). We focus on (a) and (b), since we regard both (a) and (b) much stronger than points (c) and (d). Bennett seems also to agree on that issue, since he dedicates much more space to (a) and (b) than to (c) and (d) (for c and d each less than a page).

Before we outline arguments (a) and (b), let us first describe the core idea of possible worlds semantics for conditionals. We use for that purpose CS-semantics (see also Chapters 4–5). CS-semantics and generally possible worlds semantics for conditional logics are based on the notion of truth at possible worlds. A formula α is, hence, not true simpliciter, but true at a possible world. For formulas not containing the conditional operator $\Box\!\!\rightarrow$, truth-value assignments are perfectly analogously to p.c.-semantics. The only difference to p.c.-semantics is that the truth-assignments in possible worlds semantics are relativized to possible worlds. The truth conditions of conditional formulas $\alpha \Box\!\!\rightarrow \beta$ at a world w, however, depend in CS-semantics on an accessibility relation between possible worlds w and w', relative to given subsets of the set of possible worlds W. In other words, the accessibility relation R_X is relativized to propositions $X \subseteq W$. In possible worlds semantics, for each formula α the set of possible worlds, at which α is true, is described by $\|\alpha\|$. A conditional formula $\alpha \Box\!\!\rightarrow \beta$ is, then, true at a possible world w in CS-semantics if and only if β is true at all worlds w' which are accessible from w relative to $\|\alpha\|$.

Let us now focus on argument (a). To specify (a) adequately, we have to describe some assumptions of subjective probabilistic accounts, as, for example, advocated by Adams (1965, 1966, 1977, 1975, 1986), Bennett (2003) and Edgington (2007) first. In such a framework all formulas, to which subjective probabilities are assigned, are regarded as propositions, which can either be true or false. A probability assigned to a formula α is, then, interpreted as the probability that α is true (see Section 2.1.4). Moreover, this approach endorses the Stalnaker thesis, namely that the probability of a conditional $\alpha \Box\!\!\rightarrow \beta$ equals the (subjective) conditional probability $P(\beta|\alpha)$. D. Lewis' (1976) triviality result, then, shows that a (subjective) probabilistic semantics for conditionals, as described by Adams (1965, 1966, 1977, 1975, 1986), cannot be extended to a full language $\mathcal{L}_{\mathrm{KL}}$, which allows for (i) conjunctions of conditional formulas (e.g. $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\gamma \mathrel{\Box\!\!\rightarrow} \delta)$) and (iii) conjunctions of conditional formulas and unconditional formulas (e.g. $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge \alpha$) (cf. D. Lewis, 1976, p. 304, see Section 3.6.1). We shall also show that admittance of iterations of conditional formulas (e.g. $\alpha \mathrel{\Box\!\!\rightarrow} (\beta \mathrel{\Box\!\!\rightarrow} \gamma)$) leads to counter-intuitive consequences (see Section 3.6.2) in Adams' approach. Schurz's approach (1997b, 1998) is not so much affected by D. Lewis' (1976) triviality result, since he does not assign probabilities to conditionals, but only associates conditional probabilities in the sense that $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ hold only if the respective conditional probability is sufficiently high (Schurz, 1997b, p. 536; Schurz, 1998, p. 85).

One way out of this dilemma is an approach taken by Adams (1965, 1966, 1977, 1975, 1986). He restricts the language in such a way that D. Lewis' triviality result cannot apply. For that purpose Adams does not allow for boolean combinations of conditionals and iterations of conditionals. Based on the fact that the conditional logic language cannot arguably be the full language – otherwise it is prone to D. Lewis' triviality result – Bennett, Adams, Edgington and Gibbard argue that boolean combinations of conditional formulas and nestings of conditional formulas do not represent propositions. Bennett, Adams, Edgington and Gibbard go even one step further to argue that also simple non-nested conditionals formulas do represent propositions (Bennett, 2003, p. 94; see also Adams, 1975, p. 7). Since in his view only propositions have truth-values (see Section 2.1.4), Bennett (2003) concludes that conditionals do not have truth-values.

We agree that, given one accepts all assumptions made by Bennett and others, it is plausible to conclude that conditionals do not have to truth-values. It is, however, quite another question (i) whether we should accept Bennett, Adams, Edgington and Gibbard's assumptions and (ii) whether these assumptions also apply to possible worlds semantics. Point (ii) is much less clear as it might seem first. This is due to the fact that in possible worlds semantics such as CS-semantics does not draw on the notion of probability, which is essential to D. Lewis' (1976) triviality result. So, how can D. Lewis' triviality result apply to possible worlds semantics, such as CS-semantics? We address the latter question in some detail in this Sections 3.5.4 and 3.6.4.

Regarding (b) the case is different. Bennett presupposes for his Gibbardian stand-off argument (i) a specific type of Ramsey-test interpretation of conditionals and (ii) a consistency criterion. With respect to point (ii) we distinguish between a (A) weak consistency criterion and (B) a strong consistency criterion. For a set of conditionals is inconsistent exactly if it contains the conditional $\top \mathrel{\Box\!\!\rightarrow} \bot$, where $\top$ and $\bot$ are defined as $p \vee \neg p$ and $p \wedge \neg p$, respectively. Bennett (2003), however, employs the strong consistency criterion (B), which renders any set of formulas that contains for given formulas α and β both $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ and $\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta$ inconsistent. Note here that while (A) is valid in most conditional logics, such as Adams (1965, 1966, 1977) and Kraus et al.'s (1990; see also Lehmann & Magidor, 1992), (B) is not. Bennett concludes from his argument that only a subjective probabilistic semantics allows to account for indicative semantics adequately. We discuss his argument in some detail and inquire in particular whether assumptions (i) holds generally and under which conditions (B) is warranted for possible worlds semantics and probabilistic semantics. We conclude that neither (i) nor (B) are conclusive for the following reasons: First, there exists semantics for conditionals, which do not employ the subjective Ramsey-test, such as objective frequency-based semantics for conditionals (see Section 3.4). Second, the only conditional logic system we are aware of which endorses a weakened version of consistency criterion (B) is Adams (1975). This system is, however, a default logic and, hence, suffers from the drawbacks of non-monotonic logics described in Section 2.2.6.

3.2 Ramsey-Test Interpretations and Possible Worlds Semantics for Conditionals

In this section we discuss different Ramsey-test interpretations starting from Ramsey's original proposal. We identify what we regard the core feature of the Ramsey-test and contrast it with further optional components. Our discussion is, then, aimed (i) to shed light on the relation between the Ramsey-test and the Stalnaker thesis, (ii) the question whether a consistency requirement is an essential part of the Ramsey-test and (iii) how Bennett's (2003) version of the Ramsey-test relates to other versions of the Ramsey-test proposed in the literature. Points (ii) and (iii) serve, then, as basis for our discussion of D. Lewis' (1976) triviality result

(Section 3.6) and Bennett's (2003) Gibbardian stand-off argument (Section 3.7).

3.2.1 Ramsey's Original Proposal

The Ramsey-test receives its name due to a footnote by Ramsey (1965, p. 247). In this footnote Ramsey (1965) expresses an intuition that is employed by others, such as Stalnaker (1968) and Bennett (2003, p. 28) to justify a subjective probabilistic approach to conditionals. There are, however, several versions of the Ramsey-test, which differ quite strongly from each other. We can, for example, distinguish between (i) a probabilistic version (Bennett, 2003, p. 28), (ii) a possible worlds version (Stalnaker, 1968, p. 101f) and (c) a belief revision version (cf. Lindström & Rabinowicz, 1995, p. 147f). All three versions refer to Ramsey's footnote, but differ significantly in what elements of Ramsey's original exposition they take into account.

We, furthermore, distinguish between Ramsey-test interpretations and ordering approaches (cf. Makinson, 1993). In an ordering approach conditionals are analyzed on the basis of rankings of entities. Conditionals receive truth-values (probability value) w.r.t. those entities, which are minimal according to that ranking, if there exists such minimal entities. In the literature both approaches are often not clearly distinguished. For example, Stalnaker (1968) describes Ramsey-test interpretations for possible worlds semantics in some detail. Despite this fact, Stalnaker (1968, p. 104f) interprets his semantics rather in terms of orderings of possible worlds. Note, however, that a Ramsey-test interpretation of his semantics is clearly possible.

Ramsey-test interpretations refer, at least indirectly, to the above-mentioned footnote by Ramsey (1965, p. 247). To evaluate conditionals, Ramsey (1965) suggested in that footnote the following procedure:

> "If two people are arguing 'If p will q?' and are both in doubt as to p, they are adding p hypothetically to their stock of knowledge and arguing on that basis about q; so that in a sense 'If p, q' and 'If $p, \bar{q}$' are contradictories. We can say they are fixing their degrees of belief in q given p." (Ramsey, 1965, p. 247)

The expression $\bar{p}$ stands for the negation of p (cf. Ramsey, 1965, p. xviii). The passage can be divided into the following three parts: (a) *If two people are arguing 'If p will q?' and are both in doubt as to p, they are adding p hypothetically to their stock of knowledge and arguing on that basis about q'*, (b) *so in a sense 'If p, q' and 'If p, $\bar{q}$' are contradictories* and (c) *We can say they are fixing their degrees of belief in q given p*.

Part (a) seems to describe the basic idea of the Ramsey-test and might be roughly summarized the following way: To evaluate a conditional, first hypothetically suppose that the antecedent holds and, then, test whether the consequent holds according to that supposition, too.[1] Passages (b) and (c) refer to optional components. Part (b) describes a consistency requirement. According to that requirement you are not allowed to arrive at both 'If p, q' and 'If p, $\neg q$'. Hence, according to the procedure described before, one is not permitted to conclude when supposing p that both q and $\neg q$ hold. This suggests a strong consistency criterion that makes two conditionals $\alpha \boxright \beta$ and $\alpha \boxright \neg\beta$ contradictory (cf. Section 3.1). We shall, however, postpose this discussion of point (b) to Section 3.2.5.

Moreover, passage (c) suggests a (subjective) probabilistic interpretation, since a subjective probability assessment of a specific statement is often considered to represent a degree of belief of an agent in that proposition (see Section 3.4). This reference to subjective probability is not surprising, since Ramsey is also regarded a pioneer of a subjective approach to probability (cf. Schurz, 2008, p. 101). The probabilistic conditional logic systems of Adams (1965, 1966, 1977, 1975, 1986) follow Ramsey insofar as they use a subjective probabilistic framework. They, in particular, define the probability of conditionals as the respective conditional probabilities. Observe that not all probabilistic conditional logic semantics take this approach. Schurz (1997b, 1998), for example, uses for his conditional logic semantics rather objective frequency-based probabilities. The Ramsey-test, which is inherently subjective, is, thus, not directly applicable to Schurz's (1997b, 1998) approach, although Schurz associates conditionals with their conditional probabilities (see Section 3.1).

[1] Although the basic intuition underlying the Ramsey-test seems to be clear, an adequate formulation is not so easy to achieve (Bennett, 2003, p. 28f).

In the next subsections we will discuss Stalnaker's (1968) and the Bennett's (2003) version of the Ramsey-test. Both versions of the Ramsey-test incorporate element (a), but differ with respect to elements (b) and (c). Stalnaker (1968, p. 101f) focuses much on component (b), but ignores component (c). This is not surprising, since he aims to justify a possible-worlds semantics that does not rely on a probability semantics. Bennett (2003, pp. 28–30), however, focuses on element (c) and does not discuss element (b) explicitly. This is to be surprising, as Bennett (2003) draws on a consistency requirement for his Gibbardian stand-off argument against a truth-value semantics for conditionals. Let us, however, first focus on the basic idea underlying ordering approaches.

3.2.2 Stalnaker's Version of the Ramsey Test, Stalnaker Semantics, Set Selection Semantics and Chellas-Segerberg Semantics

Stalnaker (1968) suggests that one's deliberation about a conditional should follow the following lines:

> "add the antecedent (hypothetically) to your stock of knowledge (or beliefs), and then consider whether or not the consequent is true. Your belief about the conditional should be the same as your hypothetical belief, under this condition, about the consequent" (Stalnaker, 1968, p. 101)

This essentially corresponds to part (a) of Ramsey test in the preceding section. Stalnaker (1968), in addition, provides following more detailed account of the Ramsey-test procedure:

> "First, add the antecedent (hypothetically) to your stock of beliefs; second, make whatever adjustments are required to maintain consistency (without modifying the hypothetical belief in the antecedent); finally, consider whether or not the consequent is then true" (Stalnaker, 1968, p. 102)

The version of the Ramsey-test differs from the former account insofar as the latter, in addition, includes consistency criterion (b).

We now specify Stalnaker models (short: St-models) and extensions of Stalnaker model (see Stalnaker, 1968, p. 103f and Stalnaker & Thomason, 1970, pp. 25–28). We use here the full conditional logic language $\mathcal{L}_{\mathrm{KL}}$ (see Section 3.1):

Definition 3.1. *Let $\mathcal{PP} = \{p_1, p_2, \dots\}$ be the set of atomic propositions in language $\mathcal{L}_{\mathrm{KL}}$. Then, $\mathcal{M}_{St} = \langle W, \lambda, f, V\rangle$ is a Stalnaker models (short: St-model) iff*
a) W is a set of possible worlds, which contains λ and at least one more possible world,
b) f is a function such that f: $form_{\mathrm{KL}} \times W \backslash \{\lambda\} \to W$
c) V is a valuation function such that V: $\mathcal{PP} \times W \to \{0, 1\}$ and $V(p, \lambda) = 1$ for all $p \in \mathcal{PP}$
d) For all formulas α, β of $\mathcal{L}_{\mathrm{KL}}$ and worlds $w \in W\backslash\{\lambda\}$ holds:
(i) $f(\alpha, w) \subseteq \|\alpha\|^{\mathcal{M}_{St}}$
(ii) *if* $w \in \|\alpha\|^{\mathcal{M}_{St}}$, *then* $f(\alpha, w) = w$
(iii) *if* $f(\alpha, w) \subseteq \|\beta\|^{\mathcal{M}_{St}}$ *and* $f(\beta, w) \subseteq \|\alpha\|^{\mathcal{M}_{St}}$ *then* $f(\alpha, w) = f(\beta, w)$

Here, the values '0' and '1' are interpreted as 'true' and 'false', respectively. The parameter λ refers to the absurd world, which is a distinguished element among the worlds in W. The set *form*$_{\mathrm{KL}}$ refers to set of all formulas of the language $\mathcal{L}_{\mathrm{KL}}$. Moreover, $\|\alpha\|^{\mathcal{M}_{\mathrm{St}}}$ is defined as $\{w \,|\, V(\alpha, w) = 1$ such that $\mathcal{M}_{\mathrm{St}} = \langle W, \lambda, f, V\rangle\}$. Stalnaker (1968, p. 103f) and Stalnaker and Thomason (1970, pp. 25–28) employ for the specification of Stalnaker models in addition to the parameters W, λ, f and V described in Definition 3.1 an accessibility relation R. This accessibility relation is, however, not needed (cf. D. Lewis, 1973/2001, p. 78) and is, hence, not used in our specification of Stalnaker models. We moreover, limited the function f to sets of ordered pairs $form_{\mathcal{L}_{\mathrm{KL}}} \times W\backslash\{\lambda\}$ rather than sets of ordered pairs $form_{\mathcal{L}_{\mathrm{KL}}} \times W$, since we will specify below extensions of St-models – in line with Stalnaker (1968) and Stalnaker and Thomason (1970) –in such a way that the truth conditions for formula α w.r.t. the absurd world do not draw on the world selection function f. This is the case, since at the absurd world λ all formulas α are true. Note, furthermore, that condition d) of Definition 3.1 essentially involves the valuation function V. Let us now define extensions of Stalnaker models:

Definition 3.2. *Let $\mathcal{M}_{St} = \langle W, \lambda, f, V\rangle$ be a Stalnaker-model, as described in Definition 3.1, and let $\mathcal{PP}$ be the set of atomic propositions $\mathcal{PP} = \{p_1, p_2, \dots\}$. Then, V^* is an extension of V to arbitrary formulas of $\mathcal{L}_{\mathrm{KL}}$ iff*
(a) $\forall p \in \mathcal{PP}\ \forall w \in W$: $V^*(p, w) = 1$ iff $V(p, w) = 1$
(b) for all formulas α of $\mathcal{L}_{\mathrm{KL}}$ it is the case that $V(\alpha, \lambda) = 1$
(c) for all α, β and $w \in W\backslash\{\lambda\}$ holds:

$$\vDash_w^{\mathcal{M}_{St}} \neg\alpha \quad \textit{iff} \quad \nvDash_w^{\mathcal{M}_{St}} \alpha \qquad (V_\neg)$$
$$\vDash_w^{\mathcal{M}_{St}} \alpha \vee \beta \quad \textit{iff} \quad \vDash_w^{\mathcal{M}_{St}} \alpha \ \textit{or}\ \vDash_w^{\mathcal{M}_{St}} \beta \qquad (V_\vee)$$
$$\vDash_w^{\mathcal{M}_{St}} \alpha \mathbin{\Box\!\!\rightarrow} \beta \quad \textit{iff} \quad \vDash_{f(\alpha,w)}^{\mathcal{M}_{St}} \beta \qquad (V_{\Box\!\rightarrow})$$

In Definition 3.2 the expressions $\vDash_w^{\mathcal{M}_{\mathrm{St}}} \alpha$ and $\nvDash_w^{\mathcal{M}_{\mathrm{St}}} \alpha$ indicate that $V^*(\alpha, w) = 1$ and $V^*(\alpha, w) = 0$ are the case, respectively, for world w in model $\mathcal{M}_{\mathrm{St}}$. Note that at the absurd world λ all formulas of language $\mathcal{L}_{\mathrm{KL}}$ are true. We shall, however, use V and V^* of Stalnaker models and their extensions indiscriminately.

This world λ is needed in order to deal with conditionals, which logically inconsistent antecedents. The formula $\perp \mathbin{\Box\!\!\rightarrow} \perp$ – where $\perp$ is defined as $p \wedge \neg p$ – is, for example, in Stalnaker's' account true since by point d.iii of Definition 3.1 the value $f(\perp, w)$ of a possible world in a St-model $\mathcal{M}_{\mathrm{St}} = \langle W, \lambda, f, V\rangle$ cannot be another possible world in W, but must be the absurd world λ. At λ every formula is according to Definition 3.2 true, including $\perp$. Thus, $\perp \mathbin{\Box\!\!\rightarrow} \perp$ turns out to be true at world w according to Definition 3.1 and condition $\mathrm{V}_{\Box\!\rightarrow}$ of Definition 3.2. Furthermore, also conditionals of the form $\alpha \mathbin{\Box\!\!\rightarrow} \perp$, where α is a p.c.-consistent formula, can be true at a possible world w in a Stalnaker model $\mathcal{M}_{\mathrm{St}} = \langle W, \lambda, f, V\rangle$. A formula $\alpha \mathbin{\Box\!\!\rightarrow} \perp$ is true at a world w in a Stalnaker model $\mathcal{M}_{\mathrm{St}}$ iff f assigns to w and α the absurd world λ.

Let us now interpret Stalnaker's (1968) formal semantics of conditionals in terms of the Ramsey-test. Definitions 3.1 and 3.2 give us – in line with Stalnaker (1968) and Stalnaker and Thomason (1970) – that the selection function f in a Stalnaker model $\mathcal{M}_{\mathrm{St}} = \langle W, \lambda, f, V\rangle$ determines for each formula α and world w in $W - \{\lambda\}$ a world w' in W, which results from hypothetically putting the antecedent into your stock of beliefs. The conditional $\alpha \mathbin{\Box\!\!\rightarrow} \beta$ is, then, true at w if and only if β is true at world w', which is selected for w.

As we saw above, Stalnaker (1968) gives two versions of his Ramsey-test interpretation of conditionals: One includes the consistency criterion (b) and the other does not. Interestingly, Stalnaker's and Thomason's semantics does not include a consistency criterion. This is due to the absurd world λ, which can be assigned to a standard world w in a Stalnaker model $\mathcal{M}_{\mathrm{St}} = \langle W, \lambda, f, V \rangle$ and arbitrary antecedents α. Since f is interpreted as providing the world, which results from hypothetically putting the antecedent into one's stock of beliefs, and all formulas, including contradictory ones, are true at the absurd world, Stalnaker's semantics does not have a consistency requirement. As a result, conditional formulas of the form $\alpha \mathrel{\Box\!\!\rightarrow} \perp$ are true at some worlds in some Stalnaker models and some (e.g. $\perp \mathrel{\Box\!\!\rightarrow} \perp$) are even true at all worlds in all Stalnaker models.

There are variants of Stalnaker's semantics, which avoid the inclusion of the absurd world λ (e.g. Nute & Cross, 2001, p. 9f). In our eyes the most elegant one is the following: Instead of assigning by the function f possible words to pairs of possible worlds and formulas of language $\mathcal{L}_{\mathrm{KL}}$, we can assign to those pairs sets of possible worlds, which are either singletons or otherwise empty. The truth condition for conditional formulas in Definition 3.2 is, then, changed so that $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ is true at a possible world w in a Stalnaker model $\mathcal{M}_{\mathrm{St}} = \langle W, f, V \rangle$ iff β is true at all worlds, which f selects for the ordered pair α and w (cf. D. Lewis, 1973/2001, p. 58).

We can generalize this alternative semantics not just to empty and singleton sets of possible worlds, but also to arbitrary sets of possible worlds. The modified function f' assigns, then, for pairs of possible worlds and formulas of $\mathcal{L}_{\mathrm{KL}}$ sets of possible worlds. D. Lewis (1973/2001, p. 58) calls functions of type f' 'set-selection functions'. Set-selection models can, then, be formally defined the following way:

Definition 3.3. *Let $\mathcal{PP} = \{p_1, p_2, \dots\}$ be the set of atomic propositions in language $\mathcal{L}_{\mathrm{KL}}$. Then, $\mathcal{M}_{St} = \langle W, f', V \rangle$ is a set-selection model iff*
a) W is a non-empty set of possible worlds
b) f is a function such that $f\colon form_{\mathrm{KL}} \times W \to \mathcal{P}(W)$
c) V is a valuation function such that $V\colon \mathcal{PP} \times W \to \{0, 1\}$

The expression $\mathcal{P}(W)$ refers to the power set of W. Extensions of set-selection functions are, then, defined the following way:

Definition 3.4. *Let $\mathcal{M}_{set} = \langle W, f', V \rangle$ be a set-selection model, as described in Definition 3.3, and let $\mathcal{PP}$ be the set of atomic propositions $\mathcal{PP} = \{p_1, p_2, \dots\}$. Then, V^* is an extension of V to arbitrary formulas of $\mathcal{L}_{\text{KL}}$ iff*
(a) $\forall p \in \mathcal{PP}\ \forall w \in W$: $V^*(p, w) = 1$ iff $V(p, w) = 1$
(c) for all α, β and $w \in W$ holds:

$$\vDash_w^{\mathcal{M}_{set}} \neg\alpha \quad \textit{iff} \quad \nvDash_w^{\mathcal{M}_{set}} \alpha \qquad (V_\neg)$$
$$\vDash_w^{\mathcal{M}_{set}} \alpha \vee \beta \quad \textit{iff} \quad \vDash_w^{\mathcal{M}_{set}} \alpha \ \textit{or} \ \vDash_w^{\mathcal{M}_{set}} \beta \qquad (V_\vee)$$
$$\vDash_w^{\mathcal{M}_{set}} \alpha \mathrel{\Box\!\!\rightarrow} \beta \quad \textit{iff} \quad \forall w'(w' \in f(w, \alpha) \Rightarrow \vDash_{w'}^{\mathcal{M}_{set}} \beta) \qquad (V_{\Box\rightarrow})$$

We will again use V and V^* use, henceforth indiscriminately. Set selection models as specified by Definition 3.3 and 3.4 use essentially the same truth conditions for conditional formulas as the modified semantics for Stalnaker models described above. Observe, furthermore, that the notion of set-selection functions is not only more general than Stalnaker models in terms of allowing for assigning arbitrary sets of possible worlds to ordered pairs of possible worlds and formulas, but also omits the specifications in point d) of Definition 3.1.

The set-selection function semantics is highly similar to the Chellas-Segerberg semantics (Chellas, 1975; Segerberg, 1989) described and investigated in Chapters 4 through 7. There are, however, two main differences between set selection semantics on the one hand and Chellas-Segerberg semantics on the other hand. The less important difference lies in the fact that in Chellas-Segerberg semantics an accessibility relation between pairs of possible worlds and formulas rather than a set selection function, as specified in Definition 3.3, is used. It is, however, not too difficult to see that we can replace set-selection functions f' in set-selection models by accessibility relations R' and, vice versa, without changing the truth-values of formulas at any possible world: Suppose that f' in a given set-selection model $\mathcal{M}_{\text{set}} = \langle W, f', V \rangle$ assigns to w and α the set of possible worlds X. We can, then, encode that information by an accessibility relation R' in such a way that all worlds in the set X are accessible from w relative to α. Furthermore, let R' be an accessibility relation as described above, which is specified w.r.t. ordered pairs of possible worlds in a set of possible worlds W and formulas. Then, R' gives us for arbitrary formulas α and worlds w in W the sets of worlds $X \subseteq W$, which are accessible by R'. Hence, we can encode the information in the accessibility relation R' regarding arbitrary worlds w and formulas α by a selection function f', which assigns the set X to w and α.

The second, more important difference between set-selection semantics and Chellas-Segerberg semantics (Chellas, 1975; Segerberg, 1989) is, that in Chellas-Segerberg semantics propositions rather than formulas are used, where a proposition $\|\alpha\|^{\mathcal{M}}$ is identified with the set of possible worlds, at which α is true in a particular model $\mathcal{M}$. This implies that in Chellas-Segerberg semantics a three-place accessibility relation between pairs of possible worlds and sets of possible worlds X rather than between sets of possible worlds and formulas is employed. In models of Chellas-Segerberg semantics a conditional formula $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ is true at a possible world w iff β is true at all possible worlds w', which are accessible from w by R relative to $\|\alpha\|^{\mathcal{M}}$. Although the use of formulas rather than propositions for the specification of accessibility relation seems to make only a small difference (cf. D. Lewis, 1973/2001, p. 60), it has a strong impact on the formal properties of the semantics (see Chapters 4–6), among those that a standard version of completeness can be proven for selection-models that use only formulas, but not for models, which use propositions instead (cf. Chapters 4 and 6). For a further discussion of advantages of Chellas-Segerberg semantics see Section 4.1.

Finally, we can also interpret the Chellas-Segerberg semantics in terms of the Ramsey-test. We shall, however, postpone our interpretation of Chellas-Segerberg semantics in terms of the Ramsey-test and discuss this issue in detail in Section 7.1.

3.2.3 Ordering Semantics. D. Lewis (1973/2001), Kraus et al. (1990) and Related Semantics

In this section we will give a general characterization of what we call "ordering semantics" and, then, describe particular instances of this type of semantics, such as D. Lewis' (1973/2001) systems of spheres semantics and alternative semantics, which are directly based on an ordering relation, such as D. Lewis (1973/2001, 48–50), Burgess (1981), Kraus et al. (1990) and Lehmann and Magidor (1992). Our discussion of ordering semantics will serve as basis for a comparison in the next section between ordering semantics on the one hand and Stalnaker (1968; Stalnaker & Thomason, 1970) semantics, set selection semantics and Chellas Segerberg semantics (Chellas, 1975; Segerberg, 1989) on the other hand (see preceding section).

Note here that ordering semantics are used for different purposes: Whereas D. Lewis (1973/2001) and Burgess (1981) interpret their semantics in terms of counterfactual conditionals (see Section 1.2.1), Kraus et al. (1990) and Lehmann and Magidor (1992) employ their semantics to account for normic conditionals (see Section 1.2.2). Interestingly, both types of approaches draw on essentially the same semantic intuition (cf. Makinson, 1993; Nejdl, 1992).

Let us, first, outline the basic assumptions of orderings semantics. It is an essential feature of ordering semantics that they employ rankings of possible worlds. These are either rankings of truth-valued possible worlds, such as in D. Lewis (1973/2001), Kraus et al. (1990) and Lehmann and Magidor (1992), or else rankings of possible worlds according to the numerical values of probability functions (cf. Goldszmidt & Pearl, 1996, p. 58). Due to time and space constraints we will here focus exclusively on rankings of truth-valued possible worlds.

Let us now describe the systems of spheres semantics of D. Lewis (1973/2001). Let us, first, describe the language, in which D. Lewis (1973/2001) formulates his systems of spheres semantics: D. Lewis (1973/2001) uses the full language $\mathcal{L}_{\mathrm{KL}}$ (see Section 3.1; cf. Section 4.2.1). Hence, he allows for boolean combinations of conditional formulas (e.g. $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge \gamma$) and nestings of conditional formulas (e.g. $\alpha \mathrel{\Box\!\!\rightarrow} (\beta \wedge (\gamma \mathrel{\Box\!\!\rightarrow} \delta))$). Lewis frames and Lewis models (D. Lewis, 1973/2001, p. 13f and p. 119) can, then, be defined the following way:

Definition 3.5. *$\mathcal{F}_L = \langle W, \$ \rangle$ is a Lewis frame iff*

(a) W is a non-empty set of possible worlds and

(b) $\$$ is a function, such that $\$: W \to \mathcal{P}(\mathcal{P}(W))$ and the following property hold ($\$_w$ is the set, which is assigned to w by $\$$):

(i) $\varnothing \notin \$_w$ and

(ii) $\$_w \subseteq \mathcal{P}(W)$ and for all $S, S' \in \$_w$ holds: $S \subseteq S'$ or $S' \subseteq S$.

Here, $\mathcal{P}(X)$ denotes the power set of a set X. We call a $\$_w$, which is based on a Lewis frame $\mathcal{F}_L = \langle W, \$ \rangle$, 'system of spheres (of world w)'. The elements $S, S', \ldots \in \$_w$ are spheres (of world w).

Note that D. Lewis (1973/2001) requires in his original definition of Lewis frames, in addition to the specifications in Definition 3.5, that all systems of spheres $\$_w$ are (a) closed under unions and (b) closed under (non-empty) intersections. We omit criterion (a) and (b), since D. Lewis (1973/2001, p.119, Footnote) argues that they are neither needed for his soundness results nor for his completeness results. We, moreover, require that any system of spheres $\$_w$ in a Lewis frame does not contain the empty set. This condition is not included in D. Lewis (1973/2001). The exclusion of the empty set as a member of systems of spheres $\$_w$ for Lewis frame $\mathcal{F}_L = \langle W, \$ \rangle$, however, does not impact the truth conditions w.r.t. Lewis frames, which are described below (cf. D. Lewis, 1973/2001, p. 15). Moreover, Lewis (1973/2001) admits that allowing for the inclusion of the empty set in systems of spheres $\$_w$ is unintuitive. D. Lewis (1973/2001), further, indicates that he allows the empty set as a member of systems of spheres $\$_w$ only for the sake of technical convenience (p. 15).

The spheres in a system of spheres (for w) $\$_w$ determine an ordering of possible worlds. In Lewis frames the ordering of possible worlds in a system of spheres $\$_w$ is based on the similarity of possible worlds in W to the world $w \in W$. The system of spheres $\$_w$ in a Lewis frame $\mathcal{F}_L = \langle W, \$ \rangle$, hence, describes a ranking of possible worlds w.r.t. similarity to the (actual) world w (cf. D. Lewis, 1973/2001, pp. 13–16), where the lower the rank of a world w' in some $\$_w$ the more similar w' is to w. Note that in D. Lewis' semantics the systems of spheres $\$_w$ for $w \in W$ need not encompass all possible worlds in W for a Lewis frame $\mathcal{M}_L = \langle W, \$ \rangle$. Furthermore, ranks in systems of spheres are determined the following way: For a systems of sphere $\$_w$ world $w' \in W$ is of strictly lower rank than world $w'' \in W$ iff there exist spheres S and S' in $\$_w$, such that $w' \in S'$, $w'' \in S$, $S' \subset S$ and $w'' \notin S$. Moreover, for a system of spheres $\$_w$ worlds w and $w' \in W$ are of equal rank iff there exists a sphere S in $\$_w$, such that $w', w'' \in S$ and there exists no sphere S', such the following holds: $S' \subset S$ and either (i) w' in S', but not w'' in S' or (ii) w'' in S', but not w' in S'.

In order to allow for a more substantive interpretation of systems of spheres $\$_w$ in terms of similarity of possible worlds, D. Lewis (1973/2001) adds the following condition to Lewis frames:

$C_{\text{Centering}}$ $\quad \forall w \in W(\{w\} \in \$_w)$

The condition $C_{\text{Centering}}$ is called 'Centering Condition'. The centering condition gives us that the world $w \in W$ is the only world in the innermost sphere for any systems of spheres $\$_w$ in any Lewis model $\mathcal{L}_L = \langle W, \$, V \rangle$. Since by condition b.i of Definition 3.5 any system of spheres $\$_w$ for any world in a Lewis model $\mathcal{M}_L = \langle W, \$, V \rangle$ is not allowed to contain the empty set, it follows by the centering condition that any system of spheres $\$_w$ must contain the set $\{w\}$. Furthermore, condition b.ii of Definition 3.5 implies, then, that $\{w\}$ is the innermost sphere of any Lewis model, for which the centering condition holds. This condition seems reasonable given D. Lewis' (1973/2001) interpretation of systems of spheres $\$_w$ in terms of similarity orderings: The world w is then maximally similar to itself and it is the only world, which is maximally similar to itself (D. Lewis, 1973/2001, p. 14f).

Let us now describe the notion of Lewis models:

Definition 3.6. *Let $\mathcal{F}_L = \langle W, \$ \rangle$ be a Lewis frame as described in Definition 3.5 and let $\mathcal{PP}$ be the set of atomic propositions $\mathcal{PP} = \{p_1, p_2, \ldots\}$. Then, $\mathcal{M}_L = \langle W, \$, V \rangle$ is a Lewis model iff V is a valuation function such that $V\colon \mathcal{PP} \times W \to \{0, 1\}$.*

Here, the values '0' and '1' are again interpreted as 'true' and 'false', respectively. Let us now define extensions of Lewis models:

Definition 3.7. *Let $\mathcal{M}_L = \langle W, \$, V \rangle$ be a Lewis model, as described in Definition 3.6, and let $\mathcal{PP}$ be the set of atomic propositions $\mathcal{PP} = \{p_1, p_2, \ldots\}$. Then V^* is an extension of V to arbitrary formulas of $\mathcal{L}_{\text{KL}}$ iff (a) $\forall p \in \mathcal{PP}\, \forall w \in W$: $V^*(p, w) = 1$ iff $V(p, w) = 1$ and (b) for all α, β and $w \in W$ holds:*

$$\vDash_w^{\mathcal{M}_L} \neg\alpha \quad \textit{iff} \quad \nvDash_w^{\mathcal{M}_L} \alpha \tag{$V_\neg$}$$

$$\vDash_w^{\mathcal{M}_L} \alpha \vee \beta \quad \textit{iff} \quad \vDash_w^{\mathcal{M}_L} \alpha \textit{ or } \vDash_w^{\mathcal{M}_L} \beta \tag{$V_\vee$}$$

$$\begin{aligned} \vDash_w^{\mathcal{M}_L} \alpha \,\Box\!\!\rightarrow \beta \quad \textit{iff} \quad & (a)\ \neg\exists S \in \$_w \exists w' \in S\, (\vDash_{w'}^{\mathcal{M}_L} \alpha) \textit{ or} \\ & (b)\ \exists S \in \$_w \forall w' \in S\, (\vDash_{w'}^{\mathcal{M}_L} \alpha \to \beta) \end{aligned} \tag{$V_{\Box\!\rightarrow}$}$$

In Definition 3.7 the expressions $\vDash_w^{\mathcal{M}_L} \alpha$ and $\nvDash_w^{\mathcal{M}_L} \alpha$ abbreviate that $V^*(\alpha, w) = 1$ and $V^*(\alpha, w) \neq 1$, respectively. We will, henceforth, refer to Lewis models and extensions of Lewis models indiscriminately. Let us call a world w in a Lewis model $\mathcal{M}_L = \langle W, \$, V\rangle$ an α-world iff α is true at w. We, moreover, refer to a sphere S in $\$_w$ as α-permitting sphere iff there exists a world $w' \in S$ such that w' is an α-world. Then, according to Definitions 3.6 and 3.7 a conditional formula $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ is true in a Lewis model $\mathcal{M}_L = \langle W, \$, V\rangle$ at a world w iff either (a) there is no α-permitting sphere in $\$_w$ or (b) there exists an α-permitting sphere S in $\$_w$, such that $\alpha \to \beta$ is true at every world in S (cf. D. Lewis, 1973/2001, p. 16). The logical operator $\to$ is again the material implication.

Before continue with a general discussion of ordering semantics, let us, first, consider the following restrictions for Lewis frames:

$C_{\text{P-Cons}}$ $\quad \forall w \in W(\$_w \neq \varnothing)$
C_{MP} $\quad \forall w \in W(\exists S \in \$_w \wedge \forall S' \in \$_w(w \in S'))$
C_{CS} $\quad \forall w \in W(\exists S \in \$_w \Rightarrow \{w\} \in \$_w)$

Here ‘P-Cons’, ‘MP’ and ‘CS’ stand for ‘Probabilistic Consistency’, ‘Modus Ponens’ and ‘Conjunctive Sufficiency’, respectively. These restrictions of Lewis frames correspond to the following principles:

MP $\quad (\alpha \mathrel{\Box\!\!\rightarrow} \beta) \to (\alpha \to \beta)$
E $\quad \alpha \wedge \beta \to (\alpha \mathrel{\Box\!\!\rightarrow} \beta)$
P-Cons $\quad \neg(\top \mathrel{\Box\!\!\rightarrow} \bot)$

By correspondence we mean that whenever condition C_α applies to a Lewis model $\mathcal{M}_L = \langle W, \$, V\rangle$ then α is true at all worlds in $\mathcal{M}_L$ and vice versa. The formulas $\top$ and $\bot$ abbreviate the formulas $p \vee \neg p$ and $p \wedge \neg p$, respectively.

We mention here conditions C_{MP} and C_{CS}, since both conditions are conjointly equivalent to centering condition $C_{\text{Centering}}$. C_{MP} is, moreover, sometimes called ‘weak centering condition’ (D. Lewis, 1973/2001, p. 120). Condition C_{MP} gives us that there exists a sphere in $\$_w$ and that w is a world of any sphere S in a W. This implies that w is in the innermost sphere in $\$_w$, if an innermost sphere exists. The principle MP allows for a variant of modus ponens with the conditional formulas $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ (rather than with the material implication $\alpha \to \beta$). MP gives us that we can conclude β from $\alpha \;\Box\!\!\rightarrow \beta$ and α. The condition C_{CS} on the other hand makes sure that if there exists a sphere S in $\$_w$ for some world w in a model $\mathcal{M}_L = \langle W, \$, V \rangle$, then $\{w\}$ is in $\$_w$. The principle CS allows for inferring $\alpha \;\Box\!\!\rightarrow \beta$ from the conjunction $\alpha \wedge \beta$. Note that both principles CS and MP are valid in p.c. We will discuss CS and MP in more detail in Section 3.3.

Principle P-Cons and condition $C_{\mathrm{P-Cons}}$ are interesting insofar, as P-Cons seems to be a plausible principle, which represents a minimal consistency condition in the following sense: One should not arrive at a contradiction $\bot$, when one considers parts of an ordering of worlds in a systems of spheres $\$_w$, which is determined by a tautology $\top$ (Lewis models give us just that), the more as $\top$ is true at all possible worlds in all Lewis models. Despite this intuition, however, P-Cons is in general not valid in Lewis models. Condition $C_{\mathrm{P-Cons}}$ indicates where it fails: $\top \;\Box\!\!\rightarrow \bot$ and also all formulas of the form $\alpha \;\Box\!\!\rightarrow \bot$ are true at a world w in a Lewis model $\mathcal{M}_L = \langle W, \$, V \rangle$ if $\$_w$ is the empty set. In this case the truth conditions of Definition 3.7 hold trivially for any conditional formula. We will, however, postpone a discussion of principle P-Cons and other types of consistency conditions to Section 3.2.5.

Let us, now, describe an alternative characterization of systems of spheres by means of an accessibility relation (relational Lewis models) suggested also by D. Lewis (1973/2001, pp. 48–50). Our alternative characterization uses the notion of Lewis frames and models described in Definitions 3.5–3.7 and does not incorporate the centering condition.

Variants of the relational Lewis semantics are also endorsed by Lehmann and Magidor (1992), Burgess (1981) and Delgrande (1998). Kraus et al. (1990) and Burgess (1981), in addition, investigate weaker semantics than relational Lewis semantics. Burgess (1981, p. 76f and p. 82) and D. Lewis (1973/2001, p. 48f) specify a type of relational Lewis model and use for that purpose a three-place accessibility relation S, which specifies systems of spheres for arbitrary worlds w that assigns systems of spheres to possible worlds w. Let us now define relational Lewis frames, models and extensions accordingly (cf. also Lehmann & Magidor, 1992, p. 21):

Definition 3.8. *$\mathcal{F}_{rL} = \langle W, S \rangle$ is a relational Lewis frame (short: rL-frame) iff* *(a)* *W is a non-empty set of possible worlds and*

(b) *S is a relation on three-place relationship on W, such that for all w, w′ and w″ ∈ W holds:*

(i) $w' S_w w'$ *(Reflexivity)*

(ii) $w' S_w w'' \wedge w'' S_w w''' \Rightarrow w' S_w w'''$ *(Transitivity)*

(iii) $w' S_w w'' \Rightarrow w''' S_w w'' \vee w' S_w w'''$ *(Restricted Connectivity)*

The accessibility relation S gives us an ordering of worlds. In our terminology world $w' \in W$ is of lower rank than world $w'' \in W$ (w.r.t. world $w \in W$) iff $w' S_w w''$, but not $w'' S_w w'$, whereas w' and $w'' \in W$ are of equal rank (w.r.t. world $w \in W$) iff $w' S_w w''$ and $w'' S_w w'$. Moreover, a world $w' \in W$ is accessible (w.r.t. world $w \in W$) iff there exists a world $w'' \in W$, such that either $w' S_w w''$ or $w'' S_w w'$.

Relational Lewis frames $\mathcal{M}_{\mathrm{rL}} = \langle W, S \rangle$ correspond insofar to Lewis frames $\mathcal{M}_L = \langle W, \$ \rangle$, as we can reconstruct the system of spheres function \$ in terms of the accessibility relation S and vice versa. Given we have a systems of spheres \$ (relative to a set of possible worlds W), then two worlds w', $w'' \in W$ are in the same sphere S in $\$_w$ for a world $w \in W$ iff both $w' S_w w''$ and $w'' S_w w'$. Moreover, it holds for worlds w', $w'' \in W$ that $w' S_w w''$ but not $w'' S_w w'$ for $w \in W$ iff w' and w'' are in spheres S and $S' \in \$_w$, respectively, such that $S \subset S'$. This way we can reconstruct Lewis models from relational Lewis models.

The type of ordering described in Definition 3.8 is also called ‘ranked’ as opposed to preferential orderings of possible worlds (Lehmann & Magidor, 1992, p. 21). In preferential orderings all specifications of 3.8 except for b.iii hold (Lehmann & Magidor, 1992, p. 21).

Our specification of the ordering relation in Definition 3.8 is, however, more in line with Lehmann and Magidor (1992) than D. Lewis (1973/2001). In D. Lewis (1973/2001, p. 48) instead of condition b.iii (restricted connectivity) the following stronger condition is used:

Connectivity $\quad \forall w, w', w'' \in W (w' S_w w'' \vee w'' S_w w')$

The condition connectivity is stronger, since it implies – contrary to restricted connectivity – that any two worlds w' and w'' in W stand in the relationship S_w for any world $w \in W$. D. Lewis (1973/2001, p. 48), however, distinguishes, then, between maximal worlds (relative to a world w) and non-maximal worlds among those, for which S_w holds. The non-maximal worlds are exactly those worlds that stand in our terminology in the relation S to w and w' w.r.t. w, viz. to worlds w' that there exists a world w'' such that either $w'S_w w''$ or $w''S_w w'$. The maximal worlds in W, however, are those worlds $w' \in W$ in a Lewis frame $\mathcal{M}_L = \langle W, \$ \rangle$, which are not accessible w.r.t. a world w in D. Lewis' systems of spheres semantics $\$_w$: There is no sphere S in $\$_w$, such that $w' \in S$. We use Definition 3.8 rather than D. Lewis' (1973/2001) characterization, since the former is somewhat more perspicuous. Let us now define relational Lewis models and their extensions:

Definition 3.9. *Let $\mathcal{F}_{rL} = \langle W, S \rangle$ be a relational Lewis frame as described in Definition 3.8 and let $\mathcal{PP}$ be the set of atomic propositions $\mathcal{PP} = \{p_1, p_2, \ldots\}$. Then, $\mathcal{M}_L = \langle W, \$, V \rangle$ is a relational Lewis model (rL-model) iff V is a valuation function such that $V\colon \mathcal{PP} \times W \to \{0, 1\}$.*

Definition 3.10. *Let $\mathcal{M}_{rL} = \langle W, S, V \rangle$ be a relational Lewis model, as described in Definition 3.9, and let $\mathcal{PP}$ be the set of atomic propositions $\mathcal{PP} = \{p_1, p_2, \ldots\}$. Then V^* is an extension of V to arbitrary formulas of $\mathcal{L}_{\mathrm{KL}}$ iff (a) $\forall p \in \mathcal{PP}\ \forall w \in W$: $V^*(p, w) = 1$ iff $V(p, w) = 1$ and (b) for all α, β and $w \in W$ holds:*

$$
\begin{array}{lll}
\vDash_w^{\mathcal{M}_{rL}} \neg\alpha & \textit{iff} & \nvDash_w^{\mathcal{M}_{rL}} \alpha \qquad (V_\neg) \\
\vDash_w^{\mathcal{M}_{rL}} \alpha \vee \beta & \textit{iff} & \vDash_w^{\mathcal{M}_{rL}} \alpha \ \textit{or}\ \vDash_w^{\mathcal{M}_{rL}} \beta \qquad (V_\vee) \\
\vDash_w^{\mathcal{M}_{rL}} \alpha \mathrel{\Box\!\!\rightarrow} \beta & \textit{iff} & \textit{(a)}\ \neg\exists w'(\exists w''(w'S_w w'' \vee w''S_w w') \wedge \vDash_{w'}^{\mathcal{M}_{rL}} \alpha)\ \textit{or} \\
 & & \textit{(b)}\ \exists w'(\vDash_{w'}^{\mathcal{M}_{rL}} \alpha \wedge \forall w''(w''S_w w' \Rightarrow \vDash_{w''}^{\mathcal{M}_{rL}} \alpha \to \beta) \qquad (V_{\Box\!\rightarrow})
\end{array}
$$

In Definition 3.7 the expressions $\neg$, $\vee$, $\wedge$ and $\Rightarrow$ stand for meta-language negation, disjunction, conjunction and material implication symbols. The expressions $\vDash_w^{\mathcal{M}_L} \alpha$ and $\nvDash_w^{\mathcal{M}_L} \alpha$ abbreviate that $V^*(\alpha, w) = 1$ and $V^*(\alpha, w) \neq 1$, respectively. We will, henceforth, refer to relational Lewis models and their extensions indiscriminately. A conditional $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ is, then, according to Definition 3.7 true at a world w iff either (i) there is no α-world w', which is accessible w.r.t. w, or (ii) there exists an α-world w' and for all worlds of equal or lower rank the formula $\alpha \to \beta$ holds (cf. D. Lewis, 1973/2001, p. 49).

It is important to note that Lehmann and Magidor (1992) and Kraus et al. (1990) do not interpret the relation S in their semantics in terms of similarity of possible worlds in an objective sense, as D. Lewis (1973/2001) does, but in terms of ranks of normality according to an agent's beliefs (Kraus et al., 1990, p. 169, cf. Section 1.2.2), where the lower the rank of a world w the more normal w is (w.r.t. the agent's beliefs). The ranks of normality are, then, used to determine expectancies of agents: What is most normal – given the information an agent has – is to be expected by the agent (cf. Kraus et al., 1990, p. 173f).

Moreover, contrary to D. Lewis (1973/2001) and Burgess (1981), Kraus et al. (1990) and Lehmann and Magidor (1992) use an absolute ordering relation S. In Kraus et al. (1990) and Lehmann and Magidor (1992) this relation is a two place relationship, which is not relativized to a world w, but indicates that the rank of a world $w' \in W$ is lower than or equal to the rank of a world $w'' \in W$ just in case $w'Sw''$ (Kraus et al., 1990, p. 181f; Lehmann & Magidor, 1992, p. 7f, p. 21; see also Makinson, 1993, p. 347f).[2] Note, however, that Kraus et al. (1990) and Lehmann and Magidor (1992) do not use the full language $\mathcal{L}_{\mathrm{KL}}$ (see Section 2.2.7), but languages, which do not allow for nestings of conditionals (e.g. $p \mathrel{\Box\!\!\rightarrow} (q \mathrel{\Box\!\!\rightarrow} r)$ is not a formula in their approach). In Kraus et al. (1990) and Lehmann and Magidor (1992) at maximum only boolean combinations of non-nested conditionals (e.g. $\neg(p \mathrel{\Box\!\!\rightarrow} q)$) are considered formulas (cf. Section 2.2.7). Boolean combinations of conditional formulas are called "first degree formulas" (Makinson, 1993, p. 348). Interestingly, the use of an absolute ordering relation S compared to a relativized but otherwise identical ordering relation S' renders no additional first-order formulas valid (Makinson, 1993, p. 348f). By validity we mean here, truth at all worlds in all models of the pre-specified type.

Kraus et al. (1990) and Burgess (1981) also investigate weaker types of models than described by Definitions 3.5–3.7 on the one hand and 3.8–3.10 on the other hand. They use for that purpose weaker notions of relational Lewis frames (see Definition 3.8) by giving up parts of the restriction on the accessibility relation in point (b) of Definition 3.8. In addition, Kraus et al. (1990, p. 182 and p. 187) also investigate relational Lewis models, which do rely on orderings of possible worlds, but rather on ordering of non-empty sets of possible worlds.

[2]Although Kraus et al. (1990) use the symbol $<$ to describe the relation S, they allow $<$ to be asymmetric, viz. that for some worlds $w, w' \in W$, both wSw' and $w'Sw$ hold (Kraus et al., 1990, p. 181, Definition 3.6 and p.183, Definition 3.14).

Let us, finally, discuss a characteristic aspect of D. Lewis' (1973/2001) systems of spheres semantics, namely how D. Lewis (1973/2001) deals with infinite sequences of α-permitting spheres. For mnemonic reasons, we shall repeat here the informal summary of the truth conditions described in Definition 3.7: A conditional formula $\alpha \;\Box\!\!\rightarrow \beta$ is true in a Lewis model $\mathcal{M}_L = \langle W, \$, V \rangle$ at a world w iff either (a) there is no α-permitting sphere in $\$_w$ or (b) there exists an α-permitting sphere S in $\$_w$, such that $\alpha \rightarrow \beta$ is true at every world in S.

The inclusion requirement b.ii of Definition 3.5 guarantees, then, in case that (b) holds that $\alpha \rightarrow \beta$ is true at all worlds in spheres S' such that $S' \subseteq S$. This property allows D. Lewis (1973/2001) also to deal with infinite descending sequences of α-permitting spheres. A system of spheres $\$_w$ contains an infinite sequence of α-permitting spheres $S_1, S_2, \ldots$ in $\$_w$ in a Lewis model $\mathcal{M}_S = \langle W, \$, V \rangle$ iff S_1, $S_2, \ldots$ are α-permitting spheres and it holds that $S_1 \supset S_2 \supset \ldots$ for every such S_i ($i \in \mathbb{N}$) (cf. Schurz, 1998, p. 83). In such a system of spheres $\$_w$, hence, there exists no smallest α-permitting sphere, since for any α-permitting sphere S there exists an α-permitting sphere S', such that $S' \subset S$. D. Lewis' semantics, however, allows us to deal with these cases in an intuitive way: We saw above that if there exists in that sequence of α-permitting spheres a sphere S, such that $\alpha \rightarrow \beta$ is true at all worlds w' in S, then $\alpha \rightarrow \beta$ is true at all spheres $S' \subseteq S$. Since, however, a conditional $\alpha \;\Box\!\!\rightarrow \beta$ is according to Definition 3.7 true at a world w – given that there are α-permitting spheres in $\$_w$ – if and only if there exists such a "limit" sphere S in $\$_w$. The approach of Lewis parallels in the case of infinite sequences of α-permitting spheres the use of limits of functions in mathematical calculus. A conditional $\alpha \;\Box\!\!\rightarrow \beta$ is in this case true at a world w – given there exist α-permitting spheres in $\$_w$ – if and only it is guaranteed that the formula $\alpha \rightarrow \beta$ is true at all worlds w' in spheres S, which are "arbitrarily close" to the limit.

Note that infinite sequences of α-permitting spheres arise only in certain types of Lewis models, but not in others. Adams (1977), for example, proves equivalence of entailment for his probabilistic conditional logic semantics in Adams (1965, 1966, 1977) on the one hand and D. Lewis' (1973/2001) system of spheres semantics on the other hand. Adams (1977) restricts himself in his investigation to finite Lewis models (where a Lewis models $\mathcal{M}_L = \langle W, \$, V \rangle$ is finite iff W contains only a finite number of worlds). In finite Lewis models, however, trivially no infinite sequence of α-permitting spheres can arise.[3] Note that we will discuss Adams' (1977) results in more detail in Section 3.4.3.

D. Lewis (1973/2001) discusses also the so-called "limit assumption". This assumption makes sure that no infinite sequence of α-permitting spheres can arise. This is done by requiring for any formula α in language $\mathcal{L}_{\mathrm{KL}}$ and any world w in any Lewis model $\mathcal{M}_L = \langle W, \$, V\rangle$ that there always exists a smallest α-permitting sphere in $\$_w$, if there exists a world w' in some sphere S in $\$$, such that α is true at w'. Formally, the limit assumption amounts to the following: $\forall \alpha \in form_{\mathcal{L}_{\mathrm{KL}}} \forall w \in W(\exists S(S \in \$_w \wedge \exists w'(w' \in S \wedge \vDash_{w'}^{\mathcal{M}_L} \alpha) \Rightarrow \exists S(S \in \$_w \wedge \exists w'(w' \in S \wedge \vDash_{w'}^{\mathcal{M}_L} \alpha) \wedge \neg\exists S' \exists w''(S' \in \$_w \wedge w'' \in S' \wedge \vDash_{w''}^{\mathcal{M}_L} \alpha \wedge S' \subset S)))$.

If we add the limit assumption to our definition of Lewis models, we can simplify the truth conditions of $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ at a world w in a Lewis model $\mathcal{M}_L = \langle W, \$, V\rangle$ by requiring that (i) either there is no α-permitting sphere in $\$_w$ or (ii) if there is a α-permitting sphere S in $\$$, then $\alpha \rightarrow \beta$ is true at all worlds w' in S (D. Lewis, 1973/2001, p. 19f). Note that the limit assumption does not make additional formulas valid in Lewis models. More precisely, there are no formulas that hold at all worlds in Lewis models, for which the limit assumption holds, compared to Lewis models for which the limit assumption does not hold (cf. D. Lewis, 1973/2001, p. 121).

D. Lewis (1973/2001), however, argues that the limit assumption is not warranted in many cases: Consider, for example, the conditional "If there were a line which is strictly longer than 2cm, then X would be the case". Given the limit assumption we, however, cannot represent the antecedent of that conditional adequately in Lewis models, since the limit assumption requires that there is no infinite sequence of α-permitting spheres. Such a sequence would, however, be required since for any length larger than 2cm, we can still find a length, which is still larger, but closer to 2cm (cf. D. Lewis, 1973/2001, p. 20). D. Lewis (1973/2001), argues that in order to cope with this type of conditionals, it is not appropriate to make the limit assumption.

The semantics of Kraus et al. (1990) and Lehmann and Magidor (1992) ex-

[3] Furthermore, contrary to D. Lewis (1973/2001) Adams (1977), uses the restricted language $\mathcal{L}_{\mathrm{KL}}$, which allows only for p.c.-formulas that do not contain a conditional operator, or alternatively conditional formulas $\alpha \mathrel{\Box\!\!\rightarrow} \beta$, where α and β are formulas, which do not contain a conditional operator (cf. Section 3.4.3 and 4.2.1).

plicitly employ variants of the limit assumption. Kraus et al. (1990, p. 182) and Lehmann and Magidor (1992, p. 7) call this variant of the limit assumption 'smoothness condition'. Accordingly, they employ a variant of the simplified version of D. Lewis' (1973/2001) truth-conditions described above. Kraus et al. (1990) and Lehmann and Magidor (1992), however, have good reasons to use the limit assumption. First, they intend to describe indicative normic conditionals rather than counterfactual conditionals (cf. Sections 1.2.1 and 1.2.2). They presuppose that normality ranks are assigned to worlds (or sets of worlds) by agents and that these ranks determine the agents' expectancies (see above). It seems, however, hardly plausible to presuppose that real agents represent infinite sequences of α-permitting spheres. So, for their semantics the limit assumption (which excludes these cases) is rationally justified. Second, even if we consider a counterfactual semantics, which is not interpreted in terms of agents' subjective states, the limit assumption might be warranted from a pragmatic perspective: In order to arrive at computationally tractable accounts of counterfactuals we cannot allow for infinite descending sequences α-worlds, since otherwise the respective algorithm might not terminate.

3.2.4 Contrasting Ramsey Test Interpretations of Conditionals and Ordering Semantics

In this section we will contrast Ramsey-test interpretations, such as described in Section 3.2.2, with the core idea of ordering semantics. We argue that Stalnaker semantics (1968; Stalnaker & Thomason, 1970), set selection function semantics and Chellas-Segerberg semantics (Chellas, 1975; Segerberg, 1989; see Sections 3.1 and 3.2.2) are more naturally interpreted in terms of the Ramsey-test, while ordering semantics, such as described in D. Lewis (1973/2001), Burgess (1981), Kraus et al. (1990) and Lehmann and Magidor (1992) are based on a principally different semantic idea. Note that both types of semantics are often referred to quite indiscriminately (e.g. D. Lewis, 1973/2001, p. 57f and p. 77; Gärdenfors, 1979, p. 381; Makinson, 1993, p. 340, p. 343).

The core idea that underlies the ordering semantics of D. Lewis (1973/2001), Burgess (1981), Kraus et al. (1990) and Lehmann and Magidor (1992) is that possible worlds (used in the semantics) are ranked according to a criterion, such as similarity to the actual world (D. Lewis, 1973/2001) or normality preferences of agents (Kraus et al., 1990). While in Lewis' (1973/2001) approach for each world a system of spheres is used, Burgess (1981), Kraus et al. (1990) and Lehmann and Magidor (1992) employ an absolute ordering relation between possible worlds. In all cases possible worlds (or sets of possible worlds) are ranked on the basis of a global criterion. The antecedent of a conditional only gives us that segment of the ordering of possible world, which is used in order to determine the truth of a conditional.

The Ramsey-test, however, does not draw on rankings of possible worlds according to a global criterion. Rather each conditional is determined by putting the antecedent to one's stock of beliefs and determine on that basis the truth a conditional (at a possible world) (see Section 3.2.1). Hence, proceeding by the Ramsey-test allows for a step-by-step process rather than a global one-time ranking, as ordering semantics presuppose: In order to find out whether a conditional is true at a possible world w in an ordering semantics, we have to provide a general ordering of a subset of possible worlds (w.r.t. the actual world w) independently of the antecedent of the respective conditional. In that sense a Ramsey-test approach is much more parsimonious, since for a Ramsey-test of a conditional (w.r.t to possible worlds semantics) we only have to take into account the possible worlds, which are relevant for the antecedent of that conditional. In an ordering semantics the ranking of possible worlds (w.r.t. the actual world w) gives us the truth of any conditional (at world w). Except for trivial cases, such an ordering of possible worlds involves by far more information than which is required to assess a conditional by means of the Ramsey-test. Furthermore, observe that basic Stalnaker semantics (that is Definition 3.1 without condition d), set selection functions (see Definition 3.3 and Chellas-Segerberg semantics (Chellas, 1975; Segerberg, 1989; see Section 3.2.2) allow for plausible and pure Ramsey interpretation, but not so much for an interpretation in terms of ordering semantics.

Prima facie, one might suspect that Stalnaker (1968) suggests for Stalnaker semantics a pure interpretation in terms of the Ramsey-test, the more as he introduces this idea into the modern discussion. Surprisingly, Stalnaker (1968) does not do that. In his interpretation of Stalnaker semantics, Stalnaker (1968) mixes elements of a Ramsey-test interpretation with elements of ordering semantics: In his informal description of his semantics, Stalnaker (1968, p. 104) essentially draws on the notion of rankings of possible worlds according to similarity (to the actual world) and argues that his truth conditions require that the world $w' \in W \backslash \{\lambda\}$ selected by f for $w \in W \backslash \{\lambda\}$ and α in a Stalnaker model $\mathcal{M}_{\mathrm{St}} = \langle W, \lambda, f, V \rangle$ must differ minimally from the actual world w (p. 104). By these means Stalnaker motivates conditions d.ii and d.iii in Definition 3.1, since these ensure – conjointly with the other conditions in Definition 3.1 – that possible worlds are ordered according to some criterion, such as similarity (Stalnaker, 1968, p. 105f).

We agree that Stalnaker's semantics (see Definitions 3.1 and 3.2) cannot solely be justified on the basis of the Ramsey-test, even if we omit point d in Definition 3.1. The main reason why this is the case is that Stalnaker (1968) presupposes that " a possible world is the ontological analogue of a stock of hypothetical beliefs" (p. 102). Accordingly, the selection function f in a Stalnaker model $\mathcal{M}_{\mathrm{St}} = \langle W, \lambda, f, V \rangle$ chooses for each $w \in W \backslash \{\lambda\}$ and each formula α a single world $w' \in W$. Since at any standard possible world w' (a world that is not the absurd world λ), either β or $\neg\beta$ is true for all formulas β, it follows that in Stalnaker's account stocks of hypothetical beliefs must be negation-complete, in the sense that for every formula β either β or else $\neg\beta$ is (hypothetically) believed. The only exception is the case, in which f chooses the absurd world λ. Since every formula is true at λ, the absurd world λ represents an inconsistent stock of hypothetical beliefs.

It is, however, hardly plausible that hypothetically putting the antecedent α to one's stock of beliefs results in all cases either in an inconsistent or a negation-complete stock of hypothetical beliefs. This would, for example, imply that one has – in order to find out whether an arbitrary conditional is true – (A) to hypothetically believe that either (i) Goldbach's conjecture is true or (ii) that Goldbach's conjecture is not true, or otherwise (B) arrive at an inconsistent stock of hypothetical beliefs.

Note that selection function semantics (see Definitions 3.3 and 3.4) and Chellas Segerberg semantics do not make the assumption, that hypothetically putting a formula to one's stock of hypothetical beliefs always results either in an inconsistent stock of beliefs or a negation-complete stock of beliefs. Thus, these types of semantics are more plausible candidates for a pure interpretation of conditionals in terms of the Ramsey-test. Chellas-Segerberg semantics is more promising than set selection semantics, since the former but not the latter uses propositions (sets of possible worlds) rather than formulas. Chellas-Segerberg semantics allows, hence, for a more natural and language-independent characterization in terms of frames rather than models (cf. Section 3.2.2; see also Section 4.1). We will not enquire this issue any further in this context and postpone an interpretation of Chellas-Segerberg semantics in terms of a modified Ramsey-test to Section 7.1.

3.2.5 The Consistency Requirement and The Principle of Conditional Excluded Middle

In this section we will discuss different consistency criteria (see criterion b) for the Ramsey-test, as described in Section 3.2.1. For that purpose we use the full conditional logic language $\mathcal{L}_{\mathrm{KL}}$. This language allows for arbitrary nestings (e.g. $\alpha \mathrel{\Box\!\!\rightarrow} (\beta \wedge (\gamma \mathrel{\Box\!\!\rightarrow} \delta)))$ and boolean combinations (negation, disjunction etc.) of arbitrary formulas, including conditional formulas $\alpha \mathrel{\Box\!\!\rightarrow} \beta$. We shall first describe different versions of consistency criteria and, then, discuss the role of consistency criteria in Stalnaker's own semantics (Stalnaker models, see Definitions 3.1 and 3.2).

Let us now focus on the notion of conditional consistency. In general a set of formulas Γ is said to be inconsistent if it has all formula of the language as theorems. By conditional consistency we mean, however, something different, namely that (i) certain types of conditionals are not allowed to have a consequent formula, which is inconsistent, or (ii) that two or more consequents of conditional formulas with the same antecedent contradict each other. In the case of (i) and (ii), if a formula set Γ contains such formulas, then Γ becomes inconsistent. A weak conditional consistency criterion is the principle P-Cons ("Probabilistic Consistency") described in Sections 3.1 and 3.2.3. For mnemonic reasons we repeat this principle here:

P-Cons $\neg(\top \mathrel{\Box\!\!\rightarrow} \bot)$

The expressions $\top$ and $\bot$ abbreviate $p \vee \neg p$ and $p \wedge \neg p$, respectively. So, why should we consider P-Cons a conditional consistency criterion? The answer is that P-Cons makes sure – given reasonable assumptions – that whenever all formulas of the form $\alpha \boxright \beta$ are true, then all formulas of the language are true. Note that this is not in all conditional logics the case. P-Cons is, for example, not valid in the class of all Lewis models (see Definitions 3.5–3.7): For instance, a Lewis model $\mathcal{M}_L = \langle W, \$, V \rangle$, such that $W = \{w\}$, $\$_w = \varnothing$ makes all conditionals $\alpha \boxright \beta$ true at world w (regardless how we choose V). The above interpretation of P-Cons is, however, warranted given the following theorems and rules:

LLE if $\vdash \alpha \leftrightarrow \beta$ and $\alpha \boxright \gamma$, then $\beta \boxright \gamma$
RW if $\vdash \alpha \rightarrow \beta$ and $\gamma \boxright \alpha$, then $\gamma \boxright \beta$
CM $(\alpha \boxright \gamma) \wedge (\alpha \boxright \beta) \rightarrow (\alpha \wedge \beta \boxright \gamma)$

'LLE', 'RW' and 'CM' abbreviate 'Left Logical Equivalence', 'Right Weakening' and 'Cautious Monotonicity', respectively. Note that LLE, RW and CM or variants are valid in a conditional logic semantics, such as Stalnaker models (Definitions 3.1 and 3.2), Lewis models (Definitions 3.5–3.7) and other semantics (e.g. Adams, 1965, 1966, 1977, 1986; Schurz, 1997b, 1998; Hawthorne & Makinson, 2007).

Let us now show how P-Cons guarantees that not all conditionals are true in a pre-specified semantics. For that purpose we prove the following lemma:

Lemma 3.11. *Let* **L** *be a logic, which contains LLE+RW+CM, and suppose that* Γ *is a set of formulas. Then,* $\Gamma \vdash_{\mathbf{L}} \top \boxright \bot$ *iff* $\Gamma \vdash_{\mathbf{L}} \alpha \boxright \beta$ *for arbitrary* α *and* β.

Proof. "$\Leftarrow$": Trivial.
"$\Rightarrow$": By Lemma 3.12.

$\square$

Lemma 3.12. *LLE+RW+CM+P-Cons* $\Rightarrow \alpha \boxright \beta$

Proof.

1.	$\top \,\Box\!\!\rightarrow \bot$	given
2.	$\top \,\Box\!\!\rightarrow \alpha$	1, RW
3.	$\top \,\Box\!\!\rightarrow \beta$	1, RW
4.	$\top \wedge \alpha \,\Box\!\!\rightarrow \beta$	2, 3, CM
5.	$\alpha \,\Box\!\!\rightarrow \beta$	4, LLE

$\square$

Observe that P-Cons is already implied by bridge principles, such as MP ("Modus Ponens"). We repeat principle MP for mnemonic reasons:

MP $\quad (\alpha \,\Box\!\!\rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)$

The following lemma gives us that MP implies P-Cons:

Lemma 3.13. *MP* $\Rightarrow$ *P-Cons*

Proof.

1.	$(\top \,\Box\!\!\rightarrow \bot) \rightarrow (\top \rightarrow \bot)$	MP
2.	$\neg(\top \,\Box\!\!\rightarrow \bot)$	1, p.c.

$\square$

There are, however, stronger consistency criteria, namely the following:

CNC	$\neg((\alpha \,\Box\!\!\rightarrow \beta) \wedge (\alpha \,\Box\!\!\rightarrow \neg\beta))$
CNC′	$\neg(\alpha \,\Box\!\!\rightarrow \bot)$
RCNC	if $\nvdash_{p.c.} \neg\alpha$ then $\neg((\alpha \,\Box\!\!\rightarrow \beta) \wedge (\alpha \,\Box\!\!\rightarrow \neg\beta))$
RCNC′	if $\nvdash_{p.c.} \neg\alpha$ then $\neg(\alpha \,\Box\!\!\rightarrow \bot)$

Here 'CNC' and 'RCNC' abbreviate 'conditional non-contradiction' and 'restricted conditional non-contradiction', respectively. CNC′ and RCNC′ are variants of principles CNC and RCNC, respectively. CNC says that any two conditionals formulas $\alpha \,\Box\!\!\rightarrow \beta$ and $\alpha \,\Box\!\!\rightarrow \neg\beta$ contradict each other without further qualification. RCNC states that two conditional formulas $\alpha \,\Box\!\!\rightarrow \beta$ and $\alpha \,\Box\!\!\rightarrow \neg\beta$ contradict each other if α is consistent. Furthermore, CNC′ states that no conditional is allowed to have an inconsistent consequent, while RCNC′ gives us that no conditional formula with a consistent antecedent formula has an inconsistent consequent formula. The difference between CNC′ and P-Cons lies in the fact that, if CNC′ is a theorem of a logic **L**, then for formula set Γ a single conditional with an inconsistent consequent formula in Γ suffices to make Γ **L**-inconsistent. For RCNC′ only conditionals with inconsistent antecedent formulas are exempted.

We can easily prove that CNC and CNC′ on the one hand and RCNC and RCNC′ on the other hand are equivalent given reasonable restrictions. The reasonable restrictions are that RW and the following principle are valid:

AND $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \beta \wedge \gamma)$

Note that AND is in a range of conditional logic systems valid (e.g. Adams, 1965, 1966, 1977, 1986; Schurz, 1997b, 1998). Let us now prove that CNC is given RW and AND equivalent to CNC′:

Lemma 3.14. *RW+AND ⇒ (CNC ⇔ CNC′)*

Proof. By Lemmata 3.15 and 3.16. □

Lemma 3.15. *RW+CNC ⇒ CNC′*

Proof.

1.	$\neg((\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta))$	given
2.	$\neg\neg(\alpha \mathrel{\Box\!\!\rightarrow} \bot)$	Ass IP (Indirect Proof)
3.	$\alpha \mathrel{\Box\!\!\rightarrow} \bot$	2, p.c.
4.	$\alpha \mathrel{\Box\!\!\rightarrow} \beta$	3, RW
5.	$\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta$	3, RW
6.	$(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta)$	4, 5, p.c.
7.	$((\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta)) \wedge \neg((\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta))$	6, 1, p.c.
8s.	$\neg(\alpha \mathrel{\Box\!\!\rightarrow} \bot)$	2–6, IP

□

Lemma 3.16. *AND+CNC′ ⇒ CNC*

Proof.

1.	$\neg(\alpha \mathrel{\Box\!\!\rightarrow} \bot)$	given
2.	$\neg\neg((\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta))$	1, Ass IP
3.	$(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta)$	2, p.c.
4.	$(\alpha \mathrel{\Box\!\!\rightarrow} \bot)$	3, AND
5.	$(\alpha \mathrel{\Box\!\!\rightarrow} \bot) \wedge \neg(\alpha \mathrel{\Box\!\!\rightarrow} \bot)$	4, 1, p.c.
6.	$\neg((\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta))$	2–5, IP

□

The proof for the equivalence of RCNC and RCNC′ is perfectly analogous to the proof of Lemma 3.14. On the basis of the equivalence results for RCNC and RCNC′ on the one hand and CNC and CNC′ on the other hand, we will henceforth not specifically refer to the variants RCNC′ and CNC′.

Note that Bennett (2003, p. 84) uses the principle CNC for his Gibbardian stand-off argument (see Section 3.7). We will discuss both CNC and RCNC in more detail in Sections 3.5.3 and 3.7.

Let us now focus on consistency criteria in Stalnaker's (1968; Stalnaker & Thomason, 1970) system. Although Stalnaker (1968) suggests a consistency criterion, he is not very specific about consistency criteria in his semantics. He argues w.r.t. the Ramsey-test that "one cannot legitimately reach an impossible conclusion from a consistent assumption" (Stalnaker, 1968, p. 104). This suggests a consistency criterion in line with RCNC. Interestingly, both RCNC and CNC are, however, not valid in Stalnaker' (1968; Stalnaker & Thomason, 1970 system. Prima facie, Stalnaker (1968) seems to include a consistency criterion in his formalism. He refers to an accessibility relation R in his notion of Stalnaker models (cf. Definitions 3.1 and 3.2), to "make sure" that the impossible world λ – at which all formulas, including $\bot$, are true – is chosen only if there is no world that is accessible, at which the antecedent formula is true. Stalnaker's interpretation of the accessibility relation R is, however, not warranted, since R is simply ineffective in his formalism. In Section 3.2.2 we, hence, followed D. Lewis (1973/2001) in omitting this parameter in our specification of Stalnaker-models. Note here that the great majority of conditional logic systems makes neither RCNC nor CNC valid (e.g. Stalnaker, 1968; Stalnaker & Thomason, 1970; D. Lewis, 1973/2001; Adams, 1965, 1966, 1977, 1986; Schurz, 1998). Notable exceptions are Adams (1975) and Schurz (1997b). Since the precondition of RCNC refers to non-derivability conditions, the reliance on RCNC makes the resulting system a default logic in the sense of Section 2.2.

Stalnaker's semantics, however, makes the following principles valid:

CEM $\quad (\alpha \mathbin{\Box\!\!\rightarrow} \beta) \vee (\alpha \mathbin{\Box\!\!\rightarrow} \neg\beta)$
CEM′ $\quad (\alpha \mathbin{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \mathbin{\Diamond\!\!\rightarrow} \beta)$

'CEM' abbreviates here 'Conditional Excluded Middle'. CEM′ is a variant of CEM. Note that CEM′ is on the basis of $\mathrm{Def}_{\Diamond\!\rightarrow}$ ($\alpha \mathbin{\Diamond\!\!\rightarrow} \beta =_{\mathrm{df}} \neg(\alpha \mathbin{\Box\!\!\rightarrow} \neg\beta)$, see Section 4.2.1) p.c.-equivalent to CEM. To make the difference still more pronounced, note that CEM is also p.c.-equivalent to the following formula:

CEM″ $\quad \neg(\neg(\alpha \mathbin{\Box\!\!\rightarrow} \beta) \wedge \neg(\alpha \mathbin{\Box\!\!\rightarrow} \neg\beta))$

Hence, CNC states that both $\alpha \mathbin{\Box\!\!\rightarrow} \beta$ and $\alpha \mathbin{\Box\!\!\rightarrow} \neg\beta$ cannot both be true, while CEM, CEM′ and CEM″ give us that both conditionals cannot be false. Note that an interpretation of conditional formulas in terms of the material implication makes CEM, CEM′ and CEM″ valid, but not CNC. Although Stalnaker models validate neither RCNC nor CNC, they do so for the weaker consistency criterion P-Cons. This is due to fact that Stalnaker models make MP true and MP implies P-Cons by Lemma 3.13.

Approaches, which enact the consistency requirement in terms of CNC or RCNC, have to make sure that the consequent of every conditional (with a consistent antecedent) is consistent. In case the supposition of the antecedent implies at an inconsistency, adjustments have to be made to make the hypothetical stock of beliefs consistent. In the most typical case there are, however, multiple ways, in which we can adjust inconsistent suppositions. The adjustment is such cases not uniquely determined. In order to arrive at a clear criterion, approaches, which enact such a consistency criterion have to deal in some way or other with different methods in adjusting inconsistent suppositions. Let us finally discuss Bennett's (2003) Ramsey-test criterion:

> "To evaluate $A \rightarrow C$, I should (1) take the set of probabilities that constitutes my present belief system, and add to it a probability = 1

> for A; (2) allow this addition to influence the rest of the system in the most natural, conservative manner; and then (3) see whether what results from this includes a high probability for C." (Bennett, 2003, p. 29)

In Bennett's terminology $A \rightarrow C$ correspond to our $\alpha \,\square\!\!\rightarrow \beta$ (Bennett, 2003, p. 15). Particularly interesting is here condition (2). The adjustment of one's own belief system "in the most natural, conservative manner" rather hints at an ordering approach. Bennett seems to suppose here that such a unique natural and conservative way of revision is in available in all cases. We, however, strongly believe that one has to say more on that issue than just to presuppose that a unique way of adjustment always exists.

3.2.6 A General Ramsey Test Requirement?

Let us now summarize the results of our discussion of the Ramsey-test in the preceding sections and, then, based on these results formulate the core idea of the Ramsey-test. We saw in Sections 3.2.1 and 3.2.2 that the core aspect of the Ramsey-test of conditionals amounts to the following: Hypothetically putting the antecedent to your stock of beliefs and, then to check whether the consequent is true (aspect a, see also Section 3.2.2).

Moreover, we observed that two further aspects of the Ramsey-test are optional, namely (b) a consistency requirement and (c) a probabilistic interpretation. Point (b) states that the stock of beliefs, which results from adding the antecedent to your stock of beliefs is consistent and point (c) requires the Ramsey-test to be formulated in terms of subjective (viz. person-relative or agent-relative) probabilities. We argued in Section 3.2.4 and 3.2.5 that there are semantics, which allow for a pure Ramsey-test interpretation that are truth-valued possible worlds semantics and that do not incorporate aspect (b) and (c).

Let us now discuss what the core assumptions of any Ramsey-test interpretation are: First, the Ramsey-test is subjective in the sense that it refers to some person-relative or agent-relative stock of (hypothetical) beliefs. In that way it differs for D. Lewis' (1973/2001) interpretation of his semantics: In Lewis models the basic notion is overall similarity of possible worlds. The Ramsey-test, however, shares the subjective component, for example, with the ordering semantics of Kraus et al. (1990) and Lehmann and Magidor (1992), which interpret their semantics in terms of preference orderings of normality by agents.

Second, the Ramsey-test refers to stocks of beliefs. Stocks of beliefs, however, do not in general correspond to single possible worlds in possible worlds semantics, as Stalnaker (1968, p. 102) argues: Putting the problem of inconsistent stocks of beliefs aside, the representation of stocks of beliefs by single possible worlds would require that stocks of beliefs are negation-complete in the sense that either we accept α or $\neg\alpha$ for any formula α (see Section 3.2.4). Negation completeness, however, does not follow if we represent stocks of beliefs by sets of possible worlds rather than single possible worlds. Hence, the ontological analogue to a stock of (hypothetical) beliefs is a set of possible worlds rather than a single possible world.

The use of stocks of (hypothetical) beliefs, however, implies that we not only arrive at sets of possible worlds in possible worlds semantics by putting the antecedent hypothetically to our stock of beliefs, but also that we start from a set of possible worlds rather than a single possible world. In possible worlds semantics, however, the evaluation of formulas is done for each possible world and not for sets of possible worlds (cf. the semantics in Sections 3.2.2 and 3.2.3). To accommodate for such a Ramsey-test interpretation we, however, need not change possible worlds semantics, but can reinterpret possible worlds semantics in a specific way: In Chellas-Segerberg semantics (Chellas, 1975; Segerberg, 1989), for example, we can achieve this goal by requiring that a conditional formula $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ is true w.r.t. a set of possible world X just in case $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ is true at all possible worlds $w \in X$. Note that we will discuss this issue in more detail for Chellas Segerberg semantics in Section 7.1.

3.3 Distinguishing Indicative and Counterfactual Conditionals

Let us, first, repeat the criteria for distinguishing counterfactual conditionals from indicative conditionals in natural language from Section 2.1.4. We saw that there are at least two ways, in which this can be done. In the first approach (approach A) the difference between indicative and counterfactual conditionals is drawn on the basis of the mood (indicative vs. subjunctive). We employ here the second approach that follows the intuition that counterfactuals are "counter to the facts" (approach B). According to approach (B) we can further distinguish between (B1) "genuine" counterfactuals and (B2) tentative counterfactuals. Counterfactuals of the first type are counterfactuals, for which the antecedent is false, whereas counterfactuals of type (B2) include counterfactuals whose antecedent is merely improbable (see Section 2.1.4).

3.3.1 Indicative Conditionals

We shall now focus on indicative conditionals. In Section 1.2 we distinguished between indicative conditionals (in the broader sense) and normic conditionals. For mnemonic reasons we repeat here example E10, which is a clear case of a normic conditional:

E10 Fishes are normally cold-blooded.

Normic conditionals differ from indicative conditionals (in the narrower sense) insofar as the antecedent does not always guarantee that the consequent holds. We, however, saw in Section 1.4 that it is hardly plausible to distinguish between indicative conditionals in a narrow sense and normic conditionals on the basis of linguistic facts alone (such as by inclusion of a modifier 'normally' or 'probably'). Given our argumentation in Section 1.2.2 it seems, hence, plausible that we treat both types of conditionals as indicative conditionals. We, moreover, saw that inference S6′ is not valid. Inference S6′ corresponds to the following principle:

MP $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \rightarrow \beta)$

Note that we discussed principle MP ("Modus Ponens") already in Section 3.2.3 for Lewis frames and models (Definitions 3.5–3.7). We saw, moreover, in Section 3.2.3 that principle MP corresponds to the weak centering condition C_{MP}.

Principle MP is, however, not valid for indicative conditionals, since for normic conditionals (e.g. example E10) it might well happen that $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ and α are the case, but β is not. Since we intend to include in our logical analysis all types of indicative conditionals, principle MP should not be treated as a valid principle for indicative conditionals.

Let us now take a closer look at counterfactuals. If we use the narrower definition (B1), then MP is valid. This is due to the fact that the antecedent of a counterfactual is – according to that definition – always false. Hence, $\alpha \to \beta$ is always true and MP must hold. Note that this is not the case if we instead use criterion (B2). Here the antecedent of the counterfactual is merely improbable, but need not be false. Let us now take a look at a further principle:

CS $\quad (\alpha \wedge \beta) \to (\alpha \mathrel{\Box\!\!\rightarrow} \beta)$

We encountered principle CS ("Conditional Sufficiency") already in Section 3.2.3 in the context of Lewis models. We, furthermore, saw that it corresponds to the second centering condition C_{CS}. Does CS, then, hold for indicative conditionals?

We argue that principle CS is no valid principle for indicative conditionals (which include normic conditionals). Let us take a look at example E14 and E15 from Section 1.2.2. For mnemonic reasons we repeat here both examples:

E14 If specimen 214 is a fish, then it is probably cold-blooded.
E15 If specimen 214 is a fish, then it is normally cold-blooded.

Principle CS fails for both examples E14 and E15, since specimen 214 being a fish and being coldblooded does not suffice to imply either normic conditional E14 or E15. For counterfactual conditionals we have to distinguish again between both criteria (B1) and (B2). Given criterion (B1), the antecedent α of counterfactual conditional $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ is always false and, hence, $\alpha \wedge \beta$ is false for any counterfactual conditional. It follows that CS is valid trivially for all counterfactual conditionals interpreted in terms of (B1). Criterion (B2), however, does not give us this result, since it only requires that the antecedent of a counterfactual is merely improbable.

In section 1.2.2 we describe a further reason why MP might not be regarded a normatively adequate principle for indicative conditionals. We saw that a perfectly acceptable set of propositions becomes inconsistent if we accept MP:

If specimen 213 is a mammal, it is normally viviparous.
If specimen 213 is an anteater, it is normally not viviparous.
Specimen 213 is an anteater and a mammal.

These propositions are most plausibly represented by following set of formulas: $\{\alpha \mathrel{\Box\!\!\rightarrow} \gamma, \beta \mathrel{\Box\!\!\rightarrow} \neg\gamma, \alpha \wedge \beta\}$. From $\alpha \mathrel{\Box\!\!\rightarrow} \gamma$ and $\beta \mathrel{\Box\!\!\rightarrow} \neg\gamma$ follows by MP that $\alpha \rightarrow \gamma$ and $\alpha \rightarrow \neg\gamma$, respectively. Those two formulas however, imply together with $\alpha \wedge \beta$ that $\gamma \wedge \neg\gamma$. Note that the above set of propositions does not represent a marginal case, but that perfectly parallel cases are present in the sciences (see also Section 1.2.2).

3.3.2 Criteria for Counterfactual Conditionals

Based on our discussion, we, hence, conclude that MP and CS do not hold for a logic of indicative conditionals. Note in that context that principles CS and MP are valid in some logics for indicative conditionals (i.e. Adams, 1965, 1966, 1977, 1975; Stalnaker, 1968; Stalnaker & Thomason, 1970; see also Section 3.2.2) and in some not (i.e. Kraus et al., 1990; Lehmann & Magidor, 1992; cf. Section 3.2.3). Concerning counterfactual conditionals MP and CS seem to valid. This, however, holds only on criterion (B1) and not for criterion (B2). This wider criterion is not strong enough to guarantee that both MP and CS hold. In D. Lewis' (1973/2001) preferred logic for counterfactuals, however, both principles are valid (see Section 3.2.3).

Note that both principles MP and CS are bridge principles. This type of formula postulates a (fixed) relationship between conditional and unconditional formulas. We restricted ourselves to the discussion of MP and E, since those two principles seem to be the most plausible candidates for drawing a dividing line between the logic of indicative conditionals and counterfactual conditionals. Due to our description of counterfactuals in terms of criterion (B1) and (B2) it is, hence, plausible to locate the difference indicative and counterfactual conditionals in bridge principles. Note in that context that accepting CS and MP for a possible worlds semantics recovers partial truth functionality in the following sense: Due to both principles the truth-value of the antecedent and the consequent determines the truth-value of the conditional formula for the first two rows of the truth table (where the antecedent is true) (see also Section 3.5.4). Note here that criterion (A) – which locates the difference between indicative and counterfactuals conditionals in the mood of the sub-clauses – does not directly give us reasons for drawing the difference between indicative and counterfactual conditionals on the basis of any logical properties.

3.3.3 Subjective or Objective Interpretations of Indicative and Counterfactual Conditionals?

We saw in Section 2.1.4 that in philosophical approaches to conditionals often indicative and counterfactual conditionals are given different logical and philosophical analyses. Although, for example, D. Lewis (1973/2001, p. 3) admits that the term 'counterfactuals' is too narrowly construed for his investigation of conditional logics, D. Lewis' (1973/2001) discussion clearly shows that he focuses on counterfactual conditionals rather than indicative or normic conditionals. Moreover, Bennett (2003) accepts a Ramsey-test interpretation of indicative conditionals, but rejects such an interpretation for counterfactual conditionals. He argues that the following counterfactual does not pass the Ramsey-test:

"If Yeltsin had been in control of Russia and of himself, Chechnya would have achieved independence peacefully." (Bennett, 2003, p. 30)

Unfortunately, Bennett (2003) only states that the Ramsey-test fails for this particular counterfactual conditional, but does not give general reasons why this is the case. (Note that Bennett just gives that one example and neither discusses this issue any further nor does he refer to other sources.) In addition, Adams (1970, p. 92) argues that his probabilistic semantics (1965, 1966, 1977; see also Adams, 1977; Schurz, 1996, 1997b, 1998, 2005) is only applicable to indicative conditionals, but not to counterfactuals.

In the literature several approaches exist where to draw the line between counterfactuals on the one hand and indicative conditionals on the other hand (cf. Bennett, 2003, Chapter 22 and 23). These approaches do not stop at the criteria (A) and (B1) and (B2) discussed at the beginning of this section and Section 2.1.4, but in addition try to work out the commonalities and differences of indicative and counterfactual conditionals. We will not survey these approaches here, but focus instead on some general observations that seem to underlie many of these approaches. Let us, for that purpose, focus on the examples of Adams (1970, p. 90) (cf. Section 2.1.4):

E18 If Oswald hadn't shot Kennedy in Dallas, then no one else would have.

E19 If Oswald didn't shoot Kennedy in Dallas, then no one else did.

We saw in Section 2.1.4 that E18 is naturally accepted as true, but E19 is not. What is, then, the difference in truth conditions for E18 and E19? In example E19 we seem to take into account our own knowledge about the assassination of Kennedy and its historic context. Hence, it is natural to read E19 in a subjective, agent-relative way. This is, however, not the case for E18. Here we focus on the event of the assassination and rather abstract from our knowledge regarding Kennedy's assassination and its historic context. This difference of interpretation of E18 and E19 also seems to underlie Adams' (1970, p. 92) argumentation, since he suggests an analysis of conditionals such as example E19 but not of conditionals of type E18 in terms of his subjective probabilistic semantics (Adams, 1965, 1966, 1977).

What do we make, then, of this subjective aspect of indicative conditionals, such as E19? We saw in Section 3.2.6 that the Ramsey-test has an essentially subjective component by requiring that conditionals are evaluated relative to an agent's hypothetical beliefs. It is, hence, natural to provide an interpretation of indicative conditionals in terms of the Ramsey-test. Moreover, this subjective element is missing in conditionals of type E18. Note, however, that a Ramsey-test interpretation is not the only possible interpretation of indicative conditionals: For example, Kraus et al. (1990) also suggest an interpretation of (normic) indicative conditionals in terms of agents' subjective ordering of "normality" preferences (see Section 3.2.3), which is based on a different semantic idea compared to the Ramsey-test (see Section 3.2.4).

How about interpretations of counterfactual conditionals, such as E18? We saw that E18 does not rely on one's particular assumption about Kennedy's assassination and its historic context. It is, hence, not surprising that D. Lewis (1973/2001) suggests an analysis of counterfactuals, such as E18, in terms of objective criteria such as similarity of possible worlds (see Section 3.2.3). For D. Lewis this is a part of a larger philosophical project: First, D. Lewis (1973/2001) bases his analysis of counterfactuals on the notion of similarity between possible worlds. Then, he uses his counterfactual semantics in order to provide an analysis of other philosophical concepts, such as causality (D. Lewis, 1973) and alethic necessity D. Lewis (1973/2001, p. 22f, see also 137–142).

By alethic necessity we refer to states of affairs, which are not just true, but necessarily so, as opposed to necessity in deontic/normative way ('It ought to be the case that ...') and or an epistemic way ('Person *A* knows [believes] that ...'). Observe that we can explicate alethic necessity in different ways, such as necessity according to logical laws, physical laws etc. (D. Lewis, 1973/2001, p. 7f; see also Schurz, 1997a, p. 8 and p. 18). These notions of alethic necessity, however, share that they refer to necessity of truth rather than necessity according to beliefs or normative principles.[4]

Note that causality and alethic necessity are very naturally conceived as being objective. So an analysis in terms counterfactuals, which is in turn based on an objective criterion, such as similarity of possible worlds, is in that sense a perfect fit. Moreover, in order to find out whether an alethic necessity statement, such as 'It is necessarily the case that $2 + 2 = 4$', is true, we often engage in counterfactual reasoning, such as 'If things were different, would $2 + 2$ still be 4?'. So, by these means D. Lewis (1973/2001) gives us a natural and coherent analysis of alethic necessity and counterfactual conditionals.

