Controlled algebra for simplicial rings and the algebraic K-theory assembly map

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Summary

The algebraic K-theory Farrell-Jones Conjecture is a conceptional approach to calculate the algebraic K-groups of a group ring $R[G]$ where R is a simplicial ring with unit and G an infinite group. It states that the *assembly map*

$$h_n^G(E_{\mathcal{VC}yc}G, \mathbf{K}_R) \to h_n^G(\mathrm{pt}, \mathbf{K}_R)$$

is an isomorphism for all $n \in \mathbb{Z}$. The target calculates to $K_n(R[G])$, the nth algebraic K-group of the group ring $R[G]$. The conjecture has deep connections to geometric topology. For example for G a torsion-free group and $R = \mathbb{Z}$ it predicts the vanishing of the Whitehead group $\mathrm{Wh}(G)$ of G and hence by the famous s-cobordism theorem of Barden-Mazur-Stallings and Smale the triviality of each h-cobordism over a differentiable manifold of dimension ≥ 5 with fundamental group G. The conjecture is known for a large class of groups but the general case is open.

Recently Bartels, Lück and Reich proved the Farrell-Jones Conjecture for G a word-hyperbolic group in the sense of Gromov and R an arbitrary discrete ring. This prompts the question about an extension of it to a more general kind of rings. In this thesis we formulate the Farrell-Jones Conjecture for simplicial rings. Simplicial rings are a homotopy theoretic generalization of discrete rings and play itself an important role in the investigation of higher-dimensional manifolds. Our work requires a whole new set of tools. We prove, among other things:

Theorem A. *Let R be a simplicial ring. For each free G-equivariant control space X there is a category*

$$\mathcal{C}^G(X, R)$$

of G-equivariant controlled simplicial R-modules over X. It has the structure of a category with cofibrations and weak equivalences, so its algebraic K-theory is defined.

Theorem B. *Let R be a simplicial ring. The functor $h^G(-, \mathbf{K}_R)$ from G-CW-complexes to spectra, defined as*

$$Z \longmapsto \mathbb{K}^{-\infty}(g_\infty w\mathcal{C}^G(Z^{cc}, R)),$$

is a G-equivariant homology theory. Its coefficients are the non-connective algebraic K-theory spectra $G/H \mapsto \mathbb{K}^{-\infty}(R[H])$.

This description allows the following conclusion:

Theorem C. *The assembly map is an isomorphism if and only if*

$$\mathbb{K}^{-\infty}\big(w\mathcal{C}^G\big((E_{\mathcal{VC}yc}G)^{cc}, R\big)\big)$$

is contractible.

This opens the door to attack the conjecture for word-hyperbolic groups and simplicial rings with techniques known from the case of discrete rings. To establish the theorems we considerably generalize the theory of "controlled modules" over a discrete ring used by Bartels-Lück-Reich and introduce homotopy theoretic methods into the subject. As a crucial step we obtain a definition of the non-connective algebraic K-theory spectrum of any simplicial ring.

Zusammenfassung

Die Farrell-Jones-Vermutung für algebraische K-Theorie ist ein konzeptioneller Ansatz zur Bestimmung der algebraischen K-Gruppen eines Gruppenringes $R[G]$, wobei R ein simplizialer Ring mit Eins und G eine unendliche Gruppe ist. Die Vermutung besagt, dass die *Assembly-Abbildung*

$$h_n^G(E_{\mathcal{VC}yc}G, \mathbf{K}_R) \to h_n^G(\mathrm{pt}, \mathbf{K}_R)$$

für alle $n \in \mathbb{Z}$ ein Isomorphismus ist. Die rechte Seite berechnet sich zu $K_n(R[G])$, der $n$ten algebraischen K-Gruppe des Gruppenringes $R[G]$. Die Vermutung hat tiefe Verbindungen zur geometrischen Topologie. So folgt aus ihr für G eine torsionsfreie Gruppe und $R = \mathbb{Z}$, dass die Whitehead-Gruppe $\mathrm{Wh}(G)$ von G trivial ist. Der berühmte s-Kobordismussatz von Barden-Mazur-Stallings und Smale impliziert dann die Trivialität jedes h-Kobordismus über einer differenzierbaren Mannigfaltigkeit der Dimension ≥ 5 mit Fundamentalgruppe G. Die Vermutung ist für eine große Klasse von Gruppen bekannt, aber der allgemeine Fall ist offen.

