Cohomology and homology of PSL_2 over imaginary quadratic integers with general coefficients $M_{n,m}(\mathcal{O}_K)$ and Hecke eigenvalues

Inaugural–Dissertation

zur
Erlangung des Doktortitels
der
Mathematisch–Naturwissenschaftlichen Fakultät
der
Heinrich–Heine–Universität Düsseldorf

vorgelegt von

CHRISTINE PRIPLATA

aus Dresden

Düsseldorf
2000

Gedruckt mit Genehmigung der Mathematisch–Naturwissenschaftlichen
Fakultät der Heinrich–Heine–Universität Düsseldorf

Referent: Prof. Dr. Fritz Grunewald
Korreferent: Prof. Dr. Joachim Schwermer

Tag der mündlichen Prüfung: 3.7.2000

The probability of success
is difficult to estimate;
but if we never search,
the chance of success is zero.

("Search for Interstellar Communications", Nature, vol.184)

***

At a French airport one day,
the customs official looked
suspiciously at Hitchcock's passport,
in which his occupation
was simply listed as "Producer".
"What do you produce?" he asked.
"Gooseflesh," Hitchcock replied.

(British anecdote)

Contents

Introduction

In my thesis I study the cohomology and homology of certain arithmetic groups $\Gamma \leq PSL_2(\mathcal{O}_K)$ with coefficients in the Γ-modules $M = M_{n,m}(\mathcal{O}_K)$. Here K is an imaginary quadratic number field and $\mathcal{O}_K$ denotes the ring of integers in K. The Γ-modules $M_{n,m}(\mathcal{O}_K)$ are the tensor products of the Γ-modules of homogeneous polynomials in two variables of degree n respectively m with coefficients in $\mathcal{O}_K$, the first Γ-module coming with the standard action, the second with the complex conjugate one. Geometrically, the game plays on the 3-dimensional hyperbolic space $\mathbb{H}^3$ (take the model $\mathbb{C} \times \mathbb{R}^+$), i. e. $PSL_2(\mathbb{C})$ acts on $\mathbb{H}^3$ as a group of isometries and $PSL_2(\mathcal{O}_K)$ acts on $\mathbb{H}^3$ via the embedding $\mathcal{O}_K \subset \mathbb{C}$. The coset space $\Gamma\backslash\mathbb{H}^3$ has the structure of a topological 3-manifold for any discrete subgroup Γ of $PSL_2(\mathbb{C})$. If Γ is torsionfree the quotient gets the structure of a hyperbolic 3-dimensional Riemannian manifold from $\mathbb{H}^3$.
The cohomology and homology modules I study are finitely generated $\mathcal{O}_K$-modules that contain ℓ-torsion classes (ℓ a prime) and so-called *automorphic* classes (nontorsion). Furthermore they admit certain natural endomorphisms T_π $(0 \neq \pi \in \mathcal{O}_K)$ called Hecke operators. Then it is of special interest to investigate the properties of the eigenvalue systems for these Hecke operators acting on the cohomology or homology modules. For a more comprehensive introduction to the subject I would like to refer to Section 1.3, which should also give some more understanding of the specific questions I investigated.

At first I mainly tried to get a general idea of the occuring ℓ-torsions. For that I studied the modules $H^2(\Gamma, M)$, $H^1(\Gamma, M)$ and in particular $H_1(\Gamma, M)$ using different approaches. With the help of extensive computer calculations on the basis of several Computer Algebra Systems I produced a lot of tables showing many interesting torsion phenomena. The computations were carried out on several fast LINUX work stations.
As a central part of my thesis I then studied the Hecke operators T_π (π a prime element now) on $H_1(\Gamma, M)$ for $\Gamma = PSL_2(\mathbb{Z}[i])$, in particular on ℓ-torsion classes. These Hecke operators on $H_1(\Gamma, M)$ were easier to handle than Hecke operators on $H^2(\Gamma, M)$, and on the other hand $H_1(\Gamma, M)$ contained more interesting torsion classes than $H^1(\Gamma, M)$. Again I carried out a lot of computer calculations to determine simultaneous eigenvectors (eigenclasses) for many T_π with π of small norm, which led to interesting congruences for the Hecke eigenvalues a_π and to several insights about the eigenclasses in the so-called *free part* $H_1(\Gamma, M)$ modulo torsion. To produce tables for the structure of the cohomology or homology modules for certain series of $M_{n,m}(\mathcal{O}_K)$ the computations sometimes went on several days, but for the extreme cases of computing Hecke eigenvalues on the so-called large torsion classes (see below) it could take a week to get a single eigenvalue for prime elements π of large norm. But here the point was to make it possible to get at least some.

To compute the cohomology and homology modules I mentioned above it was important to choose suitable approaches which would allow to break the methods down to a nice algorithmic realization. I used a more topological construction to compute $H^2(\Gamma, M)$ starting with a fundamental cellular domain whereas I chose the direct group cohomological approach for $H^1(\Gamma, M)$ and $H_1(\Gamma, M)$. An advantage of the latter method is that one can directly work with the data in the presentations of the groups Γ, which allows to switch quite easily between different arithmetic groups Γ in the computer programs.

It follows a more detailed description of the content of each chapter. In the first chapter I collect the fundamental notions I need throughout the thesis. In particular Section 1.1 contains a collection of presentations of arithmetic groups $\Gamma \leq PSL_2(\mathcal{O}_K)$, which are quite scattered in the literature but form the basis for my computations in Chapter 3 and 4. The fundamental coefficient modules $M_{n,m}(\mathcal{O}_K)$ are introduced in Section 1.2. In addition to this introduction I discuss some more background in Section 1.3, summarize some work done before and give some remarks how my results fit in there.
The next three chapters are organized in a similar way. In the second chapter I compute $H^2(\Gamma, M)$ for $\Gamma = PSL_2(\mathbb{Z}[i])$ and $H^2(\Gamma'_\infty, M)$ for many coefficient modules $M = M_{n,m}(\mathbb{Z}[i])$, where $\Gamma'_\infty := \left\langle \left(\begin{smallmatrix} 1 & 0 \\ \mathbb{Z}[i] & 1 \end{smallmatrix} \right) \right\rangle$. The general idea I use goes back to E. Mendoza. He constructed a suitable 2-dimensional deformation retract of $\mathbb{H}^3$ with a cellular Γ-action (see [Me]). More precisely, I use the formulas for the second cohomology developed in [Th] based on [Me]. Section 2.1 gives a short summary of that approach. The strategy for the actual computation of the module invariants and the algorithmic realization is explained in Section 2.2. In particular this section serves as a basis for working out a conceptual setup for the computer calculations. Therefore it contains many remarks which are relevant for the following chapters as well (representations of the modules, choice of the Computer Algebra Systems and certain algorithms). Most of the programs are written in MAPLE. In general I used the Smith algorithm to compute the module invariants but I also determined a lot of torsion by computation modulo ℓ. The computational results and some conclusions are contained in Section 2.3. The main observation from the tables is that really large ℓ-torsions (ℓ large) occur in the global cohomology $H^2(\Gamma, M)$ but not in $H^2(\Gamma'_\infty, M)$ for the same coefficient modules M. For $M = M_{n,m}(\mathbb{Z}[i])$ I want to say from now on that an ℓ-torsion is *large* if ℓ is greater than $\max(n, m)$, otherwise I want to call the ℓ-torsion *small.* Furthermore it turned out later that $H^2(\Gamma, M)$ seems to be closely related to $H_1(\Gamma, M)$.
The first cohomology is studied in Chapter 3. Here I use the direct possibility to express the first group cohomology as derivations modulo principal derivations. This allows to compute $H^1(\Gamma, M)$ directly from the presentation of Γ. The transfer of the relations from the presentations of the groups Γ into the modules M^r (r the number of generators of Γ) is explained in Section 3.1 and is carried out for $\Gamma = PSL_2(\mathbb{Z}[i])$

there. The more algorithmically oriented realization of the computation of the module invariants for $H^1(\Gamma, M)$ is explained in Section 3.2. Furthermore I discuss some observations I got from computations with the matrix realizations coming from the principal derivations in Section 3.3. The computer experiments suggest that the torsion is already encoded in these matrices. So I decided to collect the corresponding tables from these computations for several arithmetic groups $\Gamma \leq PSL_2(\mathcal{O}_K)$ in Section 3.4. The aspects behind this should be studied in more detail now. In general I didn't find any large torsion classes for $H^1(\Gamma, M)$ so that I concentrated on $H_1(\Gamma, M)$ in the end.
The group homological approach for $H_1(\Gamma, M)$ is developed in Section 4.1. More precisely, I consider the related $\mathcal{O}_K$-module Λ_Γ, where $H_1(\Gamma, M)$ is the kernel of a certain map from Λ_Γ to M. Since M is free abelian all the torsion of Λ_Γ sits already in $H_1(\Gamma, M)$, and it is enough to concentrate on Λ_Γ for the torsion. Again one can compute Λ_Γ directly from the presentation of Γ. The generators of Λ_Γ satisfy several relations coming from the relations in the presentation of Γ. The transfer of these relations is described in Section 4.2 and is carried out there for many groups Γ. The algorithmic realization is described in Section 4.3, and a choice of the results is given in Section 4.4. Here I start with $\Gamma = PSL_2(\mathbb{Z}[i])$ and some tables analogous to the case of $H^2(\Gamma, M)$ and then selected the groups $\Gamma = PSL_2(\mathbb{Z}[\sqrt{-2}])$, $\Gamma = PSL_2(\mathbb{Z}[\frac{1+\sqrt{-3}}{2}])$, the figure-8 knot complement group Γ_8 and the link complement group $\Gamma_{-7}(6, 4)$ with some smaller tables. Again really huge torsion occurs but in particular the same large ℓ-torsions appear for corresponding coefficient modules in $H^2(\Gamma, M)$ and $H_1(\Gamma, M)$ for $\Gamma = PSL_2(\mathbb{Z}[i])$ in all computed cases. Some comments on that observation are also contained in Section 4.4. Furthermore the results for $H_1(\Gamma, M)$ in the case of $\Gamma = PSL_2(\mathbb{Z}[i])$ formed the basis for the following computations with the Hecke operators.

Finally Chapter 5 is devoted to Hecke operators acting on Λ_G. It turns out from general theory that Λ_G is generated for any group G by the elements of the form $(g-1) \otimes_G \mathrm{m}$ (g a generator of G and m a generator of M). A general formula for the Hecke operators T_π on Λ_G giving $T_\pi((g-1) \otimes_G \mathrm{m})$ with $0 \neq \pi \in \mathcal{O}_K$ for the groups $G = PSL_2(\mathcal{O}_K)$ is derived in Section 5.1 (see Theorem 5.3). Then I developed explicit formulas for certain T_π in the case of $\Gamma = PSL_2(\mathbb{Z}[i])$ with its generators A, B and U (see Propositions 5.7–5.9). Based on them I computed simultaneous eigenclasses for many T_π (π of small norm) for several modules $M_{n,m}(\mathbb{Z}[i])$. The algorithmic realization is explained in Section 5.3. I carried out the computations for several ℓ-torsion parts $\Lambda_\Gamma^{\mathrm{tors}} \otimes_{\mathbb{Z}} \mathbb{F}_\ell$ but also for the free parts $\Lambda_\Gamma / \Lambda_\Gamma^{\mathrm{tors}}$. The results and conclusions are contained in Section 5.4.
The analysis of the results from the computer calculations led to the following observations. Again I found a different behaviour for the small and the large ℓ-torsion classes. Up to now I obtained congruences for the Hecke eigenvalues a_π in all cases

of small ℓ-torsion I considered, including congruences to twists of the eigenvalues for the free part (if one occured, i. e. for $n + m + 2 \equiv 0 \bmod 4$). In the cases without such a free part I found congruences of the a_π to twists of the eigenvalues for the free part in certain other homology modules. Note that all these congruences are established from the computational results and therefore only hold for many π of small norm up to now.
For example, I analysed the 5-torsion in the case of the module $M_{10,0}(\mathbb{Z}[i])$ and found $a_\pi \equiv \pi \cdot (N(\pi) + 1) \bmod 5$. The eigenvalue of the free part is $\pi^{11} + \overline{\pi}$ here, and we have $a_\pi \equiv \pi^2(\pi^{11} + \overline{\pi}) \bmod 5$. Similar congruences showed up for the 5-torsion in the case $M_{16,0}(\mathbb{Z}[i])$, for the 3-torsion and 5-torsion for $M_{7,1}(\mathbb{Z}[i])$, for the 13-torsion for $M_{18,0}(\mathbb{Z}[i])$ and so on. Surprisingly, it was possible to identify the numbers $a_\pi \cdot \pi^{-5}$ for the 5-torsion in the case $M_{10,0}(\mathbb{Z}[i])$ with numbers b_π (traces of Frobenii) in a list of elliptic curves over $\mathbb{Q}(i)$ in J. Cremona's thesis [Cr1]. I also managed to compute some Hecke eigenvalues on large torsion classes. So I treated e. g. the 661-torsion for $M_{40,0}(\mathbb{Z}[i])$ and the 137-torsion for $M_{44,0}(\mathbb{Z}[i])$. In contrast to the small torsions I didn't find such conguences here.
From the computations I also got hints about the structure of a general eigenclass in the free part of $H_1(\Gamma, M)$. Using a homology test program and the programs for the Hecke operators I could finally identify a general candidate in Λ_Γ for the modules $M_{n,m}(\mathbb{Z}[i])$. Then I showed that this candidate sits indeed in $H_1(\Gamma, M)$, and I proved the eigenvalue equations (with the eigenvalues $\pi^{n+1} + \overline{\pi}^{m+1}$) for the series $M_{n,0}(\mathbb{Z}[i])$ and $M_{n,1}(\mathbb{Z}[i])$ (see Propositions 5.16 and 5.18). Also I could prove in the general case of modules $M_{n,m}(\mathbb{Z}[i])$ that the so-called $(B-1)$-part always vanishes (see Lemma 5.19). Furthermore I obtained several results about torsion classes. All that is put together in Section 5.5.
The original motivation for this thesis was of arithmetic nature (see also Section 1.3 and the suggestions for further work in Chapter 6). So the final hope or goal would be a suggestion for a formulation of an analogue of Serre's conjecture in the imaginary quadratic case which would relate certain Hecke eigenclasses to appropriate Galois representations mod ℓ.

Finally, it is a pleasure for me to thank, most of all, Prof. Fritz Grunewald for suggesting this wonderful topic to me, for numerous stimulating discussions and his constant support. I'm also obliged to Colin Stahlke and Michael Stoll for several helpful suggestions and especially for their advice in the world of Computer Algebra Systems. Then I want to thank Fritz, Colin, Michael, Wolfgang Huntebrinker and Sascha Rogmann for providing a nice and creative atmosphere around me.
Furthermore I would like to thank Prof. Günter Harder for many insights I got from his Automorphic Forms Seminar and for several very helpful conversations. Last, but not least, I wish to thank Richard Taylor and his students since I could really profit in the end from the inspiration I got during my stays in his research group in Cambridge, U.K., and Harvard.

1 Preliminaries and general setting

In this chapter we present the basic concepts and notations we want to use throughout the thesis. This covers several remarks about arithmetic groups, and in particular the introduction of the groups $PSL_2(\mathcal{O}_K)$ and its congruence subgroups acting on hyperbolic three space.
First of all Section 1.1 puts together a collection of suitable presentations for the groups, which form the basis for our explicit computations in Chapter 3 and 4.
In Section 1.2 we introduce the modules $M_{n,m}(\mathcal{O}_K)$ which will be the general coefficient systems for the group cohomology and homology we want to consider.
Finally Section 1.3 summarizes some work done before and contains several remarks how our results fit in there.

1.1 Arithmetic groups over imaginary quadratic integers

There are various concepts of arithmetic groups and one has to take care about the subtle differences, see e. g. [Bo], [PR], [Se1], [Se2] and [GP1], [GP2]. For us it is suitable to take the following definition due to Harish–Chandra. Let G be a linear algebraic group (subgroup of $GL_n(\mathbb{C})$) defined over $\mathbb{Q}$. We set $G(\mathbb{Z}) := G \cap GL_n(\mathbb{Z})$. A subgroup $\Gamma \leq G(\mathbb{Q})$ is then called an *arithmetic subgroup* of $G(\mathbb{Q})$ if Γ is commensurable with $G(\mathbb{Z})$, i. e. $\Gamma \cap G(\mathbb{Z})$ has finite index in both Γ and $G(\mathbb{Z})$.
Now we start with the algebraic group SL_2/K for an imaginary quadratic field K and let $G/\mathbb{Q} = R_{K/\mathbb{Q}}(SL_2/K)$ be the restriction of scalars. The group $G(\mathbb{Z}) = SL_2(\mathcal{O}_K)$ is an arithmetic subgroup of $G(\mathbb{Q}) = SL_2(K)$. For technical reasons we always want to consider the arithmetic subgroup $PSL_2(\mathcal{O}_K) = SL_2(\mathcal{O}_K)/\{\pm Id\}$ from now on.
We write $K = \mathbb{Q}(\sqrt{-d})$, $d > 0$ a squarefree integer, for the imaginary quadratic number fields. The ring of integers in K will always be denoted by $\mathcal{O}_K$. All these rings are Dedekind rings. For more details we refer to fundamental books about algebra or algebraic number theory, see e. g. [Ar] or [Na].
There are exactly five Euclidean cases among the rings of integers $\mathcal{O}_K$, i. e. for $d = 1, 2, 3, 7, 11$. If one switches from the classical notion of being norm Euclidean to a concept of generalized Euclidean domain there are still no more $\mathcal{O}_K$ in the list, see e. g. [Ch] or [Na]. Furthermore there are only three more rings $\mathcal{O}_K$ with class number $h_K = 1$ or, in other words, with $\mathcal{O}_K$ being a principal ideal domain. We want to denote the ideal class group by Cl_K. In general we mainly want to consider finitely generated modules over the Dedekind rings $\mathcal{O}_K$. A good summary of their structure is given in [Na] and from a more computational point of view in [Co2] and [Co3].
We also need the concept of a congruence group (see e. g. [EGM 3]). Given a nonzero ideal $\mathfrak{n} \subset \mathcal{O}_K$, the *full congruence group of level* $\mathfrak{n}$ in $PSL_2(\mathcal{O}_K)$ is defined

by

$$P\Gamma(\mathfrak{n}) \;:=\; \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathcal{O}_K) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \bmod \mathfrak{n} \right\}.$$

A discrete subgroup $\Gamma \leq PSL_2(\mathbb{C})$ which is $PSL_2(\mathbb{C})$-conjugate to a group containing $P\Gamma(\mathfrak{n})$ for some non-zero ideal $\mathfrak{n} \subset \mathcal{O}_K$ is called a *congruence subgroup* with respect to $PSL_2(\mathcal{O}_K)$. Note that if $\Gamma \leq PSL_2(\mathbb{C})$ is a congruence subgroup then there exits an element $\gamma \in PSL_2(\mathbb{C})$ so that the index of $\gamma\Gamma\gamma^{-1} \cap PSL_2(\mathcal{O}_K)$ in $PSL_2(\mathcal{O}_K)$ is finite. An important example of a congruence group is given for any non-zero ideal $\mathfrak{n} \subset \mathcal{O}_K$ by

$$\Gamma_0(\mathfrak{n}) \;:=\; \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathcal{O}_K) \mid c \in \mathfrak{n} \right\}.$$

There are more examples of groups commmensurable with $PSL_2(\mathcal{O}_K)$ which are of interest for us. So it can happen that the 4-dimensional K-algebra $M(2,K)$ has several $GL_2(K)$-conjugacy classes of maximal $\mathcal{O}_K$-orders. An example is given by

$$M(\mathcal{O}_K, \mathfrak{a}) \;:=\; \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathcal{O}_K) \mid a, d \in \mathcal{O}_K,\ c \in \mathfrak{a},\ b \in \mathfrak{a}^{-1} \right\}$$

where $\mathfrak{a} \subset \mathcal{O}_K$ is a non-zero ideal. From each of these orders $M(\mathcal{O}_K, \mathfrak{a})$ we get the following cofinite subgroup of $PSL_2(\mathbb{C})$:

$$PSL_2(\mathcal{O}_K, \mathfrak{a}) \;:=\; \{\, g \in M(\mathcal{O}_K, \mathfrak{a}) \mid det(g) = 1 \,\} / \{\pm Id\}.$$

$PSL_2(\mathcal{O}_K, \mathfrak{a})$ is commensurable with $PSL_2(\mathcal{O}_K)$. For number theoretical applications like, e. g., the study of Hecke operators it is also necessary to consider in addition to $PSL_2(\mathcal{O}_K)$ the groups $PSL_2(\mathcal{O}_K, \mathfrak{a})$ where $\mathfrak{a}$ runs through a system of representatives for Cl_K/Cl_K^2. A study of explicit fundamental domains for these groups is carried out in [Sch] which leads to presentations as in the standard cases. For the general geometric background, which is the theory of groups acting on hyperbolic three space, we refer to [EGM 3] giving a comprehensive treatment. So let us only mention that we want to use the upper half space model of hyperbolic three space and want to define $\mathbb{H}^3 := \mathbb{C} \times]0, \infty[= \{(z, r) \;:\; z \in \mathbb{C},\ r > 0\}$. The group $PSL_2(\mathbb{C})$ of complex (2×2)-matrices with determinant one modulo its center $\{\pm Id\}$ has a natural action on $\mathbb{H}^3$. Furthermore we have that $PSL_2(\mathcal{O}_K)$ is a discrete subgroup of $PSL(\mathbb{C})$, which has finite covolume but is not cocompact. For more details about the construction of fundamental domains or a discussion of the cusps see also [EGM 3].

Next we put together a collection of presentations for several groups $\Gamma \leq PSL_2(\mathcal{O}_K)$. They form the basis for our computations of the group cohomology and homology

in Chapter 3 and 4. For that we need in particular very small presentations, i. e. with few generators and few relations. Hence we tried to find the best suitable presentations. On the other hand we changed the letters of the generators from the references in such a way that we could use them again in as many presentations as possible. Later on this becomes very helpful for the reuse of certain relations and for their well-organized realization in the computer programs. The collection of the groups $PSL_2(\mathcal{O}_K)$ includes all cases with Euclidean $\mathcal{O}_K$, one case of class number 1 which is not Euclidean, two cases of class number 2 and one example of class number 4. In the cases of class number greater than 1 we also cover the groups associated to the non-trivial ideal classes as described above. So we will get a nice picture about what happens with the cohomology and homology under these different assumptions and about the problems which arise.
At first we want to consider the five Euclidean cases. Let us start with the number field $K = \mathbb{Q}(i)$ with its ring of integers $\mathcal{O}_K = \mathbb{Z}[i]$. From [EGM 3, Ch.7] we have

Proposition 1.1 *Let*

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}.$$

Then the following is a presentation of $PSL_2(\mathbb{Z}[i])$:

$$PSL_2(\mathbb{Z}[i]) = \langle A, B, U \mid R_1 = R_2 = R_3 = R_4 = R_5 = R_6 = 1 \rangle$$

with
$R_1 = B^2$, $R_2 = (AB)^3$, $R_3 = AUA^{-1}U^{-1}$, $R_4 = (BUBU^{-1})^3$, $R_5 = (BU^2BU^{-1})^2$, $R_6 = (AUBAU^{-1}B)^2$.

Remark 1.2 Note that we give all relations a number here, which will be used when we refer to these relations in the following chapters. So, relation R_n will be called Relation n later. This convention will be used for all presentations we consider.

We go on with the field $K = \mathbb{Q}(\sqrt{-2})$ with its ring of integers $\mathcal{O}_K = \mathbb{Z}[\sqrt{-2}]$. Here we set $\omega := \sqrt{-2}$. From [Sw] we have

Proposition 1.3 *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}.$$

Then the following is a presentation of $PSL_2(\mathbb{Z}[\sqrt{-2}])$:

$$PSL_2(\mathbb{Z}[\sqrt{-2}]) = \langle A, B, U \mid R_1 = R_2 = R_3 = R_4 = 1 \rangle$$

with
$R_1 = B^2$, $R_2 = (AB)^3$, $R_3 = AUA^{-1}U^{-1}$, $R_4 = (BU^{-1}BU)^2$.

Next we have the field $K = \mathbb{Q}(\sqrt{-3})$ with its ring of integers $\mathcal{O}_K = \mathbb{Z}\,[\,\frac{1+\sqrt{-3}}{2}\,]$. We set $\omega := (-1+\sqrt{-3})/2$. From [EGM 3, Ch.7] we have

Proposition 1.4 *Let*

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}.$$

Then the following is a presentation of $PSL_2(\,\mathbb{Z}\,[\,\frac{1+\sqrt{-3}}{2}\,]\,)$:

$$PSL_2(\,\mathbb{Z}\,[\,(1+\sqrt{-3})/2\,]\,) \;=\; \langle\, A, B, U \mid R_1 = R_2 = R_3 = R_4 = R_5 = R_6 = 1\,\rangle$$

with
$R_1 = B^2$, $R_2 = (AB)^3$, $R_3 = AUA^{-1}U^{-1}$, $R_4 = (AUBU^{-2}B)^2$, $R_5 = (AUBU^{-1}B)^3$, $R_6 = A^2UBU^{-1}BUBUBU^{-1}B$.

For the field $K = \mathbb{Q}(\sqrt{-7})$ we have the ring of integers $\mathcal{O}_K = \mathbb{Z}\,[\,\frac{1+\sqrt{-7}}{2}\,]$. Then we set $\omega := (1+\sqrt{-7})/2$. From [Sw] we have

Proposition 1.5 *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}.$$

Then the following is a presentation of $PSL_2(\,\mathbb{Z}\,[\,\frac{1+\sqrt{-7}}{2}\,]\,)$:

$$PSL_2(\,\mathbb{Z}\,[\,(1+\sqrt{-7})/2\,]\,) \;=\; \langle\, A, B, U \mid R_1 = R_2 = R_3 = R_4 = 1\,\rangle$$

with
$R_1 = B^2$, $R_2 = (BA)^3$, $R_3 = AUA^{-1}U^{-1}$, $R_4 = (BAU^{-1}BU)^2$.

The field $K = \mathbb{Q}(\sqrt{-11})$ has the ring of integers $\mathcal{O}_K = \mathbb{Z}\,[\,\frac{1+\sqrt{-11}}{2}\,]$. Now we set $\omega := (1+\sqrt{-11})/2$. Again we have from [Sw]

Proposition 1.6 *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}.$$

Then the following is a presentation of $PSL_2(\,\mathbb{Z}\,[\,\frac{1+\sqrt{-11}}{2}\,]\,)$:

$$PSL_2(\,\mathbb{Z}\,[\,(1+\sqrt{-11})/2\,]\,) \;=\; \langle\, A, B, U \mid R_1 = R_2 = R_3 = R_4 = 1\,\rangle$$

with
$R_1 = B^2$, $R_2 = (BA)^3$, $R_3 = AUA^{-1}U^{-1}$, $R_4 = (BAU^{-1}BU)^3$.

We go on with one more field of class number 1 but with a ring of integers, which is no longer Euclidean. Among the fields $\mathbb{Q}(\sqrt{-d})$ with that property it is the one with the smallest d. The field is $K = \mathbb{Q}(\sqrt{-19})$ its the ring of integers is $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$. We set $\omega := (1+\sqrt{-19})/2$. From [Sw] we have

Proposition 1.7 *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1-\omega & 2 \\ 2 & \omega \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}.$$

Then the following is a presentation of $PSL_2(\mathbb{Z}[\sqrt{-19}])$:

$$PSL_2(\mathbb{Z}[(1+\sqrt{-19})/2]) \quad = \quad \langle\, A, B, C, U \mid R_1 = R_2 = R_3 = R_4 = R_5 = R_6 = R_7 = 1 \,\rangle$$

with
$R_1 = B^2$, $R_2 = (AB)^3$, $R_3 = AUA^{-1}U^{-1}$, $R_4 = C^3$, $R_5 = (CA^{-1})^3$, $R_6 = (BC)^2$, $R_7 = (BA^{-1}UCU^{-1})^3$.

Now we consider the first field of class number 2, which is $\mathbb{Q}(\sqrt{-5})$. It has the ring of integers $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$. Here we set $\omega := \sqrt{-5}$. From [Sw] we have

Proposition 1.8 *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -\omega-4 & -2\omega \\ 2\omega & \omega-4 \end{pmatrix}$$

and

$$D = \begin{pmatrix} -\omega & 2 \\ 2 & \omega \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}.$$

Then the following is a presentation of $PSL_2(\mathbb{Z}[\sqrt{-5}])$:

$$PSL_2(\mathbb{Z}[\sqrt{-5}]) \quad = \quad \langle\, A, B, C, D, U \mid R_1 = R_2 = \ldots = R_8 = 1 \,\rangle$$

with
$R_1 = B^2$, $R_2 = (AB)^3$, $R_3 = AUA^{-1}U^{-1}$, $R_4 = A^2$, $R_5 = (BD)^2$, $R_6 = (BUDU^{-1})^2$, $R_7 = AC^{-1}A^{-1}BCB$, $R_8 = AC^{-1}A^{-1}UDU^{-1}CD$.

As we explained above there is one more relevant group associated to an ideal which is not principal. In [Sch] it is explained how one can find presentations in these cases, and a kind of algorithmic procedure for that is given. So we consider here the case of the non-trivial ideal class represented by $\mathfrak{a} =< 2, 1-\sqrt{-5} >$ in $\mathbb{Z}[\sqrt{-5}]$. We set $\omega := \sqrt{-5}$. From [Sch] we have

Proposition 1.9 *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ \frac{1-\omega}{2} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}.$$

Then the following is a presentation of $PSL_2(\mathbb{Z}[\sqrt{-5}], \mathfrak{a})$:

$$PSL_2(\mathbb{Z}[\sqrt{-5}], \mathfrak{a}) = \langle A, C, D, U \mid R_1 = R_2 = R_3 = R_4 = R_5 = 1 \rangle$$

with
$R_1 = CDC^{-1}D^{-1}$, $R_2 = (AC^{-1})^2$, $R_3 = AUA^{-1}U^{-1}$, $R_4 = (DU^{-1})^3$,
$R_5 = (CD^{-1}UA^{-1})^3$.

It follows one more example with class number 2. We take $\mathbb{Q}(\sqrt{-10})$ with its ring of integers $\mathcal{O}_K = \mathbb{Z}[\sqrt{-10}]$. Here we set $\omega := \sqrt{-10}$. From [EGM 3, Ch.7] going back to [Fl] we have

Proposition 1.10 *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -\omega & 3 \\ 3 & \omega \end{pmatrix}, \quad D = \begin{pmatrix} \omega - 1 & -4 \\ 3 & \omega + 1 \end{pmatrix}$$

and

$$E = \begin{pmatrix} \omega & 3 \\ 3 & -\omega \end{pmatrix}, \quad F = \begin{pmatrix} 11 & 5\omega \\ 2\omega & -9 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}.$$

Then the following is a presentation of $PSL_2(\mathbb{Z}[\sqrt{-10}])$:

$$PSL_2(\mathbb{Z}[\sqrt{-10}]) = \langle A, B, C, D, E, F, U \mid R_1 = R_2 = \ldots = R_{11} = 1 \rangle$$

with
$R_1 = B^2$, $R_2 = (BA)^3$, $R_3 = C^2$, $R_4 = E^2$, $R_5 = (BC)^2$, $R_6 = (BE)^2$,
$R_7 = C^{-1}AD^{-1}EAD$, $R_8 = U^{-1}E^{-1}UFCF^{-1}$, $R_9 = D^{-1}E^{-1}B^{-1}DU^{-1}DBCD^{-1}U$,
$R_{10} = D^{-1}B^{-1}ADC^{-1}U^{-1}EDA^{-1}BD^{-1}U$,
$R_{11} = U^{-1}DB^{-1}A^{-1}D^{-1}UFD^{-1}BADF^{-1}$.

Again we also consider the group PSL_2 for the other ideal class. Here we take as a representative $\mathfrak{b} =< 2, \sqrt{-10} >$ in $\mathbb{Z}[\sqrt{-10}]$ and set again $\omega := \sqrt{-10}$. From [Sch] we have

Proposition 1.11 *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}, \quad E = \begin{pmatrix} -2 & \frac{-\omega}{2} \\ -\omega & 2 \end{pmatrix}$$

and

$$F = \begin{pmatrix} -3 & -1-\frac{\omega}{2} \\ 2-\omega & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \frac{\omega}{2} \\ 0 & 1 \end{pmatrix}.$$

Then the following is a presentation of $PSL_2(\mathbb{Z}[\sqrt{-10}], \mathfrak{b})$:

$$PSL_2(\mathbb{Z}[\sqrt{-10}], \mathfrak{b}) \quad = \quad \langle\, A, C, D, E, F, U \mid R_1 = R_2 = \ldots = R_{11} = 1 \,\rangle$$

with
$R_1 = CDC{-}1D^{-1}$, $R_2 = E^2$, $R_3 = AUA^{-1}U^{-1}$, $R_4 = (CA^{-1})^2$, $R_5 = F^3$,
$R_6 = (FE)^2$, $R_7 = (DEU^{-1})^2$, $R_8 = (FC^{-1}EA)^2$, $R_9 = (DF^{-1}U^{-1})^3$,
$R_{10} = (CF^{-1}A^{-1})^3$, $R_{11} = (CDF^{-1}A^{-1}U^{-1})^3$.

We finally come to an example of class number 4. We choose $\mathbb{Q}(\sqrt{-14})$ with its ring of integers $\mathbb{Z}[\sqrt{-14}]$. Here we put $\omega := \sqrt{-14}$. From [Sch] we have

Proposition 1.12 *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \omega & -5 \\ 3 & \omega \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 1+\omega \\ 1-\omega & 4 \end{pmatrix}$$

and

$$E = \begin{pmatrix} -5+4\omega & -23 \\ 4-\omega & 7+\omega \end{pmatrix}, \quad F = \begin{pmatrix} 13 & 6\omega \\ -2\omega & 13 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}.$$

Then the following is a presentation of $PSL_2(\mathbb{Z}[\sqrt{-14}])$:

$$PSL_2(\mathbb{Z}[\sqrt{-14}]) \quad = \quad \langle\, A, B, C, D, E, F, U \mid R_1 = R_2 = \ldots = R_9 = 1 \,\rangle$$

with
$R_1 = B^2$, $R_2 = (BA)^3$, $R_3 = AUA^{-1}U^{-1}$, $R_4 = (A^{-1}C^{-1}BDBAD^{-1}C)^2$,
$R_5 = (A^{-1}CD^{-1}ABDBC^{-1})^2$, $R_6 = D^{-1}CE^{-1}A^{-3}DC^{-1}A^3E$,
$R_7 = CB^{-1}C^{-1}FC^{-1}BCF^{-1}$,
$R_8 = C^{-1}DA^{-1}B^{-1}D^{-1}B^{-1}CAE^{-1}A^{-2}CBD^{-1}BA^{-1}DC^{-1}A^3E$,
$R_9 = ACB^{-1}D^{-1}B^{-1}A^{-1}DC^{-1}AFA^{-1}C^{-1}BDBAD^{-1}CA^{-1}F^{-1}$.

The ideal class group Cl_K for $K = \mathbb{Q}(\sqrt{-14})$ has order 4 but Cl_K/Cl_K^2 has only order 2. So we find only one more representative $\mathfrak{c}$, which leads to a group which is not conjugate to the standard one we already considered (see also [Sch]). Hence we only take the ideal class represented by $\mathfrak{c} =< 3, 1+\sqrt{-14} >$ here. We put again $\omega := \sqrt{-14}$. From [Sch] we have

Proposition 1.13 *Let*

$$A=\begin{pmatrix}1&1\\0&1\end{pmatrix},\quad B=\begin{pmatrix}-2&\frac{\omega-1}{3}\\1+\omega&2\end{pmatrix},\quad C=\begin{pmatrix}1&0\\3&1\end{pmatrix},\quad D=\begin{pmatrix}-1&0\\1+\omega&-1\end{pmatrix}$$

and

$$E=\begin{pmatrix}-3-\omega&-4\\6&3-\omega\end{pmatrix},\quad F=\begin{pmatrix}-3+\omega&-3\\2+2\omega&-3+\omega\end{pmatrix},\quad U=\begin{pmatrix}1&\frac{1-\omega}{3}\\0&1\end{pmatrix}.$$

Then the following is a presentation of $PSL_2(\mathbb{Z}[\sqrt{-14}],\mathfrak{c})$:

$$PSL_2(\mathbb{Z}[\sqrt{-14}],\mathfrak{c}) \;=\; \langle\, A,B,C,D,E,F,U \mid R_1=R_2=\ldots=R_9=1\,\rangle$$

with
$R_1=B^2$, $R_2=CDC^{-1}D^{-1}$, $R_3=AUA^{-1}U^{-1}$, $R_4=(CA^{-1})^3$, $R_5=(DBU^{-1})^2$,
$R_6=F^{-1}AE^{-1}A^{-1}UFEU^{-1}$, $R_7=(CBE^{-1}A^{-1}UBU^{-1}AEA^{-1})^3$,
$R_8=(AEU^{-1}DBE^{-1}A^{-1}UBD^{-1})^2$,
$R_9=DC^{-1}BU^{-1}AEBD^{-1}UE^{-1}F^{-1}CBE^{-1}A^{-1}UBU^{-1}AEA^{-1}F$.

For our studies of the so-called cohomology or homology of the boundary in the case of the group $\Gamma=PSL_2(\mathbb{Z}[i])$ we want to introduce its two subgroups Γ_∞ and Γ'_∞ now. Γ_∞ ist the subgroup of lower triangular matrices in $PSL_2(\mathbb{Z}[i])$ and Γ'_∞ the subgroup of lower triangular matrices in $PSL_2(\mathbb{Z}[i])$ with diagonal entries one, so Γ'_∞ forms a subgroup of index two in Γ_∞. We have the presentations:

$$\Gamma_\infty \;=\; \langle\, A,D,U \mid AUA^{-1}U^{-1}=D^2=(AD)^2=(UD)^2=1\,\rangle \tag{1.1}$$

and

$$\Gamma'_\infty \;=\; \langle\, A,U \mid AUA^{-1}U^{-1}=1\,\rangle \tag{1.2}$$

with A, U as for $PSL_2(\mathbb{Z}[i])$ in Proposition 1.1 and $D=\left(\begin{smallmatrix}i&0\\0&-i\end{smallmatrix}\right)$.

Let now again $K=\mathbb{Q}(\sqrt{-3})$ with its ring of integers $\mathcal{O}_K=\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$. Again we set $\omega:=(-1+\sqrt{-3})/2$ and let

$$A=\begin{pmatrix}1&1\\0&1\end{pmatrix},\quad U=\begin{pmatrix}1&0\\-\omega&1\end{pmatrix}.$$

Then we have the famous group

$$\Gamma_8 \;=\; \langle\, A,U \mid [A^{-1},U][A,U]AU^{-1}=1\,\rangle \tag{1.3}$$

sitting inside $PSL_2(\mathbb{Z}[(1+\sqrt{-3})/2])$. The group Γ_8 is a torsion-free subgroup of index 12 and is in particular a congruence subgroup of $PSL_2(\mathbb{Z}[(1+\sqrt{-3})/2])$.

It holds that $\Gamma_8\backslash\mathbb{H}^3$ is homeomorphic to the complement $S^3\backslash K_8$, where S^3 is the 3-sphere and K_8 the figure 8-knot (see [Rl]). Note that this is the only arithmetic knot complement but there are still more arithmetic link complements.
Later on we carry out some computations for three link complement groups we took from [GH]. They are all subgroups of index 6 in $PSL_2(\mathbb{Z}[(1+\sqrt{-7})/2])$. We chose the groups $\Gamma_{-7}(6,4)$, $\Gamma_{-7}(6,5)$ and $\Gamma_{-7}(6,6)$. The generators of these groups are given in [GH]. They are expressed in A, B and U as in Proposition 1.5. Using that information and the presentation of $PSL_2(\mathbb{Z}[(1+\sqrt{-7})/2])$ we found the following presentations for the link complement groups using MAGMA:

$$\begin{aligned}&\Gamma_{-7}(6,4) \;=\\ &\langle\, X,Y,Z \mid [X,Z] = Z^{-1}YX^{-1}Z^{-1}YZ^{-1}Y^{-1}ZY^{-1}ZXY^{-1}ZY = 1\,\rangle\end{aligned} \tag{1.4}$$

with $X = BA^2B$, $Y = UA^{-1}$, $Z = BUA^{-1}B$,

$$\begin{aligned}&\Gamma_{-7}(6,5) \;=\\ &\langle\, X,Y,Z \mid Y^{-1}[X,Y^{-1}]XY^{-1}[X,Y]XY[X^{-1},Y]X^{-1}Y[X^{-1},Y^{-1}]X^{-1} = 1\,\rangle\end{aligned} \tag{1.5}$$

with $X = ABA$ and $Y = AUA^{-1}$, and

$$\begin{aligned}&\Gamma_{-7}(6,6) \;=\\ &\langle\, X,Y,Z \mid Y^{-1}[X,Y]XY[X,Y^{-1}]XY[X^{-1},Y^{-1}]X^{-1}Y^{-1}[X^{-1},Y]X^{-1} = 1\,\rangle\end{aligned} \tag{1.6}$$

with $X = ABA$ and $Y = AUA^{-2}$.

1.2 General coefficients – the modules $M_{n,m}(\mathcal{O}_K)$

As a fundamental notion for the whole thesis we will now introduce the modules $M = M_{n,m}(\mathcal{O}_K)$. Let us already emphasize here that we assume that $n+m$ is even throughout the whole thesis. The modules $M_{n,m}(\mathcal{O}_K)$ will form the coefficient systems for the group homology or cohomology of our arithmetic groups $\Gamma \leq PSL_2(\mathcal{O}_K)$. They arise from the finite dimensional irreducible rational representations $M_{n,m}(K)$ of the algebraic group $G/\mathbb{Q}$ introduced in Section 1.1 when one chooses the appropriate $SL_2(\mathcal{O}_K)$-invariant $\mathcal{O}_K$-lattices. Note that M will both carry the structure of a Γ-module and of an $\mathcal{O}_K$-module. Let now

$$M_n(\mathcal{O}_K) \;:=\; \left\{ \sum_{i=0}^{n} a_i\, x^{n-i}y^i \;:\; a_i \in \mathcal{O}_K \right\},$$

which is a free $\mathcal{O}_K$-module of rank $n+1$. As a standard basis we can take $e_0 = x^n$, $e_1 = x^{n-1}y$, …, $e_n = y^n$. For example, we get $M_0 = \mathcal{O}_K$ and $M_1 = \mathcal{O}_K x + \mathcal{O}_K y$.

Next let

$$\overline{M}_m(\mathcal{O}_K) \;:=\; \left\{ \sum_{j=0}^{m} b_j\, u^{m-j} v^j \;:\; b_j \in \mathcal{O}_K \right\},$$

which is a free $\mathcal{O}_K$-module of rank $m+1$.
Let now $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL_2(\mathcal{O}_K)$. Then we set

$$\gamma(f)(x,y) \;:=\; f(ax+cy, bx+dy) \quad \text{for } f \in M_n(\mathcal{O}_K)$$

and

$$\gamma(f)(u,v) \;:=\; f(\overline{a}u+\overline{c}v, \overline{b}u+\overline{d}v) \quad \text{for } f \in \overline{M}_m(\mathcal{O}_K),$$

where the bar stands for complex conjugation. An easy computation shows that this gives a left Γ-action on $M_n(\mathcal{O}_K)$ respectively $\overline{M}_m(\mathcal{O}_K)$. We can now define the module

$$M_{n,m}(\mathcal{O}_K) \;:=\; M_n(\mathcal{O}_K) \otimes_{\mathcal{O}_K} \overline{M}_m(\mathcal{O}_K)$$

sitting inside $\mathcal{O}_K[x,y,u,v]$. $M_{n,m}(\mathcal{O}_K)$ is a free $\mathcal{O}_K$-module of rank $k=(n+1)(m+1)$ with the basis $e_{ij} = x^{n-i}y^i \otimes u^{m-j}v^j$ ($i=0,\dots,n$ and $j=0,\dots,m$). Later on we don't like to use these double indices for the basis elements anymore. So let us fix a suitable order for the k basis elements $e_1,\dots,e_k$ from now on. For that we start with $i=0$ going through $j=0,\dots,m$, continue with $i=1$ in the same way up to $i=n$. So we end up with $e_1 = x^n \otimes u^m, \dots, e_{m+1} = x^n \otimes v^m, \dots, e_k = y^n \otimes v^m$.
The left Γ-action basically induces to the modules $M_{n,m}(\mathcal{O}_K)$. The only thing we have to care about is that $-Id$ has to act trivally to give a $PSL_2(\mathcal{O}_K)$-action. A short computation shows that this is only satisfied for $n+m$ even, and therefore we only want to consider these left $PSL_2(\mathcal{O}_K)$-modules from now on. Note that we then have an action of the subgroups $\Gamma \leq PSL_2(\mathcal{O}_K)$ on the modules $M_{n,m}(\mathcal{O}_K)$ as well.