[4] D. Lewis (1973/2001, p. 8) discusses only one non-alethic interpretations of necessity, namely deontic necessity. D. Lewis' ideas for this type of necessity are, however, described in a separate subchapter (namely Chapter 5.1), where Chapter 5 is labeled 'Analogies'.

3.4 Fundamental Issues of Probabilistic Approaches to Conditional Logic

Before we describe the conditional logic systems of Adams (1965, 1966, 1977, 1986) and Adams (1975), let us first discuss the following fundamental issues of probabilistic conditional logics: (i) the difference between subjective and objective frequency-based probabilistic semantics, (ii) taking either unconditional or conditional (open) formulas as basic and (iii) the language, in which the conditional logic is formulated. Point (i) aids to emphasize a point made in our discussion of Bennett's Gibbardian stand-off argument (Section 3.7), namely that there are semantics for conditionals, which are not based on the Ramsey-test. Points (ii) and (iii), then, support our discussion of D. Lewis' (1976) triviality result (Section 3.6).

3.4.1 Subjective and Objective Probabilistic Semantics

We can distinguish at least two fundamental approaches in probability theory, namely a subjective and an objective frequency-based account of probabilities (Schurz, 2008, p. 99). Both approaches satisfy the Kolmogorov axioms (Kolmogorov 1950; see Schurz, 2008, p. 101f), but differ crucially in the interpretation of probabilities (cf. Schurz, 2008, p. 99). It is, hence, not surprising that we can distinguish between subjective and objective frequency-based probabilistic semantics for conditional logics.

In both, the subjective probabilistic conditional logics of Adams (1965, 1966, 1977, 1975, 1986) and the objective frequency-based conditional logics described in Schurz (1997b, 2005), indicative conditionals are analyzed in terms of a conditional operator, such as '$\Box\!\!\rightarrow$'. In the subjective approach of Adams (1965, 1966, 1977, 1975, 1986) indicative conditionals are represented in terms of conditional formulas $\alpha \Box\!\!\rightarrow \beta$, where α and β stand for the antecedent and the consequent of the natural language conditional, respectively. Formulas α and β are either (a) formulas of a f.o.l.-language, which do not contain free individual variables or else (b) formulas of a p.c.-language. Approach (a) makes only sense, if we in general allow for quantification over individual variables in non-conditional formulas. This is neither presupposed in the discussion of conditionals in Chapter 1 nor is it an element of Adams' (1965, 1966, 1977, 1975, 1986) conditional logic systems. Hence, in both cases approach (b) is used.

In an objective frequency-based approach, such as Schurz (1997b; see also Bacchus, 1990, Chapters 3 and 4), natural language conditionals are represented by conditional formulas of the form $\alpha[x] \;\Box\!\!\rightarrow \beta[x]$. The antecedent formula $\alpha[x]$ and consequent formula $\beta[x]$ are formulas, which contain only monadic predicates and x is the only free variable occurring in $\alpha[x]$ and $\beta[x]$ (Schurz, 1997b, p. 538).

In the subjective and objective approach probabilities are assigned to closed and open formulas, respectively. In the subjective approach the assignments express an agent's degree of belief in the proposition described by the formula (cf. Schurz, 2008, p. 99). Note that the formula does not describe a type of event (or a type of state of affairs), but rather a particular instance of an event at a particular time at a particular place (or in other words a fully determined state of affairs). In the objective approach probabilities, which are assigned to open formulas, are interpreted as the relative frequencies of the properties expressed by the open formulas (i.e. $\alpha[x]$) or their relative frequencies in the long run (cf. Schurz, 2008, p. 99). For this approach to be applicable, the events or instances described by a formula $\alpha[x]$ have to be in some sense "repeatable". Hence, the frequency-based approach cannot directly account for single-case probabilities, which refer to specific instances occurring only once at a particular time at a particular place (or which, alternatively, refer to fully determined state of affairs). We shall, however, below discuss a (partial) solution to this problem based on the principle of total evidence.

Conditional logic systems with a subjective or a frequency-based probabilistic semantics for conditionals do not allow for an equally natural interpretation of all conditionals. In Section 1.2.2 two types of normic conditionals in natural language were introduced – quasi-universally-quantified (quq) conditionals and their propositional versions. Instances E8 ('Most fishes are cold-blooded.') and E14 ('If specimen 214 is a fish, then it is probably cold-blooded.') are examples for the quq-case and the propositional case, respectively. Since conditionals like E14 refer to fully determined state of affairs, the propositional case seems most apt for a subjective probabilistic interpretation. In a similar vain, since E8 appears to refer to frequencies, the objective frequency-based interpretation seems most natural.

In our description of subjective and objective frequency-based approaches, we did not use a full f.o.l.-version. For the subjective and the objective frequency-based approach, we employed a p.c.-language and a monadic f.o.l.-language, respectively. For both the subjective and the frequency-based approach a full probabilistic f.o.l.-version for a conditional logic is not easy to achieve (Schurz, 1997b, p. 538). In particular, the relation between quantified conditionals such as E8 and single-case conditionals such as E14 is hard to specify. On the one hand there exists no deterministic relationship between both types of conditionals, since the quq-conditional E8 does not strictly imply its instance E14 (see Section 1.2.2). On the other hand they are not unrelated, as Adams' (1998, p. 285f) account suggest. There exists a logical relationship between both types of probabilistic conditionals. This relationship is, however, not strictly deterministic. Schurz (2008, p. 101 and 115f; see also Schurz, 2005, p. 42; 1997b, p. 538) suggests accounting for the relationship between subjective probabilities and frequency-based probabilities by means of the (non-monotonic) principle of total evidence, which goes back to Carnap (1962, p. 211). Note, however, that Schurz does not directly specify the relation between subjective probabilistic conditionals, such E14, and their frequency-based version, such as E8. He rather describes the subjective probability of *unconditional* facts on the basis of *conditional* frequency-based facts. Schurz's (2008, 116) solution amounts to the following: Given we have a finite set of facts and individuals, we can determine the subjective probability of a fact $\alpha[a]$ from a set of objective probability conditionals the following way: $P_{\text{subj}}(\gamma[a] \mid \alpha[a] \wedge \beta[b_1, \ldots, b_n] \wedge P(\gamma[x]/\alpha[x]) = r) = r$, where $\alpha[a]$ summarizes our total factual knowledge about individual a and $\beta[b_1, \ldots, b_n]$ is the complete factual knowledge about other individuals $b_1, \ldots, b_n$. Alternatively, one can use only the statistically relevant factual knowledge pertaining to the individuals. This is due to the fact that statistically irrelevant facts do not bear on inferences drawn from these facts. Observe that in our terminology α, β, γ are only allowed to contain monadic predicates.

The principle of total evidence determines probability P_{subj} by the narrowest reference class available on the basis of our total knowledge. According to this approach, hence, our degrees of beliefs in a proposition $\alpha[a]$ are determined by objective facts and either our total knowledge or alternatively our complete statistically relevant knowledge.

3.4.2 Conditional or Unconditional Probabilities as Primitive

Both the subjective accounts of Adams (1965, 1966, 1975, 1977) and the objective frequency-based account of Schurz (1997b, 1998, 2005) agree in the way they assign probabilities to conditional formulas. In the Adams' (1965, 1966, 1977, 1975) approaches $P(\alpha \mathrel{\Box\!\!\to} \beta)$ is equated with $P(\beta \mid \alpha)$.[5] In Schurz's (1997b, 1998, 2005) approach $P(\alpha[x] \mathrel{\Box\!\!\to} \beta[x])$ is used as an abbreviation for the conditional probability $P(\beta[x] \mid \alpha[x])$. Schurz (1997b, 1998, 2005) is, however, very careful in the sense that he does not equate the probability with the respective conditional probability, but rather associates conditionals with conditional probabilities. In this account a conditional is the case only if the respective conditional probability is high (Schurz, 1997b, p. 536; Schurz, 1998, p. 85; Schurz, 2005, p. 38).

In the account of Adams, hence, the following two components are effectively employed: (1) Probabilities of conditional formulas are equated with the respective conditional probabilities and (2) conditional probabilities are defined by means of unconditional probabilities. Component (1) is also called the Stalnaker thesis (Bennett, 2003, p. 58). This thesis is an essential assumption of D. Lewis' (1976) well-known triviality result (see Chapter 3.6). We discuss this issue at length in Chapter 3.6.

Moreover, Bennett (2003, p. 55) argues that component (1) is the core idea of the Ramsey-test (cf. also Adams, 1975, p. 3). There are, however, strong reasons to assume that this is not the case: First, Stalnaker (1968) describes a version of the Ramsey-test, which does not rely on probabilities, but which is targeted for possible world semantics. Hence, a probabilistic interpretation in terms of (1) is

[5] Adams (1965, 1966, 1977, 1975) does not, strictly speaking, use the notion of conditional probability in his approach, but defines $P(\alpha \mathrel{\Box\!\!\to} \beta)$ directly by means of $P(\alpha \wedge \beta)$ and $P(\beta)$. He, however, does so in accordance with the usual definitions of conditional probabilities (see Sections 3.5.1 and 3.5.3). Hence, his account essentially equates the probability of conditional formulas with the respective conditional probability.

not a necessary feature of the Ramsey-test (see Chapter 3.2).[6] Second, objective frequency-based approaches like Schurz (1997b) do not allow for a natural interpretation of conditionals in terms of the Ramsey-test. They, instead, employ a frequency-based interpretation of conditionals (see above). Hence, component (1) is also not a sufficient feature of the Ramsey-test.

Let us now address point (2). Adams effectively defines conditional probabilities by means of unconditional probabilities. This is, however, not the only possible approach. Hawthorne (1996, p. 191), for example, takes conditional probabilities as basic (by means of Popper functions). He can – but need not – define unconditional probabilities on the basis of conditional probabilities, for example by equating $P(\alpha) = P(\alpha \mid \top)$ or, alternatively, by equating $P(\alpha[x]) = P(\alpha[x] \mid \top[x])$. Here $\top$ and $\top[x]$ represent a logically true formula and a logically true formula based solely on monadic predicates and with a single free variable x, respectively.

Note two important points here. First, such a definition is by no means innocuous, but renders – given a sufficiently rich language – additional logical principles valid (see Chapter 3.3). Second, we might choose (a) not to presuppose a fixed relationship between conditional and unconditional formulas or (b) to restrict the set of formulas in such a way that only conditional formulas of the form $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ are admitted, where α and β do not contain an instance of the conditional operator $\mathrel{\Box\!\!\rightarrow}$ (cf. language $\mathcal{L}_{\text{rrKL}}$, Appendix 4.2.2).

3.4.3 The Status of Conditionals and the Language of a Probabilistic Conditional Logic

In a majority of conditional logics with a possible worlds semantics (i.e. Stalnaker, 1968; D. Lewis, 1973/2001), a full conditional logic language is used: A two-place conditional operator '$\mathrel{\Box\!\!\rightarrow}$' is employed, such that any recombination of conditional and unconditional formulas is permitted. Such a full language is language $\mathcal{L}_{\text{KL}}$, as specified in Section 4.2.1.

In the probabilistic approaches of Adams (1965, 1966, 1977, 1975) on the one hand and Adams (1986) and Schurz (1998) on the other hand more restricted

[6] To our knowledge Stalnaker (1968) was the first to introduce the Ramsey-test for conditionals into the conditional logic literature and the philosophy of conditionals literature.

languages are employed. Adams (1965, 1966, 1977, 1975) employ a specific language, which allows for both conditional and non-conditional formulas (formulas without a conditional operator), but is restricted insofar as it (a) neither admits boolean combinations (by means of p.c.-connectives) of conditional formulas nor (b) allows for nestings of conditionals.[7] The following two formulas are examples of boolean combinations of conditional formulas: $(p_1 \boxright p_2) \wedge (p_3 \boxright p_4)$ and $\neg(p_1 \boxright p_2)$. Moreover, the formula $p_1 \boxright (p_2 \wedge (p_3 \boxright p_4))$ is an example for a nested conditional formula. The notions *boolean combination* and *nestedness* are defined and discussed in Section 4.2.1 w.r.t. the language $\mathcal{L}_{\mathrm{KL}}$.

Let us describe the language of Adams (1965, 1966, 1977) in some more detail. Formulas of $\mathcal{L}_{\mathrm{KL}^-}$ are either purely propositional formulas (without a conditional operator) or contain exactly one conditional operator '$\boxright$', which is the formula's main connective. Adams' (1965, 1966, 1977) language corresponds to our language $\mathcal{L}_{\mathrm{KL}^-}$, which is described in Section 4.2.1 (cf. Adams, 1965, p. 184; Adams, 1966, p. 270).[8] The symbol '$-$' in '$\mathcal{L}_{\mathrm{KL}^-}$' indicates that $\mathcal{L}_{\mathrm{KL}^-}$ is a restricted version of language $\mathcal{L}_{\mathrm{KL}}$.

Furthermore, Adams (1986) use a language, which allows both (simple) conditional formulas of the form $\alpha \boxright \beta$ and disjunctions of conditional formulas (e.g. $(p_1 \boxright p_2) \vee (p_3 \boxright p_4)$) in the conclusion of arguments, but not in premises. So, Adams (1986) essentially draws on the language $\mathcal{L}_{\mathrm{rKL}}$, which admits (a′) in general for disjunctions of conditionals and also endorses (b). '$\mathcal{L}_{\mathrm{rKL}}$' stands here for 'restricted version of language $\mathcal{L}_{\mathrm{KL}}$'. Language $\mathcal{L}_{\mathrm{rKL}}$ is also formally specified in Section 4.2.1. Note, however, that Adams (1986) uses language $\mathcal{L}_{\mathrm{rKL}^*}$ in a restricted way, since he limits disjunctions of conditional formulas to the conclusion of inferences. We shall also ignore here that Adams (1986) allows for empty disjunctions (disjunctions with 0 disjuncts) and postpose a discussion of this issue to the next section. Moreover, observe that Schurz (1998, p. 84f) shows (see also Section 4.2.1) that allowing for inferences with disjunctions in the an-

[7]Adams (1975) adds the requirement that any conditional formula in his language must have a p.c.-consistent antecedent formula (p. 46). We will discuss this issue in Section 3.5.3.

[8]Note that in Adams (1965, 1966, 1977) the propositional part of his language seems to contain '$\neg$' (negation), '$\wedge$' (conjunction) and '$\vee$' (disjunction) as primitives, while '$\rightarrow$' (material implication) seems to be defined as a meta-language abbreviation (Adams, 1965, p. 184). We defined the language $\mathcal{L}_{\mathrm{KL}^-}$ in a slightly different way. Note also that Adams (1965, 1966, 1977) is not too explicit about the specifics of his propositional system and its language.

tecedent is inessential insofar as any valid inference in Adam's (1986) system with disjunctive premises can be transformed into *sets* of inferences, which have only disjunctions in the conclusion. This result does, however, not imply that we can also express inferences by *single* inferences – rather than sets of inferences – in Adams' (1986) restricted language.

Kraus et al. (1990), Lehmann and Magidor (1992), Hawthorne (1996) and Hawthorne and Makinson (2007), however, employ a conditional logic language, which is as expressive as language $\mathcal{L}_{\mathrm{rKL}}$. These approaches use conditional assertions of the form $\alpha \mid\!\sim \beta$ and their negations $\alpha \not\mid\!\sim \beta$ rather than conditional formulas (cf. Section 2.2.7). Since '$\mid\!\sim$' is regarded a meta-language symbol, these conditional assertions represent no (object language) formulas, but rather statements about formulas in the object language. Kraus et al. (1990), Lehmann and Magidor (1992), Hawthorne (1996) and Hawthorne and Makinson (2007) also employ a second-order meta-language inference relation (cf. Hawthorne & Makinson, 2007, pp. 251ff; Kraus et al., 1990, p. 177). This second-order inference relation is, however, often not explicitly spelled out (see Section 2.2.7).

We can reconstruct the conditional logic language of Kraus et al. (1990), Lehmann and Magidor (1992), Hawthorne (1996) and Hawthorne and Makinson (2007) in terms of an object language account. Since these approaches focus on conditional assertions (and their negations) only, we can specify a language, in which only conditional formulas and their negations are admissible. We define such a language, namely language $\mathcal{L}_{\mathrm{rKL}^*}$, formally in Section 4.2.1. All formulas of language $\mathcal{L}_{\mathrm{rKL}^*}$ are either of the form $\alpha \,\Box\!\!\rightarrow \beta$ or of the form $\neg(\alpha \,\Box\!\!\rightarrow \beta)$, where α and β are not allowed to contain conditional operators. Note that both languages $\mathcal{L}_{\mathrm{rKL}}$ and $\mathcal{L}_{\mathrm{rKL}^*}$ are equally expressive (cf. Schurz, 1998, p. 84), as we shall discuss in more detail in Section 4.2.1.

We have to stress two important points here. The first point focuses on the argument by Bennett, Edgington, Adams and Gibbard that conditionals are not propositions (Bennett, 2003, p. 94; see Sections 3.1 and 2.1.4). Bennett does not clarify what he means by 'propositions'. His discussion, however, shows that he associates at least three properties with the notion of a proposition, namely (1) having truth-value and (2) being embeddable into more complex structures, for example by means of conjunctions etc. (Bennett, 2003, p. 95) and (3) being the object of probability assignments in a subjective probabilistic framework. As we shall argue here, points (1)–(3) cannot be equivalent to each other and should, therefore, not be treated as being tantamount to each other.

Let us, first, address the relationship between points (1) and (2). We can, for example, allow for boolean combinations of conditionals (point 2), as it is done in languages $\mathcal{L}_{\mathrm{rKL}}$ and $\mathcal{L}_{\mathrm{rKL}^*}$. In such a language simple conditionals are embeddable in larger structures, namely either in disjunctions or negations (of conditional formulas). A more extreme example is language $\mathcal{L}_{\mathrm{KL}}$, in which all boolean combinations of conditionals are allowed. The admittance of boolean combinations of conditionals, however, does not per se imply that conditional formulas have truth-values: If we intend to give a semantic interpretation of conditionals in a compositional way (cf. Connolly, Fodor, Gleitman, & Gleitman, 2007, p. 2f; Schurz, 2007, Section 4.5), then the meaning of simple conditionals has to determine the meaning of the disjunctions of conditionals and negations of conditionals. This, however, does not imply that in such a case simple conditionals *must* have truth-values. For example, in a probabilistic framework one could alternatively rely on the notion of probability. We do not suggest that one should take such an approach. Observe, however, that it is one of the motivations of the introduction of Popper-functions that Popper-functions do not rely on deductive logic, which includes the reliance on the notion of truth (Hawthorne, 1996, p. 190f).

In addition, (1) does not imply (2). The fact that conditional formulas have truth-values does, for example, not imply that conditional formulas can be parts of boolean combinations. If we, for example, restrict the language of standard p.c. to language $\mathcal{L}_{\mathrm{KL}^-}$ and accept the extensional semantics of p.c. for the analysis of conditionals, we assign truth-values to conditional formulas, but do not allow for embeddings of conditionals in more complex structures. Observe following important point here. In order to assign truth-values to conditionals, we have to refer to expressions of a specific language. When we represent natural language conditionals only by conditional assertions of the form $\alpha \mathrel{|\!\sim} \beta$, neither an assignment of truth-values to conditionals assertions is possible in the object language nor can one embed conditional assertions in more complex object language formulas. Hence, in this case both approaches (1) and (2) are not possible w.r.t. the object language. If we use language $\mathcal{L}_{\mathrm{KL}^-}$, we do not enact (2) for the object language, but might enact (1). Since Adams (1965, 1966, 1977) employs the language $\mathcal{L}_{\text{KL}^-}$, he explicitly excludes aspect (2) for the object language, but does not preclude (1). In the approaches of Kraus et al. (1990), Lehmann and Magidor (1992), Hawthorne (1996) and Hawthorne and Makinson (2007), however, both aspects (1) and (2) are precluded for the object language, as they use only conditional assertions. This is remarkable, since Kraus et al. (1990) and Lehmann and Magidor (1992) employ a possible worlds semantics and, thus, use the notions of truth and falsehood as basis for the analysis of conditionals in natural language. The latter fact, however, contradicts Bennett's (2003, p. 95) position, since he argues that (2) is an advantage of probabilistic semantics over truth-value based semantics.

Let us now address point (3). The line of argumentation against truth-value approaches by, for example, Bennett (2003) and Adams (1975) is based on the fact that – given a subjective probabilistic framework, as described by Adams (1965, 1966, 1977, 1975, 1986) – the full language $\mathcal{L}_{\text{KL}}$ cannot be used. Since the formulas, to which probabilities are assigned, are regarded as propositions, and a non-restricted language (containing arbitrary boolean combinations of conditional formulas) runs into D. Lewis' (1976) triviality result, it is argued that conditional formulas are no propositions and, hence cannot in general have truth-values (see Sections 3.1 and 3.6.1).

Let us, for the sake of the present discussion, presuppose that – contrary to our discussion above – (2) is tantamount to (1), viz. that any proposition X described by a specific language has a truth-value, and vice versa. Even in that case, (3) does not give us that conditionals do not in general have truth-values. Such an inference is problematic for at least two reasons: First, we are not aware of any formal result, which shows that truth-value semantics, such as Chellas-Segerberg semantics (see Chapters 4–7), presupposes a subjective probabilistic framework as described by Adams (1965, 1966, 1977, 1975, 1986). Such a result would, however, be needed in order to show that conditional in general cannot have truth-values. D. Lewis' (1976) proof, however, gives us only the following result: Provided (i) we enact a subjective probabilistic account as Adams (1965, 1966, 1977, 1975, 1986) and (ii) use the full language $\mathcal{L}_{\text{KL}}$, then the probabilistic systems becomes trivial. Lewis' (1976) result, however, does not extend to approaches, for which (ii) does not hold.

D. Lewis' (1976) triviality result is, hence, not a genuine problem of truth-value approaches, but rather of probabilistic approaches. D. Lewis (1976) concludes in a similar vain that "it was not the connection between truth and probability that led to my triviality results, but only the application of standard probability theory to the probability of conditionals" (p. 304).

Moreover, a closer look at D. Lewis' (1976) triviality result shows that Adams' (1975, p. 35) and Bennett's (2003, p. 94) approach – to reject that conditionals have truth-values (NTV, no truth-value) – does not line up perfectly with D. Lewis' triviality results, as Bennett (2003, p. 77) seems to presuppose. D. Lewis' (1976, p. 304) discussion shows that we do not have to exclude any boolean combination of conditionals, but only have to make sure that no conjunction involving conditional formulas is expressible in our language (see Section 3.6.1). Hence, we can allow either for (i) negations of conditionals (e.g. language $\mathcal{L}_{\mathrm{rKL}}$) or (ii) disjunctions of conditionals (e.g. language $\mathcal{L}_{\mathrm{rKL}}$), but not both. It should be, however, clear that NTV alone can hardly account for disjunctions or negations of conditionals. Accordingly, Adams (1986, p. 260) justifies his use of disjunctions of conditionals (although only in the meta-language) not just by NTV, but also by his notion of probabilistic enthymemes. We shall, however, not go into further detail regarding this issue.

Second, we argue that it is quite counter-intuitive to assume NTV on the basis of a subjective probabilistic semantics. Let us, for that purpose, presuppose – in line with Adams (1975) and Bennett (2003) – (i) a subjective probabilistic framework, as described by Adams (e.g.   1965), and (ii) accept NTV. Furthermore, consider the following statements:

(A) Specimen 214 is a cold-blooded fish.
(B) If specimen 214 is a fish, then it is probably cold-blooded. (E14)

We can analyze (A) and (B) in a subjective probabilistic framework and assign to both (A) and (B) subjective probabilities. Adam's probabilistic framework implies, then, that (A) has an (objective) truth-value, while on the basis of NTV conditionals, such as (B), in general do not have a truth-value. Assumptions (i) and (ii) imply that conditional statements must have a different metaphysical status than unconditional statements. In terms of the correspondence theory of truth there might be no facts, which correspond to conditional statements, although there are always facts, which correspond to unconditional statements. This implication, however, seems hardly acceptable from a philosophy of science stance – both from a realist and an anti-realist perspective. So, what went wrong? In our eyes NTV is to blame here. A probabilistic framework as in e.g. Adams (1965) does not give us that conditionals are not propositions and, hence, do not have truth-values.

Furthermore, observe that the restriction of a conditional logic's language has strong implications for the relation of the proof theory and the model theory of the conditional logic. Most typically for conditional logics with a possible worlds semantics formulated in the full language, the canonical model technique is used (e.g. Stalnaker & Thomason, 1970, p. 33–38; D. Lewis, 1973, p. 133f). This technique relies on maximally consistent sets. A typical property of such sets is that for all formulas α of the language either α or $\neg\alpha$ is an element of such a set (the maximality property, see Hughes & Cresswell, 1984, p. 19). In languages, such as $\mathcal{L}_{\mathrm{rrKL}}$, no negations of conditional formulas are permitted. Hence, this technique is not directly applicable to conditional logics formulated in that language. A more dramatic example for the impact of the language's expressivity on soundness and completeness proofs gives Adams (1977). On the basis of language $\mathcal{L}_{\mathrm{KL}^-}$, Adams (1977) shows that validity in his probabilistic semantics (Adams, 1965, 1966, 1977) and validity in all finite D. Lewis' (1973/2001) models (see Definitions 3.5–3.7), which satisfy the centering condition C (see Section 3.2.3). This holds not for the full language $\mathcal{L}_{\mathrm{KL}}$. In this language the proof theory for D. Lewis' semantic variant is system **VC** (D. Lewis, 1973, p. 132). Logic **VC** makes a version of rational monotonicity (RM) valid (see Section 2.2.7). RM is a theorem of Adams' (cf. 1986, p. 260) and Schurz's (1998, p. 84) system, but not of Adams' (1965, 1966, 1977) proof-theory (cf. Schurz, 1998, p. 84). For the restricted language $\mathcal{L}_{\mathrm{KL}^-}$, however, both semantics agree, since RM is simply not expressible in the language $\mathcal{L}_{\mathrm{KL}^-}$. (This is due to the fact that any equivalent version of RM has to refer in some way to negated conditional formulas or alternatively to a disjunctive conditional formula, while both types of formulas are not expressible in $\mathcal{L}_{\mathrm{KL}^-}$.)

3.4.4 A Motivation for the Restriction of Languages in Probabilistic Semantics

In the previous section we observed that many conditional logics with a probabilistic semantics (i.e. Adams, 1965, 1966, 1977, 1975; Hawthorne, 1996; Hawthorne & Makinson, 2007) are formulated in a language, which neither allows for nestings of conditional formulas nor for free boolean combinations of conditionals. In the literature at least two reasons for the restriction of a conditional logic's language are given. The first reason concerns D. Lewis' (1976) triviality result. We will, however, postpone a more detailed discussion of this point to Section 3.6.

A second and more direct reason for the restriction of a conditional logic's language is based on the argument that natural language conditionals do not lend themselves easily into an analysis of nested conditionals and boolean combinations of conditionals. Adams (1965, p. 181f), for example, argues that it is often unclear, how to interpret disjunctions of conditionals in natural language (see also Adams, 1975, p. 32, see also Section 3.5.3). If a restricted language like $\mathcal{L}_{\mathrm{KL}^-}$ is used, problematic cases, such as disjunctions of natural language conditionals, are excluded from a formal analysis, since they are not representable in $\mathcal{L}_{\mathrm{KL}^-}$.

Adams (1965, p. 181) further argues that disjunctions of conditionals correspond to meta-assertions rather than assertions. Adams' argument is strengthened by the fact that a justification of assertions (denials) of conditionals in terms of rational betting behavior does not directly work for boolean combinations of conditionals (cf. Adams, 1965, p. 181, p. 183). Note that Adams' argument for assertions of boolean combinations of conditionals being meta-assertions presupposes an analysis of indicative conditionals in terms of a subjective probabilistic semantics such as Adams (1965). A justification of assertions (denials) of conditionals is, however, a specific problem of subjective accounts of probabilities. These approaches have to justify that the application of their formalisms is indeed rational. For objective frequency-based approaches no such justification is needed (Schurz, 2008, p. 144).

In our opinion there is – except if we presuppose a subjective probabilistic framework for a conditional logic – little reason to assume that boolean combinations and nestings of conditionals on the one hand and unconditional statements are of different sort. This is due to the fact that we are able to formulate boolean combinations of conditionals in natural language. Adams (1965, p. 181, Example F10) even does so. In addition, one can also easily find examples for nested conditionals in natural language, such as the following:

E20 If the cup broke if it was dropped, it was fragile. (Gibbard, 1980, p. 235)
E21 If a Republican wins the election, then if it's not Reagan who wins, it will be Anderson. (McGee, 1985, p. 462)

Most notable, we do not see any reasons to suppose assertions of sentences, such as E20 and E21, have a different status than assertions of non-conditional statements, except if we presuppose a subjective probabilistic analysis of conditionals in line with Adams (1965). In sum, we do not see a strong argument – independent from a subjective probabilistic approach to conditionals – to accept that nestings and boolean combinations of conditional assertions have a different status compared to unconditional statements.

3.5 Adams' P-Systems

In this section we describe the conditional logic **P** and variants of it. In Section 3.5.1 we will, first, discuss Adams' (1965, 1966, 1977) original system, which we will call 'system $\mathbf{P}^*$' and, then, discuss the related systems $\mathbf{P}^+$ (Adams, 1986; Schurz, 1998), system **P** (Schurz, 2005; see also Kraus et al., 1990; Lehmann & Magidor, 1992). Second, we compare in Section 3.5.2 the probabilistic threshold semantics by Hawthorne and Makinson (2007) and Hawthorne (1996) to Adams' probabilistic validity criterion described in Section 3.5.1. We, then, address a further variant of system **P**, namely system $\mathbf{P}_\epsilon$ (Adams, 1975; Schurz, 1997b), in a separate section (Section 3.5.3), since this system is – contrary to systems **P** and $\mathbf{P}^*$ – also a genuine default logic in the sense of Section 2.2.

Note that the **P**-systems are standard subjective probabilistic logics for indicative conditionals. There are other types of probabilistic semantics (Leitgeb, 2004, Chapter 9), but interestingly a range of basic probabilistic validity criteria converge, insofar as they are support the same types of inferences (Leitgeb, 2004, Theorem 70, p. 179). We, hence, do not see the point of describing these alternative validity criteria, but focus instead on standard probabilistic semantics along the lines of Adams' probabilistic semantics.

Adams' standard systems, then, serve as basis for some philosophers (e.g. Adams, 1975; Bennett, 2003; Gibbard, 1980) to argue that conditionals do not have truth-values (short: NTV, no truth-value; cf. Bennett, 2003, p. 94). Interestingly, Bennett (2003) does not discuss the differences between these different versions of system **P** and variants, but rather treats Adams' formal approach as a uniform system (cf. p. 129). This is problematic, since Adams (1975) but neither system $\mathbf{P}^*$ (Adams, 1965, 1966, 1977) nor system **P**+ (Adams, 1986) nor system **P** (Kraus et al., 1990; Lehmann & Magidor, 1992) possesses a consistency criterion as described in Sections 3.1 and 3.2.

We, hence, investigate in Section 3.5.4 the relationship between possible worlds semantics, such as Lewis models (see Definitions 3.5–3.7) and CS-semantics (see Chapters 4–7), on the one hand and the probabilistic semantics of Adams (1965, 1966, 1977) and Adams (1975) and Schurz (1997b) on the other hand. We focus then on the question, whether a probabilistic semantics in line with Adams has a direct bearing on possible worlds semantics.

3.5.1 The Systems P, $\mathbf{P}^*$ and $\mathbf{P}^+$

In this section we describe the conditional logic systems **P** (e.g. Schurz, 2005), $\mathbf{P}^*$ (e.g. Adams, 1965, 1966, 1977) and $\mathbf{P}^+$ (e.g. Adams, 1986; Schurz, 1998). Each of these probabilistic conditional logics draws on the same basic probabilistic model theory. Systems **P**, $\mathbf{P}^*$ and $\mathbf{P}^+$, however, differ w.r.t. which types of formulas are admitted in the respective language. This restriction on the language's expressibility has, then, a profound impact on the relation of proof-theory and model-theory. We use the probabilistic semantics of Adams' (1965, 1966, 1977) system $\mathbf{P}^*$ as starting point for our discussion of the model theory of systems **P**, $\mathbf{P}^*$ and $\mathbf{P}^+$. Moreover, the proof-theoretic side of system **P**, as described by Kraus et al. (1990) and Lehmann and Magidor (1992), serves a basis for our discussion of the proof theories of systems **P**, $\mathbf{P}^*$ and $\mathbf{P}^+$. We, then, discuss the central **P**-theorem described by Schurz (2005) for different restrictions of the conditional logic's language.

Let us now focus on the semantics for system $\mathbf{P}^*$ (Adams, 1965, 1966, 1977). To formulate system $\mathbf{P}^*$, Adams (1965, 1966, 1977) uses the restricted language $\mathcal{L}_{\mathrm{KL}^-}$ (see Section 4.2.2). In this language only non-conditional formulas (formulas without instances of the conditional operator) or formulas of the form $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ exist, where α and β are p.c.-formulas. As basis of the semantics serve probability assignments, which can be specified the following way:

Definition 3.17. *P is a probability function over* $\mathcal{L}_{\mathrm{KL}^-}$ *iff*
a) P: $form_{\mathcal{L}_{\mathrm{KL}^-}} \rightarrow \mathbb{R}$
b) for all $\alpha, \beta \in form_{\mathcal{L}_{\mathrm{prop}}} \subseteq \mathcal{L}_{\mathrm{KL}^-}$ *holds:*

i) $0 \leq P(\alpha) \leq 1$*, and* $P(\top) = 1$
ii) if $\vDash \alpha \rightarrow \beta$*, then* $P(\alpha) \leq P(\beta)$
iii) if $\vDash \neg(\alpha \wedge \beta)$*, then* $P(\alpha \vee \beta) = P(\alpha) + P(\beta)$
iv) if $P(\alpha) \neq 0$*, then* $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = P(\alpha \wedge \beta)/P(\alpha)$*, and if* $P(\alpha) = 0$*, then* $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = 1$

Here $\mathbb{R}$ refers to the set of real numbers and $form_{\mathcal{L}_{\mathrm{prop}}} \subseteq \mathcal{L}_{\mathrm{KL}^{-\mathrm{cons}}}$ describes the set of formulas of language $\mathcal{L}_{\mathrm{KL}^{-\mathrm{cons}}}$, which do not contain an instance of a conditional operator. Definition 3.17 modifies the Kolmogorov axioms of conditional probabilities insofar as it assigns probability 1 to a conditional formula if the probability of its antecedent formula is 0 (Adams, 1965, p. 185; Adams, 1966, p. 272f). In clause b.iv $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta)$ is directly defined by $P(\alpha \wedge \beta)/P(\alpha)$ in case $P(\alpha) > 0$. Note that the conditional probability $P(\beta|\alpha)$ is usually defined as $P(\alpha \wedge \beta)/P(\alpha)$ for $P(\alpha) > 0$ (Feller, 1968, p. 115). To capture clause b.iv of Definition 3.17 adequately, we can, then, in turn define the probability $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta)$ as $P(\alpha|\beta)$ for $P(\alpha) > 0$. Hence, Adams (1965, 1966, 1977) in effect defines the probability of conditionals as their respective conditional probabilities and in addition takes unconditional probabilities as primitives (see Section 3.4.2). We, however need not split up the definition $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta)$ of clause b.iv of Definition 3.17 in the way Adams does. We can instead define $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta)$ as $P(\beta|\alpha)$ and, then, define $P(\beta|\alpha)$ as $P(\alpha \wedge \beta)/P(\beta)$ for $P(\alpha) > 0$ and as 1 for $P(\alpha) = 0$. The latter procedure can, hence, be used to describe Adams' (1965, 1966, 1977) approach in way such that $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta)$ is defined as $P(\beta|\alpha)$ without any qualification.

Moreover, Definition 3.17 draws on p.c.-validities and a p.c-consequence relation. This notion is, then, used to define the probability of conjuncts and negations of purely propositional formulas (cf. Adams, 1965, p. 184). (The probability of the conjunct in clause b.iv of Definition 3.17 is determined that way.) Since Definition 3.17 relies on p.c.-validity and a p.c.-consequence relation, the definition is, hence, not purely probabilistic. It is one motivation of the so-called Popper functions to remedy this drawback (see Hawthorne, 1996, p. 190f, see also Section 3.4.2).

Adams (1965, 1966, 1977), then, goes on to introduce the following probabilistic validity criterion (short: p-validity criterion) for his system $\mathbf{P}^*$:

Definition 3.18. *(p-Valdity)*
Let $\alpha \in form_{\mathcal{L}_{\mathrm{KL}^-}}$, suppose that Γ be a finite set of formulas of language $\mathcal{L}_{\mathrm{KL}^-}$ and that ϵ, $\delta \in \mathbb{R}$. Then, α is a probabilistic consequence of Γ (short: $\Gamma \vDash_p \alpha$) iff $\forall \epsilon > 0, \exists \delta > 0, \forall P$ in $\mathcal{L}_{\mathrm{KL}^-}$: If $\forall \beta \in \Gamma$ holds that $P(\beta) \geq 1 - \delta$, then $P(\alpha) \geq 1 - \epsilon$.

This criterion says intuitively that for any arbitrarily high probability assignment (unequal 1) a probability value $1 - \epsilon$ of the conclusion there exists a threshold $1 - \delta$ for all premises, such that if the probability of every single premise is higher than $1 - \delta$, then the probability of the conclusion is higher than $1 - \epsilon$. So, we can guarantee – given an inference is p-valid – arbitrarily high probabilities (unequal to 1) for the conclusion provided sufficiently high (but less than 1) probabilities of all premises (Adams, 1966, p. 273f).

Let us now discuss the proof theories of systems $\mathbf{P}$, $\mathbf{P}^*$ and $\mathbf{P}^+$ and, then, describe the corresponding semantics for systems $\mathbf{P}$ and $\mathbf{P}^+$. For a more perspicuous treatment of the proof theory of conditional logics $\mathbf{P}$, $\mathbf{P}^*$ and $\mathbf{P}^+$, we will, first, start with an axiomatization of system $\mathbf{P}$ by Kraus et al. (1990) and Lehmann and Magidor (1992). Kraus et al. (1990) and Lehmann and Magidor (1992) describe system $\mathbf{P}$ in a language, which is as expressive as $\mathcal{L}_{\mathrm{rKL}}$ (see Section 3.4.3). (In $\mathcal{L}_{\mathrm{rKL}}$ only negated or simple conditional formulas are allowed (see Section 3.4.3).) We will, however, formulate system $\mathbf{P}$ rather w.r.t. language $\mathcal{L}_{\mathrm{KL}^-}$, which admits only non-conditional formulas or simple conditional formulas:

Definition 3.19. *System $\mathbf{P}$ for language $\mathcal{L}_{\mathrm{KL}^-}$ is the smallest logic, which is closed under the following rules and axioms (cf. Lehmann & Magidor, 1992, p. 5f):*

LLE	*if* $\alpha \mathrel{\Box\!\!\rightarrow} \gamma$ *and* $\vdash \alpha \leftrightarrow \beta$*, then* $\beta \mathrel{\Box\!\!\rightarrow} \gamma$
RW	*if* $\gamma \mathrel{\Box\!\!\rightarrow} \alpha$ *and* $\vdash \alpha \rightarrow \beta$*, then* $\gamma \mathrel{\Box\!\!\rightarrow} \beta$
*AND**	*if* $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ *and* $\alpha \mathrel{\Box\!\!\rightarrow} \gamma$ *then* $\alpha \mathrel{\Box\!\!\rightarrow} \beta \wedge \gamma$
Refl	$\alpha \mathrel{\Box\!\!\rightarrow} \alpha$
*CM**	*if* $\alpha \mathrel{\Box\!\!\rightarrow} \gamma$ *and* $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ *then* $\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$
*Or**	*if* $\alpha \mathrel{\Box\!\!\rightarrow} \gamma$ *and* $\beta \mathrel{\Box\!\!\rightarrow} \gamma$ *then* $\alpha \vee \beta \mathrel{\Box\!\!\rightarrow} \gamma$

'LLE', 'RW', 'Refl' and 'CM' abbreviate here 'Left Logical Equivalence', 'Right Weakening', 'Reflexivity' and 'Cautious Monotonicity'. The asterix '*' in 'AND*', 'CM*' and 'Or*' indicates that the respective principle is formulated as a rule rather than an axiom, in contrast to our axiom-versions for the full language $\mathcal{L}_{\mathrm{KL}}$, which we will discuss in Chapters 4–7. Note here that our use of the term 'logic' presupposes that the respective proof-theoretic system is closed under p.c.-consequences. Moreover, as we shall prove in Section 7.3 (cf. Theorem 7.63), System **P*** differs from System **P** insofar it includes the further rules:

Det*	if $\top \mathrel{\Box\!\!\rightarrow} \alpha$, then α
Cond*	if α, then $\top \mathrel{\Box\!\!\rightarrow} \alpha$

Det* ("Detachment) and Cond* ("Conditionalization") represent again rule versions of the principle Det* and Cond*, respectively, which we shall discuss in Section 7.3. Note that Det and Cond are – as we shall prove in Section 7.3 (see Theorem 7.61) – on the basis of System **P** logically equivalent to the following two principles:

MP*	if $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ and α then β
CS*	if $\alpha \wedge \beta$ then $\alpha \mathrel{\Box\!\!\rightarrow} \beta$

MP* and CS* are rule version of principle MP ("Modus Ponens") and CS ("Conjunctive Sufficiency"), respectively. We discussed MP and CS already in Sections 3.2.3 and 3.3. Note that MP, CS, Det and Cond are bridge principles, which describe a fixed relationship between conditional formulas and non-conditional formulas. Let us now define the proof theory of system **P***:

Definition 3.20. *System* **P*** *for language* $\mathcal{L}_{\mathrm{KL}^-}$ *is the smallest logic, which contains* **P** *and Cond** *and Det***.*

Adams (1966, Theorem 7.1 and 7.2, p. 292) proved that system $\mathbf{P}^*$ is sound and complete for the language $\mathcal{L}_{\mathrm{KL}^-}$ w.r.t. the probabilistic validity criterion described in Definition 3.18. Note, however, that this completeness results only holds up, if we limit inferences $\Gamma \vdash_{\mathbf{P}^*} \alpha$ in system $\mathbf{P}^*$ to finite sets of formulas Γ.

System $\mathbf{P}^+$ (Adams, 1986) differs from system $\mathbf{P}^+$ insofar as we use the language $\mathcal{L}_{\mathrm{rKL}^*}$ rather than language $\mathcal{L}_{\mathrm{KL}^-}$. Language $\mathcal{L}_{\mathrm{rKL}^*}$ allows for simple conditional formulas and disjunctions of conditional formulas, but not for non-conditional formulas. In Adams' (1986) original proposal, the language contains a further restriction, insofar as disjunctive conditionals are only allowed as the conclusion of inferences, but not as premises (see Section 3.4.3). In Adams (1986), then, in addition to the rules and axioms of system $\mathbf{P}$, system $\mathbf{P}^+$ includes the following rule:

dRM* if $\alpha \mathrel{\Box\!\!\rightarrow} \gamma$ then $(\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta) \vee (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$

'dRM*' stands for 'Rule Version of Disjunctive Version of Rational Monotonicity' (cf. Schurz, 1998, p. 84). In the full conditional logic $\mathcal{L}_{\mathrm{KL}}$ dRM is p.c.-equivalent to the following rule:

RM* if $\alpha \mathrel{\Box\!\!\rightarrow} \gamma$ and $\neg(\alpha \mathrel{\Box\!\!\rightarrow} \beta)$ then $(\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$

'RM' stands here for 'Rational Monotonicity'. Note that RM is expressible in language $\mathcal{L}_{\mathrm{rKL}}$, which allows for negations of simple conditional formulas. System $\mathbf{P}^+$ can, then defined the following way:

Definition 3.21. *System* $\mathbf{P}^+$ *for language* $\mathcal{L}_{\mathrm{rKL}^*}$ *is the smallest logic, which contains* $\mathbf{P}$ *and dRM**.

Adams (1986, p. 161) defines, then, the probability of conditionals $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ on the basis of Definition 3.18. We ignore here that Adams (1986, p. 259) specifies empty disjunctions (disjunctions with 0 disjuncts) as logical falsehoods and presuppose that all formulas of his language are simple conditional formulas or disjunctions of conditional formulas. If we do not make this move and admit non-conditional formulas in his language (such as $\perp$), his system would become incomplete w.r.t. his semantics, since he does not have rules or axioms, which make sure that if arbitrary formulas conditionals formulas are true, then the set becomes inconsistent insofar as $\perp$ follows from such a set.

Schurz (1998) uses a variation of system $\mathbf{P}^+$, which effectively includes in addition to rules and axioms of system $\mathbf{P}^+$ the following principle:

TR* if $\neg\alpha \,\Box\!\!\to \alpha$ then α

'TR*' abbreviates "Rule Version of Total Reflexivity". Schurz (1998, p. 84) uses in fact a p.c.-equivalent variant of TR called 'Poss' ("Possibility"). We will, however, not prove this equivalence here. $\neg\alpha \,\Box\!\!\to \alpha$ is also sometimes defined as $\Box\alpha$, which is in turn interpreted as 'it is necessary that α'. We will postpone a discussion whether such a terminology is justified to Section 7.2.3.

Schurz (1998) escapes the difficulties of Adams (1986) – described above – for a language, which allows for non-conditional formulas besides simple conditional formulas and disjunctions of conditional formulas: In case arbitrary conditionals $\alpha \,\Box\!\!\to \beta$ are valid, system $\mathbf{P}^+$ implies by LLE, RW and CM* (see Lemma 3.11) that also $\top \,\Box\!\!\to \perp$ is valid, where $\top$ and $\perp$ abbreviate $p\vee\neg p$ and $p\wedge\neg p$, respectively. TR, however, implies on the basis of the rules and axioms of system $\mathbf{P}$ and $\top \,\Box\!\!\to \perp$ the inconsistent formula $\perp$:

Lemma 3.22. *LLE+TR+*$\top \,\Box\!\!\to \perp \Rightarrow$ *P-Cons*

Proof.

1.	$\top \,\Box\!\!\to \perp$	given
2.	$\neg\perp \,\Box\!\!\to \perp$	1, LLE
3.	$\perp$	2, TR

$\Box$

We indicate in this thesis derivability of principles by $\Rightarrow$. We, furthermore, use on the notion of object language proof described in in Section 4.2.6. Let us now discuss the Central $\mathbf{P}$-Theorem (Schurz, 2005), which connects the proof-theoretic side of system $\mathbf{P}$ with probabilistic semantics and possible worlds semantics. For that purpose we will restrict ourselves to language $\mathcal{L}_{\mathrm{rrKL}}$, in which every formula has the form $\alpha \,\Box\!\!\to \beta$, where α and β do not contain any conditional operator. The Central $\mathbf{P}$-Theorem of (Schurz, 2005) can be described as follows:

Theorem 3.23. *(Central* **P***-Theorem, cf. Schurz, 2005, p. 43)*
Let α and Γ be a formula of $\mathcal{L}_{\mathrm{rrKL}}$ and a finite set of formulas of language $\mathcal{L}_{\mathrm{rrKL}}$. Then, the following four conditions are equivalent (Schurz, 2005, p. 43):
(1) (Calculus) $\Gamma \vdash_{\mathbf{P}} \alpha$
(2) (Normal world semantics) Γ implies α w.r.t. is valid in all finite, ranked Lewis models.
(3) (Infinitesimal probability semantics) α is a p-consequence of Γ.
(4) (Non-infinitesimal probability semantics) For all probability functions P holds that $\sum_{\beta \in \Gamma} u_P(\beta) \leq u_P(\alpha)$ (the Uncertainty Sum Rule).

By finite ranked models we mean relational Lewis models as described by Definitions 3.8–3.10. The non-infinitesimal probability semantics described by the uncertainty sum rule refers to the uncertainty of formulas for a probability P. The uncertainty $U_P(\cdot)$ is defined as $1 - P(\cdot)$. The non-infinitesimal probability semantics allows us to – contrary to the p-validity criterion – to determine valid inferences in system **P** in terms of a finite number of steps.

We can also state alternative versions of the central **P**-theorem for different languages. If we use language $\mathcal{L}_{\mathrm{KL}^-}$ rather than language $\mathcal{L}_{\mathrm{rrKL}}$, then, the derivability relation in point (1) of Theorem 3.23 has to refer to system $\mathbf{P}^*$ rather than system **P** (Adams, 1966, Theorem 7.1 and Theorem 7.2, p. 292). One has, in addition, to change (3) in such a way that it employs finite ranked models with centering axioms rather than the class of all finite ranked models (Adams, 1977, p. 188)

Let us now focus on a version of the central **P** theorem for language $\mathcal{L}_{\mathrm{rKL}^*}$ with the additional restriction that disjunctions (with more than one disjunct) are allowed only in the conclusion of inferences. Then, the derivability relation in (1) of Theorem 3.23 has to refer to system $\mathbf{P}^+$ (Adams, 1986, Meta-Metatheorem 1, p. 261; cf. Schurz, 1998, Theorem 3, p. 96). Moreover, we have to modify condition (2). Adams (1986, Meta-Metatheorem 4, p. 269) shows that it suffices to limit (2) to finite ranked models, for which C_{MP} (see Section 3.2.3) holds. We conjecture here that $C_{\mathrm{P-Cons}}$ (see Section 3.2.3) suffices for that purpose. In addition, point (4) has to be changed in the following way: The sum of the uncertainties of the premises $\sum_{\beta \in \Gamma} u_P(\beta)$ has to be less or equal to the product of the uncertainties of the disjuncts $\prod_{1 \leq i \leq n} u_P(\beta_i)$, $n \in \mathbb{N}$, for disjunctive conclusions $\beta_1 \vee \ldots \vee \beta_n$ (cf.

Adams, 1986, p. 261; cf. Schurz, 1998, p. 86).

Finally, we conjecture that for language $\mathcal{L}_{\mathrm{rKL}}$ – which contains only simple conditional formulas and negated conditional formulas – we have to modify in (1) the proof-theoretic derivability relation to **P**+PCons. We, furthermore, argue that (2) has to be modified to refer to finite ranked Lewis models, for which condition $C_{\mathrm{P-Cons}}$ holds.

3.5.2 Threshold Semantics

Adams' probabilistic validity criteria are not the only ones discussed in the literature. Hawthorne and Makinson (2007, p. 220), for example, employ a threshold validity criterion instead. They define $\alpha \mid\!\sim_{P,r} \beta$ to hold iff either $P(\alpha) = 0$ or $P(\beta \,|\, \alpha) \geq r$. Here Hawthorne and Makinson (2007) put a conditional probability interpretation into the definition of $\alpha \mid\!\sim_{P,r} \beta$. This criterion carries, then, through to their second-order consequence relation (which they do not mention explicitly). Adams on the other hand interprets conditionals as formulas of type $\alpha \;\Box\!\!\rightarrow\; \beta$. On a semantic level he associates these formulas only with the respective conditional probabilities and gives a criterion of applicability, then, on the level of his consequence relation $\models_p$. The second-order consequence relation of Hawthorne and Makinson (2007) and the consequence relation of Adams (1965, 1966, 1977), hence, effectively use different criteria. This can also be seen by the following observation: In the Adams approach the inference from $\alpha \;\Box\!\!\rightarrow\; \beta$ and $\alpha \;\Box\!\!\rightarrow\; \gamma$ to $\alpha \;\Box\!\!\rightarrow\; \beta \wedge \gamma$ is p-valid (see Axiom 7 of Adams, 1965, p. 189 and Adams, 1966, p. 277). Note that this inference corresponds to AND of Kraus et al. (1990, p. 173, see also Section 4.2.6). Given the threshold-criterion of Hawthorne and Makinson (2007) this inference is, however, not warranted (see Hawthorne & Makinson, 2007, p. 252).

3.5.3 Adams' (1975) System $\mathbf{P}_\epsilon$ and Schurz's (1997b) Modification

The starting point for Adams (1975) modified account lies in the following two observations, one being syntactically and the other being semantically motivated.

We describe the syntactic deviation first and, then, focus on Adams' (1975) semantic modification.

A Syntactic Deviation

For the first point, consider the following sentences from natural language:

E22 If Adam goes shopping, he is happy.
E23 If Adam goes shopping, he is not happy.
E24 It is not the case, that if Adam goes shopping, he is happy.

Adams (1965, p. 181) observed that we use statements such as E23 rather than statements such as E24 to negate statements such as E22. Note that E24 is not directly representable in the language $\mathcal{L}_{\text{KL}^-}$ of Adams (1965, 1966, 1977, 1975), since negations of conditional formulas are not formulas of language $\mathcal{L}_{\text{KL}^-}$. E22 and E23, however, are representable as $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ and $\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta$, respectively. Adams' observation, hence, suggests that $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta)$ is inconsistent. Adams (1975), then, took this as starting point to render both statements E22 and E23 inconsistent (cf. Adams, 1975, p. 46, p. 56). So, our informal discussion suggests that the following formula could be valid in Adams' (1975) system $\mathbf{P}_\epsilon$ (given the full language $\mathcal{L}_{\text{KL}}$):

CNC $\neg((\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta))$

Here 'CNC' abbreviates 'Conditional Non-Contradiction'. Note that in the Adams (1965, 1966, 1977, 1975) systems $\mathbf{P}^*$ and $\mathbf{P}_\epsilon$ CNC is in fact not a formula, since it draws on boolean combinations of conditional formulas. We can, however, express this principle in Adams' system as the following inference rule (Adams, 1975, p. 61):

CNC' $\alpha \mathrel{\Box\!\!\rightarrow} \beta, \alpha \mathrel{\Box\!\!\rightarrow} \neg\beta \vdash_p \gamma$

Although CNC and the other principles discussed below are not directly representable in the language of Adams (1965, 1966, 1977, 1975), we nevertheless use the full language $\mathcal{L}_{\text{KL}}$ for the following reasons: In our opinion the specifics of these principles are easier to understand in the less restricted language. Moreover, we can more easily compare different systems if we use the same representation format. For that purpose, however, full language versions seem most appropriate. We will, however, for each principle discussed in this section refer to translations into $\mathcal{L}_{\mathrm{KL}^-}$, if there exist any.

Let us now focus on the principle CNC. Note that this formula is p.c.-equivalent to the following axiom:

D $\quad (\alpha \,\Box\!\!\rightarrow \beta) \rightarrow (\alpha \,\Diamond\!\!\rightarrow \beta)$.

The principle D draws on our Definition $\mathrm{Def}_{\Diamond\!\rightarrow}$ we specifies $\alpha \,\Diamond\!\!\rightarrow \beta$ as $\neg(\alpha \,\Box\!\!\rightarrow \neg\beta)$ (see Section 4.2.1). Our name D derives from the fact that the principle represents a generalization from the standard axiom D in normal modal logic (cf. Hughes & Cresswell, 1996/2003, p. 43f).[9] CNC is moreover, equivalent, given some the basic conditional logic principles AND and the rule RW (see Table 4.7), to the following principle:

CNC* $\quad \neg(\alpha \,\Box\!\!\rightarrow \bot)$[10]

Note that for virtually all indicative proof-theoretic conditional logic systems in the literature both AND and RW are theorems, including the system of Adams (Axioms 7 and 3 of Adams, 1965, p. 189 and Adams, 1966, p. 277) and our basic proof-theoretic system **CK** (see Table 4.7).[11] The formula $\bot$ abbreviates $p \wedge \neg p$ (see Section 4.2.1). We will provide here proofs for both equivalences: (a) CNC and D provided $\mathrm{Def}_{\Diamond\!\rightarrow}$ (see Theorem 3.24) and (b) CNC and CNC* given AND and RW (see Theorem 3.27). The principles AND and RW are the following (see also Section 3.5.1):

RW $\quad$ if $\vdash \alpha \rightarrow \beta$ and $\gamma \,\Box\!\!\rightarrow \alpha$, then $\gamma \,\Box\!\!\rightarrow \beta$
AND $\quad (\alpha \,\Box\!\!\rightarrow \beta) \wedge (\alpha \,\Box\!\!\rightarrow \gamma) \rightarrow (\alpha \,\Box\!\!\rightarrow \beta \wedge \gamma)$

'RW' stands here for 'Right Weakening'. Let us now prove the following theorem:

[9] This can be seen more clearly, if we interpret the conditional operator $\alpha \,\Box\!\!\rightarrow \beta$ as $[\alpha]\beta$, where $[\alpha]$ describes a type of unary "modal operator" (see Chellas, 1975, p. 138).

[10] In the language $\mathcal{L}_{\mathrm{KL}^-}$ CNC* can be represented by the following rule: $\alpha \,\Box\!\!\rightarrow \bot \vdash_p \beta$.

[11] One conditional logic approach, which does not enact AND, but RW is the threshold semantics approach of Hawthorne and Makinson (2007, see also Section 3.5.1).

Theorem 3.24. *$Def_{\Diamond\!\!\rightarrow} \Rightarrow (CNC \Leftrightarrow D)$*

Proof. By Lemma 3.25 and 3.26. □

Lemma 3.25. *$Def_{\Diamond\!\!\rightarrow}$+$CNC \Rightarrow D$*

Proof.

1.	$\vdash \neg((\alpha \,\Box\!\!\rightarrow \beta) \wedge (\alpha \,\Box\!\!\rightarrow \neg\beta))$	given
2.	$\vdash (\alpha \,\Box\!\!\rightarrow \beta) \rightarrow \neg(\alpha \,\Box\!\!\rightarrow \neg\beta)$	1, p.c.
3.	$\vdash (\alpha \,\Box\!\!\rightarrow \beta) \rightarrow (\alpha \,\Diamond\!\!\rightarrow \beta)$	2, $\text{Def}_{\Diamond\!\!\rightarrow}$

□

Lemma 3.26. *$Def_{\Diamond\!\!\rightarrow}$+$D \Rightarrow CNC$*

Proof.

1.	$\vdash (\alpha \,\Box\!\!\rightarrow \beta) \rightarrow (\alpha \,\Diamond\!\!\rightarrow \beta)$	given
2.	$\vdash (\alpha \,\Box\!\!\rightarrow \beta) \rightarrow \neg(\alpha \,\Box\!\!\rightarrow \neg\beta)$	1, $\text{Def}_{\Diamond\!\!\rightarrow}$
3.	$\vdash \neg((\alpha \,\Box\!\!\rightarrow \beta) \wedge (\alpha \,\Box\!\!\rightarrow \neg\beta))$	2, p.c.

□

We will now show the equivalence of CNC and CNC* provided RW and AND hold:

Theorem 3.27. *RW+$AND \Rightarrow (CNC \Leftrightarrow CNC^*)$*

Proof. By Lemma 3.28 and 3.29. □

Lemma 3.28. *$CNC \Rightarrow CNC^*$*

Proof.

1.	$\vdash \neg((\alpha \,\Box\!\!\rightarrow \beta) \wedge (\alpha \,\Box\!\!\rightarrow \neg\beta))$	given
2.	$\vdash \neg\neg(\alpha \,\Box\!\!\rightarrow \bot)$	ass IP (Indirect Proof)
3.	$\quad\vdash \alpha \,\Box\!\!\rightarrow \bot$	2, p.c.
4.	$\quad\vdash \alpha \,\Box\!\!\rightarrow \beta$	3, RW
5.	$\quad\vdash \alpha \,\Box\!\!\rightarrow \neg\beta$	3, RW
6.	$\quad\vdash (\alpha \,\Box\!\!\rightarrow \beta) \wedge (\alpha \,\Box\!\!\rightarrow \neg\beta)$	4, 5, p.c.
7.	$\vdash \neg(\alpha \,\Box\!\!\rightarrow \bot)$	IP, 2-6, 2, 6

□

Lemma 3.29. *RW+AND+CNC* $\Rightarrow$ CNC*

Proof.

1.	$\vdash \neg(\alpha \mathrel{\Box\!\!\rightarrow} \bot)$	given
2.	$\vdash \neg\neg((\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta))$	ass IP
3.	$\vdash (\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta)$	2, p.c.
4.	$\vdash \alpha \mathrel{\Box\!\!\rightarrow} \beta$	3, p.c.
5.	$\vdash \alpha \mathrel{\Box\!\!\rightarrow} \neg\beta$	3, p.c.
6.	$\vdash \alpha \mathrel{\Box\!\!\rightarrow} \beta \wedge \neg\beta$	4, 5, AND
7.	$\vdash \alpha \mathrel{\Box\!\!\rightarrow} \bot$	6, RW
8.	$\vdash \neg((\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta))$	IP, 2-7, 2, 7

□

Note that given these equivalences D, CNC and CNC* represent a type of consistency requirement. Given a specific antecedent α not both β and $\neg\beta$ are allowed to hold. Moreover, given any antecedent no contradictory formula is permitted to hold.

Despite the informal discussion above, both the principles D, CNC and CNC* and their translations into the language $\mathcal{L}_{KL^-}$ neither hold for the Adams (1965, 1966, 1977, 1975) systems nor for any other conditional logic system we are aware of. This is due to another conditional logic axiom:

Refl $\quad \alpha \mathrel{\Box\!\!\rightarrow} \alpha.$

This axiom roughly states that if α is the case, then α is the case. Although there are some interpretations for specific types of conditionals, such as conditional obligation, for which this principle might not hold (e.g. Spohn, 1975, pp. 248–250; Bonati, 2005, p. 74f), we are not aware of any approach targeted for indicative and counterfactual conditionals, which does not enact principle Refl.

The problem here is that given the very basic conditional logic principle RW (see Section 4.2.6) – enacted by virtually all conditional logic approaches (e.g. Adams, 1965, 1966, 1977, 1975; Kraus et al., 1990; Lehmann & Magidor, 1992) – both Refl and CNC are inconsistent:[12]

[12] A parallel proof can be given for the restricted language $\mathcal{L}_{KL^-}$ of Adams (1965, 1966, 1977, 1975) w.r.t. restricted versions of CNC and Refl.

Theorem 3.30. *Given RW, the principles CNC and Refl are p.c.-inconsistent.*

Proof. We proof this theorem with the instance $\perp$.

1.	$\vdash \perp \,\square\!\!\rightarrow \perp$	Refl
2.	$\neg((\perp \,\square\!\!\rightarrow \alpha) \wedge (\perp \,\square\!\!\rightarrow \neg\alpha))$	CNC
3.	$\vdash \perp \,\square\!\!\rightarrow \alpha$	1, RW
4.	$\vdash \perp \,\square\!\!\rightarrow \neg\alpha$	1, RW
5.	$\vdash (\perp \,\square\!\!\rightarrow \alpha) \wedge (\perp \,\square\!\!\rightarrow \neg\alpha)$	3, 4, p.c.
6.	$\vdash ((\perp \,\square\!\!\rightarrow \alpha) \wedge (\perp \,\square\!\!\rightarrow \neg\alpha)) \wedge \neg((\perp \,\square\!\!\rightarrow \alpha) \wedge (\perp \,\square\!\!\rightarrow \neg\alpha))$	5,4 p.c.

□

The inconsistency is, however, easier to avoid than it might first seem. The only problematic case is the one, in which a logically false formula (i.e. $\perp$) is the antecedent of a conditional formula. To exclude this case, we can, hence, use the following rule:

RCNC if $\nvdash_{p.c.} \neg\alpha$ then $\neg((\alpha \,\square\!\!\rightarrow \beta) \wedge \neg(\alpha \,\square\!\!\rightarrow \neg\beta))$.[13]

Here $\nvdash_{p.c.} \alpha$ describes the precondition that α is not p.c.-derivable. Hence, $\nvdash_{p.c.} \alpha$ ensures that the antecedent of a formula is p.c.-consistent (see Sections 2.2.3 and 2.2.5). As RCNC is a non-trivial non-monotonic rule, any proof-theoretic system endorsing this principle results – provided sufficiently reasonable conditions – in a default logic (see Section 2.2.4). Adams (1975, p. 61, Rule R7) uses a version of this rule in the specification of his proof theory (cf. Schurz, 1998, p. 84)[14], such that his system is effectively a default-logic. Since default logic approaches are not unproblematic with respect to their axiomatization (see Section 2.2.5), also the Adams (1975) system is prone to the same types of drawbacks. Despite this fact, we are not aware that this issue for the Adams (1975) system is addressed in the literature.

[13] The following rule RCNC′ is a translation of RCNC into $\mathcal{L}_{\mathrm{KL}^-}$: if $\alpha \,\square\!\!\rightarrow \beta$, $\alpha \,\square\!\!\rightarrow \neg\beta$ and $\nvdash \neg\alpha$, then γ. Note that in Adams' language $\mathcal{L}_{\mathrm{KL}^-}$ the formula α is not allowed to contain an instance of a conditional operator. Otherwise α could neither be an antecedent formula nor could its negation be a formula of that language. This implies that the precondition $\nvdash \neg\alpha$ guarantees propositional consistency of α.

[14] Strictly speaking, Adams (1975, p. 61) uses following rule RCNC″: if $\alpha \,\square\!\!\rightarrow \beta$ and $\nvdash \alpha$, $\vdash \neg(\alpha \wedge \beta)$ then γ. Note that RCNC″ is equivalent to RCNC′ from Footnote 13 given Refl, AND and RW. Moreover, Refl, AND and RW hold for Adams' (1975) system.

In addition to a version of RCNC, Adams (1975) imposes following further requirement: He accepts conditional formulas of language $\mathcal{L}_{\mathrm{KL}^-}$ for his language $\mathcal{L}^{\mathrm{cons}}_{\mathrm{KL}^-}$ (see also next section) only, if the antecedent of the conditional formula is p.c.-consistent (p. 46, cf. our Footnote 13). Due to that fact he has to adopt the rules of his proof theory in such a way that he allows for conditional formulas only in case they have a consistent antecedent. This seems, moreover, to be the motivation for the addition of consistency conditions for his other rules (p. 60f). Note here that this is not only way possible. Schurz (1997b) suggests an alternative way we will sketch below.

Note, however, that Adams in his earlier approach (Adams, 1965, 1966, 1977) neither employs the non-monotonic rule RCNC nor variants of it (see Schurz, 1998, p. 84). Since Adams (1965, 1966, 1977) uses only monotonic rules (see Adams, 1965, p. 189 and Adams, 1966, p. 277), his 1965/1966 system is not a default-logic, and, hence, does not suffer from the possible drawbacks of a default logic (see Section 2.2.5). Most importantly the inference relation is still monotonic, although the conditional operator $\Box\!\!\rightarrow$ is not. (The non-monotonicity is, hence, pushed inside the conditional operator.) Note that the proof-theoretic system **CK** described in Section 4.7 shares the property of having a monotonic inference relation.

The Semantic Modification

In this section we will, first, describe the probabilistic validity criterion of Adams (1975) and, then, discuss Schurz's (1997b) alternative account.

Let us, first, focus on Adams' (1975) validity criterion. The first important difference between the semantics of Adams (1975) and the semantics of Adams (1965, 1966, 1977) is the fact that Adams (1975) modifies the definition of probability functions described in Definition 3.17. We saw in Section 3.5.1 that he deviated from Kolmogorov's account in specifying the conditional probability $P(\beta\,|\,\alpha) = 1$ in case $P(\alpha) = 0$ (see Definition 3.17.b.iv). Adams (1965, p. 176f) points out that he endorsed this approach on behalf of technical completeness and simplicity.

Although Adams' argumentation suggests that this step is to some extent arbitrary, there is a certain intuition guiding it: Since p.c.-inconsistent statements, such as $\perp$, have to be assigned zero-probability, it follows that all conditional formulas with p.c.-inconsistent antecedents have probability 1. In particular both $P(\alpha\,|\,\perp) = 1$ and $P(\neg\alpha\,|\,\perp) = 1$ hold. Since for any formula α it is the case that α and $\neg\alpha$ are p.c. consequences of $\perp$, this specification seems rather adequate.

Adams (1975, p. 48, p. 50), then, uses following modified version of Definition 3.17:

Definition 3.31. *P is a probability function over $\mathcal{L}^{\text{cons}}_{\text{KL}^-}$ iff*
a) P is a partial function from $form_{\mathcal{L}^{\text{cons}}_{\text{KL}^-}}$ to $\mathbb{R}$
b) for all $\alpha, \beta \in form_{\mathcal{L}_{\text{prop}}} \subseteq \mathcal{L}^{\text{cons}}_{\text{KL}^-}$ holds:

i) $0 \leq P(\alpha) \leq 1$, *and* $P(\top) = 1$
ii) if $\vDash \alpha \rightarrow \beta$, *then* $P(\alpha) \leq P(\beta)$
iii) if $\vDash \neg(\alpha \wedge \beta)$, *then* $P(\alpha \vee \beta) = P(\alpha) + P(\beta)$
iv) if $P(\alpha) > 0$, *then* $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = P(\alpha \wedge \beta)/P(\alpha)$

The expression $form_{\mathcal{L}_{\text{prop}}} \subseteq \mathcal{L}^{\text{cons}}_{\text{KL}^-}$ refers here to the set of formulas of $\text{KL}^{-\text{cons}}$, which do not contain an instance of a conditional operator. P is only a partial function on the whole set $form_{\mathcal{L}^{\text{cons}}_{\text{KL}^-}}$, since for $P(\alpha) = 0$ the probability $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta)$ is undefined. Definition 3.31 differs in this respect from Definition 3.17. A further difference between Definition 3.31 and Definition 3.17 is the fact that Definition 3.31 is restricted to formulas of the language $\mathcal{L}^{\text{cons}}_{\text{KL}^-}$. The set of formulas *form* of this language is the set of formulas $form_{\mathcal{L}_{\text{KL}^-}}$ with the additional requirement that the antecedent of any conditional formula has to be p.c.-consistent (see previous section). Since Adams (1975) uses Definition 3.31 rather than Definition 3.17, his approach differs from his accounts in Section 3.5.1 in at least two ways: (a) Any antecedent of a conditional formula is p.c.-consistent and (b) $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta)$ is defined only if $P(\alpha) \neq 0$.

Adams (1975, p. 49), then, proceeds to define the following two notions:

Definition 3.32. *Suppose that α and $\beta \in \mathcal{L}_{\text{prop}} \subseteq \mathcal{L}^{\text{cons}}_{\text{KL}^-}$ and that P is a probability function according to Definition 3.31 over language $\mathcal{L}_{\text{KL}^-}$. Then, P is proper for $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ iff $P(\alpha) > 0$.*

Definition 3.33. *For all formulas $\alpha \in form_{\mathcal{L}^{\text{cons}}_{\text{KL}^-}}$ and formula sets $\Gamma \subseteq form_{\mathcal{L}^{\text{cons}}_{\text{KL}^-}}$ and probability functions P w.r.t. $\mathcal{L}^{\text{cons}}_{\text{KL}^-}$, as specified in Definition 3.31, holds: P is proper for Γ iff P is proper for all $\alpha \in \Gamma$.*

Definition 3.32 gives us that a probability function P is proper for a conditional formula $\alpha \mathrel{\Box\!\!\to} \beta$ iff $P(\alpha) > 0$. Definition 3.33 extends this definition to sets of formulas. Adams (1975, p. 57), then, employs this notion to specify a modified criterion of valid inference:

Definition 3.34. *(p^*-Validity)*
Let $\alpha \in form_{\mathcal{L}^{\text{cons}}_{\text{KL}^-}}$, $\Gamma \subseteq form_{\mathcal{L}^{\text{cons}}_{\text{KL}^-}}$ and $\epsilon, \delta \in \mathbb{R}$. Then, α is probabilistic consequence* of Γ (short: $\Gamma \vDash_{p^*} \alpha$) iff $\forall \epsilon > 0, \exists \delta > 0, \forall P$ over $\mathcal{L}^{\text{cons}}_{\text{KL}^-}$, which are proper for Γ and α, the following is the case: If $\forall \beta \in \Gamma$ holds that $P(\beta) \geq 1 - \delta$, then $P(\alpha) \geq 1 - \epsilon$.

In Definition 3.34 the notion of properness is used to restrict probability functions. According to Definition 3.34 no probability is assigned to a conditional formula $\alpha \mathrel{\Box\!\!\to} \beta$ if $P(\alpha) = 0$. Adam's reference to his notion of properness excludes those cases: In his p^*-criterion (Definition 3.34) the properness of α and Γ guarantees that neither the antecedent of α (if it is a conditional formula) nor any antecedent of a conditional formula in Γ receives a zero-probability assignment. Note that in addition only conditional formulas with a consistent antecedent are permitted in Definition 3.34 (due to the restrictions of $\mathcal{L}^{\text{cons}}_{\text{KL}^-}$).

Let us now describe Schurz's (1997b) alternative characterization of Adams' (1975) probabilistic semantics. Schurz (1997b) uses Definition 3.17 rather than Definition 3.31 as basis. Hence, he uses language $\mathcal{L}_{\text{KL}^-}$ rather than the restricted language $\mathcal{L}^{\text{cons}}_{\text{KL}^-}$. Schurz (1997b, p. 539), then, defines the notion of properness the following way:

Definition 3.35. *Suppose that α and $\beta \in \mathcal{L}_{\text{prop}} \subseteq \mathcal{L}_{\text{KL}^-}$ and that P is a probability function according to Definition 3.17 over language $\mathcal{L}_{\text{KL}^-}$. Then, P is proper for $\alpha \mathrel{\Box\!\!\to} \beta$ iff the following holds: either $\vdash_{p.c.} \neg\alpha$ or $P(\alpha) > 0$.*

Definition 3.36. *For all formulas $\alpha \in form_{\mathcal{L}_{\text{KL}^-}}$ and formula sets $\Gamma \subseteq form_{\mathcal{L}_{\text{KL}^-}}$ and probability functions P w.r.t. $\mathcal{L}_{\text{KL}^-}$, as specified in Definition 3.17, holds: P is proper for Γ iff P is proper for all $\alpha \in \Gamma$.*

Schurz (1997b) modifies Adams' (1975) notion of properness in such a way that all probability functions over language $\mathcal{L}_{\mathrm{KL}^-}$ are proper for all conditionals with inconsistent antecedent formulas. This move allows Schurz to extend his system to arbitrary formulas of language $\mathcal{L}_{\mathrm{KL}^-}$.

Let us now describe Schurz's Schurz (1997b, p. 539) alternative version of p* validity (which we will call 'p′-validity'):

Definition 3.37. *(p′-Validity)*
Let $\alpha \in form_{\mathcal{L}_{\mathrm{KL}^-}}$, $\Gamma \subseteq form_{\mathcal{L}_{\mathrm{KL}^-}}$ *and* $\epsilon, \delta \in \mathbb{R}$. *Then,* α *is probabilistic consequence′ of* Γ *(short:* $\Gamma \vDash_{p'} \alpha$*) iff* $\forall \epsilon > 0, \exists \delta > 0, \forall P$ *over* $\mathcal{L}_{\mathrm{KL}^-}$*, which are proper for* Γ *and* α*, the following is the case: If* $\forall \beta \in \Gamma$ *holds that* $P(\beta) \geq 1 - \delta$*, then* $P(\alpha) \geq 1 - \epsilon$.

Schurz's modification of the notions of properness and p*-validity makes all conditional formulas with inconsistent formulas such as $\perp \mathrel{\Box\!\!\rightarrow} \alpha$ valid, but otherwise does not change Adams' (1975) p*-validity criterion. Most notably, in both versions, Adams' (1975) account and Schurz's (1997b) modification, the inference RCNC is valid.

One might argue that in Adams' (1975) approach CNC holds and that we, hence, effectively apply CNC rather than RCNC. It is true that in Adams' (1975) semantics, due to its restriction to language $\mathcal{L}^{\mathrm{cons}}_{\mathrm{KL}^-}$, CNC is valid. However, in order to apply Adams' (1975) system to conditionals in natural language, we effectively have to apply RCNC and cannot restrict ourselves to CNC. To draw inferences in the Adams (1975) system for natural language conditionals of the form $\alpha \mathrel{\Box\!\!\rightarrow} \beta$, we, first, have to determine whether its antecedent α is p.c.-consistent. Otherwise we are not allowed to represent the conditional in Adams' (1975) language $\mathcal{L}^{\mathrm{cons}}_{\mathrm{KL}^-}$. This p.c.-consistency check is, however, equivalent to the non-derivability precondition of RCNC (see previous section). Moreover, our discussion of Schurz's alternative p′-validity criterion shows that the restriction of Adams (1975) to language $\mathcal{L}^{\mathrm{cons}}_{\mathrm{KL}}$ is in no way essential.

3.5.4 Possible Worlds Semantics and Truth-Assignments in the Adams Approaches

In this section we aim to show, where and in which way probabilistic semantics, such as Adams (1965, 1966, 1977) and Adams (1975), differ from systems with possible worlds semantics, such Chellas-Segerberg (CS) semantics (Chellas, 1975; Segerberg, 1989; see Chapters 4–5). This is important, since it might be argued that the probabilistic approaches of Adams (1965, 1966, 1977) and Adams (1975) are more general than any truth-value accounts and that, hence, truth-value accounts can be described within Adams' probabilistic approaches adequately. (Adams (1975, pp. 5–7), for example, seems to argue that way.) This assumption, however, is – as we shall see – not warranted.

Why might one be led to believe that truth-value approaches can be appropriately described in the Adams (1965, 1966, 1977, 1975) approaches in the first place? First, we can describe truth-values, as Adams (1965, 1966, 1977, 1975) suggests, by using only the values 1 (truth) and 0 (falsity) (Adams, 1966, p. 297, Definition 10.1; Adams, 1975, p. 47). Accordingly, atomic formulas of the languages $\mathcal{L}_{\mathrm{KL}^-}$ (for Adams, 1965, 1966, 1977) and $\mathcal{L}^{\mathrm{cons}}_{\mathrm{KL}^-}$ (for Adams, 1975) are assigned the truth-values 1 and 0. The values for conditional formulas and boolean combinations of non-conditional and formulas are, then, calculated based on these assignments by Definition 3.17 and Definition 3.31. Note that in $\mathcal{L}^{\mathrm{cons}}_{\mathrm{KL}^-}$ conditionals with a probability assignment 0 for an consistent antecedent formula remain undefined. Then, for non-conditional formulas in both languages, p.c. truth-value assignments and those according to the definitions agree (cf. Adams, 1966, p. 275; Adams, 1975, p. 49).

A second reason, why we might be led to believe that the probabilistic approaches by Adams are more general, is based on following strict consequence criterion (Adams, 1965, p. 185; Adams, 1966, p. 274):

Definition 3.38. *Let $\mathcal{L}_{\mathrm{KL}^-}$ be as specified in Section 4.2.1, let $\alpha \in form_{\mathcal{L}_{\mathrm{KL}^-}}$ and $\Gamma \subseteq form_{\mathcal{L}_{\mathrm{KL}^-}}$. Then, α is a strict consequence of Γ (short: $\Gamma \vDash^s_p \alpha$) iff $\forall P$ over $\mathcal{L}_{\mathrm{KL}^-}$ the following is the case: If $\forall \beta \in \Gamma$ holds that $P(\beta) = 1$, then $P(\alpha) = 1$.*

Based on this criterion Adams (1966), then, shows that his notion of strict consequence and p.c.-entailment coincide (p. 274f, Theorem 1.1). Hence, in a sense p.c is the limiting case for Adams' (1965, 1966, 1977) approach. Observe here that p-validity in Adam's (1965, 1966, 1977) probabilistic semantics coincides with validity in finite Lewis models with centering axioms (Adams, 1975; see Section 3.5.1).

Let us now discuss how material implications $\alpha \to \beta$ and conditional formulas $\alpha \mathrel{\Box\!\!\to} \beta$ are related to each other in probabilistic systems, such as Adams (1965, 1966, 1977, 1975). For arbitrary probability assignments in the 3.17 the probability of $P(\alpha \to \beta)$ is the upper bound of the probability $P(\alpha \mathrel{\Box\!\!\to} \beta)$, as the following lemma shows:

Lemma 3.39. *Let be P defined as in Definition 3.17. Then, the following holds:* $P(\alpha \to \beta) \geq P(\alpha \mathrel{\Box\!\!\to} \beta)$.

Proof. We distinguish two cases here: (i) $P(\alpha \wedge \neg\beta) > 0$ and (ii) $P(\alpha \wedge \neg\beta) = 0$.

(A) Suppose (i) and assume for an indirect proof that (1) $P(\alpha \to \beta) < P(\alpha \mathrel{\Box\!\!\to} \beta)$. Since by (i) it is the case that $P(\alpha \wedge \neg\beta) > 0$, it follows by Definition 3.17.b.i–iv that (2) $1 - P(\alpha \wedge \neg\beta) < \frac{P(\alpha\wedge\beta)}{P(\alpha)}$. By Definition 3.17.b.ii and iii we get that (3) $1 < \frac{P(\alpha\wedge\beta)}{P(\alpha\wedge\beta)+P(\alpha\wedge\neg\beta)} + P(\alpha \wedge \neg\beta)$. We abbreviate $P(\alpha \wedge \beta)$ and $P(\alpha \wedge \neg\beta)$ as X and Y respectively. Thus, (3) is equivalent to (4) $1 < \frac{X}{X+Y} + Y$. It follows from (4) that (5) $X + Y < X + Y(X + Y)$ and, hence, (6) $0 < -Y + XY + Y^2$ is the case. Since by assumption (i) and the definition of Y it is the case that $Y > 0$, this implies that (7) $X + Y > 1$. Thus, we get (8) $P(\alpha \wedge \beta) + P(\alpha \wedge \neg\beta) > 1$. This, however, contradicts points b.i–iii of Definition 3.17. Hence, it follows that $P(\alpha \to \beta) \geq P(\alpha \mathrel{\Box\!\!\to} \beta)$.

(B) Suppose that (ii). Then, by Definition 3.17.b.i–iii it is the case that $P(\alpha \to \beta) = 1$. Since $P(\alpha \mathrel{\Box\!\!\to} \beta)$ is a probability assignment proper in the sense of Definition 3.17, it follows by point b.i of Definition 3.17 that $P(\alpha \to \beta) \geq P(\alpha \mathrel{\Box\!\!\to} \beta)$. □

In addition, for probability values of 1, the following stronger relationship holds:

Lemma 3.40. *Let be P defined as in Definition 3.17. Then, the following holds:* $P(\alpha \mathrel{\Box\!\!\to} \beta) = 1$ *iff* $P(\alpha \to \beta) = 1$

Proof. "$\Leftarrow$": Immediate by Lemma 3.39.

"$\Rightarrow$": Let $P(\alpha \to \beta) = 1$ be the case. We treat the cases (i) $P(\alpha) = 0$ and (ii) $P(\alpha) > 0$ separately. Suppose that (i) is the case. Then, by Definition 3.17.b.iv it follows trivially that $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = 1$. Suppose that (ii) holds. Then, by Definition 3.17.b.iv we have (1) $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = \frac{P(\alpha\wedge\beta)}{P(\alpha)}$. We get by Definition 3.17.b.i–iii that (2) $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = \frac{P(\alpha\wedge\beta)}{P(\alpha\wedge\beta)+P(\alpha\wedge\neg\beta)}$. Since from assumption (i) it follows that $P(\alpha \wedge \neg\beta) = 0$, we have $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = \frac{P(\alpha\wedge\beta)}{P(\alpha\wedge\beta)} = 1$. (Note that by (i) and $P(\alpha \wedge \beta) = P(\alpha)$ it follows that also $P(\alpha \wedge \beta) > 0$.) □

There is a further connection between conditional formulas and the material implication in the Adams (1965, 1966, 1977, 1975) approach and Schurz' alternative account of $\mathbf{P}_\epsilon$ semantics of Adams (1975). Let us, for that purpose take a look at Table 3.1. In Table 3.1 we describe quasi truth-assignments in the probabilistic semantics of Adams (1965, 1966, 1977), Adams (1975) and Schurz (1997b). These quasi truth-assignments concur with definitions 3.17 and 3.31, but assign to non-conditional formulas of languages $\mathcal{L}_{\mathrm{KL}^-}$ and $\mathcal{L}_{\mathrm{KL}^{-\mathrm{cons}}}$ only the values 0 and 1. We see in Table 3.1 that both the material implication and the representation of truth-values in Adams' probabilistic framework concur. In Adams' (1975) approach, however, both values, for which the antecedent formula is assigned the value 0 are undefined. Suppose that α and β are p.c.-formulas. Then, in Adams' (1965, 1966, 1977) and Schurz's (1997b) framework the probability value of a conditional formula $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ is a function of the probability values for the conjunction of the antecedent and the consequent $\alpha \wedge \beta$ and the consequent β. We will call this property – the property that probability values of $\alpha \wedge \beta$ and α determine the probability value of $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ – 'extended value functionality (of conditional formulas)'. Observe that, however, when α and β are p.c.-formulas, the probabilities $P(\alpha)$ and $P(\beta)$ do not in general determine $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta)$, $P(\alpha \wedge \beta)$ and $P(\alpha \vee \beta)$, but only $P(\neg\alpha)$ and $P(\neg\beta)$.

Let us now focus on truth assignments in possible worlds semantics, such as Lewis models (D. Lewis, 1973/2001; see Section 3.2.3) and CS-semantics. Note that in these approaches in general only the values 1 (truth) and 0 are available. In the most extreme case the truth-value of a conditional $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ is in no truth-table case determined by truth-value of the antecedent and consequent formulas α and β. Among those systems are D. Lewis' (1973/2001) semantics as described by Lewis models in Definitions 3.5–3.7, the basic Chellas-Segerberg (CS) semantics (see Chapters 4–7; see also Section 3.2.2) and the ordering semantics by Kraus et

Table 3.1
Quasi Truth-Values for Conditionals in (1) Adams (1965, 1966, 1975) and Schurz (1997b), (2) Adams (1975) and Truth-Values for Conditionals in (3) Lewis (1973/2001) Models

$V_{1\text{–}3}(\alpha)$	$V_{1\text{–}3}(\beta)$	$V_{1\text{–}3}(\alpha \rightarrow \beta)$	$V_1(\alpha \mathrel{\Box\!\!\rightarrow} \beta)$	$V_2(\alpha \mathrel{\Box\!\!\rightarrow} \beta)$	$V_3(\alpha \mathrel{\Box\!\!\rightarrow} \beta)$
1	1	1	1	1	$0/1$
1	0	0	0	0	$0/1$
0	1	1	1	undef.	$0/1$
0	0	1	1	undef.	$0/1$

V_1 and V_2 are quasi-truth values in Adams (1965, 1966, 1975) and Schurz (1997b) on the one hand and Adams (1975) on the other hand, respectively. Quasi-truth-values result from restricting Definitions 3.17 and 3.31 to the values $\{0, 1\}$, respectively. For V_3 truth-values in Lewis (1973/2001) models (Definitions 3.5–3.7) are used. Note that our definition of Lewis models does not make any bridge principles valid. The expressions 'undef.' and '$0/1$' indicate that the value is undefined and may be 0 or 1, respectively.

al. (1990) and Lehmann and Magidor (1992) (cf. Section 3.2.3). These semantics have in common that they do not contain bridge principle, viz. principles, which suppose a fixed relationship between conditional and unconditional formulas (see Section 4.2.1). Note that in the systems described by Kraus et al. (1990) and Lehmann and Magidor (1992) bridge principles cannot even be expressed since they consider only conditional assertions or negations thereof (see Section 3.4.3).

In possible worlds semantics without bridge principles, hence, no nonconditional formula can determine the value of a conditional formula. Hence, possible worlds semantics do not in general have the extended value-functionality of conditional formulas. In fact in semantics without bridge principles, such as Lewis models (see Definitions 3.5–3.7), neither the truth-value of formulas α and β nor the truth-value of α and $\alpha \wedge \beta$ determines the truth-value of $\alpha \mathrel{\Box\!\!\rightarrow} \beta$. The truth-value of disjuncts, conjuncts and negated formulas, however, determines the truth-value of disjunction, conjunctions and negations, respectively.

There exists also a number of systems with a possible worlds semantics, which make bridge principles valid. Among those are Stalnaker models (1968; Stalnaker & Thomason, 1970; see Definition 3.1) and Lewis models that satisfy the centering conditions (see Section 3.2.3). We discussed in previous sections, such as Section 3.2.3, already the two important bridge principles MP ($(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \rightarrow \beta)$) and CS ($\alpha \wedge \beta \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \beta)$). Both principles are also valid in Adams' system $\mathbf{P}^*$ (Adams, 1965, 1966, 1977) and the extended Adams (1975) system $\mathbf{P}_\epsilon$ (see Sections 3.5.1 and 3.5.3). The conditions MP and CS, however, give us neither value functionality nor extended value functionality: in case α is false, neither the truth-values α and β nor α and $\alpha \wedge \beta$ determine the truth-value of $\alpha \mathrel{\Box\!\!\rightarrow} \beta$.

As we saw above the lack of bridge principles leads to the fact that we neither have truth-functionality nor extended truth-functionality of conditionals. If we add CS to a conditional logic without bridge principles, then the truth of both the antecedent formula α and the consequent formula β leads to the truth of the conditional formula $\alpha \mathrel{\Box\!\!\rightarrow} \beta$. Likewise, if MP is valid in a conditional logic system, then the truth of the antecedent formula α and the falsehood of the consequent β lead to the falsehood of the conditional formula $\alpha \mathrel{\Box\!\!\rightarrow} \beta$. If we add both CS and MP, then the resulting truth-table for conditionals resembles the one quasi truth-values in Adams' (1975) approach. There exists, however, an essential difference between both cases: In Adams (1975) quasi truth-assignments of conditionals are undefined in the third and fourth row, while in the possible worlds accounts the truth-value of conditional formulas is merely undetermined by the truth-values of antecedent and consequent formulas. In Schurz's (1997b) alternative version of Adams' (1975) $\mathbf{P}_\epsilon$ semantics also conditionals with inconsistent antecedent formulas are assigned probability values. These assignments concur for the values 0 and 1 perfectly with the probability assignments of Adams (1965, 1966, 1977).

We might still add a third bridge principle to a conditional logic, namely $\neg\alpha \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \beta)$ (EFQ, S1$'$). If we do so as described before, then in the third and fourth row of the truth-table the conditional is determined as being true. Hence, if we add MP, CS and EFQ, then our semantics collapses with the extensional semantics of the material implication. With this additional assumption we, thus, obtain both value-functionality and extended value functionality. The principle EFQ holds in the Adams (1965, 1966, 1977) system only with the strict validity criterion, but not with the probabilistic validity criterion. The other two principles CS and MP are, however, also p-valid in the systems of Adams (1965, 1966, 1977) and Adams (1975). Probabilistic approaches such as Adams (1965, 1966, 1977) escape EFQ here, since only for the special case where the probability of the antecedent α

is zero (and hence $P(\neg\alpha) = 1$), the probability $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = 1$ (see above). In general, however, $\neg\alpha$ does not guarantee an arbitrarily high probability (smaller than 1) for $\alpha \mathrel{\Box\!\!\rightarrow} \beta$. Hence, p-validity fails for this inference. (Analogously for p*-validity.) For possible worlds semantics no such escape is possible, since 0 and 1 are the only assignments possible.