Kürzlich zeigten Bartels, Lück und Reich die Farrell-Jones Vermutung wenn G eine wort-hyperbolische Gruppe im Sinne Gromovs und R ein diskreter Ring ist. Das wirft die Frage nach einer Erweiterung der Vermutung für eine allgemeinere Art von Ringen auf. In dieser Dissertation formulieren wir die Farrell-Jones Vermutung für simpliziale Ringe, diese können als homotopietheoretische Verallgemeinerung von diskreten Ringen angesehen werden und spielen selbst eine wichtige Rolle in der Untersuchung höher-dimensionaler Mannigfaltigkeiten. Unsere Untersuchen benötigen eine Reihe neuer Methoden und Konstruktionen. Unter anderem zeigen wir:

Theorem A. *Sei R ein simplizialer Ring. Für jeden freien G-äquivarianten Kontrollraum X gibt es eine Kategorie*

$$\mathcal{C}^G(X, R)$$

von G-äquivarianten kontrollierten simplizialen R-Moduln über X. Diese hat die Struktur einer Kategorie mit Kofaserungen und schwachen Äquivalenzen, womit ihre algebraische K-Theorie definiert ist.

Theorem B. *Sei R ein simplizialer Ring. Der Funktor $h^G(-, \mathbf{K}_R)$ von G-CW-Komplexen nach Spektren, der als*

$$Z \longmapsto \mathbb{K}^{-\infty}(g_\infty w \mathcal{C}^G(Z^{cc}, R)),$$

definiert wird, ist eine G-äquivariante Homologietheorie. Ihr Koeffizientenspektrum sind die nicht-zusammenhängenden algebraischen K-Theorie-Spektren $G/H \mapsto \mathbb{K}^{-\infty}(R[H])$.

Diese Beschreibung erlaubt die folgende Schlussfolgerung:

Theorem C. *Die Assembly-Abbildung ist genau dann ein Isomorphismus, wenn das Spektrum*

$$\mathbb{K}^{-\infty}\big(w\mathcal{C}^G\big((E_{\mathcal{VC}yc}G)^{cc}, R\big)\big)$$

zusammenziehbar ist.

Diese Formulierung ermöglicht die Vermutung für wort-hyperbolische Gruppen und simpliziale Ringe mit Methoden aus dem Fall der diskreten Ringe anzugreifen. Zum Beweis der obigen Theoreme verallgemeinern wir die von Bartels-Lück-Reich genutzte Theorie der „kontrollierten Moduln“ über diskreten Ringen und führen homotopietheoretische Techniken in das Gebiet ein. Als wichtigen Zwischenschritt erhalten wir eine Definition des nicht-zusammenhängenden algebraischen K-Theorie-Spektrums eines simplizialen Ringes.

Contents

Introduction

The algebraic K-theory Farrell-Jones Conjecture for a group G and a simplicial ring R (with unit) states that the assembly map

$$h_n^G(E_{\mathcal{VC}yc}G, \mathbf{K}_R) \to h_n^G(\mathrm{pt}, \mathbf{K}_R) \tag{†}$$

is an isomorphism for all $n \in \mathbb{Z}$. The target calculates to $K_n(R[G])$, the nth algebraic K-group of the group ring $R[G]$. The source is the G-equivariant homology theory with coefficients in the G-equivariant non-connective algebraic K-theory spectrum of R, evaluated on the classifying space for the family of virtually cyclic subgroups of G.