1.3 Context of the problems

Throughout this thesis we study the cohomology and homology of the considered arithmetic groups from a group cohomological resp. group homological point of view. The Hecke operators are considered in that context as well.
Nevertheless we want to emphasize that there are quite different definitions for the cohomology of arithmetic groups, which all lead to deep general results in their own way. In this sense, our case of PSL_2 over imaginary quadratic fields is often covered as a special case. For a general introduction to the subject we refer to [Ha3] and

[BW], for a good survey with many references see [LS]. Fundamental facts about the theory can also be found in [Se1], [Se2], [PR] and [Bo].
Apart from group cohomology one can consider Čech or sheaf cohomology, de Rham cohomology or the relative Lie algebra cohomology. The latter two variants involve coefficients over $\mathbb{R}$ or $\mathbb{C}$ and represent the analytical or so-called automorphic part of the theory. All these cohomologies are connected by comparison isomorphisms. Here we only want to mention the one we use in Chapter 2 (see e. g. [Ha3]). For that let $\Gamma \leq PSL_2(\mathcal{O}_K)$ be an arithmetic group as introduced in Section 1.1 and let $X = \Gamma\backslash\mathbb{H}^3$ be the corresponding arithmetic quotient. Let M be one of the Γ-modules we introduced in Section 1.2 and let $\widetilde{M}$ be the associated local coefficient system on X (where the orders of the finite subgroups of Γ are assumed to be inverted in the modules M resp. $\widetilde{M}$). Then we have $H^*(\Gamma, M) \cong H^*(X, \widetilde{M})$ for the group cohomology of Γ and the Čech cohomology of X.
An important aspect of the theory is that one actually studies automorphic forms when one studies the cohomology of arithmetic groups in the analytical setting. In the case of our general coefficient modules over $\mathcal{O}_K$ instead of K or $\mathbb{C}$ we leave that analytical interpretation and consider torsion classes as well. But still we want to call the nontorsion classes *automorphic*. The other way round we can interpret automorphic forms as cohomology classes. Classically, this is expressed by the Eichler-Shimura isomorphism, which states for $\Gamma = SL_2(\mathbb{Z})$ and the modules $M_n(\mathbb{C})$ that the sequence

$$0 \longrightarrow S_{n+2}(\Gamma, \mathbb{C}) \oplus \overline{S}_{n+2}(\Gamma, \mathbb{C}) \longrightarrow H^1(\Gamma, M_n(\mathbb{C})) \longrightarrow H^1(\Gamma_\infty, M_n(\mathbb{C})) \longrightarrow 0$$

is exact, where $S_{n+2}(\Gamma, \mathbb{C})$ and $\overline{S}_{n+2}(\Gamma, \mathbb{C})$ are the spaces of the holomorphic resp. antiholomorphic cusp forms of weight $n+2$ for Γ. A detailed analysis of the group cohomology of $SL_2(\mathbb{Z})$ and its arithmetical applications is carried out in [Hab], for the Eisenstein class in the first cohomology of $SL_2(\mathbb{Z})$ see also [Wa].
A main focus is the understanding of the Hecke operators acting on the cohomology (see e. g. [AS1], [AS2] and [As]). Their eigenvalues give e. g. the local data to relate certain eigenclasses to more arithmetical objects like Galois representations or elliptic curves. A good summary of the various relations between modular forms, Galois representations and abelian varieties can be found in [DDT], for several arithmetical applications of modular forms we also refer to [Ri1]–[Ri3], [SD] and [Se3], [Se4].
For GL_2 over $\mathbb{Q}$ many of these connections are already worked out. Let us only mention the recent proof of the modularity of all elliptic curves over $\mathbb{Q}$ known as the Shimura–Taniyama conjecture (see [Wi] and [TW] and recent work of F. Diamond and R. Taylor). Furthermore J.-P. Serre (see [Se5]) established a strong conjecture describing how Galois representations into $GL_2(\overline{\mathbb{F}}_\ell)$ can be modular. In its qualitative version this conjecture states: Every irreducible continuous odd representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\overline{\mathbb{F}}_\ell)$, also called Galois representation mod ℓ, is modular. Later J.-P. Serre worked out a precise recipe to actually pinpoint the space of cusp

forms where one should find the corresponding modular form. So he has an exact guess for the level, the weight and the character of the expected form. This very general conjecture is far from being proved but still has enriched the picture of the whole theory a lot. For more details see also [Da], [Di] and [Ed].
Now it is quite natural to study similar questions in the case of GL_2 over imaginary quadratic fields. The automorphic forms in this context are certain real analytic functions living on the hyperbolic three space. The locally symmetric spaces $\Gamma\backslash\mathbb{H}^3$ are arithmetic 3-orbifolds. And here we see the fundamental difference to the case of GL_2 over $\mathbb{Q}$ and other cases – there is no complex structure on the quotients, so we don't have an obvious algebraic variety to exploit (like Shimura varieties in other cases). Even more, the algebraic varieties in the other cases are defined over number fields and so we also miss this direct link to arithmetic, and at first glance we mainly have to rely on methods from geometry, group theory and analysis.
The project of analysing the situation for imaginary quadratic fields was initiated by F. Grunewald and J. Mennicke in the 70's by explicit computations over $\mathbb{Q}(\sqrt{-d})$ for small d, e. g. by producing tables of newforms of weight two for certain groups $\Gamma_0(\mathfrak{p}) \leq PSL_2(\mathbb{Z}[i])$ and tables of corresponding elliptic curves over $\mathbb{Q}(i)$. Here the modular forms are interpreted as homology classes of the considered arithmetic group with constant coefficients $\mathbb{Z}$. The computations use methods from combinatorial group theory. For more details we refer to [GM], [GHM], [EGM 1] and [EGM 2].
Many more computations for larger d were done by J. Cremona and his students using modular symbols (see [Cr1]–[Cr3]). This is a more geometric method and covered constant coefficients $\mathbb{C}$ only. For a treatment of the homology with constant coefficients including the torsion see [SV] and [Vo].
In 1994 R. Taylor et al. (see [HST] and [Tay]) proved, roughly speaking, that for all modular forms belonging to the cohomology of the arithmetic quotients in the imaginary quadratic case there exists a corresponding system of compatible λ-adic Galois representations (for all primes outside a set of Dirichlet density zero). The theorem covers the case of general weight but only the situation over $\mathbb{C}$, not including any torsion phenomena. Up to now there is no idea for a construction of a corresponding elliptic curve out of the modular form of weight two as we have it over $\mathbb{Q}$ (Eichler). On the other hand, if there are candidates of corresponding elliptic curves, Galois representations and modular forms, then a check for the actual coincidence is possible using the so-called Faltings-Serre method (cp. [Tay]).
Many important insights into the structure of the cohomology of arithmetic groups $\Gamma \leq SL_2(\mathcal{O}_K)$, or about the GL_2 case in general, are due to G. Harder. For example he proved that the de Rham cohomology of $\Gamma\backslash\mathbb{H}^3$ decomposes as a direct sum of a cuspidal and an Eisenstein part and that this decomposition is respected by the Hecke algebra. He also gives a precise description of the action of the Hecke algebra on the Eisenstein part, see [Ha1], [Ha2] and [Ha4] and also [GS]. For the analytical background in the case of $SL_2(\mathbb{C})$ see also [Bl]. It contains e. g. an explicit des-

cription of the Eichler-Shimura isomorphism in the imaginary quadratic case and also the vanishing theorem $H^1_{\text{cusp}}(\Gamma, M_{n,m}(\mathbb{C})) = 0$ for $n \neq m$ and the groups Γ we consider. Therefore the modules $M_{n,m}(\mathbb{C})$ with $n = m$ remain of special interest. The Eisenstein cohomology of $SL_2(\mathbb{Z}[i])$ is considered in [Kö]. A topological model which is very helpful for explicit computations of the cohomology is given in [Me]. Using that some computations of the second cohomology of $\Gamma = PSL_2(\mathbb{Z}[i])$ with coefficients in the series of modules $M_{n,n}(\mathbb{Z}[i])$ are carried out in [Th]. Further explicit computations of the cohomology of $PGL_2(\mathbb{Z}[i])$ and $GL_2(\mathbb{Z}[\zeta_6])$ for the special series $M_{n,0}$ of the coefficient modules are done in [Ca] and [Fe], also including the determination of certain denominators of Eisenstein classes and a discussion of bounds for the torsion in the first global cohomology and in the first and second cohomology of the boundary (no large ℓ-torsions).
For a good summary of the work done in the imaginary quadratic case see Chapter 7 of [EGM 3]. It also includes an example for mod ℓ Hecke eigenclasses computed by F. Grunewald which suggests an analogue of Serre's conjecture for imaginary quadratic fields. In several cases such a conjecture for constant coefficients is also discussed in [Fi1] and [Fi2]. The computations are based on the method of modular symbols for coefficients $\mathbb{F}_\ell$, which is not worked out for more general coefficients in that case.
The relations to arithmetic formed the original motivation for my thesis. In order to investigate a correspondence between certain Hecke eigenvalue systems and Galois representations mod ℓ one can start with special examples of Galois representations and can try to identify corresponding (co)homology classes. This is the approach in [Fi1] and [Fi2]. Here only the first homology with constant coefficients $\mathbb{F}_\ell$ was considered, but also for congruence subgroups $\Gamma_1(N)$ of $PGL_2(\mathcal{O}_K)$. In contrast to that we started a quite systematic investigation of the cohomology and homology of arithmetic groups $\Gamma \leq PSL_2(\mathcal{O}_K)$ with general coefficients $M_{n,m}(\mathcal{O}_K)$ to get in particular a good overview of the occuring torsion classes. In a next step we started with a quite general setup for studying Hecke operators T_π on $H_1(\Gamma, M)$ and worked it out explicitly for $\Gamma = PSL_2(\mathbb{Z}[i])$. Then we computed simultaneous Hecke eigenclasses (for many T_π) in many cases of $H_1(\Gamma, M)$, in particular on certain ℓ-torsion classes but also on the automorphic classes.
The general questions about such Hecke eigenvalue systems are: how do they vary when one changes the group Γ or the module M or both, resp. how are they related for different homology modules (or even inside the same homology module), what are the relations between automorphic and ℓ-torsion classes, what can be said about congruence properties and which arithmetical objects (e. g. Galois representations mod ℓ) could correspond to the Hecke eigenclasses? Our computational results provide several insights in that direction. Having established the machinary on the homology side a next step would be to analyse the Galois side in more detail.

2 Computation of the second cohomology $H^2(\Gamma, M)$

The second group cohomology $H^2(PSL_2(\mathbb{Z}[i]), M_{n,m}(\mathbb{Z}[i]))$ should become our first playground to get an impression about the existence and distribution of torsion for general coefficients $M_{n,m}(\mathbb{Z}[i])$. It also serves as a basis for working out a conceptual setup for the computer calculations with these general modules.
At first the choice between a group cohomological or a more topological approach had to be made. The second group cohomology is already quite complicated to handle. On the other hand we have a nice method through the work of E. Mendoza (see [Me]). The main idea is to study the cohomology of a group Γ with finite virtual cohomological dimension through an associated finite dimensional contractible topological space X with a proper Γ-action.
In our case of $\Gamma = PSL_2(\mathbb{Z}[i])$ the standard candidate for X is the hyperbolic three space $\mathbb{H}^3$, where we would have a suitable fundamental domain. However, this is still quite complicated. Furthermore $\Gamma\backslash\mathbb{H}^3$ is not compact. Now, the virtual cohomological dimension of $PSL_2(\mathbb{Z}[i])$ is two. So the question arises whether one could construct a contractible space of dimension two with proper Γ-action. This is the problem, which was solved by E. Mendoza. He constructed a Γ-invariant 2-dimensional deformation retract I_K of $\mathbb{H}^3$ (K an imaginary quadratic field, so that all $PSL_2(\mathcal{O}_K)$ are covered), so that the quotient of I_K by any subgroup of Γ of finite index is compact, I_K has a natural CW structure with a cellular Γ-action and the quotient $\Gamma\backslash I_K$ is a finite CW complex. In our case this leads to a fundamental cellular domain with one 2-cell, four 1-cells and four 0-cells. The formulas for the second cohomology become quite easy then, since one only has to take into account the edges. For the first cohomology one would also have to take care of the vertices which makes it more complicated again.
Since this approach was already used in [Th] in a computation for $M = M_{n,m}(\mathbb{Z}[i])$ with $n = m$ we could start with the setup as developed there. So we just sketch the construction in Section 2.1 and present the formulas one can derive for $H^2(\Gamma, M)$ and $H^2(\Gamma'_\infty, M)$. Section 2.2 describes the computation of their module invariants and the realization in a computer program. Section 2.3 collects and discusses our results.

2.1 The formulas for $H^2(\Gamma, M)$ and $H^2(\Gamma'_\infty, M)$

Our goal is to compute the second group cohomology of $\Gamma = PSL_2(\mathbb{Z}[i])$ with coefficients in the Γ-modules $M = M_{n,m}(\mathbb{Z}[i])$. As already mentioned we want to use a cellular decomposition for that. Since we mainly want to sketch the construction from [Me] and [Th] and plan to study group cohomology in Chapter 3 in more detail we want to refer for the definition of group cohomology to Section 3.1 (see also Section

4.1 for group homology) or to [Wei]. What we need is a suitable projective resolution of $S := \mathbb{Z}[\,i\,][\frac{1}{6}]$ (note that we have to invert the orders of the stabilizers to have the comparison isomorphism for the cohomologies and that we want to consider the modules M over S as well). After applying the invariants functor one gets a complex whose cohomology groups are the cohomology groups $H^q(\Gamma, M)$. Using the cellular decomposition from [Me] yields a much better projective resolution as the standard way due to Eilenberg-MacLane.
Mendoza's construction of the deformation retract is based on G. Harder's reduction theory for arithmetic groups. A central notion is the so-called *distance from a cusp*. We don't want to work that out here. For details see [Me] and for a good overview we also refer to [SV]. So I_K for an imaginary quadratic field K is defined as the set of all points in $\mathbb{H}^3$, which lie in the minimal sets of (at least) two different cusps. Now, a finite subcomplex F of I_K is called a fundamental cellular domain for $\Gamma = PSL_2(\mathcal{O}_K)$ if $I_K = \Gamma \cdot F$ and if points in open induced 2-cells are not Γ-equivalent. Then we have from [Me] or [Th], that

$$f \;=\; \left\{(z,\zeta) \in \mathbb{H}^3 :\ |z|^2 + \zeta^2 = 1,\ \ 0 \le Re(z) \le \frac{1}{2},\ \ 0 \le Im(z) \le \frac{1}{2}\right\}$$

is a fundamental celular domain for the action of $PSL_2(\mathbb{Z}[\,i\,])$ on $I_{\mathbb{Q}(i)}$. The four vertices are $c_1 = (0,1)$, $c_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $c_3 = (\frac{1+i}{2}, \frac{\sqrt{2}}{2})$ and $c_4 = (\frac{i}{2}, \frac{\sqrt{3}}{2})$. The four egdes are $k_1 = \{(z,\zeta) \in f : Im(z) = 0\}$, $k_2 = \{(z,\zeta) \in f : Re(z) = \frac{1}{2}\}$, $k_3 = \{(z,\zeta) \in f : Im(z) = \frac{1}{2}\}$ and $k_4 = \{(z,\zeta) \in f : Re(z) = 0\}$. Hence $I_{\mathbb{Q}(i)}$ is a 2-dimensional CW complex. We write $C_0 := \{gc_j :\ g \in PSL_2(\mathbb{Z}[\,i\,]),\ \ j = 1, \ldots 4\}$ for the set of 0-cells, $C_1 := \{gk_j :\ g \in PSL_2(\mathbb{Z}[\,i\,]),\ \ j = 1, \ldots 4\}$ for the set of closed 1-cells and $C_2 := \{gf :\ g \in PSL_2(\mathbb{Z}[\,i\,])\}$ for the set of closed 2-cells. Furthermore we need the stabilizers for the edges:

$$\mathrm{Stab}_\Gamma(k_1) = \left\langle \sigma_1 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\rangle = \mathbb{Z}/2\mathbb{Z},\ \ \mathrm{Stab}_\Gamma(k_2) = \left\langle \tau := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle = \mathbb{Z}/3\mathbb{Z},$$

$$\mathrm{Stab}_\Gamma(k_3) = \left\langle \tau_1 := \begin{pmatrix} -1 & i \\ i & 0 \end{pmatrix} \right\rangle = \mathbb{Z}/3\mathbb{Z},\ \ \mathrm{Stab}_\Gamma(k_4) = \left\langle \sigma := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle = \mathbb{Z}/2\mathbb{Z}.$$

The stabilizers for all other 1-cells are conjugate to these stabilizers.
Now one can form the free S-modules $S[C_0]$, $S[C_1]$ and $S[C_2]$ over the sets C_0, C_1 and C_2 and can define appropriate $S\Gamma$-module homomorphisms d_0, d_1 and d_2 between them. One finally gets (see [Th], Ch.2) that for $\Gamma = PSL_2(\mathbb{Z}[\,i\,])$ the sequence

$$0 \longrightarrow S[C_2] \xrightarrow{d_2} S[C_1] \xrightarrow{d_1} S[C_0] \xrightarrow{d_0} S$$

is a projective resolution of S, consisting of finitely generated $S\Gamma$-modules. Note that the sequence is exact because of the contractability of $I_{\mathbb{Q}(i)}$. For $\Gamma = PSL_2(\mathbb{Z}[\,i\,])$

the modules $S[C_2]$ and $S[C_1]$ have a quite simple structure. One gets (see [Th]) that $S[C_2] \cong S\Gamma$ and $S[C_1] \cong \bigoplus_{j=1}^4 S[\mathrm{Stab}_\Gamma(k_j)\backslash\Gamma]$. This gives $\mathrm{Hom}_\Gamma(S[C_2], M) \xrightarrow{\sim} M$, which sends ϕ to $\phi(f)$ and

$$\mathrm{Hom}_\Gamma(S[C_1], M) \quad \xrightarrow{\sim} \quad \bigoplus_{j=1}^{4} M^{\mathrm{Stab}_\Gamma(k_j)},$$

which sends ϕ to $(\phi(k_1), \dots, \phi(k_4))$. With the right choice of orientation this gives for the boundary map $d_2 : S[C_2] \longrightarrow S[C_1]$ that $d_2(f) = k_1 + \ldots + k_4$. Hence we get the associated coboundary map (apply $\mathrm{Hom}_\Gamma(\,\cdot\,, M)$):

$$d^1 \quad : \quad \bigoplus_{j=1}^{4} M^{\mathrm{Stab}_\Gamma(k_j)} \longrightarrow M,$$

which sends $(m_1, \dots, m_4)$ to $m_1 + \ldots + m_4$. Now we take the definition of $H^q(\Gamma, M)$ and get:

$$\begin{aligned} H^2(\Gamma, M) &= M/(M^{\mathrm{Stab}_\Gamma(k_1)} + \ldots + M^{\mathrm{Stab}_\Gamma(k_4)}) \\ &= M/\left(\left(\sum_{\gamma\in\mathrm{Stab}_\Gamma(k_1)} \gamma\right) M + \ldots + \left(\sum_{\gamma\in\mathrm{Stab}_\Gamma(k_4)} \gamma\right) M\right). \end{aligned}$$

Using our stabilizers for the edges we end up with the following formula (see [Th]):

Proposition 2.1 *Let $\Gamma = PSL_2(\mathbb{Z}[\,i\,])$ and $M = M_{n,m}(\mathbb{Z}[\,i\,][\frac{1}{6}])$. Then*

$$H^2(\Gamma, M) \quad = \quad M/((1+\sigma)M + (1+\sigma_1)M + (1+\tau+\tau^2)M + (1+\tau_1+\tau_1^2)M).$$

We are also interested in the cohomology $H^2(\Gamma'_\infty, M)$, which is a variant of the so-called cohomology of the boundary. For the definition of Γ'_∞ see Section 1.1. Here the computation is a lot easier since all stabilizers are trivial. Let again $M = M_{n,m}(\mathbb{Z}[\,i\,])$ again. The fundamental cellular domain is

$$\tilde{f} \quad = \quad \{z \in \mathbb{C} : \; 0 \le Re(z) \le 1, \;\; 0 \le Im(z) \le 1\}\,.$$

The edges of $\tilde{f}$ are $\tilde{k}_1 = \{z \in \tilde{f} : Im(z) = 0\}$, $\tilde{k}_2 = \{z \in \tilde{f} : Re(z) = 0\}$, $\tilde{k}_3 = \left(\begin{smallmatrix} 1 & i \\ 0 & 1 \end{smallmatrix}\right) \tilde{k}_1$ and $\tilde{k}_4 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \tilde{k}_2$, and the vertices are $\tilde{c}_1 = 0$, $\tilde{c}_2 = 1$, $\tilde{c}_3 = 1 + i$ and $\tilde{c}_4 = i$. As for Γ we define the sets $\tilde{C}_0$, $\tilde{C}_1$ and $\tilde{C}_2$ of 0-cells, 1-cells and 2-cells, form for $\tilde{S} = \mathbb{Z}[\,i\,]$ free $\tilde{S}$-modules $\tilde{S}[\tilde{C}_0]$, $\tilde{S}[\tilde{C}_1]$ and $\tilde{S}[\tilde{C}_2]$, which become $\tilde{S}\Gamma'_\infty$-modules, and after a suitable choice of orientation define the boundary maps. Then we get the sequence

$$0 \longrightarrow \tilde{S}[\tilde{C}_2] \xrightarrow{\tilde{d}_2} \tilde{S}[\tilde{C}_1] \xrightarrow{\tilde{d}_1} \tilde{S}[\tilde{C}_0] \xrightarrow{\tilde{d}_0} \tilde{S},$$

which is a projective resolution of $\tilde{S}$ consisting of finitely generated free $\tilde{S}\Gamma'_\infty$-modules. Now we find (see [Th]) that $\tilde{S}[\tilde{C}_2] \cong \tilde{S}\Gamma'_\infty$ and $\tilde{S}[\tilde{C}_1] \cong \tilde{S}{\Gamma'_\infty}^2$, which gives us the maps $\mathrm{Hom}_{\Gamma'_\infty}(\tilde{S}[\tilde{C}_2], M) \xrightarrow{\sim} M$ sending ϕ to $\phi(\tilde{f})$ and $\mathrm{Hom}_{\Gamma'_\infty}(\tilde{S}[\tilde{C}_1], M) \xrightarrow{\sim} M^2$, which sends ϕ to $(\phi(\tilde{k}_1), \phi(\tilde{k}_2))$. The analysis of the boundary map $\tilde{d}_2$ leads to $\tilde{d}^1 : M^2 \longrightarrow M$, sending (m_1, m_2) to $(1 - \left(\begin{smallmatrix} 1 & -i \\ 0 & 1 \end{smallmatrix}\right))m_1 + (1 - \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right))m_2$. We set $\rho_1 := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\rho_2 := \left(\begin{smallmatrix} 1 & i \\ 0 & 1 \end{smallmatrix}\right)$. Then we finally get (see [Th]):

Proposition 2.2 *Let $\Gamma = PSL_2(\mathbb{Z}[\,i\,])$ and $M = M_{n,m}(\mathbb{Z}[\,i\,])$. Then*

$$H^2(\Gamma'_\infty, M) \quad = \quad M/((\rho_1 - 1)M + (\rho_2 - 1)M).$$

2.2 The strategy for computing the module invariants

Now we want to compute explicitly the invariants of the modules $H^2(\Gamma, M)$ and $H^2(\Gamma'_\infty, M)$. Since it is technically more complicated to carry out the computations over $\mathbb{Z}[\,i\,][\frac{1}{6}]$ we prefer to compute over $\mathbb{Z}[\,i\,]$ only. Hence we won't find the right parts for the 2-torsion and 3-torsion (and therefore don't list them in our tables), but our main focus is on higher torsion anyway. The modules $H^2(\Gamma, M)$ and $H^2(\Gamma'_\infty, M)$ are finitely generated (of finite type) over $\mathbb{Z}[\,i\,]$. For such modules the following theorem (see e. g. [EMS]) holds:

Theorem 2.3 *A module T of finite type over a principal ideal domain A is isomorphic to a direct sum of a finite number of cyclic modules. A cyclic module is either isomorphic to A or decomposes further as a direct sum of cyclic modules of the form $A/(\pi^e)$, where π is a prime element. The representation of a module as a direct sum of such modules is unique.*

The A-module T can be represented in the form $T = A^\nu/N$ with $N \subseteq A^\nu$, where N is free and finitely generated as well. If we choose a basis for A^ν and express N in that basis, then our representation just means that T is defined by the associated system of linear equations. More precisely, let $A^\nu = Ae_1 \oplus \ldots \oplus Ae_\nu$ and let $N = (u_1, \ldots, u_\mu)$ be generated by $u_1, \ldots, u_\mu$, then we find c_{ij} so that $u_i = \sum_{j=1}^{\nu} c_{ij}e_j$, and the associated system of linear equations is $\sum_{j=1}^{\nu} c_{ij}e_j = 0$ for $i = 1, \ldots, \mu$, i. e. the matrix (c_{ij}) corresponds to the module N. Over a principal ideal domain, any matrix can be reduced to diagonal form by multiplying both sides by unimodular matrices. If we have an Euclidean algorithm in the ring A, then these multiplications can be performed by the standard row and column transformations (interchanging of two rows or columns or adding a multiple of one to another). These operations allow us to find a system of generators for the module T such that the matrix (c_{ij}) is diagonal. If $a_1, \dots, a_l$ are the diagonal elements which are not zero, then we have

$$T \;\cong\; A/(a_1) \oplus \dots \oplus A/(a_l) \oplus A^{\nu-l}.$$

From Proposition 2.1 we know that $H^2(\Gamma, M) = M/N$, where $M \cong \mathbb{Z}[i]^k$ is a module of rank $k = (n+1)(m+1)$. For $H^2(\Gamma'_\infty, M)$ we are in a similar situation. So we have to represent N by a matrix like (c_{ij}) from above, where the columns of that matrix generate the submodule N. In the following we will use the notation NMAT for such a matrix. The capital letters and the appendix MAT should stress the more algorithmic aspect of these specific representations for the cohomology modules. Then we have

$$\mathrm{rank}(H^2(\Gamma, M)) \;=\; \mathrm{corank}(\mathrm{NMAT}) \;:=\; k - \mathrm{rank}(\mathrm{NMAT}),$$

and the torsion of $H^2(\Gamma, M)$ can be read off from the diagonal elements. The diagonal matrix we described is not unique, since we have a certain freedom through the elementary transformations. So we want to choose a unique form (up to units) from now on. We say that an $(r \times s)$-matrix D is in Smith normal form (SNF) if D is an extended diagonal matrix (the diagonal does not necessary end in the lower right corner), so that any diagonal element always divides the next lower diagonal element. Let now B be an $(r \times s)$-matrix with entries in a principal ideal domain A. Then there exists a unique matrix in Smith normal form D (up to units), such that $D = U \cdot B \cdot V$ with $U \in GL_r(A)$ and $V \in GL_s(A)$. The entries d_i on the diagonal of D (which are not zero) are the elementary divisors. For the Smith normal form over $\mathbb{Z}$ and principal ideal domains see [Co1], for a more general setup over Dedekind rings we refer to [Co2] or [Co3], and we come back to that later in Chapter 4 (for the theoretical background see also [Na]).

What we have to do now is to build up the matrix NMAT for $H^2(\Gamma, M)$. Here we use the k basis elements e_i of $M = M_{n,m}(\mathbb{Z}[i]) \cong \mathbb{Z}[i]^k$ as introduced in Section 1.2 and have to apply the action of the generators of the stabilizers to these basis elements. From now on we want to call this basis for short the monomial basis. Through the described base change we transform the matrix NMAT into the Smith normal form. Then we get a basis that we want to call Smith basis. We have

$$H^2(\Gamma, M) = M/N = M/((1+\sigma)M + (1+\sigma_1)M + (1+\tau+\tau^2)M + (1+\tau_1+\tau_1^2)M).$$

The matrix NMAT represents N when its columns generate N. We see that we have four summands that contribute. Since M has rank k we get a $(k \times 4k)$-matrix. The $4k$ columns form 4 blocks belonging to σ, σ_1, τ and τ_1. Each block consits of k vectors of length k since the actions in our formula (like e. g. $(1+\sigma)$) have to be applied to each of the k basis vectors e_i of length k. Since we use similar realizations in Chapter 3 and 4 again, we want to explain one example in more detail here to make really clear how we build up such matrices. To avoid confusion let us emphasize that we use the notation e_i for both the monomials in $M_{n,m}(\mathbb{Z}[i])$ and their representations as vectors in $\mathbb{Z}[i]^k$.

For the example let us take the module $M_{4,0}(\mathbb{Z}[i])$, so we have $m = 0$ and $k = (n+1) = 5$. The basis of M is formed by $e_1 = x^4$, $e_2 = x^3y$, $e_3 = x^2y^2$, $e_4 = xy^3$ and $e_5 = y^4$. Now $e_1 = x^4$ is represented by the vector $(1,0,0,0,0)^t$, e_2 is represented by $(0,1,0,0,0)^t$ and so on. We also have to compute how σ, σ_1, τ and τ_1 act on x and y (because of $m = 0$ we don't have u and v here). We get $\sigma(x) = y$, $\sigma(y) = -x$, $\sigma_1(x) = iy$, $\sigma_1(y) = ix$, $\tau(x) = x + y$, $\tau(y) = -x$, $\tau^2(x) = y$, $\tau^2(y) = -(x + y)$, $\tau_1(x) = -x + iy$, $\tau_1(y) = ix$, $\tau_1^2(x) = iy$ and $\tau_1^2(y) = ix + y$. The final matrix NMAT will be a (5×20)-matrix with entries in $\mathbb{Z}[i]$ then. The first (5×5)-block belongs to σ. Here we have $\sigma(e_1) = e_5$ and then $(1+\sigma)(e_1) = e_1+e_5 = (1,0,0,0,1)^t$, which just gives the first column of NMAT. With $\sigma(e_2) = -e_4$ and $(1 + \sigma)(e_2) = e_2 - e_4 = (0,1,0,-1,0)^t$ we then get the second column of NMAT and so on. The same has to be done for σ, σ_1, τ and τ_1 to fill the other blocks. For example, we find $(1 + \tau + \tau^2)(e_1) = (2,4,6,4,2)^t$ which would be the first column of the third block and we get $(1 + \tau_1 + \tau_1^2)(e_1) = (2,-4i,-6,4i,2)^t$ giving the first column of the last block. Now we can determine the rank of NMAT and the elementary divisors. We get rank(NMAT) $= 4$ and therefore

$$\operatorname{rank}(H^2(\Gamma, M_{4,0}(\mathbb{Z}[i]))) \;=\; 1,$$

and we find only 2-torsion in that case. If one analyses the matrices NMAT one finds quite a few symmetries and opportunities to simplify the matrices but in general it becomes a long computation by hand to actually determine the elementary divisors. Since we also can't control the binomial coefficients we get from powers of terms like $x + y$ it seems to be extremely difficult to actually proof something about the rank in general in that way, even if we have a very simple pattern as we will see later.

If we treat a more general module $M = M_{n,m}(\mathbb{Z}[i])$ we also have to deal with the variables u and v. Here we just have to choose a suitable order for the basis (see Section 1.2) and go on as described above. The computation of $H^2(\Gamma'_\infty, M)$ follows the same strategy. From Proposition 2.2 we have

$$H^2(\Gamma'_\infty, M) \;=\; M/((\rho_1 - 1)M + (\rho_2 - 1)M).$$

So we end up with a $(k \times 2k)$-matrix containing the two $(k \times k)$-blocks coming from ρ_1 and ρ_2. The action of Γ'_∞ gives $\rho_1(x) = x$, $\rho_1(y) = x + y$, $\rho_2(x) = x$ and $\rho_2(y) = ix+y$ and then one has to build up the corresponding matrix NMAT column by column as above.

Now, our goal became to go through several series of $M_{n,m}(\mathbb{Z}[i])$ to see whether there is more than 2-torsion and 3-torsion, to find out whether there would be a pattern for the torsion and which phenomena would occur. Of course, we were also

interested in the pattern for the ranks. For $n \neq m$ the ranks are quite well described by analytical methods but it is not clear what happens for $M_{n,n}(\mathbb{Z}[i])$ in general. Another point was to compare the torsion results for $H^2(\Gamma, M)$ and $H^2(\Gamma'_\infty, M)$.
For that we had to realize our strategy in a computer program. Here we first had to find the right balance between getting enough relevant data and a reasonable amount of time for developing the software. On the other hand we had in mind that we would like to do other computations, e. g. for $H^1(\Gamma, M)$, and that we would like to use the general setup for the following computations of Hecke eigenvalues as well without inventing everything again. So it seemed to be useful to realize everything in a quite conceptual way from the very beginning. A large part of the computation involves linear algebra for modules, number theory and group theory and one needs a good handling of the data. So it seemed to be quite clear to use a Computer Algebra System for the computations which is not too specialized. Since only MAPLE was quite generally available on our computers in the department we decided to start the development there to see how it works and perhaps to combine it with the use of other Computer Algebra Systems for algebraic number theory (like PARI or KANT) or group theory (like GAP) and, if promising, to transform it to another system later. Another very good possibility would have been MAGMA since it covers more or less the mixture of things we need, seems to be very efficient and is really prepared for more advaned algebraic computations, but at the beginning it was not available for us, and it was still very much in development. One more important reason to start with MAPLE was that it has an implementation of the Smith algorithm over $\mathbb{Z}[i]$ which was just what we needed. In MAGMA there is still only a realization over $\mathbb{Z}$ but the implementation for the general case of Dedekind rings is planned. So we could avoid the realization of the Smith algorithm and could develop quite fast under MAPLE. Finally, it turned out that we could perform the computations quite far and could get enough data to see significant patterns.
So we could compute the Smith normal form for $H^2(\Gamma, M)$ for $M = M_{n,0}(\mathbb{Z}[i])$ up to $n = 40$ and got similar series for $m \neq 0$. Since the matrices for Γ'_∞ are much smaller we could do all these computations for $H^2(\Gamma'_\infty, M)$ as well. A central point in the program is the construction of the matrix NMAT. For that we needed a good realization of the monomial basis for $M_{n,m}(\mathbb{Z}[i])$ and a conceptual realization for the action of Γ on $M_{n,m}$ which could be used for other rings of integers $\mathcal{O}_K$ as well. For larger n, m it takes already quite long to build up these matrices and MAPLE uses an enormous amount of memory for that. Later we realized the same procedure in MATHEMATICA for a test and it was amazingly faster. The critical step is then the Smith algorithm. In MAPLE the transformation matrices U and V get huge entries which limits the possibilities of computation in the end. So one idea is to use the LLL-algorithm in between to simplify the columns but that is limited as well. We used it with success in several computations of Hecke eigenvalues for $H_1(\Gamma, M)$, since we needed the matrix U for further computations. Recently we saw that MAGMA

can perform the Smith algorithm over $\mathbb{Z}$ for large matrices in a quite short time and that U has small entries. So one can get at least an overview of the elementary divisors, because one can just interpret the $\mathbb{Z}[i]$-lattice as $\mathbb{Z}$-lattice. It also seems to be useful to do further computations over general Dedekind rings with MAGMA (or KANT), when the algorithms will be realized. In special cases we could also realize them before and could enlarge some of the tables we got. A main ingredient for the Smith algorithm is the realization of a kind of Euclidean algorithm. With the right Smith algorithm at hand we could quite easily perform the whole computation for other groups $\Gamma = PSL_2(\mathcal{O}_K)$ or congruence subgroups of them. For that we would need the fundamental cellular domain as to be found e. g. in [Me]. Then we need the stabilizers, can produce similar formulas for $H^2(\Gamma, M)$, have to build up the NMAT in a similar way and can use a kind of Smith algorithm to determine the module invariants. For the present we have it only for Euclidean rings or principal ideal domains but in [Co2] a more general approach for all Dedekind rings is described which should be analysed in view of our situation. Note that all these algorithmic remarks also concern the computations made in Chapter 3, 4 and 5 as well as the suggestions for further work in Chapter 6 and are therefore a bit longer here.
Another nice idea for detecting the torsion is to compute the rank of the matrix NMAT modulo prime ideals. We used this idea to perform the computations much further than mentioned above. We compute the K-rank and the rank modulo the prime ideal. If there is a difference we just detect as many ℓ-torsions as the prime element appears as a factor in different diagonal elements (refering to the $\mathbb{Z}$-structure of the considered $\mathcal{O}_K$-modules here). Note that we always have to consider two prime ideals in the cases of split ℓ, which is covered by using the two roots of -1 in $\mathbb{F}_\ell$ as substitutes for i. The rank differences for both roots were always the same in the following computed cases, so we want to fix here, that we count them for one root only. Actually, we don't need the K-rank since the full rank appears for most of the primes. With that computation modulo ℓ we continued e. g. the list for $H^2(\Gamma, M_{n,0}(\mathbb{Z}[i]))$ up to $n = 80$, but we could have gone further as well. We checked the first thousend primes (which is up to 7919) and found a lot of large and interesting torsion. Recently G. Harder showed us the list in [Ca] where a computation for the special series $M_{n,0}(\mathbb{Z}[i])$ was done for $PGL_2(\mathbb{Z}[i])$ (based on sheaf cohomology) using a C-program, and so we additionally checked case by case some of the really huge primes, which then all appeared in our case too. On the other hand we found several torsion which didn't appear in the list of [Ca] because we computed over $PSL_2(\mathbb{Z}[i])$. Using the Smith algorithm over $\mathbb{Z}$ in MAGMA would give all these huge torsions as well and probably one would find some more for PSL_2. We also computed several series for $m \neq 0$ and in particular we treated the cases $M_{n,n}(\mathbb{Z}[i])$. We summarize all results in tables in Section 2.3 and discuss the phenomena we see.