Our discussion shows where probabilistic approaches such as Adams (1965, 1966, 1975) and possible worlds semantics deviate from each other. The probabilistic approaches retain extended value-functionality of conditionals from p.c., when restricted to values 1 and 0. In other words, it holds that $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = 1$ iff $P(\alpha) = 0$ or $P(\beta) = 1$ for definitions of probability functions, as specified in Definition 3.17. For probability functions in line with Definition 3.31, in case of $P(\alpha) = 0$ the probability value of $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ is not defined. Hence, one might say that probabilistic conditional logics generalize p.c.-semantics by extending the number of possible assignments to p.c.-formulas. In possible worlds semantics the bivalent assignment is retained from p.c. and the value-functionality w.r.t. boolean combinations of non-conditional formulas. What differs, however, is that both value-functionality and extended value functionality of conditional formulas are given up instead. On the basis of these observations, it seems inappropriate to try to describe possible words approaches within the framework of probabilistic conditional logics such as Adams (1965, 1966, 1975) and Schurz (1997b).

Despite this fact there exist arguments against possible worlds approaches, which take that road. Adams (1975, p. 5–7), for example, aims to show that truth-value approaches and in particular possible worlds approaches such as Stalnaker (1968; see Section 3.2.2) have absurd consequences. In particular, he intends to demonstrate that for any such approach there exist no propositions α, β and γ, and quasi truth-value assignments t_1, t_2 and t_3 such that $t_1(\alpha) \neq t_2(\alpha)$, $t_1(\beta) \neq t_3(\alpha)$ and $t_2(\gamma) \neq t_3(\alpha)$. For his proof to go through, Adams (1975) presupposes – among other things – that truth-values of conditional formulas in truth-value semantics, including possible worlds semantics, can be adequately described by quasi truth-value assignments as described above. One might argue that if Adams uses his (1975) approach, then, his argument has some force, at least for possible worlds semantics with bridge principles. We disagree. Not defined values and not determined values are completely distinct cases. Any argument, which is based on one approach, might not generalize to the other approach. Moreover, the restriction of a conditional logic to language $\mathcal{L}^{\text{cons}}_{\text{KL}^-}$ for $\mathbf{P}_\epsilon$ semantics such as Adams (1975) is inessential, as our discussion of the alternative approach to $\mathbf{P}_\epsilon$ semantics by Schurz (1997b) in the previous section shows. In addition, we could still escape the partial extended value functionality by making the bridge principles CS and MP invalid.

3.6 Lewis' (1976) Triviality Results

The aims of the present section are as follows: We will, first, describe both versions of D. Lewis' (1976) triviality result (Section 3.6.1). Then, we discuss why extensions of a conditional logic language, which allow for nestings of conditionals, are problematic (Section 3.6.2). We finally discuss the implications of these results for probabilistic conditional logic systems (Section 3.6.3) and finally investigate, which bearing these results have on possible worlds semantics (Section 3.6.4), such as Lewis (1973/2001) models (see Definitions 3.5–3.7) and CS-semantics (see Chapters 4–7). The latter investigation is needed, since D. Lewis' (1976) triviality results are sometimes considered to be lethal for conditionals having truth-values. Bennett (2003), for example, argues that D. Lewis' (1976) triviality results count against truth-value approaches to conditionals, since the thesis that conditionals in general do not have truth-values has the "unique power to protect the Equation [the Stalnaker thesis] from the 'triviality' proofs of Lewis and others" (Bennett, 2003, p. 103). Note that, since possible worlds semantics, such as Stalnaker (1968), Stalnaker and Thomason (1970), Kraus et al. (1990), Lehmann and Magidor (1992) and Chellas-Segerberg semantics (Chellas, 1975; Segerberg, 1989; see Chapters 4–6) essentially draw on the notion of truth, one would prima facie also expect that D. Lewis' (1976) triviality results are also a decisive argument against the usefulness of these semantics. Observe here that D. Lewis' (1976) triviality result is not formulated in a truth-value context, but w.r.t a probabilistic semantics. So, a direct bearing of D. Lewis' (1976) against possible worlds semantics would be rather surprising.

3.6.1 Lewis'Proofs

For his triviality proofs, D. Lewis (1976, p. 299) effectively uses the following definition of probabilities:

Definition 3.41. *P is a probability function over $\mathcal{L}_{\mathrm{KL}}$ iff*
a) P: form of $\mathcal{L}_{\mathrm{KL}} \to \mathbb{R}$
b) for all $\alpha, \beta \in$ form of $\mathcal{L}_{\mathrm{KL}}$ holds:
i) $0 \leq P(\alpha) \leq 1$, *and* $P(\top) = 1$
ii) if $\models \alpha \to \beta$, *then* $P(\alpha) \leq P(\beta)$
iii) if $\models \neg(\alpha \wedge \beta)$, *then* $P(\alpha \vee \beta) = P(\alpha) + P(\beta)$
iv) if $P(\beta) > 0$, *then* $P(\beta | \alpha) = P(\alpha \wedge \beta)/P(\beta)$

Note that in our terminology the expression $\top$ (in Definition 3.41.b.ii) abbreviates $p \vee \neg p$ (cf. Section 4.2.1). Hence, in order to guarantee that every probability function specified by Definition 3.41 assigns probability 1 to all logically true formulas, we need point b.ii in Definition 3.41.

Moreover, Definition 3.41 differs from Definition 3.31 (see Section 3.5.3), insofar as in Definition 3.41 employs the language $\mathcal{L}_{\mathrm{KL}}$ rather than the restricted language $\mathcal{L}^{\mathrm{cons}}_{\mathrm{KL}^-}$. Furthermore, in condition b.iv of Definition 3.41, no probabilities of conditional formulas (i.e. $\alpha \mathrel{\Box\!\!\to} \beta$) are defined, but conditional probabilities.

At the center of the proof stands the so-called "Stalnaker thesis" (cf. Stalnaker, 1970). We can distinguish at least the following three versions of this thesis:

Assumption 3.42. *(Unrestricted Stalnaker Thesis)*
For all probability functions P over language $\mathcal{L}_{\mathrm{KL}}$ and all $\alpha, \beta, \gamma \in form_{\mathcal{L}_{\mathrm{KL}}}$ with $P(\gamma) > 0$ holds: $P_\gamma(\alpha \mathrel{\Box\!\!\to} \beta) = P_\gamma(\beta | \alpha)$ if $P_\gamma(\alpha) > 0$.

Assumption 3.43. *(Very Restricted Stalnaker Thesis)*
There exists a probability function P over language $\mathcal{L}_{\mathrm{KL}}$ and for all $\alpha, \beta \in form_{\mathcal{L}_{\mathrm{KL}}}$ and some formula $\gamma \in form_{\mathcal{L}_{\mathrm{KL}}}$ with $P(\gamma) > 0$ holds: $P_\gamma(\alpha \mathrel{\Box\!\!\to} \beta) = P_\gamma(\beta | \alpha)$ if $P_\gamma(\alpha) > 0$.

Assumption 3.43*. *(Restricted Stalnaker Thesis)*
There exists a probability function P over language $\mathcal{L}_{\mathrm{KL}}$, such that for all $\alpha, \beta, \gamma \in form_{\mathcal{L}_{\mathrm{KL}}}$ with $P(\gamma) > 0$ holds: $P_\gamma(\alpha \mathrel{\Box\!\!\to} \beta) = P_\gamma(\beta | \alpha)$ if $P_\gamma(\alpha) > 0$.

For a given probability function P over language $\mathcal{L}_{\mathrm{KL}}$, the expression $P_\gamma(.)$ describes a probability function over $\mathcal{L}_{\mathrm{KL}}$, which results from P by conditionalizing on γ, provided $\gamma \in form_{\mathcal{L}_{\mathrm{KL}}}$ and $P(\gamma) > 0$. Lemma 3.44 (see below) shows, then, that $P_\gamma(\alpha) = P(\alpha\,|\,\gamma)$ for arbitrary formulas α if it is the case that $P(\gamma) > 0$. Note that P is also regarded as a probability function, which results from P by conditionalization, since it holds that $P(\cdot) = P_\top(\cdot)$.

All versions of the Stalnaker thesis define probabilities of conditionals by means of conditional probabilities. They, however, do so to varying degrees of generality. While the unrestricted Stalnaker thesis (Assumption 3.42) effectively equates the probability $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta)$ with $P(\beta\,|\,\alpha)$, given $P(\alpha) > 0$, for all probability functions, the restricted Stalnaker thesis (Assumption 3.6.1*) states only that there exists a probability function P, such that for all probability functions P' resulting from conditionalization of P it holds that $P'(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = P'(\beta\,|\,\alpha)$, provided $P'(\alpha) > 0$. The very restricted Stalnaker thesis (Assumption 3.6.1) is weaker than the restricted version, since it only states that there exists a probability function P, such that for some probability function P', which results from conditionalization, it holds that $P'(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = P'(\beta\,|\,\alpha)$, provided $P'(\alpha) > 0$. This, however, gives us only that there exists a probability function P, such that $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = P(\beta\,|\,\alpha)$, provided $P(\alpha) > 0$.

D. Lewis (1976) shows that the unrestricted Stalnaker thesis leads to a triviality result for all probability functions (first triviality theorem). He proves in addition that any probability function, for which the description in the restricted Stalnaker thesis (Assumption 3.6.1*) applies, must itself be trivial (second triviality theorem). Only the very restricted Stalnaker thesis does not lead to any sort of triviality result. It is very weak insofar it gives us only that there is a single probability function P, such that $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = P(\beta\,|\,\alpha)$ provided $P(\alpha) > 0$. It should be clear that the very restricted Stalnaker thesis does not provide us with more than a toy version of a probabilistic conditional logic semantics.

Let us now describe both triviality theorems of D. Lewis. We provide, a range of lemmata, on which both triviality results draw:

Lemma 3.44. *(Conditionalization Lemma)*
Let P be a probability function over language $\mathcal{L}_{\mathrm{KL}}$ and let α be an arbitrary formula of $\mathcal{L}_{\mathrm{KL}}$. Then, there is a probability function P_α over $\mathcal{L}_{\mathrm{KL}}$, such that for any

formula $\beta \in form_{\mathcal{L}_{\mathrm{KL}}}$ *it holds that* $P_\alpha(\beta) = (\beta \,|\, \alpha)$, *provided* $P(\alpha) > 0$ *is the case.*

Proof. Let P be a probability function over language $\mathcal{L}_{\mathrm{KL}}$ and let $\alpha \in form_{\mathcal{L}_{\mathrm{KL}}}$ be such that $P(\alpha) > 0$. Due to the latter fact, $P(\beta \,|\, \alpha)$ is defined for all $\beta \in form_{\mathcal{L}_{\mathrm{KL}}}$. It is trivially the case for all formulas $\gamma, \delta \in form_{\mathcal{L}_{\mathrm{KL}}}$ and all probability assignments P' over $\mathcal{L}_{\mathrm{KL}}$ that $P'(\delta \,|\, \gamma)$ obeys points a and b.i-b.iii in Definition 3.41 if $P'(\gamma) > 0$. Hence, the probabilities assigned by $P(\beta \,|\, \alpha)$ for all $\beta \in form_{\mathcal{L}_{\mathrm{KL}}}$ can be described by an unconditional probability function P'' over $\mathcal{L}_{\mathrm{KL}}$. Thus, an unconditional probability function P_α over $\mathcal{L}_{\mathrm{KL}}$ exists, namely P'', such that $P_\alpha(\beta) = P(\beta \,|\, \alpha)$ for all $\beta \in form_{\mathcal{L}_{\mathrm{KL}}}$.

□

Lemma 3.45. *For any probability function* P *of* $\mathcal{L}_{\mathrm{KL}}$ *and any formulas* $\alpha, \beta \in form_{\mathcal{L}_{\mathrm{KL}}}$, *such that* $P(\alpha \wedge \beta), P(\alpha \wedge \neg\beta) > 0$, *holds:* $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = P(\alpha \mathrel{\Box\!\!\rightarrow} \beta \,|\, \beta) \cdot P(\beta) + P(\alpha \mathrel{\Box\!\!\rightarrow} \beta \,|\, \neg\beta) \cdot P(\neg\beta)$.

Proof.

1.	$P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = P((\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge \top)$	Def 3.41.b.ii
2.	$= P((\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\beta \vee \neg\beta))$	Def 3.41.b.ii
3.	$= P(((\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge \beta) \vee ((\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge \neg\beta))$	Def 3.41.b.ii
4.	$= P((\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge \beta) + P((\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge \neg\beta)$	Def 3.41.b.iii
5.	$= P(\alpha \mathrel{\Box\!\!\rightarrow} \beta \,|\, \beta) \cdot P(\beta) + P(\alpha \mathrel{\Box\!\!\rightarrow} \beta \,|\, \neg\beta) \cdot P(\neg\beta)$	Def 3.41.b.iv $P(\alpha \wedge \beta), P(\alpha \wedge \neg\beta) > 0$

□

Note that Lemma 3.45 explicitly draws on conjunctions involving conditional formulas. Let us now prove the essential step in D. Lewis' triviality proof:

Lemma 3.46. *Let* P *be an arbitrary probability function over* $\mathcal{L}_{\mathrm{KL}}$. *Furthermore, suppose that either (i) the unrestricted Stalnaker thesis (Assumption 3.42) holds for probability functions* P' *over* $\mathcal{L}_{\mathrm{KL}}$ *or, alternatively, (ii)* P *is one of the probability functions, which is described in the restricted Stalnaker thesis (Assumption 3.6.1*). Then, for any* $\alpha, \beta, \gamma \in form_{\mathcal{L}_{\mathrm{KL}}}$ *it is the case that* $P(\beta \mathrel{\Box\!\!\rightarrow} \gamma \,|\, \alpha) = P(\gamma \,|\, \alpha \wedge \beta)$ *provided* $P(\alpha \wedge \beta) > 0$ *holds.*

Proof. Let P be a probability function over $\mathcal{L}_{\mathrm{KL}}$ and let α, β, γ be arbitrary formulas of language $\mathcal{L}_{\mathrm{KL}}$. Furthermore, suppose that either (i) the unrestricted

Stalnaker thesis (Assumption 3.42) holds for all probability functions P' over $\mathcal{L}_{\text{KL}}$, or (ii) P is one of the probability functions, which is described in the restricted Stalnaker thesis (Assumption 3.6.1*). Let, moreover, $P(\alpha \wedge \beta) > 0$ be the case. Since $P(\alpha \wedge \beta) > 0$, it follows that $P(\beta \mathrel{\Box\!\!\rightarrow} \gamma | \alpha)$ is defined, and we have by Lemma 3.44 that there exists a probability function P_α, such that $P(\beta \mathrel{\Box\!\!\rightarrow} \gamma | \alpha) = P_\alpha(\beta \mathrel{\Box\!\!\rightarrow} \gamma)$. Moreover, as $P(\alpha) > 0$, we get by Definition 3.41.b.iv that $P(\beta | \alpha) = P(\beta \wedge \alpha)/P(\alpha)$. Since $P(\alpha \wedge \beta) > 0$ and $P(\alpha) > 0$, it follows that $P(\beta \wedge \alpha)/P(\alpha) > 0$ and, hence, $P(\beta | \alpha) > 0$ results. Since $P(\alpha) > 0$ is the case, this implies by definition of P_α that $P_\alpha(\beta) > 0$. This result implies that $P_\alpha(\gamma | \beta)$ is defined. By (i) or, alternatively, by (ii) we get that $P_\alpha(\beta \mathrel{\Box\!\!\rightarrow} \gamma) = P_\alpha(\gamma | \beta)$. Thus, $P(\beta \mathrel{\Box\!\!\rightarrow} \gamma | \alpha) = P_\alpha(\gamma | \beta)$ is the case.

Due to $P_\alpha(\beta) > 0$ and Definition 3.41.b.iv we have $P_\alpha(\gamma | \beta) = P_\alpha(\gamma \wedge \beta)/P_\alpha(\beta)$. As $P(\alpha) > 0$, we obtain by the definition of P_α that $P_\alpha(\gamma \wedge \beta) = P(\gamma \wedge \beta | \alpha)$ and $P_\alpha(\beta) = P(\beta | \alpha)$. Hence, it follows that $P_\alpha(\gamma \wedge \beta)/P_\alpha(\beta) = P(\gamma \wedge \beta | \alpha)/P(\alpha | \beta)$. Since $P(\alpha) > 0$, the latter term, however, equals on the basis of Definition 3.41.b.iv $(P(\gamma \wedge \beta \wedge \alpha) \cdot P(\alpha))/((P(\alpha) \cdot P(\alpha \wedge \beta))$, which is due to Definition 3.41.b.i and ii equal to $P(\gamma \wedge \alpha \wedge \beta)/P(\alpha \wedge \beta)$. Hence, $P_\alpha(\gamma \wedge \beta)/P_\alpha(\beta) = P(\gamma \wedge \alpha \wedge \beta)/P(\alpha \wedge \beta)$ and $P_\alpha(\gamma | \beta) = P(\gamma \wedge \alpha \wedge \beta)/P(\alpha \wedge \beta)$ follows. Since $P(\alpha \wedge \beta) > 0$, we obtain by Definition 3.41.b.iv that $P_\alpha(\gamma | \beta) = P(\gamma | \alpha \wedge \beta)$ and, hence, $P(\beta \mathrel{\Box\!\!\rightarrow} \gamma | \alpha) = P(\gamma | \alpha \wedge \beta)$ is the case.

□

Observe that conditions (i) and (ii) in Lemma 3.46 draw on the unrestricted and the restricted Stalnaker assumption, respectively. Let us, finally, prove D. Lewis' triviality theorems:

Theorem 3.47. *(Lewis' Triviality Theorems)*
Let P be a probability function over $\mathcal{L}_{\text{KL}}$. Furthermore, suppose that either (i) the unrestricted Stalnaker thesis (Assumption 3.42) holds for probability functions P' over $\mathcal{L}_{\text{KL}}$ or, alternatively, (ii) P is one of the probability functions, which is described in the restricted Stalnaker thesis (Assumption 3.6.1). Then, for all α, $\beta \in form_{\mathcal{L}_{\text{KL}}}$ such that $P(\alpha \wedge \beta)$, $P(\alpha \wedge \neg\beta) > 0$ the following is the case: $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = P(\beta)$.*

Proof. Let P be a probability function over $\mathcal{L}_{\text{KL}}$. Suppose, in addition, that either (i) the unrestricted Stalnaker thesis (Assumption 3.42) holds for probability functions P' over $\mathcal{L}_{\text{KL}}$ or that (ii) P is one of the probability functions, which is described in the restricted Stalnaker thesis (Assumption 3.6.1*). Furthermore, suppose that $\alpha, \beta \in form_{\mathcal{L}_{\text{KL}}}$ and that $P(\alpha \wedge \beta) > 0$ and $P(\alpha \wedge \neg\beta) > 0$ are the case. Due to the latter two facts, we get by Lemma 3.45 that $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = P(\alpha \mathrel{\Box\!\!\rightarrow} \beta | \beta) \cdot P(\beta) + P(\alpha \mathrel{\Box\!\!\rightarrow} \beta | \neg\beta) \cdot P(\neg\beta)$. Since by assumption it is the case that $P(\alpha \wedge \beta), P(\alpha \wedge \neg\beta) > 0$ and either (i) or (ii) holds, it follows by Lemma 3.46 both that $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta | \beta) = P(\beta | \alpha \wedge \beta)$ and that $P(\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta | \beta) = P(\neg\beta | \alpha \wedge \beta)$. Hence, $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = P(\beta | \alpha \wedge \beta) \cdot P(\beta) + P(\beta | \alpha \wedge \neg\beta) \cdot P(\neg\beta)$ is the case.

Since $P(\alpha \wedge \beta) > 0$, we have by Definition 3.41.b.vi that $P(\beta | \alpha \wedge \beta) = P(\beta \wedge \alpha \wedge \beta)/P(\alpha \wedge \beta)$. As $\beta \wedge \alpha \wedge \beta$ is p.c. equivalent to $\alpha \wedge \beta$, we get by Definition 3.41.b.ii that $P(\beta \wedge \alpha \wedge \beta) = P(\alpha \wedge \beta)$. Hence, $P(\beta \wedge \alpha \wedge \beta)/P(\alpha \wedge \beta) = 1$ and, thus, $P(\beta | \alpha \wedge \beta) = 1$. Moreover, as $P(\alpha \wedge \neg\beta) > 0$, we have by Definition 3.41.b.iv that $P(\beta | \alpha \wedge \neg\beta) = P(\beta \wedge \alpha \wedge \neg\beta)/P(\alpha \wedge \neg\beta)$. Note that $\beta \wedge \alpha \wedge \neg\beta$ is inconsistent. Thus, by Definition 3.41.b.i-iii it follows that $P(\beta \wedge \alpha \wedge \neg\beta) = 0$. Hence, we have $P(\beta \wedge \alpha \wedge \neg\beta)/P(\alpha \wedge \neg\beta) = 0$ and, thus, $P(\beta | \alpha \wedge \neg\beta) = 0$. Both $P(\beta | \alpha \wedge \beta) = 1$ and $P(\beta | \alpha \wedge \neg\beta) = 0$ imply that $P(\beta | \alpha \wedge \beta) \cdot P(\beta) + P(\beta | \alpha \wedge \neg\beta) \cdot P(\neg\beta) = 1 \cdot P(\beta) + 0 \cdot P(\neg\beta) = P(\beta)$. Hence, we get $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = P(\beta)$. □

Lewis' (1976) first triviality result is described by condition (i) in Theorem 3.47 and Lewis' second triviality result is specified by condition (ii) in the same theorem.

3.6.2 Triviality due to Nestings and Iterations of Conditionals?

The aim of the present section is to show that the admission of nested conditionals (e.g. $\alpha \mathrel{\Box\!\!\rightarrow} (\beta \wedge (\gamma \mathrel{\Box\!\!\rightarrow} \delta))$) or iterated conditionals (e.g. $\alpha \mathrel{\Box\!\!\rightarrow} (\beta \mathrel{\Box\!\!\rightarrow} \gamma)$) has counter-intuitive consequences for a probabilistic semantics. We will focus here on iterated conditional formulas, since these are a subclass of nested conditional formula, but not vice versa. Moreover, We shall for our argument presuppose a probabilistic semantics as described by Adams (1965, 1966, 1977, 1975), and extends the language also to nested conditionals or iterated conditionals. We will, hence, formulate our proofs in language $\mathcal{L}_{\text{KL}}$, which allows for both nestings and iterations of conditionals. For our argument neither (i) the unrestricted nor (ii) the restricted Stalnaker thesis suffice, since we also have to draw for that purpose on a validity criterion, such as 3.18 or 3.34, which has in addition to be modified to allow for both non-conditional formulas and iterations of conditional formulas.

Let us start with unrestricted Stalnaker thesis, as endorsed in the approach of Adams (1965, 1966, 1977, 1975), and the restricted Stalnaker thesis (see Section 3.6.1) and, then, focus on Adams' probabilistic semantics. We shall, for that purpose, first consider the following two inferences, which draw on iterated formulas:

Ex* if $\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$ then $\alpha \mathrel{\Box\!\!\rightarrow} (\beta \mathrel{\Box\!\!\rightarrow} \gamma)$
Im* $\alpha \mathrel{\Box\!\!\rightarrow} (\beta \mathrel{\Box\!\!\rightarrow} \gamma)$ then $\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$

'Ex' and 'Im' stand for 'Importation' and 'Exportation', respectively. The asterix indicates that the respective principles are described here as rules rather than in terms of axioms. Lemma 3.46 gives us on the basis of (i) the unrestricted Stalnaker assumption or (ii) any probability function P, which concurs with the Stalnaker thesis, as described in Definition 3.6.1, that the following is the case: $P(\beta \mathrel{\Box\!\!\rightarrow} \gamma | \alpha) = P(\gamma | \alpha \wedge \beta)$ provided $P(\alpha \wedge \beta) > 0$ holds. Since we explicitly allow for iterations of conditionals, it follows by both (i) and (ii) for P that $P(\beta \mathrel{\Box\!\!\rightarrow} \gamma | \alpha) = P(\alpha \mathrel{\Box\!\!\rightarrow} (\beta \mathrel{\Box\!\!\rightarrow} \gamma))$ and $P(\gamma | \alpha \wedge \beta) = P(\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$. Hence, for $P(\alpha \wedge \beta) > 0$ it is the case that $P(\alpha \mathrel{\Box\!\!\rightarrow} (\beta \mathrel{\Box\!\!\rightarrow} \gamma)) = P(\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$.

Note here that Ex* and Im* are valid in Adams' (1965, 1966, 1977, 1975, 1986) probabilistic semantics in case we allow also for iterations of conditional formulas. The present result is, however, somewhat stronger than what is required by both the p-validity criterion 3.18 and the p*-validity criterion 3.34. Both criteria are already met if we can assign arbitrarily high probabilities to formula $\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$ on the basis of formula $\alpha \mathrel{\Box\!\!\rightarrow} (\beta \mathrel{\Box\!\!\rightarrow} \gamma)$ (or vice versa). These validity criteria, however, do not require that both probabilities must be equal.

If we extend probabilistic semantics, such as Adams' (1965, 1966, 1977, 1975), to languages, which allow for nestings or iterations of conditionals, we run into problems. This is due to the fact that the following inference becomes valid in such a semantics:

Triv′ $\alpha \rightarrow \beta$ iff $\alpha \mathrel{\Box\!\!\rightarrow} \beta$

Triv′ ("Triviality"), however, renders all conditional formulas logically equivalent with the respective material implications. Such a result can be regarded as extremely counter-intuitive, since it is the main motivation of conditional logics to provide an alternative to the material implication (see Chapter 1). Let us now see how Triv′ can be obtained from Ex on basis of the following inferences:

Refl	$\alpha \mathrel{\Box\!\!\to} \alpha$
RW	if $\vdash \alpha \to \beta$ and $\gamma \mathrel{\Box\!\!\to} \alpha$, then $\gamma \mathrel{\Box\!\!\to} \beta$
MP*	if $(\alpha \mathrel{\Box\!\!\to} \beta)$ then $(\alpha \mathrel{\Box\!\!\to} \beta)$

Note that Refl, RW and MP* are all valid in the probabilistic semantics of Adams (1965, 1966, 1977, 1975; see Section 3.5). Let us now formulate and prove the following theorem:

Theorem 3.48. *RW+Refl+MP+Ex⇒ Triv′*

Proof. MP trivially implies Triv$'^{\Rightarrow}$. Moreover, Lemma 3.49 gives us on the basis of RW, Refl, MP* and Ex* the inference Triv$'^{\Leftarrow}$. □

Here 'Triv$'^{\Rightarrow}$' and 'Triv$'^{\Leftarrow}$' stand for the left-to-right direction and the right-to-left direction of Triv′, respectively. Let us now provide the proof for Triv$'^{\Rightarrow}$.

Lemma 3.49. *RW+Refl+MP+Ex ⇒ Triv$'^{\Rightarrow}$.*

Proof.

1.	$\alpha \to \beta$	given
2.	$(\alpha \to \beta) \wedge \alpha \mathrel{\Box\!\!\to} (\alpha \to \beta) \wedge \alpha$	Refl
3.	$(\alpha \to \beta) \wedge \alpha \mathrel{\Box\!\!\to} \beta$	2, RW
4.	$(\alpha \to \beta) \mathrel{\Box\!\!\to} (\alpha \mathrel{\Box\!\!\to} \beta)$	3, Ex
5.	$\alpha \mathrel{\Box\!\!\to} \beta$	4, 1, MP

□

The proof of Lemma 3.49 draws strongly on a similar proof by McGee (1985, p. 466). By Theorem 3.48 the validity of the inferences Im* and Ex*, hence, implies – in case RW, Refl and MP* are also valid – that we have a material implication analysis of conditionals. This observation is the more problematic, since the principles RW and Refl are considered cornerstones of almost every conditional logic in the literature, including Stalnaker models (see Definition 4.19), Lewis models (see Definitions 3.5–3.7), the ordering semantics of Kraus et al. (1990) and Lehmann and Magidor (1992; see Section 3.2.3) and the probabilistic threshold-semantics by Hawthorne and Makinson (2007) and Hawthorne (1996). Moreover, MP is a valid principle in many conditional logics for indicative (e.g. Stalnaker models) and counterfactual conditionals (e.g. Lewis models with centering conditions).

One can use the above proofs to argue in line with McGee (1989) for Ex* and against MP*. The problem in a probabilistic framework as Adams (1965, 1966, 1977, 1975) is that this can only be achieved by restricting the language in such a way that MP* is not expressible. Observe here that MP* is equivalent in Adams' probabilistic semantics to the following principle, as we shall show in Section 7.3 (Theorem 7.61):

Det* if $\top \mathrel{\Box\!\!\rightarrow} \beta$, then α

It is, however, easy to prove that, for example, in Adams' (1965, 1966, 1977) probabilistic semantics it holds that $P(\alpha \mid \top) = P(\alpha)$. In the same semantics, when the language is extended to iterations or nestings of conditionals, it is also the case for $P(\alpha \wedge \beta) > 0$ that $P(\alpha \mathrel{\Box\!\!\rightarrow} (\beta \mathrel{\Box\!\!\rightarrow} \gamma)) = P(\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$. Given these equalities it, hence, seems equally arbitrary to accept either one MP* or Ex*, and reject the other.

3.6.3 Probabilistic Semantics and Restriction of the Language

Let us, first, (i) summarize the implications of D. Lewis' (1976) triviality results (see Section 3.6.1) and (ii) our results regarding iterated conditional formulas for probabilistic semantics, such as Adams (1965, 1966, 1977, 1975). Second, we will discuss, in which way the probabilistic system systems **P**, $\mathbf{P}^*$, $\mathbf{P}^+$ and $\mathbf{P}_\epsilon$ (see Sections 3.5.1 and 3.5.3) avoid both types of counter-intuitive consequences.

D. Lewis' (1976) first triviality result gives us that any probabilistic conditional logic system, which endorses Definition 3.41 and accepts the unrestricted Stalnaker thesis, runs into the triviality result, namely that for $P(\alpha) > 0$, it follows that $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = P(\beta)$. Furthermore, D. Lewis' (1976) second triviality result shows that any probability function P, which is based on Definition 3.41

and for which the restricted Stalnaker thesis holds, runs into the same triviality result. Only when a very weak Stalnaker thesis, such as Definition 3.43 but not the stronger version holds, we do not get D. Lewis' (1976) triviality result. Definition 3.43 is, however, too weak, insofar as it only makes sure that for a given probability function P all conditionalized probability functions P_γ, such that $P(\gamma) > 0$ and $P_\gamma(\alpha) > 0$ it is the case that $P_\gamma(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = P_\gamma(\beta\,|\,\alpha)$. So, the very weak Stalnaker thesis does not guarantee that for given probability function P the conditional probability equals the probability of the respective conditional for all conditionalizations of P. This, can, however, be regarded as a minimum requirement for a probabilistic semantics for a conditional logic (cf. Section 3.6.1).

The conditional logic systems $\mathbf{P}$, $\mathbf{P}^*$, $\mathbf{P}^+$ and $\mathbf{P}_\epsilon$ endorse a definition of probability semantics in line with definition 3.41 and the strong Stalnaker thesis (Assumption 3.42). Both points can be directly read off from Definition 3.17. (Note that in Schurz's (1997b) modified semantics for $\mathbf{P}_\epsilon$ is also based on Definition 3.17.) The probabilistic conditional logic systems $\mathbf{P}$, $\mathbf{P}^*$, $\mathbf{P}^+$ and $\mathbf{P}_\epsilon$ escape the triviality result by restricting their languages. In the case of $\mathbf{P}$, $\mathbf{P}^*$ and $\mathbf{P}^+$ languages $\mathcal{L}_{\mathrm{rrKL}}$, $\mathcal{L}_{\mathrm{KL}^-}$ and $\mathcal{L}_{\mathrm{rKL}^*}$ rather than language $\mathcal{L}_{\mathrm{KL}}$ are employed, respectively (see Section 3.5.1). Languages $\mathcal{L}_{\mathrm{rrKL}}$, $\mathcal{L}_{\mathrm{KL}^-}$ and $\mathcal{L}_{\mathrm{rKL}^*}$ have in common that they do not allow for conjunctions involving conditional formulas. It is exactly this feature that blocks D. Lewis' (1976) triviality results in these systems (D. Lewis, 1976, p. 304). Note that D. Lewis (1976) draws only on conjunctions of conditional formulas (e.g. $\alpha \mathrel{\Box\!\!\rightarrow} \beta$) and unconditional formulas (β; see Lemma 3.45), but not directly on conjunctions of conditional formulas. Conjunctions involving conditional formulas are, however, as problematic as conjunctions involving conditional and non-conditional formulas. This is due to the fact that by Definition 3.41 it holds that $P(\alpha) = P(\alpha\,|\,\top)$. Even if we exempt non-conditional formulas from our language and use Popper functions (see Section 3.4.2), it is to be expected that $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = P(\top \mathrel{\Box\!\!\rightarrow} \beta)$ holds. Such a result is, however, again problematic, since it also trivializes probabilities of conditionals.

In sum, D. Lewis' (1976) triviality results imply that, given we accept (i) the unrestricted Stalnaker thesis, as implied by Definition 3.17, or (ii) the restricted Stalnaker thesis, we cannot in general allow for arbitrary boolean combination of conditional formulas (with conditional and unconditional formulas). When we endorse the languages $\mathcal{L}_{\mathrm{rKL}}$ and $\mathcal{L}_{\mathrm{rKL}^*}$ rather than $\mathcal{L}_{\mathrm{KL}}$, we can avoid D. Lewis' (1976) triviality result by following means: Languages $\mathcal{L}_{\mathrm{rKL}}$ and $\mathcal{L}_{\mathrm{rKL}^*}$ allow either for disjunctions of conditional formulas or negations of conditional formulas, but not both, respectively. The restrictions of languages $\mathcal{L}_{\mathrm{rKL}}$ and $\mathcal{L}_{\mathrm{rKL}^*}$, however, do not imply that we cannot represent inferences involving boolean combinations of formulas in an indirect way. Schurz (1998, p. 84f), for example, suggests that we can use instead sets of inferences, which are – given full language $\mathcal{L}_{\mathrm{KL}}$ – equivalent to single inferences with arbitary boolean combinations. Schurz's (1998) formal results imply that we can express by these means indirectly inferences involving boolean combination of conditionals, also when we apply weaker languages such as $\mathcal{L}_{\mathrm{rKL}}$ and $\mathcal{L}_{\mathrm{rKL}^*}$ (see Sections 3.4.3 and 4.2.1).

Our results in Section 3.6.2, then, show that not just boolean combinations involving conditional formulas, but also nestings and iterations of conditional formulas are problematic, given one applies in addition a validity criterion, as in Adams (1965, 1966, 1977, 1986). Interestingly, the languages of Adams (1965, 1966, 1977, 1986), Schurz (2005) and Schurz (1998) avoid these problems, insofar as they do not allow for nestings and iterations of conditional formulas.

Let us now discuss the restriction of language in systems **P**, **P*** and **P**+. We regard all these restrictions too strong for both an empirically or normative adequate theory of conditionals. First, we can in fact produce boolean combinations and nestings of conditionals in natural language and make sense out of them (see Section 3.4.3). Second, a full normative account of conditionals should allow for boolean combinations and nestings of conditionals, since otherwise we cannot account for many logically interesting properties (relating to conditionals). This holds the more as we are not aware of any plausible argument why on a normative level conditionals should in principle not be combinable or be nestable.

Sometimes it is argued that the lack of embedding of conditionals in Adams' (1965, 1966, 1975) accounts is a merit of his approach (Bennett, 2003, p. 95, p. 104, see als Section 3.4.3) and would be advantage compared to truth-value approaches, such as the material implication analysis (p. 95). We, however, doubt that. Note that we can represent embedded conditionals only in a very restricted way (e.g. only disjunctions of conditional formulas in Adams, 1986) and cannot in principle account for nested and iterated conditionals in Adams' restricted languages. The language is simply not expressive enough. We can use quasi-conjunctions or quasi-disjunctions etc. as described in Adams (1975, p. 46f). The problem of this approach is, however, that quasi-disjunctions, etc. are not disjunctions of conditional formulas, but conditional formulas of a specific sort, which – given some background assumptions – are equivalent to disjunctions, etc. of conditional formulas. The problem is, hence, that we have to use rather strong systems of conditional logic in order that these equivalences hold. The presuppositions and their application can, however, at best only partially be discussed within Adams' systems, since his languages are not expressive enough. We, hence, think that systems with a full language such as $\mathcal{L}_{\mathrm{KL}}$ have an advantage over conditional logic systems with a more restricted language. First, we can represent the inferences needed for quasi-conjunctions, etc. quite freely in such a system and discuss them within the system. Second, we can, then, say when and where pragmatic features do not allow for nestings and boolean combinations of conditionals. In systems with restricted languages, such as $\mathcal{L}_{\mathrm{KL}^-}$ (Adams, 1965, 1966) and $\mathcal{L}_{\mathrm{KL}^-}^{\mathrm{cons}}$ (Adams, 1975), such a move is not possible. In those systems we have to reason on a case to case basis (Bennett, 2003, p. 95) and rather informally (cf. Adams, 1975, pp. 31–37). Hence, by this method the logical properties of the inferences are rather obscured than made more perspicuous.

Note in addition that the restricted use of a language is not specific to probabilistic approaches to indicative conditionals. For example, systems **P** of Kraus et al. (1990) and Lehmann and Magidor (1992) is effectively formulated in the language $\mathcal{L}_{\mathrm{rKL}^*}$ (see Section 4.2.1). There exists, however, a sound and complete possible worlds semantics for the proof-theory of system **P** (see Section 3.2.3; see Kraus et al., 1990, Theorem 5.18, p. 196). We, hence, see no prima facie reason, why it is not possible to construct a possible world semantics for the systems of Adams (1965, 1966, 1977) and Adams (1975) systems exactly for his languages $\mathcal{L}_{\mathrm{KL}^-}$ and $\mathcal{L}_{\mathrm{KL}^-}^{\mathrm{cons}}$, respectively. In fact, our discussion in Section 3.5.1 showed that there exist possible worlds semantics, which make the same formulas valid as the probabilistic semantics for system $\mathbf{P}^*$ and $\mathbf{P}^+$. So, if one argues that it is an advantage of Adams' approach that he uses a more restricted language, she has to be aware that this is not specific Adams' logical systems but can replicated also in terms of a possible worlds semantics.

3.6.4 Lewis' Triviality Result and Truth-Value Accounts

We saw in the introductory part of this section that Bennett (2003) argues against truth-value accounts on the basis of D. Lewis' (1976) triviality result. This observation is further strengthened by the fact that he describes the triviality result and variants of it as the third route to NTV ("No Truth-Value") (Bennett, 2003, p. 103). NTV is the thesis that conditionals are neither propositions nor do they in general have truth-values (Bennett, 2003, p. 94). Bennett then goes on to argue that NTV has the "unique power to protect the Equation [the probability of a conditional equals its conditional probability] form the 'triviality' proofs of Lewis and others" (p. 103). Note that Bennett's argument has several hidden assumptions, since otherwise it is clearly mistaken, for the following two reasons: First, probabilities of conditionals and conditional probabilities do not have direct applicability in any purely truth-valued approach, such as p.c. or systems with possible worlds semantics. These are, as their name indicates, based on the notion of truth and falsity and not probabilities. Second, neither we nor Bennett (2003, p. 60–63) nor D. Lewis (1976) uses the concept of truth in any essential way in the proof of the triviality result. So, how can D. Lewis' triviality result tell us anything about truth-value accounts?

Note that Bennett (2003, p. 103) indicates that only NTV makes an escape from the triviality result possible. Since, as we observed, the proof for the triviality result relies on a probabilistic framework, one possible interpretation is that Bennett regards the probability approach as the more fundamental approach. First, it might be argued that we can account for truth-value approaches within a probabilistic framework such as Adams (1965, 1966, 1975, 1977) and that, hence, the probabilistic approach is the more fundamental one. Note, however, that such an interpretation is not plausible for possible worlds semantics, since those reject the value functionality and extended value functionality of conditionals in probabilistic semantics (see Section 3.5.4).

Second, the lack of embeddings of conditionals in larger linguistic structures might be regarded as evidence that the probabilistic approach by Adams is empirically more adequate than truth-value accounts. Bennett (2003, p. 104) seems to argue partially that way. As we saw in the last section, there exist truth-value approaches such as Kraus et al. (1990) and Lehmann and Magidor (1992), which use also a restricted language. In addition, there seems no reason why one should not be able to provide a possible worlds semantics exactly for Adams' (1965, 1966, 1977, 1975, 1986) proof-theoretic systems, which is also based on the same language as the Adams systems. Moreover, the discussion of variants of Adams' system **P** in Section 3.5.1 shows that at least for the system **P**, **P*** and **P**+ possible worlds semantics exists, w.r.t. which the respective proof-theoretic system is sound and complete. So, even if the triviality results applies to systems with possible worlds semantics, it might still be possible as in the probabilistic framework to escape the triviality result.

Some passages of Bennett (2003) and Adams (1975) indicate that problem here is not so much the notion of truth applied to conditionals, but rather the notion of probability of truth. Bennett (2003), for example, argues that the main problem of truth-value approaches is that they "all assume that indicative conditionals are normal propositions whose probability is a probability of being true" (p. 59). Adams (1975), for example associates with truth-value approaches that "the probability of a proposition is the same as the probability that it is true" (p. 2, italics removed).[15] A closer look at Bennett's (2003, p. 103) argumentation and D. Lewis' triviality proof reveals that it is not so much the notion of truth that is problematic, but rather the assignment of probability to compound propositions. In truth-value approaches using the full language $\mathcal{L}_{\mathrm{KL}}$ conditional proposition can be freely embedded into larger structures. In the restricted languages $\mathcal{L}_{\mathrm{KL}^-}$ and $\mathcal{L}_{\mathrm{rKL}^*}$ this is, however, not possible. D. Lewis' (1976) triviality proof shows that using language $\mathcal{L}_{\mathrm{KL}}$ rather than its restricted versions makes probability assignments – provided Assumption 3.42 or 3.6.1* hold – trivial. The probability of the truth of a proposition is, hence, rather associated with assigning probabilities to the full language and not with the notion of truth. We should, therefore, again distinguish between the following two properties of (normal) propositions (see Section 3.4.3): (a) having truth-value and (b) being freely embeddable into larger structures, for example by means of conjunctions etc. (Bennett, 2003, p. 95).

[15]Note that Adams (1975) uses the term 'truth-conditional' to describe truth-value approaches. He explicitly includes possible worlds approaches such as Stalnaker (1968) under truth-conditional approaches (p. 7).

What Bennett seems to address on the basis of D. Lewis' triviality result is factor (b), but not factor (a).

3.6.5 Conclusion

Note that we could not identify any reason why D. Lewis' (1976) triviality results counts against truth-value approaches and in particular possible worlds semantics, such as CS-semantics (Chellas, 1975; Segerberg, 1989). The problem is instead specific to probabilistic semantics for conditional logics. Probabilistic approaches increase the number of admissible values for formulas. They, however, retain extended value functionality from the truth-functional material implication approach (cf. Section 3.5.4). Due to the extended value functionality, probabilistic semantics seems not to be expressive enough to account for compounds and nestings of conditionals given assumptions 3.42. Possible worlds semantics allow similar to the material implication approach only for truth-value for formulas. They, however, differ from the material implication approach and probabilistic approaches insofar as they give up both value functionality and extended value functionality. This does not seem to create the same problems as possible worlds semantics. So, we are not aware of any similar triviality result for possible worlds semantics. Due to those facts we cannot sensibly argue that the probabilistic approach is the more general one of both, the more as we can (but need not) restrict the language in an analogous way for possible worlds semantics as for probabilistic semantics.

3.7 Bennett's Argument against Truth-Value Semantics and Objective Probabilistic Semantics

In his Chapters 6 and 7 Bennett (2003) provides an intricate argument against truth-value approaches and objective probabilistic approaches to conditionals. For that purpose he considers pairs of conditionals of the following form:

(a) $\alpha \mathrel{\Box\!\!\rightarrow} \beta$

(b) $\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta$

His main line of argument is that for certain types of situations – he calls them Gibbardian stand-offs (c.f. Gibbard, 1980, p. 231f) – both conditionals (a) and (b) cannot be assigned any truth-values in a non-arbitrary way. In addition, no viable objective probability assignment is possible in those cases. Since – as he argues – a vast majority of conditionals with false antecedent are of this type (Bennett, 2003, p. 87), an account based on truth-values or objective probability assignments is not able to describe conditionals in natural language adequately.

We, first, reconstruct Bennett's intricate argument against truth-value interpretations of conditionals. Then, we describe in which way Bennett's argument fails for truth-value account as in possible worlds semantics. Then we also address in which way objective frequency based approaches (see Section 3.4.2) escape Bennett's argument and, finally, we aim to show that his argument applies to subjective probabilistic approaches such as Adams (1975), which Bennett does not seem to consider.

3.7.1 Bennett's Gibbardian Stand-Offs Argument

Let us now take a look at those Gibbardian stand-offs. These are situations, in which one is equally justified in endorsing a conditional of the form (a) or (b). Bennett (2003) takes those from Gibbard (1980, p 231f). For the sake of clarity he, however, provides his own example for a Gibbardian stand-off situation:

> "Top Gate holds back water in a lake behind a dam; a channel running down from it splits into two distributaries, one (blockable by East Gate) running eastwards and the other (blockable by West Gate) running westwards. The gates are connected as follows: if east lever is down, opening Top Gate will open East Gate so that the water will run eastwards; and if west lever is down, opening Top Gate will open West Gate so that the water will run westwards. On the rare occasions when both levers are down, Top Gate cannot be opened because the machinery cannot move three Gates at once.
>
> Just after the lever-pulling specialist has stopped work, Wesla knows that west lever is down, and thinks 'If Top Gates opens, all the water will run westwards'; Esther knows that east lever is down, and

thinks 'If Top Gate opens, all the water will run eastwards'." (Bennett, 2003, p. 85)

The following pair of conditionals represent – so Bennett – given the background story a Gibbardian stand-off:

E24 If Top Gates opens, all the water will run westwards.
E25 If Top Gate opens, all the water will run eastwards.

Note that, strictly speaking, conditionals E24 and E25 do not correspond to the formulas (a) and (b): Neither the consequent of E22 is a natural language negation of E25, nor the other way around. The best we might achieve – using side constraints of the story – is that both conditionals cannot simultaneously be true. Note that substituting the consequent of E24 by a natural language negation of a consequent of E25 (or the other way around) does not work here. Let us, however, for the sake of Bennett's argument abstract from this difficulty here and suppose that the consequent of E24 can be substituted for a natural language negation of E25 (and the other way around).[16] The Gibbardian stand-off situation gives us, then, that E24 and E25 correspond to (a) and (b) and are both equally justified by the background story.

For the conditional formulas (a) and (b) in principle the following truth-value combinations are possible (Bennett, 2003, p. 94):

	(a)	(b)
(1)	1	1
(2)	1	0
(3)	0	1
(4)	0	0

Here '1' and '0' correspond to 'true' and 'false' respectively. If we take all information of a Gibbardian stand-off situation into account, both conditionals (a)

[16]We do not argue that Gibbardian stand-offs do not exist. We, however, believe that completely parallel, but plausible Gibbardian stand-offs with conditionals directly corresponding to the formulas (a) and (b) are extremely difficult to construct. Note that Bennett's difficulties to come up with such a Gibbardian stand-off counts against the empirical significance of such situations.

and (b) are equally justified. Then, situations (2) and (3) cannot arise. Bennett (2003) goes, then, on to argue that – given we take all information into account – neither condition (1) nor condition (4) can be the case either. Here the principles CNC (see Section 3.5.3) and CEM″ (“Conditional Excluded Middle”, see Section 3.2.5) contradict condition (1) and (4), respectively. To make the discussion more perspicuous, let us repeat both principles here:

CEM″ $\neg(\neg(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge \neg(\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta))$
CNC $\neg((\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta))$

To argue against conditions (1) and (4), Bennett (2003, p. 84) only refers to principle CNC directly. To argue that (4) is not possible, Bennett rather uses the specifics of the example. Note that principle CNC corresponds to the strong consistency criterion discussed in Sections 3.1 and 3.2.5.

For his argument Bennett distinguishes between objective and subjective approaches to conditionals. Both differ insofar as objective approaches take the whole truth of the situation into account (as we did above), while subjective approaches only consider an agent’s beliefs and most typically do *not* use the whole truth (see Bennett, 2003, p. 80f).[17]

Bennett divides his argument into two parts. In the first part he argues against objective (truth-value) approaches (pp. 80–88), in the second part he addresses subjective truth-value accounts (pp. 88–93). Let us denote the first and the second part of his arguments by *A* and *B*, respectively and let us now focus part A of Bennett’s argument. Based on their respective beliefs Wesla and Esther are justified to believe conditionals E24 and E25, respectively. (The example is specifically designed that this is the case.) If one takes the whole information into account one would be justified to endorse E24 and E25, which correspond to (a) and (b). According to CNC, however, (a) and (b) cannot both be true. Bennett (2003, p. 83–88), thus, concludes that objective truth-value accounts cannot describe Gibbardian stand-offs adequately.

Let us now focus on part B of Bennett’s argument. According to this argument a subjective truth-value account runs into following problem: It has to relativize

[17]Note that is debatable whether an objective approach in Bennett’s terminology is applicable at all, since it is doubtful whether any worldly agent can know the whole truth about any state of affairs.

its truth-value according to a belief system, otherwise it is not clear to which subjective position it refers to (p. 88f). Bennett (2003, pp. 89–91) argues that one can use for that purpose either a self-description expression (i.e. indicating that it is “my belief system”) or alternatively a fixed-reference term (i.e. Wesla’s belief system). Since in both subjective approaches Wesla’s and Esther’s belief systems are considered separately, (a) or (b) can be true, but both need not be true. So the subjective approach escapes contradicting CNC.

Bennett (2003), however, argues that both subjective truth-value approaches fail for other reasons. The self-reference approach cannot work, since in natural language conditionals are most typically not used that way. In many situations assertions of conditionals do not serve as reports on the agent’s beliefs (what the self-description approach predicts). Instead they often are used as claims that the conditionals in fact hold (Bennett, 2003, p. 90). (Note that Bennett phrases his argumentation here in terms of probabilities rather than truth.)

Due to Bennett (2003) the fixed-reference approach is also bound to fail. In that approach any conditional Wesla asserts has a fixed reference, namely Wesla’s belief system. If someone asks ‘If Top Gate opens, where does the water go?’, Wesla would answer ‘west’ and Esther ‘east’. Both just refer to their fixed belief system. So, how is communication between Esther and Wesla possible, if both always refer to distinct belief systems when asserting conditionals (Bennett, 2003, p. 91)? Note here that in both approaches – the self-reference and fixed-reference account – it is not possible to pick out the same belief system for both Wesla and Esther, since this would result again in a belief system, in which two conditionals of the form (a) and form (b) are both asserted. This would, however, contradict CNC.

3.7.2 Truth-Value Accounts

Why should Bennett’s argument hold for truth-value approaches? Due to the pivotal importance of CNC to Bennett’s argument, Bennett addresses only those truth-value accounts for conditionals, for which CNC is valid. Since CNC does not hold for the material implication approach (Bennett, 2003, p. 84), Bennett’s argument does not apply to it. Bennett, then, explicitly focuses on possible worlds approaches as described in Davis (1979). According to Bennett (2003, p. 84) those approaches tie the truth of a conditional $\alpha \Box\!\!\rightarrow \beta$ to the truth of β at a certain world. Bennett, then, argues that, since β and $\neg\beta$ cannot both be true at a single possible world, CNC is valid in those approaches (p. 84). The problem with Bennett's reasoning here is that it is based on false premises.

Davis (1979, p. 544) explicitly refers for his account of indicative conditionals to Stalnaker's (1968) approach. In his informal discussion Stalnaker argued that a consistency-adjustment should be used in his version of the Ramsey-test (see Section 3.2.5). His own system (Stalnaker, 1968, Stalnaker & Thomason, 1970), however, does not endorse CNC (see Section 3.2.5). This can be seen from the fact that Stalnaker uses for his semantics also an absurd world λ, in which any proposition is true (see Section 3.2.2). So, in this absurd world both β and $\neg\beta$ are true.

We are, in addition, not aware of any conditional logic system, in which CNC is valid in its strict form. This is due to the fact that it contradicts the principle Refl, which all conditional logics we are aware of, endorse (see Section 3.5.3). That leaves the principle RCNC, which restricts principle CNC to cases, in which the antecedent is p.c.-consistent (see Section 3.5.3). In standard systems of (indicative and counterfactual) conditional logics such as Adams (1965, 1966, 1977), Stalnaker (1968), Stalnaker and Thomason (1970) and D. Lewis (1973/2001) neither CNC nor RCNC is valid, including system **CK** (see Section 4.2.6). Hence, Bennett does not address theses approaches here.

Adams (1975) and Schurz (1997b) are the only conditional logic systems we are aware of which make RCNC valid. Note that CNC neither holds in Adams (1975) nor in Schurz (1997b) (see 3.5.3). These approaches, however, deviate from standard approaches to conditionals insofar as they are default logics in the sense of Section 2.2.

3.7.3 Objective Probabilistic Approaches

Bennett (2003) also uses part A of his argument against objective probabilistic accounts of conditionals. For that purpose he discusses objective Ramsey-test interpretations (pp. 78-80) and arrives – in his discussion – at the only in principle viable objective Ramsey-test interpretation. This Ramsey-test interpretation takes the whole truth into account. This approach parallels Bennett's subjective version of the Ramsey-test insofar as these do not take the whole truth into account, but everything the agent believes (p. 80). Bennett, then, argues that also this approach is misguided. Applying the objective Ramsey-test – which involves a consistency adjustment if necessary (Bennett, 2003, p. 80) – Bennett argues that both conditional probabilities $P(\beta|\alpha)$ and $P(\neg\beta|\alpha)$ cannot be greater than .5 and that, hence, both cannot be rationally accepted by at single agent at a single occasions. Hence, CNC holds, Bennett (2003, p. 84) argues.

In Section 3.4.1 we described an objective frequency-based approach. Although we can also employ conditional probabilities for a semantic of conditionals formulas (i.e. Schurz, 1997b, 2005), this approach on the one hand does not follow a Ramsey-test interpretation and on the other hand does not need a justification in terms of the Ramsey-test (see 3.4.2). Hence, Bennett's argument does not apply to this type of semantics.

3.7.4 Subjective Probabilistic Approaches

In addition, Bennett does not provide reasons why his argumentation against objective approaches described above does not apply for subjective probabilistic accounts as well. Note that Bennett does not exempt subjective probabilistic interpretations of conditionals from the list of systems, for which CNC applies. He, furthermore, seems to incorporate a consistency requirement for his Ramsey-test interpretation (see Section 3.2.5). Hence, Bennett sees Adams' (1975) system as a plausible candidate for an appropriate subjective probabilistic account of conditionals (cf. also Bennett, 2003, p. 129). We rather use here Adams' p-validity criterion and consistency criterion, since Bennett (2003) explicitly uses Adams conditional logic (p. 127) and does not discuss threshold approaches such as Hawthorne and Makinson (2007) (see also Section 3.4).

In Adams (1975) the principle RCNC is, however, valid (see Section 3.5.3). Accordingly any set of formulas containing (a) and (b) where the antecedent of (a) and (b) are p.c.-consistent is p-inconsistent.[18]. Moreover, we saw in Section

[18] Informally, a set of conditionals is p-consistent in Adams (1975) we can assign arbitrarily

3.2.5 that a version of CEM″ holds in Adams system if we allow the language for negations of conditional formulas. (We have to extend the language, otherwise we cannot express this principle directly in that system (see Section 3.2.5).) Accordingly, then either (a) or (b) has to be endorsed. Hence, only (a) or (b) but not both may p-consistently be endorsed in Adams' (1975) (extended) logical system. Note that the situation is such that, if all facts are taken into account no conditional (a) and (b) should be assigned a higher probability than the other one. These are, however, the only p-consistent situations possible. So the only escape possible in terms of not taking in the whole truth of the situation is to use a subjective approach, which leads to part B of Bennett's argument.

In a subjective approach it is possible to take only an agent's belief state into account and *not* the whole truth. This way subjective approaches can avoid contradicting CNC in Gibbardian stand-offs. Such an approach, however, falls to the same criticism as the subjective truth-value approaches in part B of Bennett's argument: In order to succeed, the subjective probabilistic approaches also have to refer to the reasoner's belief system. Otherwise it is not clear who's beliefs they refer to. It cannot refer to the whole truth, since otherwise it would contradict CNC and RCNC. Parallel to part B of Bennett's argument described above, the self-reference can be done via self-description (i.e. indicating that it is "my belief system") or a fixed-reference (i.e. Wesla's belief system). Both, however, run into the same difficulties as we saw in Section 3.7.1.

Note here that only subjective probabilistic approaches, which endorse CNC and RCNC are in danger of being prone to Bennett's argument. The Adams (1975) system is clearly one of them, since it employs a consistency criterion. In Adams' system $\mathbf{P}^*$ (Adams, 1965, 1966, 1977) and system $\mathbf{P}^+$ (Adams, 1986; Schurz, 1998; see Section 3.5.1), however, no such assumption is made. Hence, both systems escape Bennett's criticism.

high probabilities smaller than 1 to all members of that set, such that the probability assignment is proper (Adams, 1975, p. 51) (cf. also 3.5.3)

3.7.5 Summary

We showed that Bennett's (2003) argument addresses only approaches, which employ strong consistency conditions (cf. Section 3.2.5). Strong consistency conditions are, however, not an element of the great majority of conditional logic systems such as Adams (1965, 1966, 1977), Stalnaker (1968), Stalnaker and Thomason (1970), D. Lewis (1973/2001) and system **CK** (see Section 4.2.6). Furthermore, Bennett's Gibbardian stand-off argument does not address objective frequency-based approaches to conditionals as Schurz (1997b, 2005). Interestingly the only system we are aware of, for which the argument seems to have some force, is the subjective probabilistic approach of Adams (1975).

3.8 Conclusion

We discussed and defended in the previous section an truth-value account in terms of possible worlds semantics. We observed that two main arguments against truth-value analyses described by Bennett (2003, p. 102–104) do not seem to be decisive against possible worlds analyses of indicative conditionals as described in Chapters 4–7.

Bennett's argument (b) is, however, not plausible for the following reasons: First, there are types of semantics for conditionals, which do not rely on Ramsey-test interpretations, such as objective frequency-based interpretations of conditionals. Second, such a consistency criterion is neither a necessary part of the Ramsey-test – regardless whether probabilistically interpreted or described in terms of possible worlds – nor of the Stalnaker thesis. In fact, a wide range of conditional logics with possible worlds semantics such as CS-semantics (Chellas, 1975; Segerberg, 1989) or alternatives (Stalnaker, 1968; D. Lewis, 1973/2001; Kraus et al., 1990; Lehmann & Magidor, 1992), and probabilistic semantics (e.g. Adams, 1965, 1966, 1977) do not have such a consistency criterion. Among the much less common conditional logic system are Adams (1975) and Schurz (1997b). The latter type of conditional logic system, however, deviates from other conditional logics insofar as it is in addition a non-monotonic default logic.

Regarding (a) we agree that D. Lewis' (1976) triviality result poses a problem, given one accepts a subjective probabilistic semantics, which includes the Stalnaker thesis, namely that the probability of a conditional equals the respective conditional probability (formally: $P(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = P(\beta|\alpha)$). It is, however, quite a different question whether D. Lewis' triviality result bears on possible worlds semantics, such as CS-semantics directly, viz. if one does not accept a subjective probabilistic framework and the assumptions made by Bennett and others.

Part II

Formal Results for Chellas-Segerberg Semantics

Chapter 4

Formal Framework

4.1 Why Chellas-Segerberg Semantics?

In this and the following chapters we investigate and discuss the Chellas-Segerberg (CS) semantics (Chellas, 1975; Segerberg, 1989) from a formal and a philosophical perspective. We, however, have not yet fully answered the question why we pick CS-semantics rather than an alternative possible worlds semantics for conditionals, such as Lewis (1973/2001) models (see Section 3.2.3), the relational semantics of Burgess (1981), Kraus et al. (1990) and Lehmann and Magidor (1992) (see Section 3.2.3) or the propositional neighborhood semantics also described by Chellas (1975, pp. 144–147). Let us, hence, provide some reasons for choosing CS-semantics.

First, the proof-theoretic system corresponding to basic CS-semantics is by far weaker than the minimal systems investigated by D. Lewis (1973/2001), Burgess (1981), Kraus et al. (1990) and Lehmann and Magidor (1992). In particular, CS-semantics allows to describe weaker systems than those accounted for in these alternative semantics. In addition, it is possible to describe strong conditional logic systems within CS-semantics by superimposing additional axioms. We shall in this thesis even describe conditional logics, which make conditionals as strong as the material implication (see Section 7.3.4).

A second reason for choosing CS-semantics is that this semantics allows – as we shall see – for a simple, but intuitive interpretation of conditionals in terms of both a modified Ramsey-test and an objective alethic interpretation (see Section 7.1). The modified Ramsey-test interpretation can, then, be used to account for both indicative and counterfactual conditionals (see Section 3.3). In contrast, ordering semantics, such as in D. Lewis (1973/2001), Burgess (1981), Kraus et al. (1990) and Lehmann and Magidor (1992), and the semantics of Stalnaker (1968) employ an ordering semantics approach and, therefore, do not lend themselves easily into a Ramsey-test interpretation (see Section 3.2). Moreover, we do not see that the propositional neighborhood semantics (Chellas, 1975, pp. 144–147) also allows for a natural interpretation in terms of the Ramsey-test.

Finally, CS-semantics is also interesting from a formal point of view, since it deviates in essential ways from standard Kripke semantics and its multi-modal extensions (see Section 4.3.1–4.3.3). Moreover, the formal properties of CS-semantics are, except for Chellas (1975) and Segerberg (1989), virtually unexplored. Before we, however, focus on CS-semantics, let us first describe some proof-theoretic notions used in this thesis.

4.2 Proof-Theoretic Notions

4.2.1 Languages $\mathcal{L}_{\mathrm{KL}}$, $\mathcal{L}_{\mathrm{KL}^-}$, $\mathcal{L}_{\mathrm{rKL}}$, $\mathcal{L}_{\mathrm{rKL}^*}$ and $\mathcal{L}_{\mathrm{rrKL}}$

In this section we define several conditional logic languages. We start with the full language $\mathcal{L}_{\mathrm{KL}}$ and describe, then, the more restricted languages $\mathcal{L}_{\mathrm{KL}^-}$, $\mathcal{L}_{\mathrm{rKL}}$, $\mathcal{L}_{\mathrm{rKL}^*}$ and $\mathcal{L}_{\mathrm{rrKL}}$. Here KL, KL$^-$, rKL, rKL* and rrKL stand for 'conditional logic' and 'qualified conditional logic', 'restricted conditional logic' 'variant of the restricted conditional logic' and 'more restricted conditional logic', respectively. All languages are based on the same propositional vocabulary, but differ with respect to (w.r.t.) the type of expressions, which count as formulas of the respective language.

For our languages we specify the following meta-language expressions: $\neg$ ("Negation"), $\wedge$ ("Conjunction"), $\vee$ ("Disjunction"), $\Rightarrow$ ("Material Implication"), $\Leftrightarrow$ ("Material Equivalence"), $\forall$ ("Universal quantification"), $\exists$ ("Existential quantification"), $=$ ("Identity"), $=_{\mathrm{df}}$ ("Identity by definition") and $\neq$ ("Negation of Identity"). In addition, the meta-language symbols $w, w', w'', \ldots, w^*, w^{**}, \ldots$ (individual variables) and X, Y, Z, X_1, X_2, ..., Y_1, Y_2, ... (variables ranging over sets of individuals) are used. All meta-language expressions serve as abbreviations for respective natural language phrases.

Let us now focus on the vocabulary of our object languages: The propositional variables $p, q, \ldots, p_1, p_2, \ldots, p_1, p_2, \ldots$ denote atomic propositional variables (we restrict ourselves to formal languages with countably but infinitely many atomic propositions); The expressions $\mathcal{PP}$ and *form* denotes the set of all atomic propositional variables (or primitive propositions) and the set of all formulas in a given language, respectively; our propositional logical constants are: $\neg$ (negation) and $\vee$ (disjunction). The symbol $\Box\!\!\rightarrow$ is a two-place modal operator, also called 'conditional operator'.

The expressions $\alpha, \beta, \ldots, \alpha_0, \alpha_1, \alpha_2, \ldots$ stand for arbitrary object language formulas of the respective language. We also use these expressions to refer to formula schemata rather than formulas. Although the former expressions are strictly speaking axiom schemata, for the sake of brevity we shall often refer to them as formulas. In addition, the auxiliary symbols '(' and ')' are used. (We employ parentheses also in the meta-language.) The expressions $\wedge$ (conjunction), $\rightarrow$ (material implication or subjunction), $\leftrightarrow$ (bisubjunction) $\top$ (verum), $\bot$ (falsum) and $\Diamond\!\!\rightarrow$ (conditional possibility operator) are meta-language abbreviations for formulas in the object language. This way we can restrict ourselves to the discussion of a small range of primitive symbols, but use the meta-language abbreviations to make the discussion and proofs more perspicuous.

In particular, following definitions are assumed: $\text{Def}_{\wedge}$: $\alpha \wedge \beta =_{\text{df}} \neg(\neg\alpha \vee \neg\beta)$; $\text{Def}_{\rightarrow}$: $\alpha \rightarrow \beta =_{\text{df}} \neg\alpha \vee \beta$; $\text{Def}_{\leftrightarrow}$: $\alpha \leftrightarrow \beta =_{\text{df}} (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$; $\text{Def}_{\top}$: $\top =_{\text{df}} p \vee \neg p$; $\text{Def}_{\bot}$: $\bot =_{\text{df}} p \wedge \neg p$; $\text{Def}_{\Diamond\!\rightarrow}$: $\alpha \Diamond\!\!\rightarrow \beta =_{\text{df}} \neg(\alpha \Box\!\!\rightarrow \neg\beta)$. The definitions apply to formulas in a language $\mathcal{L}$, if the expressions, by which they are defined, can occur in formulas of $\mathcal{L}$. We will sometimes use the concept of boolean combinations of formulas in $\mathcal{L}$. Boolean combinations of formulas are combinations of formulas by means of the connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$ and $\leftrightarrow$ (where formulas with the connectives $\wedge$, $\rightarrow$ and $\leftrightarrow$ are again meta-language abbreviations of corresponding object language formulas).

Let us now describe the set of formulas of the languages $\mathcal{L}_{\text{KL}}$, $\mathcal{L}_{\text{KL}^-}$, $\mathcal{L}_{\text{rKL}}$, $\mathcal{L}_{\text{rKL}}$ and $\mathcal{L}_{\text{rrKL}}$. The full language $\mathcal{L}_{\text{KL}}$ resembles p.c. insofar as it allows the conditional operator $\Box\!\!\rightarrow$ to combine in formulas in any way the material implication $\rightarrow$ recombines in p.c. In particular, any type of boolean combination of conditionals (e.g. $(\alpha \Box\!\!\rightarrow \beta) \wedge \gamma$) and nestings of conditionals (e.g. $\alpha \Box\!\!\rightarrow (\beta \wedge (\gamma \Box\!\!\rightarrow \delta))$) are formulas of $\mathcal{L}_{\mathrm{KL}}$. The set of formulas of the language of $\mathcal{L}_{\mathrm{KL}}$ can be formally described the following way:

(a) $p, q, \ldots, p_1, p_2, \ldots$ are formulas of $\mathcal{L}_{\mathrm{KL}}$.
(b) If α and β are formulas of $\mathcal{L}_{\mathrm{KL}}$, then $(\neg\alpha)$ and $(\alpha \vee \beta)$ are formulas of $\mathcal{L}_{\mathrm{KL}}$.
(c) if α and β are formulas of $\mathcal{L}_{\mathrm{KL}}$, then $(\alpha \Box\!\!\rightarrow \beta)$ is a formula of $\mathcal{L}_{\mathrm{KL}}$.
(d) No other expression is a formula of $\mathcal{L}_{\mathrm{KL}}$.

The languages $\mathcal{L}_{\mathrm{KL}^-}$, $\mathcal{L}_{\mathrm{rKL}}$, $\mathcal{L}_{\mathrm{rKL}^*}$ and $\mathcal{L}_{\mathrm{rrKL}}$ differ from $\mathcal{L}_{\mathrm{KL}}$ insofar as they do not allow to represent nested conditionals: Any conditional formula $\alpha \Box\!\!\rightarrow \beta$ in languages $\mathcal{L}_{\mathrm{KL}^-}$, $\mathcal{L}_{\mathrm{rKL}}$, $\mathcal{L}_{\mathrm{rKL}^*}$ and $\mathcal{L}_{\mathrm{rrKL}}$ requires α and β not to contain any instance of the conditional operator $\Box\!\!\rightarrow$. The language $\mathcal{L}_{\mathrm{KL}^-}$ differs from the languages $\mathcal{L}_{\mathrm{rKL}}$, $\mathcal{L}_{\mathrm{rKL}^*}$ and $\mathcal{L}_{\mathrm{rrKL}}$ insofar as $\mathcal{L}_{\mathrm{KL}^-}$ allows for both conditional and unconditional formulas, while the set of formulas of $\mathcal{L}_{\mathrm{rKL}}$, $\mathcal{L}_{\mathrm{rKL}^*}$ and $\mathcal{L}_{\mathrm{rrKL}}$ do not contain formulas without instances of the conditional operator $\Box\!\!\rightarrow$. Hence, for example, $p \wedge q$ is a formula of the language $\mathcal{L}_{\mathrm{KL}^-}$, but not of the languages $\mathcal{L}_{\mathrm{rKL}}$, $\mathcal{L}_{\mathrm{rKL}^*}$ and $\mathcal{L}_{\mathrm{rrKL}}$. Language $\mathcal{L}_{\mathrm{KL}^-}$, however, neither allows for boolean combinations of conditional formulas on the one hand and conditional and unconditional formulas on the other hand. The language $\mathcal{L}_{\mathrm{KL}^-}$ is, then, formally described the following way:

(a^-) $p, q, \ldots, p_1, p_2, \ldots$ are formulas of $\mathcal{L}_{\mathrm{prop}}$.
(b^-) If α and β are formulas of $\mathcal{L}_{\mathrm{prop}}$, then $(\neg\alpha)$ and $(\alpha \vee \beta)$ are formulas of $\mathcal{L}_{\mathrm{prop}}$.
(c^-) No other expression is a formula of $\mathcal{L}_{\mathrm{prop}}$.
(d^-) If α is a formula of $\mathcal{L}_{\mathrm{prop}}$, then α is a formula of $\mathcal{L}_{\mathrm{KL}^-}$.
(e^-) If α and β are formula of $\mathcal{L}_{\mathrm{prop}}$, then $(\alpha \Box\!\!\rightarrow \beta)$ is a formula of $\mathcal{L}_{\mathrm{KL}^-}$.
(f^-) No other expression is a formula of $\mathcal{L}_{\mathrm{KL}^-}$.

We could specify languages, which also include boolean combinations of conditional formulas and boolean combinations of conditional and unconditional formulas. We will, however, not employ these type of languages in this thesis and shall, hence, not discuss these types of languages any further.

The languages $\mathcal{L}_{\mathrm{rKL}}$, $\mathcal{L}_{\mathrm{rKL}^*}$ and $\mathcal{L}_{\mathrm{rrKL}}$ differ from each other to which extent they allow for boolean combinations, where $\mathcal{L}_{\mathrm{rKL}}$, $\mathcal{L}_{\mathrm{rKL}^*}$ and $\mathcal{L}_{\mathrm{rrKL}}$ allow for any boolean combination of conditional formulas, only for negations of conditional formulas and no boolean combinations of conditional formulas, respectively. The set of formulas of $\mathcal{L}_{\mathrm{rKL}}$ can accordingly be determined by the following set of clauses:

(a$_1$)–(c$_1$) Conditions (a$^-$)–(c$^-$), respectively, from above.

(d$_1$) If α, β, γ and δ are formulas of $\mathcal{L}_{\mathrm{prop}}$, then $(\alpha \mathrel{\Box\!\!\rightarrow} \beta)$, and $((\alpha \mathrel{\Box\!\!\rightarrow} \beta) \vee (\gamma \mathrel{\Box\!\!\rightarrow} \delta))$ are formulas of $\mathcal{L}_{\mathrm{rrKL}}$.

(e$_1$) Nothing else is a formula of $\mathcal{L}_{\mathrm{rKL}}$.

For the specification of the set of formulas of language $\mathcal{L}_{\mathrm{rKL}^*}$ we have to replace conditions (d$_1$) and (e$_1$) in the definition of formulas of $\mathcal{L}_{\mathrm{rKL}}$ by the following clauses d$_2$ and e$_2$, respectively:

(d$_2$) If α, β are formulas of $\mathcal{L}_{\mathrm{prop}}$, then $(\alpha \mathrel{\Box\!\!\rightarrow} \beta)$ and $(\neg(\alpha \mathrel{\Box\!\!\rightarrow} \beta))$ are formulas of $\mathcal{L}_{\mathrm{rKL}^*}$.

(e$_2$) Nothing else is a formula of $\mathcal{L}_{\mathrm{rKL}^*}$.

For definition of formulas of language $\mathcal{L}_{\mathrm{rrKL}}$ we analogously have to replace conditions d$_1$ and e$_1$ in the definition of the set of formulas of $\mathcal{L}_{\mathrm{rKL}}$ by the following clauses d$_3$ and e$_3$, respectively:

(d$_3$) If α and β are formulas of $\mathcal{L}_{\mathrm{prop}}$, then $(\alpha \mathrel{\Box\!\!\rightarrow} \beta)$ is a formula of $\mathcal{L}_{\mathrm{rrKL}}$.

(e$_3$) No other expression is a formula of $\mathcal{L}_{\mathrm{rrKL}}$.

Let us now discuss, which systems in the conditional logic literature draw on the languages $\mathcal{L}_{\mathrm{KL}}$, $\mathcal{L}_{\mathrm{KL}^-}$,$\mathcal{L}_{\mathrm{rKL}}$, $\mathcal{L}_{\mathrm{rKL}^*}$ and $\mathcal{L}_{\mathrm{rrKL}}$: D. Lewis' (1973/2001, p. 118) counterfactual logic and the indicative and counterfactual conditional logic of Stalnaker (1968, p. 105) and Stalnaker and Thomason (1970, p. 24) are formulated w.r.t. the full language $\mathcal{L}_{\mathrm{KL}}$, whereas Adams' (1965, p. 184; 1966, p. 270; 1977, p. 186f) conditional logic system is based on language $\mathcal{L}_{\mathrm{KL}^-}$. Adams (1975, p. 46), however, uses a modified version of language $\mathcal{L}_{\mathrm{KL}^-}$, namely language $\mathcal{L}^{\mathrm{cons}}_{\mathrm{KL}^-}$, in which all conditional formulas are required to have p.c.-consistent antecedents. In contrast, Adams (1986) effectively employs language $\mathcal{L}_{\mathrm{rKL}^*}$: He allows only for inferences of type $\{\alpha_1 \wedge \ldots \wedge \alpha_m\} \vdash \beta_1 \vee \ldots \vee \beta_n$ (p. 259), where $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$

are simple conditionals and it is the case that $m, n \in \mathbb{N}_0$ (p. 256f).[1] Schurz (1998, p. 84f), however, shows that in a probabilistic system S this type of inference schema can also be expressed in language $\mathcal{L}_{\mathrm{rKL}}$, which allows (a′) in general for disjunctions of simple conditionals and also endorses (b). '$\mathcal{L}_{\mathrm{rKL}}$' stands here for 'restricted version of language $\mathcal{L}_{\mathrm{KL}}$'. Language $\mathcal{L}_{\mathrm{rKL}}$ is also formally specified in Section 4.2.1 and allows only for simple and negated conditionals). For that purpose Schurz (1998) has to assume that in S (a) the deduction theorem holds, which states the following: $\Gamma \vdash_{\mathbf{L}} \alpha$ holds iff $\exists \Gamma_f \subseteq \Gamma$: $\vdash_{\mathbf{L}} \bigwedge \Gamma_f \rightarrow \alpha$, where Γ refers to an arbitrary set of formulas, Γ_f denotes a finite set of formulas and $\bigwedge X$ stands for the conjunction of all elements in the set of formulas X in a pre-specified order (see also Schurz, 2002a, p. 453). Moreover, Schurz (1998) has to presuppose (b) that all p.c.-valid inferences are S-valid. Inferences of type $\{\alpha_1 \wedge \ldots \wedge \alpha_m\} \vdash \beta_1 \vee \ldots \vee \beta_n$ are, then, S-valid exactly if all inferences in $\{\{\alpha_1 \wedge \ldots \wedge \alpha_m\} \vdash \beta_1, \ldots, \{\alpha_1 \wedge \ldots \wedge \alpha_m\} \vdash \beta_n\}$ are S-valid. Moreover, Schurz's (1998, p. 84f) proof gives us also all inferences in $\mathcal{L}_{\mathrm{rKL}^*}$ in a probabilistic system S can not just be expressed by sets of inferences in $\mathcal{L}_{\mathrm{rKL}}$, but rather by (simple) inferences in $\mathcal{L}_{\mathrm{rKL}}$: Any inference in $\mathcal{L}_{\mathrm{rKL}}$ (with a finite set of premises) is valid iff inference of type $\{\alpha_1 \wedge \ldots \wedge \alpha_m \wedge \beta_1 \wedge \ldots \wedge \beta_n\} \vdash \gamma$ with $m, n \in \mathbb{N}_0$ and specific formulas $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \alpha_n, \gamma$ is valid iff $\{\alpha_1 \wedge \ldots \wedge \alpha_m\} \vdash \beta_1 \vee \ldots \vee \beta_n \vee \gamma$ is valid. This equivalence result, however, presupposes again that (a) and (b) hold.

W.r.t. to the probabilistic system S points (a) and (b) are assumed to hold and we, in addition, presuppose (c), namely that in the probabilistic system S inferences are restricted to a finite number of premises. Criterion (c) is presupposed for probabilistic system, such as system $\mathbf{P}^{+}$, since these are non-compact (Schurz, 1998, p. 84). Finally we require that (d) $\top \mathrel{\Box\!\!\rightarrow} \top$ is valid in system S. On the basis of our assumptions regarding inferences of a probabilistic system S in the language $\mathcal{L}_{\mathrm{rKL}}$ take in general the form $\Gamma_f \vdash \beta$, where Γ_f represents a finite set of formulas in $\mathcal{L}_{\mathrm{rKL}}$. Hence, any inference in such a system can be described by $\alpha \vdash_{\mathbf{L}} \beta$, where α is the conjunction of all elements in Γ (formally: $\alpha = \bigwedge \Gamma$). Moreover, since language $\mathcal{L}_{\mathrm{rKL}}$ allows only for boolean combinations of condi-

[1] In case of $m = 0$ the conjunctive premise is interpreted as $\top \mathrel{\Box\!\!\rightarrow} \top$ and in case of $n = 0$ the disjunctive conclusion is interpreted as $\top \mathrel{\Box\!\!\rightarrow} \bot$. We will, however, ignore this complication here (see also Section 3.4.3).

tional formulas, we can transform α and β into disjunctive and conjunctive normal forms $\bigvee_i \bigwedge_j \alpha_{ij}$ and $\bigwedge_k \bigvee_l \beta_{kl}$, respectively, where $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, o\}$ and $l \in \{1, \ldots, p\}$ and α_{ij} and β_{kl} represent simple or negated conditional formulas.[2] This observation implies that any inference in a probabilistic system S can be represented in the following equivalent form: $\bigvee_i \bigwedge_j \alpha_{ij} \vdash_{\mathbf{L}} \bigwedge_k \bigvee_l \beta_{kl}$. This inference is valid in a probabilistic system $\mathbf{L}$ iff $\{\bigwedge_j \alpha_{1j} \vdash_{\mathbf{L}} \bigvee_l \beta_{1l}, \ldots, \bigwedge_j \alpha_{mj} \vdash_{\mathbf{L}} \bigvee_l \beta_{ol}\}$ is valid in $\mathbf{L}$ iff $\{(\bigwedge_j \alpha_{1j}) \wedge (\bigwedge_l \neg\beta_{1l}) \vdash_{\mathbf{L}} \neg(\top \mathrel{\Box\!\!\rightarrow} \top), \ldots, (\bigwedge_j \alpha_{pj}) \wedge (\bigwedge_l \neg\beta_{ol}) \vdash_{\mathbf{L}} \neg(\top \mathrel{\Box\!\!\rightarrow} \top)\}$. Hence for all probabilistic systems, for which assumptions (a)–(d) hold (e.g. $\mathbf{P}^+$), languages $\mathcal{L}_{\mathrm{rKL}}$ and $\mathcal{L}_{\mathrm{rKL}^*}$ are equally expressive. Note in that context that Schurz (1998) neither uses language $\mathcal{L}_{\mathrm{rKL}}$ nor $\mathcal{L}_{\mathrm{rKL}^*}$, since he allows for unconditional formulas in his language (p. 85). Moreover, since in Schurz's (1998) language inferences can involve disjunctions of conditionals (p. 85), his language is in fact less restricted than language $\mathcal{L}_{\mathrm{KL}^-}$ (which does not allow for any boolean combinations of conditional formulas).

The language $\mathcal{L}_{\mathrm{rKL}^*}$ is effectively used by Lehmann and Magidor (1992, p. 5; see Section 2.2.7). Hence, Lehmann and Magidor (1992) can also represent inferences of probabilistic systems w.r.t. language $\mathcal{L}_{\mathrm{KL}^-}$ or inferences described by Adams (1986) In contrast, Kraus et al. (1990) do not explicitly indicate whether they effectively employ the language $\mathcal{L}_{\mathrm{rKL}^*}$ or the language $\mathcal{L}_{\mathrm{rrKL}}$. In particular they do state whether they regard negations of conditionals also as formulas of the object language or only as a meta-language abbreviation (cf. Footnote 13 in Section 2.2.7). Note that the use of the restricted languages $\mathcal{L}_{\mathrm{KL}^-}$, $\mathcal{L}_{\mathrm{rrKL}}$ and $\mathcal{L}_{\mathrm{rrKL}}$ has a profound impact on a system's soundness and completeness properties (see Section 3.4.3). We, however, feel that the use of the full language has several advantages over a more restricted version (see Section 3.6.3). Hence, we formulate the model-theory and proof-theory investigated in Chapters 6 and 5 in the full language $\mathcal{L}_{\mathrm{KL}}$.

Furthermore, for the sake of perspicuity, we omit in all languages outer bracket

[2]Our representation of disjunctive [conjunctive] normal forms presupposes that all conjunctions [disjunctions] of simple and negated formulas in the disjunctive [conjunctive] normal form have the same length m [p]. The classical description of conjunctive and disjunctive normal forms does not presuppose that. In order to transform the classical versions of conjunctive and disjunctive normal forms into our version, we can fill up "empty" slots in disjunctions and conjunctions with the formulas $\neg(\top \mathrel{\Box\!\!\rightarrow} \top)$ and $\top \mathrel{\Box\!\!\rightarrow} \top$, respectively.

and other brackets according to following rules: The expressions $\neg$ binds strongest. The operator $\Box\!\!\rightarrow$ is less strong, but bind stronger than both $\wedge$ and $\vee$, which in turn bind stronger than both connectives $\Box\!\!\rightarrow$ and $\rightarrow$.

Let us now describe some terminology used throughout this thesis. We start with the definition of scopes of conditional operators. The antecedent scope of a conditional operator $\Box\!\!\rightarrow$ in language $\mathcal{L}$ is defined as the formula α of $\mathcal{L}$, which precedes $\Box\!\!\rightarrow$. The consequent scope of a conditional operator $\Box\!\!\rightarrow$ in a language $\mathcal{L}$ is defined as the formula α of $\mathcal{L}$, which is preceded by $\Box\!\!\rightarrow$. Note that an antecedent scope and a consequent scope must always exist for any conditional operator occurring in formulas of the languages $\mathcal{L}_{\mathrm{KL}}$, $\mathcal{L}_{\mathrm{rKL}}$ and $\mathcal{L}_{\mathrm{rrKL}}$. We often call the antecedent scope and the consequent scope of a conditional operator its 'antecedent formula' and its 'consequent formula', respectively. Let us now focus on the concepts of iteration and nestedness for $\mathcal{L}_{\mathrm{KL}}$ (note that both iterated and nested formulas are not allowed for the languages $\mathcal{L}_{\mathrm{rKL}}$ and $\mathcal{L}_{\mathrm{rrKL}}$): A formula α in the language $\mathcal{L}_{\mathrm{KL}}$ is iterated iff there exist formulas β, γ and δ of $\mathcal{L}_{\mathrm{KL}}$, such that $\alpha = (\beta \Box\!\!\rightarrow (\gamma \Box\!\!\rightarrow \delta))$ or $\alpha = ((\beta \Box\!\!\rightarrow \gamma) \Box\!\!\rightarrow \delta)$. A formula α in the language $\mathcal{L}_{\mathrm{KL}}$ is nested, if there exist formulas β and γ of $\mathcal{L}_{\mathrm{KL}}$, such that $\alpha = (\beta \Box\!\!\rightarrow \gamma)$ and either β or γ contains an instance of $\Box\!\!\rightarrow$. From both definitions follows immediately that any formula of $\mathcal{L}_{\mathrm{KL}}$, which is iterated is also nested. The converse, however, does not hold, since, for example, the formula $p \Box\!\!\rightarrow \neg(p \Box\!\!\rightarrow r)$ of $\mathcal{L}_{\mathrm{KL}}$ is nested, but not iterated. Furthermore, in the literature often left-nestedness and right-nestedness are distinguished. We can also make those notions precise as follows: A formula α in $\mathcal{L}_{\mathrm{KL}}$ is left-nested iff there are formulas β and γ of $\mathcal{L}_{\mathrm{KL}}$, such that $\alpha = (\beta \Box\!\!\rightarrow \gamma)$ and β contains an instance of $\Box\!\!\rightarrow$. A formula α in $\mathcal{L}_{\mathrm{KL}}$ is right-nested iff there are formulas β and γ of $\mathcal{L}_{\mathrm{KL}}$, such that $\alpha = (\beta \Box\!\!\rightarrow \gamma)$ and γ contains an instance of $\Box\!\!\rightarrow$.

In this thesis we often use the term 'bridge principle'. This notion can be made more precise following way: A formula (schema) α is a bridge principle in $\mathcal{L}_{\mathrm{KL}}$ iff it contains the conditional operator $\Box\!\!\rightarrow$ and there exists a formula β of $\mathcal{L}_{\mathrm{KL}}$, for which the following holds: (a) β is contained in α, (b) β does not contain any occurrence of $\Box\!\!\rightarrow$, (c) β lies outside the antecedent scope and the consequent scope of any conditional operator, and (d) α is not logically equivalent to a conjunction of formulas, which do not satisfy (a)–(c). Typical examples for bridge principles are the formulas $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \rightarrow \beta)$ (MP, "Modus Ponens") and $\alpha \wedge \beta \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \beta)$ (CS; cf. Nute, 1980, p. 52 and p. 40, Nute & Cross, 2001, p. 10 and p. 14; Nute & Cross, 2001, Nute, 1980; see Table 5.1). We include condition (d) to make sure that, for example, no principles of the form $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge \gamma$ are regarded as bridge-principles (cf. Schurz, 1997a, p. 91f). The addition of (d) is essential, since formulas, such as $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge \gamma$, are equivalent to sets of formulas (e.g. $\{\alpha \mathrel{\Box\!\!\rightarrow} \beta, \gamma\}$), which do not contain bridge principles (cf. Schurz, 1997a, p. 92).

Note that we cannot express any type of bridge principle (as defined here) in the languages $\mathcal{L}_{\text{rKL}}$ and $\mathcal{L}_{\text{rrKL}}$ directly. We can, however, sometimes use a rule of inference in the language $\mathcal{L}_{\text{rKL}}$, which corresponds to a bridge principle in the full language $\mathcal{L}_{\text{KL}}$ (cf. Section 3.5.3). The inferences 'if $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ then $\alpha \rightarrow \beta$' and 'if $\alpha \wedge \beta$ then $\alpha \mathrel{\Box\!\!\rightarrow} \beta$' are, for example, rules in language $\mathcal{L}_{\text{rKL}}$, which correspond to MP and E, respectively. Note that we do not distinguish between both types of bridge principles (formulas on the one hand and rules on the other hand) in the discussion, but use the labels 'MP', 'E', etc. and 'bridge principle' indiscriminately. It is, furthermore, interesting to note that no bridge principles (in that wide sense) can exist for language $\mathcal{L}_{\text{rrKL}}$, since all formulas of that language have to contain at least one conditional operator.

4.2.2 Logics

In this section we define a number of syntactic notions. These syntactic notions are not restricted to particular system, but rather to apply to all systems discussed in the thesis (except if specified otherwise). This enables us to communicate many concepts more efficiently.

First, we use the notion of a logic **L** in two ways: On the one hand we refer by 'logic' to a *logical systems* with specific axioms and rules of derivation (see below); on the other hand, we associate with this term a *set of formulas*, which are valid in such systems. In the latter sense we always presuppose that a logic **L** is closed under propositional consequence and substitution of non-logical symbols (cf. Schurz, 2002a, p. 447; see also next section). Our use of that term in general should not lead to ambiguities. When we fear that it does, we specify our use of the term accordingly. Note also that all logical systems considered here – if not specified otherwise – refer to the full conditional language $\mathcal{L}_{KL}$.

Let A, B, ... denote sequences of formulas and Γ, Δ and Σ sets of formulas of language $\mathcal{L}$. A proof in a system **L** is, then, a finite sequence of formulas, for which every formula in that sequence is either an instance of an axiom schema of or follows by the rules of **L** from preceding formulas. Both the set of axioms and rules are specified in such a way that they are effectively enumerable. A formula α is derivable in system **L** (short: $\vdash_{\mathbf{L}} \alpha$) if it is there exists a proof for α in **L**. $\Gamma \vdash_{\mathbf{L}} \alpha$ (α is derivable from Γ in **L**) iff the following holds: if $\vdash_{\mathbf{L}} \beta$ for all $\beta \in \Gamma$, then $\vdash_{\mathbf{L}} \alpha$.

Let us now take a closer look at the notions of axioms and rules. A rule can be described as an ordered pair $\langle P, C\rangle$, where P is a set of derivability or non-derivability claims (c.f. Schurz, 1996, p. 205) and C is a derivability claim. Derivability claims and non-derivability claims have the form $\vdash \alpha$ and $\nvdash \alpha$, respectively (see also Section 2.2). A rule might be read intuitively following way: If α_1, α_2, ... are derivable and β_1, β_2, ... are not derivable, then γ is derivable. A rule $\langle P, C\rangle$ is monotonic iff P contains only derivability claims; a rule $\langle P, C\rangle$ is non-monotonic iff P contains at least one non-derivability claim. Please note that our notion of rule is somewhat broader than the usual notion, which is restricted to monotonic rules.