It is known that for discrete rings R the assembly map (†) is an isomorphism for a large class of groups, the recent result [BLR08] shows this for word-hyperbolic groups. The conjecture for a group G, together with a variant for L-theory, implies a wide range of other well-known conjectures in geometric topology and algebra. Most notably is perhaps the Borel Conjecture, which states that closed aspherical manifolds of dimension ≥ 5 with fundamental group G are topologically rigid. This means, each two closed manifolds of dimension ≥ 5 with fundamental group isomorphic to G whose universal covers are contractible are homeomorphic. An algebraic conjecture implied by the Farrell-Jones Conjecture is e.g. the Kaplansky Conjecture, which states that for G torsion-free and a field F of characteristic zero the only idempotents in the group ring $F[G]$ are 0 and 1. See [LR05] for a broad overview.

The recent success on the Farrell-Jones Conjecture with coefficients in a discrete ring prompts the question about an extension of it to a more general kind of rings. In this thesis we formulate the Farrell-Jones Conjecture for simplicial rings. This requires a whole new set of tools. We prove (as Proposition 3.3):

Theorem A. *Let R be a simplicial ring. For each free G-equivariant control space X there is a category*

$$\mathcal{C}^G(X, R)$$

of G-equivariant controlled simplicial R-modules over X. It has the structure of a category with cofibrations and weak equivalences, so its algebraic K-theory is defined.

We explain the notions later. If Z is a G-CW-complex we get a G-equivariant control space Z^{cc}. It is the space $Z \times G \times [1, \infty)$ together with continuous control conditions. We define it in Definition 7.3. We show (as Theorem 7.7):

Theorem B. *Let R be a simplicial ring. The functor $h^G(-,\mathbf{K}_R)$ from G-CW-complexes to spectra, defined as*

$$Z \longmapsto \mathbb{K}^{-\infty}(g_\infty w\mathcal{C}^G(Z^{cc}, R)),$$

is a G-equivariant homology theory. Its coefficients are the non-connective algebraic K-theory spectra $G/H \mapsto \mathbb{K}^{-\infty}(R[H])$.

This description allows the following conclusion (Lemma 7.33):

Theorem C. *The assembly map (†) is an isomorphism if and only if*

$$\mathbb{K}^{-\infty}\big(w\mathcal{C}^G\big((E_{\mathcal{VC}yc}G)^{cc}, R\big)\big)$$

is contractible.

We follow the work of [BLR08], so our work makes it possible to attack the Farrell-Jones Conjecture for simplicial rings with the same methods which proved to be successful for the case of discrete rings.

We refrain from explaining the assembly map here but refer to the introduction of Chapter 7 for the definition of a G-equivariant homology theory and to Section 7.6 for the definition of the assembly map.

Motivational Background. Simplicial rings are important in the study of Waldhausen's A-theory ([Wal85]), an early result is Goodwillie's calculation of the rationalized homotopy fiber of the A-theory of a map of topological spaces [Goo86]. Waldhausen's A-theory of a space X, its long name being *algebraic K-theory of topological spaces* of X, is a variant of algebraic K-theory of the group ring $\mathbb{Z}[\pi_1(X)]$, where π_1 denotes the fundamental group. It takes into account the whole homotopy type of X and is deeply related to the study of higher-dimensional manifolds. It can be viewed as the algebraic K-theory of the ring spectrum $\mathbb{S}[G(X)]$, the "group ring" over the sphere spectrum $\mathbb{S}$, where $G(X)$ is the Kan Loop Group of X. The Kan Loop Group of a connected topological space X is a topological group which has the homotopy type of the loop space of X. There is an assembly map for A-theory. It is proved in [CPV98], using techniques related to the techniques of [BLR08], that for a certain class of groups, smaller than the one mentioned above, the assembly map for A-theory is injective. This uses results of Vogell [Vog90, Vog95]. As ring spectra are a further generalization of simplicial rings, the results in this thesis can be viewed as a step to unify this two approaches.

Next we motivate the main notions of this thesis before we give an outline.