2.3 Results for $\Gamma = PSL_2(\mathbb{Z}[i])$ and Γ'_∞

Let us start with the results for the ranks for $M_{n,m}(\mathbb{Z}[i])$. Recall that we have $n+m$ even (see Section 1.2). We first computed the series $H^2(\Gamma, M_{n,0}(\mathbb{Z}[i]))$ and found (for $n > 0$ up to $n = 150$):

$$\mathrm{rank}(H^2(\Gamma, M_{n,0}(\mathbb{Z}[i])) \;=\; \begin{cases} 1 & \text{for } n \equiv 0 \bmod 4, \\ 0 & \text{for } n \equiv 2 \bmod 4 \end{cases}$$

and

$$\mathrm{rank}(H^2(\Gamma'_\infty, M_{n,0}(\mathbb{Z}[i]))) \;=\; 1.$$

We got a similar picture for the general $M = M_{n,m}(\mathbb{Z}[i])$. For $n \neq m$ with $n, m > 0$ we found that the rank of $H^2(\Gamma, M)$ is always 0 for $n+m \equiv 2 \bmod 4$ with n, m even and for $n+m \equiv 0 \bmod 4$ with n, m odd. The rank is 1 for $n+m \equiv 0 \bmod 4$ with n, m even and $n+m \equiv 2 \bmod 4$ with n, m odd. For the group Γ'_∞ we always found rank 1 (also for $n = m$). This fits quite well in the picture of the vanishing theorem and the analytical understanding of the Eisenstein part of the cohomology.
In the interesting case of $M = M_{n,n}(\mathbb{Z}[i])$ we found for the ranks of $H^2 = H^2(\Gamma, M)$:

n	0	2	4	6	8	10	12	14	16	18	20	22	24	26
rkH^2	0	1	1	1	1	2	1	2	2	2	2	3	2	3

Table 1: n even

n	1	3	5	7	9	11	13	15	17	19	21
rkH^2	1	1	2	3	3	4	5	5	6	7	7

Table 2: n odd

Since we didn't need the Smith normal form for further computations (in contrast to the computations of the Hecke eigenvalues for the first homology), we only want to show one table with the precise elementary divisors we computed. In the other cases we want to present the results we got from the computations modulo ℓ, because we need them mainly to get an overview and to compare them with other results, e. g. for the first homology.
Let us start with the Smith normal form for $M_{n,0}(\mathbb{Z}[i])$. Here $c \times d$ means that we have c entries d on the diagonal. We found:

n	elementary divisors with factorization
2	2
4	2
6	$2, \quad (1+i) \cdot 2 \cdot 3$
8	$1+i, \; 2 \cdot 7$
10	$2 \times 2, (1+i) \cdot 2, \; 2^3 \cdot 5^2$
12	$1+i, \; 2, \; 2 \cdot 3 \cdot 11$
14	$1+i, \; 2 \times 2, \; (1+i) \cdot 2, \; (1+i) \cdot 2^2 \cdot 5 \cdot 7$
16	$1+i, \; 3 \times 2, 2 \cdot 3 \cdot 5^2 \cdot 11 \cdot 13$
18	$1+i, \quad 2 \times 2, \quad 2 \times (1+i) \cdot 2, 2^4 \cdot 3^3 \cdot 5 \cdot 7$
20	$2 \times (1+i), \; 3 \times 2, (1+i) \cdot 2^2 \cdot 3 \cdot 5 \cdot 13 \cdot 17 \cdot 19$
22	$1+i, \; 3 \times 2, 2 \times (1+i) \cdot 2, (1+i) \cdot 2^2 \cdot 5, \quad 2^6 \cdot 3 \cdot 5^2 \cdot 11$
24	$2 \times (1+i), \; 4 \times 2, \; 2 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 23$
26	$2 \times (1+i), \; 3 \times 2, 2 \times (1+i) \cdot 2, 2^3 \cdot 5, \; (1+i) \cdot 2^3 \cdot 3 \cdot 5^2 \cdot 11 \cdot 13^2$
28	$2 \times (1+i), \; 5 \times 2, 2 \cdot 3 \cdot 5, \; (1+i) \cdot 2^2 \cdot 3^4 \cdot 5^4 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
30	$2 \times (1+i), \; 3 \times 2, 3 \times (1+i) \cdot 2, \; (1+i) \cdot 2^3 \cdot 3 \cdot 5,$ $(1+i) \cdot 2^5 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
32	$3 \times (1+i), \; 5 \times 2, 2 \cdot 5^2,$ $(1+i) \cdot 2^2 \cdot 3^2 \cdot 5^2 \cdot 13 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31$
34	$2 \times (1+i), \; 4 \times 2, 3 \times (1+i) \cdot 2, (1+i) \cdot 2^2 \cdot 5, \; 2^5 \cdot 5,$ $(1+i) \cdot 2^5 \cdot 3^2 \cdot 5^2 \cdot 13 \cdot 17^2 \cdot \mathbf{151}$
36	$3 \times (1+i), \; 6 \times 2, 2 \cdot 3^2 \cdot 5^2, \; 2^2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 23 \cdot 29 \cdot 31$
38	$3 \times (1+i), \; 4 \times 2, 3 \times (1+i) \cdot 2, \; (1+i) \cdot 2^2 \cdot 5, \; (1+i) \cdot 2^3 \cdot 3 \cdot 5,$ $2^7 \cdot 3^2 \cdot 5^2 \cdot 13 \cdot 17 \cdot 19 \cdot 29$
40	$3 \times (1+i), \; 7 \times 2, \; 2 \cdot 5, \; 2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 13,$ $(1+i) \cdot 2^3 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot \mathbf{661}$

Table 3: Elementary divisors for $H^2(PSL_2(\mathbb{Z}[i]), M_{n,0}(\mathbb{Z}[i]))$

From the computation mod ℓ we got the following tables for $M_{n,m}(\mathbb{Z}[i])$. Here we just give for each n a list of pairs $[\ell, c_\ell]$, where ℓ is a prime and c_ℓ is the rank difference one gets from the computation mod ℓ. From now on we don't count the 2-torsion and 3-torsion anymore (orders of the finite subgroups in Γ). We present tables for the series $M_{n,0}(\mathbb{Z}[i])$ up to $M_{n,5}(\mathbb{Z}[i])$. The tables start with larger n since $M_{n,m}(\mathbb{Z}[i]) \cong M_{m,n}(\mathbb{Z}[i])$ and we also add a table for $M_{n,n}(\mathbb{Z}[i])$. Finally, we present tables for Γ'_∞ in the cases $M_{n,0}(\mathbb{Z}[i])$ and $M_{n,1}(\mathbb{Z}[i])$.
The main observation is that we find really large torsion classes (large ℓ) in the global second cohomology $H^2(\Gamma, M)$ but all these torsions don't appear for Γ'_∞, where we only find primes up to $\max(n, m)$. The ℓ-torsion classes for ℓ greater than $\max(n, m)$ we want call *large* from now on, the other ones *small*. We also mark the large primes ℓ with bold print in the tables.
Furthermore $H^2(\Gamma, M)$ is closely related to $H_1(\Gamma, M)$. For more detailed comments and analogous tables we refer to Section 4.4. We also computed Hecke eigenvalues on many torsion classes for $H_1(\Gamma, M)$. Again there seems to be an obvious difference between the large and the small torsion classes concerning congruence properties. For that we refer to Section 5.4.

n	ℓ-torsion up to $\ell = 7919$ and for several large extra primes
2	
4	
6	
8	[7,1]
10	[5,1]
12	[11,1]
14	[5,1], [7,1]
16	[5,1], [11,1], [13,1]
18	[5,1], [7,1]
20	[5,1], [13,1], [17,1], [19,1]
22	[5,2], [11,1]
24	[5,1], [7,1], [11,1], [17,1], [19,1], [23,1]
26	[5,2], [11,1], [13,1]
28	[5,2], [13,1], [17,1], [13,1], [23,1]
30	[5,2], [7,1], [11,1], [13,1]
32	[5,2], [13,1], [19,1], [23,1], [29,1], [31,1]
34	[5,3], [13,1], [17,1], [**151**,1]
36	[5,2], [7,1], [11,1], [17,1], [23,1], [29,1], [31,1]

n	ℓ-torsion up to $\ell = 7919$ and for several large extra primes
38	[5,3], [13,1], [17,1], [19,1], [29,1]
40	[5,3], [7,2], [11,1], [13,2], [17,1], [23,1], [29,1], [31,1], [37,1], [**661**,1]
42	[5,3], [7,2], [13,1], [17,1], [19,1], [**641**,1]
44	[5,3], [13,2], [17,1], [29,1], [31,1], [37,1], [41,1], [43,1], [**67**,1], [**137**,1]
46	[5,4], [7,1], [11,1], [13,1], [17,1], [19,1], [23,1], [**139**,1]
48	[5,3], [13,1], [29,1], [31,1], [37,1], [41,1], [43,1], [47,1]
50	[5,4], [11,1], [13,1], [17,1], [19,1], [23,1], [**59547091**,1]
52	[5,4], [7,1], [13,2], [17,2], [29,1], [31,1], [37,1], [41,1], [43,1], [47,1], [**3011**,1]
54	[5,4], [7,1], [13,2], [17,1], [19,1], [23,1], [**163**,1]
56	[5,4], [7,3], [11,1], [13,2], [17,2], [31,1], [37,1], [41,1], [43,1], [47,1], [53,1], [**461**,1]
58	[5,5], [7,1], [13,2], [17,1], [23,1], [29,1], [**367**,1], [**945929**,1]
60	[5,4], [7,1], [11,2], [13,1], [17,2], [19,1], [29,1], [37,1], [41,1], [47,1], [53,1], [59,1], [**1650371**,1]
62	[5,5], [13,2], [17,1], [23,1], [29,1], [31,1], [**26387**,1]
64	[5,5], [7,2], [11,1], [13,2], [17,1], [19,1], [29,1], [37,1], [41,1], [43,1], [47,1], [53,1], [59,1], [61,1], [**197**,1], [**103979**,1]
66	[5,5], [7,1], [11,1], [13,2], [17,1], [23,1], [29,2], [31,1], [**19920917**,1]
68	[5,5], [7,2], [13,3], [17,2], [19,1], [29,2], [37,1], [41,1], [43,1], [47,1], [53,1], [59,1], [61,1], [67,1], [**503**, 1], [**1297**,1], [**1531**,1]
70	[5,6], [7,1], [11,2], [13,2], [17,2], [29,1], [31,1], [**429901**,1]
72	[5,5], [7,2], [13,2], [17,2], [19,1], [23,1], [29,1], [41,1], [43,1], [47,1], [53,1], [59,1], [61,1], [67,1], [71,1]
74	[5,6], [7,1], [11,1], [13,2], [17,2], [29,1], [31,1], [37,1]
76	[5,6], [7,1], [13,3], [17,2], [23,1], [29,1], [37,1], [41,1], [43,1], [47,1], [53,1], [59,1], [61,1], [67,1], [71,1], [73,1], [**179**,1], [**41193114818503**,1]
78	[5,6], [7,1], [13,3], [17,2], [19,1], [29,1], [31,1], [37,1], [**381**,1], [**631**,1]
80	[5,6], [7,2], [11,1], [13,3], [17,1], [23,1], [29,1], [37,1], [43,1], [47,2], [**97**,1], [**2647**,1], [**3347**,1]

Table 4: ℓ-torsion in $H^2(PSL_2(\mathbb{Z}[i]), M_{n,0}(\mathbb{Z}[i]))$

n	ℓ-torsion up to $\ell = 7919$
3	
5	
7	[5,1], [7,1]
9	
11	[5,1], [7,1], [11,1]
13	[5,1]
15	[5,2], [7,1], [11,1], [13,1]
17	[5,1], [7,1]
19	[5,2], [11,1], [13,1], [17,1], [19,1]
21	[5,2], [7,1], [**59**,1]
23	[5,2], [7,1], [11,1], [13,1], [17,1], [19,2], [23,1], [**37**,1]
25	[5,2], [7,1], [11,1]
27	[5,3], [7,1], [11,1], [13,1], [17,1], [19,1], [23,1], [**139**,1], [**347**,1]
29	[5,2], [7,1], [11,1], [13,1], [**73**,1], [**239**,1]
31	[5,3], [7,1], [13,1], [17,1], [19,1], [23,1], [29,1], [31,1], [**83**,1], [**293**,1]
33	[5,3], [7,2], [11,1], [13,1], [**47**,1], [**53**,1], [**113**,1], [**191**,1]
35	[5,4], [7,2], [11,1], [13,1], [17,1], [19,1], [23,1], [29,1], [31,1], [**101**,1], [**523**,1], [**5333**,1]
37	[5,3], [7,1], [11,1], [13,1], [17,1], [**211**,1], [**5087**,1]
39	[5,4], [7,1], [11,1], [13,2], [17,1], [19,1], [23,1], [29,1], [31,1], [37,1], [41,1], [**769**,1]
41	[5,4], [11,1], [13,1], [17,1], [19,1], [41,1], [**1747**,1]
43	[5,4], [7,1], [11,1], [13,2], [17,1], [19,1], [23,1], [29,1], [31,1], [37,1] [41,1], [43,1]
45	[5,4], [7,1], [11,1], [13,1], [17,1], [19,1], [**47**,1], [**6361**,1]
47	[5,5], [7,2], [11,1], [13,2], [17,1], [19,1], [23,1], [29,1], [31,1], [37,1], [41,1], [43,1], [47,2]
49	[5,4], [7,3], [11,1], [13,1], [17,1], [19,1], [23,1], [37,1], [**1759**,1]
51	[5,5], [7,1], [11,1], [13,2], [17,2], [19,1], [23,1], [29,1], [31,1], [37,1], [41,1], [43,1], [47,1], [**79**,1], [**233**,1]

Table 5: ℓ-torsion in $H^2(PSL_2(\mathbb{Z}[\,i\,]), M_{n,1}(\mathbb{Z}[\,i\,]))$

n	ℓ-torsion up to $\ell = 7919$
4	
6	[5,1]
8	
10	[5,1], [7,1]
12	[5,1]
14	[5,1], [7,1], [11,1], [13,1]
16	[5,1], [7,1], [**197**,1]
18	[5,2], [7,1], [13,1], [17,1], [**53**,1]
20	[5,2], [7,1]
22	[5,2], [7,2], [11,1], [13,1], [17,1], [19,1], [**43**,1], [**599**,1]
24	[5,2], [7,1], [11,1], [**31**,1], [**2053**,1]
26	[5,3], [7,2], [11,1], [13,1], [17,1], [19,1], [23,1], [**47**,1]
28	[5,2], [7,3], [11,1], [13,1], [**89**,1], [**107**,1], [**829**,1]
30	[5,3], [7,1], [11,1], [13,1], [17,1], [19,1], [23,1], [29,1], [**569**,1]
32	[5,3], [7,2], [11,1], [13,1]
34	[5,3], [7,1], [11,2], [13,1], [17,1], [19,1], [23,1], [29,1], [31,1], [**61**,1], [**2039**,1]
36	[5,3], [7,1], [11,1], [13,1], [17,1], [23,1], [31,1], [**41**,1]
38	[5,4], [7,2], [11,3], [13,1], [17,1], [19,1], [23,1], [29,1], [31,1], [37,1], [**61**,1], [**613**,1], [**1523**,1]
40	[5,4], [7,2], [11,1], [13,1], [17,1], [19,1], [**59**,1], [**967**,1], [**6089**,1]
42	[5,4], [7,3], [11,1], [13,2], [17,1], [19,2], [23,1], [29,1], [31,1], [37,1], [41,1]
44	[5,4], [7,2], [11,2], [13,1], [17,1], [19,1], [41,1], [**811**,1], [**7253**,1]
46	[5,5], [7,1], [11,1], [13,2],[17,1], [19,1], [23,1], [29,1], [31,1], [37,1], [41,1], [43,1], [**1453**,1], [**2437**,1]
48	[5,4], [7,1], [11,3], [13,1], [17,1], [19,1], [23,1], [**1373**,1]
50	[5,5], [7,2], [11,1], [13,2], [17,1], [19,1], [23,1], [29,1], [31,1], [37,1], [41,1], [43,1],[47,1], [**103**,1], [**359**,1]

Table 6: ℓ-torsion in $H^2(PSL_2(\mathbb{Z}[i]), M_{n,2}(\mathbb{Z}[i]))$

n	ℓ-torsion up to $\ell = 7919$
5	[5,1]
7	
9	[5,1],[7,1]
11	[5,1]
13	[5,1], [7,1], [11,1], [13,1], [**23**,1]
15	[5,1], [7,1]
17	[5,2], [7,1], [11,1], [13,1], [17,1], [**37**,1], [**257**,1]
19	[5,1], [7,2], [**3319**,1]
21	[5,2], [7,3], [11,1], [13,1], [17,1], [19,1], [**59**,1], [**941**,1]
23	[5,2], [7,1], [11,1], [**199**,1], [**283**,1], [**487**,1]
25	[5,3], [7,2], [11,1], [13,2], [17,1], [19,1], [23,1]
27	[5,2], [7,1], [11,2], [13,1]
29	[5,3], [7,2], [11,1], [13,1], [17,1], [19,1], [23,1], [29,1]
31	[5,3], [7,2], [11,1], [13,1], [23,1], [**43**,1], [**701**,1]
33	[5,3], [7,2], [11,2], [13,1], [17,1], [19,1], [23,1], [29,1], [31,1]
35	[5,3], [7,4], [11,1], [13,1], [17,1], [23,1], [**127**,1], [**1451**,1], [**5237**,1]
37	[5,4, [7,2], [11,3], [13,2], [17,1], [19,1], [23,1], [29,1], [31,1], [37,1]
39	[5,3], [7,2], [11,1], [13,2], [17,1], [19,1], [**179**,1], [**367**,1]
41	[5,4], [7,2], [11,2], [13,2], [17,1], [19,1], [23,1], [29,1], [31,1], [37,1], [41,1], [**97**,1]
43	[5,4], [7,2], [11,1], [13,1], [17,1], [19,2]
45	[5,5], [7,3], [11,2], [13,2], [17,1], [19,1], [23,1], [29,1], [31,1], [37,1], [41,1], [43,1], [**103**,1]
47	[5,4], [7,3], [11,2], [13,1], [17,1], [19,1], [23,1]
49	[5,5], [7,4], [11,3], [13,3], [17,1], [19,1], [23,1], [29,1], [31,1], [37,1], [41,1], [43,1], [47,1], [**199**,1], [**3041**,1], [**6911**,1]
51	[5,5], [7,3], [11,4], [13,2], [17,1], [19,1], [23,2], [**101**,1]

Table 7: ℓ-torsion in $H^2(PSL_2(\mathbb{Z}[\,i\,]), M_{n,3}(\mathbb{Z}[\,i\,]))$

n	ℓ-torsion up to $\ell = 7919$
6	
8	[5,1], [7,1]
10	[5,1]
12	[5,1], [7,2], [11,1], [**23**,1]
14	[5,1], [7,2], [**139**,3]
16	[5,2], [7,1], [11,2], [13,1]
18	[5,1], [7,1], [**29**,1], [**53**,1], [**61**,1]
20	[5,2], [7,1], [11,1], [13,1], [17,1], [19,1], [37,1], [**127**,1]
22	[5,2], [7,1], [11,1], [**709**,1]
24	[5,2], [7,2], [11,1], [13,1], [17,1], [19,2], [23,1], [31,1], [**127**,1], [**4289**,1]
26	[5,2], [7,2], [11,2], [13,1], [**43**,1], [**877**,1]
28	[5,3], [7,4], [11,1], [13,1], [17,1], [19,1], [23,2], [**977**,1], [**3343**,1]
30	[5,3], [7,2], [11,1], [13,1], [**113**,1], [**233**,1], [**307**,1], [**5477**,1]
32	[5,3], [7,2], [11,1], [13,1], [17,1], [19,1], [23,2], [29,1], [31,1], [**733**,1]
34	[5,3], [7,2], [11,1], [13,1], [17,1], [**2467**,1], [**7243**,1]
36	[5,4], [7,3], [11,2], [13,1], [17,1], [19,1], [23,2], [29,1], [31,3]
38	[5,3], [7,3], [11,2], [13,1], [17,1], [19,1], [29,1], [**61**,1], [**2713**,1], [**6883**,1]
40	[5,4], [7,3], [11,4], [13,2], [17,1], [19,1], [23,1], [29,1], [31,1], [37,1], [**83**,1]

Table 8: ℓ-torsion in $H^2(PSL_2(\mathbb{Z}\,[\,i\,]), M_{n,4}(\mathbb{Z}\,[\,i\,]))$

n	ℓ-torsion up to $\ell = 7919$
7	[5,2], [7,2]
9	[5,1]
11	[5,3], [7,1], [11,1]
13	[5,2]
15	[5,4], [7,1], [11,2], [13,1], [**43**,1], [**691**,1]
17	[5,3], [7,1], [13,1], [**167**,1]
19	[5,4], [7,2], [11,1], [13,3], [17,1], [19,1], [**23**,1], [**31**,1], [**3119**,1]
21	[5,4], [7,3], [17,1], [**367**,1]
23	[5,5], [7,2], [11,1], [13,1], [17,2], [19,1], [23,2], [**83**,1], [**2789**,1]
25	[5,5], [7,1], [11,1], [23,1], [**1171**,1]
27	[5,6], [7,2], [11,2], [13,1], [17,2], [19,1], [23,1], [31,1], [**491**,1]
29	[5,5], [7,2], [11,3], [13,2], [**73**,1],[**137**,1]
31	[5,7], [7,3], [11,1], [13,3], [17,1], [19,2], [23,1], [29,1], [31,1]
33	[5,6], [7,3], [11,1], [13,3], [29,1], [31,1], [**1303**,1]
35	[5,8], [7,5], [11,4], [13,1], [17,1], [19,1], [23,2], [29,2], [**71**,1], [**439**,1]

Table 9: ℓ-torsion in $H^2(PSL_2(\mathbb{Z}\,[\,i\,]), M_{n,5}(\mathbb{Z}\,[\,i\,]))$

n	ℓ-torsion up to $\ell = 97$
0	
1	
2	
3	
4	
5	
6	[5,3]
7	[7,1]
8	[5,2], [7,3]
9	
10	[5,3], [7,2]
11	[5,2], [11,1]
12	[5,3], [7,2], [11,3]
13	[5,2]
14	[5,3], [7,1], [11,2], [13,3]
15	[5,4], [7,3]
16	[5,7], [7,5], [11,4], [13,2]
17	[5,4], [7,2]
18	[5,8], [7,4], [11,2], [13,2], [17,2], [**19**,2], [**23**,2]
19	[5,4], [7,2], [13,2], [19,1]
20	[5,10], [7,3], [11,2], [13,2], [17,2], [19,3]
21	[5,10], [7,3]
22	[5,10], [7,7], [11,1], [13,2], [17,2], [19,2]
23	[5,8], [7,5], [11,3], [23,1]
24	[5,10], [7,13], [11,5], [13,2], [17,2], [19,2], [23,3]
25	[5,12], [7,4], [11,2], [13,2], [17,2]
26	[5,17], [7,7], [11,6], [13,3], [17,2], [19,3], [23,4]
27	[5,14], [7,4], [11,2], [13,2]
28	[5,16], [7,8], [11,6], [13,5], [17,2], [19,6], [23,2], [**29**, 2]
29	[5,14], [7,7], [11,2], [13,2]

Table 10: ℓ-torsion in $H^2(PSL_2(\mathbb{Z}[i]), M_{n,n}(\mathbb{Z}[i]))$

n	ℓ-torsion up to $\ell = 7919$
2	
4	
6	[5,1]
8	[5,1], [7,1]
10	[5,2], [7,1]
12	[5,2], [7,1], [11,1]
14	[5,2], [7,1], [11,1], [13,1]
16	[5,3], [7,1], [11,1], [13,1]
18	[5,3], [7,1], [11,1], [13,1], [17,1]
20	[5,4], [7,1], [11,1], [13,1], [17,1], [19,1]
22	[5,4], [7,1], [11,1], [13,1], [17,1], [19,1]
24	[5,4], [7,1], [11,1], [13,1], [17,1], [19,1], [23,1]
26	[5,5], [7,1], [11,1], [13,2], [17,1], [19,1], [23,1]
28	[5,5], [7,1], [11,1], [13,2], [17,1], [19,1], [23,1]
30	[5,6], [7,1], [11,1], [13,2], [17,1], [19,1], [23,1], [29,1]
32	[5,6], [7,1], [13,2], [17,1], [19,1], [23,1], [29,1], [31,1]
34	[5,6], [11,1], [13,2], [17,2], [19,1], [23,1], [29,1], [31,1]
36	[5,7], [7,1], [11,1], [13,2], [17,2], [19,1], [23,1], [29,1], [31,1]
38	[5,7], [7,1], [11,1], [13,2], [17,2], [19,1], [23,1], [29,1], [31,1], [37,1]
40	[5,8], [7,1], [11,1], [13,3], [17,2], [19,1], [23,1], [29,1], [31,1], [37,1]
42	[5,8], [7,1], [11,1], [13,3], [17,2], [19,1], [23,1], [29,1], [31,1], [37,1], [41,1]
44	[5,8], [7,1], [11,1], [13,3], [17,2], [19,1], [23,1], [29,1], [31,1], [37,1], [41,1], [43,1]
46	[5,9], [7,1], [11,1], [13,3], [17,2], [19,1], [23,1], [29,1], [31,1], [37,1], [41,1], [43,1]
48	[5,9], [11,1], [13,3], [17,2], [19,1], [23,1], [29,1], [31,1], [37,1], [41,1], [43,1], [47,1]
50	[5,10], [7,1], [11,1], [13,3], [17,2], [19,1], [23,1], [29,1], [31,1], [37,1], [41,1], [43,1], [47,1]
52	[5,10], [7,1], [11,1], [13,4], [17,3], [19,1], [23,1], [29,1], [31,1], [37,1], [41,1], [43,1], [47,1]

n	ℓ-torsion up to $\ell = 7919$
54	[5,10], [7,1], [13,4], [17,3], [19,1], [23,1], [29,1], [31,1], [37,1], [41,1], [43,1] [47,1], [53,1]
56	[5,11], [7,2], [11,1], [13,4], [17,3], [23,1], [29,1], [31,1], [37,1], [41,1], [43,1] [47,1], [53,1]
58	[5,11], [7,2], [11,1], [13,4], [17,3], [19,1], [23,1], [29,2], [31,1], [37,1], [41,1], [43,1], [47,1], [53,1], [59,1]
60	[5,12], [7,2], [11,1], [13,4], [17,3], [19,1], [23,1], [29,2], [31,1], [37,1], [41,1], [43,1], [47,1], [53,1], [59,1]
62	[5,12], [7,1], [11,1], [13,4], [17,3], [19,1], [23,1], [29,2], [31,1], [37,1], [41,1], [43,1], [47,1], [53,1], [59,1], [61,1]
64	[5,12], [7,2], [11,1], [13,4], [17,3], [19,1], [23,1], [29,2], [31,1], [37,1], [41,1], [43,1], [47,1], [53,1], [59,1], [61,1]
66	[5,13], [7,2], [11,1], [13,5], [17,3], [19,1], [23,1], [29,2], [31,1], [37,1], [41,1], [43,1], [47,1], [53,1], [59,1], [61,1]
68	[5,13], [7,2], [11,1], [13,5], [17,4], [19,1], [29,2], [31,1], [37,1], [41,1], [43,1] [47,1], [53,1], [59,1], [61,1], [67,1]
70	[5,14], [7,2], [11,1], [13,5], [17,4], [19,1], [23,1], [29,2], [31,1], [37,1], [41,1], [43,1], [47,1], [53,1], [59,1], [61,1], [67,1]
72	[5,14], [7,2], [11,1], [13,5], [17,4], [19,1], [23,1], [29,2], [31,1], [37,1], [41,1], [43,1], [47,1], [53,1], [59,1], [61,1], [67,1], [71,1]
74	[5,14], [7,2], [11,1], [13,5], [17,4], [19,1], [23,1], [29,2], [31,1], [37,2], [41,1], [43,1], [47,1], [53,1], [59,1], [61,1], [67,1], [71,1], [73,1]
76	[5,15], [7,1], [13,5], [17,4], [19,1], [23,1], [29,2], [31,1], [37,2], [41,1], [43,1] [47,1], [53,1], [59,1], [61,1], [67,1], [71,1], [73,1]
78	[5,15], [7,2], [11,1], [13,6], [17,4], [19,1], [23,1], [29,2], [31,1], [37,2], [41,1], [43,1], [47,1], [53,1], [59,1], [61,1], [67,1], [71,1], [73,1]
80	[5,16], [7,2], [11,1], [13,6], [17,4], [19,1], [23,1], [29,2], [31,1], [37,2], [41,1], [43,1],[47,1], [53,1], [59,1], [61,1], [67,1], [71,1], [73,1], [79,1]

Table 11: ℓ-torsion in $H^2(\Gamma'_\infty, M_{n,0}(\mathbb{Z}[i]))$

n	ℓ-torsion up to $\ell = 7919$
1	
3	
5	[5,1]
7	[5,1], [7,1]
9	[5,1], [7,1]
11	[5,2], [7,1], [11,1]
13	[5,2], [7,1], [11,1], [13,1]
15	[5,3], [7,2], [11,1], [13,1]
17	[5,3], [7,2], [11,1], [13,1], [17,1]
19	[5,3], [7,2], [11,1], [13,1], [17,1], [19,1]
21	[5,4], [7,2], [11,1], [13,1], [17,1], [19,1]
23	[5,4], [7,2], [11,2], [13,1], [17,1], [19,1], [23,1]
25	[5,5], [7,2], [11,2], [13,1], [17,1], [19,1], [23,1]
27	[5,5], [7,1], [11,2], [13,2], [17,1], [19,1], [23,1]
29	[5,5], [7,2], [11,2], [13,2], [17,1], [19,1], [23,1], [29,1]
31	[5,6], [7,2], [11,2], [13,2], [17,1], [19,1], [23,1], [29,1], [31,1]
33	[5,6], [7,2], [11,2], [13,2], [17,1], [19,1], [23,1], [29,1], [31,1]
35	[5,7], [7,2], [11,2], [13,2], [17,2], [19,1], [23,1], [29,1], [31,1]
37	[5,7], [7,2], [11,2], [13,2], [17,2], [19,1], [23,1], [29,1], [31,1], [37,1]
39	[5,7], [7,2], [11,2], [13,3], [17,2], [19,2], [23,1], [29,1], [31,1], [37,1]
41	[5,8], [7,1], [11,2], [13,3], [17,2], [19,2], [23,1], [29,1], [31,1], [37,1], [41,1]
43	[5,8], [7,2], [11,1], [13,3], [17,2], [19,2], [23,1], [29,1], [31,1], [37,1], [41,1], [43,1]
45	[5,9], [7,2], [11,2], [13,3], [17,2], [19,2], [23,1], [29,1], [31,1], [37,1], [41,1], [43,1]
47	[5,9], [7,2], [11,2], [13,3], [17,2], [19,2], [23,2], [29,1], [31,1], [37,1], [41,1], [43,1], [47,1]
49	[5,9], [7,2], [11,2], [13,3], [17,2], [19,2], [23,2], [29,1], [31,1], [37,1], [41,1], [43,1], [47,1]
51	[5,10], [7,2], [11,2], [13,3], [17,3], [19,2], [23,2], [29,1], [31,1], [37,1], [41,1], [43,1], [47,1]

Table 12: ℓ-torsion in $H^2(\Gamma'_\infty, M_{n,1}(\mathbb{Z}[\,i\,]))$

3 Study of the first cohomology $H^1(\Gamma, M)$

To compute the first cohomology of Γ we could again start with a cellular domain as in Chapter 2, but this would become more complicated. On the other hand, the first group cohomology is easy to get as the quotient of the derivations modulo the principal derivations. Then the computation just involves the data directly given by the presentation of Γ. This can be realized quite well in computer programs and makes it possible to treat different groups quite easily. Furthermore one can define Hecke operators on the group cohomology and group homology in a nice way but the transfer to the topological side of Chapter 2 is quite complicated. That was the second reason for prefering this more algebraic approach.
In Section 3.1 we explain the general construction and carry out the first steps in the case of $\Gamma = PSL_2(\mathbb{Z}[\,i\,])$. Note that we want to distinguish between more general constructions and applications to specific arithmetic groups by the notations G resp. Γ for the groups in this and the next two chapters. Then we describe in Section 3.2, how the invariants of the cohomology modules (rank and torsion) can be computed. More details about the computations we carried out for several arithmetic groups and some experimental observations about the torsion are discussed in Section 3.3, and our computational results are put together in Section 3.4.

3.1 The group cohomological approach and $\Gamma = PSL_2(\mathbb{Z}[\,i\,])$

Let G be a group and M be a left G-module (abelian group M on which G acts by additive maps from the left). For $g \in G$ and $\mathrm{m} \in M$ we write $g \cdot \mathrm{m}$ for the action of g on m. Note that we also assume for our computation that M carries the structure of an $\mathcal{O}_K$-module. Let further $R = \mathcal{O}_K G$ be the group ring of G over $\mathcal{O}_K$. As an $\mathcal{O}_K$-module on which G acts $\mathcal{O}_K$-linearly from the left, M is a left R-module. To describe the first cohomology of G with coefficients in M we start with the standard approach as e. g. given in [Wei] or [Gr]. For that let $M^G = \{\mathrm{m} \in M : g \cdot \mathrm{m} = \mathrm{m} \text{ for all } g \in G\}$ be the invariant submodule of the R-module M. The invariant submodule functor is left exact and we have $M^G \cong \mathrm{Hom}_R(\mathcal{O}_K, M)$. Then one can define the qth cohomology with coefficients in the R-module M as the qth right derived functor of the invariants functor M^G. Then one gets $H^q(G, M) \cong \mathrm{Ext}^q_R(\mathcal{O}_K, M)$ and therefore $H^q(G, M)$ has the structure of an $\mathcal{O}_K$-module. But now we can express the first cohomology in a more suitable way.

Definition 3.1 A *derivation* of G in a left G-module M is a map $f : G \longrightarrow M$ satisfying

$$f(gh) \;=\; f(g) + gf(h)$$

for all $g, h \in G$.

We write $\mathrm{Der}(G, M)$ for the familiy of all these derivations from G in M. Since M is an $\mathcal{O}_K$-module $\mathrm{Der}(G, M)$ becomes an $\mathcal{O}_K$-module as well in the obvious way via $(f + \tilde{f})(g) = f(g) + \tilde{f}(g)$ and $(c \cdot f)(g) = c \cdot f(g)$ for all $g \in G$. For $\mathrm{m} \in M$ we further define $f_{\mathrm{m}}(g) := g \cdot \mathrm{m} - \mathrm{m}$. Obviously f_{m} is again a derivation. Such f_{m} are called *principal derivations* of G in M, and it holds e. g. $f_{\mathrm{m}+\mathrm{n}} = f_{\mathrm{m}} + f_{\mathrm{n}}$. Therefore he set $\mathrm{PDer}(G, M) = \{f_{\mathrm{m}} : \mathrm{m} \in M\}$ of principal derivations forms a submodule of the module $\mathrm{Der}(G, M)$. In particular we get $\mathrm{PDer}(G, M) \cong M/M^G$. Finally one can derive the following proposition (see e. g. [Wei]).

Proposition 3.2 $H^1(G, M) \cong \mathrm{Der}(G, M)/\mathrm{PDer}(G, M)$.

If we now have a presentation of the group G we can use the following construction to prepare an explicit computation of $H^1(G, M)$. For that let $g_1, \dots, g_r$ be a set of generators of G. Then we can consider the map

$$\Phi \ : \ \mathrm{Der}(G, M) \longrightarrow M^r$$

given by $\Phi(f) = (f(g_1), \dots, f(g_r))$ for $f \in \mathrm{Der}(G, M)$. This map Φ is injective, because two different derivations $f \neq \tilde{f}$ with $f(g_1) = \tilde{f}(g_1), \dots, f(g_r) = \tilde{f}(g_r)$ would lead to $f(g) = \tilde{f}(g)$ for all $g \in G$ (just express g as a word in $g_1, \dots, g_r$ and its inverses and use the rule in the definition of a derivation), so this cannot happen. It is easy to see that Φ is also an $\mathcal{O}_K$-module homomorphism and therefore we get $\Phi(\mathrm{Der}(G, M)) \cong \mathrm{Der}(G, M)$. Furthermore we can restrict Φ to $\mathrm{PDer}(G, M)$, which sends f_{m} to $((g_1 - 1)\mathrm{m}, \dots, (g_r - 1)\mathrm{m})$ for $\mathrm{m} \in M$ (note that $(g_i - 1) \in R$). Since $\mathrm{PDer}(G, M)$ is a submodule of $\mathrm{Der}(G, M)$ we have $\Phi(\mathrm{PDer}(G, M)) \cong \mathrm{PDer}(G, M)$. This gives

Proposition 3.3 $H^1(G, M) \cong \Phi(\mathrm{Der}(G, M))/\Phi(\mathrm{PDer}(G, M))$.

We set $C := \Phi(\mathrm{Der}(G, M))$ and $D := \Phi(\mathrm{PDer}(G, M))$. Now, our goal is to describe explicitly the submodules C and D of M^r and to represent them by certain matrices to finally compute the module invariants. As we saw above the whole construction is determined by the presentation of the group G. A derivation is determined by its values on the generators of the group. It remains to express the relations in the group inside the module M^r. The transfer of these relations is given through Φ, which sends the 1 in $\mathrm{Der}(G, M)$ to 0 in M^r. So C consits of all tupels $(\mathrm{m}_1, \dots, \mathrm{m}_r) \in M^r$ satisfying the relations given by $f(w_1) = f(w_2) = \ldots = f(w_s) = 0$, where $w_1 = w_2 = \ldots = w_s = 1$ are the relations in the group G and $f \in \mathrm{Der}(G, M)$. Hence we have an explicit description of the submodule C of the free module M^r via a set of relations. The description of the submodule D was already given in a suitable explicit form through $\Phi(f_{\mathrm{m}}) = ((g_1 - 1)\mathrm{m}, \dots, (g_r - 1)\mathrm{m})$. So we are able to represent both modules C and D through certain matrices and can determine the invariants we are interested in. The description of these matrices and the steps to compute the invariants will be discussed for the example $\Gamma = PSL_2(\mathbb{Z}[i])$ in Section 3.2. As a preperation for that computation we derive the relations in the module M^r now. For that first remember that $\Gamma = PSL_2(\mathbb{Z}[i]) = \langle A, B, U \mid R_1 = R_2 = R_3 = R_4 = R_5 = R_6 = 1 \rangle$ with A, B, U and $R_1, \ldots, R_6$ as in Proposition 1.1. Note that the module M is now one of the modules $M_{n,m}(\mathbb{Z}[i])$ of rank $k = (n+1)(m+1)$ as introduced in Section 1.2. The map Φ is

$$\Phi \ : \ \mathrm{Der}(G, M) \longrightarrow M \times M \times M,$$

sending f to $(f(A), f(B), f(U))$. We express $(\alpha, \beta, \delta)^t \in M^3$ via $\alpha = \sum_{i=1}^k \alpha_i e_i$, $\beta = \sum_{i=1}^k \beta_i e_i$ and $\delta = \sum_{i=1}^k \delta_i e_i$ in the basis of $M_{n,m}(\mathbb{Z}[i])$ (see Section 1.2) and go through the six relations now. In the first case we have $f(B^2) = f(B) + Bf(B) = (1+B)f(B) = 0$. This gives

Formula 3.4 (Relation 1) $\quad (1+B)\beta \ = \ 0.$

Note that $(1+B)\beta$ means that one has to evaluate the action of the identity on $\beta \in M$ and the action of B on $\beta \in M$ and to add the two results in M (since $(1+B)$ is in R but not in G). One easily sees that

$$f(g^3) \ = \ (1 + g + g^2)f(g) \tag{3.1}$$

for $g \in G$. With $f(AB) = f(A) + Af(B)$ we get

$$\begin{aligned} f((AB)^3) &= (1 + AB + (AB)^2)(f(A) + Af(B)) \\ &= (1 + AB + (AB)^2)f(A) + (A + ABA + (AB)^2A)f(B), \end{aligned}$$

which gives

Formula 3.5 (Relation 2) $\quad (1 + AB + (AB)^2)\alpha + (A + ABA + (AB)^2A)\beta \ = \ 0.$

We also find $f(g^{-1}) = -g^{-1}f(g)$. So we get

$$\begin{aligned} f(AUA^{-1}U^{-1}) &= f(AU) + AUf(A^{-1}U^{-1}) \\ &= f(AU) + AU(f(A^{-1}) + A^{-1}f(U^{-1})) \\ &= f(A) + Af(U) + AU(-A^{-1}f(A) - A^{-1}U^{-1}f(U)) \\ &= (1 - AUA^{-1})f(A) + (A - AUA^{-1}U^{-1})f(U), \end{aligned}$$

which gives

Formula 3.6 (Relation 3) $\quad (1 - AUA^{-1})\alpha + (A - AUA^{-1}U^{-1})\delta \ = \ 0.$

Next we find

$$\begin{aligned} f(BUBU^{-1}) &= f(BU) + BUf(BU^{-1}) \\ &= f(BU) + BU(f(B) + Bf(U^{-1})) \\ &= f(B) + Bf(U) + BU(f(B) - BU^{-1}f(U)) \\ &= (1 + BU)f(B) + (B - BUBU^{-1})f(U), \end{aligned}$$

which gives with (3.1)

Formula 3.7 (Relation 4) $T(1 + BU)\beta + T(B - BUBU^{-1})\delta \;=\; 0$

with $T := 1 + BUBU^{-1} + (BUBU^{-1})^2$.

We go on with

$$\begin{aligned} f(BU^2BU^{-1}) &= f(BU^2) + BU^2f(BU^{-1}) \\ &= f(B) + Bf(U^2) + BU^2(f(B) + Bf(U^{-1})) \\ &= f(B) + B(f(U) + Uf(U)) + BU^2(f(B) - BU^{-1}f(U)) \\ &= (1 + BU^2)f(B) + (B + BU - BU^2BU^{-1})f(U). \end{aligned}$$

In combination with $f(g^2) = (1 + g)f(g)$ this leads to

Formula 3.8 (Relation 5) $T(1 + BU^2)\beta + T(B + BU - BU^2BU^{-1})\delta \;=\; 0$

with $T := 1 + BU^2BU^{-1}$.

We finish with

$$\begin{aligned} & f(AUBAU^{-1}B) \\ &= f(AUB) + AUBf(AU^{-1}B) \\ &= f(A) + Af(B) + AUB(f(A) + Af(U^{-1}B)) \\ &= f(A) + Af(U) + AUf(B) + AUB(f(A) - AU^{-1}f(U) + AU^{-1}f(B)) \\ &= (1 + AUB)f(A) + (AU + AUBAU^{-1})f(B) + (A - AUBAU^{-1})f(U), \end{aligned}$$

which gives

Formula 3.9 (Relation 6)

$$T(1 + AUB)\alpha + T(AU + AUBAU^{-1})\beta + T(A - AUBAU^{-1})\delta \;=\; 0$$

with $T := 1 + AUBAU^{-1}B$.

Remark 3.10 One finds a similar pattern for the formulas above as described in more detail for the first homology at the end of Section 4.2. To extract the sum for the part $f(g)$ one has to take into account all contributions of letters g and g^{-1} in the word. In case of the letter g one has to take the word up to g not including g with positive sign, and for g^{-1} one has to take the word up to g^{-1} including g^{-1} with negative sign.

3.2 Computation of the module invariants

Our goal is to compute the first cohomology $H^1(\Gamma, M) \cong C/D$. We explain how the module invariants can be determined. To be more transparent we want to describe the method in more detail for $\Gamma = PSL_2(\mathbb{Z}[i])$ but it can be carried out for other Γ in the same way. For that let $L := M^3$. The first task is to represent the submodules C and D of L in a suitable matrix form. Since $M = M_{n,m}(\mathbb{Z}[i])$ has rank $k = (n+1)(m+1)$ we have that L is a free $\mathbb{Z}[i]$-module of rank $3k$. To apply the Smith algorithm as for the second cohomology in Chapter 2 it is in particular necessary to represent D as a submodule of the free module C, e. g. we have to identify the sublattice D in the lattice C inside L in a precise way. Note that we also use the notation e_i of the monomials for their representation as vectors in $\mathbb{Z}[i]^k$ (as we did in the construction of the matrix NMAT in Section 2.2).
We first consider the module C and represent it by a matrix RMAT given by the relations in M^3 we established in the last section. Then we find for $(\alpha, \beta, \delta)^t \in M^3$

$$(\alpha,\beta,\delta)^t = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \\ \beta_1 \\ \vdots \\ \beta_k \\ \delta_1 \\ \vdots \\ \delta_k \end{pmatrix} = \sum_{i=1}^{k} \alpha_i \begin{pmatrix} e_i \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sum_{i=1}^{k} \beta_i \begin{pmatrix} 0 \\ \vdots \\ 0 \\ e_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sum_{i=1}^{k} \delta_i \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ e_i \end{pmatrix}.$$

In matrix form the relations in M^3 just mean that

$$\text{RMAT} \cdot \big(\alpha_1, \dots, \alpha_k,\ \beta_1, \dots, \beta_k,\ \delta_1, \dots, \delta_k\big)^t = 0$$

with

$$\text{RMAT} = \begin{pmatrix} 0 & R_1(\beta)e_1 \cdots R_1(\beta)e_k & 0 \\ \vdots & \vdots & \vdots \\ R_6(\alpha)e_1 \cdots R_6(\alpha)e_k & \cdots & R_6(\delta)e_1 \cdots R_6(\delta)e_k \end{pmatrix}$$

and $R_1(\beta) = (1+B)$ as in Formula 3.4, $R_6(\alpha) = T(1+AUB)$ and $R_6(\delta) = T(A - AUBAU^{-1})$ with T as in Formula 3.9 above. Note that RMAT is a $(6k \times 3k)$-matrix then, but it has not yet the right form for our computation. For the application of the Smith algorithm we need a matrix representation for C, where the columns generate the submodule C of L. To get such a matrix we have to compute the kernel of the map described by the matrix RMAT. This leads to a matrix, which we want to call CMAT in the following. On the other hand we get the matrix for D already in a quite suitable form, since Φ sends a principal derivation f_{m} to $((A-1)\mathrm{m}, (B-1)\mathrm{m}, (U-1)\mathrm{m})$. So we can represent D by the following matrix DMAT:

$$\mathrm{DMAT} \;=\; \begin{pmatrix} (A-1)e_1 & \dots & (A-1)e_k \\ \vdots & & \vdots \\ (U-1)e_1 & \dots & (U-1)e_k \end{pmatrix}.$$

This is a $(3k \times k)$-matrix and its columns just generate D. But it remains the problem to find D in C. For that note that any column in DMAT is in the span of the columns of CMAT. Therefore we are done if we can express each column of the matrix DMAT as a linear combination of columns of the matrix CMAT, which can be realized by a base change (solving the linear systems). This gives a final matrix representing the module C/D that we want to call CDMAT. To that matrix we can apply the Smith algorithm, and the elementary divisors in the Smith normal form describe the torsion part of $H^1(\Gamma, M) \cong C/D$. Of course, if we are only interested in the rank it is enough to work with the matrices RMAT and DMAT. Since $\mathrm{rank}(H^1(\Gamma, M)) = \mathrm{rank}(\mathrm{CMAT}) - \mathrm{rank}(\mathrm{DMAT})$ and $\mathrm{rank}(\mathrm{CMAT}) = \mathrm{corank}(\mathrm{RMAT}) := 3k - \mathrm{rank}(\mathrm{RMAT})$ we don't need to determine the matrix CDMAT.
In the case of $\mathbb{Z}[\,i\,]$-modules we could again use the Smith algorithm over $\mathbb{Z}[\,i\,]$ in Maple. So we realized our strategy of computation in a Maple program to get a first overview of the torsion and the of picture for the ranks. Later on we did some experiments with the computation mod ℓ which led to the observation that the whole torsion could be encoded in the matrix DMAT already, which would simplify the computation a lot. Furthermore one can treat many other groups very fast then. We want to discuss that in more detail in the next section.