4.2.3 Non-Monotonicity

A monotonic logic is, then, a logic, which can be described by monotonic rules only. The basic idea behind the restriction to monotonic rules is the fact that if we add monotonic rules to monotonic systems, then the set of theorems cannot decrease. More formally the following holds for monotonic systems **L**: if $\Gamma \vdash_{\mathbf{L}} \beta$, then $\Gamma \cup \{\alpha\} \vdash_{\mathbf{L}} \beta$. (We call the derivability relation of a system **L** monotonic iff **L** is monotonic.) The upshot of this is that in monotonic systems no rule prevents another rule from applying. For non-monotonic systems this is not the case. Since the conclusion of a non-monotonic rule depends on non-derivability claims, the addition of axioms can result in the retraction of a conclusion. The non-derivability conditions of non-monotonic rules are to blame here. In monotonic systems this can, hence, not be the case (see Section 2.2.4). Please note that the addition of axioms to a monotonic logic system cannot make the monotonic system non-monotonic. This is due to the fact that axioms are also rules, namely rules of type $\langle P, C \rangle$ with P being the empty set. This means, in other words, that axioms are rules, which are applied unconditionally. Hence, axioms do not have non-derivability conditions, and, hence, adding them to monotonic system does not make those systems non-monotonic.

Although we investigate in this thesis non-monotonic properties of conditionals, our formal investigation in Chapters 6 and 5 is limited to monotonic logics in the above sense. We do that by specifying the logical properties of the conditional operator $\Box\!\!\rightarrow$ in such a way that $\Box\!\!\rightarrow$ is non-monotonic, but the inference relation $\vdash$ is still monotonic. That our inference relation for all systems described in Chapters 6 and 5 is monotonic (in the above sense) can be seem from the fact that we only use axioms and monotonic rules. Hence, our system is no default logic in the sense specified in Section 2.2.4. Note also that due to that fact, our system does not suffer from the difficulties of default logic approaches (see Section 2.2.5).

4.2.4 Consistency and Maximality

We first define the notions of consistency and maximality: A set of formulas Γ is **L**-consistent iff $\Gamma \nvdash \bot$, where $\nvdash$ denotes non-derivability. A set of formulas Γ is **L**-maximal iff there is no set of formulas Δ, such that $\Gamma \subset \Delta$ and Δ is **L**-consistent. Note that a **L**-maximal set of formulas needs not be consistent. Hence, we define the notion of maximally **L**-consistent sets of formulas: Γ is maximally **L** consistent iff it is **L** consistent and **L**-maximal.

4.2.5 A Propositional Basis for Conditional Logics

The lattice of conditional logics investigated in the following chapters are all closed under propositional consequence. The set of propositional tautologies can be axiomatized by the following set of axioms plus of the additional rules *modus ponens* and *substitution* (specified below):

1. $p \rightarrow (q \rightarrow p)$
2. $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$
3. $(\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p)$

Let us now describe the propositional rules of inference. Modus ponens tells us that if $\alpha \rightarrow \beta$ and α are the case, then β holds. The substitution rule allows for the uniform substitution of any atomic proposition by an arbitrary formula in any derivable formula. Formally, if $\vdash \alpha$ then $\vdash s(\alpha)$, where $s(\alpha)$ refers to the simultaneous and uniform substitution of atomic propositions by some arbitrary formulas in the formula α (cf. Schurz, 2002a, p. 448f). The substitution rule is, however, not truth-preserving but only validity-preserving: It does not suffice for the application of the substitution rule to presuppose that α is true, but we have to presuppose that α is in fact valid or derivable (cf. Schurz, 2002a, p. 448f). Observe, moreover, that the substitution rule does not in general hold for default logics (see Section 2.2.5).

Moreover, since we do not have defined expressions in the object language (see Section 4.2.1), we do not need an additional rule of replacement, which guarantees that these definitions hold. Note that we use in Chapters 5–7 axiom schemata rather than axioms. Hence, we do not need to employ the substitution rule explicitly. The only propositional rule, which is not presupposed so far, is the rule of modus ponens. We will however – as it is standard in the conditional logical literature – not refer to p.c.-rules and p.c.-axioms explicitly, but use them quite freely.

4.2.6 System CK

System **CK** is then syntactically characterized the following way:

Definition 4.1. *System* **CK** *is the smallest logic containing LLE+RW+AND+LT (see Table 4.1).*

Our characterization of system **CK** (and henceforth all definitions of conditional logics) presupposes closure under p.c.-rules and p.c.-axioms. So, system **CK** is LLE+RW+AND, LT plus axioms of propositional logic and the rule modus ponens. Note also that the deduction theorem holds for system **CK** and the stronger

Table 4.1
Rules and Axioms of System CK

LLE	if $\vdash \alpha \leftrightarrow \beta$ and $\alpha \mathrel{\Box\!\!\rightarrow} \gamma$, then $\beta \mathrel{\Box\!\!\rightarrow} \gamma$
RW	if $\vdash \alpha \rightarrow \beta$ and $\gamma \mathrel{\Box\!\!\rightarrow} \alpha$, then $\gamma \mathrel{\Box\!\!\rightarrow} \beta$
AND	$(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \beta \wedge \gamma)$
LT	$\alpha \mathrel{\Box\!\!\rightarrow} \top$

Note. 'LLE', 'RW' and 'LT' stand for 'Left Logical Equivalence', 'Right Weakening' and 'Logical Truth', respectively.

systems S discussed in Chapters 5–7 (see Section 4.2.1).[3] The deduction theorem allows us, for example, to conclude from LLE the following: If $\alpha \leftrightarrow \beta$ is derivable in S, then $(\alpha \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\beta \mathrel{\Box\!\!\rightarrow} \gamma)$ is derivable in S. We can, furthermore, by the same theorem infer from AND that if $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \gamma)$ is derivable in S, then $(\alpha \mathrel{\Box\!\!\rightarrow} \beta \wedge \gamma)$ is derivable in S. We shall, however, neither prove the deduction theorem for system **CK** nor for the stronger systems discussed in this thesis.

We also provide from time to time object language proofs of lemmata and theorems associated with system **CK**. For the sake of perspicuity we employ natural deduction proofs rather than axiomatic proofs. The natural deduction elements, however, refers to the p.c.-part of the respective systems only. Premises and presuppositions of conditional proofs are indicated by 'given'. Moreover, assumptions for proofs by cases and indirect proofs are marked and numbered by expressions, such as 'ass 1 (2), proof by cases' and 'ass IP', respectively. Furthermore, not too obvious steps referring to p.c. are indicated by 'p.c.'. Examples of object language proofs can be found below.

Observe that our axiomatization of system **CK** represents a variation of the axiomatization by Segerberg (1989, p. 158). There are two main differences: First, our axiomatization is more in line with Kraus et al. (1990). In particular we use RW instead of the rule RLE (right logical equivalence) plus axiom CW (Consequent Weakening), as Segerberg (1989) does:

RLE if $\vdash \beta \leftrightarrow \gamma$ and $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ then $\alpha \mathrel{\Box\!\!\rightarrow} \gamma$
CW $(\alpha \mathrel{\Box\!\!\rightarrow} \beta \wedge \gamma) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \gamma)$

[3] Note, however, that the deduction theorem does not hold for system $\mathbf{P}_\epsilon$ of Adams (1975).

Lemma 4.2 shows that RLE + CW are logically equivalent to RW given system **CK**:

Lemma 4.2. *RW* $\Leftrightarrow$ *RLE+CW*

Proof. By Lemmata 4.3 through 4.5. $\square$

Lemma 4.3. *RLE+CW* $\Rightarrow$ *RW*

Proof.

1.	$\vdash \alpha \rightarrow \beta$	given
2.	$\gamma \mathrel{\square\!\!\rightarrow} \alpha$	given
3.	$\vdash \alpha \leftrightarrow \alpha \wedge \beta$	1, p.c.
4.	$\gamma \mathrel{\square\!\!\rightarrow} \alpha \wedge \beta$	3, 2, RLE
5.	$(\gamma \mathrel{\square\!\!\rightarrow} \alpha) \wedge (\gamma \mathrel{\square\!\!\rightarrow} \beta)$	4, CW
6.	$(\gamma \mathrel{\square\!\!\rightarrow} \beta)$	5, p.c.

$\square$

Lemma 4.4. *RW* $\Rightarrow$ *RLE*

Proof.

1.	$\vdash \beta \leftrightarrow \gamma$	given
2.	$\alpha \mathrel{\square\!\!\rightarrow} \beta$	given
3.	$\vdash \beta \rightarrow \gamma$	1, p.c.
4.	$\alpha \mathrel{\square\!\!\rightarrow} \gamma$	3, 2, RW

$\square$

Lemma 4.5. *RW* $\Rightarrow$ *CW*

Proof.

1.	$\alpha \mathrel{\square\!\!\rightarrow} \beta \wedge \gamma$	given
2.	$\alpha \mathrel{\square\!\!\rightarrow} \beta$	1, RW
3.	$\alpha \mathrel{\square\!\!\rightarrow} \gamma$	1, RW
4.	$(\alpha \mathrel{\square\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\square\!\!\rightarrow} \gamma)$	2, 3, p.c.

$\square$

The second difference between Segerberg's (1989) axiomatization and our axiomatization of conditional logic systems lies in the fact that we formulate principles in terms of axioms rather than rules whenever possible. For many principles from Tables 5.1 and 5.2 this works, because we use the full language $\mathcal{L}_{\mathrm{KL}}$ and employ an inference relations $\vdash$, which is monotonic (see Section 4.2.1). Note that our treatment of logical principles allows for an easier application in terms of correspondence and completeness proofs.

4.2.7 Alternative Axiomatizations of System CK

Let us now turn to an alternative axiomatization of system **CK**. Originally, Chellas (1975, p. 137f) defined system **CK** in terms of the propositional part plus LLE plus the following rule RCK:

RCK if $\vdash \beta_1 \wedge \cdots \wedge \beta_n \rightarrow \gamma$ and $(\alpha \mathrel{\Box\!\!\rightarrow} \beta_1) \wedge \cdots \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \beta_n)$, then $\alpha \mathrel{\Box\!\!\rightarrow} \gamma$, $n \geq 0$

For $n = 0$ the material implication $\beta_1 \wedge \cdots \wedge \beta_n \rightarrow \gamma$ is identified with its consequent formula γ (see Chellas, 1975, Footnote 6). We can easily see that Chellas' axiomatizations and the axiomatization described in Table 4.1 agree. Hence, we have the following theorem:

Theorem 4.6. *System* **CK** *can be axiomatized by RCK+LLE.*

Proof. By Lemma 4.7. □

Lemma 4.7. *RCK ⇔ RW + LT + AND*

Proof. By Lemmata 4.8–4.11. □

Lemma 4.8. *RCK ⇒ RW (see Nejdl, 1992, Theorem 2.1)*

Proof.

1.	$\alpha \mathrel{\Box\!\!\rightarrow} \beta$	given
2.	$\vdash \beta \rightarrow \gamma$	given
3.	$(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \gamma)$	2, RCK
4.	$\alpha \mathrel{\Box\!\!\rightarrow} \gamma$	3, 1, p.c.

□

Lemma 4.9. *RCK ⇒ AND (see Nejdl, 1992, Theorem 2.4)*

Proof.

1.	$\alpha \mathrel{\Box\!\!\rightarrow} \beta$	given
2.	$\alpha \mathrel{\Box\!\!\rightarrow} \gamma$	given
3.	$\vdash \beta \wedge \gamma \rightarrow \beta \wedge \gamma$	p.c.
4.	$(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \beta \wedge \gamma)$	3, RCK
5.	$\alpha \mathrel{\Box\!\!\rightarrow} \beta \wedge \gamma$	4, 1, 2, p.c.

□

By Lemma 4.10 we can derive LT from RCK focusing on the case $n = 0$:

Lemma 4.10. *RCK ⇒ LT*

Proof.

1.	$\vdash \top$	p.c.
2.	$\alpha \mathrel{\Box\!\!\rightarrow} \top$	1, RCK ($n = 0$)

□

Furthermore, the following lemma holds:

Lemma 4.11. *RW+LT+AND ⇒ RCK*

Proof.

for $n = 0$:

1.	$\vdash \gamma$	given
2.	$\vdash \top \leftrightarrow \gamma$	1, p.c.
3.	$\alpha \mathrel{\Box\!\!\rightarrow} \top$	LT
4.	$\vdash \top \rightarrow \gamma$	2, p.c.
5.	$\alpha \mathrel{\Box\!\!\rightarrow} \gamma$	3, 4, RW

for $n > 0$:

1.	$\vdash \beta_1 \wedge \cdots \wedge \beta_n \rightarrow \gamma$	given
2.	$(\alpha \mathrel{\Box\!\!\rightarrow} \beta_1) \wedge \cdots \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \beta_n)$	given
3.	$\alpha \mathrel{\Box\!\!\rightarrow} \beta_1 \wedge \cdots \wedge \beta_n$	2, $n - 1$ times AND
4.	$\alpha \mathrel{\Box\!\!\rightarrow} \gamma$	3,1, RW

□

4.3 Model-Theoretic Notions

Let us now describe the basic Chellas-Segerberg (CS) semantics. The basic notions investigated in Chapters 6 and 5. The CS-semantics is a possible worlds semantics and represents a generalization of standard Kripke semantics as, for example, described in Hughes and Cresswell (1996/2003, p. 38) and Blackburn et al. (2001, pp. 16–18). In the CS-semantics, however, not a single accessibility relation among possible worlds in W is used as in standard Kripke semantics, but rather a multitude of such accessibility relations. The basic ideal of this semantics is that we use for each antecedent α of a conditional formula $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ an accessibility relation R, which is relativized to α. Let in this context 'R_α' denote an accessibility relation R relativized to α. The formula $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ is, then, true at a possible world w in W if and only if at all possible worlds w', which are accessible to w by R_α the consequent β is true. In the CS-semantics, however, the accessibility relation R is not directly relativized to formulas (i.e. α), but instead to subsets of W, at which the formulas (i.e. α) is true. We denote the sets of possible worlds, at which α is true by '$\|\alpha\|$'. In our terminology the set $\|\alpha\|$ is the proposition, which is expressed by α. Propositions are, hence, subsets of the set of possible worlds W. Moreover, we write '$R_{\|\alpha\|}$' and 'R_X' for an accessibility relation relativized to a proposition $\|\alpha\|$ or a proposition X, respectively. In this approach we can, then, define frames in the tradition of Kripke frames (Hughes & Cresswell, 1996/2003, p. 38), which results in ordered pairs of a non-empty set of possible worlds W and an accessibility relation R defined on members and subsets of W. This approach avoids direct reference to formulas on the level of frames and seems, hence, more natural. This definition of frames, moreover, parallels the notion of Kripke frames, for which no such relativization to propositions is needed (cf. Hughes & Cresswell, 1996/2003, p. 38). Moreover, the use of sets of possible worlds makes naturally LLE (described in Section 4.2.6) valid. Due to the specification of valuations V on possible worlds (see below), any two formulas being logically equivalent describe the same set of possible worlds.

Let us now turn to the notions of frames and models investigated in Chapters 6 and 5. We distinguish between two types of frames and models. The most general notions of frames and models are Chellas frames and models. A less general, but useful notion are Segerberg frames and Segerberg models. Note that both types of frames and models are based on the ideas described above. For the sake of perspicuity, let us first introduce the notions of Chellas frames and models and then discuss, in which way Segerberg frames and models might be a more appropriate notions regarding soundness and completeness.

4.3.1 Chellas Frames and Chellas Models

The basic units of the CS-semantics investigated in this thesis are Chellas frames and Chellas models. These are generalizations of the ideas described before to Kripke frames and Kripke models (Hughes & Cresswell, 1996/2003, p. 38). To our knowledge they were first formally described by Chellas (1975, p. 134f) and can be defined as follows:

Definition 4.12. *$\mathcal{F}_C = \langle W, R \rangle$ is a Chellas frame iff*
(a) W is a non-empty set of possible worlds and
(b) R is a relation on $W \times W \times Pow(W)$

Definition 4.13. *Let $\mathcal{F}_C = \langle W, R \rangle$ be a Chellas frame as described in Definition 4.12. Then, $\mathcal{M}_C = \langle W, R, V \rangle$ is a Chellas model iff V is a valuation function from $\mathcal{PP} \times W$ to $\{0, 1\}$*

If it holds that $\mathcal{M}_C = \langle W, R, V \rangle$ and $\mathcal{M}'_C = \langle W, R, V' \rangle$ are such that $V = V'$, we will say that both Chellas models $\mathcal{M}_C$ and $\mathcal{M}'_C$ are elementary equivalent.

Definition 4.14. *Let $\mathcal{M}_C = \langle W, R, V \rangle$ be a Chellas model, as described in Definition 4.13. Then V^* is an extension of V to arbitrary formulas iff (a) $\forall p \in \mathcal{PP}\ \forall w \in W$: $V^*(p, w) = 1$ iff $V(p, w) = 1$ and (b) for all α, β and $w \in W$ holds:*

$\vDash_w^{\mathcal{M}_C} \neg\alpha$	*iff*	$\nvDash_w^{\mathcal{M}_C} \alpha$	*($V_\neg$)*
$\vDash_w^{\mathcal{M}_C} \alpha \vee \beta$	*iff*	$\vDash_w^{\mathcal{M}_C} \alpha \curlyvee \vDash_w^{\mathcal{M}_C} \beta$	*($V_\vee$)*
$\vDash_w^{\mathcal{M}_C} \alpha \mathrel{\Box\!\!\rightarrow} \beta$	*iff*	$\forall w'(wR_{\Vert\alpha\Vert}w' \Rightarrow \vDash_{w'}^{\mathcal{M}_C} \beta)$	*($V_{\Box\!\rightarrow}$)*

Here, $Pow(S)$ denotes the power set of the set S, namely the set of subsets of S. Moreover, $\vDash_w^{\mathcal{M}} \alpha$ and $\nvDash_w^{\mathcal{M}} \alpha$ stand for $V^*(\alpha, w) = 1$ and $V^*(\alpha, w) = 0$ for V as specified in $\mathcal{M}$, respectively. Intuitively, $\vDash_w^{\mathcal{M}} \alpha$ and $\nvDash_w^{\mathcal{M}} \alpha$ are to be interpreted as α being true at world w in model $\mathcal{M}$ and being false at world w in model $\mathcal{M}$, respectively. When we refer to a world w in a model $\mathcal{M}$, we mean in fact a world $w \in W$, such that $\mathcal{M} = \langle W, R, V \rangle$. Note that $\|\alpha\|^{\mathcal{M}}$ is defined for a CS-model $\mathcal{M} = \langle W, R, V \rangle$ the following way: $\|\alpha\|^{\mathcal{M}} =_{\text{df}} \{w \in W \mid \vDash_w^{\mathcal{M}} \alpha\}$ (Def$_{\|.\|}$). Strictly speaking only the extension of V, namely V^*, generalizes to arbitrary formulas. Since, however, V determines V^* in a unique way, we will often use V and V^* indiscriminately. Moreover, we sometimes write $\|\alpha\|$ when the context makes it clear, to which model $\|\alpha\|$ refers.

The function V is a valuation function, which assigns truth values to atomic formulas of the language $\mathcal{L}_{\text{KL}}$ relative to possible worlds. Conditions $\text{V}_{\neg}$, $\text{V}_{\vee}$ give us truth conditions for boolean combinations of formulas. For truth conditions of conditional formulas, Chellas models refer to the three-place accessibility relation R. This accessibility relation in Chellas models is a relation R between possible worlds relativized to sets of possible worlds. To indicate that a relation R holds between $w, w' \in W$ and $X \subseteq W$ we henceforth write wR_Xw'.

4.3.2 A Discussion of Chellas Models and Frames

Note that Chellas frames [models] differ from Kripke frames [models] (e.g. Hughes & Cresswell, 1996/2003, p. 38; Blackburn et al., 2001, p. 16–18) and its multimodal extensions (e.g. Blackburn et al., 2001, p. 20) in following three ways: (a) For any Kripke model $\mathcal{M} = \langle W, R, V \rangle$ the cardinality of the set of accessibility relations R is always greater than the cardinality of W. This is due to the fact that for any subset $X \subseteq W$, there exists an accessibility relation R_X between possible worlds in W. Note that any subset of W is element of power set of W. The power set $\mathcal{P}(W)$, however, has higher cardinality than W, except if $W = \varnothing$ holds. The latter fact is, however, excluded by condition (a) of Definition 4.13. In contrast, Kripke semantics for normal modal logic – where only a single accessibility relation is used – does not have this property (Hughes & Cresswell, 1996/2003, p. 38; Blackburn et al., 2001, pp. 16–18). Even in multimodal explorations of modal logic this is not the case. Either a constant finite number of n-ary accessibility relations is investigated (cf. Blackburn et al., 2001, p. 20) or a constant, but potentially infinite number of accessibility relations (cf. Schurz, 1997a, p.

166). These investigations do, however, not require that the number of accessibility relations in a frame is strictly greater than the number of possible world in that frame.

(b) The second difference lies in the use of the accessibility relations. From a proof-theoretic perspective we can regard $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ as a type of modality where $\alpha \mathrel{\Box\!\!\rightarrow} \cdot$ expresses the modal operator. Chellas (1975, p. 138f) introduced for this purpose the unofficial terminology $[\alpha]\beta$. In this semantics the association between R_X and a modal operator $[\alpha]$ is only determined on the basis of the valuation function V. Hence, for Chellas frames $\mathcal{F}_C = \langle W, R, \rangle$ it is undetermined, which accessibility relation R_X is associated with the modalities $[\alpha]$, $[\beta]$, etc. Again, this differs from Kripke frames and their multi-modal extensions (cf. Blackburn et al., 2001, p. 20; Schurz, 1997a, p. 166; Gabbay et al., 2003, p. 21). For this types of frames there is a predetermined connection between modal operators and accessibility relations.[4]

(c) The third difference lies in the fact that the relativization of the accessibility relation R is in Chellas frames $\mathcal{F}_C = \langle W, R\rangle$ is stronger than what is actually needed for the truth conditions in Chellas models. The relativization is done for every subset X of W. So, there exists an accessibility relation for each subset X of W. However, the truth-conditions for Chellas models can refer in $V_{\Box\!\rightarrow}$ only to subsets of W, for which a formula α exists, such that $\|\alpha\|^{\mathcal{M}_S}$. We call this type of subsets of the Chellas model $\mathcal{M}_C$'representable sets in $\mathcal{M}_C$' and give following formal definition:

Definition 4.15. *Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a Chellas model. Then, a subset $X \subseteq W$ is syntactically representable in $\mathcal{M}_S$ (by formulas of $\mathcal{L}_{\mathrm{KL}}$) iff there exists a formula α (of $\mathcal{L}_{\mathrm{KL}}$), such that $X = \|\alpha\|^{\mathcal{M}_C}$.*

So, (c1) the truth conditions of Chellas models $\mathcal{M}_C = \langle W, R, V\rangle$ (Definition 4.13) do not refer to non-representable subsets in $\mathcal{M}_C$, but (c2) there may exist non-representable subsets X in $\mathcal{M}_C$, for which accessibility relations R_X are defined.

[4]What resembles CS-semantics to some extent is standard semantics for dynamic logic (Gabbay et al., 2003, p. 63). Here accessibility relations are relativized to actions. Note, however, that actions are *not* represented by subsets of possible worlds as given by the valuation function (see above).

Let us, first, focus on (c2). In general not all subsets X of W of a Chellas model $\mathcal{M}_C = \langle W, R, V\rangle$ are syntactically representable. For a subset X in $\mathcal{M}_C$ to be syntactically representable there has to exist a single formula, which is true at all and only the worlds contained in X. If we find a finite collection of formulas Γ, which are true at all and only those worlds in X, then X is representable, since there exists such a formula, namely the conjunction of the finite collection of formulas Γ. Since any formula of $\mathcal{L}_{\mathrm{KL}}$ can be described in terms of a conjunctive normal form by atomic propositional variables and conditional formulas, the search for syntactically representable formulas boils down to finding finite sets of formulas, which conjointly represent the respective set. It should, however, be clear that particularly if W is infinite, not for all subsets of W such a finite collection of formulas might exist.

To make this more plausible, let us take a look at the proof-theoretic analogue of semantic representability. This is (propositional) finite axiomatizability of theories, namely axiomatizability by a finite set of eigenaxioms. A set of possible worlds X might be associated with the set of formulas, which are true and only true at the possible worlds in w. Due to the definitions of Chellas models sets of formulas of this kind are closed under propositional consequence and, hence, represent propositional theories. It should be clear that not all theories of this type are axiomatizable by a finite set of eigenaxioms.

Let us now focus on (c1). To make (c1) more pronounced we proof Lemma 4.16:

Lemma 4.16. *Let $\mathcal{M}_C = \langle W, R, V\rangle$ and $\mathcal{M}'_C = \langle W, R', V'\rangle$ be Chellas models. Suppose (a) that $V = V'$ and suppose (b) that for all syntactically representable sets $X \subseteq W$ in $\mathcal{M}_C$ and all $w, w' \in W$ holds that wR_Xw' iff wR'_Xw'. Then, for arbitrary formulas α of $\mathcal{L}_{\mathrm{KL}}$ and all worlds $w \in W$ holds:$\vDash_w^{\mathcal{M}_C} \alpha$ iff $\vDash_w^{\mathcal{M}'_C} \alpha$.*

Proof. Proof by induction on the construction of formulas. Condition (a) gives us that $\mathcal{M}_C$ and $\mathcal{M}'_C$ are elementary equivalent. Hence, it holds for all for all atomic propositions $p \in \mathcal{PP}$ and all worlds $w \in W$: $\vDash_w^{\mathcal{M}_C} p$ iff $\vDash_w^{\mathcal{M}'_C} p$.

Non-Atomic Propositional Case: By induction hypothesis $V(\alpha, w) = V'(\alpha, w)$ and $V(\beta, w) = V'(\beta, w)$ for arbitrary worlds $w \in W$ and given formulas α and β. By Definition 4.13 it follows that for any boolean combination γ of α and β it holds that $V(\gamma, w) = V'(\gamma, w)$.

Modal Case ("$\Leftrightarrow$"): By induction hypothesis $\vDash_w^{\mathcal{M}_C} \alpha$ iff $\vDash_w^{\mathcal{M}'_C} \alpha$, and $\vDash_w^{\mathcal{M}_C} \beta$ iff $\vDash_w^{\mathcal{M}'_C} \beta$ for arbitrary worlds $w \in W$ and given formulas α and β. Suppose for an arbitrary $w \in W$ that $\vDash_w^{\mathcal{M}_C} \alpha \mathrel{\Box\!\!\rightarrow} \beta$. Then, by $V_{\Box\!\rightarrow}$ it follows that (i) $\forall w' \in W(wR_{\|\alpha\|^{\mathcal{M}_C}} w' \Rightarrow \vDash_{w'}^{\mathcal{M}_C} \beta)$. The set $X = \|\alpha\|^{\mathcal{M}_C}$ is representable in $\mathcal{M}_C$. By the induction hypothesis hold both $\|\alpha\|^{\mathcal{M}_S} = \|\alpha\|^{\mathcal{M}'_S}$ and the fact that for all w: $\vDash_w^{\mathcal{M}_C} \beta$ iff $\vDash_w^{\mathcal{M}'_C} \beta$. Hence, it follows by (b) that (ii) $\forall w' \in W(wR_{\|\alpha\|^{\mathcal{M}'_C}} w' \Rightarrow \vDash_{w'}^{\mathcal{M}'_C} \beta)$ iff (i). By $V_{\Box\!\rightarrow}$ condition (ii) holds iff $\vDash_w^{\mathcal{M}'_C} \alpha \mathrel{\Box\!\!\rightarrow} \beta$ is the case. □

Lemma 4.16 shows that any two Chellas models, which are elementary equivalent, concur w.r.t. the extension of their valuation functions. This lemma, hence, shows that any accessibility relations R_X, where X is a non-representable subset in a Chellas model, does not have an impact on the truth of formulas at any possible world in that model. Hence, any of these R_X are irrelevant in the sense that they do not contribute to the determination of the truth of formulas at any world in that Chellas model. Despite this fact, however, all those R_X are defined in Chellas models and Chellas frames and we saw for the discussion of (c2).

Let us summarize the result of the discussion of this section. We saw in this section that Chellas frames $\mathcal{F}_C = \langle W, R\rangle$ and Chellas models $\mathcal{M}_C = \langle W, R, V\rangle$ deviate from Kripke frames and Kripke models (and their multi-modal extensions) in three ways: (a) For any Chellas frame [model] there are strictly more accessibility relations R_X than possible worlds in W. (b) The association between modal operators $[\alpha]$ and the accessibility relation is not fixed on the level of frame, but only determined in Chellas models. (c1) Truth of formulas depends only on accessibility relations R, which are relativized to syntactically representable sets in the Chellas model. However, (c2) accessibility relations w.r.t. to syntactically non-representable subsets in that model are also defined.

4.3.3 Segerberg Frames and Segerberg Models

We saw in the previous section that in Chellas models $\mathcal{M}_C = \langle W, R, V\rangle$ only syntactically representable sets (in that model) determine the truth-value of formulas at a world, whereas non-representable sets in such a model are not relevant in that context (Lemma 4.16, Point c1). The set of syntactically representable sets is essentially determined by the valuation function V and cannot be determined on the level of frames. Since for Chellas models no restriction on the valuation function exists, for any subset of X of W the accessibility relation R_X is defined, since it might potentially be representable.

This is, however, not the case for Segerberg models and frames. For this type of frame and model an additional parameter P is employed. Hence, a Segerberg frame is a triple $\mathcal{F}_S = \langle W, R, P\rangle$ and a Segerberg model is a quadruple $\mathcal{M}_S = \langle W, R, P, V\rangle$. The parameter P is, then, defined to be a subset of the power-set $\mathcal{P}(W)$ and to be closed under the logical operations (w.r.t. negation, disjunction and conditional operator). Segerberg frames and Segerberg models, hence, follow general frames and general frame based models (cf. Hughes & Cresswell, 1996/2003, p. 167; Blackburn et al., 2001, p. 28f) rather than Kripke frames. They owe their name to Segerberg, since they are to our knowledge first defined by Segerberg (1989, p.160f). Models on general frames are restricted in such a way that all syntactically representable sets in the corresponding Chellas model $\mathcal{F}_C = \langle W, R, V\rangle$ have to be in that set P. Segerberg models share this property. They, however, differ from general frames in one important respect: In addition, the accessibility relation R is not defined on $W\times W\times\mathcal{P}(W)$, but rather on $W\times W\times P$. So, the accessibility relation is relativized to a (possibly proper) subset of $\mathcal{P}(W)$. Hence, an additional parameter is used to restrict the accessibility relation R_X. In Kripke frames with n-ary modal operators [models] (cf. Blackburn et al., 2001, p. 20) and their multi-modal extensions (cf. Schurz, 1997a, p. 166; Gabbay et al., 2003, p. 21) this is not the case.

Let us now define the notions of Segerberg frames and Segerberg models formally. Note that we define the notion Segerberg models w.r.t. the set of admissible valuations on Segerberg frames (Definition 4.18). Definition 4.18, hence, gives us the restrictions of Chellas models discussed informally earlier.

Definition 4.17. $\mathcal{F}_S = \langle W, R, P\rangle$ *is a Segerberg frame iff*
(a) W is a non-empty set of possible worlds,
(b) R is a relation on $W \times W \times P$*,*
(c) $P \subseteq Pow(W)$ *is such that for all* $X, Y \subseteq W$ *holds:*

$\varnothing \in P$ *(Def$_{P_\varnothing}$)*

if $X \in P$ *then* $-X \in P$ *(Def$_{P_-}$)*

if $X, Y \in P$ *then* $X \cap Y \in P$ *(Def$_{P_\cap}$)*

if $X, Y \in P$ *then* $\{w \in W \mid \forall w' \in W(wR_X w' \Rightarrow w' \in Y)\} \in P$ *(Def$_{P_{Mod}}$)*

Note that the conditions on (c) in Definition 4.17 are used to provide closure conditions for logical operators $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\Box\!\!\rightarrow$ and $\Diamond\!\!\rightarrow$ regarding the set P. These closure conditions are needed to ensure that all propositions in a Segerberg model are in P. On the other hand (c) ensures that validity in a Segerberg frame is closed under substitution. Let us now focus on the notion of Segerberg models:

Definition 4.18. *Let $\mathcal{F}_S = \langle W, R, P\rangle$ be a Segerberg frame and let the valuation V and its extension V^* be defined w.r.t. W and R, as described in Definition 4.17 and Definitions 4.13 and 4.14, respectively. The valuation V is, then, admissible in $\mathcal{F}_S$ iff for all formulas α and the extension V^* of V holds: $\|\alpha\|^{\mathcal{M}_C} = \{w \in W \mid V^*(\alpha, w) = 1\} \in P$.*

Definition 4.19. *$\mathcal{M}_S = \langle W, R, P, V\rangle$ is a Segerberg model iff*
(a) $\langle W, R, P\rangle$ is a Segerberg frame as described in Definition 4.17
(b) V is a valuation function w.r.t. W and R as described in Definition 4.13 and
(c) V is admissible in $\langle W, R, P\rangle$.

Theorem 4.20. *Let $\mathcal{F}_S = \langle W, R, P\rangle$ be a Segerberg frame, as described in Definition 4.17, and let the valuation function V be such that $\mathcal{M}_C = \langle W, R, V\rangle$ is a Chellas model according to Definition 4.13. Then, V is admissible in $\mathcal{F}_S$ iff for all $p \in PP$ holds: $\|p\|^{\mathcal{M}_C} = \{w \in W \mid V(p, w) = 1\} \in P$.*

Proof. The left-to-right direction is trivial and follows directly from Definitions 4.17, 4.18, 4.13 and 4.14. Hence, we only prove the right-to-left direction.

Proof by induction on the construction of formulas: Let $\mathcal{F}_S = \langle W, R, P\rangle$ be a Segerberg frame (Definition 4.17), and let the valuation function V be such that $\mathcal{M}_C = \langle W, R, V\rangle$ is a Chellas model (Definition 4.13). Suppose that $\|p\|^{\mathcal{M}_C} \in P$. In order for V to be admissible, we have to show that for all formulas α and the extension V^* of V holds (see Definition 4.14): $\|\alpha\|^{\mathcal{M}_C} \in P$. (We shall henceforth use V and V^* indiscriminately.) For atomic propositions this holds by supposition. Hence, the non-atomic propositional case and the modal case remain. By induction hypothesis it is the case that $\|\beta\|^{\mathcal{M}_C}, \|\gamma\|^{\mathcal{M}_C} \in P$ for given formulas β and γ.

$\alpha = \neg\beta$: By induction hypothesis it holds that $\|\beta\|^{\mathcal{M}_C} \in P$. Def_{P_-} of Definition 4.17 gives us that $W - \|\beta\|^{\mathcal{M}_C} \in P$. Thus, due to $\mathrm{V}_\neg$ and $\mathrm{Def}_{\|\cdot\|}$ it holds that $\|\neg\beta\|^{\mathcal{M}_C} \in P$.

$\alpha = \beta \wedge \gamma$: By induction hypothesis $\|\beta\|^{\mathcal{M}_C}$, $\|\gamma\|^{\mathcal{M}_C} \in P$ is the case. By $\mathrm{Def}_{P_\cap}$ of Definition 4.17 we get $\|\beta\|^{\mathcal{M}_C} \cap \|\gamma\|^{\mathcal{M}_C} \in P$. Hence, due to $\mathrm{V}_\wedge$ and $\mathrm{Def}_{\|\cdot\|}$ we get $\|\beta \wedge \gamma\|^{\mathcal{M}_C} \in P$.

$\alpha = \beta \mathrel{\Box\!\!\rightarrow} \gamma$: By induction hypothesis, it holds that $\|\beta\|^{\mathcal{M}_C}$, $\|\gamma\|^{\mathcal{M}_C} \in P$. Due to $\mathrm{Def}_{P_{Mod}}$ of Definition 4.17, it follows that $\{w \in W \mid \forall w' \in W(wR_{\|\beta\|}w' \Rightarrow w' \in \|\gamma\|)\} \in P$. By $V_{\Box\!\rightarrow}$ and $\mathrm{Def}_{\|\cdot\|}$ this implies that $\|\beta \mathrel{\Box\!\!\rightarrow} \gamma\|^{\mathcal{M}_C} \in P$. □

Note here that Segerberg frames [models] are more general than Chellas frames [models] in the following sense: For any Chellas frame $\mathcal{F}_C = \langle W, R\rangle$ exists a Segerberg frame $\mathcal{F}_S = \langle W, R', P\rangle$, such that $P = \mathcal{P}(W)$ and $R = R'$. Moreover, any valuation V on $\mathcal{F}_C$ is also admissible in $\mathcal{F}_S$ and vice versa. Note, moreover, that for any valuation function V admissible in a Segerberg frame $\mathcal{F}_S = \langle W, R, P\rangle$ the Segerberg model $\mathcal{M}_S = \langle W, R, P, V\rangle$ has following property: All syntactically representable subsets of W in the corresponding Chellas model are in P and R_X is defined for all syntactically representable sets X in that model. So, all accessibility relations w.r.t. syntactically representable subsets in that model are defined. The parameters V and P might, however, be defined in such a way that only syntactically representable subsets of W are in P. In this case R is restricted to the set of syntactically representable subsets. Due to these facts we can construct for each Chellas model $\mathcal{M}_C = \langle W, R, V\rangle$ a Segerberg model $\mathcal{M}_S = \langle W, R', P, V\rangle$, such that R' is restricted to the subsets, which are syntactically representable in $\mathcal{M}_C$. By these means we can remedy points (a) and (c2) discussed in the previous sections.

4.3.4 Validity, Logical Consequence and Satisfiability

In this section we define different notions of validity, logical consequence and satisfiability w.r.t. to Chellas and Segerberg frames [models]. We indicate different versions of these definitions by means of square brackets. We abbreviate Chellas frames [models] and Segerberg frames [models] by $\mathcal{F}_C$, $\mathcal{F}'_C$, $\mathcal{F}''_C$, … [$\mathcal{M}_C$, $\mathcal{M}'_C$,

$\mathcal{M}''_C, \ldots]$ and $\mathcal{F}_S, \mathcal{F}'_S, \mathcal{F}''_S, \ldots [\mathcal{M}_S, \mathcal{M}'_S, \mathcal{M}''_S, \ldots]$, respectively. Classes of Chellas frames [models] and Segerberg frames [models] frames are denoted by $\mathbf{F}_C, \mathbf{F}'_C$, $\mathbf{F}''_C, \ldots [\mathbf{M}_C, \mathbf{M}'_C, \mathbf{M}''_C, \ldots]$ and $\mathbf{F}_S, \mathbf{F}'_S, \mathbf{F}''_S, \ldots [\mathbf{M}_S, \mathbf{M}'_S, \mathbf{M}''_S, \ldots]$, respectively. A Chellas model $\mathcal{M}_C = \langle W, R, V\rangle$ is based on a Chellas frame $\mathcal{F}_C = \langle W', R'\rangle$ iff $W = W'$ and $R = R'$. A Segerberg model $\mathcal{M}_S = \langle W, R, P, V\rangle$ is based on a Segerberg frame $\mathcal{F}_S = \langle W', R', P'\rangle$ iff $W = W'$, $R = R'$, $P = P'$ and V is admissible in $\mathcal{F}_S$. A Segerberg frame $\mathcal{F}_S = \langle W, R, P\rangle$[a Segerberg model $\mathcal{M}_S = \langle W, R, P, V\rangle$] is based on a Chellas frame $\mathcal{F}_S = \langle W', R'\rangle$ iff $W = W'$ and $R =\downarrow_P R'$. The expression $\downarrow_P R'$ refers to the restriction of R' to elements of P. Finally, a Chellas frame $\mathcal{F}_C = \langle W, R\rangle$ [a Segerberg frame $\mathcal{F}_C = \langle W, R, P\rangle$] is a frame for a system $\mathbf{L}$ iff all theorems of $\mathbf{L}$ are valid on $\mathcal{F}_C$ [$\mathcal{F}_S$].

A formula α is valid *in* a Chellas model $\mathcal{M}_C = \langle W, R, V\rangle$ (short: $\vDash_{\mathcal{M}_C} \alpha$) [*in* a Segerberg model $\mathcal{M}_S = \langle W, R, P, V\rangle$ (short: $\vDash_{\mathcal{M}_S} \alpha$)] iff for all $w \in W$: $\vDash^{\mathcal{M}_C}_w \alpha$ [$\vDash^{\mathcal{M}_S}_w \alpha$]. A formula α is valid *on* a Chellas frame $\mathcal{F}_C = \langle W, R\rangle$ (short: $\vDash_{\mathcal{F}_C} \alpha$) [*on* a Segerberg frame $\mathcal{F}_S = \langle W, R\rangle$ (short: $\vDash_{\mathcal{F}_S} \alpha$)] iff α is valid in all Chellas models based on $\mathcal{F}_C$ [in all Segerberg models based on $\mathcal{F}_S$]. A formula α is valid w.r.t. a class of Chellas models $\mathbf{M}_C$ (short: $\vDash_{\mathbf{M}_C} \alpha$) [w.r.t. a class of Segerberg models $\mathbf{M}_S$ (short: $\vDash_{\mathbf{M}_S} \alpha$)] iff $\vDash_{\mathcal{M}_C} \alpha$ for all $\mathcal{M}_C \in \mathbf{M}_C$ [$\vDash_{\mathcal{M}_S} \alpha$ for all $\mathcal{M}_S \in \mathbf{M}_S$]. A formula α is valid w.r.t. a class of Chellas frames $\mathbf{F}_C$ (short: $\vDash_{\mathbf{F}_C} \alpha$) [w.r.t. a class of Segerberg frames $\mathbf{F}_S$ (short: $\vDash_{\mathbf{F}_S} \alpha$)] iff $\vDash_{\mathcal{F}_C} \alpha$ for all $\mathcal{F}_C \in \mathbf{F}_C$ [$\vDash_{\mathcal{F}_S} \alpha$ for all $\mathcal{F}_S \in \mathbf{F}_S$].

A formula α follows from a formula set Γ in a world w in a Chellas model $\mathcal{M}_C$ (short: $\Gamma \vDash^{\mathcal{M}_C}_w \alpha$) [in a Segerberg model $\mathcal{M}_S$ (short: $\Gamma \vDash^{\mathcal{M}_S}_w \alpha$)] iff $(\forall\beta \in \Gamma$: $\vDash^{\mathcal{M}_C}_w \beta) \Rightarrow \vDash^{\mathcal{M}_C}_w \alpha$ [$(\forall\beta \in \Gamma$: $\vDash^{\mathcal{M}_S}_w \beta) \Rightarrow \vDash^{\mathcal{M}_S}_w \alpha$].[5] A formula α follows from a formula set Γ in a Chellas model $\mathcal{M}_C$ (short: $\Gamma \vDash_{\mathcal{M}_C} \alpha$) [in a Segerberg model

[5] Blackburn et al. (2001, p. 31f) define and discuss consequence relations in modal logic. (This is not too often done in the modal logic literature.) The above notions correspond to their notion of local semantic consequence relation. This notion does in general not concur with their so-called global semantic consequence relation. The latter notion can, for example, be defined for validity in a Chellas frame [Segerberg frame] the following way: Γ implies α w.r.t. a Chellas frame $\mathcal{F}_C$ [w.r.t. a Segerberg frame $\mathcal{F}_S$] globally iff for all Chellas models $\mathcal{M}_C$ based on $\mathcal{F}_C$ holds: if $\forall\beta \in \Gamma(\vDash_{\mathcal{M}_C} \beta)$ then $\vDash_{\mathcal{M}_C} \alpha$ [for all Segerberg models $\mathcal{M}_S$ based on $\mathcal{F}_S$ holds: if $\forall\beta \in \Gamma(\vDash_{\mathcal{M}_S} \beta)$ then $\vDash_{\mathcal{M}_S} \alpha$]. Note that we can apply this notion also to classes of Chellas and Segerberg frames and models. We shall, however, not do so. Note that Blackburn et al. (2001) also use the notion of local logical consequence for their further investigation rather than their global logical consequence relation (cf. p. 32).

$\mathcal{M}_S$ (short: $\Gamma \vDash_{\mathcal{M}_S} \alpha$)] iff $\forall w$ in $\mathcal{M}_C$: $\Gamma \vDash_w^{\mathcal{M}_C} \alpha$ [$\forall w$ in $\mathcal{M}_S$: $\Gamma \vDash_w^{\mathcal{M}_S} \alpha$]. A formula α follows from a formula set Γ in a Chellas frame $\mathcal{F}_C$ (short: $\Gamma \vDash_{\mathcal{F}_C} \alpha$) [in a Segerberg frame $\mathcal{F}_S$ (short: $\Gamma \vDash_{\mathcal{F}_S} \alpha$)] iff $\forall \mathcal{M}_C$ based on $\mathcal{F}_C : \Gamma \vDash_{\mathcal{M}_C} \alpha$ [$\forall \mathcal{M}_S$ based on $\mathcal{F}_S : \Gamma \vDash_{\mathcal{M}_S} \alpha$]. A formula α follows from a formula set Γ in a class of Chellas models $\mathbf{M}_C$ (short: $\Gamma \vDash_{\mathbf{M}_C} \alpha$) [in a class of Segerberg models $\mathbf{M}_S$ (short: $\Gamma \vDash_{\mathbf{M}_S} \alpha$)] iff $\forall \mathcal{M}_C \in \mathbf{M}_C : \Gamma \vDash_{\mathcal{M}_C} \alpha$ [$\forall \mathcal{M}_S \in \mathbf{M}_S$: $\Gamma \vDash_{\mathcal{M}_S} \alpha$]. A formula α follows from a formula set Γ in a class of Chellas frames $\mathbf{F}_C$ (short: $\Gamma \vDash_{\mathbf{F}_C} \alpha$) [in a class of Segerberg frames $\mathbf{F}_S$ (short: $\Gamma \vDash_{\mathbf{F}_S} \alpha$)] iff $\forall \mathcal{F}_C \in \mathbf{F}_C : \Gamma \vDash_{\mathcal{F}_C} \alpha$ [$\forall \mathcal{F}_S \in \mathbf{F}_S$: $\Gamma \vDash_{\mathcal{F}_S} \alpha$]. Please note that the notions of logical consequence and logical validity (w.r.t. the same model, frame, class of models and class of frames) in general do not have the same level of expressiveness.

Finally we define different notions of (simultaneous) satisfiability: A set of formulas Γ is (simultaneously) satisfiable in a Chellas model $\mathcal{M}_C$ [in a Segerberg model $\mathcal{M}_S$] iff all $\alpha \in \Gamma$ are true at at least one world w in $\mathcal{M}_C$ [in $\mathcal{M}_S$]. A set of formulas Γ is (simultaneously) satisfiable in a class of Chellas models $\mathbf{M}_C$ [in a class of Segerberg models $\mathbf{M}_S$] iff all $\alpha \in \Gamma$ are true at at least some world w in a model $\mathcal{M}_C \in \mathbf{M}_C$ [in a model $\mathcal{M}_C \in \mathbf{M}_C$]. A set of formulas Γ is (simultaneously) satisfiable in a Chellas frame $\mathcal{F}_C$ [in a Segerberg frame $\mathcal{F}_S$] iff all $\alpha \in \Gamma$ are satisfiable in at least one Chellas model $\mathcal{M}_C$ based on $\mathcal{F}_C$ [in at least one Segerberg model $\mathcal{M}_S$ based on $\mathcal{F}_S$]. A set of formulas Γ is (simultaneously) satisfiable in a class of Chellas frames $\mathbf{F}_C$ [in a class of Segerberg frames $\mathbf{F}_S$] iff all $\alpha \in \Gamma$ are satisfiable in at least one model $\mathcal{M}_C$ based on $\mathcal{F}_C$, such that $\mathcal{F}_C \in \mathbf{F}_C$ [in at least one model $\mathcal{M}_S$ based on $\mathcal{F}_S$, such that $\mathcal{F}_S \in \mathbf{F}_S$].

4.3.5 Notions of Frame Correspondence

In this section, we describe and discuss notions of frame correspondence. These notions allow to characterize Chellas [Segerberg] frames in terms of axioms and frame conditions. We argue here for a characterization in terms of Chellas frames rather than Segerberg frames. This notion, then, serves as basis for the correspondence proofs in Chapter 5.

Let us now focus on Chellas frame correspondence (C-correspondence) and Segerberg frame correspondence (S-correspondence). We can define both notions following way:

Definition 4.21. *(C-Correspondence) Formula schema α corresponds to frame condition C_α w.r.t. the class of all Chellas frames $\mathbf{F}_C$ iff $\forall \mathcal{F}_C \in \mathbf{F}_C$ holds: $\vDash_{\mathcal{F}_C} \alpha \Leftrightarrow \vDash_{\mathcal{F}_C} C_\alpha$.*

Definition 4.22. *(S-Correspondence) Formula schema β corresponds to frame condition C_β w.r.t the class of all Segerberg frames $\mathbf{F}_S$ iff $\forall \mathcal{F}_S \in \mathbf{F}_S$ holds: $\vDash_{\mathcal{F}_S} \beta \Leftrightarrow \vDash_{\mathcal{F}_S} C_\beta$.*

Here, $\vDash_{\mathcal{F}_C} C_\alpha$ and $\vDash_{\mathcal{F}_S} C_\beta$ abbreviate that frame condition C_α and C_β hold for the Chellas frame $\mathcal{F}_C = \langle W, R \rangle$ and the Segerberg frame $\mathcal{F}_S = \langle W, R, P \rangle$, respectively. Frame conditions for S-frames and C-frames can both draw on the parameters W and R, whereas for frame conditions for S-frames the additional parameter P is available. As we argued in the previous section, we do not have an original interest in the parameter P. For example, the frame condition C_{T} of the principle $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow \beta$ ("T", see Table 5.2) for Chellas frames $\mathcal{F}_C = \langle W, R \rangle$ is $\forall X \subseteq W\, \forall w \in W(wR_X w)$ (see Table 5.4).[6] Rather for technical reasons (completeness proof, see Chapter 6), we have to restrict, for example, the frame condition C_{T} – which refers to Chellas frames – to the frame condition C'_{T} ($\forall X \subseteq P\, \forall w \in W(wR_X w)$), which draws in addition on the parameter P that is specific to Segerberg frames $\mathcal{F}_S = \langle W, R, P \rangle$.

Despite this fact, we prefer to give an account of frame conditions purely in terms of accessibility relations R and the set of worlds W, such as C_{T}. This enables us to describe classes of frames both by means of formulas and restrictions on the accessibility relation. We can, hence, discuss different conditional logic systems in terms of basic CS-semantics plus frame restrictions. Since this is our approach in Chapter 7, we, hence, opt for establishing C-correspondence rather than S-correspondence.

Note, however, that correspondence in terms of Chellas frames also gives us a handle on validity of formulas in terms of Segerberg frames, as the following lemma shows:

[6] We do not argue that T is a plausible principle for a conditional logic. (We think that principle T is counter-intuitive for any account of conditionals.) We, however, chose principle T, since it corresponds to one of the most simple frame conditions among the principles discussed in Chapters 5–7 (see Tables 5.1 and 5.2).

Lemma 4.23. *Let $\mathcal{F}_C = \langle W, R\rangle$ be a Chellas frame. Then, a formula α is valid on $\mathcal{F}_C$ iff it is valid on all Segerberg frames $\mathcal{F}_S$ based on $\mathcal{F}_C$.*

Proof. "⇒": Suppose that α is valid on $\mathcal{F}_C = \langle W, R\rangle$. So, we have to show that α is valid on any Segerberg frame based on $\mathcal{F}_C$. Let $\mathcal{F}_S = \langle W, R', P\rangle$ be an arbitrary Segerberg frame based on $\mathcal{F}_C$. Then, by definition $R' =\downarrow_P R$. Furthermore, let $\mathcal{M}_S = \langle W, R', P, V\rangle$ be a Segerberg model based on $\mathcal{F}_S$. So, $\mathcal{M}_S$ is based on $\mathcal{F}_C$. Since, $R' =\downarrow_P R$, due to Lemma 4.24 there is a Chellas model $\mathcal{M}_C = \langle W, R, V'\rangle$, such that $V' = V$. Hence for all formulas α in $\mathcal{L}_{\mathrm{KL}}$ and $w \in W$ holds: $V'(\alpha, w) = V(\alpha, w)$. $\mathcal{M}_C$ is, however, based on $\mathcal{F}_C$ and, hence, by assumption the formula α is valid in $\mathcal{M}_C$. So, the formula α is valid in the Segerberg model $\mathcal{M}_S = \langle W, R', P, V\rangle$. As this holds for arbitrary Segerberg models based on $\mathcal{F}_C$, the formula α is valid in this class of Segerberg frames.

"⇐": Suppose that α is valid on all Segerberg frames $\mathcal{F}_S = \langle W, R, P\rangle$ based on the Chellas frame $\mathcal{F}_C = \langle W', R'\rangle$. Then, α is valid on the Segerberg frame $\mathcal{F}_S = \langle W', R', \mathcal{P}(W')\rangle$. This particular Segerberg frame is, however, structurally equivalent to the Chellas frame $\mathcal{F}_C$, as (a) R is defined for all subsets of W and (b) valuations of Segerberg models based on $\mathcal{F}_S$ can take the same range of values as valuation functions of Chellas models based on $\mathcal{F}_C$. Hence, if α is valid in $\mathcal{F}_S$, then, it is valid in $\mathcal{F}_C$. □

Lemma 4.24. *For any Segerberg model $\mathcal{M}_S = \langle W, R, P, V\rangle$ based on a Chellas frame $\mathcal{F}_C = \langle W, R'\rangle$ there exists a Chellas model $\mathcal{M}_C = \langle W, R'', V'\rangle$, such that $\mathcal{M}_S$ is based on $\mathcal{F}_C$, $R'' = R'$ and $V' = V$.*

Proof. Let $\mathcal{M}_S = \langle W, R, P, V\rangle$ be a Segerberg model based on a Chellas frame $\mathcal{F}_C = \langle W, R'\rangle$. Note that $\mathcal{M}_C = \langle W, R', V\rangle$ is by definition a Chellas model. Let $\mathcal{M}'_C = \langle W, R', V'\rangle$ be a Chellas model such that $\mathcal{M}_C$ and $\mathcal{M}'_C$ are elementary equivalent. Then, (a) $V = V'$. Moreover, R is the restriction of R' to P. Since V is by assumption admissible in $\langle W, R, P\rangle$, all syntactically representable formulas in $\mathcal{M}_S$ are in P. Hence, R is defined w.r.t. all syntactically representable subsets of W in model $\mathcal{M}_C$. Thus, (b) R agrees w.r.t. the set of all syntactically representable subsets of W with R' and it follows that (b) $R' = R$. Hence, from (a) and (b) we get that $\mathcal{M}'_C$ is Chellas model with the desired properties. □

4.3.6 Standard and Non-Standard Chellas Models and Segerberg Frames

Note that C-Correspondence for an axiom α and its frame condition C_α does not imply that any Chellas model, which satisfies α also satisfies the frame condition C_α. Analogously to the Kripke-semantics case (Schurz, 2002a, p. 451), there are so-called non-standard Chellas models $\mathcal{M}_C = \langle W, R, V\rangle$, in which $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow \beta$, for example, is valid but the C-corresponding frame condition C_{T}– namely $\forall X \subseteq W\, \forall w \in W(wR_X w)$ – does not hold. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be such that $W = \{w_1, w_2\}, R = \{\langle w_1, w_2, \varnothing\rangle, \langle w_1, w_2, \{w_1\}\rangle, \langle w_1, w_2, \{w_2\}\rangle, \langle w_1, w_2, W\rangle, \langle w_2, w_1, \varnothing\rangle,$ $\langle w_2, w_1, \{w_1\}\rangle, \langle w_2, w_1, \{w_2\}\rangle, \langle w_1, w_2, W\rangle\}$ and that $V(p, w_1) = V(p, w_2)$ for all $p \in \mathcal{PP}$. It is easy to see that $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow \beta$ is valid in $\mathcal{M}_C$: By $V_{\Box\!\rightarrow}$ all formula $(\alpha_1 \mathrel{\Box\!\!\rightarrow} \beta), (\alpha_1 \mathrel{\Box\!\!\rightarrow} \beta), \ldots$ are true at a world w if β is true at all worlds accessible by R_X, where $X \subseteq W$. However, from our assumptions follows that for all formulas β holds that β is true at world w_1 iff β is true at w_2. Hence $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow \beta$ is the case, even if $\forall X \subseteq W\, \forall w \in W(wR_X w)$ does not hold.

In addition there are also non-standard Segerberg frames, on which, for example, $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow \beta$ is valid, but which do not satisfy the C-corresponding frame condition C_{T}: Let $\mathcal{F}_S = \langle W, R, P\rangle$ be a S-frame for which $\forall w_1, w_2 \in W(\forall X \in P(w_1 \in X \Leftrightarrow w_2 \in X) \wedge \forall Y \in P(w_1 \neq w_2 \rightarrow w_1 R_Y w_2))$ holds. It is easy to see (by the same considerations as described above) that $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow \beta$ is valid on $\mathcal{F}_S$ although C_{T} does not hold for subsets of P.

Let us now define the notion of standard S-frames:

Definition 4.25. *Let $\mathcal{F}_S = \langle W, R, P\rangle$ be a S-frame according to Definition 4.17 and let C_α be a frame condition, which C-corresponds to axiom schema α (see Definition 4.21). Then, $\mathcal{F}_S$ is standard Segerberg frame for* **CK**+α *iff C_α holds for all $Y \in P$.*

Note that we exclude by Definition 4.25 cases, such as the non-standard Segerberg frame described above. We, furthermore, take C-correspondence rather than S-correspondence here as basis, since we do not have a genuine interest in the parameter P of Segerberg frames (see previous section). Additionally, we sometime omit reference to **CK** and say that a frame $\mathcal{F}_S$ is standard w.r.t to α rather than w.r.t. **CK**+ α.

Table 4.2
Notions of Soundness and Completeness

$\mathbf{L}$ is sound w.r.t. $\mathbf{M}_S$	iff	$\forall\alpha\,(\vdash_{\mathbf{L}} \alpha \Rightarrow \vDash_{\mathbf{M}_S} \alpha)$
$\mathbf{L}$ is sound w.r.t. $\mathbf{M}_C$	iff	$\forall\alpha\,(\vdash_{\mathbf{L}} \alpha \Rightarrow \vDash_{\mathbf{M}_C} \alpha)$
$\mathbf{L}$ is sound w.r.t. $\mathbf{F}_S$	iff	$\forall\alpha\,(\vdash_{\mathbf{L}} \alpha \Rightarrow \vDash_{\mathbf{F}_S} \alpha)$
$\mathbf{L}$ is sound w.r.t. $\mathbf{F}_C$	iff	$\forall\alpha\,(\vdash_{\mathbf{L}} \alpha \Rightarrow \vDash_{\mathbf{F}_C} \alpha)$
$\mathbf{L}$ is w. (weakly) complete w.r.t. $\mathbf{M}_S{}^{\dagger}$	iff	$\forall\alpha\,(\vDash_{\mathbf{M}_S} \alpha \Rightarrow \vdash_{\mathbf{L}} \alpha)$
$\mathbf{L}$ is w. complete w.r.t. $\mathbf{M}_C{}^{\dagger}$	iff	$\forall\alpha\,(\vDash_{\mathbf{M}_C} \alpha \Rightarrow \vdash_{\mathbf{L}} \alpha)$
$\mathbf{L}$ is w. complete w.r.t. $\mathbf{F}_S{}^{\dagger}$	iff	$\forall\alpha\,(\vDash_{\mathbf{F}_S} \alpha \Rightarrow \vdash_{\mathbf{L}} \alpha)$
$\mathbf{L}$ is w. complete w.r.t. $\mathbf{F}_S^{\mathrm{st}}$	iff	$\forall\alpha\,(\vDash_{\mathbf{F}_S^{\mathrm{st}}} \alpha \Rightarrow \vdash_{\mathbf{L}} \alpha)$
$\mathbf{L}$ is w. complete w.r.t. $\mathbf{F}_C$	iff	$\forall\alpha\,(\vDash_{\mathbf{F}_C} \alpha \Rightarrow \vdash_{\mathbf{L}} \alpha)$
$\mathbf{L}$ is s. (strongly) complete w.r.t. $\mathbf{M}_S{}^{\ddagger}$	iff	$\forall\,\Gamma,\alpha\,(\Gamma \vDash_{\mathbf{M}_S} \alpha \Rightarrow \Gamma \vdash_{\mathbf{L}} \alpha)$
$\mathbf{L}$ is s. complete w.r.t. $\mathbf{M}_C{}^{\ddagger}$	iff	$\forall\,\Gamma,\alpha\,(\Gamma \vDash_{\mathbf{M}_C} \alpha \Rightarrow \Gamma \vdash_{\mathbf{L}} \alpha)$
$\mathbf{L}$ is s. complete w.r.t. $\mathbf{F}_S{}^{\ddagger}$	iff	$\forall\,\Gamma,\alpha\,(\Gamma \vDash_{\mathbf{F}_S} \alpha \Rightarrow \Gamma \vdash_{\mathbf{L}} \alpha)$
$\mathbf{L}$ is s. complete w.r.t. $\mathbf{F}_S^{\mathrm{st}}$	iff	$\forall\,\Gamma,\alpha\,(\Gamma \vDash_{\mathbf{F}_S^{\mathrm{st}}} \alpha \Rightarrow \Gamma \vdash_{\mathbf{L}} \alpha)$
$\mathbf{L}$ is s. complete w.r.t. $\mathbf{F}_C$	iff	$\forall\,\Gamma,\alpha\,(\Gamma \vDash_{\mathbf{F}_C} \alpha \Rightarrow \Gamma \vdash_{\mathbf{L}} \alpha)$

Note. The logic **L** is presupposed to be an extension of **CK**. We marked equivalent notions of completeness (see text) by † and ‡, respectively.

Let us now turn to our terminology regarding standard Segerberg frames: We abbreviate standard Segerberg frames and classes of standard Segerberg frames by $\mathcal{F}_S^{\mathrm{st}}$, $\mathcal{F}_S^{\mathrm{st}\prime}$, $\mathcal{F}_S^{\mathrm{st}\prime\prime}$, ... and $\mathbf{F}_S^{\mathrm{st}}$, $\mathbf{F}_S^{\mathrm{st}\prime}$, $\mathbf{F}_S^{\mathrm{st}\prime\prime}$, ..., respectively.[7] Moreover, models based on standard Segerberg frames (short: standard Segerberg models) are referred to by $\mathcal{M}_C^{\mathrm{st}}$, $\mathcal{M}_C^{\mathrm{st}\prime}$, $\mathcal{M}_C^{\mathrm{st}\prime\prime}$, ... Finally, the concepts of validity, logical consequence and (simultaneous) satisfiability regarding standard Segerberg frames, standard Segerberg models and classes of standard Segerberg frames are to be defined analogously to the respective concepts for (simple) Segerberg frames, (simple) Segerberg models and classes of (simple) Segerberg frames in Section 4.3.4, respectively.

[7]We do not define symbols for classes of standard Segerberg models, because these notions do not play any part in our investigation.

4.3.7 Notions of Soundness and Completeness

In the previous sections we described a range of validity and satisfiability concepts w.r.t. Chellas and Segerberg frames [models] and standard Segerberg frames. Based on the concepts of derivability, validity and logical consequence defined earlier, we can distinguish between the notions of soundness and completeness in Table 6.3. So, which among these notions are, then, the appropriate notions of soundness and completeness?

Note that we do not distinguish between weak and strong versions of soundness, since for system **CK** and all its extensions investigated in Chapters 5–7 the deduction theorem holds (cf. Section 4.2.6): $\Gamma \vdash \alpha$ is the case iff $\vdash \bigwedge \Gamma_f \rightarrow \alpha$, where Γ_f represents a finite subset of a (possibly infinite) set of formulas Γ and $\bigwedge \Gamma$ stands for the conjunction of all elements of Γ (cf. Schurz, 2001a, p. 454f, see also Sections 4.2.1). So, derivability of formulas from formula sets (strong version) is equivalent to derivability of formulas (weak version).

W.r.t. completeness concepts we can, however, distinguish between strong and weak versions. Note that all strong versions (e.g. w.r.t. $\mathbf{F}_C$) imply the respective weak versions (e.g. w.r.t. $\mathbf{F}_C$) (cf. Schurz, 2002a, p. 454f). We shall, hence, in this section focus on strong completeness versions.

Strong (s.) completeness w.r.t. classes of Segerberg models $\mathbf{M}_S$ is equivalent to s. completeness w.r.t. classes of Chellas models: Every Segerberg model $\mathcal{M}_S = \langle W, R, P, V\rangle$ is a Chellas model $\mathcal{M}_S = \langle W, R, V\rangle$, since by Definition 4.19 V is a valuation function w.r.t. $\langle W, R\rangle$. Moreover, for every Chellas model $\mathcal{M}_C = \langle W, R, V\rangle$ a Segerberg model $\mathcal{M}_S = \langle W, R, P_V, V\rangle$ exists (with the same W, R, and V), where $P_V = \{\|\alpha\|^{\mathcal{M}_C} \mid \mathcal{M}_C = \langle W, R, V\rangle$ and α is a formula of $\mathcal{L}_{\mathrm{KL}}\}$. So, if a logic **L** is s. complete w.r.t. a class of Segerberg models $\mathbf{M}_S$, then it is s. complete w.r.t. a class of Chellas models $\mathbf{M}_C$, as construed above, and vice versa.

We can, however, easily show that s. completeness w.r.t. classes of Chellas models $\mathbf{M}_C$ is trivial in the sense that every logic **L**, which is an extension of **CK**, is, then, complete w.r.t classes of Chellas models. This is due to the fact that for every logic **L**, which is an extension of **CK**, (except the inconsistent one) one can define a canonical model $\mathcal{M}^c$ (see Section 6.1.2; cf. Schurz, 2002a, p. 455f), such that all formulas in **L** are valid in $\mathcal{M}^c$. So, all logics **L** are strongly complete w.r.t.

to a class of Chellas models, namely the class containing the respective canonical model (or in case of the inconsistent logic the empty class of Chellas models) (cf. Hughes & Cresswell, 1996/2003, p. 159).

On first glance one might think that s. completeness w.r.t. to Segerberg frames is a stronger notion than s. completeness w.r.t Chellas models. This is however – as we shall prove below – not the case. A closer look reveals that this is not too surprising: Segerberg frames can in fact be viewed as a generalization of general frames in Kripke semantics (see Schurz, 2002a, p. 461) to CS-semantics. S. completeness w.r.t to general frames in Kripke semantics is, however, equivalent to s. completeness w.r.t. to Kripke models (see Schurz, 2002a, p. 461).

We shall, however, prove a stronger result w.r.t. CS-semantics, namely that a logic $\mathbf{L}$ is strongly characterized w.r.t. a class of Segerberg frames $\mathbf{F}_S$ iff it is strongly characterized w.r.t. a class of Chellas models $\mathbf{M}_C$, where strong characterization is defined as follows: We say that a logic $\mathbf{L}$ is strongly characterized w.r.t a class of Chellas models $\mathbf{M}_C$ [w.r.t. a class of Segerberg frames $\mathbf{F}_S$] iff $\mathbf{L}$ is sound and strongly complete w.r.t. $\mathbf{M}_C$ [w.r.t. $\mathbf{F}_S$]. Let us prove now the following theorem:

Theorem 4.26. *A logic* $\mathbf{L}$ *is strongly characterized w.r.t. a class of Chellas models* $\mathbf{M}_C$ *(short: strong* $\mathbf{M}_C$*-characterization) iff* $\mathbf{L}$ *is strongly characterized w.r.t. a class of Segerberg frames* $\mathbf{F}_S$ *(short: strong* $\mathbf{F}_S$*-characterization).*

Proof. By Lemmata 4.27 and 4.28. □

Lemma 4.27. *A logic* $\mathbf{L}$ *is strongly characterized w.r.t. a class of Chellas models* $\mathbf{M}_C$ *if* $\mathbf{L}$ *is strongly characterized w.r.t. a class of Segerberg frames* $\mathbf{F}_S$*.*

Proof. To prove strong characterization of a logic $\mathbf{L}$ w.r.t. a class of Chellas models ($\mathbf{M}_C$-characterization), we show that (a) arbitrary $\mathbf{L}$-consistent formula sets are satisfiable in a Chellas model $\mathcal{M}_C$, and that (b) $\mathbf{L}$ is valid in that given model $\mathcal{M}_C$. By (a), (b) and the consistency lemma (Lemma Schurz, 2002a, p. 455; see also Lemma 6.4) this implies that (i) $\mathbf{L}$ is strongly complete w.r.t. a class of Chellas models $\mathbf{M}_C$ and that (ii) $\mathbf{L}$ is sound w.r.t. to the same class of Chellas models $\mathbf{M}_C$. Hence, we shall conclude by definition of strong $\mathbf{M}_C$-characterization, that $\mathbf{L}$ is strongly characterized by a class of Chellas models $\mathbf{M}_C$.

Suppose that a logic **L** is strongly characterized w.r.t. a class of Segerberg frames $\mathbf{F}_S$. By the notion of strong $\mathbf{F}_S$-characterization, logic **L** is sound w.r.t. $\mathbf{F}_S$ and strongly complete w.r.t. $\mathbf{F}_S$. This implies on the basis of the consistency lemma (Schurz, 2002a, p. 455; see also Lemma 6.4) that (c) if Γ is a **L**-consistent formula set, then Γ is satisfiable in some Segerberg model $\mathcal{M}_S = \langle W, R, P, V \rangle$ based on a Segerberg frame $\mathcal{F}_S = \langle W, R, P \rangle$, so that (d) all formulas in **L** are valid on $\mathcal{F}_S$.

Let Γ be a **L**-consistent set of formulas. Then, it follows by (c) that Γ is satisfiable in some Segerberg model $\mathcal{M}_S = \langle W, R, P, V \rangle$ based on a Segerberg frame $\mathcal{F}_S = \langle W, R, P \rangle$. This implies by Definition 4.19 and 4.13 that $\langle W, R, V \rangle$ is a Chellas model. Hence, (a′) Γ is satisfiable in $\mathcal{M}_C = \langle W, R, V \rangle$. Moreover, by (d) all formulas in the formula set **L** are valid on $\mathcal{F}_S$ and, hence, in $\mathcal{M}_S$. Since the valuation function is the same for $\mathcal{M}_S$ and $\mathcal{M}_C$ and the parameters W and R are identical in $\mathcal{M}_S$ and $\mathcal{M}_C$, it follows that (b′) all formulas of **L** are valid in $\mathcal{M}_C$. Points (a′) and (b′) imply by the above considerations that **L** is strongly $\mathbf{M}_C$-characterized. □

Lemma 4.28. *A logic* **L** *is strongly characterized w.r.t. a class of Chellas models* $\mathbf{M}_C$ *only if* **L** *is strongly characterized w.r.t. a class of Segerberg frames* $\mathbf{F}_S$.

Proof. To prove strong characterization of a logic **L** w.r.t. a class of Segerberg frames ($\mathbf{F}_S$-characterization), we show that (a) arbitrary **L**-consistent formula sets are satisfiable in a Segerberg model $\mathcal{M}_S = \langle W, R, P, V \rangle$ and that (b) **L** is valid on the Segerberg frame $\mathcal{F}_S = \langle W, R, P \rangle$, on which $\mathcal{M}_S$ is based. By the consistency lemma (Lemma Schurz, 2002a, p. 455; see also Lemma 6.4) points (a) and (b) imply that (i) **L** is strongly complete w.r.t. a class of Segerberg frames $\mathbf{F}_S$ that (ii) **L** is sound w.r.t. to the same class of Segerberg frames $\mathbf{F}_S$. Hence, we can conclude by definition of strong $\mathbf{F}_S$-characterization, that **L** is strongly characterized by a class of Segerberg frames $\mathbf{F}_S$.

Let logic **L** be strongly characterized w.r.t. a class of Chellas models $\mathbf{M}_C$. Then, on the basis of the notion of strong $\mathbf{M}_C$-characterization, it follows that logic **L** is sound w.r.t. $\mathbf{M}_S$ and strongly complete w.r.t. $\mathbf{M}_S$. This implies by the consistency lemma (Schurz, 2002a, p. 455; see also Lemma 6.4) that (c) if Γ is a **L**-consistent formula set, then Γ is satisfiable in some Chellas model $\mathcal{M}_C$, so that (d) all formulas in **L** are valid in $\mathcal{M}_C$.

Suppose that Γ is a **L**-consistent set of formulas. Then, (c) implies that Γ

is satisfiable in some Chellas model $\mathcal{M}_C = \langle W, R, V \rangle$. We define now P_V as $\{\|\alpha\|^{\mathcal{M}_C} \mid \mathcal{M}_C = \langle W, R, V \rangle$ and α is a formula of $\mathcal{L}_{\mathrm{KL}}\}$. It is, then, easy to show that $\mathcal{M}_S = \langle W, R, P_V, V \rangle$ is a Segerberg model. Since $\mathcal{M}_S$ and $\mathcal{M}_C$ agree in the parameters W, R and V, it follows that (a′) Γ is satisfiable in $\mathcal{M}_S$. Hence, it remains to be shown that $\mathbf{L}$ is valid on $\mathcal{F}_S$, where $\mathcal{F}_S = \langle W, R, P_V \rangle$.

Let $\mathcal{M}_S = \langle W, R, P_V, V' \rangle$ be an arbitrary Segerberg model based on $\mathcal{F}_S = \langle W, R, P_V \rangle$. Moreover, let α be an arbitrary formula, such that $\alpha \in \mathbf{L}$. Then, α is by point (d) valid in $\mathcal{M}_C$. Since the definition of a logic implies that $\mathbf{L}$ is closed under substitution (see Section 4.2.2), it follows that also $s(\alpha)$ is valid in $\mathcal{M}_C = \langle W, R, V \rangle$ for any substitution instance $s(\alpha)$ of α. This means that for all substitution instances $s_1(\alpha)$, $s_2(\alpha)$, ... and worlds $w \in W$ holds that $V(\alpha, w) = V(s_1(\alpha), w) = V(s_2(\alpha), w) = \ldots$ Now, let V and V' be determined as above, and suppose that $(*)$ $V(s(\alpha), w) = V'(\alpha, w)$ holds for every world w in W, every formula α of $\mathcal{L}_{\mathrm{KL}}$ and some substitution instance $s(\alpha)$ of α. Then, it follows that $V(\alpha, w) = V'(\alpha, w)$ for all $w \in W$. So, the formula α is also valid in $\mathcal{M}_S = \langle W, R, P_V, V' \rangle$. Hence, $(*)$ implies that (b′) all formulas $\alpha \in \mathbf{L}$ are also valid in $\mathcal{F}_S$.

We shall now prove $(*)$ by induction on the construction of formulas. However, rather than proving $(*)$ directly we show that $\|s(\alpha)\|^{\mathcal{M}_C} = \|\alpha\|^{\mathcal{M}_S}$, which – as one can easily establish – implies $(*)$ by the definitions of $\mathcal{M}_C$, $\mathcal{M}_S$ and $\mathrm{Def}_{\|\cdot\|}$.

Induction Basis: Let p_i be an arbitrary propositional formula ($i \in \mathbb{N}$). Then, $\|p_i\|^{\mathcal{M}_S} \in P_V$ by Definition 4.19 and 4.18. P_V is, however, by definition the subset of W, such that for all elements X of P_V holds that $X = \|\alpha_i\|^{\mathcal{M}_C}$ for some formula α_i in language $\mathcal{L}_{\mathrm{KL}}$ ($i \in \mathbb{N}$). Hence, it is the case that $\|p_i\|^{\mathcal{M}_S} = \|\alpha_i\|^{\mathcal{M}_C}$ for some formula α_i of $\mathcal{L}_{\mathrm{KL}}$. Since every formula α_i of $\mathcal{L}_{\mathrm{KL}}$ is a substitution instance of an atomic formula p_i, it holds that $\|p_i\|^{\mathcal{M}_S} = \|s(p_i)\|^{\mathcal{M}_C}$ for some substitution instance of p_i.

Induction Steps: Due to the definition of s, $V_\neg$ and $\mathrm{Def}_{\|\cdot\|}$ it is the case that $\|s(\neg\alpha)\|^{\mathcal{M}_C} = \|\neg s(\alpha)\|^{\mathcal{M}_C} = W - \|s(\alpha)\|^{\mathcal{M}_C}$. By induction hypothesis it holds that $\|s(\alpha)\|^{\mathcal{M}_C} = \|\alpha\|^{\mathcal{M}_S}$. Thus, it follows that $\|s(\neg\alpha)\|^{\mathcal{M}_C} = W - \|\alpha\|^{\mathcal{M}_S}$. Since $V_\neg$ and $\mathrm{Def}_{\|\cdot\|}$ give us that $W - \|\alpha\|^{\mathcal{M}_S} = \|\neg\alpha\|^{\mathcal{M}_S}$, we get $\|s(\neg\alpha)\|^{\mathcal{M}_C} = \|\neg\alpha\|^{\mathcal{M}_S}$.

By definition of s, $V_\wedge$ and $\mathrm{Def}_{\|\cdot\|}$ we have that $\|s(\alpha \wedge \beta)\|^{\mathcal{M}_C} = \|s(\alpha) \wedge s(\beta)\|^{\mathcal{M}_C} = \|s(\alpha)\|^{\mathcal{M}_C} \cap \|s(\beta)\|^{\mathcal{M}_C}$. By induction hypothesis, we have $\|s(\alpha)\|^{\mathcal{M}_C} = \|\alpha\|^{\mathcal{M}_S}$ and

$\|s(\beta)\|^{\mathcal{M}_C} = \|\beta\|^{\mathcal{M}_S}$. This implies that $\|s(\alpha \wedge \beta)\|^{\mathcal{M}_C} = \|\alpha\|^{\mathcal{M}_S} \cap \|\beta\|^{\mathcal{M}_S}$. Since by $V_\wedge$ and $\mathrm{Def}_{\|\cdot\|}$ it is the case that $\|\alpha\|^{\mathcal{M}_S} \cap \|\beta\|^{\mathcal{M}_S} = \|\alpha \wedge \beta\|^{\mathcal{M}_S}$, we get that $\|s(\alpha \wedge \beta)\|^{\mathcal{M}_C} = \|\alpha \wedge \beta\|^{\mathcal{M}_S}$.

On the basis of the definition of s, $V_{\Box\!\!\rightarrow}$ and $\mathrm{Def}_{\|\cdot\|}$ we have $\|s(\alpha \Box\!\!\rightarrow \beta)\|^{\mathcal{M}_C} = \|s(\alpha) \Box\!\!\rightarrow s(\beta)\|^{\mathcal{M}_C} = \{w \mid \forall w'(wR_{\|s(\alpha)\|^{\mathcal{M}_C}} w' \Rightarrow w' \in \|s(\beta)\|^{\mathcal{M}_C})$. The induction hypothesis gives us that $\|s(\alpha)\|^{\mathcal{M}_C} = \|\alpha\|^{\mathcal{M}_S}$ and that $\|s(\beta)\|^{\mathcal{M}_C} = \|\beta\|^{\mathcal{M}_S}$. Hence, we get $\|s(\alpha \Box\!\!\rightarrow \beta)\|^{\mathcal{M}_C} = \{w \mid \forall w'(wR_{\|\alpha\|^{\mathcal{M}_S}} w' \Rightarrow w' \in \|\beta\|^{\mathcal{M}_S})$. As $V_{\Box\!\!\rightarrow}$ and $\mathrm{Def}_{\|\cdot\|}$ imply that $\{w \mid \forall w'(wR_{\|\alpha\|^{\mathcal{M}_S}} w' \Rightarrow w' \in \|\beta\|^{\mathcal{M}_S}) = \|\alpha \Box\!\!\rightarrow \beta\|^{\mathcal{M}_S}$, we get that $\|s(\alpha \Box\!\!\rightarrow \beta)\|^{\mathcal{M}_C} = \|\alpha \Box\!\!\rightarrow \beta\|^{\mathcal{M}_S}$.

□

Let us now sum up the above discussion on appropriate notions of completeness. We saw that the stronger versions are preferable to weak versions (see Table 6.3) and that (i) completeness w.r.t. classes of Segerberg models $\mathbf{M}_S$, classes of Chellas models $\mathbf{M}_C$ and classes of Segerberg frames $\mathbf{F}_S$ are equivalent and (ii) trivial in the sense that every logic **L**, which is an extension of **CK**, is then strongly frame complete. This leaves us with two further options: (a) strong completeness w.r.t. standard Segerberg frames $\mathbf{F}_C^{\mathrm{st}}$ and (b) strong completeness w.r.t. classes of Chellas frames $\mathbf{F}_C$.

Completeness w.r.t. classes of Chellas frames restricts the characterization of logical systems to purely structural conditions on W and R. Completeness w.r.t. classes of Segerberg frames takes in addition the parameter P into account, which allows us to limit the set of valuation functions of Segerberg models (hence Theorem 4.26). Since it is philosophically preferable for a logical system to admit all possible valuation of all non-logical symbols (Schurz, 2002a, p. 447), notions of soundness and completeness w.r.t. classes Chellas frames are more appropriate. Note, however, that completeness w.r.t. classes of Chellas frames meets certain obstacles not present in Kripke-semantics. In particular there are strictly more accessibility relations than possible worlds in any Chellas frame (Point a of Section 4.3.1). Moreover, accessibility relations relativized to syntactically non-representable sets are defined (Point c2 of Section 4.3.1) although only accessibility relations relativized to syntactically representable sets impact the truth of formulas at possible worlds in a model (Point c1 of Section 4.3.1). It is exactly due to those obstacles that it seems doubtful that a full characterization of the lattice of systems discussed in Chapter 5 in terms of classes of Chellas frames is viable.

Hence, our final option is completeness w.r.t to standard Segerberg frames. This notion of strong standard Segerberg completeness does not suffer from the problems associated with strong Chellas frame completeness (as we shall show in Chapter 6). Moreover, it is – as we will argue in the remainder of this section – non-trivial in the sense that not every logic **L**, which is an extension of **CK**, is complete w.r.t. standard Segerberg frames. We shall, however, not prove this point here, but rather illustrate why Theorem 4.26 does not go through for completeness w.r.t. standard Segerberg frames (rather than simple Segerberg frames).

To prove Theorem 4.26, we relied on Lemma 4.28. In the proof for this lemma we constructed for each Chellas model $\mathcal{M}_C = \langle W, R, V\rangle$ a Segerberg frame $\mathcal{F}_S = \langle W, R, P_V, V\rangle$, where $P_V = \{\|\alpha\|^{\mathcal{M}_C} \,|\, \mathcal{M}_C = \langle W, R, V\rangle$ and α is a formula of $\mathcal{L}_{\mathrm{KL}}\}$. Such a move is, however, not in general possible for standard Segerberg models. For the logic **CK**+T, for instance, where T is $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow \beta$, such a procedure is not in general possible. Given peculiar truth assignments, such as in the non-standard Chellas model $\mathcal{M}_C = \langle W, R, V\rangle$ described in Section 4.3.6, we cannot construct a standard Segerberg frame by using this definition of P_V. This is due to the fact that in $\mathcal{M}_C$ the relation R – whether restricted to P_V or not – does not satisfy C_{T}, namely it does not hold that for all worlds w it is the case that wR_Xw (see Section 4.3.6).

Chapter 5

Frame Correspondence

In the present chapter we discuss and provide formal results regarding frame correspondence w.r.t. CS-semantics. We will, for that purpose, focus exclusively on C-correspondence (i.e. correspondence w.r.t. Chellas frames), as defined in Section 4.3.5. We employ the notion of C-correspondence rather than S-correspondence (i.e. correspondence w.r.t Segerberg frames, see Section 4.3.5), since we aim to align systems based on the Chellas-Segerberg (CS) semantics both in terms axioms and restrictions on the accessibility relation only. C-correspondence seems to be the more appropriate notion for this purpose. In addition, our formal results regarding C-correspondence proofs serve as basis for the philosophical discussion in Chapter 7.

Let us now give an example for C-correspondence. The axiom $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow \beta$ (Principle T), for instance, C-corresponds to the frame condition C_{T}, namely $\forall X \subseteq W\, \forall w \in W(wR_Xw)$, where X is a subset of the set of possible worlds W. This means (by Definition 4.21) that principle T is valid in a Chellas frame $\mathcal{F}_C = \langle W, R\rangle$ iff the frame restriction C_{T} holds for $\mathcal{F}_C$.

Note that all (interesting) frame conditions in CS-semantics are essentially second-order, in the sense that they involve in addition to quantification over possible worlds also quantification over sets of possible worlds $X_1, X_2, \ldots$ This is due to fact that all accessibility relations R_X in CS-semantics are relativized to subsets of the sets of possible worlds W. In contrast, for Kripke semantics this is not the case. Rather specific classes of axioms pertaining to Kripke semantics correspond to first-order frame conditions, which involve only quantification over possible worlds (not sets of possible worlds) (cf. van Benthem, 2001, Section 2.2; cf. also Schurz, 2001a, p. 451).

Moreover, for Kripke semantics correspondence results are in a certain sense trivial, since there exists a translation procedure, which allows one to go from modal formulas to corresponding frame conditions (cf. van Benthem, 2001, Section 1).[1] In a similar vain, we can for CS-semantics translate conditional logic axioms into corresponding trivial frame conditions. These trivial frame conditions, however, are in general more complicated and from a philosophical standpoint less perspicuous than necessary. For example, the trivial frame condition, which corresponds to principle $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow \beta$ is *not* the frame condition C_{T} described above, but rather $\forall X, Y\, \forall w, w'((wR_X w' \Rightarrow w' \in Y) \Rightarrow w \in Y)$. Here the reference to Y in the trivial frame condition is in a sense superfluous, since we can prove – as we shall do – that $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow \beta$ C-corresponds to C_{T}, which does not make reference to a second set of possible worlds. The frame condition C_{T} is, hence, non-trivial insofar as it does not make unnecessary reference to subsets of the set of possible worlds W.

In this chapter we will, first, give non-trivial frame conditions for the 29 conditional logic principles (see Section 5.1). We, then, define a general translation procedure from axiom schemas to trivial frame conditions, show that this definition gives us the desired C-correspondence result and provide a definition of non-trivial frame conditions (see Section 5.2). We, finally, give correspondence results for the 29 non-trivial frame conditions (see Section 5.3). We are neither aware of any definition or discussion of trivial and non-trivial frame conditions in the literature for CS-semantics nor of any C-correspondence or S-correspondence proofs for any of the 29 conditional logic principles discussed here.

[1] Note that van Benthem (2001, Section 1) provides only examples for translations from modal axioms into corresponding frame conditions and does not give a general translation procedure.

5.1 Non-Trivial Frame Conditions for a Lattice of Conditional Logics

Table 5.1 and 5.2 list the 29 axiom schemata investigated in this and the following chapter. These conditional logic principles are partially taken from indicative conditional logics systems, such as system **P** by Kraus et al. (1990) and Lehmann and Magidor (1992), and counterfactual conditional logic systems, such as D. Lewis (1973/2001). These 29 axiom schemata are, however, not all logically independent from each other, but some of them are inter-derivable given the rules of system **CK** described in Section 4.2.6 (cf. Chapter 7). Note in this context that our terminology is in accord with the literature in non-monotonic reasoning (i.e. Kraus et al., 1990; Lehmann & Magidor, 1992) rather than the conditional logic literature (e.g. Nute, 1980; Nute & Cross, 2001).

We, first, describe axiom schemas, which are typically associated with system **P** (see Section 3.5) and its extensions, such as system **P**$^{+}$ (see Section 3.5). The expressions 'Refl', 'CM', 'CC' and 'RM' stand for 'Reflexivity', 'Cautious Monotonicity' and 'Cautious Cut' and 'Rational Monotonicity' , respectively (see Schurz, 1998, p. 82; Kraus et al., 1990, p. 177f and p. 197; Lehmann & Magidor, 1992, p. 5f and p. 18). Moreover, the formula schema $\alpha \mathrel{\Diamond\!\!\rightarrow} \beta$ in the principle RM abbreviates $\neg(\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta)$ (Def$_{\Diamond\!\rightarrow}$, Section 4.2.1). The labels 'Loop', 'Or' and 'S' stem again from Kraus et al. (1990), but the authors do not explain what the names stand for. However, the respective axiom schemas suggest that these expressions abbreviate 'Loop of Antecedent and Consequent Formulas', 'Introduction of Disjunctions in the Antecedent' and 'Shift of Conjuncts from the Antecedent to the Consequent', respectively.

Furthermore, Nute and Cross (2001, p. 10) and Nute (1980, p. 52f) do not motivate their names for the principles MOD and CEM. However, the names 'CEM' and 'MOD' presumably abbreviate 'Modality' and 'Conditional Excluded Middle', respectively. Note that the principle MOD is used in D. Lewis' (1973/2001) axiomatization of his counterfactual system **VC** (p. 132) and that CEM is specific to Stalnaker's (1968; Stalnaker & Thomason, 1970) conditional logic (cf. D. Lewis, 1973/2001, p. 79).

Moreover, the principles P-Cons and WOR are discussed in the literature on

Table 5.1
Axioms Schemas for Conditional Logics

System P	
$\alpha \mathrel{\Box\!\!\rightarrow} \alpha$	(Refl)
$(\alpha \mathrel{\Box\!\!\rightarrow} \gamma) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$	(CM)
$(\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \gamma)$	(CC)
$(\alpha_0 \mathrel{\Box\!\!\rightarrow} \alpha_1) \wedge \ldots \wedge (\alpha_{k-1} \mathrel{\Box\!\!\rightarrow} \alpha_k) \wedge (\alpha_k \mathrel{\Box\!\!\rightarrow} \alpha_0) \rightarrow (\alpha_0 \mathrel{\Box\!\!\rightarrow} \alpha_k)\ (k \geq 2)$	(Loop)
$(\alpha \mathrel{\Box\!\!\rightarrow} \gamma) \wedge (\beta \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \vee \beta \mathrel{\Box\!\!\rightarrow} \gamma)$	(Or)
$(\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma))$	(S)
Extensions of System P	
$(\neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha) \rightarrow (\beta \mathrel{\Box\!\!\rightarrow} \alpha)$	(MOD)
$(\alpha \mathrel{\Box\!\!\rightarrow} \gamma) \wedge (\alpha \mathrel{\Diamond\!\!\rightarrow} \beta) \rightarrow (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$	(RM)
$(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \vee (\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta)$	(CEM)
Axioms from Weak Probability Logic (Threshold Logic)	
$\neg(\top \mathrel{\Box\!\!\rightarrow} \bot)$	(P-Cons)
$(\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma) \wedge (\alpha \wedge \neg\beta \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \gamma)$	(WOR)
Monotonic Systems	
$(\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \delta) \wedge (\gamma \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \wedge \gamma \mathrel{\Box\!\!\rightarrow} \delta)$	(Cut)
$(\alpha \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$	(Mon)
$(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\beta \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \gamma)$	(Trans)
$(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\neg\beta \mathrel{\Box\!\!\rightarrow} \neg\alpha)$	(CP)
Bridge Principles	
$(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \rightarrow \beta)$	(MP)
$\alpha \wedge \beta \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \beta)$	(CS)
$(\neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha) \rightarrow \alpha$	(TR)
$(\top \mathrel{\Box\!\!\rightarrow} \alpha) \rightarrow \alpha$	(Det)
$\alpha \rightarrow (\top \mathrel{\Box\!\!\rightarrow} \alpha)$	(Cond)
Collapse Conditions Material Implication	
$\beta \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \beta)$	(VEQ)
$\neg\alpha \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \beta)$	(EFQ)

For abbreviations see text.