Review of controlled algebra. The proof of the Farrell-Jones Conjecture for word-hyperbolic groups uses "controlled algebra". We explain the idea of controlled algebra in a simple example. Assume we have a discrete ring R. Let $\mathbb{Z}$ be the integers and consider the standard euclidean metric on it. Let M be a projective R-module with a direct sum decomposition $\bigoplus_{i\in\mathbb{Z}} M_i$, such that each M_i is finitely generated and projective. A morphism $f\colon M \to M'$ is a matrix with row and columns indexed over $\mathbb{Z}$ and the entries at (i,j) being maps $f_{i,j}\colon M_i \to M'_j$. We say that f is *controlled* if the matrix is concentrated around the diagonal, i.e. there is an $\alpha \geq 0$ such that $f_{i,j}$ is the zero map for $|i-j| > \alpha$. This gives the *category of controlled R-modules* over $\mathbb{Z}$ with metric control. It is an additive category, so we can take its algebraic K-theory. It turns out ([PW85]) that its K-group K_{l+1} is the algebraic K-group $K_l(R)$ of R for $l \geq 0$. Further its K_0 may be non-trivial. (We usually consider the algebraic K-theory of R as a space $K(R)$, so this construction gives a 1-fold *delooping* of $K(R)$, i.e. a space D_1 with ΩD_1 homotopy equivalent to $K(R)$. The newly arising homotopy group $\pi_0 D_1$ can be non-trivial, so this delooping is *non-connective.*)

We can do this construction for $\mathbb{Z}^n$ with the standard euclidean metric instead of $\mathbb{Z}$ and obtain an n-fold non-connective delooping D_n of $K(R)$. Even more there are natural maps $D_n \to \Omega D_{n+1}$, so we even get a non-connective *spectrum.* This is the *non-connective algebraic K-theory spectrum* of R. Its non-negative homotopy groups are the ordinary algebraic K-groups of R and its negative homotopy groups are the negative algebraic K-groups defined by Bass (cf. [Ros94]).

We can replace $\mathbb{Z}^n$ by the space $\mathbb{R}^n$ together with the euclidean metric. The direct sum decomposition is then indexed over $r \in \mathbb{R}^n$, but we need a finiteness condition to get an interesting category. We require that each point in $\mathbb{R}^n$ has a neighborhood U such that the modules $M_r, r \in U$, are zero for all but finitely many r. It turns out that the *control spaces* $\mathbb{Z}^n$ and $\mathbb{R}^n$ give equivalent categories, hence we can construct the delooping also using $\mathbb{R}^n$.

We can vastly generalize the kind of "control spaces" we take as inputs. Besides using any metric space we can axiomatically define what a "control space" should be. This gives much more examples than only metric spaces. We do this in Section 1.2. For any control space we get an additive category and can take its algebraic K-theory. The proof of [BLR08] reduces the Farrell-Jones Conjecture for a discrete ring to the vanishing of the algebraic K-theory of the category of "controlled modules" over a certain control space.

Main notions of the thesis. We can take a slight variation of the above definition of a controlled module over $\mathbb{Z}$. Assume M is a free module with basis $\{e_i \mid i \in I\}$. A map $\kappa\colon I \to \mathbb{Z}$ is the same as a direct sum decomposition of M indexed over $\mathbb{Z}$ such that the basis restricts to a basis for each summand. Hence, if we restrict to free modules, we can define a controlled module over $\mathbb{Z}$ to be a free module with basis indexed over I and a map κ from I to $\mathbb{Z}$. A map $f\colon (M,\kappa) \to (M',\kappa')$ between such modules is then said to be *controlled* if there is an $\alpha \geq 0$ such that each basis element e_i in M is mapped to an element which can be written as the sum of basis elements $e'_j, j \in J$ of M' with $|\kappa(i) - \kappa'(j)| \leq \alpha$ for all $j \in J$. This is the same notion of control as defined above. We again get an additive category of which we can take the algebraic K-theory. As we restricted ourselves to free modules, we cannot expect to get the same K_0. But it is well-known that taking free modules instead of projective modules gives the same K_i for $i \geq 1$ and this is also true in this situation. Again we can apply this construction to the metric spaces $\mathbb{Z}^n$ and $\mathbb{R}^n$, where for the latter we need similar finiteness conditions as above. This also gives us a non-connective delooping of $K(R)$, and because K_0 of the category of controlled modules over $\mathbb{Z}^n$ gets mapped to K_1 of the category over $\mathbb{Z}^{n+1}$, the difference of free or projective modules does not matter, the deloopings agree up to a weak equivalence of spectra.