3.3 Other arithmetic groups

Let us first collect some rank patterns we obtained computationally. With the setup from Section 3.2 for $\Gamma = PSL_2(\mathbb{Z}[\,i\,])$ we found a similar pattern as for the second cohomology. Basically, the zeros are shifted. So we got $\mathrm{rank}(H^1(\Gamma, M_{n,m}(\mathbb{Z}[\,i\,]))) = 1$ for $n \neq m$ and $n+m+2 \equiv 0 \bmod 4$ and rank zero in the other cases for $n \neq m$. For $n = m$ we found rank zero for $n = 2, 4, 6, 8, 12$ and rank one for $n = 10, 14, 16, 18, 20$, furthermore rank one for $n = 1, 3$, rank two for $n = 5$ and rank three for $n = 7, 9$.
To compute the module invariants of $H^1(\Gamma, M)$ for other groups Γ we have to build up the corresponding matrices RMAT and DMAT along the same lines as for $\Gamma = PSL_2(\mathbb{Z}[\,i\,])$. As a preparation for that we have to determine again the

relations in M^r, which come from the presentations of the groups Γ (r the number of the generators). For example, in the the case of Γ_∞ for $\Gamma = PSL_2(\mathbb{Z}[i])$ (see (1.1)) we can reuse Formula 3.6 and have to determine the decompositions for $f(D^2)$, $f((AD)^2)$ and $f((UD)^2)$ to get the other relations. For $PSL_2(\mathbb{Z}[\frac{1+\sqrt{-3}}{2}])$ we can use Formula 3.4–3.6 as for $PSL_2(\mathbb{Z}[i])$. In addition we have to decompose $f((AUBU^{-2}B)^2)$, $f((AUBU^{-1}B)^3)$ and $f(A^2UBU^{-1}BUBU^{-1}B)$, which leads to the other three relations. For the figure-8 knot complement group Γ_8 we have to decompose $f(A^{-1}UAU^{-1}AUA^{-1}U^{-1}AU^{-1})$.
We also got $\text{rank}(H^1(\Gamma_\infty, M_{n,m}(\mathbb{Z}[i])) = 2$ for $n+m+2 \equiv 0 \bmod 4$ and rank zero in all other cases. In the case of $\Gamma = PSL_2(\mathbb{Z}[\frac{1+\sqrt{-3}}{2}])$ we found for the series $M = M_{n,0}(\mathbb{Z}[\frac{1+\sqrt{-3}}{2}])$ that $\text{rank}(H^1(\Gamma, M)) = 1$ for $n = 2q$ with $q \equiv 2 \bmod 3$ and $\text{rank}(H^1(\Gamma, M)) = 0$ in the remaining cases. In contrast to that, we found for $\Gamma = \Gamma_8$ and $M = M_{n,0}(\mathbb{Z}[i])$ that $\text{rank}(H^1(\Gamma, M)) = 1$ for all n we checked.
Furthermore we analysed $H^1(PSL_2(\mathbb{Z}), M_n(\mathbb{Z}))$ (cp. (4.4) for $PSL_2(\mathbb{Z})$). Here we computed the ranks up to $n = 150$ and present them up to $n = 60$ in Table 13. We also computed the Smith normal forms for the matrices CDMAT up to $n = 100$ and determined the factorizations of the elementary divisors. As for the second cohomology we then counted the occurence of the prime ℓ in different elementary divisors (representing the rank mod ℓ minus the $\mathbb{Q}$-rank of the corresponding matrices). We want to call these numbers ℓ-coranks of the considered matrices in the following. Then we started playing with the matrices involved and found exactly the same results by taking the matrices DMAT instead of the matrices CDMAT. This suggested that the DMAT would already encode the torsion of the first cohomology. So we continued to study that phenomenon for $\Gamma = PSL_2(\mathbb{Z}[i])$ and found the same connection (checked up to $n = 30$ for $M_{n,0}(\mathbb{Z}[i])$). Of course, such a correspondence would simplify the computations a lot, since only the generators are needed to build up the matrices DMAT. Therefore we decided to go on with that investigation for many more groups from Section 1.1 (where we computed mod ℓ) and concentrate on the results we got from these computations in Section 3.4. Note that the ℓ-coranks for the matrices DMAT for $PSL_2(\mathbb{Z})$ and $PSL_2(\mathbb{Z}[i])$ (up to $n = 30$) indeed represent the ℓ-torsions of $H^1(\Gamma, M)$. In contrast to $H^2(\Gamma, M)$ we didn't find any large ℓ-torsion in $H^1(\Gamma, M)$.
Our observation has to be studied in more detail now. Note that $H^0(\Gamma, M) \cong M^\Gamma$ and $\text{PDer}(\Gamma, M) \cong M/M^\Gamma$. What we have to do is to investigate the relation of $H^1(\Gamma, M)$ and $H^0(\Gamma, M)$ and what it gives for the description of the torsion in $H^1(\Gamma, M)$. For $\Gamma = PSL_2(\mathbb{Z})$ it is known (see [Wa], p.72, going back to Dickson) that

$$\dim_{\mathbb{F}_\ell} H^0(\Gamma, M_n(\mathbb{F}_\ell)) \;=\; \#\{(a,b) \in \mathbb{N}^2 \;:\; a(\ell+1) + b\ell(\ell-1) = n\}.$$

As one can check in Table 13 these $\mathbb{F}_\ell$-dimensions coincide with our ℓ-coranks for the matrices DMAT. Up to now we didn't work out anything like that for $PSL_2(\mathbb{Z}[i])$.

3.4 Results for several arithmetic groups

In the first table $\mathbb{Q}$-rank stands for the rank of $H^1(PSL_2(\mathbb{Z}), M_n(\mathbb{Z}))$. The ranks of several other $\mathcal{O}_K$-modules $H^1(\Gamma, M)$ were already presented in Section 3.3.

n	$\mathbb{Q}$-rank	ℓ-coranks up to $\ell = 1223$
2	1	[2,1]
4	1	[2,1], [3,1]
6	1	[2,2], [3,1], [5,1]
8	1	[2,2], [3,1], [7,1]
10	3	[2,2], [3,1]
12	1	[2,3], [3,2], [5,1], [11,1]
14	3	[2,3], [3,1], [13,1]
16	3	[2,3], [3,2], [7,1]
18	3	[2,4], [3,2], [5,1], [17,1]
20	3	[2,4], [3,2], [5,1], [19,1]
22	5	[2,4], [3,2]
24	3	[2,5], [3,3], [5,1], [7,1], [11,1], [23,1]
26	5	[2,5], [3,2], [5,1]
28	5	[2,5], [3,3], [13,1]
30	5	[2,6], [3,3], [5,1], [29,1]
32	5	[2,6], [3,3], [5,1], [7,1], [31,1]
34	7	[2,6], [3,3]
36	5	[2,7], [3,4], [5,1], [11,1], [17,1]
38	7	[2,7], [3,3], [5,1], [37,1]
40	7	[2,7], [3,4], [5,1], [7,1], [19,1]
42	7	[2,8], [3,4], [5,1], [7,1], [13,1]
44	7	[2,8], [3,4], [5,1], [43,1]
46	9	[2,8], [3,4], [5,1]
48	7	[2,9], [3,5], [5,1], [7,1], [11,1], [23,1], [47,1]
50	9	[2,9], [3,4], [5,1], [7,1]
52	9	[2,9], [3,5], [5,1]
54	9	[2,10], [3,5], [5,1], [17,1], [53,1]
56	9	[2,10], [3,5], [5,1], [7,1], [13,1]
58	11	[2,10], [3,5], [5,1], [7,1]
60	9	[2,11], [3,6], [5,2], [19,1], [29,1], [59,1]

Table 13: ℓ-coranks for the DMAT of $PSL_2(\mathbb{Z})$ and $M_n(\mathbb{Z})$

n	ℓ-coranks up to $\ell = 1223$
2	[2,1]
4	[2,1]
6	[2,2], [5,1]
8	[2,2]
10	[2,2], [3,2]
12	[2,3], [5,1]
14	[2,3], [13,1]
16	[2,3]
18	[2,4], [5,1], [17,1]
20	[2,4], [3,1], [5,1]
22	[2,4]
24	[2,5], [5,1]
26	[2,5], [5,1]
28	[2,5], [13,1]
30	[2,6], [3,1], [5,1], [29,1]
32	[2,6], [5,1]
34	[2,6]
36	[2,7], [5,1], [17,1]
38	[2,7], [5,1], [37,1]
40	[2,7], [3,1], [5,1]

Table 14: ℓ-coranks for the DMAT of $PSL_2(\mathbb{Z}[i])$ and $M_{n,0}(\mathbb{Z}[i])$

n	ℓ-coranks up to $\ell = 1223$
2	[2,1]
4	[2,1], [3,1]
6	[2,2], [3,1]
8	[2,2], [3,1]
10	[2,2], [3,1]
12	[2,3], [3,2], [11,1]
14	[2,3], [3,1]
16	[2,3], [3,2]
18	[2,4],[3,2], [17,1]
20	[2,4], [3,2], [19,1]

Table 15: ℓ-coranks for the DMAT of $PSL_2(\mathbb{Z}[\sqrt{-2}])$ and $M_{n,0}(\mathbb{Z}[\sqrt{-2}])$

n	ℓ-coranks up to $\ell = 1223$
2	
4	[3,1],
6	[3,1]
8	[3,1], [7,1]
10	[2,1], [3,1]
12	[2,1], [3,2]
14	[3,1], [13,1]
16	[3,2], [7,1]
18	[3,2]
20	[2,1], [3,2], [19,1]

Table 16: ℓ-coranks for the DMAT of $PSL_2(\mathbb{Z}[\frac{1+\sqrt{-3}}{2}])$ and $M_{n,0}(\mathbb{Z}[\frac{1+\sqrt{-3}}{2}])$

n	ℓ-coranks up to $\ell = 1223$
2	[2,1]
4	[2,1]
6	[2,2]
8	[2,2],[7,1]
10	[2,2], [3,1]
12	[2,3], [11,1]
14	[2,3]
16	[2,3], [7,1]
18	[2,4]
20	[2,4], [3,1]

Table 17: ℓ-coranks for the DMAT of $PSL_2(\mathbb{Z}[\frac{1+\sqrt{-7}}{2}])$ and $M_{n,0}(\mathbb{Z}[\frac{1+\sqrt{-7}}{2}])$

n	ℓ-coranks up to $\ell = 1223$
2	
4	[3,1]
6	[3,1], [5,1]
8	[3,1]
10	[2,1], [3,1]
12	[2,1], [3,2], [5,1], [11,1]
14	[3,1]
16	[3,2]
18	[3,2],[5,1]
20	[2,1],[3,2], [5,1]

Table 18: ℓ-coranks for the DMAT of $PSL_2(\mathbb{Z}[\frac{1+\sqrt{-11}}{2}])$ and $M_{n,0}(\mathbb{Z}[\frac{1+\sqrt{-11}}{2}])$

n	ℓ-coranks up to $\ell = 1223$
2	
4	
6	[5,1]
8	
10	[2,1]
12	[2,1], [5,1]
14	
16	
18	[5,1]
20	[2,1], [5,1]

Table 19: ℓ-coranks for the DMAT of $PSL_2(\mathbb{Z}[\frac{1+\sqrt{-19}}{2}])$ and $M_{n,0}(\mathbb{Z}[\frac{1+\sqrt{-19}}{2}])$

n	ℓ-coranks up to $\ell = 1223$
2	
4	[3,1],
6	[3,1], [5,1]
8	[3,1], [7,1]
10	[3,1]
12	[3,2], [5,1]
14	[3,1]
16	[3,2], [7,1]
18	[3,2], [5,1]
20	[3,2], [5,1]

Table 20: ℓ-coranks for the DMAT of $PSL_2(\mathbb{Z}[\sqrt{-5}])$ and $M_{n,0}(\mathbb{Z}[\sqrt{-5}])$

n	ℓ-coranks up to $\ell = 1223$
2	[2,1]
4	[2,1], [3,1]
6	[2,1], [3,1]
8	[2,1], [3,1], [7,1]
10	[2,2], [3,1]
12	[2,2], [3,2]
14	[2,2], [3,1], [13,1]
16	[2,2], [3,2], [7,1]
18	[2,2], [3,2]
20	[2,3], [3,2], [19,1]

Table 21: ℓ-coranks for the DMAT of the knot group Γ_8 and $M_{n,0}(\mathbb{Z}[\frac{1+\sqrt{-3}}{2}])$

n	ℓ-coranks up to $\ell = 1223$
2	[2,3]
4	[2,5],
6	[2,7]
8	[2,19], [7,1]
10	[2,11], [3,1]
12	[2,13], [11,1]
14	[2,15]
16	[2,17], [7,1]
18	[2,19]
20	[2,21], [3,1]
22	[2,23]
24	[2,25], [7,1], [11,1], [23,1]
26	[2,27], [5,1]
28	[2,29]
30	[2,31], [3,1], [29,1]

Table 22: ℓ-coranks for the DMAT of the link group $\Gamma_{-7}(6,4)$ and $M_{n,0}(\mathbb{Z}\,[\,\frac{1+\sqrt{-7}}{2}\,])$

n	ℓ-coranks up to $\ell = 1223$
2	[2,2]
4	[2,3],
6	[2,4]
8	[2,5], [7,1]
10	[2,6], [3,1]

Table 23: ℓ-coranks for the DMAT of the link group $\Gamma_{-7}(6,5)$ and $M_{n,0}(\mathbb{Z}\,[\,\frac{1+\sqrt{-7}}{2}\,])$

n	ℓ-coranks up to $\ell = 1223$
2	[2,2]
4	[2,3],
6	[2,4]
8	[2,5], [7,1]
10	[2,6], [3,1]

Table 24: ℓ-coranks for the DMAT of the link group $\Gamma_{-7}(6,6)$ and $M_{n,0}(\mathbb{Z}\,[\,\frac{1+\sqrt{-7}}{2}\,])$

4 The first homology $H_1(\Gamma, M)$

The conclusions in Chapter 2 and 3 show that $H^2(\Gamma, M)$ contains more interesting torsion than $H^1(\Gamma, M)$. On the other hand there are some difficulties to handle Hecke operators in our setup for $H^2(\Gamma, M)$ but later on our main focus shall be on the explicit computation of Hecke eigenvalues, in particular on the torsion part. Now classical Poincaré duality for manifolds suggested to look at $H_1(\Gamma, M)$ as well to see whether it would show phenomena similar to the ones for $H^2(\Gamma, M)$. So it became of double interest for us to study $H_1(\Gamma, M)$: to use a group homological approach for the computation to have a suitable setup for the following study of the Hecke operators and to find out how the torsion in $H^2(\Gamma, M)$ and $H_1(\Gamma, M)$ would be related.
In Section 4.1 we describe the general concept of group homology we use. Then we derive in Section 4.2 the explicit relations for the $\mathcal{O}_K$-modules Λ_Γ in several cases of $\Gamma \leq PSL_2(\mathcal{O}_K)$. The relations form the basis of our computations for $H_1(\Gamma, M)$. The computation of the module invariants of Λ_Γ and its algorithmic realization is described in Section 4.3, and Section 4.4 contains the results of our computer calculations and some conclusions. Here we had to make a choice for the tables we want to present. We concentrate on $\Gamma = PSL_2(\mathbb{Z}[\,i\,])$ and add several smaller tables for $PSL_2(\,\mathbb{Z}[\,\sqrt{-2}\,]\,)$, $\Gamma = PSL_2(\,\mathbb{Z}[\,\frac{1+\sqrt{-3}}{2}\,]\,)$, the knot complement group Γ_8 and a link complement group.

4.1 Group homology with general coefficients

We use similar notations as in Chapter 3. So let again $R = \mathcal{O}_K G$ be the group ring of the group G over $\mathcal{O}_K$ and let M be a left G-module which is also an $\mathcal{O}_K$-module. The Dedekind domain $\mathcal{O}_K$ is a commutative ring, but if G is not a commutative group, R becomes a non-commutative ring. Note that we may view the group G as embedded in R via the identification of $g \in G$ with $1 \cdot g \in R$. As an $\mathcal{O}_K$-module on which G acts $\mathcal{O}_K$-linearly from the left, we may consider M as a left R-module. For these and some of the following facts from homological algebra see e.g. [Wei], [Ev] or [Br].
Furthermore let $\varepsilon : R \longrightarrow \mathcal{O}_K$ with $\varepsilon\left(\sum_{g\in G} a_g \cdot g\right) = \sum_{g \in G} a_g$ be the augmentation ring homomorphism and let $\mathfrak{a}_G$ be the kernel of ε. $\mathfrak{a}_G$ is also called the *augmentation ideal* of R.

Lemma 4.1 *$\mathfrak{a}_G$ is a free right R-module with $\mathcal{O}_K$-basis* $\{g - 1 : g \in G, g \neq 1\}$.

PROOF: We have that $\{g-1 : g \in G, g \neq 1\} \cup \{1\}$ is a basis of $R = \mathcal{O}_K G$ as a free $\mathcal{O}_K$-module. Now it follows that $\mathfrak{a}_G$ is a free $\mathcal{O}_K$-module with basis $\{g-1 : g \in G, g \neq 1\}$, in other words

$$\mathfrak{a}_G = \left\{ \sum_{g \in G,\, g \neq 1} a_g \cdot (g-1) \quad : \; a_g \in \mathcal{O}_K \right\}.$$

$\mathfrak{a}_G$ is actually a 2-sided ideal. We just choose $\mathfrak{a}_G$ to be a right R-module. □

Now we can form the tensor product over R of the right R-module $\mathfrak{a}_G$ with the left R-module M.

Definition 4.2 *Let G, M, R and $\mathfrak{a}_G$ be as above. Then*

$$\Lambda_G \;:=\; \mathfrak{a}_G \otimes_R M.$$

To make always clear which group we consider we better write G instead of R from now on (by abuse of notation):

$$a \otimes_G \mathrm{m} \;:=\; a \otimes_R \mathrm{m}.$$

The main property we will need is

$$a \cdot g \otimes_G \mathrm{m} \;=\; a \otimes_G g \cdot \mathrm{m} \quad \text{for} \quad g \in G, \mathrm{m} \in M, a \in \mathfrak{a}_G. \tag{4.1}$$

If M is an R-module, let M_G denote the largest quotient module of M on which G acts trivially, also called the *coinvariants* of M. This coinvariants functor $-_G$ is a right exact functor.
For short one can define the qth homology $H_q(G, M)$ of the group G with coefficients in the R-module M as the qth left derived functor of M_G (cf. [Wei]). Note that this qth left derived functor $L_q(-_G)(M)$ equals $H_q(-_G)(P)$, the qth homology of the complex formed by a projective resolution $P \longrightarrow M$ after applying the coinvariants functor. Remark that this definition does not depend on the base ring k for $R = kG$ and k any commutative ring (cf. [Gr] or [Ev]). For our purposes it is convenient to follow the approach in [Gr] which basically does what we described. The construction ends up with the following proposition when we take into account the change of coefficients adapted to our situation (cf. [Gr, ch. 2.2]).

Proposition 4.3 *Let $R = \mathcal{O}_K G$. If $\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathcal{O}_K \longrightarrow 0$ is a right projective resolution of $\mathcal{O}_K$ (all P_i being right R-modules) and $X = \mathrm{Im}(P_q \longrightarrow P_{q-1})$, then, for any left R-module M, the sequence*

$$0 \longrightarrow H_q(G, M) \longrightarrow X \otimes_G M \longrightarrow P_{q-1} \otimes_G M$$

is exact.

Taking into account the exact sequence of free R-modules

$$0 \longrightarrow \mathfrak{a}_G \longrightarrow R \longrightarrow \mathcal{O}_K \stackrel{\varepsilon}{\longrightarrow} 0,$$

we get from Proposition 4.3 the exact sequence

$$0 \longrightarrow H_1(G, M) \longrightarrow \mathfrak{a}_G \otimes_G M \stackrel{\Phi}{\longrightarrow} R \otimes_G M.$$

Hence we get

Proposition 4.4 *For a group G and an R-module M we have*

$$H_1(G, M) \quad = \quad \ker(\Phi).$$

Note that Φ is an $\mathcal{O}_K$-module homomorphism but it is not a G-module homomorphism, and we have $\Phi(a \otimes_G \mathrm{m}) = a \cdot \mathrm{m}$ for $a \in \mathfrak{a}_G$ and $\mathrm{m} \in M$.
In our case G is a finitely presented group and M is a finitely generated $\mathcal{O}_K$-module. Then $H_1(G, M)$ is a finitely generated $\mathcal{O}_K$-module as well.
As we pointed out at the beginning of this chapter our main interest is to compute the torsion part of $H_1(G, M) = \ker(\Phi) = \ker(\Lambda_G \longrightarrow M)$. Observe now that M is a free abelian $\mathcal{O}_K$-module, and therefore all the torsion in the $\mathcal{O}_K$-module $\Lambda_G = \mathfrak{a}_G \otimes_G M$ sits already in $H_1(G, M)$, saying that it is enough to compute in Λ_G if one is only interested in the torsion.
Let $g_1, \ldots, g_r$ be a generating system of the group G and $e_1, \ldots, e_k$ a generating system for the $\mathcal{O}_K$-module M. Then Λ_G is generated as an $\mathcal{O}_K$-module by

$$(g_i - 1) \otimes_G e_j \quad =: \quad \lambda_{ij} \quad (i = 1, \ldots, r \quad \text{and} \quad j = 1, \ldots, k).$$

But there are also certain relations between these λ_{ij} coming from the relations which are given in the presentation of the group G. Hence a first step towards an explicit computation is to carry over these relations to the module Λ_G which will be the topic of the next section. Note that we consider groups $G \leq PSL_2(\mathcal{O}_K)$ and modules $M = M_{n,m}(\mathcal{O}_K)$ with $k = (n+1)(m+1)$ in the following.

4.2 The relations for Λ_G

Let

$$G \quad = \quad \langle\, g_1, \ldots, g_r \mid R_1 = R_2 = \ldots = R_s = 1\,\rangle$$

be a presentation of the group G, where $R_1, \ldots, R_s$ are words in the generators $g_1, g_1^{-1}, \ldots, g_r, g_r^{-1}$.

We can express the action of an element $g \in G$ on the $k = (n+1)(m+1)$ generators of the $\mathcal{O}_K$-module $M_{n,m}(\mathcal{O}_K)$ (and left G-module) as follows

$$g \cdot e_j \;=\; \sum_{\ell=1}^{k} g_{j\ell}\, e_\ell \qquad (j = 1, \dots, k \quad \text{and} \quad g_{j\ell} \in \mathcal{O}_K). \tag{4.2}$$

For the relations in the group we find (using Φ) that

$$(R_h - 1) \otimes_G \mathrm{m} \;=\; 0 \qquad \text{for } h = 1, \dots, s.$$

Then we can go through the following two steps to produce the relations between the generators λ_{ij} of the $\mathcal{O}_K$-module Λ_G.

Step 1. Decompose the words R_h into a sum of terms of the form $(g_i - 1)w$, w a word in $g_1, g_1^{-1}, \dots, g_r, g_r^{-1}$ and $i \in \{1, \dots, r\}$.

Step 2. Use the property (4.1) to shift the words w along $\otimes_G$ to the side of $\mathrm{m} \in M$ like $(g_i - 1)w \otimes_G \mathrm{m} = (g_i - 1) \otimes_G w_i \cdot \mathrm{m}$. If we now substitute the generators $e_1, \dots, e_k$ of M for m and use (4.2) for $w \cdot e_j$, we finally get explicit relations for the λ_{ij} in Λ_G.

Hence Λ_G is the quotient of the free $\mathcal{O}_K$-module ${\mathcal{O}_K}^{r \cdot k}$ of rank $r \cdot k$ by a submodule N determined by the relations of Step 2. A treatment of Step 2 and a more detailed analysis of the submodule N will follow in Section 4.3. In this section we will continue to go through Step 1 for several groups $\Gamma \leq PSL_2(\mathcal{O}_K)$ and Γ'_∞ in the case of $\Gamma = PSL_2(\mathbb{Z}[\,i\,])$ to prepare the necessary reformulations in Λ_Γ of the relations in the group Γ. Note that we switch to the notation Γ instead of G when we study concrete arithmetic groups but we further use G for general constructions.

Two general rules.
We have to solve the following decomposition problem for the group G:

Let W be a word in $g_1, g_1^{-1}, \dots, g_r, g_r^{-1}$. Find an expression for $W - 1$ of the following form:

$$(W - 1) \;=\; (g_1 - 1) \sum_{j_1=1}^{k_1} w_{1j_1} + \dots + (g_r - 1) \sum_{j_r=1}^{k_r} w_{rj_r}, \tag{4.3}$$

where the w_{ij_i} are words in $g_1, g_1^{-1}, \dots, g_r, g_r^{-1}$ as well.

The idea to find that expression is to systematically reduce the word W through extracting terms of the form $(g_i - 1)$ by a kind of recursive procedure. It is easy to see that we have indeed only two possibilities. The word W ends either with a group generator g_i or with an inverse g_i^{-1}. To cover all cases we therefore need the following two rules only. For that let W be a word as above and let g be any of the generators of G.

Rule 4.5 (Reduction for g)

$$(Wg - 1) \;=\; (W - 1)g + (g - 1).$$

Rule 4.6 (Reduction for $\mathbf{g}^{-1}$)

$$(Wg^{-1} - 1) \;=\; (W - 1)g^{-1} - (g - 1)g^{-1}.$$

Now we have to apply these two rules recursively to the word W. Each reduction step produces a new word W which equals the old word W without the last element, thus reducing the length of W by 1. The recursion ends if W equals 1. Finally one multiplies out and collects together the sum of words for every term $(g_i - 1)$. We stress these observations here in such detail for two reasons. On one hand it makes the reduction process much more transparent and shows that one can perform it for any group G as long as one has its presentation (for finishing we need a finite one of course). Therefore this approach covers the computation for $H_1(G, M)$ for all these groups. On the other hand we need exactly this reduction process during the explicit calculations with the Hecke operators again, where it also had to be realized in a computer program in the end.

Let us now present the decompositions we found for the relations of several groups $PSL_2(\mathcal{O}_K)$. Here, we can see in practice how helpful and easy to handle the general group homological approach is. We start with $\Gamma = PSL_2(\mathbb{Z}[\,i\,])$, which will be our main example concerning a detailed analysis of the torsion classes and in particular the one we will consider for the explicit computation of Hecke operators in Chapter 5. We also treat Γ'_∞ in this case and the classical group $PSL_2(\mathbb{Z})$. It follows the quite similar case $PSL_2(\mathbb{Z}[\frac{1+\sqrt{-3}}{2}])$. To complete the picture we also consider the other three Euclidean cases of $\mathcal{O}_K$. Note that there are five imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$ with that property, namly for $d = 1, 2, 3, 7$ and 11. Next, we present the smallest (refering to d) case of an imaginary quadratic field with class number 1 with $\mathcal{O}_K$ not Euclidean, which is $\mathbb{Q}(\sqrt{-19})$. There are only three more cases of class number 1, or in other words with $\mathcal{O}_K$ being a principal ideal domain, that is for $d = 43$, 67 and 163. Finally, we also show two examples of class number 2. For that we choose $\mathbb{Q}(\sqrt{-5})$ and $\mathbb{Q}(\sqrt{-10})$. Here we also treat the groups associated to the non-trivial ideal class (cp. Section 1.1). It follows the example $\mathbb{Q}(\sqrt{-14})$ of class number 4. We conclude with the group Γ_8 belonging to the figure-8 knot and with a link complement group.

The group $\mathbf{PSL_2}(\mathbb{Z}[\mathbf{i}])$.

Recall Proposition 1.1 for the presentation of $PSL_2(\mathbb{Z}[i])$. We have three generators A, B and U and six relations between them. We immediately get

Formula 4.7 (Relation 1) $\quad B^2 - 1 \;=\; (B-1)(B+1).$

In the next case we can use for simplicity the formulas $W^3 - 1 = (W-1)(W^2+W+1)$ with $W = AB$ and $(AB-1) = (A-1)B + (B-1)$ which easily give

Formula 4.8 (Relation 2) $\quad (AB)^3 - 1 \;=\; (A-1)BT + (B-1)T.$

with $\quad T := (AB)^2 + AB + 1.$

To see how the reduction process really works we just want to give one instructive example in detail. For that we mark all generators we extract step by step with bold print. We consider the relation $AUA^{-1}U^{-1} = 1$. Here we apply two times Rule 4.6, then one time Rule 4.5, multiply out and finally collect the sum of words together for each generator as follows:

$$
\begin{aligned}
& AUA^{-1}U^{-1} - 1 \\
= \;& \{AUA^{-1} - 1\}\,U^{-1} - (\mathbf{U}-\mathbf{1})\,U^{-1} \\
= \;& \{\,[\,AU - 1\,]\,A^{-1} - (\mathbf{A}-\mathbf{1})\,A^{-1}\}\,U^{-1} - (\mathbf{U}-\mathbf{1})\,U^{-1} \\
= \;& \{\,[\,(\mathbf{A}-\mathbf{1})\,U + (\mathbf{U}-\mathbf{1})\,]\,A^{-1} - (\mathbf{A}-\mathbf{1})\,A^{-1}\}\,U^{-1} - (\mathbf{U}-\mathbf{1})\,U^{-1} \\
= \;& (\mathbf{A}-\mathbf{1})\,UA^{-1}U^{-1} + (\mathbf{U}-\mathbf{1})\,A^{-1}U^{-1} - (\mathbf{A}-\mathbf{1})\,A^{-1}U^{-1} - (\mathbf{U}-\mathbf{1})\,U^{-1}.
\end{aligned}
$$

So we finally get

Formula 4.9 (Relation 3)

$$AUA^{-1}U^{-1} - 1 \;=\; (A-1)(UA^{-1}U^{-1} - A^{-1}U^{-1}) + (U-1)(A^{-1}U^{-1} - U^{-1}).$$

For the last three relations we use similar simplifications as for Relation 2 and get:

Formula 4.10 (Relation 4)

$$(BUBU^{-1})^3 - 1 \;=\; (B-1)(UBU^{-1} + U^{-1})T + (U-1)(BU^{-1} - U^{-1})T$$

with $\quad T := (BUBU^{-1})^2 + BUBU^{-1} + 1.$

Formula 4.11 (Relation 5)

$$(BU^2BU^{-1})^2 - 1 \;=\; (B-1)(U^2BU^{-1}+U^{-1})T$$
$$+(U-1)(UBU^{-1}+BU^{-1}-U^{-1})T$$

with $T := BU^2BU^{-1}+1$.

Formula 4.12 (Relation 6)

$$(AUBAU^{-1}B)^2 - 1 \;=\; (A-1)(UBAU^{-1}B+U^{-1}B)T + (B-1)(AU^{-1}B+1)T$$
$$+(U-1)(BAU^{-1}B-U^{-1}B)T$$

with $T := AUBAU^{-1}B+1$.

The group Γ'_∞ for $\Gamma = \mathbf{PSL_2}(\mathbb{Z}[\,\mathbf{i}\,])$.

Γ'_∞ is the group of lower triangular matrices in $PSL_2(\mathbb{Z}[\,i\,])$ with entries 1 on the main diagonal. For the presentation of that group we refer to Section 1.1. We have to treat only one relation which is the same as the third relation for $PSL_2(\mathbb{Z}[\,i\,])$. Therefore we only have the decomposition already given by Formula 4.9.

The group $\mathbf{PSL_2}(\mathbb{Z})$.

We have the following presentation (A and B as for $PSL_2(\mathbb{Z}[\,i\,])$):

$$PSL_2(\mathbb{Z}) \;=\; \langle\, A, B \mid (AB)^3 = B^2 = 1\,\rangle, \tag{4.4}$$

and thus we are done with Formula 4.7 and Formula 4.8.

The group $\mathbf{PSL_2}(\,\mathbb{Z}[\,\frac{1+\sqrt{-3}}{2}\,]\,)$.

Here we have three generators A, B and U and six relations as in the case of $PSL_2(\mathbb{Z}[\,i\,])$ (cf. Prop. 1.4). The first three relations are actually the same and hence the decompositions are already given by Formula 4.7, Formula 4.8 and Formula 4.9. In the other three cases we find:

Formula 4.13 (Relation 4)

$$(AUBU^{-2}B)^2 - 1 \;=\; (A-1)UBU^{-2}BT + (B-1)(U^{-2}B+1)T$$
$$+(U-1)(BU^{-2}B-U^{-2}B-U^{-1}B)T$$

with $T := AUBU^{-2}B+1$.

Formula 4.14 (Relation 5)

$$\begin{aligned}(AUBU^{-1}B)^3 - 1 &= (A-1)UBU^{-1}BT + (B-1)(U^{-1}B+1)T \\ &\quad +(U-1)(BU^{-1}B - U^{-1}B)T\end{aligned}$$

with $T := (AUBU^{-1}B)^2 + AUBU^{-1}B + 1.$

Formula 4.15 (Relation 6)

$$\begin{aligned}&A^2UBU^{-1}BUBUBU^{-1}B - 1 \\ &= (A-1)\,(AUBU^{-1}BUBUBU^{-1}B + UBU^{-1}BUBUBU^{-1}B) \\ &\quad +(B-1)\,(U^{-1}BUBUBU^{-1}B + UBUBU^{-1}B + UBU^{-1}B + U^{-1}B + 1) \\ &\quad +(U-1)\,(BU^{-1}BUBUBU^{-1}B - U^{-1}BUBUBU^{-1}B + BUBU^{-1}B \\ &\qquad + BU^{-1}B - U^{-1}B).\end{aligned}$$

The group $\mathbf{PSL_2}(\mathbb{Z}[\sqrt{-2}])$.

Again we have the three generators A, B and U but four relations now (cf. Prop. 1.3). The first three relations coincide with the first three relations in the preceding cases and hence the decompositions are given by Formula 4.7, Formula 4.8 and Formula 4.9. In the remaining case we get:

Formula 4.16 (Relation 4)

$$(BU^{-1}BU)^2 - 1 = (B-1)(U^{-1}BU + U)T + (U-1)(1 - U^{-1}BU)T$$

with $T := BU^{-1}BU + 1.$

The group $\mathbf{PSL_2}(\mathbb{Z}[\frac{1+\sqrt{-7}}{2}])$.

As for $\mathbb{Z}[\sqrt{-2}]$ we have three generators A, B and U and four relations (cf. Prop. 1.5). Relation 1 and Relation 3 coincide with the ones for $\mathbb{Z}[i]$ and hence the decompositions are given by Formula 4.7 and 4.9 again. In the other two cases we find:

Formula 4.17 (Relation 2) $\quad (BA)^3 - 1 = (A-1)T + (B-1)AT.$

with $T := (BA)^2 + BA + 1.$

Formula 4.18 (Relation 4)

$$\begin{aligned}(BAU^{-1}BU)^2 - 1 &= (A-1)U^{-1}BUT + (B-1)(AU^{-1}BU + U)T \\ &\quad +(U-1)(1 - U^{-1}BU)T\end{aligned}$$

with $T := BAU^{-1}BU + 1.$

The group $\mathbf{PSL_2}(\mathbb{Z}[\frac{1+\sqrt{-11}}{2}])$.

This case differs from the last one only in Relation 4. Hence the decompositions are given by Formula 4.7, Formula 4.9, Formula 4.17 and the following one:

Formula 4.19 (Relation 4)

$$\begin{aligned}(BAU^{-1}BU)^3 - 1 &= (A-1)U^{-1}BUT + (B-1)(AU^{-1}BU + U)T \\ &\quad +(U-1)(1-U^{-1}BU)T\end{aligned}$$

with $T := (BAU^{-1}BU)^2 + BAU^{-1}BU + 1.$

The group $\mathbf{PSL_2}(\mathbb{Z}[\frac{1+\sqrt{-19}}{2}])$.

This is the non-euclidean case of class number 1 we want to consider. We have the four generators A, B, C and U and seven relations (cf. Prop. 1.7). The first three relations are the same as for $\mathbb{Z}[i]$ and the decompositions are therefore given by Formula 4.7, Formula 4.8 and Formula 4.9. For the other four relations we get:

Formula 4.20 (Relation 4) $\quad C^3 - 1 = (C-1)(C^2 + C + 1).$

Formula 4.21 (Relation 5) $\quad (CA^{-1})^3 - 1 = (A-1)(-A^{-1})T + (C-1)A^{-1}T$

with $T := (CA^{-1})^2 + CA^{-1} + 1.$

Formula 4.22 (Relation 6) $\quad (BC)^2 - 1 = (B-1)CT + (C-1)T$

with $T := BC + 1.$

Formula 4.23 (Relation 7)

$$\begin{aligned}(BA^{-1}UCU^{-1})^3 - 1 &= (A-1)(-A^{-1}UCU^{-1})T + (B-1)A^{-1}UCU^{-1}T \\ &\quad +(C-1)U^{-1}T + (U-1)(CU^{-1} - U^{-1})T\end{aligned}$$

with $T := (BA^{-1}UCU^{-1})^2 + BA^{-1}UCU^{-1} + 1.$

The group $\mathbf{PSL_2}(\mathbb{Z}[\sqrt{-5}])$.

In our first case of class number 2 we have five generators A, B, C, D and U and eight relations (cf. Prop. 1.8). The first three decompositions are given again by Formula 4.7, Formula 4.8 and Formula 4.9. In the other five cases we find:

Formula 4.24 (Relation 4) $\quad A^2 - 1 = (A-1)(A+1).$

Formula 4.25 (Relation 5) $(BD)^2 - 1 = (B-1)DT + (D-1)T$
with $T := BD + 1$.

Formula 4.26 (Relation 6)

$$(BUDU^{-1})^2 - 1 = (B-1)UDU^{-1}T + (D-1)U^{-1}T + (U-1)(DU^{-1} - U^{-1})T$$

with $T := BUDU^{-1} + 1$.

Formula 4.27 (Relation 7)

$$\begin{aligned} AC^{-1}A^{-1}BCB - 1 &= (A-1)(C^{-1}A^{-1}BCB - A^{-1}BCB) + (B-1)(CB+1) \\ &\quad +(C-1)(B - C^{-1}A^{-1}BCB). \end{aligned}$$

Formula 4.28 (Relation 8)

$$\begin{aligned} & AC^{-1}A^{-1}UDU^{-1}CD - 1 \\ &= (A-1)(C^{-1}A^{-1}UDU^{-1}CD - A^{-1}UDU^{-1}CD) \\ &\quad +(C-1)(D - C^{-1}A^{-1}UDU^{-1}CD) + (D-1)(1 - U^{-1}CD) \\ &\quad +(U-1)(DU^{-1}CD - U^{-1}CD). \end{aligned}$$

The group $\mathbf{PSL_2}(\mathbb{Z}[\sqrt{-5}], \mathfrak{a})$ with $\mathfrak{a} =< \mathbf{2}, \mathbf{1} - \sqrt{-5} >$.

Here we have the second group for $\mathbb{Z}[\sqrt{-5}]$ coming from the other ideal class (cf. Prop. 1.9). We have the four generators A, C, D and U and five relations. Relation 3 is the same as for $\mathbb{Z}[i]$ and hence its decomposition is given by Formula 4.9. In the other four cases we get:

Formula 4.29 (Relation 1)

$$CDC^{-1}D^{-1} - 1 = (C-1)(DC^{-1}D^{-1} - C^{-1}D^{-1}) + (D-1)(C^{-1}D^{-1} - D^{-1}).$$

Formula 4.30 (Relation 2) $(AC^{-1})^2 - 1 = (A-1)C^{-1}T + (C-1)(-C^{-1})T$
with $T := AC^{-1} + 1$.

Formula 4.31 (Relation 4) $(DU^{-1})^3 - 1 = (D-1)U^{-1}T + (U-1)(-U^{-1})T$
with $T := (DU^{-1})^2 + DU^{-1} + 1$.

Formula 4.32 (Relation 5)

$$\begin{aligned}(CD^{-1}UA^{-1})^3 - 1 &= (A-1)(-A^{-1})T + (C-1)D^{-1}UA^{-1}T \\ &\quad +(D-1)(-D^{-1}UA^{-1})T + (U-1)A^{-1}T\end{aligned}$$

with $T := (CD^{-1}UA^{-1})^2 + CD^{-1}UA^{-1} + 1.$

The group $\mathbf{PSL_2}(\mathbb{Z}[\sqrt{-10}\,])$.

In this case we have the seven generators A, B, C, D, E, F and U and eleven relations (cf. Prop. 1.10). The decomposition of Relation 1 is the same as for Relation 1 in the case $\mathbb{Z}[\,i\,]$, which is given by Formula 4.7. Relation 2 is the same as Relation 2 for $d = 7$, hence the decomposition is given by Formula 4.17. Relation 3 and 4 are covered by Formula 4.7 as well if one replaces B by C and E respectively. In the remaining seven cases we get:

Formula 4.33 (Relation 5) $\quad (BC)^2 - 1 = (B-1)CT + (C-1)T$

with $T := BC + 1.$

Formula 4.34 (Relation 6) $(BE)^2 - 1 = (B-1)ET + (E-1)T$

with $T := BE + 1.$

Formula 4.35 (Relation 7)

$$\begin{aligned}C^{-1}AD^{-1}EAD - 1 &= (A-1)(D^{-1}EAD + D) + (C-1)(-C^{-1}AD^{-1}EAD) \\ &\quad +(D-1)(1 - D^{-1}EAD) + (E-1)AD.\end{aligned}$$

Formula 4.36 (Relation 8)

$$\begin{aligned}U^{-1}E^{-1}UFCF^{-1} - 1 &= (C-1)F^{-1} + (E-1)(-E^{-1}UFCF^{-1}) \\ &\quad +(F-1)(CF^{-1} - F^{-1}) \\ &\quad +(U-1)(FCF^{-1} - U^{-1}E^{-1}UFCF^{-1}).\end{aligned}$$

Formula 4.37 (Relation 9)

$$\begin{aligned}&D^{-1}E^{-1}B^{-1}DU^{-1}DBCD^{-1}U - 1 \\ &= (B-1)(CD^{-1}U - B^{-1}DU^{-1}DBCD^{-1}U) + (C-1)D^{-1}U \\ &\quad +(D-1)(U^{-1}DBCD^{-1}U + BCD^{-1}U - D^{-1}E^{-1}B^{-1}DU^{-1}DBCD^{-1}U - D^{-1}U \\ &\quad +(E-1)(-E^{-1}B^{-1}DU^{-1}DBCD^{-1}U) + (U-1)(1 - U^{-1}DBCD^{-1}U).\end{aligned}$$

Formula 4.38 (Relation 10)

$$
\begin{aligned}
&D^{-1}B^{-1}ADC^{-1}U^{-1}EDA^{-1}BD^{-1}U-1\\
=\ &(A-1)(DC^{-1}U^{-1}EDA^{-1}BD^{-1}U-A^{-1}BD^{-1}U)\\
&+(B-1)(D^{-1}U-B^{-1}ADC^{-1}U^{-1}EDA^{-1}BD^{-1}U)\\
&+(C-1)(-C^{-1}U^{-1}EDA^{-1}BD^{-1}U)\\
&+(D-1)(C^{-1}U^{-1}EDA^{-1}BD^{-1}U-D^{-1}B^{-1}ADC^{-1}U^{-1}EDA^{-1}BD^{-1}U\\
&\qquad +A^{-1}BD^{-1}U-D^{-1}U)\\
&+(E-1)DA^{-1}BD^{-1}U+(U-1)(1-U^{-1}EDA^{-1}BD^{-1}U).
\end{aligned}
$$

Formula 4.39 (Relation 11)

$$
\begin{aligned}
&U^{-1}DB^{-1}A^{-1}D^{-1}UFD^{-1}BADF^{-1}-1\\
=\ &(A-1)(DF^{-1}-A^{-1}D^{-1}UFD^{-1}BADF^{-1}\\
&+(B-1)(ADF^{-1}-B^{-1}A^{-1}D^{-1}UFD^{-1}BADF^{-1})\\
&+(D-1)(B^{-1}A^{-1}D^{-1}UFD^{-1}BADF^{-1}-D^{-1}UFD^{-1}BADF^{-1}\\
&\qquad +F^{-1}-D^{-1}BADF^{-1})\\
&+(F-1)(D^{-1}BADF^{-1}-F^{-1})\\
&+(U-1)(FD^{-1}BADF^{-1}-U^{-1}DB^{-1}A^{-1}D^{-1}UFD^{-1}BADF^{-1}).
\end{aligned}
$$

The group $\mathbf{PSL_2}(\mathbb{Z}[\sqrt{-10}\,],\mathfrak{b})$ with $\mathfrak{b}=<\mathbf{2},\sqrt{-10}>$.

Now we have six generators A, C, D, E, F and U and also eleven relations (cf. Prop. 1.11). Relation 1 equals Relation 1 in the second case of $\mathbb{Z}[\sqrt{-5}\,]$ and hence the decomposition is given by Formula 4.29. Relation 3 is again the same as for $\mathbb{Z}[\,i\,]$ and the decomposition is therefore given by Formula 4.9. For Relation 2 we can take the decomposition in Formula 4.7 if we replace B by E. In the other nine cases we find:

Formula 4.40 (Relation 4) $\quad (CA^{-1})^2-1 \ = \ (A-1)(-A^{-1})T+(C-1)A^{-1}T$

with $\quad T:=CA^{-1}+1$.

Formula 4.41 (Relation 5) $\quad F^3-1 \ = \ (F-1)(F^2+F+1)$.

Formula 4.42 (Relation 6) $\quad (FE)^2-1 \ = \ (E-1)T+(F-1)ET$.

with $\quad T:=FE+1$.

Formula 4.43 (Relation 7)

$$(DEU^{-1})^2 - 1 \;=\; (D-1)EU^{-1}T + (E-1)U^{-1}T + (U-1)(-U^{-1})T$$

with $T := DEU^{-1} + 1$.

Formula 4.44 (Relation 8)

$$\begin{aligned}(FC^{-1}EA)^2 - 1 \;=\;& (A-1)T + (C-1)(-C^{-1}EA)T + (E-1)AT \\ &+(F-1)C^{-1}EAT\end{aligned}$$

with $T := FC^{-1}EA + 1$.

Formula 4.45 (Relation 9)

$$(DF^{-1}U^{-1})^3 - 1 \;=\; (D-1)F^{-1}U^{-1}T + (F-1)(-F^{-1}U^{-1})T + (U-1)(-U^{-1})T$$

with $T := (DF^{-1}U^{-1})^2 + DF^{-1}U^{-1} + 1$.