Table 5.2
Axioms Schemas for Conditional Logics (Continued)

Traditional Extensions	
$(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \mathrel{\Diamond\!\!\rightarrow} \beta)$	(D)
$(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow \beta$	(T)
$\alpha \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} (\alpha \mathrel{\Diamond\!\!\rightarrow} \beta))$	(B)
$(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} (\alpha \mathrel{\Box\!\!\rightarrow} \beta))$	(4)
$(\alpha \mathrel{\Diamond\!\!\rightarrow} \beta) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} (\alpha \mathrel{\Diamond\!\!\rightarrow} \beta))$	(5)
Iterated Principles	
$(\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} (\beta \mathrel{\Box\!\!\rightarrow} \gamma))$	(Ex)
$(\alpha \mathrel{\Box\!\!\rightarrow} (\beta \mathrel{\Box\!\!\rightarrow} \gamma)) \rightarrow (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$	(Im)

For abbreviations see text.

weak probabilistic logics, such as the threshold logic **O** (Hawthorne & Makinson, 2007, p. 252 and Footnote 2; cf. also Section 3.4.4). Moreover, the Principle P-Cons is also employed by Schurz (2008, p. 90, p. 83). The name 'WOR' stands for 'weak OR' (Hawthorne & Makinson, 2007, p. 252), whereas we abbreviate by 'P-Cons' the name 'Probabilistic Consistency'.

The names of the monotonic principles 'Mon', 'Trans' and 'CP' stand for 'Monotonicity', 'Transitivity' and 'Contraposition', respectively. Note that Cut does not correspond to "Cut" in Kraus et al. (1990, p. 177) and Lehmann and Magidor (1992, p. 6), but translates into our notion of Cautious Cut, denoted by 'CC'. Our notion of Cut corresponds to principle (9) in Lehmann and Magidor (1992, p. 6). The names of the bridge principles MP, TR, Det, Cond, VEQ and EFQ abbreviate 'Modus Ponens', 'Total Reflexivity', 'Detachment' and 'Conditionalization', 'Verum Ex Quodlibet' and 'Ex Falso Quodlibet', respectively. The names 'MP' and 'CS' on the one hand and 'VEQ' and 'EFQ' on the other hand go back to Nute (1980, p. 52 and p. 40; see also Nute & Cross, 2001, p. 10 and p. 14) and Weingartner and Schurz (1986, p. 10), respectively. Unfortunately, neither Nute (1980) nor Nute and Cross (2001) explain what 'CS' stands for. The name 'TR' follows D. Lewis (1973/2001, p. 121). Note that Schurz's (Schurz, 1998, p. 83) principle Poss ("Possibility") is p.c.-equivalent to TR. (We will not provide a proof for this fact here.)

Axioms D, T, B, 4 and 5 represent generalizations of the respective modal logic principles in Kripke semantics (cf. Schurz, 2002a, p. 451). We generalized the corresponding frame conditions – namely the seriality, reflexivity, symmetry, transitivity and euclidicity conditions (cf. Schurz, 2002a, p. 451) – to CS-semantics (see Table 5.4) and identified the conditional logic principles D, T, B, 4 and 5 (see Table 5.2), which correspond to these generalized frame conditions. Finally, the names'Ex' and 'Im' abbreviate 'Exportation' and 'Importation', respectively. The principles Ex and Im are, for example, discussed by Arló-Costa (2001, 2007) and McGee (1989; see also McGee, 1985, p. 465).

The non-trivial frame conditions for the principles in Tables 5.1 and 5.2 can be found in Tables 5.3 and 5.4. A range of non-trivial frame conditions for conditional logic axioms were not specified in the conditional logic literature (i.e. Segerberg, 1989; Chellas, 1975). Chellas (1975, p. 142) and Segerberg (1989, p. 163) identified non-trivial frame conditions for axioms Refl, MP and Or on the one hand and CM, RM, S, Det and Cond on the other hand. Segerberg (1989) discussed non-trivial frame conditions for two further frame conditions, namely his axioms #2 and #6. Note that his axiom #2 is similar to MOD from Table 5.1. Segerberg's version might be formulated as $(\neg\alpha \mathrel{\Box\!\!\rightarrow} \bot) \rightarrow (\beta \mathrel{\Box\!\!\rightarrow} \alpha)$. The principle MOD and #2 are, however, only equivalent given Refl. Thus, their corresponding frame conditions differ. Moreover, Segerberg's axiom #6 amounts to $(\alpha \mathrel{\Diamond\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma)) \rightarrow (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$. Note that there is a related principle $(\alpha \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma)) \rightarrow (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$ (EHD, easy half of the deductions theorem) sometimes discussed in the literature (e.g. Kraus et al., 1990, p. 180). We had to omit a formal investigation of these principles due to limitations of space and time.

We, however, identified non-trivial frame conditions for a further 16 principles, namely for CC, Loop, MOD, CEM, P-Cons, WOR, Cut, Mon, Trans, CP, CS, TR, VEQ, EFQ, Ex and Im. For the remaining 5 principles D, T, B, 4 and 5 we took the reverse approach: We started with generalizations of frame conditions in Kripke semantics to CS-semantics and identified corresponding axioms (see above).

The axioms and the frame conditions form both a lattice of systems: The proof-theoretic side of this lattice is determined by system **CK** plus axioms from Tables 5.1 and 5.2 and the model-theoretic side is described by Chellas frames

Table 5.3
Axioms and Corresponding Conditions

System P	
Refl	$\forall w, w'(wR_X w' \Rightarrow w' \in X)$
CM	$\forall w(\forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_{X\cap Y} w' \Rightarrow wR_X w'))$
CC	$\forall w(\forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_X w' \Rightarrow wR_{X\cap Y} w'))$
Loop	$\forall w(\forall w'(wR_{X_0} w' \Rightarrow w' \in X_1) \wedge \ldots \wedge \forall w'(wR_{X_{k-1}} w' \Rightarrow w' \in X_k) \wedge$ $\forall w'(wR_{X_k} w' \Rightarrow w' \in X_0) \Rightarrow \forall w'(wR_{X_0} w' \Rightarrow w' \in X_k))$ $(k \geq 2)$
Or	$\forall w, w'\ (wR_{X\cup Y} w' \Rightarrow wR_X w' \vee wR_Y w')$
S	$\forall w, w'(wR_X w' \wedge w' \in Y \Rightarrow wR_{X\cap Y} w')$
Extensions of System P	
MOD	$\forall w, w'(wR_{-X} w' \Rightarrow w' \in X) \Rightarrow \forall w'(wR_Y w' \Rightarrow w' \in X)$
RM	$\forall w(\exists w'(wR_X w' \wedge w' \in Y) \Rightarrow \forall w'(wR_{X\cap Y} w' \Rightarrow wR_X w'))$
CEM	$\forall w, w', w''(wR_X w' \wedge wR_X w'' \Rightarrow w'' = w')$
Axioms from Weak Probability Logic (Threshold Logic)	
P-Cons	$\forall w \exists w'(wR_W w')$
WOR	$\forall w, w'(wR_X w' \Rightarrow wR_{X\cap Y} w' \vee wR_{X\cap -Y} w')$
Monotonic Systems	
Cut	$\forall w(\forall w'(wR_Z w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_{X\cap Z} w' \Rightarrow wR_{X\cap Y} w'))$
Mon	$\forall w, w'\ (wR_{X\cap Y} w' \Rightarrow wR_X w')$
Trans	$\forall w(\forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_X w' \Rightarrow wR_Y w'))$
CP	$\forall w(\forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_{-Y} w' \Rightarrow w' \in -X))$
Bridge Principles	
MP	$\forall w(w \in X \Rightarrow wR_X w)$
CS	$\forall w(w \in X \Rightarrow \forall w'(wR_X w' \Rightarrow w' = w))$
Poss	$\forall w(\forall w'(wR_{-X} w' \Rightarrow w' \in X) \Rightarrow w \in X)$
Det	$\forall w(wR_W w)$
Cond	$\forall w(wR_W w' \Rightarrow w' = w)$
Collapse Conditions Material Implication	
VEQ	$\forall w, w'(wR_X w' \Rightarrow w' = w)$
EFQ	$\forall w(w \in -X \Rightarrow \neg\exists w'(wR_X w'))$

For enhanced readability outer universal quantifiers regarding sets of possible worlds X, $Y, \ldots, X_1, X_2, \ldots$ are omitted. For abbreviations see text.

Table 5.4
Axioms and Corresponding Frame Conditions (Continued)

Traditional Extensions		
	D	$\forall w \exists w'(wR_X w')$
	T	$\forall w(wR_X w)$
	B	$\forall w, w'(wR_X w' \Rightarrow w'R_X w)$
	4	$\forall w, w', w''(wR_X w' \wedge w'R_X w'' \Rightarrow wR_X w'')$
	5	$\forall w, w', w''(wR_X w' \wedge wR_X w'' \Rightarrow w'R_X w'')$
Iteration Principles		
	Ex	$\forall w, w', w''(wR_X w' \wedge w'R_Y w'' \Rightarrow wR_{X\cap Y} w'')$
	Im	$\forall w, w'(wR_{X\cap Y} w' \Rightarrow \exists w''(wR_X w'' \wedge w''R_Y w'))$

For enhanced readability universal quantifiers regarding sets of possible worlds $X, Y, \ldots,$ $X_1, X_2, \ldots$ are omitted. For abbreviations see text.

(see Section 4.3.1) and (standard) Segerberg frames (see Section 4.3.3) plus frame conditions from Tables 5.3 and 5.4.

Note that we identified each principle in Table 5.1 and 5.2 with one of the non-trivial frame conditions/restrictions in Table 5.3 and 5.4. The upshot of our correspondence, soundness and completeness proofs in Chapter 5 and Chapter 6 is that the model-theoretic side of the lattices and its proof-theoretic side concur in the following sense: Take any combination of principles from Table 5.1 and 5.2 and their respective frame conditions from Table 5.3 and 5.4 and apply them to the basic proof theory and to the basic model theory, respectively. Then, a proof theoretic system results, which is (a) sound and strongly complete w.r.t the class of standard Segerberg frames described by the model theoretic system (cf. Section 4.3.7). Moreover, (b) any of the principles chosen from Table 5.1 and 5.2 and their semantic counterparts from Table 5.3 and 5.4 C-correspond to each other (cf. Section 4.3.5). Note, however, that we only have a completeness result for the whole lattice of system in terms of standard Segerberg frames. The subsets of the possible worlds (i.e. X, Y) in the frame restrictions in Tables 5.3 and 5.4 are, hence, limited to elements of the additional parameter P.

5.2 The Notions of Trivial and Non-Trivial Frame Conditions

In the beginning of this chapter we illustrated the difference between trivial and non-trivial frame conditions by using the principle T, namely $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow \beta$. Let us discuss here a further example. We focus on the principle CM, which is – unlike T – a theorem of a range of conditional logics proposed in the literature (see Sections 7.2 and 7.3). CM is the following principle: $(\alpha \mathrel{\Box\!\!\rightarrow} \gamma) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$ (see Table 5.1). Principle CM C-corresponds – as we shall prove in Section 5.3 – to the following non-trivial frame condition C_{CM} (see Table 5.3): $\forall X, Y \subseteq W\, \forall w \in W(\forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_{X\cap Y} w' \Rightarrow wR_X w'))$. C-correspondence gives us that whenever CM is valid in a Chellas frame $\mathcal{F}_C = \langle W, R\rangle$ then the frame condition C_{CM} holds for R, and vice versa.

The trivial counterpart for C_{CM} is, however, the following: $\forall X, Y, Z \subseteq W\, \forall w \in W(\forall w'(wR_X w' \Rightarrow w' \in Z) \wedge \forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_{X\cap Y} w' \Rightarrow w' \in Z))$. The latter frame condition can be read off from axiom schemata by the following informal translation procedure (cf. van Benthem, 2001, Section 1): Use for any propositional connective the corresponding meta-language connective and translate any occurrence of a formula $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ and α by $\forall w'(wR_X w' \Rightarrow w' \in Y)$ and $w \in X$, respectively, where for each axiom schema letter $\alpha, \beta, \ldots$ the same of variable $X, Y, \ldots$ is used, respectively, and each new axiom schema letter is described by a new variable. Observe that the trivial frame condition for CM, as the trivial frame condition for principle T employs – compared to their non-trivial versions – second-order language in a spurious way. We shall now specify this translation procedure in a formal way and prove that it yields C-corresponding frame conditions (Section 5.2.1) and, then, define the notion of non-trivial frame conditions (Section 5.2.2).

5.2.1 A Translation Procedure from Axiom Schemata to Trivial Frame Conditions

Let us define the translations procedure, which produces trivial frame conditions from modal formulae, where $\mathcal{L}_{\mathrm{FC}}$ refers to the first-order fragment for frame conditions described informally in Section 4.2.1. Some notes on our linguistic conventions used here: First, we distinguish strictly between axiom schema (w.r.t. the language $\mathcal{L}_{\mathrm{KL}}$, cf. Section 4.2.1) and axiom schema letters. While the former refers, for example, to all axiom schemata as described in Tables 5.1 and 5.2, the latter refers only to the letters $A_1, A_2, \ldots$ We, moreover, consider axiom schema letters as placeholders for arbitrary formulas $\alpha, \alpha_1, \alpha_2, \ldots$ of $\mathcal{L}_{\mathrm{KL}}$. Furthermore, $\alpha[A_1, \ldots, A_n]$ for $n \in \mathbb{N}$ represents a name form variable (cf. Kleene, 1952, pp. 142–144; see also Schurz, 1997a, p. 45f), such that $A_1, \ldots, A_n$ are all and the only distinct axiom letters occurring in α. Furthermore, α_1/α_2 abbreviates that all occurrences of α_2 in a given formula of language $\mathcal{L}_{\mathrm{KL}}$ are replaced by occurrences of α_1. If $\alpha = \alpha[A_1, \ldots, A_n]$, then the universal closure $\mathcal{UC}(t_w(\alpha))$ is defined as the formula $\forall w \in W\, \forall X_1, \ldots, X_n \subseteq W(t_w(\alpha))$. Let us now specify a translation function, which gives us for a formula of $\mathcal{L}_{\mathrm{KL}}$ the corresponding frame condition:

Definition 5.1. *Let $A_1, A_2, \ldots$ and $\alpha_1, \alpha_2, \ldots$ denote axiom schemata letters for the language $\mathcal{L}_{\mathrm{KL}}$ and arbitrary formulas of the language $\mathcal{L}_{\mathrm{KL}}$, respectively. Moreover, let t_w for all w be a translation function from formulas of $\mathcal{L}_{\mathrm{KL}}$ to formulas of $\mathcal{L}_{\mathrm{FC}}$ (formally, for every w is holds that t_w:$form_{\mathcal{L}_{\mathrm{KL}}} \rightarrow form_{\mathcal{L}_{\mathrm{FC}}}$). Then, t determines trivial Chellas frame conditions for formulas of $\mathcal{L}_{\mathrm{KL}}$ iff $t(\alpha) = \mathcal{UC}(t_w(\alpha))$, where t_w is determined the following way:*
a) $t_w(A_j) = w \in X_j$
b) $t_w(\neg\alpha) = \neg t_w(\alpha)$
c) $t_w(\alpha \wedge \beta) = t_w(\alpha) \wedge t_w(\beta)$
d) $t_w(\alpha \mathrel{\Box\!\!\rightarrow} \beta) = \forall w'(wR_{\{w \in W \mid t_w(\alpha)\}}w' \Rightarrow t_{w'}(\beta))$, where w' is a new variable

Theorem 5.2. *Let $\mathcal{F}_C$ denote Chellas frames and let α denote axiom schemata w.r.t. language $\mathcal{L}_{\mathrm{KL}}$. Then, $\forall\alpha\, \forall\mathcal{F}_C(\vDash_{\mathcal{F}_c} \alpha$ iff $\vDash_{\mathcal{F}_C} t(\alpha))$.*

Proof. Let $\mathcal{F}_C = \langle W, R\rangle$ be an arbitrary Chellas frame and let $A_1, \ldots, A_n$ and $\alpha_1, \ldots, \alpha_n$ be arbitrary axiom schema letters and arbitrary formulas of $\mathcal{L}_{\mathrm{KL}}$ for $n \in \mathbb{N}$, respectively. In the following $\mathcal{M}_C^{\mathcal{F}_C} = \langle W, R, V\rangle$ refers to Chellas models, which are based on $\mathcal{F}_C$. Then,
$\vDash_{\mathcal{F}_C} \alpha$ iff
$\forall\mathcal{M}_C^{\mathcal{F}_C}\forall\alpha_1, \ldots, \alpha_n\forall w(V(\alpha[\alpha_1/A_1, \ldots, \alpha_n/A_n], w) = 1)$ iff (by Lemma 5.4)[1]
$\forall\mathcal{M}_C^{\mathcal{F}_C}\forall\alpha_1, \ldots, \alpha_n\forall X_1, \ldots, X_n\forall w(X_1 = \{w \mid V(\alpha_1, w) = 1\} \wedge \ldots \wedge X_n = \{w \mid V(\alpha_n, w) =$

$1\} \Rightarrow V(\alpha[\alpha_1/A_1, \ldots, \alpha_n/A_n], w) = 1)$ iff (by Lemma 5.3)
$\forall \mathcal{M}_C^{\mathcal{F}_C} \forall \alpha_1, \ldots, \alpha_n \forall X_1, \ldots, X_n \forall w(X_1 = \{w \mid V(\alpha_1, w) = 1\} \wedge \ldots \wedge X_n = \{w \mid V(\alpha_n, w) = 1\} \Rightarrow t_w(\alpha)[X_1, \ldots, X_n])$ iff (by Lemma 5.5)[2]
$\forall X_1, \ldots, X_n \forall w(t_w(\alpha)[X_1, \ldots, X_n])$ iff
$UCt_w(\alpha)$ iff (by Definition 5.1)
$t(\alpha)$

[1]The left-to-right direction holds trivially. For the right-to-left direction Lemma 5.4 is needed and an extension of the principle $\forall X, Y(\alpha[X, Y] \Rightarrow \beta[Y]) \Rightarrow (\forall X \exists Y \alpha[X, Y] \Rightarrow \forall Y \beta[Y])$ to multiple Xs and Ys. Note that $X_1, \ldots, X_n$ do not occur in the consequent, namely in $V(\alpha[\alpha_1/A_1, \ldots, \alpha_n/A_n], w) = 1$. Lemma 5.4 gives us, then, schematically $\forall X \exists Y \alpha[X, Y]$.

[2]The right-to-left direction holds trivially. For the left-to-right direction Lemma 5.5 is required. Again an extension of the principle $\forall X, Y(\alpha[X, Y] \Rightarrow \beta[Y]) \Rightarrow (\forall X \exists Y\, \alpha[X, Y] \Rightarrow \forall Y \beta[Y])$ to multiple Xs and Ys is used. Here V and $\alpha_1, \ldots, \alpha_n$ do not occur in the consequent, namely in $t_w(\alpha)[X_1, \ldots, X_n]$. Lemma 5.5 gives us, then, schematically $\forall X \exists Y \alpha[X, Y]$. □

Lemma 5.3. *Let $\mathcal{M}_C$ be an arbitrary Chellas model, such that $\mathcal{M}_C = \langle W, R, V \rangle$, and let $A_1, \ldots, A_n$ and $\alpha_1, \ldots, \alpha_n$ be arbitrary axiom schema letters and formulas of $\mathcal{L}_{\mathrm{KL}}$, respectively. Then, $\forall X_1, \ldots, X_n \subseteq W \, \forall \alpha_1, \ldots, \alpha_n \forall w \in W(X_1 = \{w \in W \mid V(\alpha_1, w) = 1\} \wedge \cdots \wedge X_n = \{w \in W \mid V(\alpha_n, w) = 1\} \Rightarrow (V(\alpha[\alpha_1/A_1, \ldots, \alpha_n/A_n], w) = 1 \Leftrightarrow t_w(\alpha)[X_1, \ldots, X_n])$*

Proof. Let $\mathcal{M}_C$ be an arbitrary Chellas model, such that $\mathcal{M}_C = \langle W, R, V \rangle$ and let $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ be placeholders for arbitrary formulas of $\mathcal{L}_{\mathrm{KL}}$ and axiom schemata, respectively. Let, moreover, $X_1, \ldots, X_n$ be subsets of W, such that $X_1 = \{w \in W \mid V(\alpha_1, w) = 1\}, \ldots, X_n = \{w \in W \mid V(\alpha_2, w) = 1\}$, and let w, w', ... be worlds in W. Finally, let $j, k, l \in \mathbb{N}$. Proof by induction:
a) $V(\alpha_j, w) = 1$ iff $w \in \{w \in W \mid V(\alpha_j, w) = 1\}$ iff $w \in X_j$ iff (by Def. 5.1.a) $t_w(\alpha_j)$.
b) $V(\neg\beta_k, w) = 1$ iff (by $V_\neg$) not $V(\beta_k, w) = 1$ iff (by induction hypothesis) $\neg t_w(\beta_k)$ iff (by Def. 5.1.b) $t_w(\neg\beta_k)$.
c) $V(\beta_k \wedge \beta_l, w) = 1$ iff (by $V_\wedge$) $V(\beta_k, w) = 1$ and $V(\beta_l, w) = 1$ iff (by induction hypothesis) $t_w(\beta_k) \wedge t_w(\beta_l)$ iff (by Def. 5.1.c) $t_w(\beta_k \wedge \beta_l)$.
d) $V(\beta_k \mathrel{\Box\!\!\rightarrow} \beta_l, w) = 1$ iff (by $V_{\Box\!\rightarrow}$) $\forall w'(wR_{\{w \in W \mid V(\beta_k, w) = 1\}} w' \Rightarrow V(\beta_l, w') = 1)$

iff (by induction hypothesis) $\forall w'(wR_{\{w \in W \mid t_w(\beta_k)\}} w' \Rightarrow t_{w'}(\beta_l))$ iff (by Def. 5.1.d) $t_w(\beta_k \mathrel{\Box\!\!\rightarrow} \beta_l)$ $\square$

Lemma 5.4. *Let $\mathcal{F}_C = \langle W, R \rangle$ be an arbitrary Chellas frame and let $\mathcal{M}_C^{\mathcal{F}_C} = \langle W, R, V \rangle$ denote Chellas models, which are based on $\mathcal{F}_C$. Let $X_1, \ldots, X_n$ for $n \in \mathbb{N}$ be subsets of W and let $\alpha_1, \ldots \alpha_n$ stand for arbitrary formulas of $\mathcal{L}_{\mathrm{KL}}$. Then, $\forall \alpha_1, \ldots, \alpha_n \forall \mathcal{M}_C^{\mathcal{F}_C} \exists X_1, \ldots, X_n (X_1 = \{w \in W \mid V(\alpha_1, w) = 1\} \wedge \ldots \wedge X_n = \{w \in W \mid V(\alpha_n, w) = 1\})$.*

Proof. Let $\mathcal{F}_C = \langle W, R \rangle$ be an arbitrary Chellas frame and let $\mathcal{M}_C^{\mathcal{F}_C} = \langle W, R, V \rangle$ denote arbitrary models, which are based on $\mathcal{F}_C$. Moreover, let $X_1, \ldots, X_n$ be subsets of W and let $\alpha_1, \ldots \alpha_n$ stand for arbitrary formulas of $\mathcal{L}_{\mathrm{KL}}$. Since the expressions $\alpha_1, \ldots, \alpha_n$ are placeholders for arbitrary formulas of $\mathcal{L}_{\mathrm{KL}}$ and all formulas are assigned on the basis of Definition 4.13 subsets of W, it follows that $\forall \alpha_1, \ldots, \alpha_n \forall \mathcal{M}_C^{\mathcal{F}_C} \exists X_1, \ldots, X_n (X_1 = \{w \in W \mid V(\alpha_1, w) = 1\} \wedge \ldots \wedge X_n = \{w \in W \mid V(\alpha_n, w) = 1\}$. $\square$

Lemma 5.5. *Let $\mathcal{F}_C = \langle W, R \rangle$ be an arbitrary Chellas frame and let $\mathcal{M}_C^{\mathcal{F}_C} = \langle W, R, V \rangle$ denote Chellas models, which are based on $\mathcal{F}_C$. Moreover, let $X_1, \ldots, X_n$ for $n \in \mathbb{N}$ be subsets of W and let $\alpha_1, \ldots \alpha_n$ be arbitrary formulas of $\mathcal{L}_{\mathrm{KL}}$. Then, $\forall X_1, \ldots, X_n \exists \mathcal{M}_C^{\mathcal{F}_C} \exists \alpha_1, \ldots, \alpha_n (X_1 = \{w \in W \mid V(\alpha_1, w) = 1\} \wedge \ldots \wedge X_n = \{w \in W \mid V(\alpha_n, w) = 1\})$.*

Proof. Let $\mathcal{F}_C$ be an arbitrary Chellas frame and let $\mathcal{M}_C^{\mathcal{F}_C} = \langle W, R, V \rangle$ denote Chellas models based on $\mathcal{F}_C$. Furthermore, let $X_1, \ldots, X_n$ for $n \in \mathbb{N}$ be subsets of W, and let $\alpha_1, \ldots \alpha_n$ stand for formulas of $\mathcal{L}_{\mathrm{KL}}$. Then, the expressions $\alpha_1, \ldots, \alpha_n$ are placeholders for formulas of $\mathcal{L}_{\mathrm{KL}}$. A fortiori, we can specify $\alpha_1, \ldots \alpha_n$ in such a way that they are placeholders for the atomic propositions $p_1, \ldots, p_n$, respectively. Since the definition of Chellas models $\mathcal{M}_C^{\mathcal{F}_C} = \langle W, R, V \rangle$ (Definition 4.13) allows us to choose for the Vs the worlds in W, at which $p_1, \ldots p_n$ are true in an arbitrary way, we can define V in such a way that $X_1 = \{w \mid V(p_1, w) = 1\}$, $\ldots, X_n = \{w \mid V(p_n, w) = 1\}$. Hence, it follows that $\forall X_1, \ldots, X_n \exists \mathcal{M}_C^{\mathcal{F}_C} \exists \alpha_1, \ldots, \alpha_n (X_1 = \{w \in W \mid V(A_1, w) = 1\} \wedge \ldots \wedge X_n = \{w \in W \mid V(A_n, w) = 1\})$ holds. $\square$

5.2.2 A Non-Triviality Criterion

We discussed in the beginning of this section trivial and non-trivial frame conditions and argued that trivial frame conditions – compared to their non-trivial counterparts – use the first-order language $\mathcal{L}_{\mathrm{FC}}$ in a spurious way. In order to describe what we mean by "spurious way", let us repeat trivial and non-trivial frame conditions for the principle CM, namely $(\alpha \mathrel{\Box\!\!\rightarrow} \gamma) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$ (see Table 5.1):

C_{CM} $\forall X, Y \subseteq W\, \forall w \in W(\forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_{X\cap Y}w' \Rightarrow wR_X w'))$.

$C^{\mathrm{tr}}_{\mathrm{CM}}$ $\forall X, Y, Z \subseteq W\, \forall w \in W(\forall w'(wR_X w' \Rightarrow w' \in Z) \wedge \forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_{X\cap Y}w' \Rightarrow w' \in Z))$

Here C^{tr} indicates the trivial version of a frame condition. The difference between C_{CM} on the one hand and $C^{\mathrm{tr}}_{\mathrm{CM}}$ on the other hand lies in the fact that $C^{\mathrm{tr}}_{\mathrm{CM}}$ employs the addition variable Z. Note that variable Z is not a functional part of an accessibility relation symbol R. In other words no accessibility relation R described in $C^{\mathrm{tr}}_{\mathrm{CM}}$ is relativized to the subset Z. Let us also repeat trivial and non-trivial frame conditions for the axiom $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow \beta$ (Principle T, see beginning section of this chapter):

C_{T} $\forall X \subseteq W\, \forall w \in W(wR_X w)$

$C^{\mathrm{tr}}_{\mathrm{T}}$ $\forall X, Y \subseteq W\, \forall w, w'((wR_X w' \Rightarrow w' \in Y) \Rightarrow w \in Y)$

Again the trivial frame condition $C^{\mathrm{tr}}_{\mathrm{T}}$ but not the frame condition C_{T} uses a variable, namely variable Y, which is not a functional part of the accessibility relation symbol R. Let us, hence, define the notion of non-trivial frame conditions the following way:

Definition 5.6. *A frame condition C_α for a Chellas frame $\mathcal{F}_C = \langle W, R\rangle$ is non-trivial iff in C_α no reference to a variable Y over subsets of W is made unless variable Y is a functional part of the index of some accessibility relation symbols in C_α.*

One might argue that Definition 5.6 is not sufficiently strict, since then, for example, also the following frame condition for CM would be non-trivial: $\forall w(\forall w' (wR_X w' \Rightarrow w' \in Y \wedge (w' \in X \vee w' \in -X)) \Rightarrow \forall w'(wR_{X\cap Y}w' \Rightarrow wR_X w'))$. We could exclude this type of frame condition by requiring for a non-trivial frame condition C_α in addition that (a) there is no equivalent frame condition C'_α, such that C'_α has less references to a variable Y over subsets of W than C_α. We will, however, not strengthen the definition of non-trivial frame conditions in that way, since it is in general extremely hard to prove that a frame condition satisfies criterion (a).

5.3 Chellas Frame Correspondence Proofs

In this section we give correspondence proofs for the principles in Tables 5.1 and 5.2 w.r.t. the respective frame conditions in Tables 5.3 and 5.4. Note that we prove here C-correspondence. Hence the frame restrictions refer to all subset of W of any Chellas frame $\mathcal{F}_C = \langle W, R\rangle$. As basis for the proofs serves Chellas frame correspondence as defined in Section 4.3.5. In order to prove the right-to-left direction ("$\Leftarrow$"), we have to show for all worlds w in any model $\mathcal{M}_C = \langle W, R, V\rangle$ based on any frame $\mathcal{F}_C = \langle W, R\rangle$, in which C_α holds, α is true at w. Conversely, to prove the left-to-right direction ("$\Rightarrow$"), we show by contraposition that, whenever C_α does not hold in a frame $\mathcal{F}_C = \langle W, R\rangle$, α is not valid on the frame $\mathcal{F}_C$. To establish the latter result, we construct a model $\mathcal{M}_C$ based on $\mathcal{F}_C$, in which some instance of the axiom schema α is not true for some w in $\mathcal{M}_C$.

5.3.1 System P

Axiom Schema Refl. ("$\Leftarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{Refl} holds. Then, for any world $w \in W$ and set $X \subseteq W$ it is the case that $\forall w'(wR_Xw' \Rightarrow w' \in X)$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$. Then, as $\forall w'(wR_Xw' \Rightarrow w' \in X)$ holds for any $X \subseteq W$, we get $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\alpha\|)$ for every $\|\alpha\| \subseteq W$. Hence, by $V_{\Box\!\!\rightarrow}$ it follows that $\vDash_w^{\mathcal{M}_C} \alpha \mathrel{\Box\!\!\rightarrow} \alpha$.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{Refl} does not hold. Then, there exist two worlds $w, w' \in W$ and a set $X \subseteq W$, such that wR_Xw', but $w' \notin X$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|$. Then, $wR_{\|\alpha\|}w'$, but $w' \notin \|\alpha\|$. From $V_{\Box\!\!\rightarrow}$, it follows that $\nvDash_w^{\mathcal{M}_C} \alpha \mathrel{\Box\!\!\rightarrow} \alpha$. □

Axiom Schema CM. ("$\Leftarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame such that C_{CM} holds. Thus, for any world $w \in W$ and sets $X, Y \subseteq W$ it is the case that $\forall w'(wR_Xw' \Rightarrow w' \in$

$Y) \Rightarrow \forall w'(wR_{X\cap Y}w' \Rightarrow wR_X w')$. Let $\mathcal{M}_C = \langle W, R, V \rangle$ be any model based on $\mathcal{F}_C$, such that $\models_w^{\mathcal{M}_C} (\alpha \boxright \gamma) \land (\alpha \boxright \beta)$. Due to $V_{\boxright}$ it follows that $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\gamma\|)$ and $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\beta\|)$. Moreover, as $\forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_{X\cap Y}w' \Rightarrow wR_X w')$ holds for any $X, Y \subseteq W$, we can infer $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\beta\|) \Rightarrow \forall w'(wR_{\|\alpha\|\cap\|\beta\|}w' \Rightarrow wR_{\|\alpha\|}w')$ for any $\|\alpha\|, \|\beta\| \subseteq W$. The latter result and $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\beta\|)$ imply $\forall w'(wR_{\|\alpha\|\cap\|\beta\|}w' \Rightarrow wR_{\|\alpha\|}w')$. Moreover, due to $\mathrm{Def}_{\|.\|}$ we get $\forall w'(wR_{\|\alpha\wedge\beta\|}w' \Rightarrow wR_{\|\alpha\|}w')$. From this and $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\gamma\|)$ we can conclude that $\forall w'(wR_{\|\alpha\wedge\beta\|}w' \Rightarrow w' \in \|\gamma\|)$. Thus, by $V_{\boxright}$ we get $\models_w^{\mathcal{M}_C} \alpha \land \beta \boxright \gamma$.

("⇒") Let $\mathcal{F}_C = \langle W, R \rangle$ be any frame, such that C_{CM} does not hold. For this to be the case, there have to be worlds $w, w' \in W$ and sets $X, Y \subseteq W$ such that $\forall w''(wR_X w'' \Rightarrow w'' \in Y)$, $wR_{X\cap Y}w'$, but not $wR_X w'$. Let $\mathcal{M}_C = \langle W, R, V \rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|$, $Y = \|\beta\|$, $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow w'' \in \|\gamma\|)$ and $w' \notin \|\gamma\|$. It follows that $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow w'' \in \|\beta\|)$, $wR_{\|\alpha\|\cap\|\beta\|}w'$, but not $wR_{\|\alpha\|}w'$. This assignment is possible for any frame $\mathcal{F}_C$, as by assumption w' is not among the w''. Since $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow w'' \in \|\gamma\|)$ and $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow w'' \in \|\beta\|)$, it follows by $V_{\boxright}$ that $\models_w^{\mathcal{M}_C} \alpha \boxright \gamma$ and $\models_w^{\mathcal{M}_C} \alpha \boxright \beta$ and, consequently, $\models_w^{\mathcal{M}_C} (\alpha \boxright \gamma) \land (\alpha \boxright \beta)$. Moreover, since $wR_{\|\alpha\|\cap\|\beta\|}w'$ if follows by $\mathrm{Def}_{\|.\|}$ that $wR_{\|\alpha\wedge\beta\|}w'$, and $w' \notin \|\gamma\|$. Thus, by $V_{\boxright}$ we get $\not\models_w^{\mathcal{M}_C} \alpha \land \beta \boxright \gamma$. □

Axiom Schema CC. ("⇐") Let $\mathcal{F}_C = \langle W, R \rangle$ be a frame, such that C_{CC} holds. Then, for any world $w \in W$ and sets $X, Y \subseteq W$ it is the case that $\forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_X w' \Rightarrow wR_{X\cap Y}w')$. Let $\mathcal{M}_C = \langle W, R, V \rangle$ be a model based on $\mathcal{F}_C$, such that $\models_w^{\mathcal{M}_C} (\alpha \land \beta \boxright \gamma) \land (\alpha \boxright \beta)$. By $V_{\boxright}$ it follows that $\forall w'(wR_{\|\alpha\wedge\beta\|}w' \Rightarrow w' \in \|\gamma\|)$, and $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\beta\|)$. By $\mathrm{Def}_{\|.\|}$ it follows that $\forall w'(wR_{\|\alpha\|\cap\|\beta\|}w' \Rightarrow w' \in \|\gamma\|)$. Since $\forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_X w' \Rightarrow wR_{X\cap Y}w')$ for any $X, Y \subseteq W$, we get $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\beta\|) \Rightarrow \forall w'(wR_{\|\alpha\|}w' \Rightarrow wR_{\|\alpha\|\cap\|\beta\|}w')$ for every $\|\alpha\|, \|\beta\| \subseteq W$. As we have $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\beta\|)$, we get $\forall w'(wR_{\|\alpha\|}w' \Rightarrow wR_{\|\alpha\|\cap\|\beta\|}w')$. From that and $\forall w'(wR_{\|\alpha\|\cap\|\beta\|}w' \Rightarrow w' \in \|\gamma\|)$, it follows that $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\gamma\|)$. This implies with $V_{\boxright}$ that $\models_w^{\mathcal{M}_C} \alpha \boxright \gamma$.

("⇒") Let $\mathcal{F}_C = \langle W, R \rangle$ be a frame, such that C_{CC} does not hold. Then, there exist two worlds $w, w' \in W$ and sets $X, Y \subseteq W$, such that $\forall w''(wR_X w'' \Rightarrow w'' \in Y)$,

$wR_X w'$, but not $wR_{X \cap Y} w'$. Let $\mathcal{M}_C = \langle W, R, V \rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|$, $Y = \|\beta\|$, $\forall w''(wR_{\|\alpha\| \cap \|\beta\|} w'' \Rightarrow w'' \in \|\gamma\|)$ and $w' \notin \|\gamma\|$. Then, $\forall w''(wR_{\|\alpha\|} w'' \Rightarrow w'' \in \|\beta\|)$, $wR_{\|\alpha\|} w'$ and not $wR_{\|\alpha\| \cap \|\beta\|} w'$. This assignment is possible for any frame $\mathcal{F}_C$, since by assumption w' is not among the w'', such that $wR_{\|\alpha\| \cap \|\beta\|} w''$. Due to $\mathrm{Def}_{\|.\|}$ it follows that $\forall w''(wR_{\|\alpha \wedge \beta\|} w'' \Rightarrow w'' \in \|\gamma\|)$. From that and $\forall w''(wR_{\|\alpha\|} w'' \Rightarrow w'' \in \|\beta\|)$, we get by $V_{\Box\!\!\rightarrow}$ that $\models_w^{\mathcal{M}_C} \alpha \wedge \beta \Box\!\!\rightarrow \gamma$ and $\models_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \beta$. However, since $wR_{\|\alpha\|} w'$ and $w' \notin \|\gamma\|$, by $V_{\Box\!\!\rightarrow}$ we can infer $\not\models_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \gamma$. □

Axiom Schema Loop. ("⇐") Let $\mathcal{F}_C = \langle W, R \rangle$ be an arbitrary frame, such hat C_{Loop} holds. Then, for any $w \in W$ and $k \geq 2$, such that $X_0, X_1, \ldots, X_k \subseteq W$ it is the case that $\forall w'(wR_{X_0} w' \Rightarrow w' \in X_1) \wedge \ldots \wedge \forall w'(wR_{X_{k-1}} w' \Rightarrow w' \in X_k) \wedge \forall w'(wR_{X_k} w' \Rightarrow w' \in X_0) \Rightarrow \forall w'(wR_{X_0} w' \Rightarrow w' \in X_k)$. Let $\mathcal{M}_C = \langle W, R, V \rangle$ be any model based on $\mathcal{F}_C$, such that $\models_w^{\mathcal{M}_C} (\alpha_0 \Box\!\!\rightarrow \alpha_1) \wedge \ldots \wedge (\alpha_{k-1} \Box\!\!\rightarrow \alpha_k) \wedge (\alpha_k \Box\!\!\rightarrow \alpha_0)$ holds. Due to $V_{\Box\!\!\rightarrow}$ it follows that $\forall w'(wR_{\|\alpha_0\|} w' \Rightarrow w' \in \|\alpha_1\|)$, $\ldots$, $\forall w'(wR_{\|\alpha_{k-1}\|} w' \Rightarrow w' \in \|\alpha_k\|)$ and $\forall w'(wR_{\|\alpha_k\|} w' \Rightarrow w' \in \|\alpha_0\|)$. Moreover, as $\forall w'(wR_{X_0} w' \Rightarrow w' \in X_1) \wedge \ldots \wedge \forall w'(wR_{X_{k-1}} w' \Rightarrow w' \in X_k) \wedge \forall w'(wR_{X_k} w' \Rightarrow w' \in X_0) \Rightarrow \forall w'(wR_{X_0} w' \Rightarrow w' \in X_k)$ holds for any $X_0, X_1, \ldots, X_k \subseteq W$, we get $\forall w'(wR_{\|\alpha_0\|} w' \Rightarrow w' \in \|\alpha_1\|) \wedge \ldots \wedge \forall w'(wR_{\|\alpha_{k-1}\|} w' \Rightarrow w' \in \|\alpha_k\|) \wedge \forall w'(wR_{\|\alpha_k\|} w' \Rightarrow w' \in \|\alpha_0\|) \Rightarrow \forall w'(wR_{\|\alpha_0\|} w' \Rightarrow w' \in \|\alpha_k\|)$ for every $\|\alpha_0\|$, $\|\alpha_1\|$, $\ldots$, $\|\alpha_k\| \subseteq W$. Since it is the case that $\forall w'(wR_{\|\alpha_0\|} w' \Rightarrow w' \in \|\alpha_1\|) \wedge \ldots \wedge \forall w'(wR_{\|\alpha_{k-1}\|} w' \Rightarrow w' \in \|\alpha_k\|) \wedge \forall w'(wR_{\|\alpha_k\|} w' \Rightarrow w' \in \|\alpha_0\|)$, we get $\forall w'(wR_{\|\alpha_0\|} w' \Rightarrow w' \in \|\alpha_k\|)$. By $V_{\Box\!\!\rightarrow}$ it follows that $\models_w^{\mathcal{M}_C} \alpha_0 \Box\!\!\rightarrow \alpha_k$.

("⇒") Let $\mathcal{F}_C = \langle W, R \rangle$ be a frame, such that C_{Loop} does not hold. Then, there are worlds $w, w' \in W$ and sets $X_0, X_1, \ldots, X_k \subseteq W$, such that $\forall w''(wR_{X_0} w'' \Rightarrow w'' \in X_1) \wedge \ldots \wedge \forall w''(wR_{X_{k-1}} w'' \Rightarrow w'' \in X_k) \wedge \forall w''(wR_{X_k} w'' \Rightarrow w'' \in X_0)$ and $wR_{X_0} w'$, but $w' \notin X_k$. Let $\mathcal{M}_C = \langle W, R, V \rangle$ be a model based on $\mathcal{F}_C$, such that $X_0 = \|\alpha_0\|$, $\ldots$, $X_k = \|\alpha_k\|$. Then, $\forall w''(wR_{\|\alpha_0\|} w'' \Rightarrow w'' \in \|\alpha_1\|) \wedge \ldots \wedge \forall w''(wR_{\|\alpha_{k-1}\|} w'' \Rightarrow w'' \in \|\alpha_k\|) \wedge \forall w''(wR_{\|\alpha_k\|} w'' \Rightarrow w'' \in \|\alpha_0\|)$ and $wR_{\|\alpha_0\|} w'$, but $w' \notin |\alpha_k|$. As $\forall w''(wR_{\|\alpha_0\|} w'' \Rightarrow w'' \in \|\alpha_1\|) \wedge \ldots \wedge \forall w''(wR_{\|\alpha_{k-1}\|} w'' \Rightarrow w'' \in \|\alpha_k\|) \wedge \forall w''(wR_{\|\alpha_k\|} w'' \Rightarrow w'' \in \|\alpha_0\|)$, by $V_{\Box\!\!\rightarrow}$ this implies that $\models_w^{\mathcal{M}_C} \alpha_0 \Box\!\!\rightarrow \alpha_1$, $\ldots$, $\models_w^{\mathcal{M}_C} \alpha_{k-1} \Box\!\!\rightarrow \alpha_k$ and $\models_w^{\mathcal{M}_C} \alpha_k \Box\!\!\rightarrow \alpha_0$, Thus, $\models_w^{\mathcal{M}_C} (\alpha_0 \Box\!\!\rightarrow \alpha_1) \wedge \ldots \wedge (\alpha_{k-1} \Box\!\!\rightarrow \alpha_k) \wedge (\alpha_k \Box\!\!\rightarrow \alpha_0)$. Since $wR_{\|\alpha_0\|} w'$, but $w' \notin \|\alpha_k\|$, by $V_{\Box\!\!\rightarrow}$ follows that $\not\models_w^{\mathcal{M}} \alpha_0 \Box\!\!\rightarrow \alpha_k$. □

Axiom Schema Or. ($\Leftarrow$) Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{Or} holds. Then, for any world $w \in W$ and sets $X, Y \subseteq W$ it is the case that $\forall w'(wR_{X\cup Y}w' \Rightarrow wR_Xw' \vee wR_Yw')$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$, such that $\vDash_w^{\mathcal{M}_C} (\alpha \,\Box\!\!\rightarrow \gamma) \wedge (\beta \,\Box\!\!\rightarrow \gamma)$. According to $V_{\Box\!\rightarrow}$ we have $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\gamma\|)$ and $\forall w'(wR_{\|\beta\|}w' \Rightarrow w' \in \|\gamma\|)$. From that we can infer $\forall w'(wR_{\|\alpha\|}w' \vee wR_{\|\beta\|}w' \Rightarrow w' \in \|\gamma\|)$. As $\forall w'(wR_{X\cup Y}w' \Rightarrow wR_Xw' \vee wR_Yw')$ holds for any $X, Y \subseteq W$, this implies that $\forall w'(wR_{\|\alpha\|\cup\|\beta\|}w' \Rightarrow wR_{\|\alpha\|}w' \vee wR_{\|\beta\|}w')$ for every $\|\alpha\|, \|\beta\| \subseteq W$. Since $\forall w'(wR_{\|\alpha\|}w' \vee wR_{\|\beta\|}w' \Rightarrow w' \in \|\gamma\|)$ holds, we get $\forall w'(wR_{\|\alpha\|\cup\|\beta\|}w' \Rightarrow w' \in \|\gamma\|)$. By $\mathrm{Def}_{\|.\|}$ it follows that $\forall w'(wR_{\|\alpha\vee\beta\|}w' \Rightarrow w' \in \|\gamma\|)$. Due to $V_{\Box\!\rightarrow}$ this implies that $\vDash_w^{\mathcal{M}_C} \alpha \vee \beta \,\Box\!\!\rightarrow \gamma$.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that that C_{Or} does not hold. Then, there exist two worlds $w, w' \in W$, and sets $X, Y \subseteq W$, such that $wR_{X\cup Y}w'$, but neither wR_Xw' nor wR_Yw'. Thus, for C_{Or} to fail, w' is not allowed to be among the w'', such that wR_Xw'' or wR_Yw''. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|$, $Y = \|\beta\|$, $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow w'' \in \|\gamma\|)$ and $\forall w''(wR_{\|\beta\|}w'' \Rightarrow w'' \in \|\gamma\|)$, but $w' \notin \|\gamma\|$. It follows that $wR_{\|\alpha\|\cup\|\beta\|}w'$, but $w' \notin \|\gamma\|$. This assignment is possible for any frame $\mathcal{F}_C$, as by assumption w' is not among the w'', such that $wR_{\|\alpha\|}w''$ or $wR_{\|\beta\|}w''$. Since $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow w'' \in \|\gamma\|)$ and $\forall w''(wR_{\|\beta\|}w'' \Rightarrow w'' \in \|\gamma\|)$ are the case, due to $V_{\Box\!\rightarrow}$ we get $\vDash_w^{\mathcal{M}_C} \alpha \,\Box\!\!\rightarrow \gamma$ and $\vDash_w^{\mathcal{M}_C} \beta \,\Box\!\!\rightarrow \gamma$ and, thus, $\vDash_w^{\mathcal{M}_C} (\alpha \,\Box\!\!\rightarrow \gamma) \wedge (\beta \,\Box\!\!\rightarrow \gamma)$. By $\mathrm{Def}_{\|.\|}$ we can infer $wR_{\|\alpha\vee\beta\|}w'$. However, since $w' \notin \|\gamma\|$, we can conclude by $V_{\Box\!\rightarrow}$ that $\nvDash_w^{\mathcal{M}_C} \alpha \vee \beta \,\Box\!\!\rightarrow \gamma$. □

Axiom Schema S. ("$\Leftarrow$"). Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{S} holds. Then, for any $w \in W$ and $X, Y \subseteq W$ it is the case that $\forall w'(wR_Xw' \wedge w' \in Y \Rightarrow wR_{X\cap Y}w')$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$ such that $\vDash_w^{\mathcal{M}_C} \alpha \wedge \beta \,\Box\!\!\rightarrow \gamma$ holds. Then, by $V_{\Box\!\rightarrow}$ we have $\forall w'(wR_{\|\alpha\wedge\beta\|}w' \Rightarrow w' \in \|\gamma\|)$. Due to $\mathrm{Def}_{\|.\|}$ we get $\forall w'(wR_{\|\alpha\|\cap\|\beta\|}w' \Rightarrow w' \in \|\gamma\|)$. Moreover, as $\forall w'(wR_Xw' \wedge w' \in Y \Rightarrow wR_{X\cap Y}w')$ holds for any $X, Y \subseteq W$, it follows that $\forall w'(wR_{\|\alpha\|}w' \wedge w' \in \|\beta\| \Rightarrow wR_{\|\alpha\|\cap\|\beta\|}w')$ for every $\|\alpha\|, \|\beta\| \subseteq W$. With $\forall w'(wR_{\|\alpha\|\cap\|\beta\|}w' \Rightarrow w' \in \|\gamma\|)$ this implies $\forall w'(wR_{\|\alpha\|}w' \wedge w' \in \|\beta\| \Rightarrow w' \in \|\gamma\|)$. Hence, $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \notin \|\beta\| \vee w' \in \|\gamma\|)$ and, thus, due to $\mathrm{Def}_{\|.\|}$ we get $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\neg\beta\| \vee w' \in \|\gamma\|)$ and, thus, $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\beta \rightarrow \gamma\|)$. Hence, we have by $V_{\Box\!\rightarrow}$ that $\vDash_w^{\mathcal{M}_C} \alpha \,\Box\!\!\rightarrow (\beta \rightarrow \gamma)$.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_S does not hold. Then, there are some $w, w' \in W$ and $X, Y \subseteq W$, such that $wR_X w'$, $w' \in Y$, but $\neg wR_{X\cap Y}w'$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$ such that $X = \|\alpha\|$, $Y = \|\beta\|$ and $\forall w''(wR_{\|\alpha\|\cap\|\beta\|}w'' \Rightarrow w'' \in \|\gamma\|)$, but $w' \notin \|\gamma\|$. Then, $wR_{\|\alpha\|}w'$, $w' \in \|\beta\|$ and $\neg wR_{\|\alpha\|\cap\|\beta\|}w'$. This assignment is possible for any frame $\mathcal{F}_C$, since by assumption w' is not among the $w'' \in W$, such that $wR_{\|\alpha\|\cap\|\beta\|}w''$. By $\mathrm{Def}_{\|.\|}$ it follows from $\forall w''(wR_{\|\alpha\|\cap\|\beta\|}w'' \Rightarrow w'' \in \|\gamma\|)$ that $\forall w''(wR_{\|\alpha\wedge\beta\|}w'' \Rightarrow w'' \in \|\gamma\|)$. This implies by $V_{\Box\!\!\rightarrow}$ that $\models_w^{\mathcal{M}_C} \alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$. As $w' \in \|\beta\|$, but $w' \notin \|\gamma\|$, we have by $\mathrm{Def}_{\|.\|}$ that $w' \notin \|\beta \to \gamma\|$. Since $wR_{\|\alpha\|}w'$, by we get due to $V_{\Box\!\!\rightarrow}$ that $\not\models_w^{\mathcal{M}_C} \alpha \mathrel{\Box\!\!\rightarrow} (\beta \to \gamma)$. □

5.3.2 Extensions of System P

Axiom Schema MOD. ("$\Leftarrow$"). Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{MOD} holds. Then, for any $w \in W$ and set $X, Y \subseteq W$ it is the case that $\forall w'(wR_{-X}w' \Rightarrow w' \in X) \Rightarrow \forall w'(wR_Y w' \Rightarrow w' \in X)$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$ such that $\models_w^{\mathcal{M}_C} \neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha$. Then, by $V_{\Box\!\!\rightarrow}$ it follows that $\forall w'(wR_{\|\neg\alpha\|}w' \Rightarrow w' \in \|\alpha\|)$ and by $V_{\|.\|}$ this implies that $\forall w'(wR_{-\|\alpha\|}w' \Rightarrow w' \in \|\alpha\|)$. Since $\forall w'(wR_{-X}w' \Rightarrow w' \in X) \Rightarrow \forall w'(wR_Y w' \Rightarrow w' \in X)$ holds for any $X, Y \subseteq W$, we have $\forall w'(wR_{-\|\alpha\|}w' \Rightarrow w' \in \|\alpha\|) \Rightarrow \forall w'(wR_{\|\beta\|}w' \Rightarrow w' \in \|\alpha\|)$ for every $\|\alpha\|, \|\beta\| \subseteq W$. As $\forall w'(wR_{-\|\alpha\|}w' \Rightarrow w' \in \|\alpha\|)$ holds, we get $\forall w'(wR_{\|\beta\|}w' \Rightarrow w' \in \|\alpha\|)$ and, by $V_{\Box\!\!\rightarrow}$ this implies $\models_w^{\mathcal{M}_C} \beta \mathrel{\Box\!\!\rightarrow} \alpha$.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{MOD} does not hold. Then, there are $w, w' \in W$ and $X, Y \subseteq W$, such that $\forall w''(wR_{-X}w'' \Rightarrow w'' \in X)$ and $wR_Y w'$, but $w' \notin X$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$ such that $X = \|\alpha\|$, $Y = \|\beta\|$. Then, $\forall w''(wR_{-\|\alpha\|}w'' \Rightarrow w'' \in \|\alpha\|)$ and $wR_{\|\beta\|}w'$, but $w' \notin \|\alpha\|$. Moreover, by $\mathrm{Def}_{\|.\|}$ we have $\forall w''(wR_{\|\neg\alpha\|}w'' \Rightarrow w'' \in \|\alpha\|)$ and, thus, $\models_w^{\mathcal{M}_C} \neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha$. However, since $wR_{\|\beta\|}w'$, but $w' \notin \|\alpha\|$, this implies $\not\models_w^{\mathcal{M}_C} \beta \mathrel{\Box\!\!\rightarrow} \alpha$. □

Axiom Schema RM. ("$\Leftarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be any frame, such hat C_{RM} holds. Hence, for any world $w \in W$, and sets $X, Y \subseteq W$, holds that $\exists w'(wR_X w' \wedge w' \in Y) \Rightarrow \forall w'(wR_{X\cap Y}w' \Rightarrow wR_X w')$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$, such that $\models_w^{\mathcal{M}_C} (\alpha \mathrel{\Box\!\!\rightarrow} \gamma) \wedge (\alpha \mathrel{\Diamond\!\!\rightarrow} \beta)$. By $V_{\Box\!\!\rightarrow}$ and $\mathrm{Def}_{\|.\|}$ it follows that $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\gamma\|)$ and $\exists w'(wR_{\|\alpha\|}w' \wedge w' \in \|\beta\|)$. As $\exists w'(wR_X w' \wedge w' \in Y) \Rightarrow \forall w'(wR_{X\cap Y}w' \Rightarrow wR_X w')$ holds for any $X, Y \subseteq W$, it follows that

$\exists w'(wR_{\|\alpha\|}w' \wedge w' \in \|\beta\|) \Rightarrow \forall w'(wR_{\|\alpha\|\cap\|\beta\|}w' \Rightarrow wR_{\|\alpha\|}w')$ for every $\|\alpha\|, \|\beta\| \subseteq W$. Since $\exists w'(wR_{\|\alpha\|}w' \wedge w' \in \|\beta\|)$, we have $\forall w'(wR_{\|\alpha\|\cap\|\beta\|}w' \Rightarrow wR_{\|\alpha\|}w')$. Moreover, as $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\gamma\|)$, it follows that $\forall w'(wR_{\|\alpha\|\cap\|\beta\|}w' \Rightarrow w' \in \|\gamma\|)$. This gives us by $V_{\|.\|}$ that $\forall w'(wR_{\|\alpha\wedge\beta\|}w' \Rightarrow w' \in \|\gamma\|)$, and due to $V_{\Box\!\!\rightarrow}$ follows that $\models_w^{\mathcal{M}_C} \alpha \wedge \beta \Box\!\!\rightarrow \gamma$.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{RM} does not hold. Then, there are worlds $w, w' \in W$ and sets $X, Y \subseteq W$, such that $\exists w''(wR_X w'' \wedge w'' \in Y)$, $wR_{X\cap Y}w'$, but not $wR_X w'$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|$, $Y = \|\beta\|$, $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow w'' \in \|\gamma\|)$ and $w' \notin \|\gamma\|$. Then, $\exists w''(wR_{\|\alpha\|}w'' \wedge w'' \in \|\beta\|)$, $wR_{\|\alpha\|\cap\|\beta\|}w'$ and not $wR_{\|\alpha\|}w'$. This assignment is possible for any frame $\mathcal{F}_C$, as by assumption w' is not among the w''s, such that $wR_{\|\alpha\|}w''$. Since $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow w'' \in |\gamma\|)$, we have by $V_{\Box\!\!\rightarrow}$ that $\models_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \gamma$. As $\exists w''(wR_{\|\alpha\|}w'' \wedge w'' \in \|\beta\|)$, we get by $V_{\Box\!\!\rightarrow}$ and $\mathrm{Def}_{\|.\|}$ that $\models_w^{\mathcal{M}_C} (\alpha \Diamond\!\!\rightarrow \beta)$. Hence, $\models_w^{\mathcal{M}_C} (\alpha \Box\!\!\rightarrow \gamma) \wedge (\alpha \Diamond\!\!\rightarrow \beta)$. Moreover, as $wR_{\|\alpha\|\cap\|\beta\|}w'$, by $V_{\|.\|}$ this implies $wR_{\|\alpha\wedge\beta\|}w'$. However, $w' \notin \|\gamma\|$. Hence, due to $V_{\Box\!\!\rightarrow}$ it follows that $\not\models_w^{\mathcal{M}_C} \alpha \wedge \beta \Box\!\!\rightarrow \gamma$. $\Box$

Axiom Schema CEM. ("$\Leftarrow$") To show that CEM is valid in all Chellas frames, for which C_{CEM} holds, we proceed as follows: We suppose that $\models_w^{\mathcal{M}} \neg(\alpha \Box\!\!\rightarrow \neg\beta)$ for a world w in model $\mathcal{M}$, based on such a frame and demonstrate that $\models_w^{\mathcal{M}} \alpha \Box\!\!\rightarrow \beta$. Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{CEM} holds. Then, for any $w \in W$, $X \subseteq W$ it is the case that $\forall w', w''(wR_X w' \wedge wR_X w'' \Rightarrow w'' = w')$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$ and suppose that $\models_w^{\mathcal{M}} \neg(\alpha \Box\!\!\rightarrow \neg\beta)$. Then, by $V_{\Box\!\!\rightarrow}$ and $\mathrm{Def}_{\|.\|}$ it follows that there is a world w', such that $wR_{\|\alpha\|}w'$ and $\models_{w'}^{\mathcal{M}} \beta$. As $\forall w', w''(wR_X w' \wedge wR_X w'' \Rightarrow w'' = w')$ for any $X \subseteq W$, it follows that $\forall w', w''(wR_{\|\alpha\|}w' \wedge wR_{\|\alpha\|}w'' \Rightarrow w'' = w')$ for every $\|\alpha\| \subseteq W$. Since there is a world w', such that $wR_{\|\alpha\|}w'$ and $\models_{w'}^{\mathcal{M}} \beta$, this implies that $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow w'' \in \|\beta\|)$. Thus, we yield by $V_{\Box\!\!\rightarrow}$ that $\models_w^{\mathcal{M}_C} (\alpha \Box\!\!\rightarrow \beta)$.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a model, such that C_{CEM} does not hold. Then, there are $w, w', w'' \in W$ and $X \subseteq W$, such that $wR_X w', wR_X w''$, but $w'' \neq w'$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|$, $w' \notin \|\beta\|$, but $w'' \in \|\beta\|$. Then, $wR_{\|\alpha\|}w'$ and $wR_{\|\alpha\|}w''$. This assignment is possible for any frame $\mathcal{F}_C$, since by assumption $w'' \neq w'$. As there is a world w', such that $wR_{\|\alpha\|}w'$, but $w' \notin \|\beta\|$, we have by $V_{\Box\!\!\rightarrow}$ that $\not\models_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \beta$. Since it is the case that $w'' \in \|\beta\|$, it follows by $\mathrm{Def}_{\|.\|}$ that $w'' \notin \|\neg\beta\|$. By $V_{\Box\!\!\rightarrow}$ we get $\not\models_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \neg\beta$. As $\not\models_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \beta$ and $\not\models_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \neg\beta$, it follows that $\not\models_w^{\mathcal{M}_C} (\alpha \Box\!\!\rightarrow \beta) \vee (\alpha \Box\!\!\rightarrow \neg\beta)$. □

5.3.3 Axioms from Weak Probability Logic (Threshold Logic)

Axiom Schema P-Cons. ("⇐"). Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that $C_{\text{P-Cons}}$ holds. Then, for any $w \in W$ there is a $w' \in W$, such that $wR_W w'$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$. Then, since $W = \|\top\|$ for any model, we have $wR_{\|\top\|}w'$. As $\varnothing = \|\bot\|$, it follows that $w' \notin \|\bot\|$ and, hence, by $V_{\Box\!\!\rightarrow}$ this implies that $\not\models_w^{\mathcal{M}_C} \top \Box\!\!\rightarrow \bot$ and, hence, $\models_w^{\mathcal{M}_C} \neg(\top \Box\!\!\rightarrow \bot)$.

("⇒") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that $C_{\text{P-Cons}}$ does not hold. Then, for some $w \in W$ there is no w', such that $wR_W w'$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$. Since $W = \|\top\|$ for any model, this implies that there is no w', such that $wR_{\|\top\|}w'$. Hence, by $V_{\Box\!\!\rightarrow}$, it follows trivially that $\models_w^{\mathcal{M}_C} \top \Box\!\!\rightarrow \bot$. Hence, $\not\models_w^{\mathcal{M}_C} \neg(\top \Box\!\!\rightarrow \bot)$. □

Axiom Schema WOR. ("⇐"). Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that $_{\text{WOR}}$ holds. Then, for any $w \in W$ and $X, Y \subseteq W$, it is the case that $\forall w'(wR_X w' \Rightarrow wR_{X\cap Y}w' \vee wR_{X\cap -Y}w')$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$, such that $\models_w^{\mathcal{M}_C} (\alpha \wedge \beta \Box\!\!\rightarrow \gamma) \wedge (\alpha \wedge \neg\beta \Box\!\!\rightarrow \gamma)$ holds. Then, by $V_{\Box\!\!\rightarrow}$ we have $\forall w'(wR_{\|\alpha\wedge\beta\|}w' \Rightarrow w' \in \|\gamma\|)$ and $\forall w'(wR_{\|\alpha\wedge\neg\beta\|}w' \Rightarrow w' \in \|\gamma\|)$. This implies that $\forall w'(wR_{\|\alpha\wedge\beta\|}w' \vee wR_{\|\alpha\wedge\neg\beta\|}w' \Rightarrow w' \in \|\gamma\|)$. By $\mathrm{Def}_{\|.\|}$ it follows that $\forall w'(wR_{\|\alpha\|\cap\|\beta\|}w' \vee wR_{\|\alpha\|\cap-\|\beta\|}w' \Rightarrow w' \in \|\gamma\|)$. Moreover, since $\forall w'(wR_X w' \Rightarrow wR_{X\cap Y}w' \vee wR_{X\cap -Y}w')$ holds for any $X, Y \subseteq W$, we have $\forall w'(wR_{\|\alpha\|}w' \Rightarrow wR_{\|\alpha\|\cap\|\beta\|}w' \vee wR_{\|\alpha\|\cap-\|\beta\|}w')$ for every $\|\alpha\|, \|\beta\| \subseteq W$. As $\forall w'(wR_{\|\alpha\|\cap\|\beta\|}w' \vee wR_{\|\alpha\|\cap-\|\beta\|}w' \Rightarrow w' \in \|\gamma\|)$, this implies $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\gamma\|)$. By $V_{\Box\!\!\rightarrow}$ it follows that $\models_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \gamma$.

("⇒") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{WOR} does not hold. Then, for some $w, w' \in W$ and $X, Y \subseteq W$ we have $wR_X w'$, but neither $wR_{X\cap Y}w'$ nor $wR_{X\cap -Y}w'$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|$, $Y = \|\beta\|$, $\forall w''(wR_{\|\alpha\|\cap\|\beta\|}w'' \Rightarrow w'' \in \|\gamma\|)$ and $\forall w''(wR_{\|\alpha\|\cap-\|\beta\|}w'' \Rightarrow w'' \in \|\gamma\|)$, but $w' \notin \|\gamma\|$. Then, $wR_{\|\alpha\|}w'$, and neither $wR_{\|\alpha\|\cap\|\beta\|}w'$ nor $wR_{\|\alpha\|\cap\|\beta\|}w'$. This assignment is possible for any frame $\mathcal{F}_C$, since by assumption w' is not among the w'', such that $wR_{\|\alpha\|\cap\|\beta\|}w''$ or $wR_{\|\alpha\|\cap-\|\beta\|}w''$. As $\forall w''(wR_{\|\alpha\|\cap\|\beta\|}w'' \Rightarrow w'' \in \|\gamma\|)$ and $\forall w''(wR_{\|\alpha\|\cap-\|\beta\|}w'' \Rightarrow w'' \in \|\gamma\|)$ it follows by $\mathrm{Def}_{\|.\|}$ that $\forall w''(wR_{\|\alpha\wedge\beta\|}w'' \Rightarrow w'' \in$

$\|\gamma\|)$ and $\forall w''(wR_{\|\alpha\wedge\neg\beta\|}w'' \Rightarrow w'' \in \|\gamma\|)$. By $V_{\Box\!\!\rightarrow}$ this implies $\vDash_w^{\mathcal{M}_C} \alpha \wedge \beta \Box\!\!\rightarrow \gamma$ and $\vDash_w^{\mathcal{M}_C} \alpha \wedge \neg\beta \Box\!\!\rightarrow \gamma$, and, thus, $\vDash_w^{\mathcal{M}_C} (\alpha \wedge \beta \Box\!\!\rightarrow \gamma) \wedge (\alpha \wedge \neg\beta \Box\!\!\rightarrow \gamma)$. However, since $wR_{\|\alpha\|}w'$ and $w' \notin \|\gamma\|$, we have by $V_{\Box\!\!\rightarrow}$ that $\nvDash_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \gamma$. □

5.3.4 Monotonic Principles

Axiom Schema Cut. Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{Cut} holds. Then, for any world $w \in W$ and sets $X, Y, Z \subseteq W$, it is the case that $\forall w'(wR_Z w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_{X\cap Z}w' \Rightarrow wR_{X\cap Y}w')$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$, such that $\vDash_w^{\mathcal{M}_C} (\alpha \wedge \beta \Box\!\!\rightarrow \delta) \wedge (\gamma \Box\!\!\rightarrow \beta)$. Then, due to $V_{\Box\!\!\rightarrow}$ it follows that $\forall w'(wR_{\|\alpha\wedge\beta\|}w' \Rightarrow w' \in \|\delta\|)$ and $\forall w'(wR_{\|\gamma\|}w' \Rightarrow w' \in \|\beta\|)$. As $\forall w'(wR_Z w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_{X\cap Z}w' \Rightarrow wR_{X\cap Y}w')$ holds for any $X, Y, Z \subseteq W$, we get $\forall w'(wR_{\|\gamma\|}w' \Rightarrow w' \in \|\beta\|) \Rightarrow \forall w'(wR_{\|\alpha\|\cap\|\gamma\|}w' \Rightarrow wR_{\|\alpha\|\cap\|\beta\|}w')$ for every $\|\alpha\|, \|\beta\|, \|\gamma\| \subseteq W$. Since $\forall w'(wR_{\|\gamma\|}w' \Rightarrow w' \in \|\beta\|)$ holds by assumption, this implies that $\forall w'(wR_{\|\alpha\|\cap\|\gamma\|}w' \Rightarrow wR_{\|\alpha\|\cap\|\beta\|}w')$. By $\text{Def}_{\|.\|}$ we have that $\forall w'(wR_{\|\alpha\wedge\gamma\|}w' \Rightarrow wR_{\|\alpha\wedge\beta\|}w')$. Moreover, since $\forall w'(wR_{\|\alpha\wedge\beta\|}w' \Rightarrow w' \in \|\delta\|)$, we get $\forall w'(wR_{\|\alpha\wedge\gamma\|}w' \Rightarrow w' \in \|\delta\|)$, and, hence, due to $V_{\Box\!\!\rightarrow}$ this implies that $\vDash_w^{\mathcal{M}_C} \alpha \wedge \gamma \Box\!\!\rightarrow \delta$.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{Cut} does not hold. Then, there are two worlds $w, w' \in W$ and sets $X, Y, Z \subseteq W$, such that $\forall w''(wR_Z w'' \Rightarrow w'' \in Y)$, $wR_{X\cap Z}w'$, but not $wR_{X\cap Y}w'$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|$, $Y = \|\beta\|$, $Z = \|\gamma\|$, $\forall w''(wR_{\|\alpha\|\cap\|\beta\|}w'' \Rightarrow w'' \in \|\delta\|)$, but $w' \notin \|\delta\|$. Then, it follows that $\forall w''(wR_{\|\gamma\|}w'' \Rightarrow w'' \in \|\beta\|)$, $wR_{\|\alpha\|\cap\|\gamma\|}w'$, but not $wR_{\|\alpha\|\cap\|\beta\|}w'$. This assignment is possible for any frame $\mathcal{F}_C$, since by assumption w' is not among the w'' such that $wR_{\|\alpha\|\cap\|\beta\|}w''$. Since $\forall w''(wR_{\|\alpha\|\cap\|\beta\|}w'' \Rightarrow w'' \in \|\delta\|)$, by $\text{Def}_{\|.\|}$ it follows that $\forall w''(wR_{\|\alpha\wedge\beta\|}w'' \Rightarrow w'' \in \|\delta\|)$. Moreover, as $\forall w''(wR_{\|\gamma\|}w'' \Rightarrow w'' \in \|\beta\|)$, it follows by $V_{\Box\!\!\rightarrow}$ that $\vDash_w^{\mathcal{M}_C} \alpha \wedge \beta \Box\!\!\rightarrow \delta$ and $\vDash_w^{\mathcal{M}_C} \gamma \Box\!\!\rightarrow \beta$, and, consequently, $\vDash_w^{\mathcal{M}_C} (\alpha \wedge \beta \Box\!\!\rightarrow \delta) \wedge (\gamma \Box\!\!\rightarrow \beta)$ holds. Moreover, by $\text{Def}_{\|.\|}$ it is the case that $wR_{\|\alpha\wedge\gamma\|}w'$. Since $w' \notin \|\delta\|$, this implies by $V_{\Box\!\!\rightarrow}$ that $\nvDash_w^{\mathcal{M}_C} \alpha \wedge \gamma \Box\!\!\rightarrow \delta$.

□

Axiom Schema Mon. Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{Mon} holds. Then, for any world $w \in W$ and sets $X, Y \subseteq W$, it is the case that $\forall w'(wR_{X\cap Y}w' \Rightarrow wR_X w')$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$, such that $\vDash_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \gamma$. By $V_{\Box\!\!\rightarrow}$

follows that $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\gamma\|)$. As $\forall w'(wR_{X\cap Y}w' \Rightarrow wR_X w')$ holds for any $X, Y \subseteq W$, we can infer that $\forall w'(wR_{\|\alpha\|\cap\|\beta\|}w' \Rightarrow wR_{\|\alpha\|}w')$ for every $\|\alpha\|, \|\beta\| \subseteq W$. Since $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\gamma\|)$, it follows that $\forall w'(wR_{\|\alpha\|\cap\|\beta\|}w' \Rightarrow w' \in \|\gamma\|)$. By $\mathrm{Def}_{\|.\|}$ we can conclude that $\forall w'(wR_{\|\alpha\wedge\beta\|}w' \Rightarrow w' \in \|\gamma\|)$, and due to $V_{\Box\!\!\rightarrow}$ this implies $\vDash_w^{\mathcal{M}_C} \alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{Mon} does not hold. Then, there are two worlds $w, w' \in W$ and sets $X, Y \subseteq W$, such that $wR_{X\cap Y}w'$, but not $wR_X w'$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|$, $Y = \|\beta\|$, $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow w'' \in \|\gamma\|)$ and $w' \notin \|\gamma\|$. Then, $wR_{\|\alpha\|\cap\|\beta\|}w'$, but not $wR_{\|\alpha\|}w'$. This assignment is possible for any frame $\mathcal{F}_C$, since by assumption w' is not among the w'', such that $wR_{\|\alpha\|}w''$. As $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow w'' \in \|\gamma\|)$ is the case, it follows by $V_{\Box\!\!\rightarrow}$ that $\vDash_w^{\mathcal{M}_C} \alpha \mathrel{\Box\!\!\rightarrow} \gamma$. Since $wR_{\|\alpha\|\cap\|\beta\|}w'$, $\mathrm{Def}_{\|.\|}$, gives us that $wR_{\|\alpha\wedge\beta\|}w'$. Since $w' \notin \|\beta\|$ this implies by $V_{\Box\!\!\rightarrow}$ that $\nvDash_w^{\mathcal{M}_C} \alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$. $\Box$

Axiom Schema Trans. Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{Trans} holds. Then, for any world $w \in W$ and sets $X, Y \subseteq W$, it is the case that $\forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_X w' \Rightarrow wR_Y w')$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$, such that $\vDash_w^{\mathcal{M}_C} (\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\beta \mathrel{\Box\!\!\rightarrow} \gamma)$. By $V_{\Box\!\!\rightarrow}$ follows that $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\beta\|)$ and $\forall w'(wR_{\|\beta\|}w' \Rightarrow w' \in \|\gamma\|)$. As $\forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_X w' \Rightarrow wR_Y w')$ holds for any $X, Y \subseteq W$, we have $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\beta\|) \Rightarrow \forall w'(wR_{\|\alpha\|}w' \Rightarrow wR_{\|\beta\|}w')$ for every $\|\alpha\|, \|\beta\| \subseteq W$. Since $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\beta\|)$, this implies $\forall w'(wR_{\|\alpha\|}w' \Rightarrow wR_{\|\beta\|}w')$. Moreover, since it is the case that $\forall w'(wR_{\|\beta\|}w' \Rightarrow w' \in \|\gamma\|)$, it follows that $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\gamma\|)$. By $V_{\Box\!\!\rightarrow}$ we get $\vDash_w^{\mathcal{M}_C} \alpha \mathrel{\Box\!\!\rightarrow} \gamma$.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{Trans} does not hold. Then, there are worlds $w, w' \in W$ and sets $X, Y \subseteq W$, such that $\forall w''(wR_X w'' \Rightarrow w'' \in Y)$, $wR_X w'$, but not $wR_Y w'$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|$, $Y = \|\beta\|$, $\forall w''(wR_{\|\beta\|}w'' \Rightarrow w'' \in \|\gamma\|)$ and $w' \notin \|\gamma\|$. Then, we have $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow w'' \in \|\beta\|)$, $wR_{\|\alpha\|}w'$ and not $wR_{\|\beta\|}w'$. This assignment is possible for any frame $\mathcal{F}_C$, since by assumption w' is not among the w'', such that $wR_{\|\beta\|}w''$. As $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow w'' \in \|\beta\|)$ and $\forall w''(wR_{\|\beta\|}w'' \Rightarrow w'' \in \|\gamma\|)$, it follows by $V_{\Box\!\!\rightarrow}$ that $\vDash_w^{\mathcal{M}_C} \alpha \mathrel{\Box\!\!\rightarrow} \beta$ and $\vDash_w^{\mathcal{M}_C} \beta \mathrel{\Box\!\!\rightarrow} \gamma$ and, hence, $\vDash_w^{\mathcal{M}_C} (\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\beta \mathrel{\Box\!\!\rightarrow} \gamma)$. However, since $wR_{\|\alpha\|}w'$, but $w' \notin \|\gamma\|$, we get due to $V_{\Box\!\!\rightarrow}$ that $\nvDash_w^{\mathcal{M}_C} \alpha \mathrel{\Box\!\!\rightarrow} \gamma$.

$\square$

Axiom Schema CP. ("$\Leftarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be any frame, such hat C_{CP} holds. Then, for any $w \in W$, and sets $X, Y \subseteq W$ it is the case $\forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_{-Y} w' \Rightarrow w' \in -X)$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$, such that $\vDash_w^{\mathcal{M}_C} (\alpha \,\Box\!\!\rightarrow \beta)$. By $V_{\Box\!\rightarrow}$ we have $\forall w'(wR_{\|\alpha\|} w' \Rightarrow w' \in \|\beta\|)$. As $\forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_{-Y} w' \Rightarrow w' \in -X)$ holds for any $X, Y \subseteq W$, we get $\forall w'(wR_{\|\alpha\|} w' \Rightarrow w' \in \|\beta\|) \Rightarrow \forall w'(wR_{-\|\beta\|} w' \Rightarrow w' \in -\|\alpha\|)$ for every $\|\alpha\|, \|\beta\| \subseteq W$. Since $\forall w'(wR_{\|\alpha\|} w' \Rightarrow w' \in \|\beta\|)$, this implies $\forall w'(wR_{-\|\beta\|} w' \Rightarrow w' \in -\|\alpha\|)$. By $\mathrm{Def}_{\|.\|}$ we have $\forall w'(wR_{\|\neg\beta\|} w' \Rightarrow w' \in \|\neg\alpha\|)$. Hence, due to $V_{\Box\!\rightarrow}$ it follows that $\vDash_w^{\mathcal{M}_C} \neg\beta \,\Box\!\!\rightarrow \neg\alpha$.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{CP} does not hold. Then, there are worlds $w, w' \in W$ and sets $X, Y \subseteq W$, such that $\forall w''(wR_X w'' \Rightarrow w'' \in Y)$, $wR_{-Y} w'$, but $w' \notin -X$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|$, $Y = \|\beta\|$. Then, $\forall w''(wR_{\|\alpha\|} w'' \Rightarrow w'' \in \|\beta\|)$, $wR_{-\|\beta\|} w'$, and $w' \notin -\|\alpha\|$. Due to $V_{\Box\!\rightarrow}$ we have $\vDash_w^{\mathcal{M}_C} \alpha \,\Box\!\!\rightarrow \beta$. Moreover, by $\mathrm{Def}_{\|.\|}$ this implies that $wR_{\|\neg\beta\|} w'$ and $w' \notin \|\neg\alpha\|$. Hence, due to $V_{\Box\!\rightarrow}$ it follows that $\nvDash_w^{\mathcal{M}_C} \neg\beta \,\Box\!\!\rightarrow \neg\alpha$. $\square$

5.3.5 Bridge Principles

Axiom Schema MP. ("$\Leftarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{MP} holds. Then, for any $w \in W$ and $X \subseteq W$ it is the case that $w \in X \Rightarrow wR_X w$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $\vDash_w^{\mathcal{M}_C} \alpha \,\Box\!\!\rightarrow \beta$. Moreover, assume that $\vDash_w^{\mathcal{M}_C} \alpha$. Then, by $V_{\Box\!\rightarrow}$ we have $\forall w'(wR_{\|\alpha\|} w' \Rightarrow w' \in \|\beta\|)$, and $w \in \|\alpha\|$. Since $w \in X \Rightarrow wR_X w$ for any $X \subseteq W$, it follows that $w \in \|\alpha\| \Rightarrow wR_{\|\alpha\|} w$ for every $\|\alpha\| \subseteq W$, and, hence, $wR_{\|\alpha\|} w$. As it is the case that $\forall w'(wR_{\|\alpha\|} w' \Rightarrow w' \in \|\beta\|)$, this implies that $w \in \|\beta\|$, and we get $\vDash_w^{\mathcal{M}_C} \beta$.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{MP} does not hold. Then, for some $w \in W$ and $X \subseteq W$ it is the case that $w \in X$, but $\neg wR_X w$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|$, $\forall w'(wR_{\|\alpha\|} w' \Rightarrow w' \in \|\beta\|)$ and $w \notin \|\beta\|$. Then, $w \in \|\alpha\|$ and $\neg wR_{\|\alpha\|} w$. This assignment is possible for any frame $\mathcal{F}_C$, since by assumption w is not among the w's, such that $wR_{\|\alpha\|} w'$. Since $\forall w'(wR_{\|\alpha\|} w' \Rightarrow w' \in \|\beta\|)$, we get due to $V_{\Box\!\rightarrow}$ that $\vDash_w^{\mathcal{M}_C} \alpha \,\Box\!\!\rightarrow \beta$. Since by assumption $w \in \|\alpha\|$ and $w \notin \|\beta\|$, it follows $\mathrm{Def}_{\|.\|}$ that $\nvDash_w^{\mathcal{M}_C} \alpha \rightarrow \beta$. $\square$

Axiom Schema CS. ("$\Leftarrow$"). Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{CS} holds. Then, for any $w \in W$ and $X \subseteq W$ it is the case that $w \in X \Rightarrow \forall w'(wR_X w' \Rightarrow w' = w)$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$ such that $\vDash_w^{\mathcal{M}_C} \alpha \wedge \beta$. This implies that $w \in \|\alpha\|$ and $w \in \|\beta\|$. Since, $w \in X \Rightarrow \forall w'(wR_X w' \Rightarrow w' = w)$ holds for any $X \subseteq W$, it follows that $w \in \|\alpha\| \Rightarrow \forall w'(wR_{\|\alpha\|} w' \Rightarrow w' = w)$ for every $\|\alpha\| \subseteq W$. As we have $w \in \|\alpha\|$, we can infer $\forall w'(wR_{\|\alpha\|} w' \Rightarrow w' = w)$. Two cases are possible for $w \in W$: a) It cannot see any world $w' \in W$ by $R_{\|\alpha\|}$. Then, by $V_{\Box\!\!\rightarrow}$ we, trivially, get $\vDash_w^{\mathcal{M}_C} \alpha \mathrel{\Box\!\!\rightarrow} \beta$. b) It can see some world $w' \in W$. In this case, as $w' = w$ for all w' such $wR_{\|\alpha\|} w'$ and $w \in \|\beta\|$, it follows by $V_{\Box\!\!\rightarrow}$ that $\vDash_w^{\mathcal{M}_C} \alpha \mathrel{\Box\!\!\rightarrow} \beta$. Hence, $\vDash_w^{\mathcal{M}_C} \alpha \mathrel{\Box\!\!\rightarrow} \beta$ holds in both cases.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{CS} does not hold. Then, there are $w, w' \in W$ and $X \subseteq W$, such that $w \in X$, $wR_X w'$, but $w' \neq w$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|$, $w \in \|\beta\|$, but $w' \notin \|\beta\|$. Then, $w \in \|\alpha\|$ and $wR_{\|\alpha\|} w'$. This assignment is possible for any frame $\mathcal{F}_C$, since $w' \neq w$. Since, $w \in \|\alpha\|$ and $w \in \|\beta\|$, it follows by $\mathrm{Def}_{\|\cdot\|}$ that $\vDash_w^{\mathcal{M}_C} \alpha \wedge \beta$. However, as $wR_{\|\alpha\|} w'$, but $w' \notin \|\beta\|$, we get by $V_{\Box\!\!\rightarrow}$ that $\nvDash_w^{\mathcal{M}_C} \alpha \mathrel{\Box\!\!\rightarrow} \beta$. □

Axiom Schema TR. ("$\Leftarrow$"). Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{TR} holds. Then, for any $w \in W$ and set $X \subseteq W$ it is the case that $\forall w'(wR_{-X} w' \Rightarrow w' \in X) \Rightarrow w \in X$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$ such that $\vDash_w^{\mathcal{M}_C} \neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha$. Then, by $V_{\Box\!\!\rightarrow}$ we have $\forall w'(wR_{\|\neg\alpha\|} w' \Rightarrow w' \in \|\alpha\|)$. By $\mathrm{Def}_{\|\cdot\|}$ we get $\forall w'(wR_{-\|\alpha\|} w' \Rightarrow w' \in \|\alpha\|)$. Moreover, as $\forall w'(wR_{-X} w' \Rightarrow w' \in X) \Rightarrow w \in X$ holds for any $X \subseteq W$, it follows that $\forall w'(wR_{-\|\alpha\|} w' \Rightarrow w' \in \|\alpha\|) \Rightarrow w \in \|\alpha\|$ for every $\|\alpha\| \subseteq W$, and, hence, $w \in \|\alpha\|$. Thus, $\vDash_w^{\mathcal{M}_C} \alpha$.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{TR} does not hold. Then, there are $w \in W$ and $X \subseteq W$, such that $\forall w'(wR_{-X} w' \Rightarrow w' \in X)$, but $w \notin X$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|$. Then, $\forall w'(wR_{-\|\alpha\|} w' \Rightarrow w' \in \|\alpha\|)$ and $w \notin \|\alpha\|$. By $\mathrm{Def}_{\|\cdot\|}$, we have $\forall w'(wR_{\|\neg\alpha\|} w' \Rightarrow w' \in \|\alpha\|)$, and due to $V_{\Box\!\!\rightarrow}$ we get $\vDash_w^{\mathcal{M}_C} \neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha$. Moreover, since $w \notin \|\alpha\|$, it follows that $\nvDash_w^{\mathcal{M}_C} \alpha$. □

Axiom Schema Det. ("$\Leftarrow$"). Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{Det} holds. Then, for any $w \in W$ it is the case that $wR_W w$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$ such that $\vDash_w^{\mathcal{M}_C} \top \mathrel{\Box\!\!\rightarrow} \alpha$. Then, by $V_{\Box\!\!\rightarrow}$ we have $\forall w'(wR_{\|\top\|} w' \Rightarrow w' \in$

$\|\alpha\|$). By Def$_{\|.\|}$ it holds $\|\top\| = W$. Thus, from wR_Ww it follows that $wR_{\|\top\|}w$ and, hence, $w \in \|\alpha\|$. Thus, $\vDash_w^{\mathcal{M}_C} \alpha$.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{Det} does not hold. Then, there is some $w \in W$, such that $\neg wR_Ww$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$ such that $\forall w'(wR_{\|\top\|}w' \Rightarrow w' \in \|\alpha\|)$, but $w \notin \|\alpha\|$. Thus, since by Def$_{\|.\|}$ it is the case that $W = \|\top\|$, we have $\neg wR_{\|\top\|}w$. This assignment is possible for any frame $\mathcal{F}_C$, as by assumption w is not among the w', such that $wR_{\|\top\|}w'$. Hence, since $\forall w'(wR_{\|\top\|}w' \Rightarrow w' \in \|\alpha\|)$, we have by $V_{\Box\!\!\rightarrow}$ that $\vDash_w^{\mathcal{M}_C} \top \Box\!\!\rightarrow \alpha$. However, since $w \notin \|\alpha\|$, this implies $\nvDash_w^{\mathcal{M}_C} \alpha$. □

Axiom Schema Cond. ("$\Leftarrow$"). Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{Cond} holds. Then, for any $w \in W$ it is the case that $\forall w'(wR_Ww' \Rightarrow w' = w)$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$ such that $\vDash_w^{\mathcal{M}_C} \alpha$. Hence, $w \in \|\alpha\|$. Since $\forall w'(wR_Ww' \Rightarrow w' = w)$ and $W = \|\top\|$, we have $\forall w'(wR_{\|\top\|}w' \Rightarrow w' = w)$. Hence, w can see by $R_{\|\top\|}$ at most w. So it can see by $R_{\|\top\|}$ either no other world or only w. In both cases, since $w \in \|\alpha\|$, we have, thus, $\forall w'(wR_{\|\top\|}w' \Rightarrow w' \in \|\alpha\|)$, and by $V_{\Box\!\!\rightarrow}$ it follows that $\vDash_w^{\mathcal{M}_C} \top \Box\!\!\rightarrow \alpha$.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{Cond} does not hold. Then, there are $w, w' \in W$, such that wR_Ww', but $w' \neq w$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$ such that $w \in \|\alpha\|$ and $w' \notin \|\alpha\|$. Since by Def$_{\|.\|}$ it holds that $W = \|\top\|$, we have $wR_{\|\top\|}w'$. This assignment is possible for any frame $\mathcal{F}_C$, since by assumption $w' \neq w$. As $w \in \|\alpha\|$, we have $\vDash_w^{\mathcal{M}_C} \alpha$. However, since $w' \notin \|\alpha\|$, but $wR_{\|\top\|}w'$, it follows by $V_{\Box\!\!\rightarrow}$ that $\nvDash_w^{\mathcal{M}_C} \top \Box\!\!\rightarrow \alpha$. □

5.3.6 Collapse Conditions Material Implication

Axiom Schema VEQ. ("$\Leftarrow$"). Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{VEQ} holds. Then, for any world $w \in W$ and set $X \subseteq W$ it is the case that $\forall w'(wR_Xw' \Rightarrow w' = w)$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$ such that $\vDash_w^{\mathcal{M}_C} \beta$. Hence, $w \in \|\beta\|$. Since $\forall w'(wR_Xw' \Rightarrow w' = w)$ holds for any $X \subseteq W$, we have $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' = w)$ for every $\|\alpha\| \subseteq W$. There are two possible cases for w: a) There is no world $w' \in W$, such that $wR_{\|\alpha\|}w'$. Then, by $V_{\Box\!\!\rightarrow}$ it follows trivially that $\vDash_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \beta$. b) There is a world $w' \in W$, such that $wR_{\|\alpha\|}w'$. Since $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' = w)$, this implies that any world w can see by $R_{\|\alpha\|}$ is identical with w. However, $w \in \|\beta\|$.