Modules over simplicial rings are more complicated than modules over discrete rings. Recall, e.g. from Appendix A, that a simplicial ring R is a sequence of rings (with unit) $R_n, n \in \mathbb{N}$, together with structure maps, and a simplicial module M over R is a sequence of abelian groups M_n together with structure maps, where M_n is an R_n-module. To take algebraic K-theory we only consider *cellular* R-modules. They correspond to the free modules over a discrete ring. The idea is to imitate CW-complexes. So we define a *cellular* module to be a module which arises by attaching cells to the zero module. The precise definition of a cellular R-module is given in Section 1.1.

A cellular R-module has a set of cells $\diamond_R M$. They can be considered as elements in M_n for varying n and correspond to the choice of a basis of a free module over a discrete ring. Hence we define a controlled simplicial R-module over $\mathbb{Z}$ to be a cellular R-module together with a map $\kappa_R \colon \diamond_R M \to \mathbb{Z}$, where $\diamond_R M$ is the set of cells of M. Like for discrete modules and CW-complexes the image of a cell under a map of cellular simplicial R-modules is contained in a finite cellular submodule, i.e. a submodule generated by finitely many cells. This gives us a way to say when a map $f \colon (M, \kappa_R) \to (M', \kappa'_R)$ is controlled, namely we want that there is an $\alpha \geq 0$ such that the image of a cell e_i is contained in a submodule generated by cells e'_j with $|\kappa(e_i) - \kappa'(e'_j)| \leq \alpha$. But we also have structure maps, so we impose the corresponding condition that the boundary of a cell e_i of M is contained in such a submodule. This gives us the category of controlled modules over $\mathbb{Z}$. A precise definition is given in Section 1.3. This definition works for any control space X, we denote the category by $\mathcal{C}(X, R)$. For technical reasons we also require all modules to be finite-dimensional.

To take algebraic K-theory of simplicial rings one needs a more general definition than the one for additive categories. The definition we use is Waldhausen's $\mathcal{S}_\bullet$-construction. We review the necessary background briefly in Appendix B. It takes as input a *category with cofibrations and weak equivalences*, so we define a class of cofibrations and a class of weak equivalences in $\mathcal{C}(X, R)$ in Chapter 2 and start to check the axioms. The weak equivalences will be the homotopy equivalences, so we define also the notions of cylinder and homotopies. All this is summarized as Proposition 3.3 in Section 3.1. The category $\mathcal{C}(X, R)$ is the one which is studied in this thesis.

Actually so far we hid an important point, the role of the discrete group G. We want G to act freely on the control space. We further want G to act freely on each cellular module in such a way that it takes cells to cells, we call this action cell-permuting. Finally the control map κ_R should be G-equivariant. This gives the category of G-equivariant controlled simplicial modules, which we denote by $\mathcal{C}^G(X,R)$. It specializes to the one discussed before for G the trivial group. Astonishingly, albeit this is the decisive ingredient in the Farrell-Jones Conjecture, it does not play an important role in our discussions. For most arguments we can simply ignore its presence.

The idea of Germs. We go back to category of discrete controlled modules over $\mathbb{Z}$. Instead of $\mathbb{Z}$ we can also take $\mathbb{N}$ together with the euclidean metric as control space and consider the category of controlled modules over it. It turns out that this category has an Eilenberg swindle, hence its K-theory vanishes. The swindle is constructed from the map $n \mapsto n+1$ on $\mathbb{N}$, it shifts a module "to infinity", and hence we can take the sum over the shifted modules as an Eilenberg functor, as this is still a finite projective module over each $n \in \mathbb{N}$. There is a way to concentrate to "what happens at infinity".