Formula 4.46 (Relation 10)

$$(CF^{-1}A^{-1})^3 - 1 \;=\; (A-1)(-A^{-1})T + (C-1)F^{-1}A^{-1}T + (F-1)(-F^{-1}A^{-1})T$$

with $T := (CF^{-1}A^{-1})^2 + CF^{-1}A^{-1} + 1$.

Formula 4.47 (Relation 11)

$$\begin{aligned}(CDF^{-1}A^{-1}U^{-1})^3 - 1 \;=\;& (A-1)(-A^{-1}U^{-1})T + (C-1)DF^{-1}A^{-1}U^{-1}T \\ &+(D-1)F^{-1}A^{-1}U^{-1}T + (F-1)(-F^{-1}A^{-1}U^{-1})T \\ &+(U-1)(-U^{-1})T\end{aligned}$$

with $T := (CDF^{-1}A^{-1}U^{-1})^2 + CDF^{-1}A^{-1}U^{-1} + 1$.

The group $\mathbf{PSL_2}(\mathbb{Z}[\sqrt{-14}\,])$.

Finally we come to our example of class number 4 and consider the standard group first. Here we have the seven generators A, B, C, D, E, F and U and nine relations (cf. Prop. 1.12). For the first relation we can use the decomposition in Formula 4.7 again, for the second we can use Formula 4.17 and for the third just Formula 4.9. For the remaining six cases we get:

Formula 4.48 (Relation 4)

$$\begin{aligned}
&(A^{-1}C^{-1}BDBAD^{-1}C)^2 - 1 \\
&= (A-1)(D^{-1}C - A^{-1}C^{-1}BDBAD^{-1}C)T + (B-1)(DBAD^{-1}C + AD^{-1}C)T \\
&\quad +(C-1)(1 - C^{-1}BDBAD^{-1}C)T + (D-1)(BAD^{-1}C - D^{-1}C)T
\end{aligned}$$

with $T := A^{-1}C^{-1}BDBAD^{-1}C + 1.$

Formula 4.49 (Relation 5)

$$\begin{aligned}
&(A^{-1}CD^{-1}ABDBC^{-1})^2 - 1 \\
&= (A-1)(BDBC^{-1} - A^{-1}CD^{-1}ABDBC^{-1})T + (B-1)(DBC^{-1} + C^{-1})T \\
&\quad +(C-1)(D^{-1}ABDBC^{-1} - C^{-1})T \\
&\quad +(D-1)(BC^{-1} - D^{-1}ABDBC^{-1})T
\end{aligned}$$

with $T := A^{-1}CD^{-1}ABDBC^{-1} + 1.$

Formula 4.50 (Relation 6)

$$\begin{aligned}
&D^{-1}CE^{-1}A^{-3}DC^{-1}A^3E - 1 \\
&= (A-1)(A^2E + E - A^{-3}DC^{-1}A^3E - A^{-2}DC^{-1}A^3E - A^{-3}DC^{-1}A^3E) \\
&\quad +(C-1)(E^{-1}A^{-3}DC^{-1}A^3E - A^3E) \\
&\quad +(D-1)(C^{-1}A^3E - D^{-1}CE^{-1}A^{-3}DC^{-1}A^3E) \\
&\quad +(E-1)(1 - E^{-1}A^{-3}DC^{-1}A^3E).
\end{aligned}$$

Formula 4.51 (Relation 7)

$$\begin{aligned}
&CB^{-1}C^{-1}FC^{-1}BCF^{-1} - 1 \\
&= (B-1)(CF^{-1} - B^{-1}C^{-1}FC^{-1}BCF^{-1}) \\
&\quad +(C-1)(B^{-1}C^{-1}FC^{-1}BCF^{-1} - C^{-1}FC^{-1}BCF^{-1} - C^{-1}BCF^{-1} + F^{-1}) \\
&\quad +(F-1)(C^{-1}BCF^{-1} - F^{-1}).
\end{aligned}$$

Formula 4.52 (Relation 8)

$$\begin{aligned}
&C^{-1}DA^{-1}B^{-1}D^{-1}B^{-1}CAE^{-1}A^{-2}CBD^{-1}BA^{-1}DC^{-1}A^3E-1\\
&= (A-1)(E^{-1}A^{-2}CBD^{-1}BA^{-1}DC^{-1}A^3E-A^{-1}DC^{-1}A^3E+A^2E\\
&\qquad +AE-A^{-1}B^{-1}D^{-1}B^{-1}CAE^{-1}A^{-2}CBD^{-1}BA^{-1}DC^{-1}A^3E\\
&\qquad -A^{-2}CBD^{-1}BA^{-1}DC^{-1}A^3E-A^{-1}CBD^{-1}BA^{-1}DC^{-1}A^3E)\\
&\quad +(B-1)(A^{-1}DC^{-1}A^3E+D^{-1}BA^{-1}DC^{-1}A^3E\\
&\qquad -B^{-1}D^{-1}B^{-1}CAE^{-1}A^{-2}CBD^{-1}BA^{-1}DC^{-1}A^3E\\
&\qquad -B^{-1}CAE^{-1}A^{-2}CBD^{-1}BA^{-1}DC^{-1}A^3E)\\
&\quad +(C-1)(AE^{-1}A^{-2}CBD^{-1}BA^{-1}DC^{-1}A^3E+BD^{-1}BA^{-1}DC^{-1}A^3E\\
&\qquad -C^{-1}DA^{-1}B^{-1}D^{-1}B^{-1}CAE^{-1}A^{-2}CBD^{-1}BA^{-1}DC^{-1}A^3E\\
&\qquad -C^{-1}A^3E)\\
&\quad +(D-1)(A^{-1}B^{-1}D^{-1}B^{-1}CAE^{-1}A^{-2}CBD^{-1}BA^{-1}DC^{-1}A^3E\\
&\qquad +C^{-1}A^3E-D^{-1}BA^{-1}DC^{-1}A^3E\\
&\qquad -D^{-1}B^{-1}CAE^{-1}A^{-2}CBD^{-1}BA^{-1}DC^{-1}A^3E)\\
&\quad +(E-1)(1-E^{-1}A^{-2}CBD^{-1}BA^{-1}DC^{-1}A^3E).
\end{aligned}$$

Formula 4.53 (Relation 9)

$$\begin{aligned}
&ACB^{-1}D^{-1}B^{-1}A^{-1}DC^{-1}AFA^{-1}C^{-1}BDBAD^{-1}CA^{-1}F^{-1}-1\\
&= (A-1)(ACB^{-1}D^{-1}B^{-1}A^{-1}DC^{-1}AFA^{-1}C^{-1}BDBAD^{-1}CA^{-1}F^{-1}\\
&\qquad -A^{-1}DC^{-1}AFA^{-1}C^{-1}BDBAD^{-1}CA^{-1}F^{-1}-A^{-1}F^{-1}\\
&\qquad +FA^{-1}C^{-1}BDBAD^{-1}CA^{-1}F^{-1}+D^{-1}CA^{-1}F^{-1}\\
&\qquad -A^{-1}C^{-1}BDBAD^{-1}CA^{-1}F^{-1})\\
&\quad +(B-1)(DBAD^{-1}CA^{-1}F^{-1}+AD^{-1}CA^{-1}F^{-1}\\
&\qquad -B^{-1}A^{-1}DC^{-1}AFA^{-1}C^{-1}BDBAD^{-1}CA^{-1}F^{-1}\\
&\qquad -B^{-1}A^{-1}DC^{-1}AFA^{-1}C^{-1}BDBAD^{-1}CA^{-1}F^{-1})\\
&\quad +(C-1)(B^{-1}D^{-1}B^{-1}A^{-1}DC^{-1}AFA^{-1}C^{-1}BDBAD^{-1}CA^{-1}F^{-1}\\
&\qquad +A^{-1}F^{-1}-C^{-1}BDBAD^{-1}CA^{-1}F^{-1}\\
&\qquad -C^{-1}AFA^{-1}C^{-1}BDBAD^{-1}CA^{-1}F^{-1})\\
&\quad +(D-1)(C^{-1}AFA^{-1}C^{-1}BDBAD^{-1}CA^{-1}F^{-1}-D^{-1}CA^{-1}F^{-1}\\
&\qquad -D^{-1}B^{-1}A^{-1}DC^{-1}AFA^{-1}C^{-1}BDBAD^{-1}CA^{-1}F^{-1}\\
&\qquad +C^{-1}AFA^{-1}C^{-1}BDBAD^{-1}CA^{-1}F^{-1})\\
&\quad +(F-1)(A^{-1}C^{-1}BDBAD^{-1}CA^{-1}F^{-1}-F^{-1}).
\end{aligned}$$

The group $\mathbf{PSL_2}(\mathbb{Z}[\sqrt{-14}], \mathfrak{c})$ with $\mathfrak{c} =< \mathbf{3}, \mathbf{1} + \sqrt{-14} >$.

We also consider the group for the other ideal class which is not conjugate to the preceding group. Again we have the seven generators A, B, C, D, E, F and U and nine relations between them (cf. Prop. 1.13). For the first relation we get the decomposition in Formula 4.7 again, for the second one we can use Formula 4.29, for the third one Formula 4.9 and for the fourth one Formula 4.21. In the last five cases we find:

Formula 4.54 (Relation 5)

$$(DBU^{-1})^2 - 1 \quad = \quad (B-1)U^{-1}T + (D-1)BU^{-1}T + (U-1)(-U^{-1})T$$

with $\quad T := DBU^{-1} + 1.$

Formula 4.55 (Relation 6)

$$\begin{aligned} & F^{-1}AE^{-1}A^{-1}UFEU^{-1} - 1 \\ = \; & (A-1)(E^{-1}A^{-1}UFEU^{-1} - A^{-1}UFEU^{-1}) \\ & +(E-1)(U^{-1} - E^{-1}A^{-1}UFEU^{-1}) \\ & +(F-1)(EU^{-1} - F^{-1}AE^{-1}A^{-1}UFEU^{-1}) + (U-1)(FEU^{-1} - U^{-1}). \end{aligned}$$

Formula 4.56 (Relation 7)

$$\begin{aligned} & (CBE^{-1}A^{-1}UBU^{-1}AEA^{-1})^3 - 1 \\ = \; & (A-1)(EA^{-1} - A^{-1}UBU^{-1}AEA^{-1} - A^{-1})T \\ & +(B-1)(E^{-1}A^{-1}UBU^{-1}AEA^{-1} + U^{-1}AEA^{-1})T \\ & +(C-1)BE^{-1}A^{-1}UBU^{-1}AEA^{-1}T \\ & +(E-1)(A^{-1} - E^{-1}A^{-1}UBU^{-1}AEA^{-1})T \\ & +(U-1)(BU^{-1}AEA^{-1} - U^{-1}AEA^{-1})T \end{aligned}$$

with $\quad T := (CBE^{-1}A^{-1}UBU^{-1}AEA^{-1})^2 + CBE^{-1}A^{-1}UBU^{-1}AEA^{-1} + 1.$

Formula 4.57 (Relation 8)

$$\begin{aligned} & (AEU^{-1}DBE^{-1}A^{-1}UBD^{-1})^2 - 1 \\ = \; & (A-1)(EU^{-1}DBE^{-1}A^{-1}UBD^{-1} - A^{-1}UBD^{-1})T \\ & +(B-1)(E^{-1}A^{-1}UBD^{-1} - D^{-1})T \\ & +(D-1)(BE^{-1}A^{-1}UBD^{-1} - D^{-1})T \\ & +(E-1)(U^{-1}DBE^{-1}A^{-1}UBD^{-1} - E^{-1}A^{-1}UBD^{-1})T \\ & +(U-1)(BD^{-1} - U^{-1}DBE^{-1}A^{-1}UBD^{-1})T \end{aligned}$$

with $\quad T := AEU^{-1}DBE^{-1}A^{-1}UBD^{-1} + 1.$

Formula 4.58 (Relation 9)

$$\begin{aligned}
&DC^{-1}BU^{-1}AEBD^{-1}UE^{-1}F^{-1}CBE^{-1}A^{-1}UBU^{-1}AEA^{-1}F-1\\
&= (A-1)(EBD^{-1}UE^{-1}F^{-1}CBE^{-1}A^{-1}UBU^{-1}AEA^{-1}F-A^{-1}F\\
&\qquad +EA^{-1}F-A^{-1}UBU^{-1}AEA^{-1}F)\\
&\quad +(B-1)(U^{-1}AEBD^{-1}UE^{-1}F^{-1}CBE^{-1}A^{-1}UBU^{-1}AEA^{-1}F\\
&\qquad +D^{-1}UE^{-1}F^{-1}CBE^{-1}A^{-1}UBU^{-1}AEA^{-1}F\\
&\qquad +E^{-1}A^{-1}UBU^{-1}AEA^{-1}F+U^{-1}AEA^{-1}F)\\
&\quad +(C-1)(BE^{-1}A^{-1}UBU^{-1}AEA^{-1}F\\
&\qquad -C^{-1}BU^{-1}AEBD^{-1}UE^{-1}F^{-1}CBE^{-1}A^{-1}UBU^{-1}AEA^{-1}F)\\
&\quad +(D-1)(DC^{-1}BU^{-1}AEBD^{-1}UE^{-1}F^{-1}CBE^{-1}A^{-1}UBU^{-1}AEA^{-1}F\\
&\qquad -D^{-1}UE^{-1}F^{-1}CBE^{-1}A^{-1}UBU^{-1}AEA^{-1}F)\\
&\quad +(E-1)(BD^{-1}UE^{-1}F^{-1}CBE^{-1}A^{-1}UBU^{-1}AEA^{-1}F+A^{-1}F\\
&\qquad -E^{-1}F^{-1}CBE^{-1}A^{-1}UBU^{-1}AEA^{-1}F\\
&\qquad -E^{-1}A^{-1}UBU^{-1}AEA^{-1}F)\\
&\quad +(F-1)(1-F^{-1}CBE^{-1}A^{-1}UBU^{-1}AEA^{-1}F)\\
&\quad +(U-1)(E^{-1}F^{-1}CBE^{-1}A^{-1}UBU^{-1}AEA^{-1}F+BU^{-1}AEA^{-1}F\\
&\qquad -U^{-1}AEBD^{-1}UE^{-1}F^{-1}CBE^{-1}A^{-1}UBU^{-1}AEA^{-1}F\\
&\qquad -U^{-1}AEA^{-1}F).
\end{aligned}$$

The group Γ_8

The group Γ_8 has two generators and only one relation. From the presentation (1.1) we find

Formula 4.59 (Relation for the knot complement group Γ_8)

$$\begin{aligned}
&A^{-1}UAU^{-1}AUA^{-1}U^{-1}AU^{-1}-1\\
&= (A-1)(U^{-1}AUA^{-1}U^{-1}AU^{-1}+UA^{-1}U^{-1}AU^{-1}+U^{-1}\\
&\qquad -A^{-1}UAU^{-1}AUA^{-1}U^{-1}AU^{-1}-A^{-1}U^{-1}AU^{-1})\\
&\quad +(U-1)(AU^{-1}AUA^{-1}U^{-1}AU^{-1}-U^{-1}AU^{-1}-U^{-1}\\
&\qquad +A^{-1}U^{-1}AU^{-1}-U^{-1}AUA^{-1}U^{-1}AU^{-1}).
\end{aligned}$$

Several link complements groups

The link complement groups we consider are the groups $\Gamma_{-7}(6,4)$, $\Gamma_{-7}(6,5)$ and $\Gamma_{-7}(6,6)$. But we only want to give the decomposition formulas for $\Gamma_{-7}(6,4)$ here. The group has three generators and two relations. From the presentation (1.4) in Section 1.1 we find

Formula 4.60 (Relation 1 for $\Gamma_{-7}(6,4)$)

$$XZX^{-1}Z^{-1}-1 \;=\; (X-1)(ZX^{-1}Z^{-1}-X^{-1}Z^{-1})+(Z-1)(X^{-1}Z^{-1}-Z^{-1}).$$

Formula 4.61 (Relation 2 for $\Gamma_{-7}(6,4)$)

$$\begin{aligned}
&Z^{-1}YX^{-1}Z^{-1}YZ^{-1}Y^{-1}ZY^{-1}ZXY^{-1}ZY-1\\
&=(X-1)(Y^{-1}ZY-X^{-1}Z^{-1}YZ^{-1}Y^{-1}ZY^{-1}ZXY^{-1}ZY)\\
&\quad+(Y-1)(X^{-1}Z^{-1}YZ^{-1}Y^{-1}ZY^{-1}ZXY^{-1}ZY-Y^{-1}ZXY^{-1}ZY-Y^{-1}ZY\\
&\qquad\qquad+Z^{-1}Y^{-1}ZY^{-1}ZXY^{-1}ZY-Y^{-1}ZY^{-1}ZXY^{-1}ZY+1)\\
&\quad+(Z-1)(Y-Z^{-1}YX^{-1}Z^{-1}YZ^{-1}Y^{-1}ZY^{-1}ZXY^{-1}ZY+XY^{-1}ZY\\
&\qquad\qquad-Z^{-1}YZ^{-1}Y^{-1}ZY^{-1}ZXY^{-1}ZY-Z^{-1}Y^{-1}ZY^{-1}ZXY^{-1}ZY\\
&\qquad\qquad+Y^{-1}ZXY^{-1}ZY).
\end{aligned}$$

Remark 4.62 If one analyses the decomposition process of the relations carefully one discovers the pattern behind it, which actually gives a method to write down the results by hand very easily. Suppose we have to find the part of the decomposition of $W-1$ for a generator g appearing in the word W, that is, we have to find $(g-1)\sum_{j=1}^{k} w_j$ as described in (4.3). The recursive reduction process shows that we just get one word w_j for each g or g^{-1}, which occurs in W. Therefore k is the number of g and g^{-1} appearing in the word W. To find the words w_j in each case we have to do the following:

1. We scan through W from left and stop if we find any g or g^{-1}. If we found a g, then w_j is the part of W which follows after g if one scans on to the right (this part can also be 1). If we found g^{-1}, then w_j is just the part of W starting with g^{-1} and going on with the rest of W on the right side of g^{-1}. In the first case the sign of w_j is always plus, in the second case it is minus (see Rule 4.5 and Rule 4.6).
2. We repeat the first step k times, that is, for all cases of g and g^{-1} occuring in W when scanning from the left to the right.

To complete the decomposition one has to do that for all generators g of the group, which appear in the word W.
The decomposition procedure is included in the computer program for the computation of the Hecke operators (cp. Section 5.3), where it has to be performed for a great amount of cases. The recursion we already described gives one algorithmic solution of the problem, this remark suggests another variant of realization.

4.3 Computation of the module invariants

Our main goal is now to compute the torsion of the module

$$\Lambda_\Gamma \;=\; {\mathcal{O}_K}^{r\cdot k}/N$$

for several arithmetic groups Γ, where r is the number of generators of Γ and N is the submodule given by the relations described in Section 4.2. Again we have to represent the submodule N by a suitable matrix NMAT whose columns generate N. Then we can apply the Smith algorithm to determine the elementary divisors or can compute modulo ℓ (resp. modulo prime ideals) to just detect the torsion. This can be done as explained in Chapter 2 and used in a similar way in Chapter 3.
To explain the construction of the NMAT we want to consider the example of $\Gamma = PSL_2(\mathbb{Z}[i])$ with $M = M_{n,m}(\mathbb{Z}[i])$. In (4.2) we expressed the action of an element $g \in G$ on the generators of the module $M_{n,m}(\mathbb{Z}[i])$. In our special situation we express $A \cdot e_j = \sum_{\ell=1}^{k} a_{j\ell}\, e_\ell$, $B \cdot e_j = \sum_{\ell=1}^{k} b_{j\ell}\, e_\ell$ and $U \cdot e_j = \sum_{\ell=1}^{k} u_{j\ell}\, e_\ell$ with $j = 1, \ldots, k$ and $a_{j\ell},\ b_{j\ell},\ u_{j\ell} \in \mathcal{O}_K = \mathbb{Z}[i]$. At the end of Section 4.1. we defined λ_{ij} to be the $3k$ generators $(g_i - 1) \otimes_G e_j$ of Λ_G. In our special case we want to call these generators $\lambda_{A,j}$, $\lambda_{B,j}$ and $\lambda_{U,j}$.
Let us now take the first relation $B^2 = 1$ in the group Γ to see how we find the entries of the matrix NMAT. We have $(B^2 - 1) \otimes_\Gamma e_j = 0$. Using Formula 4.7, which we got from our decomposition process, this gives $(B-1)(B+1) \otimes_\Gamma e_j = 0$.
The application of property (4.1) and the linearity of the tensor product yield

$$(B-1) \otimes_\Gamma (B+1) \cdot e_j \;=\; (B-1) \otimes_\Gamma B \cdot e_j + (B-1) \otimes_\Gamma e_j \;=\; 0.$$

Now we express $B \cdot e_j$ as described above, which gives

$$(B-1) \otimes_\Gamma \sum_{\ell=1}^{k} b_{j\ell}\, e_\ell + (B-1) \otimes_\Gamma e_j \;=\; 0.$$

Using the linearity of the tensor product again we finally get

$$\sum_{\ell=1}^{k} b_{j\ell}\, (B-1) \otimes_\Gamma e_\ell + (B-1) \otimes_\Gamma e_j \;=\; \left(\sum_{\ell=1}^{k} b_{j\ell}\lambda_{B,\ell}\right) + \lambda_{B,j} \;=\; 0.$$

A similar computation has to be performed for each of the other five relations, where we also have parts for A and U. Then we are ready to build up the matrix NMAT. Since N sits in the free module $\mathbb{Z}[i]^{3k}$ we have columns of length $3k$. The first k entries come from the $(A-1)$-part, the next k entries from the $(B-1)$-part and the last k entries from the $(U-1)$-part. Each relation covers all k generators e_i of $M_{n,m}(\mathbb{Z}[i]) \cong \mathbb{Z}[i]^k$, and so we end up with a $(3k \times 6k)$-matrix containing one

$(3k \times k)$-part for each of the six relations. Note that we represent the generators e_i again by vectors in $\mathbb{Z}[\,i\,]^k$ as in the preceding chapters. The computations for the relation $B^2 = 1$ give the first k columns. Since we only have B in the relation we don't get anything in the top part (first k rows) and in the bottom part (last k rows) but only in the central block. Here we just have to take the $b_{j\ell}$ we computed. The first column would then consist of k entries 0, the entries $b_{11}+1, b_{21}, \ldots, b_{k1}$ and again k entries 0 and so on. All the entries are in $\mathbb{Z}[\,i\,]$. For example we get the following matrix NMAT for the module $M_{2,0}(\mathbb{Z}[\,i\,])$:

```
> evalm(Nmat);
```

$$
\begin{bmatrix}
0 & 0 & 0 & 2 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2+2I & -2 & 2-2I \\
0 & 0 & 0 & -2 & 1 & -2 & 2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & -1 & 2 & -1 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & I & -2 & 2+2I & -2 & 2-2I \\
0 & 0 & 0 & 2 & -1 & 2 & 0 & 0 & 0 & -4I & 2 & 4I & -4I & 4 & 8I & 4+4I & -4 & 4-4I \\
1 & 0 & 1 & 2 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & I & -2 & 2+2I & -2 & 2-2I \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & -2I & 4 & -4 & -4I & 8 & 2-2I & 2I & -2-2I \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 4I & -2 & -4I & 4I & -4 & -8I & -4 & 2-2I & 4I \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 4 & 2I & -4 & 4 & 4I & -8 & 2I & -1-I & 2
\end{bmatrix}
$$

Now we can use the Smith algorithm again to find the elementary divisors. We can also compute the rank of Λ_Γ but don't get the correct ranks of $H_1(\Gamma, M) = \ker(\Lambda_\Gamma \longrightarrow M)$ that way. On the other hand there are already results about them based on analytical methods. Nevertheless we come back to that question later when we study the Hecke operators on Λ_Γ because it is an interesting point whether one could proof something about the ranks by these algebraic methods.
The realization of our strategy in a computer program is quite similar to the ones in Chapter 2 and 3. The main aspect is an elegant construction of the matrix NMAT. So we can just take the decompositions we collected in Section 4.2 and can automate the calculation described above to perform it easily for each relation. Since the matrix gets quite large (compared to the matrices we had in Chapter 2 and 3) we don't get very far with the Smith algorithm in MAPLE. To just detect the torsion we went on with the computation modulo ℓ then. Nevertheless we needed the transfer matrices from the Smith algorithm for the Hecke computations but were able to get through with a Smith algorithm modulo ℓ as will be described in Section 5.3.

4.4 Results for several arithmetic groups

We carried out the computations for most of the groups we treated in Section 4.2. To get an overview of the occuring torsion we only want to present several longer tables found by the computations modulo ℓ as we already did in Chapter 2 and 3. A complete collection of the tables will be given elsewhere, so we also want to restrict to several examples showing the main phenomena.

First of all we present similar tables as for $H^2(\Gamma, M)$ for certain series $M_{n,m}(\mathbb{Z}[i])$ in the case of $\Gamma = PSL_2(\mathbb{Z}[i])$. Again a lot of large ℓ-torsion occured. But even more, exactly the same large primes ℓ showed up in both cases. In all cases we computed this happend with the same multiplicity one. In contrast to that the small torsions were quite different. We also computed $H_1(\Gamma_\infty, M)$ and $H_1(\Gamma'_\infty, M)$ for several modules $M = M_{n,m}(\mathbb{Z}[i])$ but don't want to give the specific tables here. We got a picture similar to the one for the second cohomology, in particular all the large torsions from $H_1(\Gamma, M)$ didn't appear there for the corresponding coefficient modules.
Of course, the question arose what deeper connection should be behind the relation we saw between $H^2(\Gamma, M)$ and $H_1(\Gamma, M)$. A kind of duality was suggested by our results but not in a direct way. Therfore a good guess seemed to be a generalization of the classical Lefschetz duality for manifolds with boundary (see e.g. [SZ]), which says for 3-dimensional manifolds that the second relative cohomology is isomorphic to the first homology (both with trivial coefficients). From recent personal communication with G. Harder and J. Rohlfs we now know that a corresponding duality holds in quite a general context. In particular, this would give in our case that $H^2_c(\Gamma, M) \cong H_1(\Gamma, M)$, where $H^2_c(\Gamma, M)$ is the cohomology with compact support. Note that we have to invert the orders of the finite subgroups of Γ in M here. Now the important point is to understand the subtle relation between $H^2_c(\Gamma, M)$ and $H^2(\Gamma, M)$. From a first analysis of the long exact cohomology sequence, several torsion results (bounds) and some knowlege about the ranks of $H^1(\Gamma_\infty, M)$ we find that it is not at all clear that $H^2_c(\Gamma, M)$ and $H^2(\Gamma, M)$ should always contain the same large torsions.
Let us now start with the tables of the ℓ-torsions for $PSL_2(\mathbb{Z}[i])$ and certain series of the modules $M_{n,m}(\mathbb{Z}[i])$. After that we continue with some smaller tables for $PSL_2(\mathbb{Z}[\sqrt{-2}])$, $\Gamma = PSL_2(\mathbb{Z}[\frac{1+\sqrt{-3}}{2}])$, the knot complement group Γ_8 and the link complement group $\Gamma_{-7}(6,4)$.

n	ℓ-torsion up to $\ell = 7919$ and for several large extra primes
0	[2,2]
2	[2,2]
4	[2,5]
6	[2,4], [3,1]
8	[2,7]
10	[2,8], [3,1], [5,1]
12	[2,9]
14	[2,10], [3,1], [5,1]
16	[2,13], [5,1]

n	ℓ-torsion up to $\ell = 7919$ and for several large extra primes
18	[2,12], [3,2], [5,1], [13,1]
20	[2,15], [5,1]
22	[2,16], [3,2], [5,2], [13,1], [17,1]
24	[2,17], [3,1], [5,1]
26	[2,18], [3,1], [5,2], [13,1], [17,1]
28	[2,21], [3,1], [5,2]
30	[2,20], [3,2], [5,2], [13,1], [17,1]
32	[2,23], [3,1], [5,2], [19,1]
34	[2,24], [3,3], [5,3], [13,1], [17,1], 29,1], [**151**,1]
36	[2,25], [3,2], [5,2], [13,1]
38	[2,26], [3,2], [5,3], [13,1], [17,1], [29,1]
40	[2,29], [3,1], [5,3], [7,1], [13,1], [17,1], [**661**,1]
42	[2,28], [3,3], [5,3], [7,1], [13,1], [17,1], [29,1], [37,1] [**641**,1]
44	[2,31], [3,2], [5,3], [13,1], [17,1], [**67**,1], [**137**,1]
46	[2,32], [3,4], [5,4], [13,2], [17,1], [29,1], [37,1], [41,1], [**139**,1]
48	[2,33], [3,2], [5,3], [13,1], [17,1], [29,1]
50	[2,34], [3,2], [5,4], [13,2], [17,1], [29,1], [37,1], [41,1], [**59547091**,1]
52	[2,37], [3,2], [5,4], [13,2], [17,1], [**3011**,1]
54	[2,36], [3,4], [5,4], [7,2], [13,2], [17,1], [29,1], [37,1], [41,1], [**163**,1]
56	[2,39], [3,3], [5,4], [7,2], [13,1], [17,1], [**461**,1]
58	[2,40], [3,4], [5,5], [7,1], [13,2], [17,2], [29,1], [37,2], [41,1], [53,1], [**367**,1], [**945929**,1]
60	[2,41], [3,2], [5,4], [11,1], [13,2], [17,1], [**1650371**,1]
62	[2,42], [3,3], [5,5], [7,1], [13,2], [17,2], [29,1], [37,1], [41,1], [53,1], [**26387**,1]
64	[2,45], [3,3], [5,5], [13,2], [17,1], [29,1], [**197**,1], [**103979**,1]
66	[2,44], [3,5], [5,5], [7,2], [13,3], [17,2], [29,1], [37,1], [41,1], [53,1], [61,1] [**19920917**,1]
68	[2,47], [3,3], [5,5], [13,2], [17,2], [29,1], [**503**,1], [**1297**,1], [**1531**,1]
70	[2,48], [3,4], [5,6], [7,1], [11,1], [13,2], [17,2], [29,1], [37,1], [41,1], [53,1], [61,1], [**429901**,1]
72	[2,49], [3,3], [5,5], [7,1], [13,2], [17,1], [29,1]
74	[2,50], [3,4], [5,6], [7,1], [13,1], [17,2], [29,1], [37,1], [41,1], [53,1], [61,1]
76	[2,53], [3,4], [5,6], [13,2], [17,2], [29,2] [**179**,1], [**41193114818503**,1]
78	[2,52], [3,5], [5,6], [7,2], [13,3], [17,2], [29,2], [37,1], [41,1], [53,1], [61,1], [73,1], [**381**,1], [**631**,1]
80	[2,55], [3,3], [5,6], [13,3], [17,2], [29,1], [37,1], [47,1], [59,1], [**97**,1], [**2647**,1], [**3347**,1]

Table 25: ℓ-torsion in $H_1(PSL_2(\mathbb{Z}[i]), M_{n,0}(\mathbb{Z}[i]))$

n	ℓ-torsion up to $\ell = 7919$
1	[2,1], [3,1]
3	[2,6], [3,1]
5	[2,7]
7	[2,10], [3,1], [5,1]
9	[2,13]
11	[2,16], [3,2], [5,1], [7,1]
13	[2,17], [3,1], [5,1]
15	[2,22], [3,3], [5,2],[7,1], [11,1], [13,1]
17	[2,23], [3,1], [5,1]
19	[2,26], [3,2], [5,2], [7,1], [11,1], [13,1],[17,1]
21	[2,29], [3,2], [5,2], [**59**,1]
23	[2,32], [3,2], [5,2], [7,1], [11,1], [13,1], [17,1], [19,1], [**37**,1]
25	[2,33], [3,3], [5,2]
27	[2,38], [3,4], [5,3], [7,1], [11,1], [13,1], [17,1], [19,1], [23,1], [**139**,1], [**347**,1]
29	[2,39], [3,2], [5,2], [13,1], [**73**,1], [**239**,1]
31	[2,42], [3,4], [5,3], [7,1], [11,1], [13,1], [17,1], [19,1], [23,1], [29,1], [**83**,1], [**293**,1]
33	[2,45], [3,2], [5,3], [7,1], [13,1], [**47**,1], [**53**,1], [**113**,1], [**191**,1]
35	[2,48], [3,4], [5,4], [7,2], [11,1], [13,1], [17,1], [19,1], [23,1], [29,1], [31,1], [**101**,1], [**523**,1], [**5333**,1]

Table 26: ℓ-torsion in $H_1(PSL_2(\mathbb{Z}[i]), M_{n,1}(\mathbb{Z}[i]))$

n	ℓ-torsion up to $\ell = 7919$
2	[2,7]
4	[2,10], [3,1]
6	[2,15]
8	[2,18], [3,1], [5,1], [7,1]
10	[2,23], [3,1], [5,1]
12	[2,26], [3,2], [5,1], [7,1], [11,1]
14	[2,31], [3,2], [5,1]
16	[2,34], [3,2], [5,2],[7,1], [11,1], [13,1], [**197**,1]
18	[2,39], [3,2], [5,1], [7,1], [**53**,1]
20	[2,42], [3,3], [5,2], [7,1], [11,1], [13,1], [17,1], [19,1]
22	[2,47], [3,3], [5,2], [7,1], [**43**,1], [**599**,1]
24	[2,50], [3,4], [5,2], [7,1], [11,1], [13,1], [17,1], [19,1], [23,1], [**31**,1], [**2053**,1]
26	[2,55], [3,3], [5,2], [7,2], [11,1], [13,1], [**47**,1]
28	[2,58], [3,5], [5,3], [7,2], [11,1], [13,1], [17,1], [19,1], [23,1], [**89**,1], [**107**,1], [**829**,1]

Table 27: ℓ-torsion in $H_1(PSL_2(\mathbb{Z}[i]), M_{n,2}(\mathbb{Z}[i]))$

n	ℓ-torsion up to $\ell = 113$
0	[2,1], [3,1]
2	[2,2]
4	[2,4], [3,1]
6	[2,4], [3,2]
8	[2,6], [3,1]
10	[2,8], [3,3]
12	[2,8], [3,3]
14	[2,10], [3,3], [11,1]
16	[2,12], [3,4], [11,1]
18	[2,12], [3,5], [5,1], [11,1]
20	[2,14], [3,4], [5,1], [11,1], [17,1]
22	[2,16], [3,6], [11,2], [17,1], [19,1]
24	[2,16], [3,6], [7,1], [11,1], [17,1], [19,1]
26	[2,18], [3,6], [11,2], [17,1], [19,1], [**29**,1], [**61**,1], [**89**,1]
28	[2,20], [3,7], [5,2], [11,2], [17,1], [19,1]
30	[2,20], [3,8], [5,3], [7,1], [11,2], [17,1], [19,1], [29,1]

Table 28: ℓ-torsion in $H_1(PSL_2(\mathbb{Z}[\sqrt{-2}]), M_{n,0}(\mathbb{Z}[\sqrt{-2}]))$

n	ℓ-torsion up to $\ell = 113$
0	[3,1]
2	[3,1]
4	[2,1]
6	[3,1]
8	[2,2], [3,1]
10	[2,2], [3,2]
12	[3,2]
14	[2,1], [3,3], [7,1]
16	[2,2], [3,2]
18	[2,1], [3,4]
20	[2,2], [3,3], [7,1], [13,1]
22	[2,3], [3,4], [7,1]
24	[2,1], [3,4]
26	[2,2], [3,5], [7,1], [13,1], [19,1]
28	[2,3], [3,4], [7,1]
30	[2,1], [3,6], [7,1]

Table 29: ℓ-torsion in $H_1(PSL_2(\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]), M_{n,0}(\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]))$

We conclude with tables for the figure-8 knot complement group Γ_8 and a link complement group for the modules $M_{n,0}$. We don't know what large primes should be here.

n	ℓ-torsion up to $\ell = 113$
0	
2	
4	[2,1], [7,1]
6	[2,2], [3,1]
8	[2,3], [3,1], [7,1], [97,1]
10	[2,3], [3,2], [7,2], [13,1]
12	[2,4], [3,2], [7,2]
14	[2,5], [3,3], [7,3], [13,1], [43,1]
16	[2,6], [3,3], [7,3], [13,1], [31,1]
18	[2,7], [3,4], [7,3], [13,1], [29,1]
20	[2,7], [3,4], [7,3], [13,1]
22	[2,8], [3,5], [7,5], [13,2], [19,1], [73,1]
24	[2,9], [3,5], [5,3], [7,4], [13,2], [19,1], [97,1]
26	[2,10], [3,6], [7,4], [13,3], [19,1], [59,1], [61,1]
28	[2,11], [3,6],[5,1], [7,6], [13,2], [19,1], [43,1], [53,1]
30	[2,11], [3,7], [5,2], [7,6], [11,1], [13,2], [19,1], [37,3]

Table 30: ℓ-torsion in $H_1(\Gamma_8, M_{n,0}(\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]))$

n	ℓ-torsion up to $\ell = 113$
0	
2	[2,4],[3,2]
4	[2,7], [3,2],[7,1]
6	[2,9], [3,2], [7,1]
8	[2,12], [3,4], [7,3]
10	[2,15], [3,5], [7,2]
12	[2,17], [3,6], [7,4], [11,2]
14	[2,20], [3,6], [7,4], [11,1]
16	[2,23], [3,8], [7,5], [11,1]
18	[2,25], [3,10], [7,5], [11,1]
20	[2,28], [3,9], [7,6], [11,1]

Table 31: ℓ-torsion in $H_1(\Gamma_{-7}(6,4), M_{n,0}(\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]))$

5 Hecke eigenvalues for $H_1(\Gamma, M)$

One important goal of this chapter is to explicitly compute sequences of Hecke eigenvalues for simultaneous eigenvectors of the Hecke operators T_π ($\pi \in \mathbb{Z}[i]$ a prime element) acting on $H_1(PSL_2(\mathbb{Z}[i]), M_{n,m}(\mathbb{Z}[i]))$. In particular we want to consider the action of T_π on certain ℓ-torsion classes. The classes in the homology modulo torsion (free part) we also want to call automorphic classes. Then it is of special interest to study the properties of the occuring systems of Hecke eigenvalues mod ℓ. This includes the search for congruences, the study of the relations between such eigenvalue systems for different homology modules $H_1(\Gamma, M)$ (changing the group, the coefficient module or both) and the relations between torsion classes and automorphic classes (e.g. lifts to characterstic zero) as well as questions of more arithmetic nature like e. g. the relation to possible Galois represenations mod ℓ. We will treat some of these aspects from a computational point of view.
As we saw in Chapter 4 we can mainly concentrate on the space Λ_G. So we start off in Section 5.1 with describing a quite general construction of Hecke operators T_π on the modules Λ_G leading to a formula for T_π on elements of Λ_G. Then we derive explicit formulas for T_π on the generators $(A-1)\otimes_\Gamma \mathrm{m}$, $(U-1)\otimes_\Gamma \mathrm{m}$ and $(B-1)\otimes_\Gamma \mathrm{m}$ of Λ_Γ in the case of the group $\Gamma = PSL_2(\mathbb{Z}[i])$. These formulas are the basis for the algorithmic realization of the computation of the Hecke eigenvalues, which is explained in Section 5.3. A choice of our computational results is presented and analysed in Section 5.4. In particular we find several interesting congruences satisfied for all primes π (of small norm) we considered.
We conclude with some more theoretical insights about classes in $H_1(\Gamma, M)$ in Section 5.5. So we could guess a general candidate w in Λ_Γ for an eigenvector of the free part (if one occurs) with the help of our computer programs and can prove that it is indeed in $H_1(\Gamma, M)$. Furthermore we can prove the eigenvalue equation $T_\pi(w) = (\pi^{r+1} + \overline{\pi}^{s+1})w$ in the cases of $s = 0$ and $s = 1$ for the modules $M_{r,s}(\mathbb{Z}[i])$ using some results from Section 5.2 and some insights about torsion classes. In all computed cases we also see that the eigenclasses are of infinite order for $r + s + 2 \equiv 0 \bmod 4$. For $s = 0$ and $s = 1$ we can also prove that the candidates in the other cases for r and s are torsion classes. In the case of general s we show that the so-called $(B-1)$-part of $T_\pi(w)$ always vanishes.

5.1 Hecke operators on Λ_G

Classically, Hecke operators were defined as an interesting set of endomorphisms on the spaces of modular forms for Fuchsian groups Γ. An important aspect was the deep relation between the Fourier coefficients of certain modular forms and corresponding Hecke eigenvalues, which was obtained first by Hecke. The Hecke

theory also gives explanations for many interesting identities and leads to important arithmetical applications. This approach is explained in most of the books about modular forms, cf. e. g. [Sh] or [Mi].
More generally, such Hecke operators are defined on spaces of automorphic forms or in the other interpretation on the cohomology of arithmetic groups. Mainly one finds that in the topological context of sheaf cohomology (cf. [Ha3] or [Ha5]). Having the appropriate comparison isomorphisms in mind, we need a construction on the group cohomology or homology in our situation. There are some variants of Hecke operators on group cohomology (cf. [RW] or [Th]), which are not suitable for us. Instead we follow the approach used in [EGM 3] for Hecke operators on Γ^{ab} (first homology of Γ with trivial coefficients $\mathbb{Z}$), which is based on the mechanism of restriction and corestriction in group cohomology and homology, cf. e. g. [Br].
Since $H_1(G, M) \subseteq \Lambda_G$ it is good enough to define and to study Hecke operators on Λ_G. Thus, the task of this section is to go through the general construction to finally derive a formula of Hecke operators T_π, expressed on the elements of Λ_G. For simplicity let us suppose in this section that G is any of the groups $PSL_2(\mathcal{O}_K)$, and let $\pi \neq 0$ be an element in $\mathcal{O}_K$, which is not necessarily a prime element here. Let further $M = M_{n,m}(\mathcal{O}_K)$. Note again that we prefer to use the notation G in the general constructions and Γ for the more explicit treatment of specific arithmetic groups since that is a standard notation.
Now we take $\delta_\pi = \left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and consider the two groups $U := \Gamma_0(\pi)$ and $U' := \Gamma^0(\pi)$. Note that $\Gamma_0(\pi) = G \cap \delta_\pi^{-1} G\, \delta_\pi$ and $\Gamma^0(\pi) = G \cap \delta_\pi\, G\, \delta_\pi^{-1}$ in our case. The groups U and U' have finite index in G. If we carry out the multiplication

$$\begin{pmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & \pi^{-1} b \\ \pi c & d \end{pmatrix},$$

we see that $\Gamma_0(\pi)$ consits of the matrices from G, where the lower left entry is divisible by π. Analogously, $\Gamma^0(\pi)$ consits of the matrices of G, where the upper right entry is divisible by π. In particular we have $\delta_\pi \Gamma_0(\pi) \delta_\pi^{-1} = \Gamma^0(\pi)$, or for short in our notation

$$\delta_\pi\, U\, \delta_\pi^{-1} = U'. \tag{5.1}$$

Then the Hecke operator T_π will be defined by the following diagram

$$\begin{array}{ccc} \Lambda_G & \xrightarrow{T_\pi} & \Lambda_G \\ \downarrow{\scriptstyle \Phi} & & \uparrow{\scriptstyle I} \\ \Lambda_U & \xrightarrow{\varepsilon_\pi} & \Lambda_{U'}. \end{array} \tag{5.2}$$

The map Φ is the transfer homomorphism, ε_π is a homomorphism induced by the conjugation with δ_π and I is induced by the inclusion. The Hecke operator T_π is then defined as the composition of these three homomorphisms, i. e.

$$T_\pi \quad = \quad I \circ \varepsilon_\pi \circ \Phi.$$

We will now see that this construction is well-defined, and we want to derive a formula for $T_\pi((g-1)\otimes_G \mathrm{m})$, where $(g-1)\otimes_G \mathrm{m}$ is in Λ_G. It is easy to see that we have to take for $I : \Lambda_{U'} \longrightarrow \Lambda_G$ just the map $I((u'-1)\otimes_{U'} \mathrm{m}) := (u'-1)\otimes_G \mathrm{m}$ with $u' \in U'$ and $\mathrm{m} \in M$, determined by the change of the group ring under the tensor product. For ε_π we get:

Lemma 5.1 *The map* $\varepsilon_\pi : \Lambda_U \longrightarrow \Lambda_{U'}$ *given by*

$$\varepsilon_\pi((u-1)\otimes_U \mathrm{m}) \quad := \quad (\delta_\pi u\, \delta_\pi^{-1} - 1)\otimes_{U'} \delta_\pi \cdot \mathrm{m}, \tag{5.3}$$

with $u \in U$, $\mathrm{m} \in M$ *and* $\pi \neq 0$ *an element in* $\mathcal{O}_K$, *is well-defined.*

Proof: First note that δ_π acts on an element $\mathrm{m} \in M$ just like the elements $g \in PSL_2(\mathcal{O}_K)$ in Section 1.2 (action of $GL_2(K)$) and because of (5.1) we have $\delta_\pi u\, \delta_\pi^{-1} \in U'$.
Now let $t_1 := (u_1u_2-1)\otimes_U \mathrm{m}$, $t_2 := (u_1-1)\otimes_U u_2\cdot\mathrm{m}$ and $t_3 := (u_2-1)\otimes_U \mathrm{m})$ with $u_1, u_2 \in U$ and $\mathrm{m} \in M$. Because of property (4.1) we have $t_1 = t_2 + t_3$. To show that ε_π is well-defined, or in other words, that the algebraic structure is preserved under ε_π, we just have to show that $\varepsilon_\pi(t_1) = \varepsilon_\pi(t_2) + \varepsilon_\pi(t_3)$.
So, on the one hand we have

$$\begin{aligned}
\varepsilon_\pi(t_2) + \varepsilon_\pi(t_3) &= \varepsilon_\pi((u_1-1)\otimes_U u_2\cdot \mathrm{m}) + \varepsilon_\pi((u_2-1)\otimes_U \mathrm{m}) \\
&= (\delta_\pi u_1\, \delta_\pi^{-1} - 1)\otimes_{U'} \delta_\pi u_2 \cdot \mathrm{m} + (\delta_\pi u_2\, \delta_\pi^{-1} - 1)\otimes_{U'} \delta_\pi \cdot \mathrm{m}.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\varepsilon_\pi(t_1) &= \varepsilon_\pi((u_1u_2-1)\otimes_U \mathrm{m}) \\
&= (\delta_\pi u_1 u_2\, \delta_\pi^{-1} - 1)\otimes_{U'} \delta_\pi \cdot \mathrm{m} \\
&= ((\delta_\pi u_1\, \delta_\pi^{-1} - 1)\, \delta_\pi u_2\, \delta_\pi^{-1} + (\delta_\pi u_2\, \delta_\pi^{-1} - 1))\otimes_{U'} \delta_\pi \cdot \mathrm{m} \\
&= (\delta_\pi u_1\, \delta_\pi^{-1} - 1)\, \delta_\pi u_2\, \delta_\pi^{-1} \otimes_{U'} \delta_\pi \cdot \mathrm{m} + (\delta_\pi u_2\, \delta_\pi^{-1} - 1)\otimes_{U'} \delta_\pi \cdot \mathrm{m}.
\end{aligned}$$

But $\delta_\pi u_2\, \delta_\pi^{-1} \in U'$ because of (5.1) and so we can apply property (4.1) again and get

$$\varepsilon_\pi(t_1) \quad = \quad (\delta_\pi u_1\, \delta_\pi^{-1} - 1)\otimes_{U'} \delta_\pi u_2 \cdot \mathrm{m} + (\delta_\pi u_2\, \delta_\pi^{-1} - 1)\otimes_{U'} \delta_\pi \cdot \mathrm{m},$$

which gives the equality we wanted to show. □

Finally let us consider the transfer map Φ. For that let the index of U in G be $\nu = [G : U]$. Then we can take $r_1, \ldots, r_\nu \in G$ as a system of representatives for G/U and have

$$G \;=\; r_1 U \,\dot{\cup}\, \ldots \,\dot{\cup}\, r_\nu U.$$

Let $g \in G$, then there are uniquely determined elements $h(g,i) \in U$ with $i \in \{1, \ldots, \nu\}$, so that

$$g \cdot r_i \;=\; r_{g(i)} \cdot h(g,i). \tag{5.4}$$

The two modules we need will be $\Lambda_G = \mathfrak{a}_G \otimes_G M$ and $\Lambda_U = \mathfrak{a}_U \otimes_U M$ in the setup of Section 4.1. Now we can state the following lemma about the transfer homomorphism Φ.