Hence, it follows by $V_{\Box\!\!\rightarrow}$ that $\vDash_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \beta$. Therefore, it holds in both cases that $\vDash_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \beta$.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{VEQ} does not hold. Then, there are $w, w' \in W$ and $X \subseteq W$, such that wR_Xw', but $w' \neq w$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$ such that $X = \|\alpha\|$ and $w' \notin \|\beta\|$. Then, $wR_{\|\alpha\|}w'$. This assignment is trivially possible for any $\mathcal{F}_C$. As $w \in \|\alpha\|$, it follows that $\vDash_w^{\mathcal{M}_C} \alpha$. However, since $w' \notin \|\beta\|$, but $wR_{\|\alpha\|}w'$, this implies by $V_{\Box\!\!\rightarrow}$ that $\nvDash_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \beta$. □

Axiom Schema EFQ. ("$\Leftarrow$"). Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{EFQ} holds. Then, for any $w \in W$ and $X \subseteq W$ it is the case that $w \in -X \Rightarrow \neg\exists w'(wR_Xw')$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$ such that $\vDash_w^{\mathcal{M}_C} \neg\alpha$. Hence, by $\mathrm{Def}_{\|.\|}$ it is the case that $w \in -\|\alpha\|$. Since $w \in -X \Rightarrow \neg\exists w'(wR_Xw')$ holds for any $X \subseteq W$, it follows that $w \in -\|\alpha\| \Rightarrow \neg\exists w'(wR_{\|\alpha\|}w')$ for every $\|\alpha\| \subseteq W$. Moreover, as $w \in -\|\alpha\|$ is the case, we get $\neg\exists w'(wR_{\|\alpha\|}w')$ and due to $V_{\Box\!\!\rightarrow}$ it trivially follows that $\vDash_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \beta$ for any β.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{EFQ} does not hold. Then, there are $w, w' \in W$ and $X \subseteq W$, such that $w \in -X$ and wR_Xw'. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$ such that $X = \|\alpha\|$ and $w' \notin \|\beta\|$. Then, $w \in -\|\alpha\|$ and $wR_{\|\alpha\|}w'$. This assignment is trivially possible for any frame $\mathcal{F}_C$. It follows by $\mathrm{Def}_{\|.\|}$ that $\vDash_w^{\mathcal{M}_C} \neg\alpha$. However, since $wR_{\|\alpha\|}w'$ and $w' \notin \|\beta\|$, this implies by $V_{\Box\!\!\rightarrow}$ that $\nvDash_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \beta$.

□

5.3.7 Traditional Extensions

Axiom Schema D. ("$\Leftarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{D} holds. Then, for any $w \in W$ and any $X \subseteq W$, there exists a $w' \in W$, such that wR_Xw'. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$, such that $\vDash_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \beta$. Then, by $V_{\Box\!\!\rightarrow}$ we get $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow w'' \in \|\beta\|)$. As for any $X \subseteq W$ there is a w', such that wR_Xw', it follows that $wR_{\|\alpha\|}w'$ for every $\|\alpha\| \subseteq W$. Moreover, as $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow w'' \in \|\beta\|)$ is the case, we get $w' \in \|\beta\|$. Since $wR_{\|\alpha\|}w'$, it follows by $V_{\Box\!\!\rightarrow}$ and $\mathrm{Def}_{\|.\|}$ that $\vDash_w^{\mathcal{M}_C} \alpha \Diamond\!\!\rightarrow \beta$.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{D} does not hold. Then, there exists a world $w \in W$ and a set $X \subseteq W$, such that there is no world w' with wR_Xw'.

Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|$. Then, there is no world w', such that $wR_{\|\alpha\|}w'$. It trivially follows by $V_{\Box\!\!\rightarrow}$ that $\vDash_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \beta$. Moreover, as by assumption there is no world w', such that $wR_{\|\alpha\|}w'$, on the basis of by $V_{\Box\!\!\rightarrow}$, $\mathrm{Def}_{\|.\|}$ and $\mathrm{Def}_{\Diamond\!\!\rightarrow}$ this implies that $\nvDash_w^{\mathcal{M}_C} \alpha \Diamond\!\!\rightarrow \beta$. $\square$

Axiom Schema T. ("$\Leftarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{T} holds. Then, for any w and $X \subseteq W$, it is the case that wR_Xw. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model, such that $\vDash_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \beta$. By $V_{\Box\!\!\rightarrow}$ it follows that $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\beta\|)$. As wR_Xw for any $X \subseteq W$, this implies that $wR_{\|\alpha\|}w$ for every $\|\alpha\| \subseteq W$. Hence, w is among the w's, so that $w \in \|\beta\|$. Thus, it is the case that $\vDash_w^{\mathcal{M}_C} \beta$.

("$\Rightarrow$") Let $\langle W, R\rangle$ be a frame, such that C_{T} does not hold. Then, there exists a world w and a set $X \subseteq W$, such that it is $\neg wR_Xw$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|, \forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\beta\|)$ and $w \notin \|\beta\|$. Then, $\neg wR_{\|\alpha\|}w$. This assignment is possible for any frame $\mathcal{F}_C$, since by assumption w is not among the w', such that $wR_{\|\alpha\|}w'$. Since $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\beta\|)$, by $V_{\Box\!\!\rightarrow}$ it follows that $\vDash_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \beta$. As $w \notin \|\beta\|$ we have that $\nvDash_w^{\mathcal{M}_C} \beta$. $\square$

Axiom Schema B. ("$\Leftarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{B} holds. Then, for any $w \in W$ and $X \subseteq W$ it is the case that $\forall w'(wR_Xw' \Rightarrow w'R_Xw)$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$, such that $\vDash_w^{\mathcal{M}_C} \alpha$. Then, it is the case that $w \in \|\alpha\|$. As $\forall w'(wR_Xw' \Rightarrow w'R_Xw)$ holds for any $X \subseteq W$, it follows that $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w'R_{\|\alpha\|}w)$ for every $\|\alpha\| \subseteq W$. Concerning w there are two possible cases: a) There is no $w' \in W$, such that $wR_{\|\alpha\|}w'$. Then, by $V_{\Box\!\!\rightarrow}$ it follows trivially that $\vDash_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow (\alpha \Diamond\!\!\rightarrow \beta)$. b) There is a $w' \in W$, such that $wR_{\|\alpha\|}w'$. Then, for all w, such that $wR_{\|\alpha\|}w'$, it holds that $w'R_{\|\alpha\|}w$. Since $w \in \|\beta\|$, we get by by $V_{\Box\!\!\rightarrow}$ and $\mathrm{Def}_{\|.\|}$ that $\vDash_{w'}^{\mathcal{M}_C} \alpha \Diamond\!\!\rightarrow \beta$. As this holds for any w', such that $wR_{\|\alpha\|}w'$, it follows by $V_{\Box\!\!\rightarrow}$ that $\vDash_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow (\alpha \Diamond\!\!\rightarrow \beta)$. Hence, in both cases, we get $\vDash_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow (\alpha \Diamond\!\!\rightarrow \beta)$.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{B} does not hold. Then, there exist worlds $w, w' \in W$ and a set $X \subseteq W$, such that wR_Xw', but $\neg w'R_Xw$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$, such that $X = \|\alpha\|, \neg\exists w''(w'R_{\|\alpha\|}w'' \wedge w'' \in \|\beta\|)$ and $w \in \|\alpha\|$. Then, $wR_{\|\alpha\|}w'$ and not $w'R_{\|\alpha\|}w$. This assignment is trivially possible for any frame $\mathcal{F}_C$. As $w \in \|\alpha\|$, we get $\vDash_w^{\mathcal{M}_C} \alpha$, and, since

$\neg\exists w''(w'R_{\|\alpha\|}w'' \wedge w'' \in \|\beta\|)$, it follows by $V_{\Box\!\rightarrow}$, $\mathrm{Def}_{\|.\|}$ and $\mathrm{Def}_{\Diamond\!\rightarrow}$ that $\nvDash_{w'}^{\mathcal{M}_C} \alpha \Diamond\!\!\rightarrow \beta$. Since $wR_{\|\alpha\|}w'$, by $V_{\Box\!\rightarrow}$ we get $\nvDash_{w}^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow (\alpha \Diamond\!\!\rightarrow \beta)$. □

Axiom Schema 4. ("$\Leftarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_4 holds. Then, for any worlds w and set $X \subseteq W$ it is the case that $\forall w', w''(wR_Xw' \wedge w'R_Xw'' \Rightarrow wR_Xw'')$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be any model based on $\mathcal{F}_C$, such that $\vDash_{w}^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \beta$. Then, by $V_{\Box\!\rightarrow}$ it follows that $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\beta\|)$. As $\forall w', w''(wR_Xw' \wedge w'R_Xw'' \Rightarrow wR_Xw'')$ holds, $X \subseteq W$, it follows that $\forall w', w''(wR_{\|\alpha\|}w' \wedge w'R_{\|\alpha\|}w'' \Rightarrow wR_{\|\alpha\|}w'')$ for every $\|\alpha\| \subseteq W$. Since $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\beta\|)$ holds, it follows that $\forall w', w''(wR_{\|\alpha\|}w' \wedge w'R_{\|\alpha\|}w'' \Rightarrow w'' \in \|\beta\|)$. There are two possible cases for w: (1) There is no w', such that $wR_{\|\alpha\|}w'$. Then, by $V_{\Box\!\rightarrow}$ it holds trivially that $\vDash_{w'}^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow (\alpha \Box\!\!\rightarrow \beta)$. (2) There are worlds w', such that $wR_{\|\alpha\|}w'$. As it holds that $\forall w', w''\ (wR_{\|\alpha\|}w' \wedge w'R_{\|\alpha\|}w'' \Rightarrow w'' \in \|\beta\|)$, we have $\forall w''(w'R_{\|\alpha\|}w'' \Rightarrow w'' \in \|\beta\|)$ for any such w'. Hence, by $V_{\Box\!\rightarrow}$ follows that $\vDash_{w'}^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \beta$. As this holds in both cases (1) and (2) for any w', such that $wR_{\|\alpha\|}w'$, we get by $V_{\Box\!\rightarrow}$ that $\vDash_{w}^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow (\alpha \Box\!\!\rightarrow \beta)$.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_4 does not hold. Then, there exist worlds $w, w', w'' \in W$ and a set $X \subseteq W$, such that wR_Xw', $w'R_Xw''$, but not wR_Xw''. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|$, $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\beta\|)$ and $w'' \notin \|\beta\|$. Then, $wR_{\|\alpha\|}w'$, $w'R_{\|\alpha\|}w''$ and not $wR_{\|\alpha\|}w''$. This assignment is possible for any frame $\mathcal{F}_C$, as by assumption w'' is not among the w', such that $wR_{\|\alpha\|}w'$. Since $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\beta\|)$, by $V_{\Box\!\rightarrow}$ it follows that $\vDash_{w}^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \beta$. However, since $w'R_{\|\alpha\|}w''$ and $w'' \notin \|\beta\|$, by $V_{\Box\!\rightarrow}$ we get $\nvDash_{w'}^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow \beta$. As $wR_{\|\alpha\|}w'$, due to $V_{\Box\!\rightarrow}$ it follows that $\nvDash_{w}^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow (\alpha \Box\!\!\rightarrow \beta)$. □

Axiom Schema 5. ("$\Leftarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_5 holds. Then, for any world $w \in W$ and set $X \subseteq W$ it is the case that $\forall w', w''(wR_Xw' \wedge wR_Xw'' \Rightarrow w'R_Xw'')$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $\vDash_{w}^{\mathcal{M}_C} \alpha \Diamond\!\!\rightarrow \beta$. Then, by $V_{\Box\!\rightarrow}$ and $\mathrm{Def}_{\|.\|}$ follows that there is a world w''', such that $wR_{\|\alpha\|}w'''$ and $w''' \in \|\beta\|$. Since, $\forall w', w''(wR_Xw' \wedge wR_Xw'' \Rightarrow w'R_Xw'')$ for any $X \subseteq W$, we get $\forall w', w''(wR_{\|\alpha\|}w' \wedge wR_{\|\alpha\|}w'' \Rightarrow w'R_{\|\alpha\|}w'')$ for every $\|\alpha\| \subseteq W$. This implies that any worlds $w', w'' \in W$, such that $wR_{\|\alpha\|}w'$ and $wR_{\|\alpha\|}w''$, can see each other including themselves. Hence, as $wR_{\|\alpha\|}w'''$ and $w''' \in \|\beta\|$, any world w', such that $wR_{\|\alpha\|}w'$, can see world w''' by $R_{\|\alpha\|}$. Thus, since $w''' \in \|\beta\|$, this implies by $V_{\Box\!\rightarrow}$

and $\mathrm{Def}_{\|.\|}$ that $\vDash_{w'}^{\mathcal{M}_C} \alpha \mathbin{\Diamond\!\!\rightarrow} \beta$. Since that holds for any w', such that $wR_{\|\alpha\|}w'$, we have by $V_{\Box\!\!\rightarrow}$ that $\vDash_{w}^{\mathcal{M}_C} \alpha \mathbin{\Box\!\!\rightarrow} (\alpha \mathbin{\Diamond\!\!\rightarrow} \beta)$.

("⇒") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_5 does not hold. Then, there exist worlds $w, w' \in W$ and a set $X \subseteq W$, such that wR_Xw', wR_Xw'', but not $w'R_Xw''$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|$, $\neg\exists w'''(w'R_{\|\alpha\|}w''' \wedge w''' \in \|\beta\|)$ and $w'' \in \|\beta\|$. Then, $wR_{\|\alpha\|}w'$, $wR_{\|\alpha\|}w''$, but not $w'R_{\|\alpha\|}w''$. This assignment is possible for any frame $\mathcal{F}_C$, since by assumption $\neg w'R_{\|\alpha\|}w''$. As $wR_{\|\alpha\|}w''$ and $w'' \in \|\beta\|$, it follows by $V_{\Box\!\!\rightarrow}$, $\mathrm{Def}_{\|.\|}$ and $\mathrm{Def}_{\Diamond\!\!\rightarrow}$ that $\vDash_{w}^{\mathcal{M}_C} \alpha \mathbin{\Diamond\!\!\rightarrow} \beta$. Moreover, since $\neg\exists w'''(w'R_{\|\alpha\|}w''' \wedge w''' \in \|\beta\|)$, by $V_{\Box\!\!\rightarrow}$, $\mathrm{Def}_{\|.\|}$ and $\mathrm{Def}_{\Diamond\!\!\rightarrow}$ follows that $\nvDash_{w'}^{\mathcal{M}_C} \alpha \mathbin{\Diamond\!\!\rightarrow} \beta$, and, since $wR_{\|\alpha\|}w'$, by $V_{\Box\!\!\rightarrow}$ we have $\nvDash_{w}^{\mathcal{M}_C} \alpha \mathbin{\Box\!\!\rightarrow} (\alpha \mathbin{\Diamond\!\!\rightarrow} \beta)$. □

5.3.8 Iteration Principles

Axiom Schema Ex. ("⇐") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{Ex} holds. Then, for any world $w \in W$ and sets $X, Y \subseteq W$ it is the case that $\forall w', w''(wR_Xw' \wedge w'R_Yw'' \Rightarrow wR_{X\cap Y}w'')$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $\vDash_{w}^{\mathcal{M}_C} (\alpha \wedge \beta \mathbin{\Box\!\!\rightarrow} \gamma)$. By $V_{\Box\!\!\rightarrow}$ we have $\forall w'(wR_{\|\alpha\wedge\beta\|}w' \Rightarrow w' \in \|\gamma\|)$. Due to $\mathrm{Def}_{\|.\|}$, it is the case that $\forall w'(wR_{\|\alpha\|\cap\|\beta\|}w \Rightarrow w' \in \|\gamma\|)$. As $\forall w', w''(wR_Xw' \wedge w'R_Yw'' \Rightarrow wR_{X\cap Y}w'')$ for any $X, Y \subseteq W$, we have $\forall w', w''(wR_{\|\alpha\|}w' \wedge w'R_{\|\beta\|}w'' \Rightarrow wR_{\|\alpha\|\cap\|\beta\|}w'')$ for every $\|\alpha\|, \|\beta\| \subseteq W$. Since $\forall w'(wR_{\|\alpha\|\cap\|\beta\|}w' \Rightarrow w' \in \|\gamma\|)$, this implies $\forall w', w''(wR_{\|\alpha\|}w' \wedge w'R_{\|\beta\|}w'' \Rightarrow w'' \in \|\gamma\|)$ and, hence, $\forall w'(wR_{\|\alpha\|}w' \Rightarrow \forall w''\ (w'R_{\|\beta\|}w'' \Rightarrow w'' \in \|\gamma\|))$. Thus, by $V_{\Box\!\!\rightarrow}$ this results in $\forall w'(wR_{\|\alpha\|}w' \Rightarrow \vDash_{w'}^{\mathcal{M}_C} \beta \mathbin{\Box\!\!\rightarrow} \gamma)$ and, hence, we have $\forall w'(wR_{\|\alpha\|}w' \Rightarrow w' \in \|\beta \mathbin{\Box\!\!\rightarrow} \gamma\|)$. By $V_{\Box\!\!\rightarrow}$ it follows that $\vDash_{w}^{\mathcal{M}_C} \alpha \mathbin{\Box\!\!\rightarrow} (\beta \mathbin{\Box\!\!\rightarrow} \gamma)$.

("⇒") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{Ex} does not hold. Then, there exist worlds $w, w', w'' \in W$ and sets $X, Y \subseteq W$, such that wR_Xw', $w'R_Yw''$, but not $wR_{X\cap Y}w''$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|$, $Y = \|\beta\|$, $\forall w'''(wR_{\|\alpha\|\cap\|\beta\|}w''' \Rightarrow w''' \in \|\gamma\|)$, and $w'' \notin \|\gamma\|$. Then, $wR_{\|\alpha\|}w'$, $w'R_{\|\beta\|}w''$, and not $wR_{\|\alpha\|\cap\|\beta\|}w''$. This assignment is possible for any frame $\mathcal{F}_C$, since by assumption w'' is not among the w''', such that $wR_{\|\alpha\wedge\beta\|}w'''$. By $\mathrm{Def}_{\|.\|}$ follows that $\forall w'''(wR_{\|\alpha\wedge\beta\|}w''' \Rightarrow w''' \in \|\gamma\|)$, and, hence, by $V_{\Box\!\!\rightarrow}$ we have $\vDash_{w}^{\mathcal{M}_C} \alpha \wedge \beta \mathbin{\Box\!\!\rightarrow} \gamma$. Moreover, since $w'R_{\|\beta\|}w''$ and $w'' \notin \|\gamma\|$, by $V_{\Box\!\!\rightarrow}$ this implies $\nvDash_{w'}^{\mathcal{M}_C} \beta \mathbin{\Box\!\!\rightarrow} \gamma$. Since

$wR_{\|\alpha\|}w'$, by $V_{\Box\!\!\rightarrow}$ follows that $\nvDash_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow (\beta \Box\!\!\rightarrow \gamma)$. ☐

Axiom Schema Im. ("$\Leftarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{Im} holds. Then, for any world $w \in W$ and sets $X, Y \subseteq W$ it is the case that $\forall w'(wR_{X\cap Y}w' \Rightarrow \exists w''(wR_Xw'' \wedge w''R_Yw'))$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $\vDash_w^{\mathcal{M}_C} (\alpha \Box\!\!\rightarrow (\beta \Box\!\!\rightarrow \gamma))$. By $V_{\Box\!\!\rightarrow}$ we have $\forall w'(wR_{\|\alpha\|}w' \Rightarrow \vDash_{w'}^{\mathcal{M}_C} \beta \Box\!\!\rightarrow \gamma)$. Moreover, by $V_{\Box\!\!\rightarrow}$ this results in $\forall w'(wR_{\|\alpha\|}w' \Rightarrow \forall w''(w'R_{\|\beta\|}w'' \Rightarrow w'' \in \|\gamma\|))$ and, hence, $\forall w', w''(wR_{\|\alpha\|}w' \wedge w'R_{\|\beta\|}w'' \Rightarrow w'' \in \|\gamma\|)$. Since $\forall w'(wR_{X\cap Y}w' \Rightarrow \exists w''(wR_Xw'' \wedge w''R_Yw'))$ holds for any $X, Y \subseteq W$, we get $\forall w'(wR_{\|\alpha\|\cap\|\beta\|}w' \Rightarrow \exists w''(wR_{\|\alpha\|}w'' \wedge w''R_{\|\beta\|}w'))$ for every $\|\alpha\|, \|\beta\| \subseteq W$. So, $wR_{\|\alpha\|\cap\|\beta\|}w' \Rightarrow \exists w''(wR_{\|\alpha\|}w'' \wedge w''R_{\|\beta\|}w')$ for any $w' \in W$. Let $wR_{\|\alpha\|\cap\|\beta\|}w'$ be the case. Then there exists a $w'' \in W$, such that $wR_{\|\alpha\|}w''$ and $w''R_{\|\beta\|}w'$. Since $\forall w', w''(wR_{\|\alpha\|}w' \wedge w'R_{\|\beta\|}w'' \Rightarrow w'' \in \|\gamma\|)$ and $wR_{\|\alpha\|}w'' \wedge w''R_{\|\beta\|}w'$, we get $w' \in \|\gamma\|$. Moreover, as we assumed $wR_{\|\alpha\|\cap\|\beta\|}w'$ for any $w' \in W$, this implies $\forall w'(wR_{\|\alpha\|\cap\|\beta\|}w' \Rightarrow w' \in \|\beta\|)$. By $\mathrm{Def}_{\|.\|}$ we have $\|\alpha\| \cap \|\beta\| = \|\alpha \wedge \beta\|$. Hence, $\forall w'(wR_{\|\alpha\wedge\beta\|}w' \Rightarrow w' \in \|\gamma\|)$ for any $w' \in W$. By $V_{\Box\!\!\rightarrow}$ this implies $\vDash_w^{\mathcal{M}_C} \alpha \wedge \beta \Box\!\!\rightarrow \gamma$.

("$\Rightarrow$") Let $\mathcal{F}_C = \langle W, R\rangle$ be a frame, such that C_{Im} does not hold. Then, there exist worlds $w, w' \in W$ and sets $X, Y \subseteq W$, such that $wR_{X\cap Y}w'$, but $\neg\exists w''(wR_Xw'' \wedge w''R_Yw')$. Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a model based on $\mathcal{F}_C$, such that $X = \|\alpha\|$, $Y = \|\beta\|$, $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow \forall w'''(w''R_{\|\beta\|}w''' \Rightarrow w''' \in \|\gamma\|))$ and $w' \notin \|\gamma\|$. Then, $wR_{\|\alpha\|\cap\|\beta\|}w''$ and $\neg\exists w''(wR_{\|\alpha\|}w'' \wedge w''R_{\|\beta\|}w')$. This assignment is possible for any frame $\mathcal{F}_C$, since if there is a $w'' \in W$, such that $wR_{\|\alpha\|}w''$, then not $w''R_{\|\beta\|}w'$. Hence, w' is not among the w'''s, such that $w''' \in \|\gamma\|$. Regarding w there are two possible cases: (1) There exists a world w'' such that $wR_{\|\alpha\|}w''$. Since $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow \forall w'''(w''R_{\|\beta\|}w''' \Rightarrow w''' \in \|\gamma\|)$, by $V_{\Box\!\!\rightarrow}$ we have $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow \vDash_{w''}^{\mathcal{M}_C} \beta \Box\!\!\rightarrow \gamma)$ and by $\mathrm{Def}_{\|.\|}$ it follows that $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow w'' \in \|\beta \Box\!\!\rightarrow \gamma\|)$. (2) There exists no w'', such that $wR_{\|\alpha\|}w''$. Then, trivially $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow w'' \in \|\beta \Box\!\!\rightarrow \gamma\|)$. Hence, in both cases (1) and (2) we have $\forall w''(wR_{\|\alpha\|}w'' \Rightarrow w'' \in \|\beta \Box\!\!\rightarrow \gamma\|)$. From that it follows due to $V_{\Box\!\!\rightarrow}$ that $\vDash_w^{\mathcal{M}_C} \alpha \Box\!\!\rightarrow (\beta \Box\!\!\rightarrow \gamma)$. Since by $\mathrm{Def}_{\|.\|}$ it holds that $\|\alpha\| \cap \|\beta\| = \|\alpha \wedge \beta\|$ and $wR_{\|\alpha\|\cap\|\beta\|}w'$, we get $wR_{\|\alpha\wedge\beta\|}w'$. Moreover, as $w' \notin \|\beta\|$ by $\mathrm{Def}_{\|.\|}$, this implies $\nvDash_w^{\mathcal{M}_C} \alpha \wedge \beta \Box\!\!\rightarrow \gamma$. ☐

Chapter 6

Soundness and Completeness Proofs for a Lattice of Conditional Logics

6.1 General Overview

In this Chapter we provide soundness and completeness proofs for the lattice of conditional logics described by system **CK** (see Section 4.2.6) plus axioms from Tables 5.1 and 5.2. As our model-theoretic basis for soundness proofs we use Chellas frames (see Section 4.3.1) and for proofs of strong completeness we employ classes of standard Segerberg frames (see Sections 4.3.6 and 4.3.7) plus the frame restrictions in Tables 5.3 and 5.4. Let us, however, first discuss the notion of completeness established in this chapter.

6.1.1 Focus of Our Completeness Proofs

Our completeness results draw on Chellas (1975) and Segerberg (1989). Chellas (1975) gave a completeness proof in terms of Chellas frames for system **CK** (pp. 139–141) and the system **CK**+Refl, system **CK**+MP and system **CK**+Refl+ MP (pp. 141–143). Chellas (1975, p. 143), however, argued that his canonical methods approach does not work for a system corresponding to the monotonic system without bridge principles **M** (cf. Section 7.2.6). Since system **M** is a point in the lattice of conditional logics described in Section 5.1, this implies that Kripke frame completeness for our lattice of conditional logic systems cannot be established by Chellas' (1975) approach.

Segerberg (1989, p. 162), however, provided a strong completeness result for some points in the lattice, namely for those systems, which result from **CK** plus a selection of the axioms Refl, CM, Or, S, RM, Det and Con.[1] For this purpose he identified C-corresponding frame restrictions for these principles. Note that all of these frame conditions are non-trivial in the sense of Definition 5.6. For the completeness proof, however, Segerberg (1989) employed standard Segerberg frames rather than Chellas frames. It is essential for these proofs that all subsets described in the frame restrictions refer to elements X of P. Hence, not only the accessibility relation R_X is exclusively defined w.r.t. elements X of P, but also any other subset X used in the frame restrictions is required to be an element of P. Moreover, the class of Segerberg frames, w.r.t. which a logic is complete, is allowed to contain only standard frames:[2] Let $\mathbf{L}$ be a logic such that $\mathbf{L} = \mathbf{CK}{+}\alpha_1{+}\alpha_2 \ldots {+}\alpha_n$ for $n \in \mathbb{N}$ and $\alpha_1, \alpha_2 \ldots, \alpha_n$ and let $C_{\alpha_1}, C_{\alpha_2}, \ldots, C_{\alpha_n}$ frame-conditions, which C-correspond to $\alpha_1, \alpha_2, \ldots, \alpha_n$. Furthermore, let $\mathcal{F}_S^{\mathrm{st}} = \langle W, R, P \rangle$ be an element of the frame class $\mathbf{F}_S^{\mathrm{st}}$, w.r.t. to which $\mathbf{L}$ is strongly frame complete. Then, all elements X of the parameter P have to satisfy the frame conditions $C_{\alpha_1}, C_{\alpha_2}, \ldots, C_{\alpha_n}$ (see Section 4.3.6). Note that strong completeness w.r.t. (simple) Segerberg frames is trivial (see Theorem 4.26), while strong completeness w.r.t. to standard Segerberg frames is not (see Section 4.3.7).

In this chapter we shall employ Segerberg's approach and use the notion of strong standard Segerberg frame completeness (see Section 4.3.7). We, however, extend Segerberg's results by proving strong standard Segerberg frame completeness results for the full lattice of conditional logics defined by Tables 5.1 and 5.2. For that purpose frame restrictions for an additional 20 axioms were identified and the respective canonicity proofs were given for all 29 principles in Tables 5.1 and 5.2.[3]

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[1] Segerberg (1989, p. 163) discusses completeness for two further axiom, his axiom #2 and #6. See the introductory part of Chapter 5 for a further discussion.

[2] Segerberg (1989) does not discuss that point. His conjectures regarding completeness proofs of extensions of **CK** (p. 163), however, suggest that he rather employs completeness proofs w.r.t. classes of standard Segerberg frames than w.r.t. classes of (simple) Segerberg frames (cf. Section 4.3.7).

[3] Segerberg (1989) did not give completeness proofs for extensions of system **CK**.

6.1.2 Discussion of Segerberg Frame Completeness and Chellas Frames Completeness Proofs

In this section we discuss strong completeness proofs for classes of standard Segerberg frames, as given by Segerberg (1989), and contrast them with strong completeness proofs for classes of Chellas frames. To prove strong completeness w.r.t classes of standard Segerberg frames, one can employ the canonical model technique (cf. Hughes & Cresswell, 1996/2003, Chapter 6) and construct a canonical model $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$. The set W^c is identified with the set of all *maximally* **L**-consistent sets. Logic **L** is constructed by system **CK** plus any number of axioms from Tables 5.1 and 5.2. The valuation function V^c is, then, defined in such a way that for all atomic propositions $p \in \mathcal{PP}$ and all worlds $w \in W^c$ holds: $V^c(p, w) = 1$ iff $p \in w$. In $\mathcal{M}^c$ the parameter P^c is specified to be the set of semantically representable subsets of W^c (cf. Segerberg, 1989, p. 162). The notion of semantically representable subsets parallels the one for syntactically representable subsets. Instead of truth at a world membership in a world is taken as basic (see Definition 6.8). The valuation function of the canonical model is, then, defined in such a way that the set of semantically representable subsets of W^c is identical with the set of syntactically representable subsets of W^c (by means of the Truth Lemma). According to the definition of $\mathcal{M}^c$, then, R^c is only defined w.r.t. the set of syntactically representable subsets of W^c. As we saw in Section 4.3.1, only accessibility relations relativized to syntactically representable subsets (and, hence, relativized to semantically representable subsets) are relevant for truth of formulas at possible worlds in a model. These accessibility relations are defined then in such a way that they allow the Truth Lemma to generalize from atomic proposition to arbitrary sets of propositions.

Canonical models $\mathcal{M}^c_C = \langle W^c_C, R^c_C{}', V^c_C \rangle$ for Chellas frame completeness agree with the canonical model $\mathcal{M}^c$ w.r.t. the parameters W^c_C and V^c_C (cf. Chellas, 1975, p. 140 and Segerberg, 1989, p. 162). They, however, differ in the accessibility relation. In Chellas models the accessibility relation has to be relativized to all subsets of W^c in an arbitrary way. Hence $R^c_X{}'$ is has to be defined for all subsets X whether they are syntactically representable or not. For syntactically representable subsets of W^c_C, however, canonical models $\mathcal{M}^c_C$ agree with the definition of the accessibility relation of the canonical Segerberg model $\mathcal{M}^c$ (cf. Chellas, 1975, p. 140). Due to that fact the specification of a canonical model $\mathcal{M}^c_C$ is not unique. Hence, we can describe a class of canonical models for Chellas frame completeness by this specification. Chellas (1975, p. 140) calls any such canonical model 'proper'. The problem for a completeness result in terms of Chellas frames are exactly the accessibility relations w.r.t. syntactically non-representable subsets of W^c_C. Note that each canonical Chellas model $\mathcal{M}^c_C$ is defined in such a way that the syntactically non-representable subsets of W^c_C are identical w.r.t. the semantically non-representable subsets. We, however, saw earlier that these types of accessibility relations are in a sense irrelevant as they do not bear on the truth of formulas in worlds in the canonical model (see Section 4.3.1). Despite this fact, they have to be defined. Note, however, that for Chellas frame completeness of extensions of **CK** the canonical model must satisfy the respective frame conditions from Tables 5.3 and 5.4, which correspond to the formulas from Tables 5.1 and 5.2 added to the system **CK**. This holds for all accessibility relations in the canonical model whether syntactically representable or not. On a general basis, however, the addition of an axiom of a system **L** does not affect the accessibility relations relativized to syntactically non-representable sets. Moreover, there seems no general way available to define these types of accessibility relation in such a way that they comply with any frame condition required by the lattice of systems. In the case of standard Segerberg frame completeness no such problem arises, since we can restrict ourselves to the set of accessibility relations, which are relativized to syntactically representable subsets of W^c_C.

In order to prove completeness of extensions **L** of system **CK** w.r.t. classes of standard Segerberg frames, we, however, presuppose only that the sets X, Y, ... referred to in the frame restrictions in Tables 5.3 and 5.4 are elements of the parameter P^c of the canonical model $\mathcal{M}_c = \langle W^c, R^c, P^c, V^c \rangle$. Hence, according to the definition of $\mathcal{M}^c$ all those sets X, Y, ... are syntactically representable in W^c. This move is necessary, as not all non-trivial frame conditions make sure that any set referred to is syntactically representable in W^c, even if the accessibility relations are relativized to only syntactically representable subsets of W^c namely to elements of P^c. Consider, for example, the frame condition for CM, namely $\forall w(\forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_{X\cap Y} w' \Rightarrow wR_X w'))$. For a canonicity result we have to show that this frame restriction holds for a canonical model, if all possible worlds in W^c contain CM. The restriction on R^c to elements of P^c, however, does not make sure that the set Y is syntactically representable and, hence, an element in P^c. This has the consequence that the element $\forall w'(wR_X w' \Rightarrow w' \in Y)$ does correspond to any conditional formula $\alpha \mathrel{\square\!\!\rightarrow} \gamma$. In this case, however, the canonical model does allow axiom CM to come into play and, hence, we have no way to ensure that the condition $\forall w'(wR_{X\cap Y}w' \Rightarrow wR_X w')$ holds, as well. To guarantee that the completeness proof holds for the whole lattice of system, we, hence, have to take (standard) Segerberg frames $\mathcal{F}_S = \langle W, R, P\rangle$ as basic and require that that all subsets of W referred to in Tables 5.3 and 5.4 are elements of P.

Let us, finally, discuss in this section the notion of canonicity:

Definition 6.1. *Let* **L** *be an extension of system* **CK***, such that* **L** $=$ **CK**$+\alpha$ *and let* C_α *be frame conditions, which C-correspond to* α*, respectively. Moreover, let* $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c\rangle$ *be the canonical model of* **L**. *Then* **L** *is canonical (w.r.t. classes of standard Segerberg frames) iff* $\langle W^c, R^c, P^c\rangle$ *satisfies* C_α *for all elements* X *of* P^c.

The canonicity property w.r.t. a logic **CK**$+\alpha$ gives us that the canonical model of **L** is based on a frame, for which the corresponding frame restriction C_α from Tables 5.3 and 5.4 holds. Note that the canonicity property generalizes the following way:

Theorem 6.2. *Let* **L** *be an extension of system* **CK***, such that* **L** $=$ **CK**$+\alpha_1+\alpha_2\ldots+\alpha_n$ *(for* $n \in \mathbb{N}$*) and let* **CK**$+\alpha_1$, **CK**$+\alpha_2$, $\ldots$, **CK**$+\alpha_n$ *be canonical (w.r.t. classes of standard Segerberg frames). Then,* **L** *is canonical (w.r.t. classes of standard Segerberg frames).*

Proof. Let **L**$'$ be an extension of system **CK**, such that **L**$'$ $=$ **CK**$+\alpha_1+\alpha_2\ldots+\alpha_n$ (for $n \in \mathbb{N}$). Furthermore, let (a) **CK**$+\alpha_1$, **CK**$+\alpha_2$, $\ldots$, **CK**$+\alpha_n$ be canonical and let (b) $C_{\alpha_1}, C_{\alpha_2}, \ldots, C_{\alpha_n}$ be frame conditions C-corresponding to $\alpha_1, \alpha_2, \ldots, \alpha_n$, respectively. Since by (a) **CK**$+\alpha_1$, **CK**$+\alpha_2$, $\ldots$, **CK**$+\alpha_n$ are canonical (w.r.t. classes of standard Segerberg frames), the frame of the canonical model $\mathcal{F}^c = \langle W^c, R_c, P_c, R_c\rangle$ for these systems satisfies C_{α_1}, C_{α_2}, $\ldots$, C_{α_n} (restricted to P^c), respectively. Moreover, by (b) C_{α_1}, C_{α_2}, $\ldots$, C_{α_n} C-correspond to α_1, α_2, $\ldots$,

α_n, respectively. Thus, (i) α_1+α_2+ $\ldots$+α_n C-corresponds to $C_{\alpha_1} \wedge C_{\alpha_2} \wedge \ldots \wedge C_{\alpha_n}$. Furthermore, system **CK** does not contain non-monotonic rules by Definition 4.1, and α_1, α_2, $\ldots$, α_n are axioms and, hence, (monotonic) one-step rules (see Section 4.2.3). Thus, the sets of possible worlds of the respective canonical model – which is determined by the set of **L**-consistent formula sets of the logic **L** in question – is never increased. Hence, every frame restriction C_{α_i} (for $1 \leq i \leq n$), which holds in frame of the canonical model $\mathcal{M}^c$ for **CK** + α_i (restricted to P^c), holds also in the frame $\mathcal{F}^c_{\mathbf{L}'}$ of the canonical model $\mathcal{M}^c_{\mathbf{L}'} = \langle W^c, R^c, P^c, V^c \rangle$ for $\mathbf{L}'$ (restricted to P^c). Since $C_{\alpha_1}, C_{\alpha_2}, \ldots, C_{\alpha_n}$ are the case for $\mathcal{F}^c_{\mathbf{L}'}$ (restricted to P^c), (ii) also $C_{\alpha_1} \wedge C_{\alpha_2} \wedge \ldots \wedge C_{\alpha_n}$ holds for $\mathcal{F}_{\mathbf{L}'}$ (restricted to P^c). By (i) and (ii) it follows that $\mathbf{L}'$ is canonical (w.r.t. classes of standard Segerberg frames). □

We shall, hence, only prove the canonicity property for system **CK** plus individual principles from Tables 5.1 and 5.2 (c.f. Blackburn et al., 2001, p. 202; see also below). Theorem 6.2 gives us that the canonicity results extend to the whole lattice of systems described by Tables 5.1 and 5.2 (cf. Section 5.1). That way strong-frame completeness for classes of standard Segerberg frames can be extended to the whole lattice of conditional logic systems.

In the remainder of this chapter we proceed as follows: First, we describe singleton frames (frames that contain a single world) for CS-semantics. Then, we discuss the soundness result for the lattice of systems. Finally, we give a completeness result in terms of standard Segerberg frames for the lattice of systems defined by Tables 5.1 and 5.2.

6.2 Singleton Frames for CS-Semantics

Before we focus on the soundness and completeness results, let us first discuss singleton Chellas frames. By singleton Chellas frames $\mathcal{F}_C = \langle W, R \rangle$ we mean frames, in which the set of possible worlds consists only of a single world w (formally: $W = \{w\}$). As in Kripke semantics, singleton frames in CS-semantics allow us to specify collapse conditions. Moreover, by the use of singleton Chellas frames we can by easy means demonstrate that there exist Chellas models (and Segerberg models), which in addition satisfy the axioms listed in Tables 5.1 and

5.2. Due to the correspondence results, it suffices to check whether one of the singleton frames satisfies the respective frame condition in Tables 5.3 and 5.4.

Let us now describe the singleton Chellas frames. We can distinguish between four (incompatible) singleton frames $\mathcal{F}_C = \langle W, R\rangle$, based on the following specifications of the accessibility relation R:

$$\begin{array}{ll} \mathcal{F}_C^1 = \langle W, R_1\rangle & R_1 = \{\langle w, w, W\rangle, \langle w, w, \varnothing\rangle\} \\ \mathcal{F}_C^2 = \langle W, R_2\rangle & R_2 = \{\langle w, w, W\rangle\} \\ \mathcal{F}_C^3 = \langle W, R_3\rangle & R_3 = \{\langle w, w, \varnothing\rangle\} \\ \mathcal{F}_C^4 = \langle W, R_4\rangle & R_4 = \varnothing \end{array}$$

One can easily show – although we shall not prove this result here – that the following characteristic axioms CA_1, CA_2, CA_3 and CA_4 are valid on the respective singleton frames $\mathcal{F}_C^1, \mathcal{F}_C^2, \mathcal{F}_C^3$ and $\mathcal{F}_C^4$:

$$\begin{array}{ll} \mathrm{CA}_1: & (\alpha \mathrel{\Box\!\!\rightarrow} \beta) \leftrightarrow \beta \\ \mathrm{CA}_2: & (\alpha \mathrel{\Box\!\!\rightarrow} \beta) \leftrightarrow (\alpha \rightarrow \beta) \\ \mathrm{CA}_3: & (\alpha \mathrel{\Box\!\!\rightarrow} \beta) \leftrightarrow (\alpha \vee \beta) \\ \mathrm{CA}_4: & \alpha \mathrel{\Box\!\!\rightarrow} \beta \end{array}$$

$\mathcal{F}_C^1$ and $\mathcal{F}_C^4$ represent generalizations of the systems Triv and Ver from Kripke semantics, respectively. We saw earlier (Section 4.3.2) that the conditional operator, when relativized to an antecedent formula α, can be interpreted as a modal operator $[\alpha]$. On the basis of that terminology the principles CA_1 and CA_4 can then be formulated as $[\alpha] \leftrightarrow \alpha$ and $[\alpha]\beta$, respectively, which correspond directly to the principles Triv and Ver in Kripke semantics, respectively (see Hughes & Cresswell, 1996/2003, p. 65 and p. 67). Moreover, in the singleton Kripke frame for Triv the world w can see itself by the accessibility relation R (see Hughes & Cresswell, 1996/2003, p. 65f), whereas in the singleton Kripke frame for Ver it cannot see any other world via R (see Hughes & Cresswell, 1996/2003, p. 66). The frames $\mathcal{F}_C^1$ and $\mathcal{F}_C^4$ correspond to the singleton frames for Triv and Ver insofar as the world w can see itself by all accessibility relations (namely $R_\varnothing$ and R_W) in the case of $\mathcal{F}_C^1$ and by no accessibility relation in the case of $\mathcal{F}_C^4$. It is important to note that system **CK**+CA_4 is not the inconsistent system: Any conditional formula is valid on the frame $\mathcal{F}_C^1$, including $\top \mathrel{\Box\!\!\rightarrow} \bot$. However, neither $\neg(\top \mathrel{\Box\!\!\rightarrow} \bot)$

(P-Cons, see Table 5.1) nor any any bridge principle – which could give us P-Cons – is valid on $\mathcal{F}_C^4$.

The singleton Chellas frame, which is from a conditional logic perspective most interesting is, however, $\mathcal{F}_C^4$. Except for the principles D and T – which hold in $\mathcal{F}_C^2$ – all principles from Tables 5.1 and 5.2 are valid on $\mathcal{F}_C^2$. Moreover, as we shall see in Section 7.3.4, all principles of the monotonic collapse of conditional logics with bridge principles – which contains all conditional logics investigated in Chapter 7 – are also valid in $\mathcal{F}_C^4$. Finally, due to space and time constraints we shall not discuss here the singleton frame $\mathcal{F}_C^3$.

6.3 Soundness w.r.t. Classes of Chellas Frames

To provide soundness results for a logic **L** – where **L** is the extension of **CK** by axioms from Tables 5.1 and 5.2 – w.r.t. Chellas frames, it suffices to prove the following two points: (a) All axioms and rules of system **CK** (see Definition 4.1) are valid in the class of all Chellas models, and (b) when a frame restriction from Table 5.3 and 5.4 holds for a Chellas frame, then the corresponding principle from Table 5.1 and 5.2 is valid on that frame.

We will not provide a proof for (a), since (a) holds trivially for arbitrary Chellas models. Moreover, (b) is exactly the right-to-left direction of C-correspondence (see Definition 4.21). C-correspondence proofs are given in Chapter 5 for all principles in Tables 5.1 and 5.2. Note, moreover, that (a) holds trivially also for all Segerberg models and that the proofs for the right-to-left direction of C-correspondence in Chapter 5 go through also w.r.t. Segerberg frames $\mathcal{F}_S = \langle W, R, P \rangle$, when the frame conditions are restricted to elements of the parameter P.

In Chapter 5 we, however, prove (b) only for individual principles from Tables 5.1 and 5.2 (i.e. by adding a single principle α to **CK**), but not for combinations of principles $\alpha_1, \ldots, \alpha_n$ ($n \in \mathbb{N}$) from those tables (i.e. by adding $\alpha_1, \ldots, \alpha_n$ to **CK**). Despite this fact, our soundness proofs generalize to the whole lattice of systems defined by Tables 5.1 and 5.2, since the following theorem holds:

Theorem 6.3. *Let* **L** *be an extension of system* **CK**, *such that* **L** = **CK**+α_1+$\alpha_2 \ldots$ +α_n *(for $n \in \mathbb{N}$) and let* **CK**+α_1, **CK**+α_2, …, **CK**+α_n *be sound (w.r.t. classes of* *standard Chellas frames). Then,* $\mathbf{L}$ *is sound (w.r.t. classes of Chellas frames).*

Proof. Let $\mathbf{L}$ be an extension of system $\mathbf{CK}$, such that $\mathbf{L} = \mathbf{CK}{+}\alpha_1{+}\alpha_2\ldots{+}\alpha_n$ (for $n \in \mathbb{N}$) and let $\mathbf{CK}{+}\alpha_1$, $\mathbf{CK}{+}\alpha_2$, $\ldots$, $\mathbf{CK}{+}\alpha_n$ be sound (w.r.t. classes of Chellas frames). Moreover, let C_{α_1}, C_{α_2}, $\ldots$, C_{α_n} be frame conditions, which C-correspond to α_1, α_2, $\ldots$, α_n, respectively. Then, by the right-to-left direction of C-correspondence (Definition 4.21) it follows that all elements of $\mathbf{CK}{+}\alpha_1$, $\mathbf{CK}{+}\alpha_2, \ldots, \mathbf{CK}{+}\alpha_n$ are valid in the classes of Chellas frames, which satisfy C_{α_1}, $C_{\alpha_2}, \ldots, C_{\alpha_n}$, respectively. Since $C_{\alpha_1}, C_{\alpha_2}, \ldots, C_{\alpha_n}$ C-correspond to axioms rather than (non-monotonic) rules, $C_{\alpha_1}, C_{\alpha_2}, \ldots, C_{\alpha_n}$ do not depend on certain formulas being non-valid in a Chellas model or a Chellas frame. Thus, a class of Chellas frames $\mathcal{F}_C$ cannot increase when $\mathcal{F}_C$ is restricted by an additional frame condition C_{α_i} (for $1 \leq i \leq n$). Hence, when $C_{\alpha_1}, C_{\alpha_2}, \ldots, C_{\alpha_n}$ hold conjointly for a class of Chellas frames $\mathbf{F}'_C$, then $\mathbf{F}'_C \subseteq \mathbf{F}_C^{C_{\alpha_i}}$ for each C_{α_i}, where $\mathbf{F}_C^{C_{\alpha_i}}$ is the set of Chellas frames, for which the frame restriction C_{α_i} holds. Hence, by the right-to-left direction of C-correspondence (Definition 4.21) we have that all elements of $\mathbf{CK}{+}\alpha_i$ are valid in $\mathbf{F}_C^{C_{\alpha_i}}$ and, hence, in $\mathbf{F}'_C$. Thus, $\mathbf{L} = \mathbf{CK}{+}\alpha_1{+}\alpha_2\ldots{+}\alpha_n$ is sound (w.r.t. classes of Chellas frames). □

6.4 Standard Segerberg Frame Completeness

In this section we prove strong completeness w.r.t. classes of standard Segerberg frames. We first discuss general lemmata for the completeness result and then focus on canonical models for this type of semantics.

6.4.1 General Principles

We directly state the following Lemmata and omit their proofs, since those are quite standard:

Lemma 6.4. *(Consistency-Lemma)*
$\mathbf{L}$ *is strongly complete w.r.t.* $\mathbf{F}_S^{\mathrm{st}}$ *iff every* $\mathbf{L}$*-consistent formula set* Γ *is satisfiable in* $\mathbf{F}_S^{\mathrm{st}}$.

Lemma 6.5. *For any maximally* **L***-consistent set* Δ *and formulas* α *and* β *the following properties hold:*
a) if $\alpha \in \Delta$ *and* $\vdash_{\mathbf{L}} \alpha \rightarrow \beta$*, then* $\beta \in \Delta$ *(Deductive Closure)*
b) either $\alpha \in \Delta$ *or* $\neg\alpha \in \Delta$ *(Maximality)*
c) if $\alpha \in \Delta$*, then* $\neg\alpha \notin \Delta$ *(Consistency)*

Proof. For 6.5a, 6.5b see proofs of Lemma 2.1f and 2.1b by Hughes and Cresswell (1984, p. 19), respectively. Lemma 6.5c follows by consistency of Δ. □

Lemma 6.6. *(Lindenbaum lemma) Let* Γ *be a* **CK***-consistent formula set. Then there is maximally* **CK***-consistent formula set* Δ*, such that* $\Gamma \subseteq \Delta$.

Proof. See proof of Theorem 2.2 by Hughes and Cresswell (1984, p. 19f). □

The consistency lemma (Lemma 6.4) shows that in order to prove strong completeness of **CK** w.r.t. the classes of standard Segerberg frames $\mathbf{F}_S^{\mathrm{st}}$, it suffices to demonstrate the following: Every **CK**-consistent formula set Γ is satisfiable in $\mathbf{F}_S^{\mathrm{st}}$. To do this we construct a canonical model $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ along the lines of Segerberg (1989). This canonical model specifies W^c as the set of all maximally **CK**-consistent formula sets. The Lindenbaum-lemma (Lemma 6.6) gives us that every **CK**-consistent formula set is a subset of a maximally **CK**-consistent set. Since the possible worlds of the canonical model are by definition all (and only) maximally **CK**-consistent sets, any **CK**-consistent set is, hence, a subset of a possible world in $\mathcal{M}^c$. The next building block of the completeness proof is the Truth Lemma (Lemma 6.11). This lemma establishes that a formula α is true at a world w iff it is a member of w. Hence, any **CK**-consistent formula set Γ is a subset of a possible word w in $\mathcal{M}^c$. Since any element of a possible world w is true at w, it follows that all formulas in Γ are true at w. Hence, there is a model, namely the canonical model $\mathcal{M}^c$, in which the formula set Γ is true. Thus, Γ is $\mathbf{F}_S^{\mathrm{st}}$-satisfiable. Finally, for basic system **CK** we would have to show that the frame of the canonical model is a frame, on which all axioms of **CK** are valid. This is, however, trivial, since by definition all models of the CS-semantics are models of the frame-class CS.

6.4.2 Canonical Models

In this section we specify the canonical model for extensions of system **CK** as defined by Tables 5.1 and 5.2. This definition and the following proofs draw on Segerberg (1989, p. 162f) and the standard technique for providing completeness results as described in Hughes and Cresswell (1996/2003, Chapter 6). We use the following definition: $|\alpha|^{\mathcal{M}^c} =_{def} \{w \in W^c \mid \alpha \in w\}$ (Def$_{|.|}$). In the following we leave out reference to the canonical model $\mathcal{M}^c$ and simplify expressions, such as $|\alpha|^{\mathcal{M}^c}$ to $|\alpha|$, since those always refer to respective canonical model. Moreover, in the following all formulas $\alpha, \beta, \ldots$ refer implicitly to formulas of the language $\mathcal{L}_{\text{KL}}$.

Definition 6.7. *A Segerberg model $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ is the canonical model for an extension* **L** *of* **CK** *iff*

a) W^c *is the class of all maximally* **L***-consistent formula sets.* *(Def$_{W^c}$)*

b) $P^c = \{X \subseteq W^c \mid \exists \alpha$ *such that* $X = |\alpha|\}$ *(Def$_{P^c}$)*

c) $\forall X \in P^c$ *s.t.* $\exists \alpha\ (X = |\alpha|)\ \forall w, w' \in W^c$: wR_Xw' *iff* $\forall \beta(\alpha \mathbin{\Box\!\!\rightarrow} \beta \in w \Rightarrow \beta \in w')$ *(Def$_{R^c}$)*

d) $\forall p \in \mathcal{PP}, w \in W^c$: $V^c(p, w) = 1$ *iff* $p \in w$ *(Def$_{V^c}$)*

Note that point c of Definition 6.7 gives us that R_X is defined for all elements $X \in P^c$ since by Definition 6.7.b every such element X is syntactically representable. Moreover, in order to construct canonical models for Chellas frame completeness a restriction of the definition of R_X to syntactically representable subsets X of W^c – as in the case of standard Segerberg frames – is not possible. Chellas (1975, p. 139), thus, uses the notion of a *class* of canonical (standard) models rather than the notion of a (standard) canonical model. Chellas characterizes his class of canonical (standard) models by specifying the accessibility relation R_X only for syntactically representable subsets X of W^c (p. 139). Canonical (standard) models in Chellas' account differ, then, in the specification of R_X for syntactically non-representable subsets of W^c.

Let us now define the notion of semantically representable subsets of W^c.

Definition 6.8. *Let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of an extension* **L** *of* **CK** *and let $X \subseteq W^c$ be the case. Then, X is semantically representable in $\mathcal{M}^c$ (by formulas of $\mathcal{L}_{\text{KL}}$) iff there exists a formula α (of $\mathcal{L}_{\text{KL}}$), such that $X = |\alpha|^{\mathcal{M}^c}$.*

Observe that P^c is the set of semantically representable subsets of W^c as described in Definition 6.8. We will now focus on the proof for Lemma 6.11. Note that this proof draws on Lemma 6.10, which in turn refers to 6.9.

Lemma 6.9. *Let* **L** *be an extension of* **CK**. *Then, if* Γ *is a* **L**-*consistent set of formulas such that* $\neg(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \in \Gamma$, *then* $\{\gamma \,|\, \alpha \mathrel{\Box\!\!\rightarrow} \gamma \in \Gamma\} \cup \{\neg\beta\}$ *is* **L**-*consistent.*

Proof. By contraposition. Let $\{\gamma \,|\, \alpha \mathrel{\Box\!\!\rightarrow} \gamma \in \Gamma\} \cup \{\neg\beta\}$ be **L**-inconsistent. Then, there exists a $n \in \mathbb{N}_0$, such that $\{\gamma_1, \ldots, \gamma_n, \neg\beta\}$ is **L**-inconsistent for $\gamma_1, \ldots, \gamma_n \in \{\gamma \,|\, \alpha \mathrel{\Box\!\!\rightarrow} \gamma \in \Gamma\}$. (In the case of $n = 0$ the set $\{\neg\beta\}$ is **L**-inconsistent.) It follows that $\vdash_{\mathbf{L}} \neg(\gamma_1 \wedge \ldots \wedge \gamma_n \wedge \neg\beta)$. This implies that $\vdash_{\mathbf{L}} \gamma_1 \wedge \ldots \wedge \gamma_n \rightarrow \beta$ holds. Logic **L** is by assumption an extension of **CK**. Hence, RW, LT and AND are theorems of **L**. By Lemma 4.11 the rule RCK is, then, admissible. When applied to the former inference step, it follows that $\vdash_{\mathbf{L}} (\alpha \mathrel{\Box\!\!\rightarrow} \gamma_1) \wedge \ldots \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \gamma_n) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \beta)$. This implies in turn that $\vdash_{\mathbf{L}} \neg((\alpha \mathrel{\Box\!\!\rightarrow} \gamma_1) \wedge \ldots \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \gamma_n) \wedge \neg(\alpha \mathrel{\Box\!\!\rightarrow} \beta))$. Hence, the set $\{\alpha \mathrel{\Box\!\!\rightarrow} \gamma_1, \ldots, \alpha \mathrel{\Box\!\!\rightarrow} \gamma_n\} \cup \{\neg(\alpha \mathrel{\Box\!\!\rightarrow} \beta)\}$ is **L**-inconsistent. Since it holds that $\gamma_1, \ldots, \gamma_n \in \{\gamma \,|\, \alpha \mathrel{\Box\!\!\rightarrow} \gamma \in \Gamma\}$, it follows that $\{\alpha \mathrel{\Box\!\!\rightarrow} \gamma_1, \ldots, \alpha \mathrel{\Box\!\!\rightarrow} \gamma_n\} \subseteq \Gamma$. This implies that $\Gamma \cup \{\neg(\alpha \mathrel{\Box\!\!\rightarrow} \beta)\}$ is **L**-inconsistent. $\square$

Lemma 6.10. *For any canonical model* $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ *of an extension* **L** *of* **CK** *and all* α, β *and all* $w \in W^c : \alpha \mathrel{\Box\!\!\rightarrow} \beta \in w$ *iff* $\forall w' \in W^c(wR^c_{|\alpha|}w' \Rightarrow \beta \in w')$.

Proof. "$\Rightarrow$": By contraposition. Suppose that for arbitrary α, β and an arbitrary $w \in M^c$ it is not the case that $\forall w' \in W^c(wR^c_{|\alpha|}w' \Rightarrow \beta \in w')$. Then, there exists a $w' \in W^c$ such that $wR^c_{|\alpha|}w'$,but $\beta \notin w'$. By Def$_{R^c}$ it follows that $\alpha \mathrel{\Box\!\!\rightarrow} \beta \notin w$.

"$\Leftarrow$": By contraposition. Suppose that for arbitrary α, β and arbitrary $w \in W^c$ it is the case that $\alpha \mathrel{\Box\!\!\rightarrow} \beta \notin w$, which implies by Lemma 6.5.b that $\neg(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \in w$. Lemma 6.9 and Def$_{W^c}$ give us that there exists a world $w' \in W^c$ such that $\{\gamma \,|\, \alpha \mathrel{\Box\!\!\rightarrow} \gamma \in w\} \cup \{\neg\beta\} \subseteq w'$. Since $\{\gamma \,|\, \alpha \mathrel{\Box\!\!\rightarrow} \gamma \in w\} \subseteq w'$, it holds that $\forall\gamma(\alpha \mathrel{\Box\!\!\rightarrow} \gamma \in w \Rightarrow \gamma \in w')$. Hence, it follows by Def$_{R^c}$ that $wR^c_{|\alpha|}w'$. However, since $\{\neg\beta\} \subseteq w'$, we have $\neg\beta \in w'$. By Lemma 6.5.c it is, hence, the case that $\beta \notin w'$. Thus, there exists a world $w' \in W^c$, such that $wR^c_{|\alpha|}w'$, but $\beta \notin w'$. $\square$

We are now ready to establish the truth lemma for the canonical model of **CK**:

Lemma 6.11. *(Truth Lemma) For the canonical model* $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ *of an extensions* **L** *of system* **CK** *and for all* α *and all* $w \in W^c$ *holds:* $\alpha \in w$ *iff* $\vDash^{\mathcal{M}^c}_w \alpha$.

Proof. By induction on the construction of α. The proof of the non-modal cases are standard and can be found in textbooks for modal logic (e.g. Hughes & Cresswell, 1984, p. 23f). The only case that differs from the standard procedure is the modal one. Hence, we restrict our proof to this case and have to prove for arbitrary α, β and $w \in W^c$ that $\alpha \mathrel{\Box\!\!\rightarrow} \beta \in w$ iff $\vDash_w^{\mathcal{M}^c} \alpha \mathrel{\Box\!\!\rightarrow} \beta$, based on the following induction hypotheses: (a) $\forall w \in W^c(\alpha \in w$ iff $\vDash_w^{\mathcal{M}^c} \alpha)$, and (b) $\forall w \in W^c(\beta \in w$ iff $\vDash_w^{\mathcal{M}^c} \beta)$.

"$\Rightarrow$": Let $\alpha \mathrel{\Box\!\!\rightarrow} \beta \in w$ be the case. Then, by Lemma 6.10 it follows that for all $w' \in W^c$ holds: $wR^c_{|\alpha|}w' \Rightarrow \beta \in w'$. By the induction hypothesis (a), $\mathrm{Def}_{|.|}$ and $\mathrm{Def}_{\|.\|}$ we get $|\alpha| = \|\alpha\|$. Moreover, due to (b) $\forall w \in W^c(\beta \in w$ iff $\vDash_w^{\mathcal{M}^c} \beta)$ holds. Hence, it is the case that $\forall w' \in W^c(wR^c_{\|\alpha\|}w' \Rightarrow \vDash_{w'}^{\mathcal{M}^c} \beta)$. By $V_{\Box\!\rightarrow}$ it follows that $\vDash_w^{\mathcal{M}^c} \alpha \mathrel{\Box\!\!\rightarrow} \beta$.

"$\Leftarrow$": By contraposition. Let $\alpha \mathrel{\Box\!\!\rightarrow} \beta \notin w$ be the case. By Lemma 6.10 it follows that there exists a world $w' \in W^c$, such that $wR^c_{|\alpha|}w'$ and $\beta \notin w'$. By the induction hypothesis (a), $\mathrm{Def}_{|.|}$ and $\mathrm{Def}_{\|.\|}$ we get $|\alpha| = \|\alpha\|$. Moreover, by (b) $|\beta| = \|\beta\|$. Thus, it follows that there exists a world $w' \in W^c$, such that $wR^c_{\|\alpha\|}w'$ and $\nvDash_{w'}^{\mathcal{M}^c} \beta$. By $V_{\Box\!\rightarrow}$ this implies that $\nvDash_w^{\mathcal{M}^c} \alpha \mathrel{\Box\!\!\rightarrow} \beta$. □

6.4.3 Canonicity Proofs for Individual Principles

In this section we give canonicity proofs (see Definition 6.1) for the principles in Table 5.1 and 5.2 w.r.t. the corresponding frame conditions in Table 5.3 and 5.4. The canonicity results establish the following for extensions **L** of system **CK**, as defined by Table 5.1 and 5.2: If a given axiom from Table 5.1 and 5.2 is added to the logic **L**, then the frame of the canonical model of logic **L** satisfies the respective frame condition (see Section 6.1.2).

In the following proofs we often draw on $\mathrm{Def}_{|.|}$, in particular for the following two equalities concerning arbitrary formulas α and β: (a) $-|\alpha| = |\neg\alpha|$ and (b) $|\alpha| \cap |\beta| = |\alpha \wedge \beta|$. Note, however, that (a) and (b) do not only draw on $\mathrm{Def}_{|.|}$, but also on Lemma 6.5. Equation (a) needs Lemma 6.5.b and Lemma 6.5.c, while Equation (b) relies on Lemma 6.5.a, Lemma 6.5.b and Lemma 6.5.c. For the sake of perspicuity we will not indicate these details in the individual canonicity proofs. Note that by assumption all subsets of possible worlds described in Tables 5.3 and 5.4 are restricted to elements of P^c.

System P

Axiom Schema Refl. Let $\mathbf{L}_{\text{Refl}}$ be an extension of **CK**, such that Refl holds and let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\text{Refl}}$. Assume that Refl is not canonical. Then, there exist $X \in P^c$ and $w, w' \in W^c$, such that $wR^c_X w'$, but $w' \notin X$. By Def$_{P^c}$ there is a formula α, such that $X = |\alpha|$. Thus, $wR^c_{|\alpha|}w'$, but $w' \notin |\alpha|$. Due to Lemma 6.10 and Def$_{|.|}$ it follows that $\alpha \mathrel{\Box\!\!\rightarrow} \alpha \notin w$. This contradicts by axiom Refl, Lemmata 6.5.a and 6.5.c the definition of $\mathcal{M}^c$. □

Axiom Schema CM. Let $\mathbf{L}_{\text{CM}}$ be an extension of **CK**, such that CM holds. Moreover, let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\text{CM}}$. Assume that CM is not canonical. Then, there exist $X, Y \in P^c$ and $w, w' \in W^c$, such that $\forall w''(wR^c_X w'' \Rightarrow w'' \in Y)$, $wR^c_{X \cap Y}w'$, but $\neg wR^c_X w'$. By Def$_{P^c}$ there are formulas α and β, such that $X = |\alpha|$ and $Y = |\beta|$. Hence, $\forall w''(wR^c_{|\alpha|}w'' \Rightarrow w'' \in |\beta|)$, $wR^c_{|\alpha| \cap |\beta|}w'$, but $\neg wR^c_{|\alpha|}w'$ hold. By Def$_{|.|}$, we get that $wR^c_{|\alpha \wedge \beta|}w'$. Since $\forall w''(wR^c_{|\alpha|}w'' \Rightarrow w'' \in |\beta|)$, Lemma 6.10 and Def$_{|.|}$ imply that $\alpha \mathrel{\Box\!\!\rightarrow} \beta \in w$. Due to Def$_{R^c}$ and $\neg wR^c_{|\alpha|}w'$ there is a formula γ, such that $\alpha \mathrel{\Box\!\!\rightarrow} \gamma \in w$, but $\gamma \notin w'$. Since $wR^c_{|\alpha \wedge \beta|}w'$, it follows by Lemma 6.10 that $\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma \notin w$. As $\alpha \mathrel{\Box\!\!\rightarrow} \beta \in w$ and $\alpha \mathrel{\Box\!\!\rightarrow} \gamma \in w$, this contradicts by axiom CM, Lemma 6.5.a the definition of $\mathcal{M}^c$. □

Axiom Schema CC. Let $\mathbf{L}_{\text{CC}}$ be an extension of **CK**, such that CC holds. Furthermore, let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\text{CC}}$. Assume that CC is not canonical. Then, there exist $X, Y \in P^c$ and $w, w' \in W^c$, such that $\forall w''(wR^c_X w'' \Rightarrow w'' \in Y)$, $wR^c_X w'$, but $\neg wR^c_{X \cap Y}w'$. By Def$_{P^c}$ there are formulas α and β, such that $X = |\alpha|$ and $Y = |\beta|$. Hence, $\forall w''(wR^c_{|\alpha|}w'' \Rightarrow w'' \in |\beta|)$, $wR^c_{|\alpha|}w'$, but $\neg wR^c_{|\alpha| \cap |\beta|}w'$. Due to Def$_{|.|}$, it follows that $\neg wR^c_{|\alpha \wedge \beta|}w'$. Since $\forall w''(wR^c_{|\alpha|}w'' \Rightarrow w'' \in |\beta|)$, Lemma 6.10 and and Def$_{|.|}$ give us that $\alpha \mathrel{\Box\!\!\rightarrow} \beta \in w$. Due to Def$_{R^c}$ and $\neg wR^c_{|\alpha \wedge \beta|}w'$ there is a formula γ, such that $\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma \in w$, but $\gamma \notin w'$. Since $wR^c_{|\alpha|}w'$, it follows by Lemma 6.10 that $\alpha \mathrel{\Box\!\!\rightarrow} \gamma \notin w$. As $\alpha \mathrel{\Box\!\!\rightarrow} \beta \in w$ and $\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma \in w$ hold, this contradicts by axiom CC and Lemma 6.5.a the definition of $\mathcal{M}^c$. □

Axiom Schema Loop. Let $\mathbf{L}_{\text{Loop}}$ be an extension of **CK**, such that Loop holds. In addition, let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\text{Loop}}$. Assume that Loop is not canonical and that, hence, for some $k \in \mathbb{N}$ with $k \geq 2$, the following holds: There exist $X_0, \ldots, X_k \in P^c$ and $w, w' \in W^c$, such that $\forall w''(wR^c_{X_0}w'' \Rightarrow$

$w'' \in X_1), \ldots, \forall w''(wR^c_{X_{k-1}}w'' \Rightarrow w'' \in X_k), \forall w''(wR^c_{X_k}w'' \Rightarrow w'' \in X_0), wR^c_{X_0}w'$, but $w' \notin X_k$. By Def$_{P^c}$ there are formulas $\alpha_0, \ldots, \alpha_k$, such that $X_0 = |\alpha_0|, \ldots, X_k = |\alpha_k|$. It follows that $\forall w''(wR^c_{|\alpha_0|}w'' \Rightarrow w'' \in |\alpha_1|), \ldots, \forall w''(wR^c_{|\alpha_{k-1}|}w'' \Rightarrow w'' \in |\alpha_k|)$, $\forall w''(wR^c_{|\alpha_k|}w'' \Rightarrow w'' \in |\alpha_0|)$, $wR^c_{|\alpha_0|}w'$ and $w' \notin |\alpha_k|$. The latter fact implies by Lemma 6.10 and Def$_{|.|}$ that $\alpha_0 \mathrel{\Box\!\!\rightarrow} \alpha_1, \ldots, \alpha_{k-1} \mathrel{\Box\!\!\rightarrow} \alpha_k$, $\alpha_k \mathrel{\Box\!\!\rightarrow} \alpha_0 \in w$ and $\alpha_0 \mathrel{\Box\!\!\rightarrow} \alpha_k \notin w$. This, however, contradicts by axiom Loop and Lemma 6.5.a the definition of $\mathcal{M}^c$. □

Axiom Schema Or. Let $\mathbf{L}_{\mathrm{Or}}$ be an extension of **CK**, such that Or holds and let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\mathrm{Or}}$. Assume that Or is not canonical. Then, there exist $X, Y \in P^c$ and $w, w' \in W^c$, such that $wR^c_{X \cup Y}w'$ and both $\neg wR^c_X w'$ and $\neg wR^c_Y w'$. By Def$_{P^c}$ there are formulas α and β, such that $X = |\alpha|$ and $Y = |\beta|$. Hence, $wR^c_{|\alpha| \cup |\beta|}w'$, $\neg wR^c_{|\alpha|}w'$ and $\neg wR^c_{|\beta|}w'$. Due to Def$_{|.|}$, it follows that $wR^c_{|\alpha \vee \beta|}w'$. Since $\neg wR^c_{|\alpha|}w'$ and $\neg wR^c_{|\beta|}w'$, we get by Def$_{R^c}$ that there are formulas γ and δ, such that $\alpha \mathrel{\Box\!\!\rightarrow} \gamma, \beta \mathrel{\Box\!\!\rightarrow} \gamma \in w$, but $\gamma, \delta \notin w'$. Hence, $\gamma \vee \delta \notin w'$. In addition, Rule RW of **CK** plus Lemma 6.5.a give us that $\alpha \mathrel{\Box\!\!\rightarrow} \gamma \vee \delta \in w$ and $\beta \mathrel{\Box\!\!\rightarrow} \gamma \vee \delta \in w$. Moreover, as $\gamma \vee \delta \notin w'$ and $wR^c_{|\alpha \vee \beta|}w'$, we get by Lemma 6.10 that $\alpha \vee \beta \mathrel{\Box\!\!\rightarrow} \gamma \vee \delta \notin w$. Since $\alpha \mathrel{\Box\!\!\rightarrow} \gamma \vee \delta \in w$ and $\beta \mathrel{\Box\!\!\rightarrow} \gamma \vee \delta \in w$, this contradicts by axiom Or and Lemma 6.5.a the definition of $\mathcal{M}^c$. □

Axiom Schema S. Let $\mathbf{L}_{\mathrm{S}}$ be an extension of **CK**, such that S holds and let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\mathrm{S}}$. Assume that S is not canonical. Then, there exist $X, Y \in P^c$ and $w, w' \in W^c$, such that $wR^c_X w'$, $w' \in Y$ and $\neg wR^c_{X \cap Y}w'$. By Def$_{P^c}$ there are formulas α and β, such that $X = |\alpha|$ and $Y = |\beta|$. Hence, $wR^c_{|\alpha|}w'$, $w' \in |\beta|$ and $\neg wR^c_{|\alpha| \cap |\beta|}w'$. Due to Def$_{|.|}$, it follows that $\neg wR^c_{|\alpha \wedge \beta|}w'$. Def$_{R^c}$ gives us that there is a formula γ, such that $\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma \in w$, but $\gamma \notin w'$. Since $w' \in |\beta|$, we get by Def$_{|.|}$ that $w' \notin |\beta \rightarrow \gamma|$. As $wR^c_{|\alpha|}w'$, Lemma 6.10 gives us that $\alpha \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma) \notin w$. Since $\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma \in w$, this contradicts by axiom S and Lemma 6.5.a the definition of $\mathcal{M}^c$. □

Extensions of System P

Axiom Schema MOD. Let $\mathbf{L}_{\mathrm{MOD}}$ be an extension of **CK**, such that MOD holds and let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\mathrm{MOD}}$. Assume that

MOD is not canonical. Then, there exist $X, Y \in P^c$ and $w, w' \in W^c$, such that $\forall w''(wR^c_{-X}w'' \Rightarrow w'' \in X)$, $wR^c_Y w'$ and $w' \notin X$. By Def$_{P^c}$ there are formulas α and β, such that $X = |\alpha|$ and $Y = |\beta|$. Hence, $\forall w''(wR^c_{-|\alpha|}w'' \Rightarrow w'' \in |\alpha|)$, $wR^c_{|\beta|}w'$ and $w' \notin |\alpha|$. By Def$_{|.|}$ it follows that $\forall w''(wR^c_{|\neg\alpha|}w'' \Rightarrow w'' \in |\alpha|)$. Hence, by Lemma 6.10 and Def$_{|.|}$ we get $\neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha \in w$ and, since $wR^c_{|\beta|}w'$ and $w' \notin |\alpha|$, Lemma 6.10 gives us that $\beta \mathrel{\Box\!\!\rightarrow} \alpha \notin w$. This contradicts by axiom MOD and Lemma 6.5.a the definition of $\mathcal{M}^c$. □

Axiom Schema RM. Let $\mathbf{L}_{\mathrm{RM}}$ be an extension of **CK**, such that RM holds and let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c\rangle$ be the canonical model of $\mathbf{L}_{\mathrm{RM}}$. Assume that RM is not canonical. Then, there exist $X, Y \in P^c$ and $w, w', w'' \in W^c$, such that on the one hand that $wR^c_X w''$ and $w'' \in Y$ and on the other hand that $wR^c_{X\cap Y}w'$ and $\neg wR^c_X w'$. By Def$_{P^c}$ there are formulas α and β, such that $X = |\alpha|$ and $Y = |\beta|$. Hence, $wR^c_{|\alpha|}w''$, $w'' \in |\beta|$, $wR^c_{|\alpha|\cap|\beta|}w'$, but $\neg wR^c_{|\alpha|}w'$. By Def$_{|.|}$, we get $wR^c_{|\alpha\wedge\beta|}w'$. Def$_{\Diamond\!\rightarrow}$ gives us that $\alpha \mathrel{\Diamond\!\!\rightarrow} \beta$ is defined as $\neg\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta$ (see Section 4.2.1). Hence, since $wR^c_{|\alpha|}w''$ and $w'' \in |\beta|$, Def$_{\Diamond\!\rightarrow}$, Lemma 6.10, Def$_{|.|}$ and imply that $\alpha \mathrel{\Diamond\!\!\rightarrow} \beta \in w$. Due to Def$_{R^c}$ and $\neg wR^c_{|\alpha|}w'$ there is a formula γ, such that $\alpha \mathrel{\Box\!\!\rightarrow} \gamma \in w$, but $\gamma \notin w'$. Since $wR^c_{|\alpha\wedge\beta|}w'$ and $\gamma \notin w'$, it follows by Lemma 6.10 that $\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma \notin w$. As $\alpha \mathrel{\Diamond\!\!\rightarrow} \beta \in w$ and $\alpha \mathrel{\Box\!\!\rightarrow} \gamma \in w$, this contradicts by axiom RM and Lemma 6.5.a the definition of $\mathcal{M}^c$. □

Axiom Schema CEM. Let $\mathbf{L}_{\mathrm{CEM}}$ be an extension of **CK**, such that CEM holds. Moreover, let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c\rangle$ be the canonical model of $\mathbf{L}_{\mathrm{CEM}}$. Assume that CEM is not canonical. Then, there exist $X \in P^c$ and $w, w' \in W^c$, such that $wR^c_X w'$, $wR^c_X w''$, but $w'' \neq w'$. By Def$_{P^c}$ there is a formula α, such that $X = |\alpha|$. Thus, $wR^c_{|\alpha|}w'$ and $wR^c_{|\alpha|}w''$. Since $w'' \neq w'$, Def$_{W^c}$ implies that there is a formula β, such that $\beta \in w'$ and $\beta \notin w''$. Hence, as $wR^c_{|\alpha|}w''$, Lemma 6.10 implies that $\alpha \mathrel{\Box\!\!\rightarrow} \beta \notin w$. Since $\beta \in w'$ it follows by Lemma 6.5.c that $\neg\beta \notin w'$. Hence, by Lemma 6.10 we have $\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta \notin w$. Since $\alpha \mathrel{\Box\!\!\rightarrow} \beta \notin w$, this contradicts by axiom CEM and Lemma 6.5.a the definition of $\mathcal{M}^c$.

□

Axioms from Probability Logic

Axiom Schema P-Cons. Let $\mathbf{L}_{\text{P-Cons}}$ be an extension of **CK**, such that P-Cons holds and let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\text{P-Cons}}$. Assume that P-Cons is not canonical. Then, there exists a world $w \in W^c$, such that $\neg\exists w'(wR^c_{W^c}w')$. Since it holds by $\text{Def}_{|\cdot|}$ and Def_{W^c} that $W^c = |\top|$, we get $\neg\exists w'(wR^c_{|\top|}w')$. Hence, it follows trivially by Lemma 6.10 that $\top \boxright \bot \in w$. This contradicts by axiom P-Cons and Lemma 6.5.c the definition of $\mathcal{M}^c$. $\square$

Axiom Schema WOR. Let $\mathbf{L}_{\text{WOR}}$ be an extension of **CK**, such that WOR holds. Moreover, let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\text{WOR}}$. Assume that WOR is not canonical. Then, there exist $X, Y \in P^c$ and $w, w' \in W^c$, such that $wR^c_X w'$, but $\neg wR^c_{X\cap Y}w'$ and $\neg wR^c_{X\cap -Y}w'$. By Def_{P^c} there are formulas α and β, such that $X = |\alpha|$ and $Y = |\beta|$. Thus, $wR^c_{|\alpha|}w'$, but $\neg wR^c_{|\alpha|\cap|\beta|}w'$ and $\neg wR^c_{|\alpha|\cap -|\beta|}w'$. By $\text{Def}_{|\cdot|}$, we get $\neg wR^c_{|\alpha\wedge\beta|}w'$ and $\neg wR^c_{|\alpha\wedge\neg\beta|}w'$. Due to Def_{R^c} it follows that there are formulas γ and δ, such that $\alpha \wedge \beta \boxright \gamma$, $\alpha \wedge \neg\beta \boxright \gamma \in w$, but $\gamma, \delta \notin w'$. Hence, $\gamma \vee \delta \notin w'$. Then, Rule RW plus Lemma 6.5.a give us that $\alpha \wedge \beta \boxright \gamma \vee \delta \in w$ and $\alpha \wedge \neg\beta \boxright \gamma \vee \delta \in w$. Moreover, as $\gamma \vee \delta \notin w'$ and $wR^c_{|\alpha|}w'$, we get by Lemma 6.10 that $\alpha \boxright \gamma \vee \delta \notin w$. Since $\alpha \wedge \beta \boxright \gamma \vee \delta \in w$ and $\alpha \wedge \neg\beta \boxright \gamma \vee \delta \in w$, this contradicts by axiom WOR and Lemma 6.5.a the definition of $\mathcal{M}^c$. $\square$

Monotonic Systems

Axiom Schema Cut. Let $\mathbf{L}_{\text{Cut}}$ be an extension of **CK**, such that Cut holds. In addition let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\text{Cut}}$. Assume that Cut is not canonical. Then, there exist X, Y, $Z \in P^c$ and $w, w' \in W^c$, such that $\forall w''(wR^c_Z w'' \Rightarrow w'' \in Y)$, $wR^c_{X\cap Z}w'$, but $\neg wR^c_{X\cap Y}w'$. By Def_{P^c} there are formulas α, β and γ, such that $X = |\alpha|$, $Y = |\beta|$ and $Z = |\gamma|$. Thus, $\forall w''(wR^c_{|\gamma|}w'' \Rightarrow w'' \in |\beta|)$, $wR^c_{|\alpha|\cap|\gamma|}w'$ and $\neg wR^c_{|\alpha|\cap|\beta|}w'$. $\text{Def}_{|\cdot|}$ gives us $wR^c_{|\alpha\wedge\gamma|}w'$ and $\neg wR^c_{|\alpha\wedge\beta|}w'$. Due to Def_{R^c} there is a formula δ, such that $\alpha \wedge \beta \boxright \delta \in w$, but $\delta \notin w'$. As $wR^c_{|\alpha\wedge\gamma|}w'$, it follows by Lemma 6.10 that $\alpha \wedge \gamma \boxright \delta \notin w$. Moreover, since $\forall w''(wR^c_{|\gamma|}w'' \Rightarrow w'' \in |\beta|)$, Lemma 6.10 implies by $\text{Def}_{|\cdot|}$ that $\gamma \boxright \beta \in w$. As $\alpha \wedge \beta \boxright \delta \in w$, but $\alpha \wedge \gamma \boxright \delta \notin w'$, this contradicts by axiom Cut and Lemma 6.5.a the definition of $\mathcal{M}^c$. $\square$

Axiom Schema Mon. Let $\mathbf{L}_{\text{Mon}}$ be an extension of **CK**, such that Mon holds and let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c\rangle$ be the canonical model of $\mathbf{L}_{\text{Mon}}$. Assume that Mon is not canonical. Then, there exist $X, Y \in P^c$ and $w, w' \in W^c$, such that $wR^c_{X\cap Y}w'$ but $\neg wR^c_X w'$. By Def$_{P^c}$ there are formulas α and β, such that $X = |\alpha|$ and $Y = |\beta|$. Hence, $wR^c_{|\alpha|\cap|\beta|}w'$, but $\neg wR^c_{|\alpha|}w'$. By Def$_{|.|}$, we get $wR^c_{|\alpha\wedge\beta|}w'$. Since $\neg wR^c_{|\alpha|}w'$, due to Def$_{R^c}$, there is a formula γ, such that $\alpha \mathrel{\Box\!\!\rightarrow} \gamma \in w$, but $\gamma \notin w'$. Hence, due to Lemma 6.10 follows that $\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma \notin w$. As $\alpha \mathrel{\Box\!\!\rightarrow} \gamma \in w$ holds, this contradicts by axiom Mon and Lemma 6.5.a the definition of $\mathcal{M}^c$. □

Axiom Schema Trans. Let $\mathbf{L}_{\text{Trans}}$ be an extension of **CK**, such that Trans holds and let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c\rangle$ be the canonical model of $\mathbf{L}_{\text{Trans}}$. Assume that Trans is not canonical. Then, there exist $X, Y \in P^c$ and $w, w' \in W^c$, such that $\forall w''(wR^c_X w'' \Rightarrow w'' \in Y)$, $wR^c_X w'$, but $\neg wR^c_Y w'$. By Def$_{P^c}$ there are formulas α and β, such that $X = |\alpha|$ and $Y = |\beta|$. Thus, $\forall w''(wR^c_{|\alpha|}w'' \Rightarrow w'' \in |\beta|)$, $wR^c_{|\alpha|}w'$, but $\neg wR^c_{|\beta|}w'$. Due to Def$_{R^c}$ there exists a formula γ, such that $\beta \mathrel{\Box\!\!\rightarrow} \gamma \in w$, but $\gamma \notin w'$. Hence, since $wR^c_{|\alpha|}w'$, we get by Lemma 6.10 that $\alpha \mathrel{\Box\!\!\rightarrow} \gamma \notin w$. Moreover, since $\forall w''(wR^c_{|\alpha|}w'' \Rightarrow w'' \in |\beta|)$, it follows by Lemma 6.10 and Def$_{|.|}$ that $\alpha \mathrel{\Box\!\!\rightarrow} \beta \in w$. As $\beta \mathrel{\Box\!\!\rightarrow} \gamma \in w$, but $\alpha \mathrel{\Box\!\!\rightarrow} \gamma \notin w$, this contradicts by axiom Trans and Lemma 6.5.a the definition of $\mathcal{M}^c$. □

Axiom Schema CP. Let $\mathbf{L}_{\text{CP}}$ be an extension of **CK**, such that CP holds and let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c\rangle$ be the canonical model of $\mathbf{L}_{\text{CP}}$. Assume that CP is not canonical. Then, there exist $X, Y \in P^c$ and $w, w' \in W^c$, such that $\forall w''(wR^c_X w'' \Rightarrow w'' \in Y)$, $wR^c_{-Y}w'$, but $w' \notin -X$. By Def$_{P^c}$ there are formulas α and β, such that $X = |\alpha|$ and $Y = |\beta|$. Hence, $\forall w''(wR^c_{|\alpha|}w'' \Rightarrow w'' \in |\beta|)$, $wR^c_{-|\beta|}w'$, but $w' \notin -|\alpha|$. Due to Def$_{|.|}$ it follows that $wR^c_{|\neg\beta|}w'$ and $w' \notin |\neg\alpha|$. Hence, as $wR^c_{|\neg\beta|}w'$ and $w' \notin |\neg\alpha|$, Lemma 6.10 and Def$_{|.|}$ imply that $\neg\beta \mathrel{\Box\!\!\rightarrow} \neg\alpha \notin w$. Since it is the case that $\forall w''(wR^c_{|\alpha|}w'' \Rightarrow w'' \in |\beta|)$, Lemma 6.10 gives us that $\alpha \mathrel{\Box\!\!\rightarrow} \beta \in w$. Since $\neg\beta \mathrel{\Box\!\!\rightarrow} \neg\alpha \notin w$, his contradicts by axiom CP and Lemma 6.5.a the definition of $\mathcal{M}^c$. □

Bridge Principles

Axiom Schema MP. Let $\mathbf{L}_{\text{MP}}$ be an extension of **CK**, such that MP holds. Moreover, let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c\rangle$ be the canonical model of $\mathbf{L}_{\text{MP}}$. Assume that MP is not canonical. Then, there exist $X \in P^c$ and $w \in W^c$, such that $w \in X$, but $\neg wR^c_X w$.