Lemma 5.2 *The map $\Phi : \mathfrak{a}_G \otimes_G M \longrightarrow \mathfrak{a}_U \otimes_U M$ given by*

$$\Phi((g-1) \otimes_G \mathrm{m}) \;:=\; \sum_{i=1}^{\nu} (h(g,i) - 1) \otimes_U r_i^{-1} \cdot \mathrm{m}, \tag{5.5}$$

with $g \in G$, $\mathrm{m} \in M$, $\nu = [G : U]$ and $h(g,i)$, r_i as above, is well-defined.

Proof: Again we have to show that Φ preserves the algebraic structure determined by the definition of the the tensor product $\mathfrak{a}_G \otimes_G M$ introduced in Section 4.1. For that let $t_1 := (g_1 g_2 - 1) \otimes_G \mathrm{m}$, $t_2 := (g_1 - 1) \otimes_G g_2 \cdot \mathrm{m}$ and $t_3 := (g_2 - 1) \otimes_G \mathrm{m}$ with $g_1, g_2 \in G$ and $\mathrm{m} \in M$. Again we have $t_1 = t_2 + t_3$ because of property (4.1). Then we have to show that $\Phi(t_1) = \Phi(t_2) + \Phi(t_3)$.
Note first that we have a representation as in (5.4):

$$g_1 g_2 \cdot r_i \;=\; r_{g_1 g_2(i)} \cdot h(g_1 g_2, i). \tag{5.6}$$

There are also representations $g_1 \cdot r_i = r_{g_1(i)} \cdot h(g_1, i)$ and $g_2 \cdot r_i = r_{g_2(i)} \cdot h(g_2, i)$ and as well a representation $g_1 \cdot r_{g_2(i)} = r_{g_1(g_2(i))} \cdot h(g_1, g_2(i))$. Furthermore it holds that $g_1(g_2(i)) = g_1 g_2(i)$ for all $g_1, g_2 \in G$ and $i \in \{1, \ldots, \nu\}$. So we replace $g_2 \cdot r_i$ in (5.6) by its representation and get

$$g_1 \cdot r_{g_2(i)} \cdot h(g_2, i) \;=\; r_{g_1 g_2(i)} \cdot h(g_1 g_2, i),$$

giving

$$r_{g_1 g_2(i)}^{-1} \cdot g_1 \cdot r_{g_2(i)} \cdot h(g_2, i) \;=\; h(g_1 g_2, i).$$

Next, we replace $g_1 \cdot r_{g_2(i)}$ by the last representation above and finally get

$$h(g_1 g_2, i) = h(g_1, g_2(i)) \cdot h(g_2, i). \tag{5.7}$$

Then we use (5.7) to express

$$h(g_1g_2,i)-1 \;=\; (h(g_1,g_2(i))-1)h(g_2,i)+(h(g_2,i)-1). \qquad (5.8)$$

Now, on the one hand we have

$$\begin{aligned}\Phi(t_2) &= \Phi((g_1-1)\otimes_G g_2\cdot \mathrm{m})\\ &= \sum_{i=1}^{\nu}(h(g_1,i)-1)\otimes_U r_i^{-1}g_2\cdot \mathrm{m} \;\; =: S\end{aligned}$$

and

$$\begin{aligned}\Phi(t_3) &= \Phi((g_2-1)\otimes_G \mathrm{m})\\ &= \sum_{i=1}^{\nu}(h(g_2,i)-1)\otimes_U r_i^{-1}\cdot \mathrm{m} \;\; =: T.\end{aligned}$$

On the other hand we have

$$\begin{aligned}\Phi(t_1) &= \Phi((g_1g_2-1)\otimes_G \mathrm{m})\\ &= \sum_{i=1}^{\nu}(h(g_1g_2,i)-1)\otimes_U r_i^{-1}\cdot \mathrm{m}\\ &= \sum_{i=1}^{\nu}(h(g_1,g_2(i))-1)h(g_2,i)\otimes_U r_i^{-1}\cdot \mathrm{m}+\sum_{i=1}^{\nu}(h(g_2,i)-1)\otimes_U r_i^{-1}\cdot \mathrm{m}\end{aligned}$$

by applying (5.8). We already see that the second sum in the last line equals T. So we are done if we can show that the first sum equals S. For that we have $g_2\cdot r_i = r_{g_2(i)}\cdot h(g_2,i)$, which can be rearranged to

$$r_{g_2(i)}^{-1}\cdot g_2 \;=\; h(g_2,i)\cdot r_i^{-1}. \qquad (5.9)$$

Now we get for the first sum

$$\begin{aligned}\sum_{i=1}^{\nu}(h(g_1,g_2(i))-1)h(g_2,i)\otimes_U r_i^{-1}\cdot \mathrm{m} &= \sum_{i=1}^{\nu}(h(g_1,g_2(i))-1)\otimes_U h(g_2,i)\,r_i^{-1}\cdot \mathrm{m}\\ &= \sum_{i=1}^{\nu}(h(g_1,g_2(i))-1)\otimes_U r_{g_2(i)}^{-1}\,g_2\cdot \mathrm{m},\end{aligned}$$

because $h(g_2,i)\in U$, and therefore we can apply property (4.1) in the first step and (5.9) in the second step.

If we now set $j := g_2(i)$, which gives $i = g_2^{-1}(j)$, we can just change the index of summation and get

$$\sum_{i=1}^{\nu}(h(g_1, g_2(i)) - 1) \otimes_U r_{g_2(i)}^{-1}\, g_2 \cdot \mathrm{m} \;=\; \sum_{j=1}^{\nu}(h(g_1, j) - 1) \otimes_U r_j^{-1}\, g_2 \cdot \mathrm{m} \;= S,$$

which gives the desired equality. □

Forming the composition of the three homomorphisms gives:

Theorem 5.3 *Let G be any of the groups $PSL_2(\mathcal{O}_K)$ and let $\pi \neq 0$ be an element in $\mathcal{O}_K$. Let then T_π be the Hecke operator defined by the diagram* (5.2) *via $T_\pi = I \circ \varepsilon_\pi \circ \Phi$, where the homomorphisms Φ, ε_π and I are constructed as above and let $(g-1) \otimes_G \mathrm{m}$ be in Λ_G with $M = M_{n,m}(\mathcal{O}_K)$. Then*

$$T_\pi((g-1) \otimes_G \mathrm{m}) \;=\; \sum_{i=1}^{\nu}(\delta_\pi h(g,i)\, \delta_\pi^{-1} - 1) \otimes_G \delta_\pi r_i^{-1} \cdot \mathrm{m},$$

where $r_1, \ldots, r_\nu$ is a system of representatives for G/U and $h(g,i) \in U$ as described above.

PROOF: Using Lemma 5.2 we get

$$\Phi((g-1) \otimes_G \mathrm{m}) \;=\; \sum_{i=1}^{\nu}(h(g,i) - 1) \otimes_U r_i^{-1} \cdot \mathrm{m}.$$

Applying now Lemma 5.1 yields

$$\varepsilon_\pi \circ \Phi((g-1) \otimes_G \mathrm{m}) \;=\; \sum_{i=1}^{\nu}(\delta_\pi h(g,i)\, \delta_\pi^{-1} - 1) \otimes_{U'} \delta_\pi r_i^{-1} \cdot \mathrm{m}.$$

Finally we switch from U' to G on the right side by using the map I and get

$$\begin{aligned} T_\pi((g-1) \otimes_G \mathrm{m}) &= I \circ \varepsilon_\pi \circ \Phi((g-1) \otimes_G \mathrm{m}) \\ &= \sum_{i=1}^{\nu}(\delta_\pi h(g,i)\, \delta_\pi^{-1} - 1) \otimes_G \delta_\pi r_i^{-1} \cdot \mathrm{m}. \end{aligned}$$

□

We also need the following lemma about the index of U in G. For simplicity we only state it for the full group $PSL_2(\mathcal{O}_K)$ and for prime elements π here. For other π one would have to count more generally the elements of the projective line over $\mathcal{O}_K/\pi\mathcal{O}_K$. One can also derive a similar formula for $G = \Gamma_0(\mathfrak{p})$.

Lemma 5.4 *Let G be $PSL_2(\mathcal{O}_K)$ and let $\pi \in \mathcal{O}_K$ be a prime element. Then*

$$[G : \Gamma_0(\pi)] = N(\pi) + 1,$$

where $N(\pi)$ is the norm of π.

PROOF: Let $\mathfrak{p} = \pi\mathcal{O}_K$ be the principal ideal generated by π. Then $\#(\mathcal{O}_K/\mathfrak{p}) = N(\pi)$. The ideal $\mathfrak{p}$ is always a prime ideal and since we are in a Dedekind ring it also holds that $\mathcal{O}_K/\mathfrak{p}$ is a finite field, which we want to denote by $\mathbb{F}_q$, the field with $q = N(\pi)$ elements.
Now we can consider the canonical group homomorphism

$$\phi : PSL_2(\mathcal{O}_K) \longrightarrow PSL_2(\mathcal{O}_K/\mathfrak{p}).$$

We have $G/\ker(\phi) \cong \phi(G)$, and from $\ker(\phi) \subseteq \Gamma_0(\pi)$ follows $\Gamma_0(\pi)/\ker(\phi) \cong \phi(\Gamma_0(\pi))$. Therefore we get

$$[G : \Gamma_0(\pi)] = [G/\ker(\phi) : \Gamma_0(\pi)/\ker(\phi)] = [\phi(G) : \phi(\Gamma_0(\pi))] = \frac{\#\phi(G)}{\#\phi(\Gamma_0(\pi))}.$$

Furthermore ϕ is surjective since $PSL_2(\mathcal{O}_K/\mathfrak{p})$ is generated unipotently as PSL_2 over a field, which gives $\#\phi(G) = \#PSL_2(\mathbb{F}_q)$. In $\phi(\Gamma_0(\pi))$ we now have to count matrices in $PSL_2(\mathbb{F}_q)$ of the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$. For even q we have $(q-1)q$ possibilities and for odd q we count $(q-1)q/2$. On the other hand we have $\#PSL_2(\mathbb{F}_q) = (q^2-1)q$ for even q and $\#PSL_2(\mathbb{F}_q) = (q^2-1)q/2$ for odd q (see [Hu]). If we divide out and use that $q = N(\pi)$ in our case we get the formula we wanted to show. □
Then we can state the following corollary from Theorem 5.3.

Corollary 5.5 *Take the assumptions from Theorem* 5.3 *but let π be a prime element in $\mathcal{O}_K$. Then*

$$T_\pi((g-1) \otimes_G \mathrm{m}) = \sum_{i=1}^{N(\pi)+1} (\delta_\pi h(g,i)\, \delta_\pi^{-1} - 1) \otimes_G \delta_\pi r_i^{-1} \cdot \mathrm{m}.$$

PROOF: Just use Lemma 5.4 and replace the index ν in the sum of Proposition 5.3 by $N(\pi) + 1$. □
We conclude with some remarks about the Hecke operators T_π.

Remark 5.6 Observe that T_π is strictly speaking the Hecke operator associated to the principal ideal generated by π. But most of all the situation gets more subtle if $\mathcal{O}_K$ has no longer class number 1. One has to consider $T_\mathfrak{a}$ for general ideals $\mathfrak{a}$ then, that is, not only for principal ideals. But with our construction we cover the principal ideals only (which, of course, give already some insight). On the other hand these don't give enough information for the Euler products in view of arithmetical applications in the end. Hence one needs formulas for the other cases as well. For that one also has to consider the groups PSL_2 for the other ideal classes. If, for example, we would be in the case of class number 2 like for $\mathbb{Z}[\sqrt{-5}]$ and the two groups would be G_1 and G_2 we would have to build up a construction for the Hecke operator as a map $T_{\mathfrak{a}} : \Lambda_{G_1} \times \Lambda_{G_2} \longrightarrow \Lambda_{G_1} \times \Lambda_{G_2}$. But still, there remain strong problems for actually performing the explicit computations then (see also Remark 5.12). Note further that it is not difficult to generalize our formulas to congruence subgroups of $PSL_2(\mathcal{O}_K)$ since we preserve the finite index of U and U' in G, but these groups depend on G then.

5.2 Explicit formulas for $\Gamma = PSL_2(\mathbb{Z}[i])$

The aim of this section is to find the explicit formulas for the Hecke operators T_π for the three generators A, B and U of the arithmetic group $\Gamma = PSL_2(\mathbb{Z}[i])$. Here, the T_π live on Λ_Γ and M is $M_{n,m}(\mathbb{Z}[i])$. We also mention the changes one would have to make to produce similar formulas for other $\Gamma = PSL_2(\mathcal{O}_K)$ or $\Gamma = \Gamma_0(\mathfrak{p})$. In general one has to produce the explicit formulas for $T_\pi((g-1) \otimes_\Gamma \mathrm{m})$ for each generator g of Γ and $\mathrm{m} \in M$.
Starting with an explicit system of representatives $r_1, \ldots, r_{N(\pi)+1}$ for $\Gamma/\Gamma_0(\pi)$, the first step is to compute the matrices $h(g,i) \in \Gamma_0(\pi)$ for $i = 1, \ldots, N(\pi)+1$. Then we have to express the result for $T_\pi((g-1) \otimes_\Gamma \mathrm{m})$ in a second step as a linear combination of the generators of Λ_Γ, which would be in our case a linear combination of $(A-1) \otimes_\Gamma e_i$, $(B-1) \otimes_\Gamma e_i$ and $(U-1) \otimes_\Gamma e_i$, where the e_i (cp. Section 1.2) run through the $k = (n+1)(m+1)$ generators of the $\mathbb{Z}[i]$-module $M_{n,m}(\mathbb{Z}[i])$. This is necessary to finally build up a matrix representation for each T_π acting on Λ_Γ or for its restriction to certain ℓ-torsion parts.
For simplicity we want to restrict to prime elements π with $N(\pi) = \pi\overline{\pi} = p$ with p being a rational prime, that is, to $\pi = 1+i$ or $\pi = a+bi$ (up to units) with $N(\pi) = a^2 + b^2 = p$ and $p \equiv 1 \bmod 4$. Then we have as a system of representatives r_i $(i = 1, \ldots, N(\pi)+1)$ for $\Gamma/\Gamma_0(\pi)$ the matrices $r_x = \left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)$ for $x = 0, \ldots, p-1$ and the matrix $r_\infty = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, using ∞ for the last index.
Let us now go through the two steps for each of the generators A, B and U.

$\mathbf{T}_\pi$ for the generator A.
In this case we have to carry out the two steps for the following formula from Corollary 5.5 with m as mentioned above:

$$T_\pi((A-1) \otimes_\Gamma \mathrm{m}) \;=\; \sum_{i=1}^{p+1} (\delta_\pi h(A,i) \delta_\pi^{-1} - 1) \otimes_\Gamma \delta_\pi r_i^{-1} \cdot \mathrm{m}.$$

Note again that we have $N(\pi) = p$ here. To avoid confusion with the i in $\mathbb{Z}[i]$ we always want to use x (including $x = \infty$) as index for the representatives from now on. So we have to find the $p+1$ matrices $h(A,x)$ determined by the system of representatives described above. We get

$$A \cdot r_x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1+x & 1 \end{pmatrix}$$

for $x = 0, \dots p-1$. Now we have to find $r_{A(x)}$ and $h(A,x)$ as described in (5.4), so that

$$A \cdot r_x = r_{A(x)} \cdot h(A,x).$$

We easily see that we can take

$$\begin{pmatrix} 1 & 0 \\ 1+x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1+x & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for $x = 0, \dots, p-2$ and

$$\begin{pmatrix} 1 & 0 \\ 1+x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}$$

for $x = p-1$. Hence we find that $h(A,x) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for $x = 0, \dots p-2$ and $h(A,x) = \left(\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix}\right)$ for $x = p-1$, and these matrices are obviously contained in $\Gamma_0(\pi)$.
In the last case we have

$$A \cdot r_\infty = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

and we see that we can take

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = r_\infty \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

which gives $h(A,\infty) = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$.
Now we have $\delta_\pi \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \delta_\pi^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so that all summands for $x = 0, \dots, p-2$ just vanish. Furthermore we get $\delta_\pi \left(\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix}\right) \delta_\pi^{-1} = \delta_\pi \left(\begin{smallmatrix} 1 & 0 \\ \pi\bar{\pi} & 1 \end{smallmatrix}\right) \delta_\pi^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ \bar{\pi} & 1 \end{smallmatrix}\right)$ and $\delta_\pi \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) \delta_\pi^{-1} = \left(\begin{smallmatrix} 1 & -\pi \\ 0 & 1 \end{smallmatrix}\right)$. We also have $r_{p-1}^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ p-1 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 1-p & 1 \end{smallmatrix}\right)$ and $r_\infty^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ as an element in $PSL_2(\mathbb{Z}[i])$.
Then we obtain

$$\begin{aligned} T_\pi((A-1) \otimes_\Gamma \mathrm{m}) &= \left(\begin{pmatrix} 1 & 0 \\ \bar{\pi} & 1 \end{pmatrix} - 1\right) \otimes_\Gamma \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1-p & 1 \end{pmatrix} \cdot \mathrm{m} \\ &+ \left(\begin{pmatrix} 1 & -\pi \\ 0 & 1 \end{pmatrix} - 1\right) \otimes_\Gamma \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathrm{m}. \qquad (5.10) \end{aligned}$$

Next, we need a linear combination of the following form:

$$\begin{aligned} & T_\pi((A-1)\otimes_\Gamma \mathrm{m}) \\ = & \sum_{i=1}^{k}\alpha_i(A-1)\otimes_\Gamma e_i+\sum_{i=1}^{k}\beta_i(B-1)\otimes_\Gamma e_i+\sum_{i=1}^{k}\gamma_i(U-1)\otimes_\Gamma e_i \\ = & (A-1)\otimes_\Gamma\sum_{i=1}^{k}\alpha_i e_i+(B-1)\otimes_\Gamma\sum_{i=1}^{k}\beta_i e_i+(U-1)\otimes_\Gamma\sum_{i=1}^{k}\gamma_i e_i. \end{aligned}$$

We mention that last rearrangement, because it gives the best suitable form for a realization in our computer program later on, which will be discussed in Section 5.3. To find the desired linear combination for (5.10) we first have to express the matrices appearing on the left side of $\otimes_\Gamma$ as a word in A, B, U, A^{-1}, B^{-1} and U^{-1}. Then we can use the decomposition process we described in Section 4.2 to extract the factors $(A-1)$, $(B-1)$ and $(U-1)$. Using the properties of the tensor product we shift the remaining sums to the other side of $\otimes_\Gamma$ and obtain the coefficients for the linear combination after applying the action of the group elements to the generator m we consider.

We now describe how one can express the matrices in the formula as words in the group generators and its inverses and how the decomposition works. For that recall that $A=\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$, $B=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ and $U=\left(\begin{smallmatrix}1&0\\i&1\end{smallmatrix}\right)$. Let us start with the matrix $\left(\begin{smallmatrix}1&0\\\overline{\pi}&1\end{smallmatrix}\right)$. We have $\pi\in\mathbb{Z}[i]$, so that we can take $\pi=a+bi$ with $a,b\in\mathbb{Z}$. This gives $\overline{\pi}=a-bi$. Since $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)^a=\left(\begin{smallmatrix}1&0\\a&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\i&1\end{smallmatrix}\right)^{-b}=\left(\begin{smallmatrix}1&0\\-bi&1\end{smallmatrix}\right)$ it is clear that we can express

$$\begin{pmatrix}1&0\\\overline{\pi}&1\end{pmatrix} = A^aU^{-b},$$

which is a word in our generators A and U and its inverses. On the other hand we have

$$\begin{pmatrix}1&-\pi\\0&1\end{pmatrix} = B\begin{pmatrix}1&0\\\pi&1\end{pmatrix}B,$$

and

$$\begin{pmatrix}1&0\\\pi&1\end{pmatrix} = A^aU^b,$$

which gives us the representation $\left(\begin{smallmatrix}1&-\pi\\0&1\end{smallmatrix}\right)=BA^aU^bB$ as a word in A, B, U and their inverses.

Applying the two rules 4.5–4.6 for the reduction process in Section 4.2 we can express $A^a-1=(A-1)A_{pol(a)}$, $U^b-1=(U-1)U_{pol(b)}$ and $U^{-b}-1=(U-1)U_{pol(-b)}$ and can therefore decompose as follows:

$$A^aU^{-b}-1 = (A-1)A_{pol(a)}U^{-b}+(U-1)U_{pol(-b)}$$

and

$$A^aU^b - 1 \;=\; (A-1)A_{pol(a)}U^b + (U-1)U_{pol(b)},$$

where $A_{pol(a)}$, $U_{pol(b)}$ and $U_{pol(-b)}$ are certain polynomials in A respectively U. These polynomials depend on the sign of a and b. For example, we have

$$A_{pol(a)} \;=\; A^{a-1} + A^{a-2} + \ldots + 1 \qquad \text{for } \; a > 0$$

and

$$A_{pol(a)} \;=\; -A^{a} - A^{a+1} - \ldots - A^{-1} \qquad \text{for } \; a < 0.$$

We assumed a to be non-zero, but in case of $a = 0$ the polynomial $A_{pol(a)}$ would just be the zero-matrix. Analogously we have to express the polynomials for U.
Finally we get

$$\begin{aligned} & BA^aU^bB - 1 \\ = \;& (A-1)A_{pol(a)}U^bB + (B-1)(A^aU^bB + 1) + (U-1)U_{pol(b)}B. \end{aligned}$$

So we are ready to write down the final formula for the Hecke operator T_π expressed on any generator $(A-1) \otimes_\Gamma \mathrm{m}$ of Λ_Γ.

Proposition 5.7 *Let T_π be the Hecke operator on Λ_Γ for $\Gamma = PSL_2(\mathbb{Z}[\,i\,])$ as defined in Section* 5.1, *where $\pi = a + bi \in \mathbb{Z}[\,i\,]$ is a prime element with $N(\pi) = p$ (p a rational prime), i. e. $p \equiv 1 \bmod 4$ or $\pi = 1 + i$ up to units. Also let $\mathrm{m} \in M_{n,m}(\mathbb{Z}[\,i\,])$ and let A, B and U be the generators of Γ as introduced in Section* 1.1. *Then*

$$\begin{aligned} & T_\pi((A-1) \otimes_\Gamma \mathrm{m}) \\ = \;\; & (A-1) \otimes_\Gamma A_{pol(a)} \left(U^{-b} \begin{pmatrix} \pi & 0 \\ 1-p & 1 \end{pmatrix} + U^bB \begin{pmatrix} 0 & \pi \\ -1 & 0 \end{pmatrix} \right) \cdot \mathrm{m} \\ + \;\; & (B-1) \otimes_\Gamma (A^aU^bB + 1) \begin{pmatrix} 0 & \pi \\ -1 & 0 \end{pmatrix} \cdot \mathrm{m} \\ + \;\; & (U-1) \otimes_\Gamma \left(U_{pol(-b)} \begin{pmatrix} \pi & 0 \\ 1-p & 1 \end{pmatrix} + U_{pol(b)}B \begin{pmatrix} 0 & \pi \\ -1 & 0 \end{pmatrix} \right) \cdot \mathrm{m}. \end{aligned}$$

PROOF: Replacing the two relevant matrices in (5.10) by the words in A, B, U and its inverses we get

$$\begin{aligned} & T_\pi((A-1) \otimes_\Gamma \mathrm{m}) \\ = \;& (A^aU^{-b} - 1) \otimes_\Gamma \begin{pmatrix} \pi & 0 \\ 1-p & 1 \end{pmatrix} \cdot \mathrm{m} + (BA^aU^bB - 1) \otimes_\Gamma \begin{pmatrix} 0 & \pi \\ -1 & 0 \end{pmatrix} \cdot \mathrm{m}. \end{aligned} \tag{5.11}$$

If we now replace the words by the decompositions we found above and use the properties of the tensor product we get our final formula. □

T$_\pi$ for the generator U.
Next, we want to consider the generator U, because we have to do here a very similar job as for the generator A. So we leave the hardest case of the generator B until the end.
Again we have to master our two steps for the following formula with analogous assumptions as for A:

$$T_\pi((U-1)\otimes_\Gamma \mathrm{m}) \;=\; \sum_{i=1}^{p+1}(\delta_\pi h(U,i)\delta_\pi^{-1}-1)\otimes_\Gamma \delta_\pi r_i^{-1}\cdot \mathrm{m}.$$

As in the case of A we want to write $h(U,x)$ instead of $h(U,i)$ from now on. At first we need the $p+1$ matrices $h(U,x)$ determined by the system of representatives for $\Gamma/\Gamma_0(\pi)$. We have

$$U\cdot r_\infty \;=\; \begin{pmatrix}1&0\\ i&1\end{pmatrix}\begin{pmatrix}0&1\\ -1&0\end{pmatrix} \;=\; \begin{pmatrix}0&1\\ -1&i\end{pmatrix}$$

and so we can take

$$\begin{pmatrix}0&1\\ -1&i\end{pmatrix} \;=\; \begin{pmatrix}0&1\\ -1&0\end{pmatrix}\begin{pmatrix}1&-i\\ 0&1\end{pmatrix} \;=\; r_\infty\cdot\begin{pmatrix}1&-i\\ 0&1\end{pmatrix},$$

which gives $h(U,\infty)=\left(\begin{smallmatrix}1&-i\\ 0&1\end{smallmatrix}\right)$.
For the other representatives we get

$$U\cdot r_x \;=\; \begin{pmatrix}1&0\\ i&1\end{pmatrix}\begin{pmatrix}1&0\\ x&1\end{pmatrix} \;=\; \begin{pmatrix}1&0\\ x+i&1\end{pmatrix}.$$

Since we already used r_∞ we should try to find an expression of the following form:

$$\begin{pmatrix}1&0\\ x+i&1\end{pmatrix} \;=\; \begin{pmatrix}1&0\\ y&1\end{pmatrix}\begin{pmatrix}1&0\\ x-y+i&1\end{pmatrix}.$$

Here, the first factor should be one of the r_x and the second should be in $\Gamma_0(\pi)$ to be one of our matrices $h(U,x)$. The second matrix will be in $\Gamma_0(\pi)$ if π divides the lower left entry $x-y+i$. So we have to find an element $y\in\{0,\dots,p-1\}$ depending on x, such that this divisibility condition is satisfied. For that let $\pi=a+bi$ with $a^2+b^2=p$ as in the case of the generator A. Since π is a prime element we have $\gcd(a,b)=1$. Our goal is to express the entry as follows:

$$x-y+i \;=\; (a+bi)(c(x)+d(x)i) \;=\; (ac(x)-bd(x))+i(ad(x)+bc(x)).$$

Using the extended Euclidean algorithm we can find some $c(x), d(x) \in \mathbb{Z}$, so that $ad(x) + bc(x) = 1$. Thus we have

$$y = x - ac(x) + bd(x). \tag{5.12}$$

Furthermore we can add any multiple of a to $c(x)$ as long as we substract the same multiple of b from $d(x)$. If we do that in (5.12), we always modify the sum by a summand $a^2 + b^2 = p$. Hence we can always choose unique $c(x)$ and $d(x)$ so that y is in the set $\{0, \dots, p-1\}$, and we want to define them by this condition. From now on we put $z := x - y$. Our consideration shows that we always have a matrix

$$h(U, x) = \begin{pmatrix} 1 & 0 \\ z+i & 1 \end{pmatrix}$$

with z depending on π and x. Then we get

$$\delta_\pi \begin{pmatrix} 1 & 0 \\ z+i & 1 \end{pmatrix} \delta_\pi^{-1} = \begin{pmatrix} 1 & 0 \\ \pi^{-1}(z+i) & 1 \end{pmatrix}$$

and

$$\delta_\pi \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \delta_\pi^{-1} = \begin{pmatrix} 1 & -\pi i \\ 0 & 1 \end{pmatrix}.$$

Hence we obtain

$$\begin{aligned} T_\pi((U-1) \otimes_\Gamma \mathrm{m}) = & \sum_{x=0}^{p-1} \left(\begin{pmatrix} 1 & 0 \\ \pi^{-1}(z+i) & 1 \end{pmatrix} - 1 \right) \otimes_\Gamma \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}^{-1} \cdot \mathrm{m} \\ & + \left(\begin{pmatrix} 1 & -\pi i \\ 0 & 1 \end{pmatrix} - 1 \right) \otimes_\Gamma \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathrm{m}. \end{aligned}$$

Next we have to go through the same business as for the generator A, that is, we have to express $T_\pi((U-1) \otimes_\Gamma \mathrm{m})$ as a linear combination of the $(A-1) \otimes_\Gamma e_i$, $(B-1) \otimes_\Gamma e_i$ and $(U-1) \otimes_\Gamma e_i$. This means, we first have to express the matrices $\left(\begin{smallmatrix} 1 & 0 \\ \pi^{-1}(z+i) & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & -\pi i \\ 0 & 1 \end{smallmatrix}\right)$ as words in A, B, U and its inverses and then have to go through our decomposition procedure.
From the consideration above we have $\pi^{-1}(z+i) = c(x) + d(x)i$, where $c(x)$ and $d(x)$ depend on x. But this dependence is not as uncontrolled as one might think. From above it is clear that we can choose unique numbers $c, d \in \mathbb{Z}$ with $ad + bc = 1$, such that $ac - bd \in \{0, \dots, p-1\}$. So we define c and d by these conditions. Then we see that we end up with two cases for x only. For $x \in \{ac - bd, \dots, p-1\}$ we can just take $c(x) = c$ and $d(x) = d$ and for $x \in \{0, 1, \dots, ac - bd - 1\}$ we have to take $c(x) = c - a$ and $d(x) = d + b$. Since the first matrix type in our formula

for $T_\pi((U-1)\otimes_\Gamma \mathrm{m})$ is already a lower triangular matrix of the right shape we can express it again as

$$\begin{pmatrix} 1 & 0 \\ \pi^{-1}(z+i) & 1 \end{pmatrix} = A^{c(x)}U^{d(x)},$$

but with the two cases for $c(x)$ and $d(x)$ only. Furthermore we have $B\left(\begin{smallmatrix} 1 & 0 \\ \pi i & 1 \end{smallmatrix}\right)B = \left(\begin{smallmatrix} 1 & -\pi i \\ 0 & 1 \end{smallmatrix}\right)$ and, since $\pi i = -b+ai$, we easily see that we can write

$$\begin{pmatrix} 1 & -\pi i \\ 0 & 1 \end{pmatrix} = BA^{-b}U^aB.$$

Then our formula has the following shape:

$$\begin{aligned} & T_\pi((U-1)\otimes_\Gamma \mathrm{m}) \\ = & \sum_{x=0}^{ac-bd-1}(A^{c-a}U^{d+b}-1)\otimes_\Gamma \begin{pmatrix} \pi & 0 \\ -x & 1 \end{pmatrix}\cdot \mathrm{m} + \sum_{x=ac-bd}^{p-1}(A^cU^d-1)\otimes_\Gamma \begin{pmatrix} \pi & 0 \\ -x & 1 \end{pmatrix}\cdot \mathrm{m} \\ & + (BA^{-b}U^aB-1)\otimes_\Gamma \begin{pmatrix} 0 & \pi \\ -1 & 0 \end{pmatrix}\cdot \mathrm{m}. \end{aligned} \tag{5.13}$$

Now we use similar decompositions for our words as in the case of the generator A. The structure of the matrix polynomials is the same, only the exponents have to be adapted. So we get $A^{c(x)}-1 = (A-1)A_{pol(c(x))}$, $U^{d(x)}-1 = (U-1)U_{pol(d(x))}$, $A^{-b}-1 = (A-1)A_{pol(-b)}$ and $U^a-1 = (U-1)U_{pol(a)}$. Then we obtain

Proposition 5.8 *Let T_π, π, Γ and* m *be as in Proposition* 5.7, *c and d as introduced above and let A, B and U be the generators of Γ. Then*

$$\begin{aligned} & T_\pi((U-1)\otimes_\Gamma \mathrm{m}) \\ = & (A-1)\otimes_\Gamma \left[\left(A_{pol(c-a)}U^{d+b}\sum_{x=0}^{ac-bd-1}\begin{pmatrix} \pi & 0 \\ -x & 1 \end{pmatrix}\right) + \left(A_{pol(c)}U^d\sum_{x=ac-bd}^{p-1}\begin{pmatrix} \pi & 0 \\ -x & 1 \end{pmatrix}\right)\right. \\ & \left. + A_{pol(-b)}U^aB\begin{pmatrix} 0 & \pi \\ -1 & 0 \end{pmatrix}\right]\cdot \mathrm{m} + (B-1)\otimes_\Gamma (A^{-b}U^aB+1)\begin{pmatrix} 0 & \pi \\ -1 & 0 \end{pmatrix}\cdot \mathrm{m} \\ & + (U-1)\otimes_\Gamma \left[\left(U_{pol(d+b)}\sum_{x=0}^{ac-bd-1}\begin{pmatrix} \pi & 0 \\ -x & 1 \end{pmatrix}\right) + \left(U_{pol(d)}\sum_{x=ac-bd}^{p-1}\begin{pmatrix} \pi & 0 \\ -x & 1 \end{pmatrix}\right)\right. \\ & \left. + U_{pol(a)}B\begin{pmatrix} 0 & \pi \\ -1 & 0 \end{pmatrix}\right]\cdot \mathrm{m}. \end{aligned}$$

PROOF: Using our decompositions we get from (5.13)

$$\begin{aligned}
&T_\pi((U-1)\otimes_\Gamma \mathrm{m})\\
&= \sum_{x=0}^{ac-bd-1} ((A-1)A_{pol(c-a)}U^{d+b} + (U-1)U_{pol(d+b)}) \otimes_\Gamma \begin{pmatrix} \pi & 0 \\ -x & 1 \end{pmatrix} \cdot \mathrm{m}\\
&+ \sum_{x=ac-bd}^{p-1} ((A-1)A_{pol(c)}U^{d} + (U-1)U_{pol(d)}) \otimes_\Gamma \begin{pmatrix} \pi & 0 \\ -x & 1 \end{pmatrix} \cdot \mathrm{m}\\
&+ (B-1)(A^{-b}U^aB+1) \otimes_\Gamma \begin{pmatrix} 0 & \pi \\ -1 & 0 \end{pmatrix} \cdot \mathrm{m}\\
&+ ((A-1)A_{pol(-b)}U^aB + (U-1)U_{pol(a)}B) \otimes_\Gamma \begin{pmatrix} 0 & \pi \\ -1 & 0 \end{pmatrix} \cdot \mathrm{m}.
\end{aligned}$$

If we now apply the linearity of the tensor product, our property (4.1) and rearrange everything we get the desired formula. □

$\mathbf{T}_\pi$ for the generator B.

Finally we have to carry out our two steps for the following formula for the generator B using analogous assumptions as for the generators A and U:

$$T_\pi((B-1)\otimes_\Gamma \mathrm{m}) = \sum_{i=1}^{p+1}(\delta_\pi h(B,i)\delta_\pi^{-1} - 1) \otimes_\Gamma \delta_\pi r_i^{-1} \cdot \mathrm{m}.$$

Again we write $h(B,x)$ instead of $h(B,i)$ from now on. So we start off with the determination of the $p+1$ matrices $h(B,x) \in \Gamma_0(\pi)$. We have

$$B \cdot r_\infty = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in $PSL_2(\mathbb{Z}[i])$, and so we can take

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = r_0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore $h(B,\infty) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and the associated summand vanishes. Hence we have to concentrate on

$$B \cdot r_x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} -x & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix}.$$

For $x=0$ we can represent

$$B \cdot r_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = r_\infty \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and therefore $h(B,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that the associated summand vanishes as well. In the other cases we can express

$$\begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ p - inv(x) & 1 \end{pmatrix} \begin{pmatrix} x & 1 \\ -px + x \cdot inv(x) - 1 & inv(x) - p \end{pmatrix},$$

where $inv(x) := x^{-1}$ in the multiplicative group $\mathbb{F}_p^*$. Since $inv(x) \in \{1, \dots, p-1\}$ the first matrix is always one of the representatives r_x with $x \in \{1, \dots, p-1\}$, and the second matrix is in $\Gamma_0(\pi)$ because its determinant is one and $x \cdot inv(x) - 1$ is always divisible by p, because $x \cdot inv(x) \equiv 1 \bmod p$. Note here that $p = \pi\overline{\pi}$, so that the lower left entry in the second matrix is indeed divisible by π. Then we have

$$\begin{aligned} &\delta_\pi \begin{pmatrix} x & 1 \\ -px + x \cdot inv(x) - 1 & inv(x) - p \end{pmatrix} \delta_\pi^{-1} \\ = &\begin{pmatrix} x & \pi \\ \pi^{-1}(-px + x \cdot inv(x) - 1) & inv(x) - p \end{pmatrix}, \end{aligned}$$

and the formula for $T_\pi((B-1) \otimes_\Gamma \mathrm{m})$ looks as follows:

$$\begin{aligned} &T_\pi((B-1) \otimes_\Gamma \mathrm{m}) \\ &= \sum_{x=1}^{p-1} \left(\begin{pmatrix} x & \pi \\ \pi^{-1}(-px + x \cdot inv(x) - 1) & inv(x) - p \end{pmatrix} - 1 \right) \otimes_\Gamma \begin{pmatrix} \pi & 0 \\ -x & 1 \end{pmatrix} \cdot \mathrm{m}. \end{aligned}$$

So we see that the relevant matrix is a quite arbitrary matrix in $PSL_2(\mathbb{Z}[i])$. Hence we first have to represent that matrix as a word in A, B, U and its inverses, which can be done by applying the Euclidean algorithm, which we have in $\mathbb{Z}[i]$. After that we have to reduce this general word minus 1 by our decomposition process from Section 4.2. That procedure will finally give us a representation for the terms left to $\otimes_\Gamma$ of the form $(A-1)\mathrm{Pol}_{A(x)} + (B-1)\mathrm{Pol}_{B(x)} + (U-1)\mathrm{Pol}_{U(x)}$, where $\mathrm{Pol}_{A(x)}$, $\mathrm{Pol}_{B(x)}$ and $\mathrm{Pol}_{U(x)}$ are quite general matrix polynomials in A, B, U and its inverses. This leads to

Proposition 5.9 *Let T_π, π, Γ and* m *be as in Proposition* 5.7 *and let A, B and U be the generators of* Γ. *Then*

$$\begin{aligned} T_\pi((B-1) \otimes_\Gamma \mathrm{m}) \quad = \quad & (A-1) \otimes_\Gamma \left[\sum_{x=1}^{p-1} \mathrm{Pol}_{A(x)} \begin{pmatrix} \pi & 0 \\ -x & 1 \end{pmatrix} \right] \cdot \mathrm{m} \\ & + (B-1) \otimes_\Gamma \left[\sum_{x=1}^{p-1} \mathrm{Pol}_{B(x)} \begin{pmatrix} \pi & 0 \\ -x & 1 \end{pmatrix} \right] \cdot \mathrm{m} \\ & + (U-1) \otimes_\Gamma \left[\sum_{x=1}^{p-1} \mathrm{Pol}_{U(x)} \begin{pmatrix} \pi & 0 \\ -x & 1 \end{pmatrix} \right] \cdot \mathrm{m}, \end{aligned}$$

where $\mathrm{Pol}_{A(x)}$, $\mathrm{Pol}_{B(x)}$ *and* $\mathrm{Pol}_{U(x)}$ *are polynomials in* A, B *and* U *and its inverses as described above.*

PROOF: After the representation of the matrix $\left(\begin{smallmatrix} x & \pi \\ \pi^{-1}(-px+x\cdot inv(x)-1) & inv(x)-p \end{smallmatrix}\right)$ as a word in A, B and U and its inverses and the application of the decomposition procedure we find:

$$\begin{aligned} &T_\pi((B-1)\otimes_\Gamma \mathrm{m}) \\ &= \sum_{x=1}^{p-1}[(A-1)\mathrm{Pol}_{A(x)} + (B-1)\mathrm{Pol}_{B(x)} + (U-1)\mathrm{Pol}_{U(x)}] \otimes_\Gamma \begin{pmatrix} \pi & 0 \\ -x & 1 \end{pmatrix} \cdot \mathrm{m}. \end{aligned}$$

Using the linearity of the tensor product and property (4.1) we then get the formula we wanted to show. □

Remark 5.10 For the generator B the most general case appeared, that is, to represent an arbitrary matrix in $PSL_2(\mathbb{Z}[i])$ as a word in A, B, U and its inverses and to apply the decomposition procedure in its general form.
Of course, we could have written down the formulas for A and U in that general version as well. But we have choosen the more explicit version to really make clear what happens in each case. On the other hand, it seems to be useful for the realization in the computer program to actually avoid the most general algorithms if one already knows the expressions to finally save computation time.
Furthermore the explicit formulas for A and U could be used with success in several proofs we give in Section 5.5.