By Def_{P^c} there is a formula α, such that $X = |\alpha|$. Hence, we have $w \in |\alpha|$ and $\neg wR^c_{|\alpha|}w$. Due to Def_{R^c}, it follows that there is a formula β, such that $\alpha \mathbin{\Box\!\!\rightarrow} \beta \in w$, but $\beta \notin w$. Hence, since $w \in |\alpha|$ and $\beta \notin w$, we get by $\mathrm{Def}_{|.|}$ that $\alpha \rightarrow \beta \notin w$. As $\alpha \mathbin{\Box\!\!\rightarrow} \beta \in w$ holds, this contradicts by axiom MP and Lemma 6.5.a the definition of $\mathcal{M}^c$. □

Axiom Schema CS. Let $\mathbf{L}_{\mathrm{CS}}$ be an extension of **CK**, such that CS holds and let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\mathrm{CS}}$. Assume that CS is not canonical. Then, there exist $X \in P^c$ and $w, w' \in W^c$, such that $w \in X$, $wR^c_X w'$, but $w' \neq w$. By Def_{P^c} there is a formula α, such that $X = |\alpha|$. Hence, we get $w \in |\alpha|$ and $wR^c_{|\alpha|}w'$. Since $w' \neq w$, by Def_{W^c} there is a formula β such that $\beta \in w$, but $\beta \notin w'$. Since $w \in |\alpha|$, this implies by $\mathrm{Def}_{|.|}$ that $\alpha \wedge \beta \in w$. However, since $wR^c_{|\alpha|}w'$ and $\beta \notin w'$, Lemma 6.10 gives us that $\alpha \mathbin{\Box\!\!\rightarrow} \beta \notin w$. Since $\alpha \wedge \beta \in w$, this contradicts by axiom CS and Lemma 6.5.a the definition of $\mathcal{M}^c$. □

Axiom Schema TR. Let $\mathbf{L}_{\mathrm{TR}}$ be an extension of **CK**, such that TR holds. Moreover, let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\mathrm{TR}}$. Assume that TR is not canonical. Then, there exist $X \in P^c$ and $w \in W^c$, such that $\forall w'(wR^c_{-X}w' \Rightarrow w' \in X)$, but $w \notin X$. By Def_{P^c} there is a formula α, such that $X = |\alpha|$. Hence, $\forall w'(wR^c_{-|\alpha|}w' \Rightarrow w' \in |\alpha|)$ and $w \notin |\alpha|$. Due to $\mathrm{Def}_{|.|}$, we get $\forall w'(wR^c_{|\neg\alpha|}w' \Rightarrow w' \in |\alpha|)$. Lemma 6.10 and $\mathrm{Def}_{|.|}$ give us, then, that $\neg\alpha \mathbin{\Box\!\!\rightarrow} \alpha \in w$. However, since $w \notin |\alpha|$, by $\mathrm{Def}_{|.|}$ it is the case that $\alpha \notin w$. As $\neg\alpha \mathbin{\Box\!\!\rightarrow} \alpha \in w$, this contradicts by axiom TR and Lemma 6.5.a the definition of $\mathcal{M}^c$. □

Axiom Schema Det. Let $\mathbf{L}_{\mathrm{Det}}$ be an extension of **CK**, for which Det holds. Furthermore, let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\mathrm{Det}}$. Assume that Det is not canonical. Then, there exists a possible world $w \in W^c$, such that $\neg wR^c_{W^c}w$. Since due to $\mathrm{Def}_{|.|}$ and Def_{W^c} it is the case that $W^c = |\top|$, we get $\neg wR^c_{|\top|}w$. By Def_{R^c}, it follows that there is a formula α, such $\top \mathbin{\Box\!\!\rightarrow} \alpha \in w$, but $\alpha \notin w$. This contradicts by axiom Det and Lemma 6.5.a the definition of $\mathcal{M}^c$. □

Axiom Schema Cond. Let $\mathbf{L}_{\mathrm{Con}}$ be an extension of **CK**, for which Con holds. In addition, let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\mathrm{Con}}$. Assume that Con is not canonical. Then, there exist $w, w' \in W^c$, such that $wR^c_W w'$, but $w' \neq w$. Since due to $\mathrm{Def}_{|.|}$ it is the case that $W^c = |\top|$, we get $wR^c_{|\top|}w'$. Moreover, as $w' \neq w$, by the Def_{W^c} it follows that there is a formula α, such that $\alpha \in w$, but $\alpha \notin w'$. However, since $wR^c_{|\top|}w'$, we get by Lemma 6.10 that $\top \mathrel{\Box\!\!\rightarrow} \alpha \notin w$. Since $\alpha \in w$, this contradicts by axiom Cond and Lemma 6.5.a the definition of $\mathcal{M}^c$. $\square$

Axiom Schema VEQ. Let $\mathbf{L}_{\mathrm{VEQ}}$ be an extension of **CK**, for which VEQ holds and let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\mathrm{VEQ}}$. Assume that VEQ is not canonical. Then, there exist $X \in P^c$ and $w, w' \in W^c$, such that $wR^c_X w'$, but $w' \neq w$. By Def_{P^c} there is a formula α, such that $X = |\alpha|$. Hence, $wR^c_{|\alpha|}w'$. Moreover, as $w' \neq w$, by Def_{W^c} it follows that there is a formula β, such that $\beta \in w$, but $\beta \notin w'$. However, since $wR^c_{|\alpha|}w'$, we get by Lemma 6.10 that $\alpha \mathrel{\Box\!\!\rightarrow} \beta \notin w$. As $\beta \in w$, this contradicts by axiom VEQ and Lemma 6.5.a the definition of $\mathcal{M}^c$. $\square$

Axiom Schema EFQ. Let $\mathbf{L}_{\mathrm{EFQ}}$ be an extension of **CK**, for which EFQ holds. Moreover, let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\mathrm{EFQ}}$. Assume that EFQ is not canonical. Then, there exist $X \in P^c$ and $w, w' \in W^c$, such that $w \in -X$, but $wR^c_X w'$. By Def_{P^c} there is a formula β, such that $X = |\beta|$. Thus, we have $w \in -|\beta|$ and $wR^c_{|\beta|}w'$. By $\mathrm{Def}_{|.|}$, it follows that $w \in |\neg\beta|$. By Def_{W^c} it is the case that that $\bot \notin w'$. Since $wR^c_{|\beta|}w'$, we get by Lemma 6.10 that $\beta \mathrel{\Box\!\!\rightarrow} \bot \notin w$. As $\neg\beta \in w$, this contradicts by axiom EFQ and Lemma 6.5.a the definition of $\mathcal{M}^c$. $\square$

Traditional Extensions

Axiom Schema D. Let $\mathbf{L}_{\mathrm{D}}$ be an extension of **CK**, for which D holds. Let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\mathrm{D}}$. Assume that D is not canonical. Then, there exist $X \in P^c$ and $w \in W^c$, such that $\neg\exists w'(wR^c_X w')$. By Def_{P^c} there is a formula α, such that $X = |\alpha|$. Hence, it is the case that $\neg\exists w'(wR^c_{|\alpha|}w')$. Hence, by Lemma 6.10 it follows trivially that $\alpha \mathrel{\Box\!\!\rightarrow} \beta \in w$. In addition, Lemma 6.10 implies that $\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta \in w$. Thus, by Lemma 6.5.a and $\mathrm{Def}_{\Diamond\!\rightarrow}$ it follows that $\alpha \mathrel{\Diamond\!\!\rightarrow} \beta \notin w$. Since $\alpha \mathrel{\Box\!\!\rightarrow} \beta \in w$, this contradicts by axiom D and Lemma 6.5.a the definition of $\mathcal{M}^c$. $\square$

Axiom Schema T. Let $\mathbf{L}_{\mathrm{T}}$ be an extension of **CK**, for which T holds. Let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\mathrm{T}}$. Assume that T is not canonical. Then, there exist $X \in P^c$ and $w \in W^c$, such that $\neg wR^c_X w$. By Def_{P^c} there is a formula α, such that $X = |\alpha|$. Hence, it follows that $\neg wR^c_{|\alpha|}w$. By Lemma 6.10 this implies that there is a formula β, such that $\alpha \;\Box\!\!\rightarrow \beta \in w$, but $\beta \notin w$. This contradicts by axiom T and Lemma 6.5.a the definition of $\mathcal{M}^c$. $\square$

Axiom Schema B. Let $\mathbf{L}_{\mathrm{B}}$ be an extension of **CK**, for which B holds. Let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c\rangle$ be the canonical model of $\mathbf{L}_{\mathrm{B}}$. Assume that B is not canonical. Then, there exist $X \in P^c$ and $w, w' \in W^c$, such that $wR^c_X w'$, but $\neg w'R^c_X w$. By Def_{P^c} there is a formula α, such that $X = |\alpha|$. Hence, we have $wR^c_{|\alpha|}w'$ and $\neg w'R^c_{|\alpha|}w$. Def_{R^c} gives us that there is a formula β, such that $\alpha \;\Box\!\!\rightarrow \beta \in w'$, but $\beta \notin w$. Hence, by Lemma 6.5.b it follows that $\neg\beta \in w$. Moreover, since $\alpha \;\Box\!\!\rightarrow \beta \in w'$, RW implies by Lemma 6.5.a that $\alpha \;\Box\!\!\rightarrow \neg\neg\beta \in w'$. Lemma 6.5.c and $\mathrm{Def}_{\Diamond\!\rightarrow}$ give us that $\alpha \;\Diamond\!\!\rightarrow \beta \notin w'$. Since $wR^c_{|\alpha|}w'$, we get by Lemma 6.10 that $\alpha \;\Box\!\!\rightarrow (\alpha \;\Diamond\!\!\rightarrow \gamma) \notin w$. As it is the case that $\gamma \in w$, this contradicts by axiom B and Lemma 6.5.a the definition of $\mathcal{M}^c$. $\square$

Axiom Schema 4. Let $\mathbf{L}_4$ be an extension of **CK**, such that 4 holds. Let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c\rangle$ be the canonical model of $\mathbf{L}_4$. Assume that 4 is not canonical. Then, there exist $X \in P^c$ and $w, w', w'' \in W^c$, such that $wR^c_X w'$, $w'R^c_X w''$, but $\neg wR^c_X w''$. By Def_{P^c} there is a formula α such that $X = |\alpha|$. Thus, $wR^c_{|\alpha|}w'$, $w'R^c_{|\alpha|}w''$, but $\neg wR^c_{|\alpha|}w''$. Due to the definition of R^c it follows that there is a formula β, such that $\alpha \;\Box\!\!\rightarrow \beta \in w$, but $\beta \notin w''$. Since $w'R^c_{|\alpha|}w''$, we get by Lemma 6.10 that $\alpha \;\Box\!\!\rightarrow \beta \notin w'$. This implies, since $wR^c_{|\alpha|}w'$, by Lemma 6.10 that $\alpha \;\Box\!\!\rightarrow (\alpha \;\Box\!\!\rightarrow \beta) \notin w$. Since $\alpha \;\Box\!\!\rightarrow \beta \in w$, this contradicts by axiom 4 and Lemma 6.5.a the definition of $\mathcal{M}^c$. $\square$

Axiom Schema 5. Let $\mathbf{L}_5$ be an extension of **CK**, such that 5 holds. Let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c\rangle$ be the canonical model of $\mathbf{L}_5$. Assume that 5 is not canonical. Then, there exist $X \in P^c$ and $w, w', w'' \in W^c$, such that $wR^c_X w'$, $wR^c_X w''$, but $\neg w'R^c_X w''$. By Def_{P^c} there is a formula α such that $X = |\alpha|$. Thus, $wR^c_{|\alpha|}w'$, $wR^c_{|\alpha|}w''$, but $\neg w'R^c_{|\alpha|}w''$. This implies that there is a formula β, such that $\alpha \;\Box\!\!\rightarrow \beta \in w'$, but $\beta \notin w''$. Let γ be a formula, such that $\gamma = \neg\beta$. It follows by Lemma 6.5.b that $\gamma \in w''$. Moreover, since $\alpha \;\Box\!\!\rightarrow \beta \in w'$, we have by RW and Lemma 6.5.a that $\alpha \;\Box\!\!\rightarrow \neg\gamma \in w'$. Hence, Lemma 6.5.c and $\mathrm{Def}_{\Diamond\!\rightarrow}$ imply that $\alpha \;\Diamond\!\!\rightarrow \gamma \notin w'$. As $wR^c_{|\alpha|}w'$, it follows by Lemma 6.10 that $\alpha \;\Box\!\!\rightarrow (\alpha \;\Diamond\!\!\rightarrow \gamma) \notin w$. However, since $wR^c_{|\alpha|}w''$ and $\gamma \in w''$, Lemma 6.10, $\mathrm{Def}_{|.|}$ and $\mathrm{Def}_{\Diamond\!\rightarrow}$ imply that $\alpha \;\Diamond\!\!\rightarrow \gamma \in w$. As

$\alpha \mathrel{\Box\!\!\rightarrow} (\alpha \mathrel{\Diamond\!\!\rightarrow} \gamma) \notin w$, this contradicts by axiom 5 and Lemma 6.5.a the definition of $\mathcal{M}^c$. $\square$

Iteration Principles

Axiom Schema Ex. Let $\mathbf{L}_{\mathrm{Ex}}$ be an extension of **CK**, such that Ex holds. Let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\mathrm{Ex}}$. Assume that Ex is not canonical. Then, there exist $X, Y \in P^c$ and $w, w', w'' \in W^c$, such that $wR^c_X w'$, $w'R^c_Y w''$, but $\neg wR^c_{X \cap Y} w''$. By Def$_{P^c}$ there are formulas α, β such that $X = |\alpha|$ and $Y = |\beta|$. Hence, $wR^c_{|\alpha|} w'$, $w'R^c_{|\beta|} w''$, but $\neg wR^c_{|\alpha| \cap |\beta|} w''$. Def$_{|.|}$, then, implies that $\neg wR^c_{|\alpha \wedge \beta|} w''$. By Def$_{R^c}$ it follows that there is a formula γ, such that $\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma \in w$, but $\gamma \notin w''$. As $w'R^c_{|\beta|} w''$, Lemma 6.10 gives us that $\beta \mathrel{\Box\!\!\rightarrow} \gamma \notin w'$. Moreover, since $wR^c_{|\alpha|} w'$, Lemma 6.10 also implies that $\alpha \mathrel{\Box\!\!\rightarrow} (\beta \mathrel{\Box\!\!\rightarrow} \gamma) \notin w$. However, as $\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma \in w$, this contradicts by axiom Ex and Lemma 6.5.a the definition of $\mathcal{M}^c$. $\square$

Axiom Schema Im. Let $\mathbf{L}_{\mathrm{Im}}$ be an extension of **CK**, such that Im holds. Let $\mathcal{M}^c = \langle W^c, R^c, P^c, V^c \rangle$ be the canonical model of $\mathbf{L}_{\mathrm{Im}}$. Assume that Im is not canonical. Then, there exist $X, Y \in P^c$ and $w, w' \in W^c$, such that $wR^c_{X \cap Y} w'$, but $\neg \exists w'' \, (wR^c_X w'' \wedge w''R^c_Y w')$. By Def$_{P^c}$ there are formulas α, β such that $X = |\alpha|$ and $Y = |\beta|$. Thus, $wR^c_{|\alpha| \cap |\beta|} w'$, but $\neg \exists w''(wR^c_{|\alpha|} w'' \wedge w''R^c_{|\beta|} w')$. Moreover, due to Def$_{|.|}$ we have $wR^c_{|\alpha \wedge \beta|} w'$. By Def$_{W^c}$ there is one formula γ, such that $\gamma \in w$, but $\gamma \notin w'$ for any world $w' \in W^c$ and $w \neq w'$. Hence, $\{w'\} = \|\gamma\|$. By the Truth Lemma follows that $\{w'\} = \|\delta\|$ and by Def$_{V^c}$ we have $\{w'\} \in P^c$. Moreover, by Def$_{V^c}$, Def$_{P_{Mod}}$ and Def$_{P_{-}}$, follows that $\{w'' \,|\, w''R^c_{|\beta|} w'\} \in P^c$ and by the Truth Lemma holds that $\{w'' \,|\, w''R^c_{|\beta\|} w'\} \in P^c$. Let γ be $\neg\delta$. Then, by Def$_{|.|}$ we get $\{w'\} = -|\gamma|$ and it follows by Lemma 6.10 for any world w'', such that w'' that $\{w'' \,|\, w''R^c_{|\beta|} w'\} \in P^c$ that $\beta \mathrel{\Box\!\!\rightarrow} \gamma \notin w''$. Moreover, since w' is the only world in the canonical model, which is not in $|\gamma|$, it follows that $\{w'' \,|\, w''R^c_{|\beta|} w'\} = -|\beta \mathrel{\Box\!\!\rightarrow} \gamma|$. Hence, $-\{w'' \,|\, w''R^c_{|\beta|} w'\} = |\beta \mathrel{\Box\!\!\rightarrow} \gamma|$. By assumption holds that $\neg \exists w''(wR^c_{|\alpha|} w'' \wedge w''R^c_{|\beta|} w')$. Thus, for any $wR^c_{|\alpha|} w''$ it is the case that $w'' \in -\{w'' \,|\, w''R^c_{|\beta|} w'\}$, and, and, hence $w'' \in |\beta \mathrel{\Box\!\!\rightarrow} \gamma|$. Since this holds for any $wR^c_{|\alpha|} w''$, it follows by Lemma 6.10 that $\alpha \mathrel{\Box\!\!\rightarrow} (\beta \mathrel{\Box\!\!\rightarrow} \gamma) \in w$. However, since $w' \notin |\gamma|$ and $wR^c_{|\alpha \wedge \beta|} w'$, this implies by Lemma 6.10 that $\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma \notin w$. Since $\alpha \mathrel{\Box\!\!\rightarrow} (\beta \mathrel{\Box\!\!\rightarrow} \gamma) \in w$ is the case, this contradicts by axiom Im and Lemma 6.5.a the definition of $\mathcal{M}^c$.

$\square$

Chapter 7

CS Semantics for Indicative and Counterfactual Conditionals

In this chapter we align the formal results described in Chapters 4–6 with the philosophical discussion in the foregoing Chapters 1–3. For that purpose we discuss the following points:

(i) We give both an objective and a subjective interpretation of basic CS-semantics (Section 7.1). The objective interpretation represents a generalization of the Kripke semantics, while the subjective interpretation is a modified Ramsey-test interpretation in the sense of Section 3.2 (see in particular Section 3.2.6). Note that our Ramsey-test interpretation does not rely on a consistency criterion, as described in Chapter 3.

(ii) We discuss individual conditional logic principles from Tables 5.1 and 5.2. Note that these principles do not hold in the basic CS-semantics and are, hence, in need of additional justification, which goes beyond the modified Ramsey-test interpretation of the basic CS-semantics. We argue in particular that CM ("Cautious Monotonicity"), CC ("Cautious Cut"), Or and RM ("Rational Monotonicity") and to some extent MP ("Modus Ponens") and CS ("Conditional Sufficiency") are plausible candidates for additional conditional logic principles. Our discussion is, however, not limited to these principles, but includes also the principles Refl ("Reflexivity"), D (a conditional logic version of axiom D from modal logic), T (a conditional logic version of axiom T from modal logic), S, MOD ("Modality"), CEM

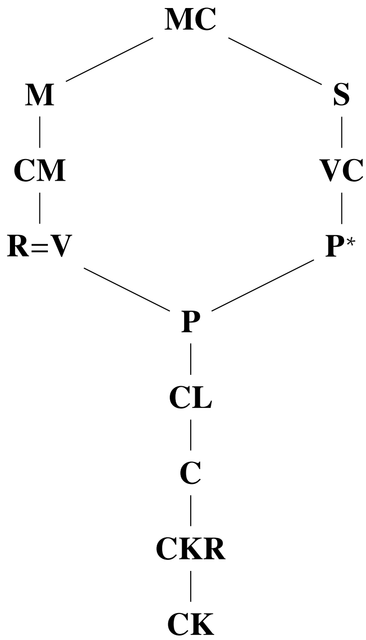

Figure 7.1
Systems Investigated in this Chapter from the Lattice of Systems described by Tables 5.1 and 5.2

("Conditional Excluded Middle"), P-Cons ("Probabilistic Consistency"), WOR ("Weak Or"), Cut, Mon ("Monotonicity"), Trans ("Transitivity"), CP ("Contraposition"), Det ("Detachment"), Cond ("Conditionalization"), VEQ ("Verum Ex Quodlibet") and EFQ ("Ex Falso Quodlibet"), and, to some extent, the principles Im ("Importation") and Ex ("Exportation"). (See Section 5.1 for a more detailed description of theses principles.) We have to limit our discussion to these principles in order not to further increase the complexity of the present discussion too much.

(iii) We show that a range of systems for indicative and counterfactual conditionals can be described by means of our lattice of conditional logics defined by the axioms in Tables 5.1 and 5.2 on the one hand and frame conditions in Tables 5.3 and 5.4 on the other hand. See Figure 7.1 for an overview over the indicative and counterfactual conditional logic systems investigated in this chapter. system **CK** (cf. Section 4.2.6) is the weakest system in this context (see Section 7.1).

System **CKR** follows (see Section 7.1), which results from adding the principle Refl ($\alpha \mathrel{\Box\!\!\rightarrow} \alpha$, see Table 5.1) to system **CK** .

We, then, focus on full language versions of systems **C** (Section 7.2.1), system **CL** (Section 7.2.2), system **P** (Section 7.2.3) and system **R** (Section 7.2.4) of Kraus et al. (1990) and Lehmann and Magidor (1992). By a full language version of a system we mean a system, which is formulated w.r.t. the full language $\mathcal{L}_{\mathrm{KL}}$ rather than a more restricted versions, such as $\mathcal{L}_{\mathrm{KL}^-}$ and $\mathcal{L}_{\mathrm{rKL}^*}$ (see Sections 3.4.3 and 4.2.1). Moreover, we account for the counterfactual systems **V** (Section 7.2.5) of D. Lewis (1973/2001). In addition, we describe full language versions of the two monotonic systems **CM** and **M** (Section 7.2.6) defined in Kraus et al. (1990).

Then, we focus on conditional logic systems with bridge principles: We, first, characterize the counterfactual systems **VC** (Section 7.3.2) of D. Lewis (1973/2001), a full language version of Adams' (1965, 1966, 1977) original system **P*** (Section 7.3.1), and Stalnaker's (1968, Stalnaker & Thomason, 1970; Section 7.3.3) system **S**. We also describe the full material collapse system **MC**, in which all conditional formulas of the form $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ are logically equivalent to material implications of the form $\alpha \rightarrow \beta$.

(iv) Finally, we give a direct characterization of systems **M** (the monotonic collapse without bridge principles, Section 7.2.6) and system **MC** (the monotonic collapse with bridge principles, Section 7.3.4) by means of frame restrictions (see Theorems 7.44 and 7.75, respectively). Both types of semantic characterizations are of interest for the investigation of conditional logics, insofar as they show, which type of semantics correspond to the limiting cases and, hence, should be avoided in conditional logic approaches.

Points (i) and (iv) are original contributions. Points (ii) and (iii) are partially original contributions, insofar as we augment the discussion of conditional logic principles and provide object language proofs for a range of principles, which are described in the literature, but for which only proof sketches are given.

The present discussion of conditional logics has some advantages over existing reviews of conditional logic such as Nejdl (1992), Kraus et al. (1990) and Lehmann and Magidor (1992). Unlike Nejdl (1992), we use the same semantic framework to discuss both weak systems of conditional logic (e.g. system **CK**) and strong systems (e.g. system **S**). Note that Nejdl (1992) describes only system **CK** by means of Chellas models [frames] and goes, then, on to discuss stronger systems such as system **P** in terms of the relational Lewis frames of Burgess (1981; see Definitions 3.8–3.10). Moreover, in contrast to Kraus et al. (1990) and Lehmann and Magidor (1992) we employ the full language $\mathcal{L}_{\mathrm{KL}}$, rather than a more restricted language, corresponding to $\mathcal{L}_{\mathrm{rrKL}}$ (cf. Section 3.4.3). Our framework, hence, makes it possible to describe bridge principles (e.g. MP) and iterated and nested principles (e.g. Im). Note that in the language of Kraus et al. (1990) and Lehmann and Magidor (1992) bridge principles and iterated and nested principles are not expressible. In addition, our semantic framework allows to characterize much weaker systems than the semantic framework of Kraus et al. (1990) and Lehmann and Magidor (1992).

7.1 The Basic CS Systems (Systems CK and CKR)

Let us, first, clarify what we mean by basic CS-semantics. We refer by that term to Chellas models and Segerberg models [and the respective notions of Chellas frames and Segerberg frames] as described in Sections 4.3.1 and 4.3.3 plus optionally the frame restriction for Refl from Table 5.3. In proof-theoretic terms these semantics can be described by system **CK** and **CKR** (system **CK**+Refl), respectively.

As basic units of our semantics serve Chellas models. A Chellas model $\mathcal{M}_C$ has the form $\mathcal{M}_C = \langle W, R, V\rangle$, where W is a non-empty set and R an accessibility relation between elements $w, w' \in W$, which is additionally relativized to subsets of W. This accessibility relation is defined for all pairs of elements w, $w' \in W$ and subsets of possible worlds X of W. Since any formula α (of $\mathcal{L}_{\mathrm{KL}}$) is true in in some subset X of W for any Chellas model $\mathcal{M}_C$, model $\mathcal{M}_C$ gives us for any formula α a natural relativization of R to formulas, namely by the subset of W, at which α is true (short: $\|\alpha\|$). A conditional $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ is, then, determined to be true at world w in model $\mathcal{M}_C$ if and only if β is true at all worlds accessible from w by the accessibility relation $R_{\|\alpha\|}$.

In Chapters 4 and 6 we investigated besides Chellas models also Segerberg models. Segerberg models differ from Chellas models insofar as an additional parameter P is employed. This parameter is, however, used mainly for technical reasons. Moreover, both Segerberg models and Chellas models ensure that for all formulas α the accessibility relation $R_{\|\alpha\|}$ is defined. In addition, they agree on the truth conditions for modal formulas. Since these are the elements, on which the following modified Ramsey-test interpretation draws, our version of the Ramsey-test is applicable to both Chellas models and Segerberg models.

7.1.1 Objective and Subjective Interpretations of CS-Semantics for Indicative and Counterfactual Conditionals

We saw in Section 3.3.3 that indicative conditionals are interpreted by taking one's current beliefs into account, while this element is in general missing when we evaluate counterfactual conditionals. Hence, in order to provide an intuitively plausible interpretation of indicative and counterfactual conditionals for CS-semantics, we have to be able to give both (a) an objective interpretation of CS-semantics and (b) a subjective, agent-relative interpretation of this semantics.

We argue here that CS-semantics can provide both (a) and (b). This is due to the fact that CS-semantics is based on general structural conditions, which allow to interpret this semantics (i) in an objective way in line with traditional alethic Kripke semantics and (ii) subjectively, by means of a modified Ramsey-test. Approach (i) gives us an objective interpretation of this semantics in terms of counterfactuals (requirement a), while approach (ii) allows for a subjective interpretation for indicative conditionals (requirement b). We will, first, describe (i) and, then, focus on (ii).

The Objective Alethic Interpretation

Let us now discuss how to interpret CS-semantics in an objective way. We discussed in Section 3.3.3 that a standard way to describe alethic necessity is by means of Kripke models (cf. D. Lewis, 1973/2001, p. 5f; Schurz, 1997a, p. 17f). In Kripke semantics an accessibility relation R^* between possible worlds is employed (Hughes & Cresswell, 1996/2003, p. 37f). This accessibility relation is, then, used to provide truth conditions of necessity formulas $\Box\alpha$ ("It is necessary that α") relative to possible worlds: A formula $\Box\alpha$ is true at a world w iff α is true at all worlds w' accessible from w by R^*. In alethic modal logic the accessibility relation R^* is interpreted in an objective way insofar as w stands in the relationship R^* to w' if and only if w' is an alternative way the world might be w.r.t. the possible world w. We can add more flesh to this characterization by specifying, in which sense R^* gives us alternative ways the world might be by, for example, requiring that these alternatives are determined purely on logical grounds or by the physical laws of our world (see Section 3.3.3). The notion of alethic necessity is so broad that it covers all these more specific interpretations, as long as these amount to "necessary truth" rather than other types of necessity (see Section 3.3.3).

The accessibility relation in CS-semantics is, however, not a two-place relation between possible worlds, but a three-place relation between pairs of possible worlds and sets of possible worlds. Thus, the accessibility relation R_X in CS-semantics represents a generalization of the accessibility relation R^* in Kripke semantics, insofar as R is relativized to some proposition X.

We shall now give an objective interpretation of the accessibility relation R_X. Let us suppose that α is a formula, such that in some Chellas model $\mathcal{M}_C = \langle W, R, V\rangle$ it holds that $X = \|\alpha\|^{\mathcal{M}_C}$. Then, all worlds $w' \in W$, which are accessible from a world $w \in W$ by R_X represent worlds that are compatible with the proposition $\|\alpha\|^{\mathcal{M}_C}$ from the perspective of world w. Our interpretation of compatibility is here to be understood in a purely alethic way (cf. Section 3.3.3): We can, for example, analyze $R^{\mathcal{M}_C}_{\|\alpha\|}$ in such a way that those worlds and only those worlds are accessible by $R^{\mathcal{M}_C}_{\|\alpha\|}$, which are compatible with the proposition $\|\alpha\|^{\mathcal{M}_C}$ on the basis of logical principles alone or on the basis of the physical laws at the world w.

In order to allow for an adequate interpretation of CS-semantics in terms of alethic modality, we, however, should not be able to arrive from a world w by $R_{\|\alpha\|^{\mathcal{M}_C}}$ at a world w', such that $\neg\alpha$ is true at w'. Since this might be the case for some Chellas models, we have to require that any Chellas model $\mathcal{M}_C = \langle W, R, V\rangle$, which is to be interpreted in an alethic way has to satisfy the following property:

C_{Refl} $\quad \forall w, w'(wR_Xw' \Rightarrow w' \in X)$

Due to the respective correspondence proof in Chapter 5, the following formula is valid in all Chellas models, for which frame condition C_{Refl} holds:

Refl $\quad \alpha \,\Box\!\!\rightarrow \alpha$

Let us now, define the result proof-theoretic system, which results from adding Refl to system **CK**:

Definition 7.1. *Logic* **CKR** *is the smallest logic containing* **CK**+*Refl.*

System **CK** is described in Definition 4.1 as LLE+RW+AND+LT. Note, however, that under certain conditions we would not like Refl to be valid in a model, for example if we interpret $R_{\|\alpha\|}$ in a deontic way. In this case $R_{\|\alpha\|}$ holds between w and $w' \in W$ if and only if w' represent a state of affairs, which is compatible with a certain pre-specified normative code, given that α holds. We will, however, not investigate deontic interpretations of CS-semantics any further in this thesis.

Let us, finally, see, in which sense the accessibility relation R_X represents a generalization of the accessibility relation R^* in Kripke semantics. We do so by reconstructing an absolute accessibility relation R^* on the basis of Chellas models $\mathcal{M}_C = \langle W, R, V \rangle$. One can achieve this by requiring (a) that wR^*w' holds (the possible world w stands in relation R^* to the possible world w') iff $\exists X(wR_Xw')$ (cf. Segerberg, 1989, p. 161). It follows from this definition that R_X for some $X \subseteq W$ is always a special case of R^*, since for any wR_Xw' it follows that wR^*w'. Interestingly, the alternative characterization of wR^*w' as $\forall X(wR_Xw')$ renders R^* void, in case we also endorse C_{Refl}: In order that wR^*w' holds it must be the case that wR_Xw' for all $X \subseteq W$, even for $X = \varnothing$. By C_{Refl} it, however, follows that for no worlds $w, w' \in W$ it is the case that $wR_\varnothing w'$.

The Modified Ramsey-Test Interpretation

We will now give a modified Ramsey-test interpretation of CS-semantics. This interpretation differs in many respects from the objective interpretation described in the previous section. It is important to note here that we only give a Ramsey-test interpretation for basic CS-semantics, namely the class of all Chellas [Segerberg] models and the class of all Chellas [Segerberg] models, for which C_{Refl} (see previous section) holds. We argue that additional restrictions on Chellas [Segerberg] models are not justified by a Ramsey-test interpretation, but must derive their intuitive import from other sources. We will, hence, discuss the intuitions of a range of important principles separately in Chapters 7.2 and 7.3. We will, however, not draw – as Stalnaker (1968) does – on the notion of similarity of possible worlds (see Section 3.2.4) but provide a pure Ramsey-test interpretation of CS-semantics.

We shall now repeat the properties our Ramsey-test interpretations has to satisfy. For the sake of perspicuity we quote again the central features of the Ramsey-test as described by Stalnaker (1968; cf. also Section 3.2):

> "add the antecedent (hypothetically) to your stock of knowledge (or beliefs), and then consider whether or not the consequent is true. Your belief about the conditional should be the same as your hypothetical belief, under this condition, about the consequent" (Stalnaker, 1968, p. 101)

We saw in Sections 3.2.6 that a stock of belief corresponds to a set of possible worlds rather than single possible worlds, as Stalnaker (1968) argues. In a Chellas model $\mathcal{M}_C = \langle W, R, V \rangle$ the accessibility relation R determines for each possible world $w \in W$ and any subset X of W a set of possible worlds, which are accessible from w by the proposition X. Suppose that there is a formula α, such that $X = \|\alpha\|^{\mathcal{M}_C}$. Then, $R_{\|\alpha\|}$ determines for a given world $w \in W$ the set of worlds, which result from revising our stock of beliefs by the proposition $\|\alpha\|$.

Since Ramsey-test interpretations are in general to be understood as a subjective terms (see Section 3.2.6), it seems sensible to presuppose that the accessibility relation R in Chellas models $\mathcal{M}_C = \langle W, R, V \rangle$ has to be read in a subjective, person-relative way in line with epistemic or doxastic logics rather than objective alethic modal logic. In epistemic and doxastic logics a modal necessity operator $\Box\alpha$ is used. A necessity formula $\Box\alpha$ is, then, interpreted as "Agent A knows that α" (Fagin, Halpern, Moses, & Vardi, 2003, p. 19; Meyer & van der Hoek, 1995, p. 7) and "Agent A believes that α" (Meyer & van der Hoek, 1995, p. 69, see also Schurz, 1997a), respectively. In standard systems of epistemic [doxastic] logic a Kripke semantics (see previous section) is used in such a way that a formula $\Box\alpha$ is true at a world w iff formula α is true at all epistemic [doxastic] alternatives to world w (Meyer & van der Hoek, 1995, p. 8). Two epistemic [doxastic] alternatives w' and w' for a possible world w (for an agent A) are interpreted in such a way that A cannot, on the basis of her knowledge [beliefs], determine, which of both worlds w' and w'' holds (Fagin et al., 2003, p. 20).

Note that there is one essential difference between epistemic and doxastic logics on the one hand and CS-semantics on the other hand. In epistemic and doxastic logics, the modal operator $\Box\alpha$ is introduced in order to be able to distinguish in a single framework state of affairs, which an agent knows [believes] and states of affairs that obtain factually (cf. Levesque, 1990, p. 269). In this type of semantics formulas, which do not contain a modal operator are understood in an objective way, whereas formulas of the form $\Box\alpha$ are interpreted subjectively.

In CS-semantics the two-place modal operator $\Box\!\!\rightarrow$ is interpreted as a conditional operator, which represents natural language conditionals. If we use the same interpretation for CS-semantics as in epistemic or doxastic logic, then conditional formulas $\alpha \Box\!\!\rightarrow \beta$ are to be understood in a subjective way (as known facts or beliefs respectively), while non-conditional formulas, such as $p \wedge q$, are interpreted as matters of facts. We, however, argue that it is in general hardly plausible that conditional and non-conditional formulas have such a different cognitive status. Such an interpretation of conditionals contradicts common sense assumptions and also seems from a philosophical perspective rather arbitrary (see Section 3.4.3).

A possible way out of this problem is the interpretation of a specific type of conditional formulas, namely conditional formulas of the form $\top \Box\!\!\rightarrow \alpha$ – where $\top$ abbreviates tautology $p \vee \neg p$ – as beliefs in a proposition α. In such an approach we, however, cannot accept that both α and $\top \Box\!\!\rightarrow \alpha$ represent the same proposition. If α and $\top \Box\!\!\rightarrow \alpha$ are to be equivalent, one has to endorse the following two principles:

Det $\quad (\top \Box\!\!\rightarrow \alpha) \rightarrow \alpha$

Cond $\quad \alpha \rightarrow (\top \Box\!\!\rightarrow \alpha)$

Here ‘Det’ and ‘Cond’ abbreviate ‘Detachment’ and ‘Conditionalization’, respectively. As we saw in Chapter 5, the principles Det and Cond correspond to the following frame conditions w.r.t. Chellas models:

C_{Det} $\quad \forall w(wR_W w)$
C_{Cond} $\quad \forall w(wR_W w' \Rightarrow w' = w)$

Conditions C_{Det} and C_{Cond} give us conjointly that each world w in a Chellas model $\mathcal{M}_C = \langle W, R, V \rangle$ can only see itself via $R_{\|\top\|^{\mathcal{M}_C}}$. Hence, whenever we accept that α is true exactly iff $\top \;\Box\!\!\rightarrow \alpha$ is, the interpretation of $\top \;\Box\!\!\rightarrow \alpha$ as an agent's belief in α, implies that we believe proposition α if and only if α is objectively true. Such an interpretation of beliefs is, however, absurd. It is important to note that unwarranted restrictions of beliefs result not only for principles Det and Cond, but also from other bridge principles, as discussed in Section 3.3 (see also Tables 5.1 and 5.2).

The problem of an epistemic or a doxastic interpretation of CS-semantics lies in the fact that a specific world (the starting world w) is interpreted as describing a state of affairs that factually obtains. We, however, need not accept this part of an epistemic/doxastic interpretation. Instead, we can construe the set of all possible worlds in a Chellas model $\mathcal{M}_C = \langle W, R, V \rangle$ as being inherently subjective. In this interpretation any world $w \in W$ is seen as a state of affairs regarded possible by an agent under some circumstances.

We also saw in Section 3.2.6 that we not only have to arrive at a (hypothetical) stock of beliefs, but that we also start with a (hypothetical) stock of beliefs. Hence, the notion of truth of conditional and non-conditional formulas at a possible world w in a Chellas model $\mathcal{M}_C = \langle W, R, V \rangle$ is not the appropriate notion. We, hence, suggested in Section 3.2.6 that we should use sets of possible worlds rather than single possible worlds as starting point. Combined with the subjective interpretation of possible worlds suggested in the previous paragraph, this implies that we use as starting point sets of possible worlds, which represent (hypothetical) subjective stocks of beliefs. So, given a set of possible worlds $X \subseteq W$ as a starting point, all formulas true in all worlds $w \in X$ are regarded true by the agent in question. The set X is, hence, the "ontological analogue" to an initial stock of beliefs (cf. Sections 3.2.4 and 3.2.6). Hypothetically putting a proposition $\|\alpha\|^{\mathcal{M}_C}$ to one's initial stock of beliefs results, then, in a new stock of (hypothetical) beliefs represented by a further set of possible worlds $Y \subseteq W$. In this approach the set Y is determined by the accessibility relation $R_{\|\alpha\|^{\mathcal{M}_C}}$ in such a way that for every $w' \in Y$ every world $w \in X$ stands in the relation $R_{\|\alpha\|}$. This condition gives us naturally that a conditional $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ is true w.r.t. the set of possible worlds X iff β is true at every world $w \in Y$.

Moreover, it seems reasonable to endorse a version of the Ramsey-test, which requires that any stock of (hypothetical) beliefs A that is updated by a proposition $\|\alpha\|$, must always arrive at possible worlds, at which α is also true. In such an interpretation of CS-semantics the formula $\alpha \mathrel{\Box\!\!\rightarrow} \alpha$ (Refl, see previous section) is always true. In order to ensure that this condition is always satisfied we can require that any Chellas (Segerberg) model, which is interpreted in such a way, has to satisfy condition C_{Refl} (see previous section).

Note, however, that there are certain interpretations of the Ramsey-test, which do not make Refl valid. In such an interpretation it is possible that hypothetically adding $\|\alpha\|$ to our stock of knowledge does not lead to the acceptance of α. In such a case we presuppose α, but we end up – on the basis of our background knowledge – rejecting α rather than accepting it. Such a scenario can arise if we, for example, endorse the following strict consistency requirement:

CNC $\quad \neg((\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta))$

'CNC' abbreviates 'Conditional Consistency Criterion'. We discussed this principle in depth in Sections 3.2.5, 3.5.3 and 3.7. We saw that, for example, Bennett (2003, p. 81) argues that CNC is valid for all conditionals. Suppose that we want to find out whether the formula $\bot \mathrel{\Box\!\!\rightarrow} \alpha$ holds for a stock of (hypothetical) beliefs described by a subset X of W in a Chellas model $\mathcal{M}_C = \langle W, R, V\rangle$. We, then, put $\|\bot\|^{\mathcal{M}_S}$ to our stock of beliefs. (Note that for any Chellas [Segerberg] model $\|\bot\|^{\mathcal{M}_S} = \varnothing$.) Due to CNC we are, however, not allowed to arrive at an absurd belief, which describes believing α and $\neg\alpha$. This, however, means that we have to reject our assumption $\bot$. Hence, even if we presuppose $\bot$, we have to arrive at a consistent formula α.

It can be seen as an advantage of CS semantics that Refl is not valid in every Chellas model, since both CNC and Refl contradict each other given minimal preconditions (see Section 3.5.3). Due to that fact, we can also describe a conditional logic with the unrestricted consistency requirement CNC. Note that principle D (a generalization of principle D from Kripke semantics) from Table 5.2 is equivalent to CNC given $\mathrm{Def}_{\Diamond\!\!\rightarrow}$ (see Section 3.5.3). Hence, we can apply C_{D} for CNC, as well. We can, however, also use the following weaker consistency criterion (cf. Section 3.2.5, see also Table 5.1):

P-Cons $\neg(\top \,\Box\!\!\rightarrow \bot)$

This principle is, for example, discussed by Hawthorne (1996, p. 194–196) and Schurz (1998, p. 88 and p. 90; see Section 3.2.5). Note that P-Cons is consistent with Refl given system **CK**. Hence, we can apply $C_{\mathrm{P-Cons}}$ in combination with C_{Refl}. This principle is, however, much weaker than CNC and ensures only in case the antecedent is a logically true formula that the consequent must be consistent (cf. Section 3.2.5)..

7.1.2 Alternative Axiomatizations of System CKR

We gave in the previous section a definition of system **CKR** in terms of **CK**+Refl (see Definition 7.1). Note, however, that there is a more parsimonious axiomatization available, namely the one given in Theorem 7.2. This is due to the fact that the following principle LT (see Table 4.1) is redundant given Refl and RW:

LT $\alpha \,\Box\!\!\rightarrow \top$

Note that we henceforth indicate derivability of principles by $\Rightarrow$. In addition, axioms of **CK**, which we rely on in an derivation are indicated by parentheses. Let us now state the following theorem:

Theorem 7.2. *System* **CKR** *can be axiomatized by LLE+RW+Refl+AND*

Proof. By Definition 4.1 and Lemma 7.3. □

Lemma 7.3. *(RW)+Refl $\Rightarrow$ LT*

Proof.

1.	$\alpha \,\Box\!\!\rightarrow \beta$	given
2.	$\alpha \,\Box\!\!\rightarrow \top$	1, RW

□

We can, however, describe the system **CKR** alternatively on the basis of the following rule:

SC if $\vdash \alpha \rightarrow \beta$, then $\vdash \alpha \,\Box\!\!\rightarrow \beta$

'SC' abbreviates for 'Supraclassicality' (see, for example, Schurz, 1998, p. 84). This rules states that any logically true material implication $\alpha \rightarrow \beta$ results in the corresponding conditional $\alpha \,\Box\!\!\rightarrow \beta$ being true. Please note that SC is not an instance of the rule RCK from Section 4.2.7. The main difference is that in RCK – unlike in SC – the material implication '$\rightarrow$' is not replaced by conditional operator '$\Box\!\!\rightarrow$'. That SC is not an instance of RCK, can also be seen from the fact that Refl is derivable from SC (see Lemma 7.5), but not from RCK. Rule SC is, moreover, not, as Schurz (1996, Footnote 3, p. 201) argues, entailed by LLE and RW. For this to hold SC would be required to be a theorem of **CK**, which it is clearly not. Let us now give the alternative axiomatization of system **CKR** on the basis of principle SC:

Theorem 7.4. *System* **CKR** *can be axiomatized by LLE+RW+SC+AND*

Proof. By Definition 4.1 and Lemma 7.5. ◻

Lemma 7.5. *(RW) ⇒ (Refl ⇔ SC)*

Proof. By Lemmata 7.6 and 7.7. ◻

Lemma 7.6. *(RW)+Refl ⇒ SC*

Proof.

1.	$\vdash \alpha \rightarrow \beta$	given
2.	$\vdash \alpha \wedge \beta \leftrightarrow \alpha$	1, prop.
3.	$\alpha \wedge \beta \,\Box\!\!\rightarrow \alpha \wedge \beta$	Refl
4.	$\alpha \,\Box\!\!\rightarrow \alpha \wedge \beta$	3, 2, LLE
5	$\alpha \,\Box\!\!\rightarrow \beta$	4, RW

◻

Lemma 7.7. *SC ⇒ Refl*

Proof.

1.	$\vdash \alpha \rightarrow \alpha$	prop.
2.	$\alpha \Box\!\!\rightarrow \alpha$	1, SC

$\Box$

7.2 Conditional Logics without Bridge Principles

In this section we will discuss the following conditional logic systems: system **C** (Section 7.2.1), system **CL** (7.2.2), system **P** (Section 7.2.3), system **R** (Section 7.2.4), system **V** (Section 7.2.5), system **CM** (Section 7.2.6) and system **M** (Section 7.2.6). Versions of system **C**, **CL**, **P**, **CM** and **M** can be found in Kraus et al. (1990). Moreover, Lehmann and Magidor (1992) discuss versions of system **P** and system **R**. We, furthermore, describe systems **P** and **R**in Section 3.5.1. We, however, use in Section 3.5.1 the expression '$\mathbf{P}^{+}$' to refer to system **R**.

Note, moreover, that the systems described in Kraus et al. (1990) and Lehmann and Magidor (1992) are not formulated in the full language (see the introductory part to this Chapter) while all of our formulations pertain to the full language $\mathcal{L}_{\mathrm{KL}}$. System **V** is in that sense exceptional, since it was already formulated by D. Lewis in the full language $\mathcal{L}_{\mathrm{KL}}$ (D. Lewis, 1973/2001, p. 132f, pp. 120–123).

7.2.1 System C

In all subsections of Section 7.2 we will focus exclusively on systems without bridge principles. These type of systems can be described by presupposing – opposed to systems with bridge principles – that no fixed relation between conditional and non-conditional formulas hold (see Section 4.2.1). We, moreover, argued earlier (Chapter 1) that any conditional logic approach aims to invalidate principles S1′–S5′. Interestingly, some of these principles are bridge principles, namely S1′ (EFQ) and S2′ (VEQ), and others are not (Principles S3′–S4′). It seems intuitive that bridge principles, such as S1′ and S2′ cannot be derived in any system without bridge principles. However, the question whether bridge principles are derivable in systems, which are axiomatizable without reference to bridge principles is far from trivial and investigated in depth in Schurz (1997a). Schurz' s

(1997a) formal investigation of the Is-Ought problem shows for a range of multi-modal systems that no bridge principles are provable when those systems are axiomatizable without reference to bridge principles (see Schurz, 1997a, Definition 8 [Special Hume Thesis], p. 72; Theorem 3, p. 118; Theorem 4, p. 121; Theorem 5, p. 124). Schurz (1997a, p. 173), furthermore, conjectures that an extension of his results also holds for two-place operators, such as the conditional operator $\Box\!\!\rightarrow$. We are, however, not aware of a proof of this result. Since, however, the above assumption is intended to only motivate our approach to discuss conditional logics with bridge principles and without bridge principles in separate sections, we will not address this issue at any depth here. Instead, we will merely postpose the discussion of EFQ and VEQ to Section 7.3 and discuss only principles, which are not bridge-principles, such as principles S3′ (Trans, "Transitivity"), S4′ (Mon, "Monotonicity") and S5′ (CP, "Contraposition") (see also Table 5.1):

Trans $(\alpha \Box\!\!\rightarrow \beta) \wedge (\beta \Box\!\!\rightarrow \gamma) \rightarrow (\alpha \Box\!\!\rightarrow \gamma)$
Mon $(\alpha \Box\!\!\rightarrow \gamma) \rightarrow (\alpha \wedge \beta \Box\!\!\rightarrow \gamma)$
CP $(\alpha \Box\!\!\rightarrow \beta) \rightarrow (\neg\beta \Box\!\!\rightarrow \neg\alpha)$

Note that Mon is the central principle here. This is due to the fact that for both **CK**+CP and **CK**+Ref+Trans principle Mon is a theorem (see Lemma 7.43 and Lemma 7.49, respectively). So, any extension of **CK**+Refl, which is non-monotonic viz. in which Mon is not a theorem, has neither Trans nor CP as theorems. Note here that Refl is often regarded a cornerstone for any conditional logic (see Section 7.1.1).

Let us now focus on system **C**. This system can be defined the following way:

Definition 7.8. *Logic* **C** *is the smallest logic containing* **CK**+*Refl+CM+CC*.

Specific to this system are the following two principles (see Table 5.1):

CM $(\alpha \Box\!\!\rightarrow \gamma) \wedge (\alpha \Box\!\!\rightarrow \beta) \rightarrow (\alpha \wedge \beta \Box\!\!\rightarrow \gamma)$
CC $(\alpha \wedge \beta \Box\!\!\rightarrow \gamma) \wedge (\alpha \Box\!\!\rightarrow \beta) \rightarrow (\alpha \Box\!\!\rightarrow \gamma)$

'CM' and 'CC' stand for 'Cautious Monotonicity' and 'Cautious Cut', respectively. The names 'Cautious Monotonicity' and 'Cautious Cut' are motivated by the fact that CM and CC represent weakened versions of the principles Mon and the principle Cut (see Section 7.2.6 and Table 5.1), respectively:

Cut $\quad (\alpha \wedge \beta \,\Box\!\!\rightarrow \delta) \wedge (\gamma \,\Box\!\!\rightarrow \beta) \rightarrow (\alpha \wedge \gamma \,\Box\!\!\rightarrow \delta)$

Interestingly CC can also be seen as a weakened version of Trans (see above). Since the relationship between CC and Trans is somewhat more perspicuous, we shall henceforth follow the discussion of Chapter 1 and focus on the relationship between CC and Trans rather than the relationship between CC and Cut.

Note that CM and CC correspond to Mon and Trans, respectively, in the following sense: Mon allows one to conclude $\alpha \wedge \beta \,\Box\!\!\rightarrow \gamma$ from $\alpha \,\Box\!\!\rightarrow \gamma$, whereas CM allows to derive $\alpha \wedge \beta \,\Box\!\!\rightarrow \gamma$ from $\alpha \,\Box\!\!\rightarrow \gamma$ and $\alpha \,\Box\!\!\rightarrow \beta$. Thus, CM is essentially a restricted version of Mon (hence the name 'Cautious Monotonicity'). Furthermore, Trans allows to infer $\alpha \,\Box\!\!\rightarrow \gamma$ from $\alpha \,\Box\!\!\rightarrow \beta$ and $\beta \,\Box\!\!\rightarrow \gamma$ while CC needs $\alpha \,\Box\!\!\rightarrow \beta$ and $\alpha \wedge \beta \,\Box\!\!\rightarrow \gamma$ in order that $\alpha \,\Box\!\!\rightarrow \gamma$ is deducible. The difference between both principles is that in the case of CC the second antecedent requires $\alpha \wedge \beta \,\Box\!\!\rightarrow \gamma$ to hold rather than only $\alpha \,\Box\!\!\rightarrow \gamma$. Note that in all counterexamples to Mon (E4, E4′ and E4″) and Trans (E3, E3′ and E3″), discussed in Sections 1.1, 1.2.1 and 1.2.2, the antecedents of Mon and Trans are satisfied, but not the stricter ones of CM and CC.

This argumentation shows that one may block counter-intuitive inferences by employing CM and CC instead of Mon and Trans, respectively. Our argumentation does, however, not show that CM and CC are justified principles on their own right. Note, however, that the principles CM and CC (as well as the Principles Or and RM [Rational Monotonicity] discussed in Sections 7.2.3 and 7.2.4) respectively) are probabilistically justified. They translate into probabilistically valid inference schemas in Adams' system **P** (Adams, 1965, 1966, 1977) and its extension $\mathbf{P}^{+}$ (Adams, 1986; Schurz, 1998; Bamber, 1994) for both a validity criterion of infinitesimal probability (see Definition 3.18) and a validity criterion with a high, but non-infinitesimal probabilities (see Theorem 3.23.4, see Section 3.5). Given these results CM and CC (and Or and RM) are, hence, statistically reliable.

We argue that in addition principles CM and CC are also in line with one's general intuitions regarding conditionals. To provide support for this thesis, let us now draw on the respective discussion of Kraus et al. (1990, p. 178f). We, however, will modify Kraus et al.'s examples in order to make the discussion more perspicuous. Let us start with a discussion of principle CM. Assume for that purpose that we believe the two following propositions:

(a1) If Fireball takes part in the race, then it will normally win.

(b1) If Fireball takes part in the race, then normally Thunderhead takes place in the race.

Are we, then, not rationally forced to believe the following as well?

(c1) If Fireball takes part in the race and Thunderhead takes place in the race, then normally Fireball wins the race?

The inference from (a1) and (b1) to (c1) is statistically reliable (see above). Moreover, it seems also quite natural to accept (c1) given that we accept (a1) and (b1). (This inference is also statistically reliable.) Furthermore, if we do not accept (c1) then it seems natural to reject (a1). The inference from (a1) and (b1) to (c1) is, however, a natural language representation of the above principle CM. A parallel argument can also be made for principle CC. Assume that we believe following two propositions:

(a2) If Fireball takes part in the race and Thunderhead takes part in the race, then Fireball will normally win.

(b2) If Fireball takes part in the race, then normally Thunderhead takes part in the race.

Would believing (a2) and (b2) not imply that we believe also the following proposition?

(c2) If Fireball takes part in the race, then normally Fireball wins the race.

Again, the inference from (a2) and (b2) to (c2) is statistically reliable (in the above sense). Moreover, it seems natural to reject (a2) (or (b2)) if we reject (c2). .

Let us now discuss the semantic representation of CM and CC in our CS-semantics. We list here the nontrivial frame restrictions for CM and CC along the nontrivial frame conditions for Mon and Trans (X and Y refer to subsets of possible worlds in a Chellas frame $\mathcal{F}_C$; see also Table 5.3):

C_{CM} $\quad \forall X, Y\, \forall w(\forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_{X\cap Y}w' \Rightarrow wR_X w'))$
C_{CC} $\quad \forall X, Y\, \forall w(\forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_X w' \Rightarrow wR_{X\cap Y}w'))$
C_{Mon} $\quad \forall X\, \forall w, w'\, (wR_{X\cap Y}w' \Rightarrow wR_X w')$
C_{Trans} $\quad \forall X, Y\, \forall w(\forall w'(wR_X w' \Rightarrow w' \in Y) \Rightarrow \forall w'(wR_X w' \Rightarrow wR_Y w'))$

The frame conditions C_{CM}, C_{CC} and C_{Trans} contain preconditions, which require worlds, which are accessible via an accessibility relation R_X, to be members of some set Y. We will call this type of frame condition 'relativized'. C_{Mon} is not relativized in that sense: It rather represents a closure condition, which does not depend on accessible worlds being members of some set and can be applied in a sense unconditionally. Note, however, that it is the relativization of C_{CM} that gives CM its intuitive import compared to Mon. In C_{Trans} the case is, however, different. Provided that C_{Refl} holds for all subsets of possible worlds X specified by the given Chellas frame $\mathcal{F}_C$ – namely that it is the case that $\forall X\, \forall w, w'(wR_X w' \Rightarrow w' \in X)$ – C_{Trans} implies C_{Mon}. Hence, the precondition C_{Trans} becomes in the presence of C_{Refl} in a sense "ineffective". This can be seen if one replaces variables 'X' and 'Y' in C_{Trans} by '$X \cap Y$' and 'X', respectively. Then, the following frame condition results:

(1) $\quad \forall X, Y\, \forall w(\forall w'(wR_{X\cap Y}w' \Rightarrow w' \in X) \Rightarrow \forall w'(wR_{X\cap Y}w' \Rightarrow wR_X w'))$

Refl, however, guarantees that the precondition of (1), namely $\forall w'(wR_{X\cap Y}w' \Rightarrow w' \in X)$, holds in all cases, and, thus, C_{Trans} implies in the presence of C_{Ref} frame condition C_{Mon}. From a proof-theoretic point of view this is mirrored by the fact that Refl+Trans imply Mon (see Lemma 7.49).

Let us now turn to the proof theory of system **C**. We will provide two alternative and more parsimonious axiomatizations of system **C** compared to Definition 7.8, as described by Theorems 7.9 and 7.11. These results generalize, for example, to the axiomatization of system **CL** (Definition 7.14), system **P** (Definition 7.15) and system **R** (Definition 7.26), since the latter systems draw in their axiomatization on system **C**. Hence, these definitions can be outlined in terms of our alternative axiomatizations of system **C** (Theorem 7.9 and 7.11). For the sake of perspicuity we, however, do not discuss alternative axiomatizations of these system on the basis of Theorem 7.9 and 7.11 for the remainder of this Chapter. Let us now describe these alternative axiomatizations of system **C**:

Theorem 7.9. *System* **C** *is axiomatizable as LLE+RW+Refl+CM+CC. (Kraus et al., 1990, p. 176).*

Proof. System **C** is by Definition 7.8 **CK**+Refl+CM+CC. Since **CK** is due to Definition 4.1 LLE+RW+AND+LT, it follows that **C** is LLE+RW+AND+LT +Refl+CM+CC. To show that **C** is LLE+RW+Refl+CM+CC, we, hence, have to prove that (a) AND and (b) LT are theorems of LLE+RW+Refl+CM+CC. Lemmata 7.3 and 7.10 give us that LT is derivable from RW+Refl and that AND is a theorem of RW+Refl+CM+CC. Hence, (a) and (b) follow. □

Lemma 7.10. *(RW)+Refl+CM+CC ⇒ AND (Kraus et al., 1990, p. 179)*

Proof.

1.	$\alpha \mathrel{\Box\!\!\rightarrow} \beta$	given
2.	$\alpha \mathrel{\Box\!\!\rightarrow} \gamma$	given
3.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$	2, 1 CM
4.	$\alpha \wedge \beta \wedge \gamma \mathrel{\Box\!\!\rightarrow} \alpha \wedge \beta \wedge \gamma$	Refl
5.	$\alpha \wedge \beta \wedge \gamma \mathrel{\Box\!\!\rightarrow} \beta \wedge \gamma$	4, RW
6.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \beta \wedge \gamma$	5, 3, CC
7.	$\alpha \mathrel{\Box\!\!\rightarrow} \beta \wedge \gamma$	6, 1, CC

□

Let us now prove the second alternative axiomatization:

Theorem 7.11. *System* **C** *is axiomatizable as LLE+SC+CM+CC.*

Proof. System **C** is by Definition 7.8 and Theorem 7.9 LLE+RW+Refl+CM+CC. To prove that **C** is LLE+SC+CC+CM, we have to show that (a) RW and (b) Refl are theorems of LLE+SC+CC+CM and that (c) SC is derivable from LLE+RW +Refl+CM+CC. Lemmata 7.12 and 7.7 give us that RW is a theorem of SC+CC and Refl is derivable from SC. Thus (a) and (b) follow. Moreover, by Lemma 7.6 SC is a theorem of LLE+RW+Refl. Hence, (c) is the case. □

Lemma 7.12. *CC+SC ⇒ RW (cf. Adams, 1966, p. 280f)*

Proof.

1.	$\alpha \mathrel{\Box\!\!\rightarrow} \beta$	given
2.	$\vdash \beta \rightarrow \gamma$	given
3.	$\vdash \alpha \wedge \beta \rightarrow \gamma$	2, p.c.
4.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$	3, SC
5.	$\alpha \mathrel{\Box\!\!\rightarrow} \gamma$	4, 1, CC

□

Observe that LLE is included in our axiomatization of system **C** in Theorem 7.11. Sometimes it is argued that the stronger systems **P** (Geffner & Pearl, 1994, p. 71; Schurz, 2005, p. 42; Schurz, 1998, p. 84) and system $\mathbf{P}_\epsilon$ (Schurz, 1994, p. 251) can be axiomatized by inference schema corresponding to SC+CM+CC+Or and SC+CM+CC+Or+RCNC, respectively, hence *without* reference to LLE. (For a discussion of the principles Or and RCNC see Sections 7.2.3 and 3.2.5.) However, Geffner and Pearl (1994, Theorem 2, p. 72) refer in their proof of LLE explicitly only to principles CM, CC and SC (which are all theorems of system **C**). There is, however, one important difference between our and Geffner and Pearl's treatment of conditionals: Geffner and Pearl (1994) conceptualize antecedents of conditionals as sets of formulas, where elements of "antecedent sets" are interpreted as "unordered conjunctions" of formulas (p. 70f). Since in Geffner and Pearl's account both $\{\alpha,\beta\} \mathrel{\Box\!\!\rightarrow} \gamma$ and $\{\beta,\alpha\} \mathrel{\Box\!\!\rightarrow} \gamma$ represent the "same" conditional, Geffner and Pearl (1994), hence, presuppose that the following principle for conditionals holds:

ECA $\quad (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\beta \wedge \alpha \mathrel{\Box\!\!\rightarrow} \gamma)$

Here 'ECA' stands for 'Exchangability of Conjuncts in the Antecedent'. Moreover, given ECA, LLE is derivable from CM, CC and SC, as the following lemma shows:

Lemma 7.13. *ECA+SC+CC+CM $\Rightarrow$ LLE (cf. Geffner & Pearl, 1994, Theorem 2, p. 72)*

Proof.

1.	$\alpha \;\Box\!\!\rightarrow \gamma$	given
2.	$\vdash \alpha \leftrightarrow \beta$	given
3.	$\vdash \alpha \rightarrow \beta$	2, p.c.
4.	$\alpha \;\Box\!\!\rightarrow \beta$	3, SC
5.	$\alpha \wedge \beta \;\Box\!\!\rightarrow \gamma$	1, 4, CM
6.	$\beta \wedge \alpha \;\Box\!\!\rightarrow \gamma$	5, ECA
7.	$\vdash \beta \rightarrow \alpha$	2, p.c.
8.	$\beta \;\Box\!\!\rightarrow \alpha$	7, SC
9.	$\beta \;\Box\!\!\rightarrow \gamma$	6, 8, CC

□

Note that ECA is essential for Lemma 7.13 to go through. We are, moreover, not aware of any alternative proof of Lemma 7.13, which does not rely on an at least implicit version of ECA.[1] We, hence, conclude that the above axiomatizations of system **P** (Geffner & Pearl, 1994, p. 71; Schurz, 2005, p. 42; Schurz, 1998, p. 84) and system $\mathbf{P}_\epsilon$ (Schurz, 1994, p. 251) have to either include ECA as a further axiom or conceptualize antecedents of conditionals as sets of formulas rather than single formulas (in the way described above).

7.2.2 System CL

System **CL** is system **C** plus the following principle Loop (see Table 5.1):

Loop $\quad (\alpha_0 \;\Box\!\!\rightarrow \alpha_1) \wedge \ldots \wedge (\alpha_{k-1} \;\Box\!\!\rightarrow \alpha_k) \wedge (\alpha_k \;\Box\!\!\rightarrow \alpha_0) \rightarrow (\alpha_0 \;\Box\!\!\rightarrow \alpha_k) \;\; (k \geq 2)$

Hence, system **CL** can be defined accordingly:

Definition 7.14. *Logic* **CL** *is the smallest logic containing* **CK**+*Refl+CM+CC+Loop (cf. Kraus et al., 1990, p. 187).*

[1]For example, in Schurz (1994, 2005) no alternative proof or reference to an alternative proof of Lemma 7.13 without ECA is provided. Moreover, the reference of Schurz (1998, p. 84) to Gärdenfors and Makinson (1994, 201) seems to be motivated by Gärdenfors and Makinson's discussion of principle SC rather than by referring to a proof, which shows that SC is equivalent to LLE+RW+AND+Refl given CC+Or+RM+TR (see Table 5.1 for a description of principles RM and TR).

This principle has been first suggested by Kraus et al. (1990, p. 187). Note that the name Loop is also appropriate from a semantical perspective. The nontrivial frame condition for Loop is the following (see Table 5.3):

C_{Loop} $\forall w(\forall w'(wR_{X_0}w' \Rightarrow w' \in X_1) \wedge \ldots \wedge \forall w'(wR_{X_{k-1}}w' \Rightarrow w' \in X_k) \wedge \forall w'(wR_{X_k}w' \Rightarrow w' \in X_0) \Rightarrow \forall w'(wR_{X_0}w' \Rightarrow w' \in X_k))$ $(k \geq 2)$

The frame condition can be interpreted in the following sense: If for a given number of sets the accessibility relations relativized to sets and the worlds accessible form a loop (by being members in the respective sets and being relativized to the respective sets), then all worlds accessibly by the first set are members of the last set.

7.2.3 System P

System **P** is defined the following way:

Definition 7.15. *Logic* **P** *is the smallest logic containing* **CK**+*Refl*+*Or*+*CM* *(Lehmann & Magidor, 1992, p. 6)*

The main difference between system **P** on the one hand and systems **CL** and **C** on the other hand is the fact that system **P** contains the additional principle Or (see Table 5.1):

Or $(\alpha \Box\!\!\rightarrow \gamma) \wedge (\beta \Box\!\!\rightarrow \gamma) \rightarrow (\alpha \vee \beta \Box\!\!\rightarrow \gamma)$

It is important to note that Or translates in probabilistically valid inferences and, thus, is probabilistically justified (see Sections 7.2.1 and 3.5). Moreover, to demonstrate the intuitive support for principle Or, let us consider the following inference, which corresponds to principle Or (Kraus et al., 1990, p. 190):

(a3) If Cathy attends the party then normally the evening will be great.
(b3) If John attends the party then normally the evening will be great.
(c3) Therefore: If Cathy or John attends the party, then normally the evening will be great.

Given that (a3) and (b3) hold, it is seem reasonable to infer (c3). Note that example (a3)-(c3) is one of many Or inferences, which provide intuitive support for the principle Or. A further motivation for including Or is that this type of inference is unproblematic insofar as it does not make the monotonic principles Mon, Trans or CP derivable (cf. Kraus et al., 1990, p. 190). This is due to the complementary relation of both principles Mon and Or. Let us for that purpose describe the nontrivial frame condition corresponding to Or (see Table 5.3):

$$C_{\mathrm{Or}} \quad \forall w, w' \left(wR_{X\cup Y}w' \Rightarrow wR_X w' \vee wR_Y w' \right)$$

This principle gives us that for any worlds w, w' in any Chellas frame $\mathcal{F}_C = \langle W, R \rangle$ and subsets X, Y of W that if $wR_X w'$, then either $wR_{X\cap Y}w'$ or $wR_{X\cap -Y}w'$ (see also Lemma 7.23 and frame restriction C_{WOR} from Table 5.3). The principle Mon on the other hand gives us for arbitrary worlds w, w' in $\mathcal{F}_C$ and X, Y being subsets of W that, if $wR_{X\cap Y}w'$ then $wR_X w'$. Note also that both frame restrictions for Mon and Or represents from a technical perspective a type of closure condition, which is not relativized in the sense of CM or CC (see Section 7.2.1). The complementary relation of Or and Mon can also be seen from a proof-theoretic perspective: While **CK**+Refl+Mon+Or allows for an axiomatization of the monotonic collapse without bridge principles (system **M**; see Definition 7.38 and Theorem 7.39), the following holds: (a) System **P** contains Or, but does not collapse with **M** (moreover Mon is not valid in system **P**) and (b) system **CM** contains Mon, but not Or.

Note that Or implies – given Refl – the principle CC (see Lemma 7.16). Hence, the axiomatization given in Definition 7.15 suffices (cf. Kraus et al., 1990, p. 190).

Lemma 7.16. *(LLE+RW+AND)+Refl+Or* $\Rightarrow$ *CC*

Proof.

1.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$	given
2.	$\alpha \mathrel{\Box\!\!\rightarrow} \beta$	given
3.	$\alpha \wedge \neg\beta \mathrel{\Box\!\!\rightarrow} \alpha \wedge \neg\beta$	Refl
4.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma \vee (\alpha \wedge \neg\beta)$	1, RW
5.	$\alpha \wedge \neg\beta \mathrel{\Box\!\!\rightarrow} \gamma \vee (\alpha \wedge \neg\beta)$	3, RW
6.	$(\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta) \mathrel{\Box\!\!\rightarrow} \gamma \vee (\alpha \wedge \neg\beta)$	4,5, Or
7.	$\alpha \mathrel{\Box\!\!\rightarrow} \gamma \vee (\alpha \wedge \neg\beta)$	6, LLE
8.	$\alpha \mathrel{\Box\!\!\rightarrow} (\gamma \vee (\alpha \wedge \neg\beta)) \wedge \beta$	7, 2, AND
9.	$\alpha \mathrel{\Box\!\!\rightarrow} \gamma$	8, RW

Note furthermore that Or is closely related to the following two principles (see Table 5.1):

S $\quad (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma))$

WOR $\quad (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma) \wedge (\alpha \wedge \neg\beta \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \gamma)$

We follow Kraus et al. (1990, p. 191) and Lehmann and Magidor (1992, p. 6) in our terminology for principle S. However, neither Kraus et al. (1990) nor Lehmann and Magidor (1992) indicate what the label ‘S’ stands for. Kraus et al. (1990, p.191) also investigate WOR (“Weak Or”, see Hawthorne & Makinson, 2007, p. 252). In Kraus et al.’s terminology the principle WOR is called ‘D’ (p. 191).

Principle WOR is an axiom of the probabilistic threshold logic **O** of Hawthorne and Makinson (2007, (p. 252), which is strictly weaker than system **P** (Hawthorne & Makinson, 2007, p. 253). The main difference between system **O** and **P** is from a proof-theoretic perspective that AND is a theorem of **P**, but not of system **O** (Hawthorne & Makinson, 2007, p. 259; see also Sections 3.5.1 and 3.2.5). Lemmata 7.17 and 7.23 give us that both principles S and WOR are logically equivalent to Or provided **CK**+Refl. The equivalence between WOR and Or, however, does not hold given the rules and axioms of system **O** (cf. Hawthorne & Makinson, 2007, p. 253). Let us now focus on the latter two lemmata:

Lemma 7.17. *(LLE+RW+AND)+Refl $\Rightarrow$ (Or $\Leftrightarrow$ S)*

Proof. By Lemmata 7.18 and 7.19. □

Lemma 7.18. *(LLE+RW+AND)+S $\Rightarrow$ Or*

Proof.

1.	$\alpha \mathrel{\Box\!\!\rightarrow} \gamma$	given
2.	$\beta \mathrel{\Box\!\!\rightarrow} \gamma$	given
3.	$(\alpha \vee \beta) \wedge (\alpha \vee \neg\beta) \mathrel{\Box\!\!\rightarrow} \gamma$	1, LLE
4.	$(\alpha \vee \beta) \mathrel{\Box\!\!\rightarrow} ((\alpha \vee \neg\beta) \rightarrow \gamma)$	3, S
5.	$(\alpha \vee \beta) \wedge (\neg\alpha \vee \beta) \mathrel{\Box\!\!\rightarrow} \gamma$	2, LLE
6.	$(\alpha \vee \beta) \mathrel{\Box\!\!\rightarrow} ((\neg\alpha \vee \beta) \rightarrow \gamma)$	5, S
7.	$(\alpha \vee \beta) \mathrel{\Box\!\!\rightarrow} ((\alpha \vee \neg\beta) \rightarrow \gamma) \wedge ((\neg\alpha \vee \beta) \rightarrow \gamma)$	4, 6, AND
8.	$(\alpha \vee \beta) \mathrel{\Box\!\!\rightarrow} \gamma$	7, RW

□

Lemma 7.19. *(LLE+RW)+Refl+Or $\Rightarrow$ S (Kraus et al., 1990, p. 191)*

Proof.

1.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$	given
2.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma)$	1, RW
3.	$\alpha \wedge \neg\beta \mathrel{\Box\!\!\rightarrow} \alpha \wedge \neg\beta$	Refl
4.	$\alpha \wedge \neg\beta \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma)$	3, RW
5.	$(\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta) \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma)$	2, 4, Or
6.	$\alpha \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma)$	5, LLE

□

The foregoing Lemma 7.17 and the following Lemma 7.20 do not only draw on the principle Refl (which is an axiom of system **O**), but also on the principle AND (which is no theorem of system **O**) to prove the logical equivalence of S and Or on the one hand and WOR and Or on the other hand.

Lemma 7.20. *(RW+AND)+Refl $\Rightarrow$ $(S \Leftrightarrow WOR)$*

Proof. By Lemmata 7.21 and 7.22. □

Lemma 7.21. *(RW+AND)+S $\Rightarrow$ WOR*

Proof.

1.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$	given
2.	$\alpha \wedge \neg\beta \mathrel{\Box\!\!\rightarrow} \gamma$	given
3.	$\alpha \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma)$	1, S
4.	$\alpha \mathrel{\Box\!\!\rightarrow} (\neg\beta \rightarrow \gamma)$	2, S
5.	$\alpha \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma) \wedge (\neg\beta \rightarrow \gamma)$	3, 4, AND
6.	$\alpha \mathrel{\Box\!\!\rightarrow} \gamma$	5, RW

□

Lemma 7.22. *(RW)+Refl+WOR ⇒ S*

Proof.

1.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$	given
2.	$\alpha \wedge \neg\beta \mathrel{\Box\!\!\rightarrow} \alpha \wedge \neg\beta$	Refl
3.	$\alpha \wedge \neg\beta \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma)$	2, RW
4.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma)$	1, RW
5.	$\alpha \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma)$	4, 3, WOR

□

The following lemma depends again not only on Refl, but also on the principle AND:

Lemma 7.23. *(LLW+RW+AND)+Refl ⇒ (WOR ⇔ Or) (Kraus et al., 1990, p. 191)*

Proof. By Lemmata 7.17 and 7.20. □

Let us finally discuss the following principle MOD ("Modality", see Table 5.1):

MOD $\quad (\neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha) \rightarrow (\beta \mathrel{\Box\!\!\rightarrow} \alpha)$

The name 'MOD' is taken from Nute and Cross (2001, p. 10) and Nute (1980, p. 52). Interestingly, Nute and Cross (2001) and Nute (1980) do not indicate what 'MOD' abbreviates. Nute and Cross' use of the name, however, suggests that 'MOD' is a proxy for 'Modality'. (We will henceforth read 'MOD' in that sense).

The principle MOD can be interpreted in the following way: Provided that α is the case even if $\neg\alpha$ holds, then α holds, no matter what is the case. In this reading the formula $\neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha$ expresses that α is necessary, in the sense that in any case α holds. Nute (1980, p. 52), accordingly, defines $\neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha$ as $\Box\alpha$, where $\Box$ is a one-place modal necessity operator (Nute, 1980, p. 16; see also Section 7.1.1). The reading of $\neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha$ as $\Box\alpha$ (α being necessary) is also suggested by an analogous interpretation of universal quantifiers in f.o.l. We can interpret $\forall x(\neg Fx \rightarrow Fx)$ the following way: Given that (an arbitrary) x is not is not a F, x is a F. This is just to say in f.o.l. that Fx holds for any x. It is, hence, not surprising that $\forall x(\neg Fx \rightarrow Fx)$ is f.o.l.-equivalent to $\forall xFx$.

However, the case of MOD is in system **CK** and in CS-semantics not perfectly analogous to the interpretation of universal quantifies in f.o.l. In order to establish an interpretation of $\neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha$ as a modal necessity in line with Kripke semantics (Hughes & Cresswell, 1996/2003, p. 38), one has to show that the truth of $\neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha$ at a world w in a Chellas model $\langle W, R, V\rangle$ guarantees that α is true at all worlds accessible by some accessibility relation R_X from w, where $X \subseteq W$. In other words, for $\Box\alpha$ the following Kripke truth-conditions should hold in $\mathcal{M}_C$, where wR^*w' holds iff $\exists X(wR_Xw')$ (see Section 7.1.1): $\Box\alpha$ is true at a world $w \in W$ iff α is true at all worlds w', such that wR^*w'.

MOD, however, corresponds to the following nontrivial frame condition (see Table 5.3):

$$C_{\text{MOD}} \quad \forall w, w'((wR_{-X}w' \Rightarrow w' \in X) \Rightarrow \forall w'(wR_Yw' \Rightarrow w' \in X))$$

The frame condition C_{MOD} does not give us that result, because the condition $\forall w'(wR_{-X}w' \Rightarrow w' \in X)$ allows that there are even formulas α and worlds w, w' in a Chellas models $\langle W, R, V\rangle$, such that $wR_{\|\neg\alpha\|}w'$ and $w' \in \|\alpha\|$. One might think that the addition of the Principle Refl ($\alpha \mathrel{\Box\!\!\rightarrow} \alpha$) makes it possible to interpret $\neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha$ as $\Box\alpha$. Let us, for that purpose, consider the nontrivial frame condition C_{Refl} (see Section 7.1.1 and Table 5.3):

$$C_{\text{Refl}} \quad \forall w, w'(wR_Xw' \Rightarrow w' \in X)$$

The frame condition C_{Refl} and $\forall w'(wR_{-X}w' \Rightarrow w' \in X)$, however, do not guarantee that $\neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha$ can be interpreted as $\Box\alpha$, since the truth of $\neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha$ at a world w in a Chellas model $\mathcal{M}_C = \langle W, R, V\rangle$ implies – given C_{Refl} – only that there are no worlds w' in $\mathcal{M}_C$, such that $wR_{\|\neg\alpha\|}w'$. There might, however, still be worlds w', which are not attainable by $R_{\|\alpha\|}$ but via some other accessibility relation $R_{\|\beta\|}$ and at which, hence, $\neg\alpha$ might be true. Thus, $\forall w'(wR_{-X}w' \Rightarrow w' \in X)$ and C_{Refl} do not make sure that formula α is true at all worlds attainable by the accessibility relation R^*. An alternative formulation of MOD investigated by Segerberg (1989, p. 158) is the following:

MOD′ $\quad (\neg\alpha \mathrel{\Box\!\!\rightarrow} \bot) \rightarrow (\beta \mathrel{\Box\!\!\rightarrow} \alpha)$

Note that MOD is logically equivalent to MOD′ given **CK**+Refl. The nontrivial frame condition corresponding to MOD′ is the following (cf. Segerberg, 1989, p. 163):

$C_{\text{MOD}} \quad \forall w(\neg\exists w'(wR_{-X}w') \Rightarrow \forall w'(wR_Y w' \Rightarrow w' \in X))$

The truth of the formula $\neg\alpha \mathrel{\Box\!\!\rightarrow} \bot$ at a world w in a Chellas model $\mathcal{M}_C = \langle W, R, V\rangle$ gives us again only that there is no world w' in $\mathcal{M}_C$, such that $wR_{\|\neg\alpha\|}w'$. It does not guarantee that α is true at all worlds attainable by accessibility R^*, as defined above. One can alternatively define $\Box\alpha$ directly the following way (based on a Chellas model $\mathcal{M} = \langle W, R, V\rangle$, cf. Segerberg, 1989, p. 161; cf. Section 7.1.1):

$V_\Box \quad \vDash_w^{\mathcal{M}_C} \Box\alpha$ iff $\forall w' \, \forall X \subseteq W(wR_X w' \Rightarrow \vDash_{w'}^{\mathcal{M}_C} \alpha)$

Observe that $V_\Box$ involves a second order quantification over subsets of the set of possible worlds W. Open problems regarding the notion of necessity include the following: (a) Can modal necessity ($\Box$) also be defined without second-order quantification? (b) What is the weakest conditional logic system, such that $(\neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha) \leftrightarrow \Box\alpha$ holds? Due to space and time limitations we have to leave these problems for further investigations.

We finally show why we included the discussion of MOD in this section rather than other sections. The principle MOD is derivable in system **P**, as the following lemma shows:

Lemma 7.24. *(LLE+RW+AND)+Refl+CM+Or* $\Rightarrow$ *MOD*

Proof.

1.	$\neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha$	given
2.	$\neg\alpha \mathrel{\Box\!\!\rightarrow} \neg\alpha$	Refl
3.	$\neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha \wedge \neg\alpha$	1,2, AND
4.	$\neg\alpha \mathrel{\Box\!\!\rightarrow} \beta$	3, RW
5.	$\neg\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \alpha$	1, 4, CM
6.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \alpha \wedge \beta$	Refl
7.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \alpha$	6, RW
8.	$(\neg\alpha \wedge \beta) \vee (\alpha \wedge \beta) \mathrel{\Box\!\!\rightarrow} \alpha$	5, 7, Or
9.	$\beta \mathrel{\Box\!\!\rightarrow} \alpha$	9, LLE

□

Finally we give an alternative axiomatization of system **P**:

Theorem 7.25. *System* **P** *is axiomatizable as LLE+SC+CC+Or+CM. (cf. Schurz, 1998, p. 84; Schurz, 2005, p. 42; Geffner & Pearl, 1994, p. 71).*

Proof. By Definition 7.15 system **P** is **CK**+Refl+Or+CM. Due to Definition 7.8 we have that system **C** is **CK**+Refl+CM+CC. Due to Definition 4.1 **CK** is LLE +RW+AND+LT. Hence, system **P** is LLE+RW+AND+LT+Refl+Or+CM and **C** is LLE+RW+AND+LT+Refl+CM+CC. By Lemma 7.16 principle CC is derivable from LLE+RW+AND+Refl+Or. Hence, since **P** and **C** differ only in where **P** contains Or system **C** contains CC, it follows that system **P** is system **C**+Or. By Theorem 7.11 we have that system **C** is LLE+SC+CC+CM. Hence, **P** is LLE+SC+CC+CM+Or. □

Note that our axiomatization of system **P** in Theorem 7.25 differs from the axiomatization given in Schurz (1998, p. 84), Schurz (2005, p. 42) and Geffner and Pearl (1994, p. 71) insofar as we in addition include the principle LLE (see comments to Theorem 7.11).

7.2.4 System R

System **R** differs from system **P** insofar as the former, but not the latter includes the following additional principle R (see Table 5.1):

RM $(\alpha \,\Box\!\!\rightarrow \gamma) \wedge (\alpha \,\Diamond\!\!\rightarrow \beta) \rightarrow (\alpha \wedge \beta \,\Box\!\!\rightarrow \gamma)$

The formula $\alpha \,\Diamond\!\!\rightarrow \beta$ abbreviates $\neg(\alpha \,\Box\!\!\rightarrow \neg\beta)$ (see $\mathrm{Def}_{\Diamond\rightarrow}$, Section 4.2.1). Principle RM differs from principle CM, insofar as it requires $\alpha \,\Diamond\!\!\rightarrow \beta$ instead of $\alpha \,\Box\!\!\rightarrow \beta$. Let us, however, postpone the discussion of principle RM and, first, describe system **R**:

Definition 7.26. *Logic* **R** *is the smallest logic containing* **CK**+*Refl+CM+OR+RM (Lehmann & Magidor, 1992, p. 19)*

Let us give here an alternative axiomatization of system **R**:

Theorem 7.27. *System* **R** *is axiomatizable as LLE+SC+CM+CC+Or+RM. (cf. Schurz, 1998, p. 84).*

Proof. By Definition 7.26 system **R** is **CK**+Refl+Or+CM+RM. Due to Definition 7.15 we have that system **P** is **CK**+Refl+Or+CM. Hence, system **R** is system **P**+RM. Theorem 7.25 gives us that system **P** is LLE+SC+CM+CC+Or. Thus, system **R** is LLE+SC+CM+CC+Or+RM. □

Theorem 7.27 gives us that system **R** by Lehmann and Magidor (1992, p. 19) and system $\mathbf{P}^+$ by Schurz (1998, p. 84; see also Section 3.5) concur, since Schurz (1998, p. 84) effectively axiomatizes system $\mathbf{P}^+$ as LLE+SC+CM+CC+Or+RM.[2]

Let us now turn to the discussion of principle RM. The principle RM corresponds to following frame condition (see Table 5.3):

C_{RM} $\forall w(\exists w'(wR_X w' \wedge w' \in Y) \Rightarrow \forall w'(wR_{X\cap Y}w' \Rightarrow wR_X w'))$

Principle RM represents – as CM – a restricted version of principle Mon (see Section 7.2.1). In case of the corresponding frame condition C_{RM}, however, an existential claim is required to be satisfied rather than a universal one as in C_{RM}. Note that we can provide a motivation of RM analogously to CM. Consider, for that purpose the following three propositions:

[2] We included here additionally LLE, since Schurz's (1998) axiomatization seems to implicitly presuppose that this principle holds (see the discussion of this issue in Section 7.2.1).

(a4) If Fireball takes part in the race, then it will normally win.

(b4) The following is not the case: If Fireball takes part in the race, then normally Thunderhead does not takes part in the race.

(c4) Therefore: If Fireball takes part in the race and Thunderhead takes place in the race, then normally Fireball wins the race.

Again principle RM is probabilistically justified based on infinitesimal and non-infinitesimal probabilistic validity criteria (see Schurz, 2005, pp. 42–45;[3] cf. Section 7.2.1; see also Section 3.5). Moreover, analogously as in the case for CM, principle RM seems to describe rationally justified inferences (cf. Lehmann & Magidor, 1992, p. 18f). Moreover, although RM might look like a default inference, it is not, since it does neither depend on non-derivability conditions nor on consistency conditions (cf. Section 2.2). The discussion of Kraus et al. (1990, p. 198) and Lehmann and Magidor (1992, p. 18f) is somewhat misleading in that respect.

Schurz (in press, Section 8.3) suggested that RM is in fact stronger than principle CM. Principle CM is, however, not a theorem of **CK**+Refl+CC+Or+RM as the following lemma shows:

Lemma 7.28. *Given* **CK**+*Refl+CC+Or+RM the principle CM is not derivable.*

Proof. To prove this lemma, we draw on the soundness result described in Chapter 6 for Chellas frames. We know by the correspondence results from Chapter 5 that, given C_{Refl}, C_{CC}, C_{Or}, C_{RM} hold for a Chellas frame $\mathcal{F}_C = \langle W, R \rangle$, then Refl, CC, Or and RM are valid on $\mathcal{F}_C$. Moreover, all **CK** theorems are valid on $\mathcal{F}_C$, including theorems derivable in **CK** on the basis of Refl, CC, Or and RM. Hence, if for $\mathcal{F}_C$ the frame restrictions C_{Refl}, C_{CC}, C_{Or} and C_{RM} are the case, but CM is not valid in $\mathcal{F}_C$, then CM is not a theorem of **CK**+Refl+CC+Or+RM. By the correspondence results, however, C_{CM} holds for a frame $\mathcal{F}_C$ iff CM is valid on $\mathcal{F}_C$. Hence, it suffices to show that there is some Chellas frame $\mathcal{F}_C$, such that C_{Refl}, C_{CC}, C_{Or} and C_{RM} hold, but C_{CM} does not.

Let $\mathcal{F}'_C = \langle W, R \rangle$ be a Chellas frame, such that $W = \{w_1, w_2\}$. Moreover, let R be $\{\langle w_1, w_2, \{w_2\} \rangle\}$. It is easy to check that C_{Refl}, C_{CC}, C_{RM} and C_{Or} apply to

[3] The principle RM corresponds in Schurz's (2005) terminology to the principle WRM (p. 42, "Weak Rational Monotonicity").

$\mathcal{F}'_C$. However, C_{CM} does not hold for $\mathcal{F}'_C$, since $\forall w_3(w_1 R_{\{w_1,w_2\}} w_3 \Rightarrow w_3 \in \{w_2\})$, $w_1 R_{\{w_1,w_2\}\cap\{w_2\}} w_2$, but not $w_1 R_{\{w_1,w_2\}} w_2$. □

Since by Lemma 7.28 principle CM is not a theorem of **CK**+Refl+Or+CC+RM, it is, hence, needed for the axiomatization of system **R**. Note that Schurz (in press, Section 8.3) discusses principle RM in the context of system $\mathbf{P}^{+}$ (see Section 3.5), which is – as we proved in Theorem 7.27 – logically equivalent to system **R**.

7.2.5 Lewis' (1973/2001) System V

The main aim of this section is to show that D. Lewis' (1973/2001) proof-theoretic system **V** is system **R** from the preceding section (see Definition 7.26). This alternative axiomatization of D. Lewis' (1973/2001) system **V** allows us, then, to describe system **V** in terms of principles described in Chapters 5 and 6 (see Tables 5.1 and 5.1). As a result, we can account for D. Lewis' (1973/2001) system **V** purely in terms of CS-semantics. Note that system **V** is sound and complete w.r.t the class of all Lewis models, as specified in Definitions 3.5–3.7 (without centering axioms).

We, finally, prove that the principles of system **R** are valid in all Lewis models, which we discussed in Section 3.2.3 (see Definitions 3.5–3.7), and discuss translations from Lewis models into Chellas models. Both points are instructive insofar as they contribute to a better understanding of the relation of CS-semantics and the systems of spheres semantics of D. Lewis (1973/2001).

Let us now focus on the alternative axiomatization D. Lewis' (1973/2001) system **V**. D. Lewis (1973/2001) does not directly give an axiomatization for his system **V** in terms of a conditional operator, but he does so for system **VC** (p. 132). system **VC** is defined as a specific system **V** plus the both centering principles MP ("Modus Ponens") and CS ("Conjunctive Sufficiency") (see D. Lewis, 1973/2001, p. 132, p. 120f; see also Section 3.2.3). The principles MP and CS, however, correspond – as we saw in Section 3.2.3 – to the frame conditions C_{MP} and C_{CS}, respectively. Furthermore, C_{MP} and C_{CS} are conjointly equivalent to the centering condition in Lewis models (see Definitions 3.5–3.7; cf. D. Lewis, 1973/2001, p. 123, p. 120f; see also Table 5.3). Observe that system **V** is sound and complete w.r.t. the class of all Lewis models described in Definitions 3.6 and 3.7, which do not include the centering conditions centering conditions C_{MP} and C_{CS} (cf. D. Lewis, 1973/2001, p. 123, p. 120ff).

D. Lewis (1973/2001) draws for the axiomatization of system **VC** on the following principles (apart from MP and CM):[4]

LV $(\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta) \vee ((\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma) \leftrightarrow (\alpha \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma))$
LE Exchange of Logical Equivalents

Here 'LV' and 'LE' abbreviate 'D. Lewis' (1973/2001) specific axiom for system **V**' and 'Logical Equivalents'. Note that the rule LE is equivalent to both rules LLE and rule RLE conjointly (cf. D. Lewis, 1973/2001, p. 132). We repeat the rules LLE and RLE from Section 4.2.6 for mnemonic reasons:

LLE if $\vdash \alpha \leftrightarrow \beta$ and $\alpha \mathrel{\Box\!\!\rightarrow} \gamma$, then $\beta \mathrel{\Box\!\!\rightarrow} \gamma$
RLE if $\vdash \beta \leftrightarrow \gamma$ and $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ then $\alpha \mathrel{\Box\!\!\rightarrow} \gamma$

Moreover, for an easier formal treatment of system **V**, we divide LV into the following two principles by employing $\mathrm{Def}_{\Diamond\!\rightarrow}$ (see Section 4.2.1):

LV′ $(\alpha \mathrel{\Diamond\!\!\rightarrow} \beta) \wedge (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma))$
LV″ $(\alpha \mathrel{\Diamond\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma)) \rightarrow (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$.

It is easy to show that the conjunction of LV′ and LV″ is p.c.-equivalent to LV given $\mathrm{Def}_{\Diamond\!\rightarrow}$. D. Lewis, then, defines **V** indirectly (as described above) in the following way:

Definition 7.29. *Logic* **V** *is the smallest logic containing RCK+LE+Refl+MOD +LV (D. Lewis, 1973/2001, p. 132, p. 123, p. 120f).*

Since we will prove that system **V** is logically equivalent with an axiomatization, which refers solely to principles in Table 5.1 (see Theorem 7.30) – for which we identified nontrivial corresponding frame conditions (see Table 5.3) – we will not discuss non-trivial frame conditions for LV, LV′ or LV″. A further motivation not to pursue that goal stems from the fact that principles LV and LV″ seem hardly

[4]We omit here for the axiomatizations of system **V** and **VC** definitions of non-primitive operators by D. Lewis (1973/2001), since we do not employ them here.

intuitive (cf. D. Lewis, 1973/2001, p. 133) and are seldomly discussed in the conditional logic literature. Hence, we see no point for including frame restrictions for LV′ or LV″, as Segerberg (1989, Principle # 6, p. 159) does for LV″. Moreover, note that we use for the proof of the equivalence of system **R** and system **V** the full language $\mathcal{L}_{\mathrm{KL}}$. In the restricted language $\mathcal{L}_{\mathrm{KL}^-}$ (see Section 4.2.1) employed by Adams (1965, 1966, 1977) D. Lewis' (1973/2001) system of sphere-semantics is sound and complete with respect to system **P** rather than **R** (Adams, 1977, pp. 188–190), due to the limited expressiveness of $\mathcal{L}_{\mathrm{KL}^-}$ (see Section 3.5). Let us now prove the equivalence of system **VC** and **R**:

Theorem 7.30. *Logic* **V** *= Logic* **R**. *(Gärdenfors, 1979, p. 393)*

Proof. By Definition 7.26 and 7.29 system **R** and **V** are defined as **CK**+Refl+CM +Or+RM and RCK+LE+Refl+MOD+LV, respectively. By Definition 4.1 **CK** is LLE+RW+AND+LT and, hence, it holds that **R** is LLE+RW+AND+LT+Refl+CM+Or+RM. As LE is defined as LLE+RLE and LV is LV′+LV″, it follows that system **V** is RCK+LLE+RLE+Refl+MOD+LV′+LV″. Moreover, Definition 4.1 and Lemma 4.6 give us that **CK** is logically equivalent to LLE+RW+AND+LT on the one hand and is RCK+LLE on the other hand. Hence, system **V** is LLE+RW +AND+LT+RLE+Refl+MOD+LV′+LV″. Since by Lemma 4.4 principle RLE is a theorem of RW, it follows that system **V** is LLE+RW+AND+LT+Refl+MOD +LV′+LV″. Hence, it remains to be shown that (a) MOD and (b) LV′ and (c) LV″ are derivable from LLE+RW+AND+LT+Refl+CM+Or+RM and that (d) CM, (e) Or and (f) RM are theorems of LLE+RW+AND+LT+Refl+MOD+LV′+LV″. Lemma 7.24 gives us that MOD is a theorem of LLE+RW+AND+Refl+CM+Or. Hence, (a) follows. Moreover, Lemma 7.31 shows that LV′ is derivable from S. Since principle S is by Lemma 7.19 a theorem of LLE+RW+Refl+Or, we get (b). Lemma 7.32 implies that LV″ is derivable from RW+AND+Refl+RM. Hence (c) is the case. Finally, as S is a theorem of RW+Refl+LV′ by Lemma 7.34 and Or is derivable from LLE+RW+AND+S by Lemma 7.18, it follows that Or is a theorem of LLE+RW+AND+Refl+LV′. Thus (e) is the case. Since by Lemma 7.35 it holds that RM is derivable from RW+LV″, (f) is the case. Lemma 7.33 gives us that CM is derivable from RW+AND+Refl+MOD+RM. Moreover, by Lemma 7.35 it holds that RM is derivable from RW+LV″. So, CM is derivable from RW+AND+Refl+MOD+LV″. Hence, (d) obtains. □

Lemma 7.31. *S ⇒ LV′*

Proof.

1.	$\alpha \diamond\!\!\rightarrow \beta$	given
2.	$\alpha \wedge \beta \,\square\!\!\rightarrow \gamma$	given
3.	$\alpha \,\square\!\!\rightarrow (\beta \rightarrow \gamma)$	2, S

□

Lemma 7.32. *(RW+AND)+Refl+RM ⇒ LV″*

Proof.

1.	$\alpha \diamond\!\!\rightarrow \beta$	given
2.	$\alpha \,\square\!\!\rightarrow (\beta \rightarrow \gamma)$	given
3.	$\alpha \wedge \beta \,\square\!\!\rightarrow (\beta \rightarrow \gamma)$	2, 1, RM
4.	$\alpha \wedge \beta \,\square\!\!\rightarrow \alpha \wedge \beta$	Refl
5.	$\alpha \wedge \beta \,\square\!\!\rightarrow (\beta \rightarrow \gamma) \wedge (\alpha \wedge \beta)$	3, 4, AND
6.	$\alpha \wedge \beta \,\square\!\!\rightarrow \gamma$	5, RW

□

Lemma 7.33. *(RW+AND)+Refl+MOD+RM ⇒ CM*

Proof.