Remark 5.11 One can treat the other cases of $\pi = p$ with $N(\pi) = p^2$ in a similar way. We would have to start with a different and larger system of representatives r_i and would have to adapt the matrices $h(g,i)$ for $g \in \{A, B, U\}$ then.

Remark 5.12 One can derive such explicit formulas in a similar way for other groups $\Gamma = PSL_2(\mathcal{O}_K)$ or congruence subgroups of them as long as $\mathcal{O}_K$ is Euclidean. One starts with the sytem of representatives for $\Gamma/\Gamma_0(\pi)$ and determines the matrices $h(g,i)$ for all generators of the group. Then one ends up with similar formulas for $T_\pi((g-1)\otimes_\Gamma \mathrm{m})$ for each generator g. The critical point arises when a situation as for our generator B appears. Then one has to represent any matrix as a word in the generators and its inverses, and for that we needed the Euclidean algorithm. So we would need another method for the other cases of class number 1, and for higher class number one needs another construction for the Hecke operators anyway as already mentioned in Remark 5.6. For some of the algorithmic aspects over general $\mathcal{O}_K$ one could use the approach for finitely generated modules over Dedekind domains already worked out in [Co2] and [Co3].

5.3 Computation of the Hecke eigenvalues

Our goal is the computation of simultaneous eigenvectors for the Hecke operators T_π on Λ_Γ for the arithmetic group $PSL_2(\mathbb{Z}[i])$. As a first step we carried out some extensive computer calculations giving results for many T_π. The programs we developed are based on the formulas we derived in Section 5.2. Inspired by the computational results and several insights we got during the computer realizations we were also led to several observations which could be proved using our formulas from Section 5.2. This will be summarized in Section 5.5. The aim of this section is the description of the steps to carry out the computer calculations. The modules and also the maps T_π are represented by certain matrices and again we want to stress the more algorithmic aspect by using capital letters and often the appendix MAT for these data structures.

Note first that we only have a representation by the matrix NMAT for the submodule N in $\Lambda_\Gamma = \mathbb{Z}[i]^{3k}/N$. Here the columns of NMAT generate the module N. Using the Smith algorithm over $\mathbb{Z}[i]$ we find the matrix $\mathrm{N_{SMITH}}$ which encodes the module invariants for Λ_Γ. The elementary divisors give the torsion in Λ_Γ and the zeros on the diagonal represent the free part.

We constructed the operators T_π on Λ_Γ but for the computation we also need a representation in a suitable matrix form. For that we have to realize T_π as a map on $\mathbb{Z}[i]^{3k}$ always having in mind that we have to consider that map modulo N in the end.

Therefore our first step is to build up the $(3k \times 3k)$-matrix for T_π which we want to call TMAT_π. This matrix has to consist of three $(3k \times k)$-blocks and has to be adapted to the shape we already chose for the matrix NMAT (see Section 4.3). So the first block belongs to $T_\pi((A-1)\otimes_\Gamma \mathrm{m})$, the second block represents $T_\pi((B-1)\otimes_\Gamma \mathrm{m})$ and the last one $T_\pi((U-1)\otimes_\Gamma \mathrm{m})$. For m we have to substitute the k generators e_i of the modules $M_{n,m}(\mathbb{Z}[i])$. To fill up the blocks we now have to use the formulas we derived in Proposition 5.7–Proposition 5.9 just expressed in the form we need. Each block consits of an $(A-1)$-part, a $(B-1)$-part and a $(U-1)$-part in an analogous form as for the matrix NMAT. To determine the entries one has to evaluate the action of the involved matrix polynomials on all the generators e_i. Of course, that can be realized in a nice way in a computer program. As one can imagine these matrices become really huge for modules $M_{n,m}(\mathbb{Z}[i])$ with larger m and n.

To give an impression of how the final matrix might look like we present as an example the matrix TMAT_π for $\pi = -1 + 2i$ in the case of the module $M_{2,0}(\mathbb{Z}[i])$ here (for the corresponding NMAT cp. Section 4.3):

```
> evalm(tpi);
```

$$\begin{bmatrix} 2+4I & 0 & 0 & 6+8I & -2+4I & -6-8I & -8-8I & 0 & 0 \\ 4-8I & 2-4I & 0 & -28+24I & 6-12I & 28-16I & 24-12I & 0 & 0 \\ 2+4I & 2+4I & 2+4I & -4-22I & 14+4I & 26I & 8+4I & -12+2I & 8+8I \\ 1 & 0 & -3-4I & -6-16I & -4-3I & 5-4I & 1 & 0 & -3-4I \\ -2+4I & 0 & 0 & 4-8I & -4+16I & 12-4I & -4-2I & 0 & 0 \\ -2-4I & -3-4I & -3-4I & 6 & -4-3I & -7-20I & 4+4I & 4-3I & -3-4I \\ 8+8I & 0 & 0 & 4+12I & 4-2I & -16-28I & 2+4I & 0 & 0 \\ 8+16I & 0 & 0 & -4-12I & 6-20I & -20+24I & 8+16I & 2-4I & 0 \\ 4I & -4I & -8-8I & -2I & 4+10I & 4+18I & 6I & -4I & 2+4I \end{bmatrix}$$

Now, the actual goal is to compute the action of T_π on certain parts of Λ_Γ. So we want to consider T_π either on certain ℓ-torsion parts of Λ_Γ or one the free part. More precisely, we have to study the induced maps

$$T_\pi^{(\ell)} \quad : \quad \Lambda_\Gamma^{\mathrm{tors}} \otimes_{\mathbb{Z}} \mathbb{F}_\ell \longrightarrow \Lambda_\Gamma^{\mathrm{tors}} \otimes_{\mathbb{Z}} \mathbb{F}_\ell,$$

where $\Lambda_\Gamma^{\mathrm{tors}}$ denotes the torsion part of Λ_Γ, respectively

$$T_\pi^{(\mathrm{free})} \quad : \quad \Lambda_\Gamma / \Lambda_\Gamma^{\mathrm{tors}} \longrightarrow \Lambda_\Gamma / \Lambda_\Gamma^{\mathrm{tors}}.$$

So we have to extract a nice matrix representation for the part of Λ_Γ we want to consider and for the maps induced by T_π. As already described, the matrix $\mathrm{N}_{\mathrm{SMITH}}$ encodes the free part and the torsion parts in a suitable way. Moreover we get the transformation matrices UT and VT with $\mathrm{N}_{\mathrm{SMITH}} = \mathrm{UT} \cdot \mathrm{NMAT} \cdot \mathrm{VT}$ from the application of the Smith algorithm (cp. Section 2.2). To treat the ℓ-torsion (ℓ a rational prime) we consider the diagonal elements d which are not prime to ℓ in $\mathbb{Z}[i]$ and set $d_\ell = \gcd_{\mathbb{Z}[i]}(d, \ell)$. Now we analyse $\mathbb{Z}[i]/d_\ell\mathbb{Z}[i]$ and determine the generators over $\mathbb{Z}[i]$ (one or two). For that we have to distinguish between different cases for the general entries. For example we get the single generator 1 for the 2-torsion in the case of the entry $1+i$, the generators 1 and i for the 5-torsion in case of the entry 5 or the generators 2 and $1+i$ for the 3-torsion in case of a mixed entry $3+3i$. We carry out this procedure for all diagonal elements contributing to the chosen ℓ-torsion. Analogously we treat the zeros representing the free part. Here we have 1 and i as generators in each case. At the end we replace each column of the matrix $\mathrm{N}_{\mathrm{SMITH}}$ contributing to the part we consider by one or two columns obtained by replacing the diagonal elements by the generators we determined. All these relevant columns are collected in a new matrix LMAT. For example, if we just find one entry d which is divisible by 5 we end up with a matrix LMAT for the 5-torsion consisting of two columns, the first one given through the replacement of d by the generator 1, the second one given through the replacement of d by the generator i.

Having described the desired part by the matrix LMAT we can now apply the map induced by T_π. Note that LMAT is a representation in the Smith basis whereas the induced map for T_π was realized by the matrix TMAT_π in the monomial basis up to now (refering to the e_i). Therefore we first have to carry out a base change into

the Smith basis for the matrix TMAT_π. This can be done by forming the matrix $\mathrm{UT}\cdot\mathrm{TMAT}_\pi\cdot\mathrm{UT}^{-1}$ since the matrix UT comes out during the determination of the Smith normal form of the matrix NMAT. The induced action of T_π on LMAT is then described by $\mathrm{LMAT}_\pi = \mathrm{UT}\cdot\mathrm{TMAT}_\pi\cdot\mathrm{UT}^{-1}\cdot\mathrm{LMAT}$.
What we finally want is to extract a special matrix $\mathrm{TMAT}_\pi^{(*)}$ representing the action of T_π just on the part we have chosen, i. e. in the basis given by the matrix LMAT. For that we would have to solve the linear system $\mathrm{LMAT}\cdot\mathrm{TMAT}_\pi^{(*)} = \mathrm{LMAT}_\pi$, where the star stands for an ℓ-torsion part or the free part. In general this system has no solution. This happens because of the interpretation of T_π by maps on $\mathbb{Z}[i]^{3k}$ instead of $\Lambda_\Gamma = \mathbb{Z}[i]^{3k}/N$. Therefore we have to change the matrix LMAT_π into a matrix LMAT_π' such the system can be solved (producing rows of zeros in LMAT_π' where we have rows of zeros in LMAT). This is possible because T_π maps the torsion part into the torsion part and the free part into the free part. Note that the columns of the matrix LMAT_π are vectors in $\mathbb{Z}[i]^{3k}$ but represent vectors in Λ_Γ and so we can always change the columns of LMAT_π modulo $\mathbb{Z}[i]$-linear combinations of the matrix $\mathrm{N}_{\mathrm{SMITH}}$ if we are in the torsion case and of a modified matrix $\mathrm{N}_{\mathrm{SMITH}}'$ if we are in the free case. We get the matrix $\mathrm{N}_{\mathrm{SMITH}}'$ from the matrix $\mathrm{N}_{\mathrm{SMITH}}$ by replacing all entries different from zero by one. So we find the matrix LMAT_π' by considering the entries in the rows of the matrix LMAT_π modulo $\mathbb{Z}[i]$-linear combinations of the entries in the corresponding rows of the matrices $\mathrm{N}_{\mathrm{SMITH}}$ or $\mathrm{N}_{\mathrm{SMITH}}'$. Of course we only get results up to conjugation for the matrices $\mathrm{TMAT}_\pi^{(*)}$ since we could have taken a different LMAT-basis at the beginning. For the free part we then solve the linear system over $\mathbb{Z}$ and for the ℓ-torsion parts we solve it over $\mathbb{F}_\ell$. So we get matrices representing the maps $T_\pi^{(\ell)}$ or $T_\pi^{(\mathrm{free})}$, and we can now look for simultaneous eigenvectors for many π.
We realized this whole process in a computer program and could perform the computation for many interesting torsion classes in the end. We also found interesting eigenvectors in the free part which are somehow related to the ones in the torsion parts, what is shown by certain congruences. But all that will be discussed in Section 5.4. The computer realization contains many complicated and tricky parts. At first many problems had to be solved for the construction of the matrix TMAT_π. For that we refer to the formulas in Section 5.2. For all three generators we need an efficient procedure which performs the decomposition procedure for a word in A, B, U and its inverses to bring it into the right shape of (4.3). For the generator B there is also the problem to represent a general matrix as a word in A, B, U and its inverses first, which can be solved by the use of the generalized Euclidean algorithm over $\mathbb{Z}[i]$. Furthermore many small tasks had to be solved as e. g. the determination of c and d in the part of U or the realization of $inv(x)$ for B and so on. In general it is not critical to build up the matrix for T_π but, of course, the computation time increases a lot when the norm of π gets large. Then we have to deal with a lot more summands in the formulas. The critical point in the computation is the determina-

tion of the Smith normal form including the determination of UT. The matrix UT gets huge entries and we even have to invert that matrix later. With the help of the LLL-algorithm we could only compute up to $n = 18$ in the series $M_{n,0}(\mathbb{Z}[i])$. Note that the Smith algorithm in MAGMA produces better matrices UT but works only over $\mathbb{Z}$ up to now.
But it is also possible to perform the whole computation modulo ℓ (Smith form mod ℓ, solving the linear system mod ℓ and so on) getting a result mod ℓ. Note that we have to consider all matrices over $\mathbb{F}_\ell$ then. So we have to fix a map from $\mathbb{Z}[i]$ to $\mathbb{F}_\ell$ by sending i to one fixed root of -1 in $\mathbb{F}_\ell$ (in the nonsplit cases we have to switch to $\overline{\mathbb{F}}_\ell$ and call the roots α and $-\alpha$ then). If we want to consider a certain ℓ-torsion we get zeros in the Smith normal form mod ℓ on the diagonal, and therefore we end up with a mixture of the ℓ-torsion part and the free part in the matrix LMAT and have to care about this during the analysis of the results. Nevertheless we then managed to compute the Hecke action on several large torsion classes as e. g. in the case of $\ell = 661$ for $M_{40,0}(\mathbb{Z}[i])$.

5.4 Analysis of the results and several congruences

In this section we present a choice of our computational results for the Hecke eigenvalues for several modules $M = M_{n,m}(\mathbb{Z}[i])$. Most of all we were interested in the induced action of the Hecke operators T_π on certain ℓ-torsion parts of Λ_Γ respectively $H_1(\Gamma, M)$ for $\Gamma = PSL_2(\mathbb{Z}[i])$. For that we studied the maps $T_\pi^{(\ell)}$ (cp. Section 5.3). Then we were looking for simultaneous eigenvectors for many $T_\pi^{(\ell)}$ and collected the corresponding sequences of eigenvalues. But of course we were also interested in simultaneous eigenvectors in the free part, which meant to study the maps $T_\pi^{(\text{free})}$. All these computations could be done within the same setup as described in Section 5.3. Since the picture for the free part can be explained quite well we only present one table as an example and discuss the pattern we see. All the other tables contain eigenvalues on certain ℓ-torsion parts.
We start with some more detailed comments on the specific data collected in the tables. Then we list a choice of tables and close with the analysis of our results. All tables contain a list of π's we chose for the computation. These π always satisfy the condition $N(\pi) = p$ (p a rational prime), since we developed the formulas for T_π in Section 5.2 for these cases only. We denote the eigenvalues of T_π on the free part of Λ_Γ for the modules $M_{n,m}(\mathbb{Z}[i])$ by $\lambda_{\pi,m,n}$. The eigenvalues on the ℓ-torsion parts are always denoted by a_π. Here we have several cases we computed with the Smith normal form over $\mathbb{Z}[i]$ and some computed mod ℓ. In the first situation we mention the eigenvectors we found for the matrices $\text{TMAT}_\pi^{(\ell)}$ and in some cases we also include these matrices. Otherwise we only mention the choice of the root of -1 in $\mathbb{F}_\ell$ for i which had to be fixed for the computation. Finally several tables

contain some more data which are relevant for the congruences we find and we want to discuss later. If we list π's or expressions in π for a choice of i the result is always meant modulo ℓ. For example, in the first table this happens in the second and in the third column and for the expression $\pi^{11}+\overline{\pi}$.

The module $M_{10,0}(\mathbb{Z}[i])$:

π	π for $i=2$	π for $i=3$	$N(\pi)+1$ mod 5	a_π for $\binom{2}{1}$ $(i=2)$	a_π for $\binom{3}{1}$ $(i=3)$	$\pi^{11}+\overline{\pi}$ $(i=2)$	$\pi^{11}+\overline{\pi}$ $(i=3)$
$1+i$	3	4	3	4	2	1	2
$1-i$	4	3	3	2	4	2	1
$-1+2i$	3	0	1	3	0	2	3
$-1-2i$	0	3	1	0	3	3	2
$3+2i$	2	4	4	3	1	2	1
$3-2i$	4	2	4	1	3	1	2
$1+4i$	4	3	3	2	4	2	1
$1-4i$	3	4	3	4	2	1	2
$-5+2i$	4	1	0	0	0	0	0
$-5-2i$	1	4	0	0	0	0	0
$-1+6i$	1	2	3	3	1	3	4
$-1-6i$	2	1	3	1	3	4	3
$5+4i$	3	2	2	1	4	4	1
$5-4i$	2	3	2	4	1	1	4
$7+2i$	1	3	4	4	2	4	3
$7-2i$	3	1	4	2	4	3	4
$-5+6i$	2	3	2	4	1	1	4
$-5-6i$	3	2	2	1	4	4	1
$-3+8i$	3	1	4	2	4	3	4
$-3-8i$	1	3	4	4	2	4	3
$5+8i$	1	4	0	0	0	0	0
$5-8i$	4	1	0	0	0	0	0
$9+4i$	2	1	3	1	3	4	3
$9-4i$	1	2	3	3	1	3	4
$-1+10i$	4	4	2	3	3	3	3
$-1-10i$	4	4	2	3	3	3	3
$3+10i$	3	3	0	0	0	0	0
$3-10i$	3	3	0	0	0	0	0

Table 32: Hecke eigenvalues a_π on the 5-torsion for $M_{10,0}(\mathbb{Z}[i])$

On the free part for $M_{10,0}(\mathbb{Z}[i])$ we found the following eigenvalues $\lambda_{\pi,10,0}$:

π	$\lambda_{\pi,10,0}$
$1+i$	$-31+31i$
$1-i$	$-31-31i$
$-1-2i$	$-6470+2644i$
$-1+2i$	$-6470-2644i$
$-2-3i$	$246044+1315914i$
$1+2i$	$6470-2644i$
$2+i$	$2644-6470i$
$2-i$	$2644+6470i$
$1-2i$	$6470+2644i$
$2+3i$	$-246044-1315914i$
$3+2i$	$1315914+246044i$
$3-2i$	$1315914-246044i$
$2-3i$	$-246044+1315914i$
$1+4i$	$-2529646+5279464i$
$4+i$	$-5279464+2529646i$
$4-i$	$-5279464-2529646i$
$1-4i$	$-2529646-5279464i$

Table 33: Hecke eigenvalues $\lambda_{\pi,10,0}$ on the free part for $M_{10,0}(\mathbb{Z}[i])$

The module $M_{16,0}(\mathbb{Z}[i])$:

π	$T_\pi^{(5)}$	π for $i=3$	π for $i=2$	$N(\pi)+1$ mod 5	a_π for $\binom{1}{1}$ $(i=3)$	a_π for $\binom{3}{1}$ $(i=2)$
$1+i$	$\begin{pmatrix}4&4\\2&1\end{pmatrix}$	4	3	3	3	2
$1-i$	$\begin{pmatrix}1&1\\3&4\end{pmatrix}$	3	4	3	2	3
$-1+2i$	$\begin{pmatrix}1&4\\2&3\end{pmatrix}$	0	3	1	0	4
$-1-2i$	$\begin{pmatrix}3&1\\3&1\end{pmatrix}$	3	0	1	4	0
$3+2i$	$\begin{pmatrix}2&2\\1&3\end{pmatrix}$	4	2	4	4	1
$3-2i$	$\begin{pmatrix}3&3\\4&2\end{pmatrix}$	2	4	4	1	4
$1+4i$	$\begin{pmatrix}1&1\\3&4\end{pmatrix}$	3	4	3	2	3
$1-4i$	$\begin{pmatrix}4&4\\2&1\end{pmatrix}$	4	3	3	3	2

Table 34: Hecke eigenvalues a_π on the 5-torsion for $M_{16,0}(\mathbb{Z}[i])$

The module $M_{7,1}(\mathbb{Z}[\,i\,])$:

π	$T_\pi^{(3)}$	a_π for $\binom{1+\alpha}{1}$ $(i=\alpha)$	a_π for $\binom{3+2\alpha}{1}$ $(i=-\alpha)$
$-1+i$	$\left(\begin{smallmatrix}0&2\\2&2\end{smallmatrix}\right)$	$1+2\alpha$	$1+\alpha$
$1+i$	$\left(\begin{smallmatrix}1&2\\2&0\end{smallmatrix}\right)$	$2+2\alpha$	$2+\alpha$
$1-i$	$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$2+\alpha$	$2+2\alpha$
$-1+2i$	$\left(\begin{smallmatrix}2&1\\1&0\end{smallmatrix}\right)$	$1+\alpha$	$1+2\alpha$
$-1-2i$	$\left(\begin{smallmatrix}0&2\\2&2\end{smallmatrix}\right)$	$1+2\alpha$	$1+\alpha$
$3+2i$	$\left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right)$	α	2α
$3-2i$	$\left(\begin{smallmatrix}2&2\\2&1\end{smallmatrix}\right)$	2α	α
$1+4i$	$\left(\begin{smallmatrix}1&2\\2&0\end{smallmatrix}\right)$	$2+2\alpha$	$2+\alpha$
$1-4i$	$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$2+\alpha$	$2+2\alpha$
$-5+2i$	$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$2+\alpha$	$2+2\alpha$
$-5-2i$	$\left(\begin{smallmatrix}1&2\\2&0\end{smallmatrix}\right)$	$2+2\alpha$	$2+\alpha$
$-1+6i$	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	1	1
$-1-6i$	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	1	1
$5+4i$	$\left(\begin{smallmatrix}0&2\\2&2\end{smallmatrix}\right)$	$1+2\alpha$	$1+\alpha$
$5-4i$	$\left(\begin{smallmatrix}2&1\\1&0\end{smallmatrix}\right)$	$1+\alpha$	$1+2\alpha$
$7+2i$	$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$2+\alpha$	$2+2\alpha$
$7-2i$	$\left(\begin{smallmatrix}1&2\\2&0\end{smallmatrix}\right)$	$2+2\alpha$	$2+\alpha$
$-5+6i$	$\left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)$	2	2
$-5-6i$	$\left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)$	2	2
$-3+8i$	$\left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right)$	α	2α
$-3-8i$	$\left(\begin{smallmatrix}2&2\\2&1\end{smallmatrix}\right)$	2α	α

Table 35: Hecke eigenvalues a_π on the 3-torsion for $M_{7,1}(\mathbb{Z}[\,i\,])$

The module $M_{7,1}(\mathbb{Z}[\,i\,])$:

π	$T_\pi^{(5)}$	π for $i=3$	π for $i=2$	$N(\pi)+1$ mod 5	a_π for $\binom{1}{1}$ $(i=3)$	a_π for $\binom{3}{1}$ $(i=2)$
$-1+i$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	2	1	3	0	0
$1+i$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	4	3	3	0	0
$1-i$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	3	4	3	0	0
$-1+2i$	$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$	0	3	1	0	3
$-1-2i$	$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$	3	0	1	3	0
$3+2i$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	4	2	4	0	0
$3-2i$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	2	4	4	0	0
$1+4i$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	3	4	3	0	0
$1-4i$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	4	3	3	0	0
$-5+2i$	$\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$	1	4	0	2	3
$-5-2i$	$\begin{pmatrix} 4 & 4 \\ 2 & 1 \end{pmatrix}$	4	1	0	3	2
$-1+6i$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	2	1	3	0	0
$-1-6i$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	1	2	3	0	0
$5+4i$	$\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$	2	3	2	4	1
$5-4i$	$\begin{pmatrix} 3 & 3 \\ 4 & 2 \end{pmatrix}$	3	2	2	1	4
$7+2i$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	3	1	4	0	0
$7-2i$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	1	3	4	0	0
$-5+6i$	$\begin{pmatrix} 3 & 3 \\ 4 & 2 \end{pmatrix}$	3	2	2	1	4
$-5-6i$	$\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$	2	3	2	4	1
$-3+8i$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	1	3	4	0	0
$-3-8i$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	3	1	4	0	0

Table 36: Hecke eigenvalues a_π on the 5-torsion for $M_{7,1}(\mathbb{Z}[\,i\,])$

The module $M_{18,0}(\mathbb{Z}[i])$:

π	a_π for $i = 5$ (tors)	a_π for $i = 5$ (free)	a_π for $i = 8$ (tors)	a_π for $i = 8$ (free)
$-1+i$	11	11	3	10
$1+i$	10	3	2	2
$1-i$	2	2	10	3
$-1+2i$	11	11	6	7
$-1-2i$	6	7	11	11
$3+2i$	0	6	6	7
$3-2i$	6	7	0	6
$1+4i$	1	12	12	1
$1-4i$	12	1	1	12
$-5+2i$	7	6	6	7
$-5-2i$	6	7	7	6
$-1+6i$	11	11	5	8
$-1-6i$	5	8	11	11
$5+4i$	10	10	12	1
$5-4i$	12	1	10	10
$7+2i$	1	1	1	1
$7-2i$	1	1	1	1
$-5+6i$	3	3	3	3
$-5-6i$	3	3	3	3
$-3+8i$	2	11	7	7
$-3-8i$	7	7	2	11
$5+8i$	2	11	10	10
$5-8i$	10	10	2	11
$9+4i$	5	5	12	1
$9-4i$	12	1	5	5
$-1+10i$	11	11	11	11
$-1-10i$	11	11	11	11

Table 37: Hecke eigenvalues a_π on the 13-torsion for $M_{18,0}(\mathbb{Z}[i])$

The modules $M_{40,0}(\mathbb{Z}[i])$ and $M_{44,0}(\mathbb{Z}[i])$:

π	a_π for $i = 106$	a_π for $i = 555$
$-1+i$	114	547
$1+i$	547	114
$1-i$	114	547
$-1+2i$	114	92
$-1-2i$	92	114
$3+2i$	84	29
$3-2i$	29	84
$1+4i$	504	596
$1-4i$	596	504
$-5+2i$	245	49
$-5-2i$	49	245
$7+2i$	178	247
$7-2i$	247	178

Table 38: Hecke eigenvalues a_π on the 661-torsion for $M_{40,0}(\mathbb{Z}[i])$

π	a_π for $i = 37$	a_π for $i = 100$
$-1+i$	38	38
$1+i$	38	38
$1-i$	38	38
$-1+2i$	62	109
$-1-2i$	109	62
$3+2i$	136	14
$3-2i$	14	136
$1+4i$	91	62
$1-4i$	62	91
$-5+2i$	79	9
$-5-2i$	9	79
$-1+6i$	52	64
$-1-6i$	64	52
$5+4i$	2	66
$5-4i$	66	2

Table 39: Hecke eigenvalues a_π on the 137-torsion for $M_{44,0}(\mathbb{Z}[i])$

Let us now analyse our computational results. What we try to do is to find patterns or relations which hold in all of our computed cases up to now.
We first discuss the situation for the free parts. Here we carried out the computations for many modules $M_{n,m}(\mathbb{Z}[i])$ and always found an eigenvector with the eigenvalue

$$\lambda_{\pi,n,m} = \pi^{n+1} + \overline{\pi}^{m+1} \quad \text{for} \quad n+m+2 \equiv 0 \mod 4.$$

In the other cases of n, m we didn't find eigenvectors in the free part. Note that we checked the cases of $n = m$ only up to $n = 5$ but for certain $n \neq m$ we treated longer series. In the case $M_{5,5}(\mathbb{Z}[i])$ we found a second eigenvector satisfying $a_\pi = a_{\overline{\pi}}$ for all π of small norm we considered. A choice of the eigenvalues $\lambda_{\pi,n,m}$ for $M_{10,0}(\mathbb{Z}[i])$ is given in Table 33. Here we can check that $\lambda_{\pi,10,0} = \pi^{11} + \overline{\pi}$ holds. In the case of $M_{5,1}(\mathbb{Z}[i])$ the corresponding list starts with $\lambda_{1+i,5,1} = -10i$, $\lambda_{-1-2i,5,1} = 114 + 40i$ and $\lambda_{-2-3i,5,1} = 2030 - 840i$ leading to $\pi^6 + \overline{\pi}^2$. Note that the eigenvalues $\pi^{n+1} + \overline{\pi}^{n+1}$ in the relevant cases of $M_{n,n}(\mathbb{Z}[i])$ are always rational integers.
In all cases we also found that the eigenvector with the eigenvalue $\pi^{n+1} + \overline{\pi}^{m+1}$ satisfies the homology condition (see Proposition 4.4). So it is not only a class in Λ_Γ but indeed in $H_1(\Gamma, M)$. In all computations we also see that it is of infinite order (not torsion). Finally we tried to describe a general representative for the eigenvector on the free part in terms of the generators of Λ_Γ, which is in our so-called monomial basis. From our computational output we found e. g. the vector

$$(A-1) \otimes_\Gamma (-xy) + (U-1) \otimes_\Gamma (-ixy)$$

for $M_{2,0}(\mathbb{Z}[i])$ and the vector

$$\begin{aligned} &(A-1) \otimes_\Gamma ((1-i)x^3y^3 + (1+i)x^2y^4 + iy^5) + (B-1) \otimes_\Gamma ((2+2i)x^3y^3) \\ &+ (U-1) \otimes_\Gamma ((1-i)x^4y^2 + (-1-i)x^3y^3 + (1+2i)x^2y^4 + (2+i)xy^5) \end{aligned}$$

for the module $M_{6,0}(\mathbb{Z}[i])$ and so on. Now one can change these vectors modulo torsion vectors to get a more transparent form but one can already see that it is difficult to extract a pattern. Using a homology test program we could finally establish a general candidate for all modules $M_{n,m}(\mathbb{Z}[i])$. Then we also proved several properties so that we would like to refer for all of that to Section 5.5.
We got already used to distinguish between so-called small and large ℓ-torsion classes. That seems to continue for the Hecke eigenvalues. Here the central observation is that some kind of congruences seems to hold for the small torsions but not for the large ones. Let us start with a closer look at the small torsions. Table 32 collects the eigenvalues a_π on the 5-torsion for the module $M_{10,0}(\mathbb{Z}[i])$. Here we find

$$a_\pi \equiv \pi \cdot (N(\pi)+1) \bmod 5 \quad \text{also} \quad \text{giving} \quad a_\pi \equiv \pi^2 \cdot (\pi^{11} + \overline{\pi}) \bmod 5,$$

which is a congruence to a twist of the eigenvalue for the free part in that case. We obtain similar results in other cases. So we find $a_\pi \equiv \pi^2 \cdot (N(\pi)+1) \bmod 5$ on the

5-torsion for $M_{16,0}(\mathbb{Z}[i])$, $a_\pi \equiv \pi \cdot (N(\pi)^2 + 1) \bmod 5$ on the 5-torsion for the module $M_{7,1}(\mathbb{Z}[i])$ as well as

$$a_\pi \equiv \pi \cdot (\pi^4 N(\pi)^2 + 1) \equiv \pi^6 \cdot (\pi^{19} + \overline{\pi}) \bmod 13$$

on the 13-torsion for $M_{18,0}(\mathbb{Z}[i])$. Note that we used the Smith normal form modulo ℓ for $M_{18,0}(\mathbb{Z}[i])$ and somehow guessed what the free part should be. Furthermore we got $a_\pi \equiv 2\pi \cdot N(\pi)^2 \bmod 3$ on the 3-torsion in the case of $M_{7,1}(\mathbb{Z}[i])$ and $a_\pi \equiv \pi \cdot (N(\pi) + 1) \bmod 3$ on the 3-torsion for the module $M_{10,0}(\mathbb{Z}[i])$ (we didn't show this table here). Note that $a_\pi \in \mathbb{F}_3[\sqrt{-1}]$ and that we have fixed the image of i in $\mathbb{F}_3[\sqrt{-1}]$ here, which is the root α or $-\alpha$ (cp. Table 35). For the expressions $2\pi N(\pi)^2$ we consider the image in $\mathbb{F}_3[\sqrt{-1}]$, and the congruence mod 3 should be understood in that sense.
In all cases of $M_{n,m}(\mathbb{Z}[i])$ with a free eigenvector we also find a congruence mod ℓ of the a_π to a twist of $\pi^{n+1} + \overline{\pi}^{m+1}$. Furthermore we don't find a congruence to just the number $\pi^{n+1} + \overline{\pi}^{m+1}$ in the other cases. But we find congruences to twists of the eigenvalues of the free parts for certain shifted homology modules (other coefficients). For example we get $a_\pi \equiv \pi^3 \cdot (\pi^{15} + \overline{\pi}) \bmod 5$ on the 5-torsion for $M_{16,0}(\mathbb{Z}[i])$ and $a_\pi \equiv \pi^3 \cdot (\pi^6 + \overline{\pi}^2) \bmod 5$ on the 5-torsion for $M_{7,1}(\mathbb{Z}[i])$ refering to the existing free eigenvectors on the free parts of $M_{14,0}(\mathbb{Z}[i])$ and $M_{5,1}(\mathbb{Z}[i])$. Via Fermat's Little Theorem we actually get congruences to such free parts in infinitely many other homology modules, which happens for the other congruences above as well.
Another observation concerns the 2-torsion. In all cases we considered we found that T_π acts nilpotently on the 2-torsion (actually we got $T_\pi \equiv 0$ for $N(\pi) > 2$).
Last but not least we extracted a surprising relation of the 5-torsion for the module $M_{10,0}(\mathbb{Z}[i])$ to the traces b_π of Frobenii at π for certain elliptic curves over $\mathbb{Q}(i)$. More precisely, we could identify the numbers $a_\pi \cdot \pi^{-5}$ with the numbers b_π mod 5 of the elliptic curves #1, #8 and #39 over $\mathbb{Q}(i)$ in Table 5.1.5 on pp. 96/97 of J. Cremona's thesis [Cr1].
Finally we analysed the 661-torsion for the module $M_{40,0}(\mathbb{Z}[i])$ and the 137-torsion for $M_{44,0}(\mathbb{Z}[i])$. In both cases there was no eigenvector on the free part. We checked quite general possibilities for congruences of a_π to polynomials in π and $\overline{\pi}$ mod ℓ but couldn't find any. In particular we checked completely the case with two monomials since that was the shape of the congruences for the small torsions. This meant to check

$$a_\pi \equiv (\pi^{\alpha}\overline{\pi}^{\beta} + \pi^{\gamma}\overline{\pi}^{\delta}) \bmod \ell \quad \text{for} \quad 0 \le \alpha, \beta < \ell \quad \text{and} \quad \gamma, \delta \in \{0, 1\}$$

but nothing was satisfied. Of course, the question remains whether the phenomena for the small and large torsions are of general nature or whether the picture is more subtle.
The choice of the computed cases and many questions we asked were motivated in view of links to arithmetic, in particular to possible Galois representations mod ℓ, which should be studied in more detail now (see also Chapter 6).

5.5 Classes in $H_1(\Gamma, M)$ and Hecke eigenvalues on the free part

First recall our assumptions and results from Section 5.2, in particular the Propositions 5.7 and 5.8. In several arguments we will also use the formulas (5.11) and (5.13). We considered the case $\Gamma = PSL_2(\mathbb{Z}[i])$, studied Hecke operators T_π on the spaces Λ_Γ and derived explicit formulas for the action of T_π on the generators $(A-1)\otimes_\Gamma \mathrm{m}$, $(B-1)\otimes_\Gamma \mathrm{m}$ and $(U-1)\otimes_\Gamma \mathrm{m}$ of Λ_Γ. In this section we rename the indices of the modules and assume $\mathrm{m} \in M = M_{r,s}(\mathbb{Z}[i])$ from now on.
Remind that we always found an eigenvector $w \in H_1(\Gamma, M)$ in the free part $\Lambda_\Gamma/\Lambda_\Gamma^{\mathrm{tors}}$ with $T_\pi(w) = (\pi^{r+1}+\overline{\pi}^{s+1})w$ for $r+s+2 \equiv 0 \bmod 4$. Then we tried to describe these automorphic classes in terms of Λ_Γ by a general expression what we called candidate for the eigenvector of the free part in the last section. The output of our programs led to expressions without any obvious pattern. But computer experiments using a homology test program in combination with the programs for the Hecke operators helped to enlight the structure of such eigenvectors. The main point was to actually find vectors satisfying the homology condition (see Proposition 4.4) since our vectors should be in $H_1(\Gamma, M)$. Doing this in a well-organized way one ends up with an eigenvector quite naturally. So it made sense to start with a presumably quite close candidate and to vary it until we would land in the first homology. Carrying out this we studied step by step the series $M_{r,0}(\mathbb{Z}[i])$, $M_{r,1}(\mathbb{Z}[i])$ and so on to extract candidates. Without a special search we could identify the vectors

$$w = (A-1)\otimes_\Gamma y^r \quad \text{and} \quad w = (U-1)\otimes_\Gamma y^r$$

in the case $M_{r,0}(\mathbb{Z}[i])$ satisfying $T_\pi(w) = (\pi^{r+1}+\overline{\pi})w$ in all computed cases. One immediately checks that the homology condition is satisfied since $(A-1)\cdot y^r = y^r - y^r = 0$ and $(U-1)\cdot y^r = 0$ (A and U send y to y).
Now $w = (A-1)\otimes_\Gamma (y^r \otimes v^s)$ seemed to be a good candidate for the general case $M_{r,s}(\mathbb{Z}[i])$. The homology condition is satisfied but as the following lemma shows w is a torsion vector for $s \geq 1$ and $r \geq 1$ (which was not zero in all computed cases). Note before that we have the relation $AU = UA$ in $\Gamma = PSL_2(\mathbb{Z}[i])$ (see Proposition 1.1) and therefore the equation $AU - A - U + 1 = UA - U - A + 1$. This gives the following relation we are going to use:

$$(A-1)\otimes_\Gamma (U-1)\mathrm{m} \;=\; (U-1)\otimes_\Gamma (A-1)\mathrm{m}. \tag{5.14}$$

Lemma 5.13 *Let $\Gamma = PSL_2(\mathbb{Z}[i])$ and let $w_1 = (A-1)\otimes_\Gamma (y^r \otimes v^s)$ and $w_2 = (U-1)\otimes_\Gamma (y^r \otimes v^s)$, which are classes in $H_1(\Gamma, M_{r,s}(\mathbb{Z}[i]))$. Then we have for $r, s \geq 1$*

$$2w_1 \;=\; 2w_2 \;=\; 0.$$

PROOF: We determine the action of A and U on a certain m $\in M_{r,s}(\mathbb{Z}[i])$ and obtain for $r \geq 1$:

$$\begin{aligned} A(xy^{r-1} \otimes v^s) &= (x+y)y^{r-1} \otimes v^s = xy^{r-1} \otimes v^s + y^r \otimes v^s, \\ U(xy^{r-1} \otimes v^s) &= (x+iy)y^{r-1} \otimes v^s = xy^{r-1} \otimes v^s + iy^r \otimes v^s. \end{aligned}$$

Then we form the actions of $(A-1)$ and $(U-1)$ on m, put them in (5.14) and get

$$i(A-1) \otimes_\Gamma (y^r \otimes v^s) = (U-1) \otimes_\Gamma (y^r \otimes v^s). \tag{5.15}$$

Then we also determine the action of A and U on another m $\in M_{r,s}(\mathbb{Z}[i])$ and get for $s \geq 1$:

$$\begin{aligned} A(y^r \otimes uv^{s-1}) &= y^r \otimes (u+v)v^{s-1} = y^r \otimes uv^{s-1} + y^r \otimes v^s, \\ U(y^r \otimes uv^{s-1}) &= y^r \otimes (u-iv)v^{s-1} = y^r \otimes uv^{s-1} - iy^r \otimes v^s. \end{aligned}$$

Analogously to above we find with (5.14)

$$-i(A-1) \otimes_\Gamma (y^r \otimes v^s) = (U-1) \otimes_\Gamma (y^r \otimes v^s). \tag{5.16}$$

Combining (5.15) and (5.16) we indeed get $2w_1 = 2w_2 = 0$. □

Our computer search finally led to the following suitable candidates which could be verified to be a free eigenvector for a couple of r. For $s = 1$ we got:

$$w = (A-1) \otimes_\Gamma (y^r \otimes iu) + (U-1) \otimes_\Gamma (y^r \otimes u).$$

Note that we could see in several examples that a vector with an $(A-1)$-part only wouldn't be possible. Next we extracted for $s = 2$:

$$w = (A-1) \otimes_\Gamma (iy^r \otimes u^2 + y^r \otimes uv) + (U-1) \otimes_\Gamma (y^r \otimes u^2 + y^r \otimes uv).$$

Then we found for $s = 3$:

$$\begin{aligned} w = {} & (A-1) \otimes_\Gamma (2iy^r \otimes u^3 + 3y^r \otimes u^2 v - 2iy^r \otimes uv^2) \\ & + (U-1) \otimes_\Gamma (2y^r \otimes u^3 + 3y^r \otimes u^2 v + 2y^r \otimes uv^2). \end{aligned}$$

For $s = 4$ we obtained:

$$\begin{aligned} w = {} & (A-1) \otimes_\Gamma (iy^r \otimes u^4 + 2y^r \otimes u^3 v - 2iy^r \otimes u^2 v^2 - y^r \otimes uv^3) \\ & + (U-1) \otimes_\Gamma (y^r \otimes u^4 + 2y^r \otimes u^3 v + 2y^r \otimes u^2 v^2 + y^r \otimes uv^3). \end{aligned}$$

That was enough to establish the following *general candidate* for $s \geq 1$:

$$\begin{aligned} w = {} & (A-1) \otimes_\Gamma \left(y^r \otimes -\sum_{k=1}^{s} (-i)^{s-k+1} \binom{s+1}{k} u^k v^{s-k} \right) \\ & + (U-1) \otimes_\Gamma \left(y^r \otimes \sum_{k=1}^{s} \binom{s+1}{k} u^k v^{s-k} \right). \end{aligned} \tag{5.17}$$

Note that we have to divide by gcd $\{\binom{s+1}{k} : k = 1, \ldots, s\}$ here to get the exact examples from above.
Now a first job is to check whether $w \in H_1(\Gamma, M)$, what we got in all computed cases.

Lemma 5.14 *Let $\Gamma = PSL_2(\mathbb{Z}[i])$, $M = M_{r,s}(\mathbb{Z}[i])$ and*

$$\begin{aligned} w &= (A-1) \otimes_\Gamma \left(y^r \otimes - \sum_{k=1}^{s} (-i)^{s-k+1} \binom{s+1}{k} u^k v^{s-k} \right) \\ &\quad + (U-1) \otimes_\Gamma \left(y^r \otimes \sum_{k=1}^{s} \binom{s+1}{k} u^k v^{s-k} \right). \end{aligned}$$

with $s \geq 1$. Then we have $w \in H_1(\Gamma, M)$.

PROOF: If we denote the element in $M_{r,s}(\mathbb{Z}[i])$ in the $(A-1)$-part by m and the one in the $(U-1)$-part by n then the homology condition says (see Proposition 4.4) that we have to verify $(A-1)\text{m} + (U-1)\text{n} = 0$. If we put the action of A and U into the expressions we find that we have to show

$$\begin{aligned} y^r \otimes T_1 &:= \left(y^r \otimes \sum_{k=1}^{s} \sum_{l=1}^{k} (-i)^{s-k+1} \binom{s+1}{k} \binom{k}{l} u^{k-l} v^{s-k+l} \right) \\ &= \left(y^r \otimes \sum_{k=1}^{s} \sum_{l=1}^{k} (-i)^{l} \binom{s+1}{k} \binom{k}{l} u^{k-l} v^{s-k+l} \right) =: y^r \otimes T_2, \end{aligned}$$

where T_1 is the contribution from the $(A-1)$-part and T_2 comes from the $(U-1)$-part. Let us start with the double sum T_1. During the summation we consider all pairs (k, l) with $1 \leq l \leq k$ for each $1 \leq k \leq s$. The first observation is that we can regroup the sum by taking all pairs (l, k) with $l \leq k \leq s$ for each $1 \leq l \leq s$, which just covers the same set of pairs. Next we find that the following formula for binomial coefficients holds:

$$\binom{s+1}{k} \binom{k}{l} = \binom{s+1}{s-l+1} \binom{s-l+1}{s-k+1}.$$

Using both we get

$$T_1 = \sum_{l=1}^{s} \sum_{k=l}^{s} (-i)^{s-k+1} \binom{s+1}{s-l+1} \binom{s-l+1}{s-k+1} u^{k-l} v^{s-k+l}.$$

Let us now substitute k by $s+1-\alpha$ and l by $s+1-\beta$ yielding

$$T_1 = \sum_{\beta=1}^{s} \sum_{\alpha=1}^{\beta} (-i)^{\alpha} \binom{s+1}{\beta} \binom{\beta}{\alpha} u^{\beta-\alpha} v^{s-\beta+\alpha}.$$

Finally we rename back α to l and β to k and get $T_1 = T_2$ as desired. □

Remark 5.15 As one can imagine, a kind of analogous candidate for $r \geq 1$ with v^s instead of y^r and a combinatorial sum in x, y can be established as well. For $r = 0$ we would have $w = (A-1) \otimes_\Gamma v^s$ then. But we don't want to go into the details here.