1.	$\alpha \,\square\!\!\rightarrow \gamma$	given
2.	$\alpha \,\square\!\!\rightarrow \beta$	given
3.	$\alpha \,\square\!\!\rightarrow \neg\beta$	ass 1, proof by cases
4.	$\quad \alpha \,\square\!\!\rightarrow \beta \wedge \neg\beta$	2, 3, AND
5.	$\quad \alpha \,\square\!\!\rightarrow \neg\alpha$	4, RW
6.	$\quad \alpha \wedge \beta \,\square\!\!\rightarrow \neg\alpha$	5, MOD
7.	$\quad \alpha \wedge \beta \,\square\!\!\rightarrow \alpha \wedge \beta$	Refl
8.	$\quad \alpha \wedge \beta \,\square\!\!\rightarrow \neg\alpha \wedge (\alpha \wedge \beta)$	6, 7, AND
9.	$\quad \alpha \wedge \beta \,\square\!\!\rightarrow \gamma$	8, RW
10.	$\neg(\alpha \,\square\!\!\rightarrow \neg\beta)$	ass 2, proof by cases
11.	$\quad \alpha \diamond\!\!\rightarrow \beta$	10, $\mathrm{Def}_{\diamond\!\rightarrow}$
12.	$\quad \alpha \wedge \beta \,\square\!\!\rightarrow \gamma$	1, 11, RM
13.	$\alpha \wedge \beta \,\square\!\!\rightarrow \gamma$	3-9,10-12, proof by cases

□

Lemma 7.34. *(RW)+Refl+LV′ ⇒ S*

Proof.

1.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$	given
2.	$\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta$	ass 1, proof by cases
3.	$\quad\alpha \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma)$	$\quad$2, RW
4.	$\neg(\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta)$	ass 2, proof by cases
5.	$\quad\alpha \mathrel{\Diamond\!\!\rightarrow} \beta$	$\quad$4, $\text{Def}_{\Diamond\!\rightarrow}$
6.	$\quad\alpha \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma)$	$\quad$5, 1, LV′
7.	$\alpha \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma)$	2-3, 4-6, proof by cases

□

Lemma 7.35. *(RW)+LV″ ⇒ RM*

Proof.

1.	$\alpha \mathrel{\Diamond\!\!\rightarrow} \beta$	given
2.	$\alpha \mathrel{\Box\!\!\rightarrow} \gamma$	given
3.	$\alpha \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma)$	2, RW
4.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$	1, 3, LV″

□

We shall now prove that the rules and axioms of system **R** hold in Lewis models (see Definitions 3.5–3.7):

Theorem 7.36. *The Rules and Axioms of system* **R** *are valid in all Lewis models.*

Proof. We have to show that the axioms and rules of system **R** hold for all worlds in all Lewis models. By Definition 7.26 system **R** is **CK**+Refl+CM+OR+RM. It is easy to prove that the rules and axioms of system **CK** (LLE+RW+LT+AND, see Definition 4.1) and Refl are true in all Lewis models. We, hence, show only that (I) CM, (II) Or and (III) RM are valid in the class of all Lewis models.

(I) We have to prove that CM, that is $(\alpha \mathrel{\Box\!\!\rightarrow} \gamma) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$, is true at an arbitrary world w in an arbitrary Lewis model $\mathcal{M}_L = \langle W, \$, V\rangle$. Suppose that (a) $\models_w^{\mathcal{M}_L} \alpha \mathrel{\Box\!\!\rightarrow} \gamma$ and (b) $\models_w^{\mathcal{M}_L} \alpha \mathrel{\Box\!\!\rightarrow} \beta$. By (a) and $V_{\Box\!\rightarrow}$ there is either (a1) no α-permitting sphere in $\$_w$ or (a2) there is an α-permitting sphere $S \in \$_w$ such that for all worlds $w' \in S$ it is the case that $\models_{w'}^{\mathcal{M}_L} \alpha \rightarrow \gamma$. In case of (a1), it follows by the definition of α-permitting spheres that there is also no $\alpha \wedge \beta$-permitting sphere in $\$_w$. Hence, we obtain trivially by $V_{\Box\!\!\rightarrow}$ that $\vDash_w^{\mathcal{M}_L} \alpha \wedge \beta \Box\!\!\rightarrow \gamma$. Suppose that (a2) is the case. By (b) it follows that either (b1) no α-permitting sphere exists in $\$_w$ or (b2) there is an α-permitting sphere $S' \in \$_w$ such that for all worlds $w' \in S'$ it is the case that $\vDash_{w'}^{\mathcal{M}_L} \alpha \rightarrow \beta$. Case (b1) is contradicted by assumption (a2). Hence, (b2) holds. Moreover, by the inclusion requirement b.ii of Definition 3.5 either (A) $S \subseteq S'$ or (B) $S' \subseteq S$. Suppose that (A) is the case. Then, since by (b2) for all worlds $w' \in S'$ it is the case that $\vDash_{w'}^{\mathcal{M}_L} \alpha \rightarrow \beta$, it follows for all worlds $w'' \in S$ that $\vDash_{w''}^{\mathcal{M}_L} \alpha \rightarrow \beta$. As by (a2) S is an α-permitting sphere, there is a world $w''' \in S$, such that $\vDash_{w'''}^{\mathcal{M}_L} \alpha$. As for all worlds $w' \in S$ it is due to (b2) the case that $\vDash_{w'}^{\mathcal{M}_L} \alpha \rightarrow \beta$, we get $\vDash_{w'''}^{\mathcal{M}_L} \alpha \wedge \beta$. Hence, S is an $\alpha \wedge \beta$-permitting sphere. By (a2) follows that that $\vDash_{w'}^{\mathcal{M}_L} \alpha \rightarrow \gamma$ for all worlds $w' \in S$. As a result, also $\vDash_{w'}^{\mathcal{M}_L} \alpha \wedge \beta \rightarrow \gamma$ is the case for all worlds $w' \in S$. Thus, since S is an $\alpha \wedge \beta$-permitting sphere, condition $V_{\Box\!\!\rightarrow}$ implies that $\vDash_w^{\mathcal{M}_L} \alpha \wedge \beta \Box\!\!\rightarrow \gamma$. Case (B) is analogous to case (A).

(II) We now show that Or, that is $(\alpha \Box\!\!\rightarrow \gamma) \wedge (\beta \Box\!\!\rightarrow \gamma) \rightarrow (\alpha \vee \beta \Box\!\!\rightarrow \gamma)$, is true at all worlds in all Lewis models. Suppose that (a) $\vDash_w^{\mathcal{M}_L} \alpha \Box\!\!\rightarrow \gamma$ and (b) $\vDash_w^{\mathcal{M}_L} \beta \Box\!\!\rightarrow \gamma$ hold for an arbitrary world w in an arbitrary model $\mathcal{M}_L = \langle W, \$, V\rangle$. By (a) and $V_{\Box\!\!\rightarrow}$ there is either (a1) no α-permitting sphere in $\$_w$ or (a2) there is an α-permitting sphere $S \in \$_w$ such that for all worlds $w' \in S$ it is the case that $\vDash_{w'}^{\mathcal{M}_L} \alpha \rightarrow \gamma$. Moreover, by (b) and $V_{\Box\!\!\rightarrow}$ there is either (b1) no β-permitting sphere in $\$_w$ or (b2) there is a β-permitting sphere $S' \in \$_w$ such that for all worlds $w' \in S'$ it is the case that $\vDash_{w'}^{\mathcal{M}_L} \beta \rightarrow \gamma$. There are four possible cases: *(i)* (a1) and (b1), *(ii)* (a1) and (b2), *(iii)* (a2) and (b1) and *(iv)* (a2) and (b2). *Case (i)*: Suppose that (a1) and (b1) are the case. Then, as there is no α-permitting sphere and no β-permitting sphere in $\$_w$, there is no $\alpha \vee \beta$-permitting sphere in $\$_w$ either. Thus, by $V_{\Box\!\!\rightarrow}$ it follows trivially that $\vDash_w^{\mathcal{M}_L} \alpha \vee \beta \Box\!\!\rightarrow \gamma$. *Case (ii)*: Let (a1) and (b2) be the case. Since by (a1) there is no α-permitting sphere in $\$_w$, it holds that for all worlds $w' \in S'$ that $\vDash_{w'}^{\mathcal{M}_L} \neg\alpha$. Hence, it follows that $\vDash_{w'}^{\mathcal{M}_L} \alpha \rightarrow \gamma$ is the case for all worlds $w' \in S'$. Moreover, by (b2) sphere S' in $\$_w$ is such for all $w' \in S'$ holds that $\vDash_{w'}^{\mathcal{M}_L} \beta \rightarrow \gamma$. It follows that for all worlds $w' \in S'$ it is the case that $\vDash_{w''}^{\mathcal{M}_L} \alpha \vee \beta \rightarrow \gamma$. Since S' is by (b2) a β-permitting sphere and any β-permitting sphere is also a $\alpha \vee \beta$-permitting sphere, we get by $V_{\Box\!\!\rightarrow}$ that $\vDash_w^{\mathcal{M}_L} \alpha \vee \beta \Box\!\!\rightarrow \gamma$. *Case (iii)*: Analogous to case (ii). *Case (iv)*: Suppose that (a2) and (b2) hold. Then, by (a2) and (b2) there are spheres S and $S' \in \$_w$, such that for all $w' \in S$

and $w'' \in S'$ it is the case that $\models_{w'}^{\mathcal{M}_L} \alpha \rightarrow \gamma$ and $\models_{w''}^{\mathcal{M}_L} \beta \rightarrow \gamma$, respectively. By the inclusion requirement b.ii of Definition 3.5 it follows that either (A) $S' \subseteq S$ or (B) $S \subseteq S'$. Suppose that (A) is the case. Then, since for all worlds $w' \in S$ it is the case that $\models_{w'}^{\mathcal{M}_L} \alpha \rightarrow \gamma$, for all world $w'' \in S'$ holds that $\models_{w''}^{\mathcal{M}_L} \alpha \rightarrow \gamma$. As by (b2) it is the case that $\models_{w''}^{\mathcal{M}_L} \beta \rightarrow \gamma$ for all worlds $w'' \in S'$, we get that $\models_{w''}^{\mathcal{M}_L} \alpha \vee \beta \rightarrow \gamma$ holds for all worlds w'' in S'. Since S' is by (b2) a β-permitting sphere, it is also an $\alpha \vee \beta$-permitting sphere. By $V_{\Box\!\!\rightarrow}$ it, hence, follows that $\models_{w}^{\mathcal{M}_L} \alpha \vee \beta \Box\!\!\rightarrow \gamma$. Case (B) is analogous to case (A).

(III) We shall now show that that principle RM, namely $(\alpha \Box\!\!\rightarrow \gamma) \wedge (\alpha \Diamond\!\!\rightarrow \beta) \rightarrow (\alpha \wedge \beta \Box\!\!\rightarrow \gamma)$ is valid in the class of all Lewis models. Suppose that (a) $\models_{w}^{\mathcal{M}_L} \alpha \Box\!\!\rightarrow \gamma$ and (b) $\models_{w}^{\mathcal{M}_L} \alpha \Diamond\!\!\rightarrow \beta$ for an arbitrary world w in an arbitrary Lewis model $\mathcal{M}_L = \langle W, \$, V \rangle$. By (a) and $V_{\Box\!\!\rightarrow}$ there is either (a1) no α-permitting sphere in $\$_w$ or (a2) there is an α-permitting sphere $S \in \$_w$ such that for all world $w' \in S$ holds that $\models_{w'}^{\mathcal{M}_L} \alpha \rightarrow \gamma$. In case of (a1), it follows by the definition of α-permitting spheres that there is also no $\alpha \wedge \beta$-permitting sphere in $\$_w$. Hence, due to $V_{\Box\!\!\rightarrow}$ this implies trivially that $\models_{w}^{\mathcal{M}_L} \alpha \wedge \beta \Box\!\!\rightarrow \gamma$. Suppose that (a2) is the case. By (b) and $\mathrm{Def}_{\Diamond\!\!\rightarrow}$ (see Section 4.2.1) it follows that $\not\models_{w}^{\mathcal{M}_L} \alpha \Box\!\!\rightarrow \neg\beta$. On the basis of Definition $V_{\Box\!\!\rightarrow}$ this implies that there is an α-permitting sphere in $\$_w$ and for all spheres $S' \in \$_w$ it holds that there is a world $w' \in S'$, such that $\not\models_{w'}^{\mathcal{M}_L} \alpha \rightarrow \beta$. Hence, each sphere $S' \in \$_w$ contains a world w', such that $\models_{w'}^{\mathcal{M}_L} \alpha \wedge \beta$. Thus, every sphere in $\$_w$ is an $\alpha \wedge \beta$-permitting sphere, including sphere S. By (a2) for all worlds $w' \in S$ it is the case that $\models_{w'}^{\mathcal{M}_L} \alpha \rightarrow \gamma$. Hence, for all $w' \in S$ it holds that $\models_{w'}^{\mathcal{M}_L} \alpha \wedge \beta \rightarrow \gamma$. Since S is an $\alpha \wedge \beta$-permitting sphere, we get by $V_{\Box\!\!\rightarrow}$ that $\models_{w}^{\mathcal{M}_L} \alpha \wedge \beta \Box\!\!\rightarrow \gamma$. □

In the proof of the preceding theorem we abbreviate truth and falsehood of a formula α at a possible world w in a Lewis model $\mathcal{M}_L = \langle W, R, V \rangle$ by $\models_{w}^{\mathcal{M}_L} \alpha$ and $\not\models_{w}^{\mathcal{M}_L} \alpha$, respectively. Note, moreover that $V_{\Box\!\!\rightarrow}$ in this proof refers to Definition 3.7 (extensions of Lewis models) rather than Definition 4.14 (extensions of Chellas models).

Let us, finally, discuss translations from Lewis models into Chellas models. We are here only interested in translations from Lewis models $\mathcal{M}_L = \langle W, \$, V \rangle$ into Chellas models $\mathcal{M}_C = \langle W, R, V \rangle$, which guarantee that for all formulas α in language $\mathcal{L}_{\mathrm{KL}}$ and all worlds $w \in W$ it is the case that $\models_{w}^{\mathcal{M}_L} \alpha$ iff $\models_{w}^{\mathcal{M}_C} \alpha$. We will call this type of translation ‘point-to-point translation’. Note that our specification of ‘point-to-point translations’ draws on the fact that the parameter V in Lewis models $\mathcal{M}_L = \langle W, \$, V\rangle$ and Chellas models $\mathcal{M}_C = \langle W, R, V\rangle$ is defined only for atomic propositions (see Definitions 3.6 and 4.13, respectively). The truth of arbitrary formulas is specified separately in so-called extension of Lewis models (see Definition 3.7) and Chellas models (see Definition 4.14).

We might suspect that we can provide a point-to-point translation from a Lewis model $\mathcal{M}_L = \langle W, \$, V\rangle$ into Chellas models $\mathcal{M}_C = \langle W, R, V\rangle$ the following way: Specify the accessibility relation R for $w, w' \in W$ and $X \subseteq W$ in such a way that wR_Xw' obtains iff $w' \in \mathrm{minS}_X$, where minS^w_X is defined as $\{w' \in W \mid \exists S \in \$_w(w' \in S \cap X \wedge \neg\exists S' \in \$_w(w' \in S' \cap X \wedge S' \subset S'))\}$. Here minS^w_X is the minimal sphere $S \in \$_w$, such that $X \cap S$ is non-empty, in case such a sphere exists. In all other cases minS^w_X is empty. We discussed in Section 3.2.3 minimal α-spheres in a systems of spheres $\$_w$ rather than minimal spheres w.r.t. propositions $X \subseteq W$. Note that a minimal α-sphere in the sense of Section 3.2.3 concurs with a minimal X-sphere (w.r.t. to w) minS^w_X in a Lewis model $\mathcal{M}_L = \langle W, \$, V\rangle$ iff $X = \|\alpha\|^{\mathcal{M}_L}$. Here the set $\|\alpha\|^{\mathcal{M}_L}$ is defined as $\{w \mid V(\alpha, w) = 1\}$ for model $\mathcal{M}_L$ (cf. Section 4.3).

There exist two main reasons why there might be no minimal α-sphere in a system of spheres $\$_w$ in a Lewis model $\mathcal{M}_L = \langle W, \$, V\rangle$. First, no α-permitting sphere might exist, viz. there is sphere S in $\$_w$, such that it contains a world w', such that α is true at world w'. Second, no minimal α-sphere might exists in a systems of spheres $\$_w$ due to the presence of infinite sequences of α-permitting spheres in $\$_w$ (see Section 3.2.3).

The infinite sequences of α-permitting spheres are, however, a problem in the point-to-point translations from Lewis models into Chellas models. The truth conditions $\mathrm{V}_{\Box\!\!\rightarrow}$ for conditional formulas $\alpha \Box\!\!\rightarrow \beta$ in Definition 4.14 guarantees that if there are no worlds w' in a Chellas model $\mathcal{M}_C = \langle W, R, V\rangle$, such that for $w \in W$ it is the case that $wR_{\|\alpha\|^{\mathcal{M}_C}}w'$, then $\alpha \Box\!\!\rightarrow \beta$ is trivially true at w. Hence, the above translation would render all conditional formulas $\alpha \Box\!\!\rightarrow \beta$ true at a world w iff there is an infinite descending chain of α-permitting spheres in $\$_w$ in the respective Lewis model $\mathcal{M}_L = \langle W, \$, V\rangle$. In a Lewis model $\mathcal{M}_L = \langle W, \$, V\rangle$, however, the presence of an infinite descending chain of α-permitting spheres in $\$_w$ for $w \in W$ does not guarantee that $\alpha \Box\!\!\rightarrow \beta$ is true at world w: In this infinite sequence of

α-permitting sphere there might be no sphere, such that $\alpha \to \beta$ is true at all worlds in that sphere. In such a case, however, the truth-condition $V_{\Box\!\!\rightarrow}$ in Definition 3.7 render the conditional $\alpha \Box\!\!\rightarrow \beta$ as false at w, while the above translation procedure would make $\alpha \Box\!\!\rightarrow \beta$ make true at w.

There remains much to say regarding the relationship of Lewis models [frames] on the one hand and Chellas models [frames] on the other hand. Due to time and space restrictions we are, however, not able pursue this issue here any further.

7.2.6 Monotonic Systems without Bridge Principles (Systems CM and M)

In this section we will discuss the two monotonic systems **CM** and **M**, which can be defined as follows:

Definition 7.37. *Logic* **CM** *is the smallest logic containing* **CK**+*Refl*+*CC*+*Mon*.

Definition 7.38. *Logic* **M** *is the smallest logic containing* **CK**+*Refl*+*Or*+*Mon*.

System **M** is the monotonic collapse without bridge principles. System **CM**, although making Mon a theorem, is not the full monotonic collapse, since Or – as can easily be seen – is not valid in Chellas frames for **CM** (cf. Kraus et al., 1990, p. 201). Note that Definitions 7.37 and 7.38 deviate from the definitions of **CM** and **M** given in Kraus et al. (1990, p. 200f) and Kraus et al. (1990, p. 202), respectively, insofar as they are more parsimonious. Kraus et al. (1990, p. 200f) characterize **CM** as all rules and axioms for system **C** (**C**=**CK**+Refl+CM+CC, see Definition 7.8) plus the following rule:

RMon if $\vdash \alpha \to \beta$ and $\beta \Box\!\!\rightarrow \gamma$, then $\alpha \Box\!\!\rightarrow \gamma$

One can easily see that – given LLE – RMon is p.c.-equivalent to Mon (see Table 5.1). Just observe that $\vdash \alpha \to \beta$ implies $\vdash \alpha \leftrightarrow (\alpha \wedge \beta)$. In addition, system **C** contains the principle CM, which can be derived trivially from Mon given the rules of **CK**.

Kraus et al. (1990, p. 202) axiomatize system **M** by system **C** (**C**=**CK**+Refl+CM+CC, see Definition 7.8) plus CP. Note, however, that Kraus et al.'s (1990, p.

202) axiomatization is redundant, since **CK**+Refl+CP imply Mon (Lemma 7.43) and Refl+Or (Lemma 7.42), which in turn imply CM and CC (Lemma 7.16), respectively. Hence, system **M** can be alternatively axiomatized as **CK**+Refl+CP. Let us now prove Theorem 7.39, which shows that our axiomatization of system **M** (**CK**+Refl+Or+Mon) and **CK**+Refl+CP are logically equivalent:

Theorem 7.39. *System* **M** *can be axiomatized by* **CK**+*Refl+CP (cf. Kraus et al., 1990, p. 202).*

Proof. By Definition 7.38 system **M** is **CK**+Refl+Or+Mon. We, hence, have to show that **CK**+Refl+CP is **CK**+Refl+Or+Mon. Lemma 7.40 gives us that Refl $\Rightarrow$ (Mon+Or $\Leftrightarrow$ CP) holds. It follows that **CK**+Refl+CP=**CK**+Refl+Mon+Or. □

Lemma 7.40. *(LLE+RW+AND)+Refl* $\Rightarrow$ *(Mon+Or* $\Leftrightarrow$ *CP)*

Proof. By Lemmata 7.41, 7.42 and 7.43. □

Lemma 7.41. *(RW+AND)+Refl+Mon+Or* $\Rightarrow$ *CP*

Proof.

1.	$\alpha \mathrel{\Box\!\!\to} \beta$	given
2.	$\alpha \wedge \neg\beta \mathrel{\Box\!\!\to} \beta$	Mon
3.	$\alpha \wedge \neg\beta \mathrel{\Box\!\!\to} \alpha \wedge \neg\beta$	Refl
4.	$\alpha \wedge \neg\beta \mathrel{\Box\!\!\to} \beta \wedge (\alpha \wedge \neg\beta)$	2, 3, AND
5.	$\alpha \wedge \neg\beta \mathrel{\Box\!\!\to} \neg\alpha$	4, RW
6.	$\neg\alpha \mathrel{\Box\!\!\to} \neg\alpha$	Refl
7.	$\neg\alpha \wedge \neg\beta \mathrel{\Box\!\!\to} \neg\alpha$	6, Mon
8.	$(\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \neg\beta) \mathrel{\Box\!\!\to} \neg\alpha$	5, 7, Or
9.	$\neg\beta \mathrel{\Box\!\!\to} \neg\alpha$	8, LLE

□

Lemma 7.42. *(LLE+RW+AND)+CP* $\Rightarrow$ *Or (Kraus et al., 1990, p. 202)*

Proof.

1.	$\alpha \mathrel{\Box\!\!\rightarrow} \gamma$	given
2.	$\beta \mathrel{\Box\!\!\rightarrow} \gamma$	given
3.	$\neg\gamma \mathrel{\Box\!\!\rightarrow} \neg\alpha$	1, CP
4.	$\neg\gamma \mathrel{\Box\!\!\rightarrow} \neg\beta$	2, CP
5.	$\neg\gamma \mathrel{\Box\!\!\rightarrow} \neg\alpha \wedge \neg\beta$	3, 4, AND
6.	$\neg(\neg\alpha \wedge \neg\beta) \mathrel{\Box\!\!\rightarrow} \neg\neg\gamma$	5, CP
7.	$\alpha \vee \beta \mathrel{\Box\!\!\rightarrow} \neg\neg\gamma$	6, LLE
8.	$\alpha \vee \beta \mathrel{\Box\!\!\rightarrow} \gamma$	7, RW

□

Lemma 7.43. *(LLE+RW)+CP ⇒ Mon (Kraus et al., 1990, p. 180f)*

Proof.

1.	$\alpha \mathrel{\Box\!\!\rightarrow} \gamma$	given
2.	$\neg\gamma \mathrel{\Box\!\!\rightarrow} \neg\alpha$	1, CP
3.	$\neg\gamma \mathrel{\Box\!\!\rightarrow} \neg\alpha \vee \neg\beta$	2, RW
4.	$\neg(\neg\alpha \vee \neg\beta) \mathrel{\Box\!\!\rightarrow} \neg\neg\gamma$	3, CP
5.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \neg\neg\gamma$	4, LLE
6.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$	5, RW

□

Model Theory

Let us now turn to a semantic characterization of system **M**. This system can, as Theorem 7.44 shows, be described by following frame restriction:

$$C_{\mathrm{M}} \quad \forall w, w'(wR_X w' \Leftrightarrow \exists Y(wR_Y w') \wedge w' \in X)$$

Frame restriction C_{M} gives, applied to semantically representable sets, the following intuitive reading: If a world w can see a world w' in a Chellas Model $\mathcal{M}_C = \langle W, R, V\rangle$ by R_Y with Y being a subset of W, then for all formulas α holds: α is true at w' (i.e. $w' \in \|\alpha\|$) if and only if it is the case that $wR_{\|\alpha\|}w'$. Let us now turn to Theorem 7.44:

Theorem 7.44. *The class of Chellas frames for system* **M** *can be characterized by the frame condition* C_{M}.

Proof. By Definition 7.38, logic **M** is **CK**+Refl+Mon+Or. Hence, all theorems of **M** are valid on a Chellas frame $\mathcal{F}_C = \langle W, R \rangle$ iff C_{Refl}, C_{Mon} and C_{Or} hold for $\mathcal{F}_C$. We, hence, have to show that C_{Refl} C_{Mon} and C_{Or} imply C_{M}, and vice versa. Let us, first, focus on (1) C_{Refl}+C_{Mon}+C_{Or} $\Rightarrow$ C_{M} and, then, show (2) C_{M} $\Rightarrow$ C_{Refl}+C_{Mon}+C_{Or}. We will abbreviate the left-to-right and the right-to-left direction of C_{M} by $C_{\text{M}}^{\Rightarrow}$ and $C_{\text{M}}^{\Leftarrow}$, respectively.

(1) C_{Refl}+C_{Mon}+C_{Or} $\Rightarrow C_{\text{M}}^{\Rightarrow}$: Suppose that worlds w, w' are arbitrary worlds in a Chellas frame $\mathcal{F}_C = \langle W, R \rangle$ and X is an arbitrary subset of W, such that wR_Xw'. It trivially holds that there is a subset Y of W, such that wR_Yw', namely X. Moreover, by C_{Refl} follows that $w' \in X$.

C_{Refl}+C_{Mon}+C_{Or} $\Rightarrow C_{\text{M}}^{\Leftarrow}$: Suppose that worlds w, w' are arbitrary worlds in a Chellas frame $\mathcal{F}_C = \langle W, R \rangle$ and let X, Y be arbitrary subsets of W, such that wR_Yw' and $w' \in X$. By C_{Or} follows that either $wR_{X\cap Y}w'$ or $wR_{-X\cap Y}w'$. However, $wR_{-X\cap Y}w'$ cannot be the case, since otherwise C_{Refl} implies that $w' \in -X \cap Y$, contradicting $w' \in X$. Hence, $wR_{X\cap Y}w'$ is the case. From this observation follows by C_{Mon} that wR_Xw'.

(2) $C_{\text{M}} \Rightarrow C_{\text{Refl}}$: Suppose that worlds w, w' are arbitrary worlds in a Chellas frame $\mathcal{F}_C = \langle W, R \rangle$ and let X be an arbitrary subset of W, such that wR_Xw'. Then, by C_{M}, it follows that $\exists Y(wR_Yw') \wedge w' \in X$. Hence, $w' \in X$ is the case.

$C_{\text{M}} \Rightarrow C_{\text{Mon}}$: Suppose that worlds w, w' are arbitrary worlds in a Chellas frame $\mathcal{F}_C = \langle W, R \rangle$ and let X, Y be arbitrary subsets of W, such that $wR_{X\cap Y}w'$. By C_{M} follows that $w' \in X \cap Y$. So $w' \in X$. Since there is a subset Z, namely $Z = X \cap Y$, such that wR_Zw', this implies by C_{M} that wR_Xw'.

$C_{\text{M}} \Rightarrow C_{\text{Or}}$: Suppose that w, w' are arbitrary worlds in a Chellas frame $\mathcal{F}_C = \langle W, R \rangle$ and let X, Y be arbitrary subsets of W, such $wR_{X\cup Y}w'$. By C_{M} it follows that $w' \in X \cup Y$. So, either $w' \in X$ or $w' \in Y$ is the case. Since $\exists Z(wR_Zw')$, namely for $Z = X \cup Y$, C_{M} implies that either wR_Xw' or wR_Yw'. □

Let us now focus on the following principle CEM ("Conditional Excluded Middle", see Table 5.1):

CEM $\quad (\alpha \mathrel{\Box\!\!\rightarrow} \beta) \vee (\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta)$

CEM is the characteristic principle of Stalnaker's system (see Sections 3.2.5 and

7.3.3). It is, however, surprising that it is not derivable in system **M**, since it is not a bridge principle. Let us for the sake of perspicuity repeat the nontrivial frame condition of CEM (see Table 5.3) here:

$C_{\text{CEM}} \quad \forall w, w', w''(wR_X w' \wedge wR_X w'' \Rightarrow w'' = w')$

The frame condition C_{CEM} requires that each accessibility relation R_X in a Chellas frame $\mathcal{F}_C = \langle W, R \rangle$ is functional in the sense that any world $w \in W$ can access at most one world $w' \in W$ via R_X. In an ordinary Chellas frame $\mathcal{F}_C = \langle W, R \rangle$, however, R_X need not be unique, i.e. there may exist worlds w, w' and w'' and a subset of possible worlds X of W, such that $wR_X w'$, $wR_X w''$, but $w'' \neq w'$. Lemma 7.45 gives us the corresponding formal result.

Lemma 7.45. *In system* **M** *the principle CEM is not derivable.*

Proof. To prove Lemma 7.45, we have to construct a Chellas frame $\mathcal{F}_C = \langle W, R \rangle$, such that C_{M}, but not C_{CEM} hold (cf. proof of Lemma 7.28).

Let $\mathcal{F}'_C = \langle W, R \rangle$ be a Chellas frame, such that $W = \{w_1, w_2, w_3\}$. Moreover, let R be $\{\langle w_1, w_2, \{w_2\}\rangle, \langle w_1, w_3, \{w_3\}\rangle, \langle w_1, w_2, \{w_2, w_3\}\rangle, \langle w_1, w_3, \{w_2, w_3\}\rangle, \langle w_1, w_2, \{w_1, w_2, w_3\}\rangle, \langle w_1, w_3, \{w_1, w_2, w_3\}\rangle, \}$. It is easy to check that C_{M} holds for $\mathcal{F}'_C$. However, C_{CEM} does not hold for $\mathcal{F}'_C$, since $wR_{\{w_2,w_3\}}w_2$ and $wR_{\{w_2,w_3\}}w_3$, but $w_2 \neq w_3$. □

This implies also that CEM is not valid in system **M**. Note, however, that CEM is derivable in system **MC** (see Lemma 7.81), as described by Definition 7.74.

Proof Theory

Let us now turn to the proof-theoretic discussion of systems **M** and **CM**. We already noted that both systems **M** and **CM** are monotonic, in the sense that Mon is a theorem of both systems. Note, however, that only **M** represents the monotonic collapse without bridge principles. Since we identify monotonic systems with systems, in which Mon is derivable, we can construct weaker monotonic systems in that sense, for example **CK**+Mon and **CK**+Refl+Mon.

For the proof-theoretic discussion of system **M** and **CM** we focus on principles Trans (also called principle S3′ in Chapter 1) and Cut (see Table 5.1). We encountered Trans and Cut in Section 7.2.1, but repeat both principles for the sake of perspicuity (see Table 5.1):

Trans $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\beta \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \gamma)$
Cut $(\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \delta) \wedge (\gamma \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \wedge \gamma \mathrel{\Box\!\!\rightarrow} \delta)$

Note that Cut does not correspond to "Cut" in Kraus et al. (1990, p. 177) and Lehmann and Magidor (1992, p. 6), but corresponds to our notion of Cautious Cut, denoted by 'CC'. Our notion of Cut, however, translates into principle (9) in Lehmann and Magidor (1992, p. 6).

In Table 5.1 we classified the principles Cut, Mon, Trans and CP under the label 'Monotonic Systems'. We, however, justified that terminology only for Mon (which is trivially derivable in a monotonic system) and CP, which implies Mon (see Lemma 7.43) given system **CK**. It, hence, remains to be shown that Cut and Trans are in the above sense monotonic, namely that Mon is derivable from Trans on the one hand and CP on the other hand. We do so in Lemma 7.49 and Lemma 7.54, respectively. Note, however, that the proofs for Lemma 7.49 and Lemma 7.54 draw on **CKR** (= **CK**+Refl) rather than only on **CK**. As, however, Refl is regarded as a centerpiece for almost all conditional logics (see Section 7.1.1), one should arguably categorize both principles Trans and Cut also as monotonic principles. We can, however, even prove something stronger based on Lemmata 7.49 and 7.54, namely that system **CM** can alternatively be axiomatized as **CK**+Refl+Trans (see Theorem 7.46) and **CK**+Refl+Cut (see Theorem 7.50).

Theorem 7.46. *System* **CM** *can be axiomatized by* **CK**+*Refl+Trans*.

Proof. By Definition 7.37 holds that **CM**=**CK**+Refl+CC+Mon. Lemma 7.47 establishes, then, that Mon and Trans are derivable from each other given **CK**+Refl+CC. □

Lemma 7.47. *(RW)+Refl+CC $\Rightarrow$ (Mon $\Leftrightarrow$ Trans)*

Proof. By Lemmata 7.48 and 7.49. □

Lemma 7.48. *Mon+CC $\Rightarrow$ Trans*

Proof.

1.	$\alpha \mathrel{\Box\!\!\rightarrow} \beta$	given
2.	$\beta \mathrel{\Box\!\!\rightarrow} \gamma$	given
3.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$	2, Mon
4.	$\alpha \mathrel{\Box\!\!\rightarrow} \gamma$	3, 1, CC

□

Lemma 7.49. *(RW)+Refl+Trans ⇒ Mon*

Proof.

1.	$\alpha \mathrel{\Box\!\!\rightarrow} \gamma$	given
2.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \alpha \wedge \beta$	Refl
3.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \alpha$	2, RW
4.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$	1, 3, Trans

□

Let us now focus on the second alternative axiomatization of system **CM**:

Theorem 7.50. *System* **CM** *can be axiomatized by* **CK**+*Refl*+*Cut*.

Proof. By Definition 7.37 holds that **CM**=**CK**+Refl+CC+Mon. Lemma 7.51 gives us that Mon and CC on the one hand and Cut on the other hand are derivable given **CK**+Refl. □

Lemma 7.51. *(LLE+RW)+Refl ⇒ (CC+Mon ⇔ Cut)*

Proof. By Lemmata 7.52 through 7.54. □

Lemma 7.52. *(LLE)+CC+Mon ⇒ Cut*

Proof.

1.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \delta$	given
2.	$\gamma \mathrel{\Box\!\!\rightarrow} \beta$	given
3.	$(\alpha \wedge \beta) \wedge \gamma \mathrel{\Box\!\!\rightarrow} \delta$	1, Mon
4.	$(\alpha \wedge \gamma) \wedge \beta \mathrel{\Box\!\!\rightarrow} \delta$	3, LLE
5.	$\gamma \wedge \alpha \mathrel{\Box\!\!\rightarrow} \beta$	2, Mon
6.	$\alpha \wedge \gamma \mathrel{\Box\!\!\rightarrow} \beta$	5, LLE
7.	$\alpha \wedge \gamma \mathrel{\Box\!\!\rightarrow} \delta$	4, 6, CC

□

Lemma 7.53. *(LLE)+Cut ⇒ CC*

Proof.

1.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$	given
2.	$\alpha \mathrel{\Box\!\!\rightarrow} \beta$	given
3.	$\alpha \wedge \alpha \mathrel{\Box\!\!\rightarrow} \gamma$	1, 2, Cut
4.	$\alpha \mathrel{\Box\!\!\rightarrow} \gamma$	3, LLE

□

Lemma 7.54. *(LLE+RW)+Refl+Cut ⇒ Mon (cf. Lehmann & Magidor, 1992, p. 6)*

Proof.

1.	$\alpha \mathrel{\Box\!\!\rightarrow} \gamma$	given
2.	$\alpha \wedge \top \mathrel{\Box\!\!\rightarrow} \gamma$	1, LLE
3.	$\beta \mathrel{\Box\!\!\rightarrow} \beta$	Refl
4.	$\beta \mathrel{\Box\!\!\rightarrow} \top$	3, RW
5.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$	2, 4, Cut

□

7.3 Conditional Logics with Bridge Principles

This section focuses on systems of conditional logics with bridge principles in the sense of Chapter 4.2.1. Our informal discussion focuses on the principles MP, CS, Cond, Det, Cond, VEQ and EFQ from Table 5.1. In addition, we give an account of Adams' (1965, 1966, 1977) original system $\mathbf{P}^*$ (see Section 3.5.1, Lewis' (1973/2001) system **VC** (see Section 3.2.3) and Stalnaker's system **S** (1968; Stalnaker & Thomason, 1970; see also Section 3.2.2) in terms of CS-semantics. We furthermore, provide a frame restriction, which characterizes the monotonic collapse in CS-semantics with bridge principles, called 'system **MC**'.

7.3.1 Adams' (1965, 1966, 1975) Original System $\mathbf{P}^*$

We introduced the principles MP ("Modus Ponens"), CS ("Conditional Sufficiency"), Cond ("Conditionalization"), Det ("Detachment") and TR ("Total Reflexivity") in Section 3.3 (see also Table 5.1). In this section we will focus on the principles MP, CS, Det and Cond. Let us for mnemonic reasons state those principles again:

MP	$(\alpha \Box\!\!\rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)$
CS	$\alpha \wedge \beta \rightarrow (\alpha \Box\!\!\rightarrow \beta)$
Det	$(\top \Box\!\!\rightarrow \alpha) \rightarrow \alpha$
Cond	$\alpha \rightarrow (\top \Box\!\!\rightarrow \alpha)$

The principles MP, CS, Det and Cond are bridge principles in the sense of our definition of bridge principles in Section 4.2.1. Bridge principles presuppose a fixed relationship between conditional and non-conditional facts (see Section 3.3). Nontrivial frame conditions corresponding to the principles MP, CS, Det and Cond can be found in Table 5.3.

Note also that the principles Det and Cond are not as weak as they might seem at first glance. In fact, Det and Cond make – given the rules and axioms of system **P** of Kraus et al. (1990; see Section 7.2.3) – the bridge principles MP and CS valid, respectively (see Theorem 7.61). Let us, however, first prove Lemmata 7.55 and 7.58:

Lemma 7.55. *$S \Rightarrow (Det \Leftrightarrow MP)$*

Proof. By Lemmata 7.56 and 7.57. □

Lemma 7.56. *(LLE)+S+Det ⇒ MP (Segerberg, 1989, Theorem 1.2.x; cf. Adams, 1998, p. 157f)*

Proof.

1.	$\alpha \Box\!\!\rightarrow \beta$	given
2.	$\top \wedge \alpha \Box\!\!\rightarrow \beta$	1, LLE
3.	$\top \Box\!\!\rightarrow (\alpha \rightarrow \beta)$	2, S
4.	$\alpha \rightarrow \beta$	3, Det

□

Lemma 7.57. *$MP \Rightarrow Det$*

Proof.

1.	$\top \,\Box\!\!\rightarrow \alpha$	given
2.	$(\top \,\Box\!\!\rightarrow \alpha) \rightarrow (\top \rightarrow \alpha)$	MP
3.	α	2, 1, prop

□

Now we show that the principles Cond and CS are logically equivalent given **CK** +CM:

Lemma 7.58. *(LLE+RW)+CM ⇒ (Cond ⇔ CS)*

Proof. By Lemmata 7.59 and 7.60. □

Lemma 7.59. *(LLE+RW)+CM+Cond ⇒ CS (cf. Adams, 1998, p. 157)*

Proof.

1.	$\alpha \wedge \beta$	given
2.	$\top \,\Box\!\!\rightarrow \alpha \wedge \beta$	1, Cond
3.	$\top \,\Box\!\!\rightarrow \alpha$	2, RW
4.	$\top \wedge \alpha \,\Box\!\!\rightarrow \alpha \wedge \beta$	2, 3, CM
5.	$\alpha \,\Box\!\!\rightarrow \alpha \wedge \beta$	4, LLE
6.	$\alpha \,\Box\!\!\rightarrow \beta$	5, RW

□

Lemma 7.60. *CS ⇒ Cond*

Proof.

1.	α	given
2.	$\top \wedge \alpha$	1, prop
3.	$\top \,\Box\!\!\rightarrow \alpha$	2, CS

□

Lemmata 7.55 and 7.58 give us that Det and MP on the one hand and Cond and CS on the other hand are logically equivalent given the rules of system **P**. We, hence, state the following theorem:

Theorem 7.61. *Given system* **P** *(a) Det and MP and (b) Cond and CS are equivalent.*

Proof. By Definition 7.15 it is the case that **P** = **CK**+Refl+Or+CM. Since **CK** = LLE+RW+AND+LT and by Lemma 7.17 AND and Or imply S, Lemma 7.55 gives us that (a) holds. Moreover, by Definition 7.15 and Lemma 7.58 it follows that (b). □

Let us now connect the present discussion to our earlier discussion of indicative and counterfactual conditionals in Chapters 1 and 3: We argued in Section 3.3 that the bridge principles MP and CS are not warranted for a logic of indicative conditional logics. We saw in particular that principle MP is problematic, although it seems to be quite intuitive prima facie. One reason for rejecting MP is the observation that MP does not allow us to describe normic conditionals, such as 'Fishes are normally cold-blooded' adequately (see Sections 1.2.2 and 3.3). Note in this context that the principles MP and E are equivalent to Det and Cond on the basis of the rules and axioms of system **P** respectively (see Theorem 7.61). Hence, our arguments from Section 3.3 imply that extensions of system **P**, which make Det or Cond valid (such as system **P***, see below), should be not regarded as an appropriate logic for indicative conditionals.

But how about counterfactual conditionals? Let us, first, repeat the criteria for counterfactual conditionals discussed in Sections 2.1.4 and 3.3. We saw in these sections that there are at least two approaches: In the first approach (approach A) the difference between indicative and counterfactual conditionals is located in the mood of the conditional sentence: The presence of the subjunctive mood (and the absence of the indicative mood) indicate(s) that a conditional is a counterfactual. In the second approach (approach B) – which we endorse here – counterfactuals are conceptualized as being "counter to the facts". W.r.t. approach (B) we can further distinguish between between (B1) "genuine" counterfactuals and (B2) tentative counterfactuals. Counterfactuals of first type are conditionals whose antecedent is false, whereas counterfactuals of type (B2) presuppose that their antecedents are merely improbable (see Section 2.1.4).

We saw in Section 3.3 that given an interpretation of counterfactuals in terms of (B1) principles MP and CS are warranted. It is, however, not clear that the interpretation of counterfactuals in terms of (B2) makes MP, CS valid principles (see Section 3.3). Observe that our arguments extend again to principles Det and

Cond for systems, which endorse the rules and axioms of system **P**.

Let us now focus on the axiomatization of Adam's (1965, 1966, 1977) original system **P**. Adams' system differs from Kraus et al.'s (1990) system insofar as it contains in addition the bridge principle PC2 (see Adams, 1966, p. 277; see also Adams, 1965, p. 189), which is p.c.-equivalent to the conjunction of Det and Cond. Since Kraus et al. (1990) do not employ bridge principles in their axiomatization of their system **P**, arguably both version of system **P** differ. We will, henceforth call Adams' original system 'system $\mathbf{P}^*$'. Adams' (1965, 1966, 1977) axiomatization of system $\mathbf{P}^*$, then, draws in addition on the following two axioms:

ED $\quad (\alpha \vee \beta \,\Box\!\!\rightarrow \gamma) \wedge (\beta \,\Box\!\!\rightarrow \neg\gamma) \rightarrow (\alpha \,\Box\!\!\rightarrow \gamma)$
RW′ $\quad (\alpha \,\Box\!\!\rightarrow \beta \wedge \gamma) \rightarrow (\alpha \,\Box\!\!\rightarrow \beta)$

'ED' stands for 'elimination of disjunctions' and RW′ represents a variation of Right Weakening. Note that RW′ is p.c. equivalent – as one can easily prove – to principle CW ("consequence weakening") described in Section 4.2.6. Let us now turn to Adams' axiomatization of system $\mathbf{P}^*$. Adams (1965, 1966, 1977) defines system $\mathbf{P}^*$, then, in the following way:

Definition 7.62. *Adams' (1965, 1966, 1977) probabilistic conditional logic* $\mathbf{P}^*$ *is the smallest logic containing LLE+RW′+AND+ED+SC+CC+Or+Det+Cond (Adams, 1965, p. 189; Adams, 1966, p. 277).*

We will show that $\mathbf{P}^*$ is in fact system **P** plus the principles MP and CS (Theorem 7.67). In order to make the proof more perspicuous we shall first prove that $\mathbf{P}^*$ without the bridge principles Det and Cond is logically equivalent to **P** (Theorem 7.63). In addition, Theorem 7.63 shows that system **P** of Kraus et al. (1990) and Lehmann and Magidor (1992) is system $\mathbf{P}^*$ without bridge principles. Let us now focus on Theorem 7.63:

Theorem 7.63. *System* **P** *(Kraus et al., 1990; Lehmann & Magidor, 1992) is LLE+RW′+AND+ED+SC+CC+Or*

Proof. Definition 7.15 gives us that system **P** is **CK**+Refl+Or+CM. Since **CK** is by Definition 4.1 LLE+RW+AND+LT and LT is implied by RW+Refl (see

Lemma 7.3), it follows that **P** is LLE+RW+AND+Refl+Or+CM. Hence, we have to show that (a) LLE, (b) RW′, (c) AND, (d) ED, (e) SC, (f) CC and (g) Or are theorems of LLE+RW+AND+Refl+Or+CM and that (h) LLE, (i) RW, (j) AND, (k) Refl, (l) Or and (m) CM are theorems of LLE+RW′+AND+ED+SC+CC+Or. Points (a), (c), (g), (h), (j) and (l) are trivial. Lemma 7.64 and 7.6 give us that RW′ is derivable from RW and that SC is a theorem of LLE+RW+Refl. Hence, (b) and (e) hold. Moreover, Refl is a theorem of SC by Lemma 7.7. Thus, (k) is the case.

It remains to be shown that (d) ED, (e) SC and (f) CC are theorems of LLE+RW+AND+Refl+Or+CM and that (i) RW and (m) CM are theorems of LLE+RW′+AND+ED+SC+CC+Or. Since Lemma 7.65 shows that ED is a theorem of LLE+RW+AND+Refl+Or+CM, (d) follows. As Lemma 7.6 implies that LLE+RW+Refl imply SC, (e) is the case. Lemma 7.16 gives us that CC is a theorem of LLE+RW+AND+Refl+Or. Hence, (f) holds. Furthermore, by Lemma 7.12 RW is derivable from SC+CC. Thus (i) holds. Finally, Lemma 7.66 gives us that CM is derivable from LLE+RW+AND+Refl+ED. Since by Lemma 7.12 and 7.7 RW is a theorem of SC+CC and Refl is a theorem of SC, it follows that CM is derivable from LLE+AND+ED+SC+CC. Hence, (m) holds. □

Lemma 7.64. *RW ⇒ RW′*

Proof.

1.	$\alpha \mathrel{\Box\!\!\rightarrow} \beta \wedge \gamma$	given
2.	$\alpha \mathrel{\Box\!\!\rightarrow} \beta$	1, RW

□

Lemma 7.65. *(LLE+RW+AND)+Refl+CM+Or ⇒ ED*

Proof.

1.	$\alpha \vee \beta \mathrel{\Box\!\!\rightarrow} \gamma$	given
2.	$\beta \mathrel{\Box\!\!\rightarrow} \neg\gamma$	given
3.	$\alpha \mathrel{\Box\!\!\rightarrow} \alpha$	Refl
4.	$\alpha \mathrel{\Box\!\!\rightarrow} \neg\gamma \vee \alpha$	3, RW
5.	$\beta \mathrel{\Box\!\!\rightarrow} \neg\gamma \vee \alpha$	2, RW
6.	$\alpha \vee \beta \mathrel{\Box\!\!\rightarrow} \neg\gamma \vee \alpha$	4, 5, Or
7.	$\alpha \vee \beta \mathrel{\Box\!\!\rightarrow} \gamma \wedge (\neg\gamma \vee \alpha)$	1, 6, AND
8.	$\alpha \vee \beta \mathrel{\Box\!\!\rightarrow} \alpha$	7, RW
9.	$(\alpha \vee \beta) \wedge \alpha \mathrel{\Box\!\!\rightarrow} \gamma$	1, 8, CM
10.	$\alpha \mathrel{\Box\!\!\rightarrow} \gamma$	9, LLE

□

Lemma 7.66. *(LLE+RW+AND)+Refl+ED ⇒ CM*

Proof.

1.	$\alpha \mathrel{\Box\!\!\rightarrow} \gamma$	given
2.	$\alpha \mathrel{\Box\!\!\rightarrow} \beta$	given
3.	$\alpha \mathrel{\Box\!\!\rightarrow} \beta \wedge \gamma$	2, 1, AND
4.	$(\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta) \mathrel{\Box\!\!\rightarrow} \beta \wedge \gamma$	3, LLE
5.	$\alpha \wedge \neg\beta \mathrel{\Box\!\!\rightarrow} \alpha \wedge \neg\beta$	Refl
6.	$\alpha \wedge \neg\beta \mathrel{\Box\!\!\rightarrow} \neg(\beta \wedge \gamma)$	5, RW
7.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \beta \wedge \gamma$	4, 6, ED
8.	$\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$	7, RW

□

Theorem 7.67. *System* **P*** *(Adams, 1965, 1966, 1977) is* **P**+*MP+CS, where system* **P** *is defined as in Definition 7.2.3.*

Proof. By Definition 7.62 **P*** is LLE+RW′+AND+ED+SC+CC+Or+Det+Cond. Theorem 7.63 give us that **P** is LLE+RW′+AND+ED+SC+CC+Or. Thus, **P*** is **P**+Det+Cond. Theorem 7.61 implies that Det and MP on the one hand and Cond and CS are logically equivalent given the rules and axioms of system **P**. Hence, it follows that **P*** is **P**+MP+CS. □

7.3.2 Lewis' (1973) System VC

In Section 3.2.3 we discussed D. Lewis' (1973/2001) systems of spheres semantics (see Definitions 3.5–3.7). Apart from this semantics description, D. Lewis (1973/ 2001) also provides a proof-theoretic characterization, system **VC**, of his systems of spheres. This proof-theoretic system **VC** is sound and complete w.r.t. to Lewis models as described by Definitions 3.5–3.7 with the centering conditions described in Section 3.2.3 (D. Lewis, 1973/2001). In this section we will not describe Lewis soundness and completeness results, but instead provide an alternative characterization of D. Lewis' (1973/2001) proof-theoretic system **VC** in terms of CS-semantics. Note that D. Lewis' (1973/2001) interprets his conditional system **VC** primarily as a system for counterfactual conditionals. The objective interpretation of CS-semantics described in Section 7.1.1 can, then, serve as an alternative interpretation to D. Lewis' ranking of possible worlds according to their similarity (cf. Section 3.2.3). Let us now characterize D. Lewis' (1973/2001) system **VC**:

Definition 7.68. *D. Lewis' (1973/2001) counterfactual system* **VC** *is the smallest logic containing V+MP+CS (D. Lewis, 1973/2001, p. 132).*

Theorem 7.69. **VC** = **R**+*MP*+*CS*

Proof. By Theorem 7.30 holds that **V** = **R**. □

Theorem 7.69 holds if we employ the full language $\mathcal{L}_{\mathrm{KL}}$. However, if we restrict our language to $\mathcal{L}_{\mathrm{KL}^-}$ (see Section 4.2.1), D. Lewis' (1973/2001) system **VC** becomes logically equivalent to system **P*** (see Adams, 1977, pp. 188–190).

7.3.3 Stalnaker and Thomason's System S

We will discuss in this section the conditional logic system of Stalnaker and Thomason (Stalnaker, 1968; Stalnaker & Thomason, 1970). We described the semantic notions of Stalnaker and Thomason's system, namely Stalnaker models (see Definition 3.1), in Section 3.2.2. In Sections 3.2.2–3.2.6 we, furthermore, discussed the intuitions underlying Stalnaker and Thomason's conditional logic and its relation to the proposed consistency criterion in the Ramsey-test (cf. Sections 3.2.1 and 3.2.2).

In this section we will focus on the proof-theoretic side of Stalnaker and Thomason's conditional logic, which we will call 'system **S**'. We provide, then, an alternative axiomatization in terms of system **CK** (see Definition 4.1) plus principles from Tables 5.1 and 5.2. By these means, we can account for Stalnaker and Thomason (1970)'s system in terms of CS-semantics.

Our description of system **S** does, however, not directly refer to Stalnaker (1968) nor to Stalnaker and Thomason (1970), but draws on D. Lewis' (1973/2001, p. 132f) proof-theoretic axiomatization of system **S**. This is due to the fact that the axiomatization of system **S** by D. Lewis (1973/2001) is more perspicuous than the axiomatizations by Stalnaker (1968) and Stalnaker and Thomason (1970). We will, moreover – unlike Stalnaker and Thomason (1970) – only focus on the propositional part of system **S**. Let us now describe the characteristic principle of system **S**, namely CEM (see also Section 7.2.6 and Table 5.1):

CEM $\quad (\alpha \mathrel{\Box\!\!\rightarrow} \beta) \vee (\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta)$

'CEM' stands for 'Conditional Excluded Middle'. cf. D. Lewis (1973/2001, p. 79 and p. 133). D. Lewis (1973/2001), then, defines system **S** as follows:

Definition 7.70. *Stalnaker's (1968) and Stalnaker and Thomason's (1970) indicative/counterfactual logic system* **S** *is system* **V**+*MP+CEM (D. Lewis, 1973/2001, p. 132f).*

Theorem 7.71. **S** = **P**+*MP+CEM*

Proof. By Definition 7.70 system **S** is **V**+MP+CEM. Theorem 7.30 gives us that **V** = **R**. Hence, it follows on the basis of Definition 7.26 that system **S** = **CK**+Refl+Or+CM+RM+MP+CEM. Lemmata 7.72 and 7.81 imply that **S** is **CK** +Refl+Or+CM+MP+CEM. Thus, since by Definition 7.15 system **P** is **CK**+ Refl+Or+CM, it follows that **S** = **P**+MP+CEM. □

We show now the remaining lemmata, on which Theorem 7.71 draws:

Lemma 7.72. *MP+CEM* ⇒ *CS*

Proof.

1.	$\alpha \wedge \beta$	given
2.	$\neg(\alpha \rightarrow \neg\beta)$	1, prop
3.	$\neg(\alpha \,\Box\!\!\rightarrow \neg\beta)$	2, MP
4.	$\alpha \,\Box\!\!\rightarrow \beta$	3, CEM

$\square$

Lemma 7.73. *CM+CEM ⇒ RM*

Proof.

1.	$\alpha \,\Box\!\!\rightarrow \gamma$	given
2.	$\alpha \,\Diamond\!\!\rightarrow \beta$	given
3.	$\neg(\alpha \,\Box\!\!\rightarrow \neg\beta)$	2, $\mathrm{Def}_{\Diamond\!\rightarrow}$
4.	$\alpha \,\Box\!\!\rightarrow \beta$	3, CEM
5.	$\alpha \wedge \beta \,\Box\!\!\rightarrow \gamma$	1,4, CM

$\square$

7.3.4 The Material Collapse System MC

Let us first define the material collapse system **MC**:

Definition 7.74. *The material collapse system* **MC** *is the smallest logic containing* **CK**+*MP*+*VEQ*+*EFQ*.

In the following we focus first on a model-theoretic characterization of system **MC** and, then, discuss the proof theoretic basis of **MC**.

Model Theory

In this section we prove Theorem 7.75 and draw for that purpose on the following frame restriction:

$C_{\mathrm{MC}} \quad \forall w, w'(wR_X w' \Leftrightarrow w' = w \wedge w \in X)$

Frame restriction C_{MC} gives, applied to semantically representable sets, the following intuitive reading: If a formula α is true at a world w in a Chellas model, then w sees only itself via accessibility relation $R_{\|\alpha\|}$. If α is false, it cannot see any world via $R_{\|\alpha\|}$. Let us, however, now turn to Lemma 7.75:

Theorem 7.75. *The class of Chellas frames for system* **MC** *can be characterized by the frame condition* C_{MC}.

Proof. By Definition 7.74, logic **MC** is **CK** plus principles MP, VEQ and EFQ. Hence, all theorems of **MC** are valid on a Chellas frame $\mathcal{F}_C = \langle W, R\rangle$ iff C_{MP}, C_{VEQ} and C_{EFQ} hold for $\mathcal{F}_C$. We have, hence, to show that C_{MP}, C_{VEQ} and C_{EFQ} imply C_{MC}, and vice versa. Let us first, focus on (1) C_{MP}+C_{VEQ}+$C_{\text{EFQ}} \Rightarrow C_{\text{MC}}$ and, then, show (2) $C_{\text{MC}} \Rightarrow C_{\text{MP}}$+$C_{\text{VEQ}}$+$C_{\text{EFQ}}$. We abbreviate the left-to-right and the right-to-left direction of C_{MC} by $C^{\Rightarrow}_{\text{MC}}$ and $C^{\Leftarrow}_{\text{MC}}$, respectively.

(1) C_{MP}+C_{VEQ}+$C_{\text{EFQ}} \Rightarrow C^{\Rightarrow}_{\text{MC}}$: Let w, w' be arbitrary worlds in a Chellas frame $\mathcal{F}_C = \langle W, R\rangle$ and let X be an arbitrary subset of W, such that wR_Xw'. By C_{VEQ} it follows that $w' = w$. Moreover, since there is a world w'' – namely w' – such that wR_Xw'', it follows by C_{EFQ} that $w \notin -X$ and, hence, $w \in X$ is the case.

C_{MP}+C_{VEQ}+$C_{\text{EFQ}} \Rightarrow C^{\Leftarrow}_{\text{MC}}$: Let w, w' be arbitrary worlds in a Chellas frame $\mathcal{F}_C = \langle W, R\rangle$ and let X be an arbitrary subset of W, such that $w' = w$ and $w \in X$. Then, C_{MP} implies that wR_Xw. Since $w = w'$ it follows that wR_Xw'.

(2) $C_{\text{MC}} \Rightarrow C_{\text{MP}}$: Let w be an arbitrary world in a Chellas frame $\mathcal{F}_C = \langle W, R\rangle$ and let X be an arbitrary subset of W, such that $w \in X$. Then, by C_{MC} follows that wR_Xw.

$C_{\text{MC}} \Rightarrow C_{\text{VEQ}}$: Let w and w' be arbitrary worlds in a Chellas frame $\mathcal{F}_C = \langle W, R\rangle$ and let X be an arbitrary subset of W, such that wR_Xw'. Then, by C_{MC} follows that $w' = w$.

$C_{\text{MC}} \Rightarrow C_{\text{EFQ}}$: Let w, w' be arbitrary worlds in a Chellas frame $\mathcal{F}_C = \langle W, R\rangle$ and let X be an arbitrary subset of W, such wR_Xw'. By C_{MC} follows that $w \in X$ and, hence, $w \notin X$ is the case. □

Proof Theory

We assumed in our terminology that for system **MC** both the material implication and the conditional coincide. To show that our assumption is justified, we, however, have to prove the following:

Theorem 7.76. $(\alpha \to \beta) \leftrightarrow (\alpha \mathrel{\Box\!\!\to} \beta)$ *is a theorem of system* **MC**.

Proof. MP gives us $(\alpha \mathrel{\Box\!\!\to} \beta) \to (\alpha \to \beta)$. VEQ and EFQ imply Triv′, namely $(\alpha \to \beta) \to (\alpha \mathrel{\Box\!\!\to} \beta)$ by Lemma 7.77. □

In the proof for theorem 7.76 we referred to following principle:

Triv′ $(\alpha \rightarrow \beta) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \beta)$

Note that Triv′ is p.c.-equivalent to the following principle:

Or-to-If: $\neg\alpha \vee \beta \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \beta)$

The principle Or-to-If is, for example, discussed in Bennett (2003, p. 139f, p. 142) and in Adams (1975, p. 11f, pp. 19–21). Although the principle Or-to-If has some intuitive appeal at first glance, Lemma 7.77 gives us that it is p.c.-equivalent to VEQ+EFQ.

Lemma 7.77. *VEQ+EFQ* $\Leftrightarrow$ *Triv′*

Proof. By Lemmata 7.78 through 7.80. □

Lemma 7.78. *VEQ+EFQ* $\Rightarrow$ *Triv′*

Proof.

1.	$\alpha \rightarrow \beta$	given
2.	$\neg\alpha$	ass 1, proof by cases
3.	$\quad\alpha \mathrel{\Box\!\!\rightarrow} \beta$	1, EFQ
4.	$\neg\neg\alpha$	ass 2, proof by cases
5.	$\quad\beta$	1, 4, prop
6.	$\quad\alpha \mathrel{\Box\!\!\rightarrow} \beta$	4, VEQ
7.	$\alpha \mathrel{\Box\!\!\rightarrow} \beta$	2-3, 3-6, proof by cases

□

Lemma 7.79. *Triv′* $\Rightarrow$ *VEQ*

Proof.

1.	β	given
2.	$\alpha \rightarrow \beta$	1, prop
3.	$\alpha \mathrel{\Box\!\!\rightarrow} \beta$	2, Triv′

□

Lemma 7.80. *Triv′* $\Rightarrow$ *EFQ*

Proof.

1.	$\neg\alpha$	given
2.	$\alpha \rightarrow \beta$	1, prop
3.	$\alpha \,\Box\!\!\rightarrow \beta$	2, Triv′

□

Both principles EFQ and VEQ, however, are hardly plausible and correspond to S1′ and S2′ from Chapter 1, respectively. We saw in Chapter 1 that both principles are counter-intuitive for both indicative and counterfactual conditionals. Moreover, since any conditional is in **MC** equivalent to a material conditional all properties of the material conditional also hold for conditional formulas. Since Mon, Trans and CP are valid for the material conditional, they hold also for all conditionals in system **MC**. In addition, CEM is derivable in **MC**, as Lemma 7.81 on the basis of Definition 7.74 implies:

Lemma 7.81. *EFQ+VEQ ⇒ CEM*

Proof.

1.	α	ass 1, proof by cases
2.	β	ass 1.1, proof by cases
3.	$\alpha \,\Box\!\!\rightarrow \beta$	2, VEQ
4.	$(\alpha \,\Box\!\!\rightarrow \beta) \vee (\alpha \,\Box\!\!\rightarrow \neg\beta)$	3, p.c.
5.	$\neg\beta$	ass 1.2, proof by cases
6	$\alpha \,\Box\!\!\rightarrow \neg\beta$	5, VEQ
7.	$(\alpha \,\Box\!\!\rightarrow \beta) \vee (\alpha \,\Box\!\!\rightarrow \neg\beta)$	6, p.c.
8.	$(\alpha \,\Box\!\!\rightarrow \beta) \vee (\alpha \,\Box\!\!\rightarrow \neg\beta)$	2-4, 5-7, proof by cases
9.	$\neg\alpha$	ass 2, proof by cases
10.	$\alpha \,\Box\!\!\rightarrow \beta$	9, EFQ
11.	$(\alpha \,\Box\!\!\rightarrow \beta) \vee (\alpha \,\Box\!\!\rightarrow \neg\beta)$	10, p.c.
12.	$(\alpha \,\Box\!\!\rightarrow \beta) \vee (\alpha \,\Box\!\!\rightarrow \neg\beta)$	1-8, 9-11, proof by cases

□

Finally, we show that system **MC** can be also axiomatized by **CK**+MP+CS+EFQ. This holds due to the following theorem:

Theorem 7.82. *EFQ* $\Rightarrow$ *(CS* $\Leftrightarrow$ *VEQ)*

Proof. By Lemmata 7.83 and 7.84. □

Lemma 7.83. *EFQ+CS* $\Rightarrow$ *VEQ*

Proof.

1.	β	given
2.	$\neg\alpha$	ass 1, proof by cases
3.	$\quad\alpha \mathrel{\Box\!\!\rightarrow} \beta$	$\quad$2, EFQ
4.	$\neg\neg\alpha$	ass 2, proof by cases
5.	$\quad\alpha \wedge \beta$	$\quad$1, 4, p.c.
6.	$\quad\alpha \mathrel{\Box\!\!\rightarrow} \beta$	$\quad$5, CS
7.	$\alpha \mathrel{\Box\!\!\rightarrow} \beta$	2-3,4-6, proof by cases

□

Lemma 7.84. *VEQ* $\Rightarrow$ *CS*

Proof.

1.	$\alpha \wedge \beta$	given
2.	β	1, p.c.
3.	$\alpha \mathrel{\Box\!\!\rightarrow} \beta$	2, VEQ

□

Chapter 8

Concluding Remarks

In this final chapter we will, first, (i) give a short overview over the problems addressed in this thesis and provide a list of further problems that may be worth being investigated. We shall, furthermore, (ii) summarize the advantages of Chellas-Segerberg (CS) semantics compared to other types of conditional logic semantics.

Let us now start with point (i). We investigated in this thesis a range of topics related to conditional logic: We gave an argument for the conditional logic project (Chapter 1), investigated the interdisciplinary ramifications of a conditional logic approach and contrasted it with default logic approaches from the non-monotonic literature (Chapter 2). We, then, described important probabilistic and possible worlds semantics and discussed the difference between indicative and counterfactual conditionals. On that basis we, furthermore, defended possible worlds approaches, such as Chellas-Segerberg (CS) semantics (Chellas, 1975; Segerberg, 1989), against criticism by Bennett (2003; see Chapter 3). In Chapters 4–5 we provided a formal account of the Chellas-Segerberg (CS) semantics – a type of possible worlds semantics for conditionals – in terms of soundness, completeness and correspondence results. Finally, in Chapter 7 we provided an objective and a subjective interpretation of CS-semantics and used the formal framework established in Chapters 4–5 to describe a range of indicative and counterfactual conditionals on the basis of CS-semantics. Note, however, that this thesis marks the beginning of a project rather than its completion. In particular, the following topics might be worth further attention:

- Providing a more detailed argument for a conditional logic project
- Investigating further arguments against truth-value analyses for conditions as, for example, described by Bennett (2003)
- Describing logical dependence and independence of the axioms from Tables 5.1 and 5.2
- Accounting for Adams' (1975) conditional default logic or/and belief revision (Alchourrón et al., 1985) by a modification of CS-semantics
- Investigating weaker possible worlds semantics for conditional logics, such as neighborhood set selection functions (e.g. Chellas, 1975, pp. 144–147; Arló-Costa, 2007, Section 3.1.2)
- Describing the notion of conditional obligation by means of a conditional operator (as described herein) plus an obligation operator
- Providing correspondence and canonicity results for further conditional logic principles, such as Negation Rationality and Disjunctive Rationality in Lehmann and Magidor (1992, p. 18)
- Investigating, for which parts of the lattice of conditional logic systems we can give completeness in terms of Chellas frames
- Describing predicate logic versions of CS-semantics
- Systematically investigating the relation of CS-semantics with other semantics for conditional logic, such as Stalnaker (1968), Stalnaker and Thomason (1970), D. Lewis (1973/2001), Burgess (1981), Kraus et al. (1990) and Lehmann and Magidor (1992)

We shall now discuss point (ii), namely the advantages of CS-semantics over alternative semantics. First, CS-semantics allows to account for a broad class of existing conditional logic systems in a uniform framework. For example, system **S** (Stalnaker, 1968; Stalnaker & Thomason, 1970), system **VC** (D. Lewis, 1973/2001; see Sections 7.3), system **R** (Lehmann & Magidor, 1992), system

P, system **CL**, system **C** (Kraus et al., 1990; see Section 7.2) and system **CK** (Chellas, 1975; Segerberg, 1989; see Section 7.1) can be described by CS-semantics. Our formal results imply that the proof-theory of these systems is sound and complete w.r.t. the class of standard Segerberg models, for which the respective conditions from Tables 5.3 and 5.4 hold. Furthermore, the proof-theory of the probabilistic systems **P**, $\mathbf{P}^*$ and $\mathbf{P}^+$ (see Section 3.5) of Adams (1965, 1966, 1977, 1986) and Schurz (1998) can be accounted for by CS-semantics. Note that some conditional logics, such as the probabilistic threshold system **O** cannot be accounted for in terms of CS-semantics, since CS-semantics validates inferences of type AND, while these are not valid in system **O** (cf. Section 3.5.2). Contrary to the aforementioned probabilistic conditional logic systems, however, CS-semantics can be described in the full language $\mathcal{L}_{\mathrm{KL}}$, which allows for both arbitrary boolean combinations and nestings of conditionals, without being prone to D. Lewis' (1976) triviality result (see Section 3.6).

Second, we demonstrated that CS-semantics can be interpreted both in terms of (A) objective alethic modality and (B) a subjective modified Ramsey-test (see Section 7.1). Since the difference between indicative and counterfactual conditionals seems to lie in the fact that indicative conditionals – in contrast to counterfactual conditionals – are interpreted relative to our subjective world knowledge (see Section 3.3), CS-semantics can plausibly serve as basis for both indicative and counterfactual conditional logics. This conclusion is, further, supported by the fact that a range of indicative conditional logics (e.g. systems **P** and **R**; Kraus et al., 1990; Lehmann & Magidor, 1992) and counterfactual conditional logics (e.g. system **VC**; D. Lewis, 1973/2001) can be described formally by CS-semantics (cf. previous paragraph). We, furthermore, do not regard it as a drawback of CS-semantics that it cannot account for all conditional logics (e.g. the probabilistic threshold system **O**). Rather CS-semantics represents a plausible compromise of a conditional logic semantics w.r.t. strength and generality: CS-semantics is on the one hand general enough to account for a wide range of conditional logic systems (see previous paragraph), while it is on the other hand strong enough to lend itself into objective and subjective interpretations in terms of (A) and (B), respectively.

Third, CS-semantics relativizes the accessibility relation R_X in both Chellas models and Segerberg models to propositions X (sets of possible worlds) rather than formulas. This allows a characterization of CS-semantics – in contrast to set-selection models and Stalnaker models (see Section 3.2.2) – in terms of purely structural conditions and, thus, gives us a more natural and more flexible semantic characterization in terms of frames rather than models (cf. Sections 3.2.2, 4.1 and 4.3).

We conclude from our summary of the advantages of CS-semantics that CS-semantics has several pros, which other formal conditional logics semantics do not possess. Given the salience of the advantages of CS-semantics, it should hence, be regarded as a live option for a semantics for both indicative and counterfactual conditionals.

References

Adams, E. (1965). The Logic of Conditionals. *Inquiry*, *8*, 166–197.

Adams, E. (1966). Probability and the Logic of Conditionals. In J. Hintikka & P. Suppes (Eds.), *Aspects of Inductive Logic* (pp. 265–316). Amsterdam: North-Holland Publishing Company.

Adams, E. (1970). Subjunctive and Indicative Conditionals. *Foundations of Language*, *6*, 89–94.

Adams, E. (1975). *The Logic of Conditionals*. Dordrecht, Netherlands: D. Reidel Publishing Company.

Adams, E. (1977). A Note on Comparing Probabilistic and Modal Logics of Conditionals. *Theoria*, *18*, 186–194.

Adams, E. (1986). On the Logic of High Probability. *Journal of Philosophical Logic*, *15*, 255–279.

Adams, E. (1998). *A Primer of Probability Logic*. Stanford: Center for the Study of Language and Information.

Alchourrón, C. E., Gärdenfors, P., & Makinson, D. (1985). On the Logic of Theory Change. Partial Meet Contraction and Revision Functions. *The Journal of Symbolic Logic*, *50*, 510–530.

Arló-Costa, H. (2001). Bayesian Epistemology and Epistemic Conditionals. On the Status of the Export–Import Laws. *The Journal of Philosophy*, *98*, 555–593.

Arló-Costa, H. (2007). The Logic of Conditionals. In E. N. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy.* Retrieved from http://plato.stanford.edu/entries/logic-conditionals/.

Bacchus, F. (1990). *Representing and Reasoning with Probabilistic Knowledge*. Cambridge, Mass.: The MIT Press.

Bamber, D. (1994). Probabilistic Entailment of Conditionals by Conditionals. *IEEE Transactions on Systems, Man, and Cybernetics*, *24*, 1714–1723.

Beck, S. R., Robinson, E. J., Carroll, D. J., & Apperly, I. A. (2006). Children's Thinking about Counterfactuals and Future Hypotheticals as Possibilities. *Child Development*, *77*, 413–426.

Bennett, J. (2003). *A Philosophical Guide to Conditionals*. Oxford: Oxford University Press.

Blackburn, P., de Rijke, M., & Venema, Y. (2001). *Modal Logic*. Cambridge: Cambridge University Press.

Bonati, R. (2005). *Nichtmonotone Systeme der deontischen Logic und ihre Bedeutung für die Behandlung bedingter Normen* [Non-Monotonic Systems of Deontic Logic and their Meaning for the Treatment of Conditional Norms] (Doctoral dissertation, University of Salzburg, Salzburg, Austria). Retrieved from http://www.ubs.sbg.ac.at/pdf/AC04907704.pdf.

Burgess, J. P. (1981). Quick Completeness Proofs for Some Logics of Conditionals. *Notre Dame Journal of Formal Logic*, *22*, 76–84.

Carnap, R. (1962). *Logical Foundations of Probability* (2nd ed.). Chicago: University of Chicago Press.

Cartwright, N. (2002). In Favor of Laws that are Not Ceteris Paribus after All. *Erkenntnis*, *57*, 425–439.

Chellas, B. F. (1975). Basic Conditional Logic. *Journal of Philosophical Logic*, *4*, 133–153.

Clocksin, W. F., & Mellish, C. S. (2003). *Programming in Prolog. Using the ISO Standard* (5th ed.). Berlin: Springer.

Connolly, A. C., Fodor, J. A., Gleitman, L. R., & Gleitman, H. (2007). Why Stereotypes Don't Even Make Good Defaults. *Cognition*, *103*, 1–22.

Davis, W. A. (1979). Indicative and Subjunctive Conditionals. *The Philosophical Review*, *88*, 544–564.

Declerck, R., & Reed, S. (2001). *Conditionals. A Comprehensive Empirical Analysis*. Berlin: Mouton de Gruyter.

Delgrande, J. P. (1987). A First-Order Conditional Logic for Prototypical Properties. *Artificial Intelligence*, *33*, 105-130.

Delgrande, J. P. (1988). An Approach to Default Reasoning Based on a First-

Order Conditional Logic: Revised Report. *Artificial Intelligence*, *36*, 63–90.

Delgrande, J. P. (1998). On First-Order Conditional Logics. *Artificial Intelligence*, *105*, 105-137.

Earman, J., Roberts, J., & Smith, S. (2002). Ceteris Paribus Lost. *Erkenntnis*, *57*, 281–301.

Edgington, D. (2007). On Conditionals. In D. M. Gabbay & F. Guenthner (Eds.), *Handbook of Philosophical Logic* (2nd ed., Vol. 14, pp. 127–221). Dordrecht: Springer.

Enderton, H. B. (2001). *A Mathematical Introduction to Logic* (2nd ed.). San Diego, CA: Harcourt/Academic Press.

Ereshefsky, M. (2007). Species. In E. N. Zalta (Ed.), *Stanford Encyclopedia of Philosophy.* Retrieved from http://plato.stanford.edu/ entries/species/#SpePlu.

Evans, J. S. B. T., Handley, S. J., & Over, D. E. (2003). Conditionals and Conditional Probability. *Journal of Experimental Psychology. Learning, Memory and Cognition*, *29*, 321–335.

Evans, J. S. B. T., & Over, D. E. (2004). *If.* Oxford: Oxford University Press.

Fagin, R., Halpern, J. Y., Moses, Y., & Vardi, M. Y. (2003). *Reasoning About Knowledge*. Cambridge, Mass.: MIT Press.

Feller, W. (1968). *An Introduction to Probability Theory and its Applications* (3rd ed.). New York: Wiley.

Gabbay, D. M., Kurucz, A., Wolter, F., & Zakharyaschev, M. (2003). *Many-Dimensional Modal Logics: Theory and Applications*. Amsterdam: Elsevier.

Gärdenfors, P. (1979). Conditionals and Changes of Belief. *Acta Philosophica Fennica*, *30*, 381–404.

Gärdenfors, P. (1986). Belief Revision and the Ramsey Test for Conditionals. *The Philosophical Review*, *95*, 81–93.

Gärdenfors, P., & Makinson, D. (1994). Nonmonotonic Inference Based Expectations. *Artificial Intelligence*, *65*, 197–245.

Geffner, H., & Pearl, J. (1994). A Framework for Reasoning with Defaults. In H. E. Kyburg Jr., R. P. Loui, & G. N. Carlson (Eds.), *Knowledge Repre-*

sentation and Defeasible Reasoning (pp. 69–87). Dordrecht, Netherlands: Kluwer Academic Publishers.

Gibbard, A. (1980). Two Recent Theories of Conditionals. In W. L. Harper, R. Stalnaker, & G. Pearce (Eds.), *Ifs. Conditionals, Belief, Decision, Chance, and Time* (pp. 211–247). Dordrecht: D. Reidel Publishing Company.

Ginsberg, M. L. (1986). Counterfactuals. *Artificial Intelligence*, *30*, 35-79.

Giordano, L., Gliozzi, V., & Olivetti, N. (2005). Weak AGM Postulates and Strong Ramsey Test. A Logical Formalization. *Artificial Intelligence*, *168*, 1–37.

Goldszmidt, M., & Pearl, J. (1996). Qualitative Probabilities for Default Reasoning, Belief Revision, and Causal Modeling. *Artificial Intelligence*, *84*, 57–112.

Grice, H. P. (1975). Logic and Conversation. In P. Cole & J. L. Morgan (Eds.), *Syntax and Semantics* (Vol. 3, pp. 41–58). New York: Academic Press.

Grice, H. P. (1989). *Studies in the Way of Words*. Cambridge, Mass.: Havard University Press.

Haegeman, L. (2003). Conditionals Clauses. External and Internal Syntax. *Mind & Language*, *18*, 317–339.

Hawthorne, J. (1996). On the Logic of Nonmonotonic Conditionals and Conditional Probabilities. *Journal of Philosophical Logic*, *25*, 185–218.

Hawthorne, J., & Makinson, D. (2007). The Qantitative/Qualitative Watershed for Rules of Uncertain Inference. *Studia Logica*, *86*, 247–297.

Helfman, G. S., Collette, B. B., Facey, D. E., & Bowen, B. W. (2009). *The Diversity of Fishes. Biology, Evolution, and Ecology* (2nd ed.). Chicester, Sussex, U.K.: Wiley-Blackwell.

Hendry, H. E., & Prokriefka, M. L. (1985). Carnapian Extensions of S5. *Journal of Philosophical Logic*, *14*, 111–128.

Horty, J. F. (2001). Nonmonotonic Logic. In L. Goble (Ed.), *The Blackwell Guide to Philosophical Logic* (pp. 336–361). Malden, MA: Blackwell.

Hughes, G. E., & Cresswell, M. J. (1984). *A Companion to Modal Logic*. London: Methuen.

Hughes, G. E., & Cresswell, M. J. (2003). *A New Introduction to Modal Logic*.

London: Routledge. (Original work published 1996)

Johnson-Laird, P. N., & Byrne, R. M. J. (2002). Conditionals. A Theory of Meaning, Pragmatics, and Inference. *Psychological Review*, *109*, 646–678.

Kleene, S. C. (1952). *Introduction to Meta-Mathematics*. Amsterdam: North-Holland Publishing Company.

Kolmogorov, A. N. (1950). *Foundations of the Theory of Probability* (N. Morrison, Trans.). New York: Chelsea.

Kratzer, A. (1977). What 'Must' and 'Can' Must and Can Mean. *Linguistics and Philosophy*, *1*, 337–355.

Kratzer, A. (1981). The Notional Category of Modality. In H.-J. Eikmeyer & H. Rieser (Eds.), *Words, Worlds, and Contexts* (pp. 38–74). Berlin: Walter de Gruyter.

Kraus, S., Lehmann, D., & Magidor, M. (1990). Nonmonotonic Reasoning, Preferential Models and Cumulative Logics. *Artificial Intelligence*, *44*, 167–207.

Lehmann, D., & Magidor, M. (1992). What Does a Conditional Knowledge Base Entail? *Artificial Intelligence*, *55*, 1–60.

Leitgeb, H. (2004). *Inference on the Low Level*. Dordrecht, Netherlands: Kluwer Academic Publishers.

Levesque, H. J. (1990). All I Know. A Study in Autoepistemic Logic. *Artificial Intelligence*, *42*, 263–309.

Lewis, C. I. (1912). Implication and the Algebra of Logic. *Mind*, *21*, 522–531.

Lewis, D. (1970). General Semantics. *Synthese*, *22*, 18–67.

Lewis, D. (1971). Completeness and Decidability of Three Logics of Counterfactual Conditionals. *Theoria*, *37*, 74–85.

Lewis, D. (1973). Causation. *The Journal of Philosophy*, *70*, 556–567.

Lewis, D. (1976). Probabilities of Conditionals and Conditional Probabilities. *The Philosophical Review*, *85*, 297–315.

Lewis, D. (1979). Counterfactual Dependence and Time's Arrow. *Nous*, *13*, 455–476.

Lewis, D. (1980). A Subjectivist's Guide to Objective Chance. In R. C. Jeffrey (Ed.), *Studies in Inductive Logic and Probability* (Vol. 2, pp. 263–293). Berkeley: University of California Press.

Lewis, D. (1997). Finkish Dispositions. *The Philosophical Quarterly*, *47*, 143–158.

Lewis, D. (2001). *Counterfactuals*. Malden, MA: Blackwell. (Original work published 1973)

Lindström, S., & Rabinowicz, W. (1995). The Ramsey Test Revisited. In G. Crocco, L. Fariñas del Cerro, & A. Herzig (Eds.), *Conditionals. From Philosophy to Computer Science* (pp. 147–191). Oxford: Oxford University Press.

Macdonald, D. W., Seifert, E. R., Balcome, K., Dunbar, R., Bininda-Emonds, O., Best, R., et al. (2007a). What is a Mammal? In D. W. Macdonald (Ed.), *The Encyclopedia of Mammals*. Oxford University Press. Oxford Reference Online. Retrieved from http://www.oxfordreference.com /views/ENTRY.html?subview=Main&entry=t227.e156 via Univ- und Landesbibliothek Düsseldorf.

Macdonald, D. W., Seifert, E. R., Balcome, K., Dunbar, R., Bininda-Emonds, O., Best, R., et al. (2007b). What is a Mammal? Species. In D. W. Macdonald (Ed.), *The Encyclopedia of Mammals*. Oxford University Press. Oxford Reference Online. Retrieved from http://www.oxfordreference.com /views/ENTRY.html?subview=Main&entry=t227.e156-ss1 via Univ- und Landesbibliothek Düsseldorf.

Makinson, D. (1993). Five Faces of Minimality. *Studia Logica*, *52*, 339–379.

Makinson, D. (1994). General Patterns in Nonmonotonic Reasoning. In D. M. Gabbay, C. J. Hogger, & J. A. Robinson (Eds.), *Handbook of Logic in Artificial Intelligence and Logic Programming. Vol. 3* (pp. 35–110). Oxford: Clarendon Press.

McCarthy, J. (1980). Circumscription. A Form of Non-Monotonic Reasoning. *Artificial Intelligence*, *13*, 27–39.

McCarthy, J. (1989). Artificial Intelligence, Logic and Formalizing Common Sense. In R. H. Thomason (Ed.), *Philosophical Logic and Artificial Intelligence* (pp. 161–190). Dordrecht, Netherlands: Kluwer Academic Publishers.

McCarthy, J., & Hayes, P. J. (1969). Some Philosophical Problems from the Standpoint of Artificial Intelligence. In B. Meltzer & D. Michie (Eds.), *Machine Intelligence 4* (pp. 463–502). Edinburgh: Edinburgh University Press.

McGee, V. (1985). A Counterexample to Modus Ponens. *The Journal of Philosophy*, *82*, 462–471.

McGee, V. (1989). Conditional Probabilities and Compounds of Conditionals. *The Philosophical Review*, *98*, 485–541.

McKenna, M. C., & Bell, S. K. (1997). *Classification of Mammals above the Species Level*. New York: Columbia University Press.

Meyer, J.-J. C., & van der Hoek, W. (1995). *Epistemic Logic for AI and Computer Science*. Cambridge: Cambridge University Press.

Nejdl, W. (1992). *The P-Systems. A Systematic Classification of Logics of Nonmonotonicity. Extended Report* (Tech. Rep.). Vienna, Austria: Technical University of Vienna.

Nute, D. (1980). *Topics in Conditional Logic*. Dordrecht, Netherlands: D. Reidel Publishing Company.

Nute, D. (1992). Basic Defeasible Logic. In L. Farinas del Cerro & M. Pentonnen (Eds.), *Intentional Logics for Programming* (pp. 125–154). Oxford: Oxford University Press.

Nute, D., & Cross, C. B. (2001). Conditional Logic. In D. M. Gabbay & F. Guenthner (Eds.), *Handbook of Philosophical Logic* (2nd ed., Vol. 3, pp. 1–98). Dordrecht, Netherlands: Kluwer Academic Publishers.

Oberauer, K., & Wilhelm, O. (2003). The Meaning(s) of Conditionals. Conditional Probabilities, Mental Models, and Personal Utilities. *Journal of Experimental Psychology. Learning, Memory and Cognition*, *29*, 680-693.

Pearl, J. (1988). *Probabilistic Reasoning in Intelligent Systems*. San Francisco: Morgan Kaufmann. (Revised Second Printing)

Pearl, J. (1990). System Z. A Natural Ordering of Defaults with Tractable Applications to Nonmonotonic Reasoning. In R. Parikh (Ed.), *Proceedings of the Third Conference on Theoretical Aspects of Reasoning about Knowledge* (pp. 121–135). San Mateo: Morgan Kaufmann.

Pelletier, F. J., Elio, R., & Hanson, P. (2008). Is Logic All in Our Heads? From Naturalism to Psychologism. *Studia Logica*, *88*, 3–66.

Pfeifer, N., & Kleiter, G. D. (in press). The Conditional in Mental Probability Logic. In M. Oaksford & N. Chater (Eds.), *Cognition and Conditionals. Probability and Logic in Human Thought*. Oxford: Oxford University Press.

Poole, D. (1988). A Logical Framework for Default Reasoning. *Artificial Intelligence*, *36*, 27–47.

Portner, P. (2009). *Modality*. Oxford: Oxford University Press.

Rafetseder, E., Cristi-Vargas, R., & Perner, J. (2010). Counterfactual Reasoning. Developing a Sense of "Nearest Possible World". *Child Development*, *81*, 376–389.

Rafetseder, E., & Perner, J. (2010). Is Reasoning from Counterfactual Antecedents Evidence for Counterfactual Reasoning? *Thinking & Reasoning*, *16*, 131–155.

Ramsey, F. P. (1965). *The Foundations of Mathematics and Other Logical Essays* (R. B. Braithwaite, Ed.). London: Routledge & Kegan Paul LTD.

Reiter, R. (1980). A Logic for Default Reasoning. *Artificial Intelligence*, *13*, 81–132.

Schurz, G. (1991). Relevant Deduction. From Solving Paradoxes Towards a General Theory. *Erkenntnis*, *35*, 391–437.

Schurz, G. (1994). Probabilistic Justification of Default Reasoning. In B. Nebel & L. Dreschler-Fischer (Eds.), *KI-94. Advances in Artificial Intelligence* (pp. 248–259). Berlin: Springer.

Schurz, G. (1996). The Role of Negation in Nonmonotonic Logic and Defeasible Reasoning. In H. Wansing (Ed.), *Negation. A Notion in Focus* (pp. 197–231). Berlin: Walter de Gruyter.

Schurz, G. (1997a). *The Is-Ought Problem*. Dordrecht, Netherlands: Kluwer Academic Publishers.

Schurz, G. (1997b). Probabilistic Default Logic Based on Irrelevance and Relevance Assumptions. In D. M. Gabbay, R. Kruse, A. Nonnengart, & H. J. Ohlbach (Eds.), *Qualitative and Quantitative Practical Reasoning* (pp. 536–553). Berlin: Springer.

Schurz, G. (1998). Probabilistic Semantics for Delgrande's Conditional Logic and a Counterexample to His Default Logic. *Artificial Intelligence*, *102*, 81–95.

Schurz, G. (2001a). Rudolf Carnap's Modal Logic. In W. Stelzner & M. Stöckler (Eds.), *Zwischen traditioneller und moderner Logik. Nichtklassische Ansätze* (pp. 365–380). Paderborn: Mentis.

Schurz, G. (2001b). What is 'Normal'? An Evolution-Theoretic Foundation for Normic Laws and Their Relation to Statistical Normality. *Philosophy of Science*, *28*, 476–497.

Schurz, G. (2002a). Alethic Modal Logics and Semantics. In D. Jacquette (Ed.), *A Companion to Philosophical Logic* (pp. 442–477). Malden, MA: Blackwell.

Schurz, G. (2002b). Ceteris Paribus Laws. Classification and Deconstruction. *Erkenntnis*, *57*, 351–372.

Schurz, G. (2004). Logic, Matters of Form, and Closure under Substitution. In L. Běhounek & M. Bílková (Eds.), *The Logica Yearbook 2004* (pp. 33–46). Prague: Filosofia.

Schurz, G. (2005). Non-Monotonic Reasoning from an Evolution-Theoretic Perspective. Ontic, Logical and Cognitive Foundations. *Synthese*, *146*, 37–51.

Schurz, G. (2007). Human Conditional Reasoning Explained by Non-Monotonicity and Probability. An Evolutionary Account. In S. Vosniadou, D. Kayser, & A. Protopapas (Eds.), *Procceedings of the EuroCogSci 2007* (pp. 628–633). Hove, East Sussex, U.K.: Lawrence Erlbaum Associates.

Schurz, G. (2008). *Einführung in die Wissenschaftstheorie* [Introduction to Philosophy of Science] (2nd ed.). Darmstadt, Germany: Wissenschaftliche Buchgesellschaft.

Schurz, G. (in press). Prototypes and their Composition from an Evolutionary Point of View. In W. Hinzen, E. Machery, & M. Werning (Eds.), *The Oxford Handbook of Compositionality.* Oxford: Oxford University Press.

Segerberg, K. (1989). Notes on Conditional Logic. *Studia Logica*, *48*, 157–168.

Spohn, W. (1975). An Analysis of Hansson's Dyadic Deontic Logic. *Journal of Philosophical Logic*, *4*, 237–252.

Stalnaker, R. C. (1968). A Theory of Conditionals. In N. Rescher (Ed.), *Studies in Logical Theory* (pp. 98–112). Oxford: Basil Blackwell.

Stalnaker, R. C. (1970). Probability and Conditionals. *Philosophy of Science*, *37*, 64–80.

Stalnaker, R. C. (1975). Indicative Conditionals. *Philosophia*, *5*, 269–286.

Stalnaker, R. C., & Thomason, R. H. (1970). A Semantical Analysis of Conditional Logic. In S. Körner (Ed.), *Practical Reason* (pp. 23–42). Oxford: Basil Blackwell.

van Benthem, J. (2001). Correspondence Theory. In D. M. Gabbay & F. Guenthner (Eds.), *Handbook of Philosophical Logic* (2nd ed., Vol. 3, pp. 325–408). Dordrecht, Netherlands: Kluwer Academic Publishers.

Weingartner, P., & Schurz, G. (1986). Paradoxes Solved by Simple Relevance Criteria. *Logique et Analyse*, *113*, 3–40.

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