Now the next step is to verify the eigenvalue equation $T_\pi(w) = (\pi^{r+1} + \overline{\pi}^{s+1})w$ for the candidate w we found. For that we have to determine $T_\pi((A-1) \otimes_\Gamma \mathrm{m})$ and $T_\pi((U-1) \otimes_\Gamma \mathrm{m})$ and can use our formulas from Section 5.2. The results are quite complicated expressions. So we could only prove the eigenvalue equations for the series $M_{r,0}$ and $M_{r,1}$ up to now. We first present the case $s = 0$ because several difficulties don't appear here. Then we go on with the case $s = 1$, where we could use Lemma 5.13. To be precise, we get a result modulo torsion here. In general a more subtle understanding of torsion vectors seems to be necessary. We will also put together some first considerations in that direction.
The application of T_π to $(A-1) \otimes_\Gamma \mathrm{m}$ and $(U-1) \otimes_\Gamma \mathrm{m}$ always produces a $(B-1)$-part (see formulas (5.11) and (5.13)), which should vanish or should be zero modulo torsion. Here we can show that this $(B-1)$-part indeed vanishes for our general candidate with $s \geq 1$.
Having established the eigenvalue equation the next step would be to show that w is indeed of infinite order in the cases $r + s + 2 \equiv 0 \bmod 4$. Up to now we know that in all computated cases only. But we can prove for $s = 0$ and $s = 1$ that the vector w for $r + s \equiv 0 \bmod 4$ is indeed a torsion class.
Let us go through the mentioned results in detail now.

Proposition 5.16 *Let $\Gamma = PSL_2(\mathbb{Z}[i])$, $M = M_{r,0}(\mathbb{Z}[i])$ with $r > 0$ and T_π as above. Then we have*

$$T_\pi((A-1) \otimes_\Gamma y^r) \quad = \quad (\pi^{r+1} + \overline{\pi})(A-1) \otimes_\Gamma y^r,$$

where $(A-1) \otimes_\Gamma y^r \in H_1(\Gamma, M)$.

Proof: From (5.11) we have

$$\begin{aligned} & T_\pi((A-1) \otimes_\Gamma \mathrm{m}) \\ = \ & (A^a U^{-b} - 1) \otimes_\Gamma \delta_\pi \begin{pmatrix} 1 & 0 \\ 1-p & 1 \end{pmatrix} \cdot \mathrm{m} + (BA^a U^b B - 1) \otimes_\Gamma \delta_\pi B \cdot \mathrm{m} \end{aligned}$$

with $\mathrm{m} \in M_{r,s}(\mathbb{Z}[i])$, $\pi = a + bi$, $N(\pi) = p$ and $\delta_\pi = \left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)$.
Using our decomposition rules, the linearity of the tensor product and property (4.1)

we then get:

$$\begin{aligned}
&T_\pi((A-1)\otimes_\Gamma \mathrm{m})\\
&= (A^a-1)\otimes_\Gamma U^{-b}\delta_\pi \begin{pmatrix} 1 & 0\\ 1-p & 1\end{pmatrix}\cdot \mathrm{m} + (U^{-b}-1)\otimes_\Gamma \delta_\pi \begin{pmatrix} 1 & 0\\ 1-p & 1\end{pmatrix}\\
&\quad + (B-1)\otimes_\Gamma A^aU^bB\delta_\pi B\cdot \mathrm{m} + (A^a-1)\otimes_\Gamma U^bB\delta_\pi B\cdot \mathrm{m}\\
&\quad + (U^b-1)\otimes_\Gamma B\delta_\pi B\cdot \mathrm{m} + (B-1)\otimes_\Gamma \delta_\pi B\cdot \mathrm{m}\\
&= (A^a-1)\otimes_\Gamma \left[U^{-b}\delta_\pi \begin{pmatrix} 1 & 0\\ 1-p & 1\end{pmatrix} + U^bB\delta_\pi B\right]\cdot \mathrm{m}\\
&\quad + (U^b-1)\otimes_\Gamma \left[B\delta_\pi B - U^{-b}\delta_\pi \begin{pmatrix} 1 & 0\\ 1-p & 1\end{pmatrix}\right]\cdot \mathrm{m}\\
&\quad + (B-1)\otimes_\Gamma \left[A^aU^bB\delta_\pi B + \delta_\pi B\right]\cdot \mathrm{m}.
\end{aligned}$$

In the following we set $\widetilde{\delta_\pi} := B\delta_\pi B = \left(\begin{smallmatrix} 1 & 0\\ 0 & \pi\end{smallmatrix}\right)$. Now we analyse the $(B-1)$-part first. We have the useful rule

$$(B-1)\otimes_\Gamma \mathrm{m} \;=\; (B-1)\otimes_\Gamma (-B)\cdot \mathrm{m}.$$

Using that we find for the $(B-1)$-part:

$$(B-1)\otimes_\Gamma (A^aU^b+B)\widetilde{\delta_\pi}\cdot \mathrm{m} \;=\; (B-1)\otimes_\Gamma (A^aU^b-1)\widetilde{\delta_\pi}\cdot \mathrm{m}.$$

If we now set $\mathrm{m} = y^r$ and take into account that $\widetilde{\delta_\pi}(y^r) = \pi^r y^r$ we end up with the expression

$$\pi^r(B-1)\otimes_\Gamma (A^aU^b-1)y^r.$$

But A and U send y to y such that the expression after the tensor product is just zero. Therefore the $(B-1)$-part vanishes.
Next we analyse the (A^a-1)-part. Using our polynomial decomposition in the preparation of Proposition 5.7 and $A_{\mathrm{pol}(a)}(y^r) = a\cdot y^r$ we get

$$\begin{aligned}
&(A^a-1)\otimes_\Gamma \left[U^{-b}\delta_\pi \begin{pmatrix} 1 & 0\\ 1-p & 1\end{pmatrix} + U^b\widetilde{\delta_\pi}\right]\cdot y^r\\
&= (A-1)\otimes_\Gamma a\left[U^{-b}\delta_\pi \begin{pmatrix} 1 & 0\\ 1-p & 1\end{pmatrix} + U^b\widetilde{\delta_\pi}\right]\cdot y^r\\
&= (A-1)\otimes_\Gamma (ay^r + a\pi^r y^r)\\
&= a(\pi^r+1)\,(A-1)\otimes_\Gamma y^r,
\end{aligned}$$

since the first expression in the square brackets sends y to y and the second one sends y to πy.
Similarily we find

$$\begin{aligned}
&(U^b-1)\otimes_\Gamma \left[\widetilde{\delta}_\pi - U^{-b}\delta_\pi \begin{pmatrix} 1 & 0 \\ 1-p & 1 \end{pmatrix}\right]\cdot y^r \\
&= (U-1)\otimes_\Gamma b\left[\widetilde{\delta}_\pi - U^{-b}\delta_\pi \begin{pmatrix} 1 & 0 \\ 1-p & 1 \end{pmatrix}\right]\cdot y^r \\
&= (U-1)\otimes_\Gamma (b\pi^r y^r - by^r) \\
&= b(\pi^r-1)\,(U-1)\otimes_\Gamma y^r.
\end{aligned}$$

So we end up with

$$T_\pi((A-1)\otimes_\Gamma y^r) \;=\; a(\pi^r+1)\,(A-1)\otimes_\Gamma y^r + b(\pi^r-1)\,(U-1)\otimes_\Gamma y^r.$$

But we also have $(A-1)(xy^{r-1}) = y^r$ and $(U-1)(xy^{r-1}) = iy^r$ (note that we need $r>0$ here). If we put that into our relation (5.14) we get

$$i(A-1)\otimes_\Gamma y^r \;=\; (U-1)\otimes_\Gamma y^r.$$

This yields

$$T_\pi((A-1)\otimes_\Gamma y^r) \;=\; [a(\pi^r+1) + ib(\pi^r-1)]\;(A-1)\otimes_\Gamma y^r.$$

We have $\pi = a+bi$ and therefore $a = (\pi+\overline{\pi})/2$ and $ib = (\pi-\overline{\pi})/2$. So we finally obtain

$$T_\pi((A-1)\otimes_\Gamma y^r) \;=\; (\pi^{r+1}+\overline{\pi})\,(A-1)\otimes_\Gamma y^r$$

as claimed. □

Remark 5.17 From the proof we see that $(U-1)\otimes_\Gamma y^r$ satisfies the eigenvalue equation as well. Of course, one could also go through $T_\pi((U-1)\otimes_\Gamma y^r)$ in a similar way as above.

We go on with the case $s=1$.

Proposition 5.18 *Let $\Gamma = PSL_2(\mathbb{Z}[i])$, $M = M_{r,1}(\mathbb{Z}[i])$ with $r \geq 1$ and T_π as above. Then we have*

$$\begin{aligned}
&T_\pi((A-1)\otimes_\Gamma (y^r\otimes iu) + (U-1)\otimes_\Gamma (y^r\otimes u)) \\
&= (\pi^{r+1}+\overline{\pi}^2)\,((A-1)\otimes_\Gamma (y^r\otimes iu) + (U-1)\otimes_\Gamma (y^r\otimes u))
\end{aligned}$$

modulo torsion, where $(A-1)\otimes_\Gamma (y^r\otimes iu) + (U-1)\otimes_\Gamma (y^r\otimes u) \in H_1(\Gamma, M)$.

Proof: In contrast to Proposition 5.16 we have to determine T_π not only on a contribution from $(A-1)$. We set $\mathrm{m}_A := y^r \otimes iu$ and $\mathrm{m}_U := y^r \otimes u$. So we have to analyse $T_\pi((A-1) \otimes_\Gamma \mathrm{m}_A)$ and $T_\pi((U-1) \otimes_\Gamma \mathrm{m}_U)$ along similar lines as in the proof of Proposition 5.16 and to put everything together in a suitable way at the end.
We start off with the first expression. Here we find from (5.11):

$$\begin{aligned} & T_\pi((A-1) \otimes_\Gamma \mathrm{m}_A) \\ = \; & (A^a - 1) \otimes_\Gamma \left[U^{-b}\gamma_\pi + U^b \widetilde{\delta}_\pi \right] \cdot \mathrm{m}_A + \; (U^b - 1) \otimes_\Gamma \left[\widetilde{\delta}_\pi - U^{-b}\gamma_\pi \right] \cdot \mathrm{m}_A + t_A, \end{aligned}$$

where $\gamma_\pi := \delta_\pi \left(\begin{smallmatrix} 1 & 0 \\ 1-p & 1 \end{smallmatrix} \right)$ and t_A stands for the contribution from $(B-1)$.
Now we find for the (A^a-1)-part:

$$\begin{aligned} & (A^a - 1) \otimes_\Gamma \left[U^{-b}(i\overline{\pi}(y^r \otimes u) + i(1-p)(y^r \otimes v)) + U^b(i\pi^r(y^r \otimes u)) \right] \\ = \; & i(A^a - 1) \otimes_\Gamma \left[\overline{\pi}(y^r \otimes (u + ibv)) + (1-p)(y^r \otimes v) + \pi^r(y^r \otimes (u - ibv)) \right] \\ = \; & i(A^a - 1) \otimes_\Gamma (\pi^r + \overline{\pi})(y^r \otimes u) + \text{ torsion} \\ = \; & ia(\pi^r + \overline{\pi})(A - 1) \otimes_\Gamma (y^r \otimes u) + \text{ torsion}. \end{aligned}$$

Note that all the contributions we collect in *torsion* are of the form $(A-1) \otimes_\Gamma (y^r \otimes v)$ with certain factors, such that we can apply Lemma 5.13. We will use that argument quite often from now on (also for $(U-1) \otimes_\Gamma (y^r \otimes v)$). In the last two steps above we used again our knowledge about the decomposition polynomials $A_{\mathrm{pol}(a)}$ (in the first step only for the torsion part).
Analogously we derive for the (U^b-1)-part:

$$\begin{aligned} & (U^b - 1) \otimes_\Gamma \left[i\pi^r(y^r \otimes u) - U^{-b}(i\,\overline{\pi}(y^r \otimes u) + i(1-p)(y^r \otimes v)) \right] \\ = \; & ib(\pi^r - \overline{\pi})(U - 1) \otimes_\Gamma (y^r \otimes u) + \text{ torsion}. \end{aligned}$$

This gives

$$\begin{aligned} & T_\pi((A-1) \otimes_\Gamma (y^r \otimes iu)) \\ = \; & a(\pi^r + \overline{\pi})(A - 1) \otimes_\Gamma \mathrm{m}_A + ib(\pi^r - \overline{\pi})(U - 1) \otimes_\Gamma \mathrm{m}_U + t_A \end{aligned}$$

modulo torsion (using Lemma 5.13).
Now we go on with the second expression $T_\pi((U-1) \otimes_\Gamma \mathrm{m}_U)$. We have $\pi = a + bi$. Recall that we can find unique numbers $c, d \in \mathbb{Z}$ with $ad + bc = 1$ and $ac - bd \in \{0, \ldots, p-1\}$ (cp. the preparation for Proposition 5.8). Then we have from (5.13) (replacing x by q):

$$\begin{aligned} & T_\pi((U-1) \otimes_\Gamma \mathrm{m}_U) \\ & = (B-1) \otimes_\Gamma (A^{-b}U^a - 1)\widetilde{\delta}_\pi \cdot \mathrm{m}_U + (U^a - 1) \otimes_\Gamma \widetilde{\delta}_\pi \cdot \mathrm{m}_U - (A^b - 1) \otimes_\Gamma A^{-b}U^a\widetilde{\delta}_\pi \cdot \mathrm{m}_U \\ & + (A^cU^d - 1) \otimes_\Gamma \sum_{q=ac-bd}^{p-1} \begin{pmatrix} \pi & 0 \\ -q & 1 \end{pmatrix} \cdot \mathrm{m}_U + (A^{c-a}U^{d+b} - 1) \otimes_\Gamma \sum_{q=0}^{ac-bd-1} \begin{pmatrix} \pi & 0 \\ -q & 1 \end{pmatrix} \cdot \mathrm{m}_U. \end{aligned}$$

Here we already applied the decomposition procedures (see Section 5.2) again. We want to denote the $(B-1)$-part by t_U in the following, and we set $\delta_q := \left(\begin{smallmatrix} \pi & 0 \\ -q & 1 \end{smallmatrix}\right)$.

First we find with $\mathrm{m}_U = y^r \otimes u$ that

$$\begin{aligned}(U^a - 1) \otimes_\Gamma \widetilde{\delta}_\pi (y^r \otimes u) &= (U^a - 1) \otimes_\Gamma \pi^r (y^r \otimes u) \\ &= a\pi^r (U-1) \otimes_\Gamma (y^r \otimes u) + \text{ torsion}\end{aligned}$$

using Lemma 5.13 again, since the corresponding contributions are all of the form $(U-1) \otimes_\Gamma (y^r \otimes v)$ with certain factors.
Then we find for the next contribution

$$\begin{aligned}-(A^b - 1) \otimes_\Gamma \begin{pmatrix} 1 & 0 \\ -b+ia & \pi \end{pmatrix} (y^r \otimes u) &= -(A^b - 1) \otimes_\Gamma \begin{pmatrix} 1 & 0 \\ i\pi & \pi \end{pmatrix} (y^r \otimes u) \\ &= -(A^b - 1) \otimes_\Gamma (\pi^r y^r \otimes (u - i\overline{\pi} v)) \\ &= -b\pi^r (A-1) \otimes_\Gamma (y^r \otimes u) + \text{ torsion} \\ &= ib\pi^r (A-1) \otimes_\Gamma \mathrm{m}_A + \text{ torsion.}\end{aligned}$$

Next we go through

$$\begin{aligned}&(A^c U^d - 1) \otimes_\Gamma \sum_{q=ac-bd}^{p-1} \delta_q (y^r \otimes u) \\ &= (A^c U^d - 1) \otimes_\Gamma \sum_{q=ac-bd}^{p-1} (y^r \otimes (\overline{\pi} u - qv)) \\ &= (A^c U^d - 1) \otimes_\Gamma (p - ac + bd) \overline{\pi} (y^r \otimes u) + \text{ torsion} \\ &= (p - ac + bd) \overline{\pi} \left[(A^c - 1) \otimes_\Gamma U^d (y^r \otimes u) + (U^d - 1) \otimes_\Gamma (y^r \otimes u) \right] + \text{ torsion} \\ &= (p - ac + bd) \overline{\pi} \left[(A^c - 1) \otimes_\Gamma (y^r \otimes (u - idv)) + (U^d - 1) \otimes_\Gamma (y^r \otimes u) \right] \\ &\quad + \text{torsion} \\ &= c(p - ac + bd) \overline{\pi}\, (A-1) \otimes_\Gamma (y^r \otimes u) + d(p - ac + bd) \overline{\pi}\, (U-1) \otimes_\Gamma (y^r \otimes u) \\ &\quad + \text{torsion} \\ &= -ic(p - ac + bd) \overline{\pi}\, (A-1) \otimes_\Gamma \mathrm{m}_A + d(p - ac + bd) \overline{\pi}\, (U-1) \otimes_\Gamma \mathrm{m}_U \\ &\quad + \text{torsion}\end{aligned}$$

All the torsion appears in a similar way as above, we used the decomposition procedures several times and analysed the action of the matrices on the module elements.

Analogously we find

$$(A^{c-a}U^{d+b} - 1) \otimes_\Gamma \sum_{q=0}^{ac-bd-1} \delta_q(y^r \otimes u)$$
$$= -i(c-a)(ac-bd)\overline{\pi}(A-1) \otimes_\Gamma \mathrm{m}_A + (d+b)(ac-bd)\overline{\pi}(U-1) \otimes_\Gamma \mathrm{m}_U + \text{ torsion.}$$

Putting everything together yields

$$\begin{aligned} &T_\pi((U-1) \otimes_\Gamma \mathrm{m}_U) \\ &= ib\pi^r(A-1) \otimes_\Gamma \mathrm{m}_A - i\overline{\pi}(c(p-ac+bd) + (c-a)(ac-bd))(A-1) \otimes_\Gamma \mathrm{m}_A \\ &\quad + a\pi^r(U-1) \otimes_\Gamma \mathrm{m}_U + \overline{\pi}(d(p-ac+bd) + (d+b)(ac-bd))(U-1) \otimes_\Gamma \mathrm{m}_U + t_U \end{aligned}$$

modulo torsion. Now we can use $p = a^2 + b^2$ and $ad + bc = 1$ to get

$$c(p-ac+bd) + (c-a)(ac-bd) = b \quad \text{and} \quad d(p-ac+bd) + (d+b)(ac-bd) = a.$$

Simplifying the sum above and combining it with the result for $T_\pi((A-1) \otimes_\Gamma \mathrm{m}_A)$ finally gives

$$\begin{aligned} &T_\pi((A-1) \otimes_\Gamma \mathrm{m}_A + (U-1) \otimes_\Gamma \mathrm{m}_U) \\ &= [a(\pi^r + \overline{\pi}) + ib(\pi^r - \overline{\pi})]\,(A-1) \otimes_\Gamma \mathrm{m}_A + [ib(\pi^r - \overline{\pi}) + a(\pi^r + \overline{\pi})]\,(U-1) \otimes_\Gamma \mathrm{m}_U \end{aligned}$$

modulo torsion. Note that $t_A + t_U = 0$ which will be proved in Lemma 5.19 for general $s \geq 1$. Using again $a = (\pi + \overline{\pi})/2$ and $ib = (\pi - \overline{\pi})/2$ we get our final formula

$$\begin{aligned} &T_\pi((A-1) \otimes_\Gamma (y^r \otimes iu) + (U-1) \otimes_\Gamma (y^r \otimes u)) \\ &= (\pi^{r+1} + \overline{\pi}^2)\,((A-1) \otimes_\Gamma (y^r \otimes iu) + (U-1) \otimes_\Gamma (y^r \otimes u)). \end{aligned}$$

modulo torsion. □

Let us also add two more examples of torsion results. We find for $s \geq 2$:

$$(A-1)(y^r \otimes u^2 v^{s-2}) = 2y^r \otimes uv^{s-1} + y^r \otimes v^s$$

and

$$(U-1)(y^r \otimes u^2 v^{s-2}) = -2iy^r \otimes uv^{s-1} - y^r \otimes v^s.$$

Then we get with relation (5.14)

$$(A-1) \otimes_\Gamma (-2iy^r \otimes uv^{s-1} - y^r \otimes v^s) = (U-1) \otimes_\Gamma (2y^r \otimes uv^{s-1} + y^r \otimes v^s).$$

Using Lemma 5.13 we then obtain

$$2(i(A-1)\otimes_\Gamma (y^r\otimes uv^{s-1})+(U-1)\otimes_\Gamma (y^r\otimes uv^{s-1}))=0$$

modulo the torsion from Lemma 5.13. In a similar way we get

$$(A-1)(y^r\otimes u^3) \;=\; 3y^r\otimes u^2v+3y^r\otimes uv^2+y^r\otimes v^3$$

and

$$(U-1)(y^r\otimes u^3) \;=\; -3iy^r\otimes u^2v-3y^r\otimes uv^2+iy^r\otimes v^3.$$

With relation (5.14) and Lemma 5.13 we then find

$$3((A-1)\otimes_\Gamma (iy^r\otimes u^2v+y^r\otimes uv^2)+(U-1)\otimes_\Gamma (y^r\otimes u^2v+y^r\otimes uv^2))=0$$

modulo the torsion from Lemma 5.13. One can establish many more torsion vectors of similar structure. A more systematic study of that could lead to a proof of the eigenvalue equation for the general candidate w in the case of $M_{r,s}(\mathbb{Z}[i])$ for $s\geq 1$.

Now we come back again to the general candidate w from (5.17). After applying T_π to $(A-1)\otimes_\Gamma \mathrm{m}$ and $(U-1)\otimes_\Gamma \mathrm{m}$ we get certain $(B-1)$-parts. For the $(B-1)$-parts in the case of our general w we get (t_A and t_U chosen as for $s=1$)

$$(B-1)\otimes_\Gamma (BA^{-b}U^aB-1)\delta_\pi B\cdot \mathrm{m}_A+(B-1)\otimes_\Gamma (A^{-b}U^a-1)\widetilde{\delta_\pi}\cdot \mathrm{m}_U \;=\; t_A+t_U,$$

where m_A and m_U denote the expressions in the $(A-1)$-part resp. $(U-1)$-part of w as for $s=1$. Then we put in the expressions for m_A and m_U and can show the following vanishing lemma. Here we set

$$S \;:=\; \sum_{k=1}^{s}\overline{\pi}^{s-k}\binom{s+1}{k}v^{s-k}\left((u-i\overline{\pi}v)^k-u^k+(-i)^{s-k+1}u^k-(-i)^{s-k+1}(u+\overline{\pi}v)^k\right).$$

Lemma 5.19 *Under the assumptions from above we have for $s\geq 1$ that*

$$t_A+t_U \;=\; (B-1)\otimes_\Gamma \pi^r y^r\otimes S \;=\; 0.$$

PROOF: It is enough to show that the sum S is zero. At first we regroup S a bit, then substitute u by X, $\overline{\pi}v$ by Y and take i instead of $-i$. So we get

$$S \;=\; \sum_{k=1}^{s}\binom{s+1}{k}Y^{s-k}((X+iY)^k-X^k+i^{s-k+1}(X^k-(X+Y)^k)).$$

Furthermore we have the identity

$$\sum_{k=1}^{s} \binom{s+1}{k} \alpha^{s-k}\beta^k = \frac{1}{\alpha}\left((\alpha+\beta)^{s+1} - \alpha^{s+1} - \beta^{s+1}\right).$$

Using that several times we find

$$\begin{aligned} S &= \frac{1}{Y}\big[(X+(1+i)Y)^{s+1} - Y^{s+1} - (X+iY)^{s+1} - (X+Y)^{s+1} + Y^{s+1} + X^{s+1} \\ &\quad + (X+iY)^{s+1} - (iY)^{s+1} - X^{s+1} - (X+(1+i)Y)^{s+1} + (iY)^{s+1} + (X+Y)^{s+1}\big] \\ &= 0. \end{aligned}$$

□

We conclude with our two results about general torsion vectors in the cases $s = 0$ and $s = 1$ for the modules $M_{r,s}(\mathbb{Z}[i])$. They cover the cases of r, s where we didn't find an eigenvector on the free part in our computer calculations and show that our general candidate w is a torsion vector then.

Proposition 5.20 *For* $\Gamma = PSL_2(\mathbb{Z}[i])$, $(A-1)\otimes_\Gamma y^r \in H_1(\Gamma, M_{r,0}(\mathbb{Z}[i]))$ *and* $r \equiv 0 \bmod 4$ *holds*

$$2(A-1)\otimes_\Gamma y^r \;=\; 0.$$

Proof: At first note that $C := \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) \in PSL_2(\mathbb{Z}[i])$. Then we have for C the action $C(y^r) = (-i)^r y^r$. Furthermore we can use our decomposition rules for $A = C^{-1}A^{-1}C$. This gives

$$\begin{aligned} (A-1)\otimes_\Gamma y^r &= (C^{-1}A^{-1}C - 1)\otimes_\Gamma y^r \\ &= -(C-1)\otimes_\Gamma (C^{-1}A^{-1}C)\cdot y^r - (A-1)\otimes_\Gamma (A^{-1}C)\cdot y^r \\ &\quad +(C-1)\otimes_\Gamma y^r \\ &= -(C-1)\otimes_\Gamma A\cdot y^r + (C-1)\otimes_\Gamma y^r - (A-1)\otimes_\Gamma (A^{-1}C)\cdot y^r \\ &= -(A-1)\otimes_\Gamma (A^{-1}C)\cdot y^r \\ &= -(-i)^r (A-1)\otimes_\Gamma y^r. \end{aligned}$$

Recall that r is even to have the Γ-action on the module $M_{r,0}(\mathbb{Z}[i])$. Then we just get $(A-1)\otimes_\Gamma y^r = -(A-1)\otimes_\Gamma y^r$ for $r \equiv 0 \bmod 4$. □

We also get

Proposition 5.21 *For* $\Gamma = PSL_2(\mathbb{Z}[i])$,

$$w \;=\; (A-1)\otimes_\Gamma (y^r \otimes iu) + (U-1)\otimes_\Gamma (y^r\otimes u) \in H_1(\Gamma, M_{r,1}(\mathbb{Z}[i]))$$

and $r+1 \equiv 0 \bmod 4$ *holds*

$$2w \;=\; 0$$

modulo the torsion from Lemma 5.13.

PROOF: As in the proof for Proposition 5.20 we can use $A = C^{-1}A^{-1}C$ and as well $U = C^{-1}U^{-1}C$. Then we find

$$\begin{aligned} w &= (C^{-1}A^{-1}C - 1) \otimes_\Gamma (y^r \otimes iu) + (C^{-1}U^{-1}C - 1) \otimes_\Gamma (y^r \otimes u) \\ &= -(C-1) \otimes_\Gamma A(iy^r \otimes u) - (A-1) \otimes_\Gamma (A^{-1}C)(iy^r \otimes u) \\ &\quad +(C-1) \otimes_\Gamma (iy^r \otimes u) - (C-1) \otimes_\Gamma U(y^r \otimes u) \\ &\quad -(U-1) \otimes_\Gamma (U^{-1}C)(y^r \otimes u) + (C-1) \otimes_\Gamma (y^r \otimes u) \\ &= -(C-1) \otimes_\Gamma (iy^r \otimes (u+v)) - i(A-1) \otimes_\Gamma ((-i)^r y^r \otimes (-iu+iv)) \\ &\quad +(C-1) \otimes_\Gamma (iy^r \otimes u) - (C-1) \otimes_\Gamma (y^r \otimes (u - iv)) \\ &\quad -(U-1) \otimes_\Gamma ((-i)^r y^r \otimes (-iu+v)) + (C-1) \otimes_\Gamma (y^r \otimes u) \\ &= -(-i)^{r+1}(A-1) \otimes_\Gamma (iy^r \otimes u) + (-i)^r (A-1) \otimes_\Gamma (y^r \otimes v) \\ &\quad -(-i)^{r+1}(U-1) \otimes_\Gamma (y^r \otimes u) - (-i)^r (U-1) \otimes_\Gamma (y^r \otimes v) \\ &= -(-i)^{r+1} \left[(A-1) \otimes_\Gamma (y^r \otimes iu) + (U-1) \otimes_\Gamma (y^r \otimes u)\right] \end{aligned}$$

modulo the torsion from Lemma 5.13, which gives the result we claimed. □

6 Suggestions for further work

Of course, many more computations for the cohomology and homology can be carried out in our setup to enrich the picture of the occuring ranks and torsions. This concerns more modules $M = M_{n,m}(\mathcal{O}_K)$ but also more groups Γ. So it seems to be useful to consider e. g. the congruence subgroups $\Gamma = \Gamma_0(\mathfrak{n}) \leq PSL_2(\mathbb{Z}[\,i\,])$ in a systematic way (for $H_1(\Gamma, M)$ and $H^1(\Gamma, M)$ and various modules M). The presentations could be established with the help of MAGMA or GAP.
For $H^1(\Gamma, M)$ the special observations for the matrix DMAT should be considered in more detail. Furthermore Hecke operators should be analysed here as well. They can be defined on $\mathrm{Der}(\Gamma, M)$ by a similar diagram as for Λ_Γ in Section 5.1 and one ends up with a similar formula as in Theorem 5.3 describing $T_\pi(f(g))$ for $f \in \mathrm{Der}(\Gamma, M)$. For an explicit realization similar problems as for Λ_Γ have to be solved. For $H_1(\Gamma, M)$ or $H^1(\Gamma, M)$ the cases of the modules $M = M_{n,n}(\mathcal{O}_K)$ should be studied in more detail. Here it is of special interest to identify candidates of Hecke eigenclasses which are not "lifted" from certain congruence subgroups $\Gamma \leq PSL_2(\mathbb{Z})$.

In particular we suggest to study $H_1(\Gamma, M)$ further. Here we think of various aspects we already pointed out at the end of Section 1.3. At first there are the questions about the properties of the Hecke eigenvalue systems. In this thesis we already found several relations between Hecke eigenvalue systems of different homology modules for $\Gamma = PSL_2(\mathbb{Z}[\,i\,])$ and we saw examples, where the eigenvalue systems couldn't be lifted to characteristic zero (automorphic classes) within the same homology modules (cp. Section 5.4). Therefore it is an important question whether such a lift always exists when one allows changes of the coefficient modules M or also of the groups Γ or both. Another intriguing question is whether one can always find corresponding Hecke eigenvalue systems between e. g. $H_1(PSL_2(\mathbb{Z}[\,i\,]), M_{n,m}(\mathbb{Z}[\,i\,]))$ and $H_1(\Gamma, M)$, where Γ is a congruence subgroup of $PSL_2(\mathbb{Z}[\,i\,])$ and $M = \mathbb{Z}[\,i\,]$ (constant coefficients or so-called weight 2 case). This would be relevant for a formulation of an analogue of Serre's conjecture since one could restrict to constant coefficients then as somehow assumed in [Fi1] and [Fi2].

Most of all the congruences should be analysed further. It might be possible to actually prove some of them with the help of our formulas from Section 5.2 and similar computations as started in Section 5.5. A further study of the torsion classes could also lead to a proof of the eigenvalue equations for all modules $M_{r,s}(\mathbb{Z}[\,i\,])$.
Also, the differences between the so-called small and large torsion classes should be studied in more detail. For that it also seems to be helpful to carry out more computations in both cases.
Finally, an investigation of the arithmetic side should be continued. A central point would be a careful analysis of possible Galois representations mod ℓ which could correspond to the Hecke eigenclasses we identified. For the small torsion classes this would lead to an explicit study of the splitting behaviour for the possible field extensions. For the large torsion classes it should be checked whether the image of the absolute Galois group in possible corresponding Galois representations mod ℓ could be full or large.
In principle, the arithmetic investigations should be carried out in close relation to the ones about the Hecke eigenvalue systems.

References

[AS1] A. Ash and G. Stevens, *Cohomology of arithmetic groups and congruences between systems of Hecke eigenvalues*, Journal Reine u. Angew. Math. **365** (1986), 192–220.

[AS2] A. Ash and G. Stevens, *Modular forms in characteristic ℓ and special values of their L-functions*, Duke Math. J. **53** (1986), 849–868.

[Ar] M. Artin, *Algebra*, Birkhäuser, 1993.

[As] A. Ash, *Galois representations attached to mod p cohomology of* $GL(n, \mathbb{Z})$, Duke Math. J. **65** (1992), 235–255.

[BW] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups and representations of reductive groups*, Annals of Math. Studies **94**, Princeton University Press, 1980.

[Bl] J. Blume, *Automorphe Formen für $SL_2(\mathbb{C})$ und eine explizite Form des Eichler–Shimura–Isomorphismus*, Diplomarbeit, Bonn, 1988.

[Bo] A. Borel, *Cohomology of arithmetic groups*, in: Proc. of the International Congress of Math., Vancouver (1974), 435–442.

[Br] K. S. Brown, *Cohomology of groups*, Graduate Texts in Math. **87**, Springer, 1982.

[Ca] J. von Campenhausen, *Kohomologie der arithmetischen Gruppe $PGL_2(\mathbb{Z}[i])$ mit Koeffizienten in den homogenen Polynomen in zwei Variablen über* $\mathbb{Z}[i, 1/6]$, Diplomarbeit, Bonn, 1993.

[Ch] P. M. Cohn, *On the structure of the GL_2 of a ring*, Publ. Math. I.H.E.S. **30** (1966), 365–413.

[Co1] H. Cohen, *A Course in Computational Algebraic Number Theory*, Graduate Texts in Math. **138**, Springer, 1993.

[Co2] H. Cohen, *Hermite and smith normal form algorithms over Dedekind domains*, Math. of Computation **65**, No. 216 (1996), 1681–1699.

[Co3] H. Cohen, *Advanced Topics in Computational Number Theory*, Graduate Texts in Math. **193**, Springer 2000.

[Cr1] J. E. Cremona, *Modular symbols*, Ph.D. thesis, Oxford University, 1981.

[Cr2] J. E. Cremona, *Hyperbolic tesselations, modular symbols, and elliptic curves over complex quadratic fields*, Compositio Math. **51** (1984), 275–324.

[Cr3] J. E. Cremona, *Modular Forms and Elliptic Curves over Imaginary Quadratic Fields*, Preprint, April 1998.

[DDT] H. Darmon, F. Diamond, and R. Taylor, *Fermat's last theorem*, in: R. Bott et al. (Eds.), Current Developments in Mathematics, 1995, International Press (1995), 1–107.

[Da] H. Darmon, *Serre's conjectures*, in: V. K. Murty (Ed.), Seminar on Fermat's Last Theorem, Canadian Mathematical Society, Conference Proceedings **17** (1995), 135–153.

[Di] F. Diamond, *The refined conjecture of Serre*, in: J. Coates and S. T. Yau (Eds.), Elliptic Curves, Modular Forms, & Fermat's Last Theorem, International Press (1995), 22–37.

[EGM 1] J. Elstrodt, F. Grunewald, and J. Mennicke, *On the group* $PSL_2(\mathbb{Z}[i])$, Journées Arithmétiques 1980, LMS Lecture Note Series **56**, 255–283.

[EGM 2] J. Elstrodt, F. Grunewald, and J. Mennicke, $PSL(2)$ *over imaginary quadratic integers*, Journées Arithmétiques 1981, Astérisque **94**, 43–60.

[EGM 3] J. Elstrodt, F. Grunewald, and J. Mennicke, *Groups Acting on Hyperbolic Space*, Springer, 1998.

[EMS] A. I. Kostrikin and I. R. Shafarevich (Eds.), *Algebra I*, Encyclopaedia of Mathematical Sciences, vol. **11**, Springer, 1990.

[Ed] B. Edixhoven, *Serre's conjectures*, in: G. Cornell, J. H. Silverman, and G. Stevens (Ed.), Modular Forms and Fermat's Last Theorem, Springer, 1997.

[Ev] L. Evans, *The Cohomology of Groups*, Oxford Mathematical Monographs, Clarendon Press, 1991.

[Fe] D. Feldhusen, *Berechnung von Nennern von Eisensteinklassen für die Gruppe* $GL_2(\mathbb{Z}[\zeta_6])$, Diplomarbeit, Bonn, 1994.

[Fi1] L. M. Figueiredo, *Serre's conjecture over imaginary quadratic fields*, Ph.D. thesis, University of Cambridge, 1995.

[Fi2] L. M. Figueiredo, *Serre's conjecture for imaginary quadratic fields*, Compositio Mat. **118** (1999), 103–122.

[Fl] D. Flöge, *Zur Struktur der PSL_2 über einigen imaginärquadratischen Zahlkörpern*, Math. Z. **183** (1983), 255–279.

[GH] F. Grunewald and U. Hirsch, *Link complements arising from arithmetic group actions*, Int. Journal of Math. **6** (1995), 337–370.

[GHM] F. Grunewald, H. Helling, and J. Mennicke, *SL_2 over complex quadratic number fields. I*, Algebra i Logika **17** (1978), 512–580.

[GM] F. Grunewald and J. Mennicke, *$SL_2(\mathcal{O})$ and elliptic curves*, Manuscript, Bielefeld University, 1973.

[GS] F. Grunewald and J. Schwermer, *A nonvanishing theorem for the cuspidal cohomology of SL_2 over imaginary quadratic integers*, Math. Ann. **258** (1981), 183–200.

[GP1] F. Grunewald and V. Platonov, *Rigidity results for groups with radical cohomology of finite groups and arithmeticity problems*, Duke Math. J. vol. **100**, no. 1 (1999), 321–358.

[GP2] F. Grunewald and V. Platonov, *On finite extensions of arithmetic groups*, C. R. Acad. Sci. Paris Sér. I Math. **325**, (1997), 1153–1158.

[Gr] K. W. Gruenberg, *Cohomological Topics in Group Theory*, Lecture Notes in Math. **143**, Springer, 1970.

[HST] M. Harris, D. Soudry, and R. Taylor, *ℓ-adic representations associated to modular forms over imaginary quadratic fields. I*, Invent. Math. **112** (1993), 377–411.

[Hab] K. Haberland, *Perioden von Modulformen einer Variablen und Gruppenkohomologie*, I–III, Math. Nachr. **112** (1983), 245–282, 283–295, 297–315.

[Ha1] G. Harder, *On the cohomology of $SL_2(\mathcal{O})$*, Lie groups and their representations, Proc. of the Summer School on Group Representations, Hilger (1975), 139–150.

[Ha2] G. Harder, *Period integrals of cohomology classes which are represented by Eisenstein series*, in: Automorphic Forms, Representation Theory and Arithmetic, Bombay Colloquium 1979, Springer, 1981.

[Ha3] G. HARDER, *Kohomologie arithmetischer Gruppen*, Lecture Notes, Bonn University, 1987/88.

[Ha4] G. HARDER, *Eisenstein cohomology of arithmetic groups, the case GL_2*, Invent. Math. **89** (1987), 37–118.

[Ha5] G. HARDER, *Eisensteinkohomologie und die Konstruktion gemischter Motive*, Lecture Notes in Math. **1562**, Springer, 1993.

[Hu] B. HUPPERT, Endliche Gruppen I, Grundlagen der math. Wissenschaften **134**, Springer, 1967.

[Kö] H. KÖNIG, *Eisenstein–Kohomologie von $SL_2(\mathbb{Z}[i])$*, Dissertation, Bonner Math. Schriften **222** (1991).

[LS] J.-P. LABESSE AND J. SCHWERMER, *Cohomology of arithmetic groups and automorphic forms*, Lecture Notes in Math. **1447**, Springer, 1990.

[Me] E. R. MENDOZA, *Cohomology of PGL_2 over Imaginary Quadratic Integers*, Dissertation, Bonner Math. Schriften **128** (1980).

[Mi] T. MIYAKE, *Modular Forms*, Springer, 1989.

[Na] W. NARKIEWICZ, *Elementary and Analytic Theory of Algebraic Numbers*, Springer-Verlag and Polish Scientific Publishers, 1990.

[PR] V. PLATONOV AND A. RAPINCHUK, *Algebraic Groups and Number Theory*, Academic Press, 1994.

[RW] Y. H. RHIE AND G. WAPLES, *Hecke operators in cohomology of groups*, J. Math. Soc. Japan **22** (1971), 431–442.

[Ri1] K. RIBET, *On ℓ-adic representations attached to modular forms*, Invent. Math. **28** (1975), 245–275.

[Ri2] K. RIBET, *A modular construction of unramified p-extensions of $\mathbb{Q}(\mu_p)$*, Invent. Math. **34** (1976), 151–162.

[Ri3] K. RIBET, *Report on mod ℓ representations of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, in: Motives, Proc. Symp. in Pure Math. **55** (2), (1994), 639–676.

[Rl] R. RILEY, *A quadratic parabolic group*, Math. Proc. Camb. Phil. Soc. **77** (1975), 281–288.

[SV] J. SCHWERMER AND K. VOGTMANN, *The integral homology of SL_2 and PSL_2 of Euclidean imaginary quadratic integers*, Comment. Math. Helvetici **58** (1983), 573–598.

[SZ] R. Stöcker and R. Zieschang, *Topologie*, Verlag B.G. Teubner, Stuttgart, 1988.

[Sch] J. Schneider, *Diskrete Untergruppen von $SL_2(\mathbb{C})$ und ihre Operation auf dem dreidimensionalen hyperbolischen Raum*, Diplomarbeit, Bonn, 1985.

[Se1] J.-P. Serre, *Arithmetic groups*, in: C.T.C. Wall (Ed.), Homological Group Theory, London Math. Soc. Lecture Notes Series **36**, Cambridge University Press (1971), 105–136.

[Se2] J.-P. Serre, *Cohomologie Des Groups Discrets*, in: Prospects in Mathematics, Annals of Math. Studies **70**, Princeton University Press, 1971, 76–169.

[Se3] J.-P. Serre, *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques*, Invent. Math. **15** (1972), 259–331.

[Se4] J.-P. Serre, *Congruences et formes modulaires (d'aprés H.P.F. Swinnerton-Dyer)*, Séminaire Bourbaki 416, Juin 1972, Lecture Notes in Math. **317** (1973), 319–338.

[Se5] J.-P. Serre, *Sur les représentations modulaires de degré* 2 *de* Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$, Duke Math. Journal **54** (1987), 179–230.

[Sh] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton University Press, 1971.

[Sw] R. G. Swan, *Generators and relations for certain special linear groups*, Advances in Math. **6** (1971), 1–77.

[SD] H. P. F. Swinnerton-Dyer, *On ℓ-adic representations and congruences for coefficients of modular forms*, in: International Summer School on Modular Functions, Antwerp, 1972, Lecture Notes in Math. **350** (1973), 1–55.

[TW] R. Taylor and A. Wiles, *Ring theoretic properties of certain Hecke algebras*, Annals of Math. **141** (1995),

[Tay] R. Taylor, *ℓ-adic representations associated to modular forms over imaginary quadratic fields. II*, Invent. Math. **116** (1994), 619–643.

[Th] C. Thiel, *Gruppenkohomologie von Untergruppen der $PSL_2(\mathbb{Z}[i])$ und Hecke–Operatoren*, Diplomarbeit, Bonn, 1988.

[Vo] K. Vogtmann, *Rational homology of Bianchi groups*, Math. Ann. **272** (1985), 399–419.

[Wi] A. WILES, *Modular elliptic curves and Fermat's last theorem*, Ann. Math., II. Ser. **141**, No. 3 (1995), 443–551.

[Wa] X. WANG, *Die Eisensteinklasse in* $H^1(SL_2(\mathbb{Z}), M_n(\mathbb{Z}))$ *und die Arithmetik spezieller Werte von L-Funktionen*, Dissertation, Bonner Math. Schriften **202** (1989).

[Wei] CH. WEIBEL, *An introduction to homological algebra*, Cambridge University Press, 1994.