In the case of a discrete ring, a controlled module over $\mathbb{N}$ can be written as a direct sum $M_{[0,i]} \oplus M_{(i,\infty)}$ for each $i \geq 0$, where the index indicates which summands are collected. We want to ignore the first summands systematically. So we say that two maps from M to N are *equivalent* if there is an i such that they agree if restricted to $M_{(i,\infty)}$. This is the *category of germs* at infinity for $\mathbb{N}$. It is again an additive category so we can take its algebraic K-theory. It turns out that this is again a 1-fold non-connective delooping of $K(R)$, i.e. K_l of this category is K_{l-1} of the ring R for $l \geq 1$.

In the case of simplicial R-modules we do not get a direct sum decomposition, but still can consider similar submodules $M_{(i,\infty)}$ of M, see Section 4.1 for a precise definition. Thus we can consider the category of germs and it turns out that we can define the notion of homotopy and hence of homotopy equivalences there. It is not clear whether there is a structure of a category with cofibrations and weak equivalences on the category of germs. But we can simply define a map in $\mathcal{C}(\mathbb{N},R)$ to be a *germwise weak equivalence* if it becomes one in the category of germs. This is a larger class of weak equivalences than the homotopy equivalences, but it also makes $\mathcal{C}(\mathbb{N},R)$ into a category with cofibrations and weak equivalences, hence we can take its K-theory.

Again this notion can be vastly generalized, we can define for any G-equivariant control space X and a set of *germ support sets* on X the germwise weak equivalences and show this gives a class of weak equivalence for $\mathcal{C}^G(X,R)$. This is carried out in Chapter 4.

Outline. Let us give a summary of the content of this thesis. We gave motivations for the crucial definitions above. All the basic definitions are contained in Chapter 1. We define the notions of cellular simplicial module, control space and controlled simplicial module over a control space in Sections 1.1 to 1.3. Section 1.5 contains the G-equivariant versions and in particular the definition of the category $\mathcal{C}^G(X,R)$ which is the object of study of this thesis. Chapter 2 is devoted to construct the structure of a category with cofibrations and weak equivalences on $\mathcal{C}^G(X, R)$, where the weak equivalences are the homotopy equivalences. The construction is finished in Section 3.1. In Sections 3.2 to 3.6 we discuss different finiteness conditions on $\mathcal{C}^G(X, R)$—in particular finite, homotopy finite and homotopy finitely dominated modules—to actually get categories with interesting K-theory. Section 3.7 discusses the connective K-theory of these categories. We prove that the finite and homotopy finite modules have equivalent K-theories and that their K-theories agree with the K-theory of the homotopy finitely dominated modules except at K_0. We show that the K-theory for the G-equivariant control space $G/1$ and the simplicial ring R is the K-theory of $R[G]$. We finish with a proof that a weak equivalence of simplicial rings induces a weak equivalence on the K-theory of the categories of controlled modules.

Chapter 4 is devoted to the discussion of germs. We show that $\mathcal{C}^G(X, R)$ together with the germwise weak equivalences is also a category with cofibrations and weak equivalences. To prove that, we have to redo most proofs of Chapter 2 for the germwise weak equivalences. In that case we only indicate the differences, we wrote Chapter 2 with that in mind.

With the homotopy equivalences and the germwise weak equivalences we have two classes of weak equivalences, hence Waldhausen's generic fibration theorem B.5 gives a homotopy fiber sequence of spaces. We analyze this homotopy fiber sequence for the germs away from a subspace in Section 5.1. This gives a coarse Mayer-Vietoris theorem for control spaces, which we discuss in Section 5.2. Finally Section 5.3 shows that a flasque shift on X like the one $n \mapsto n + 1$ on $\mathbb{N}$ gives an Eilenberg swindle on $\mathcal{C}^G(X, R)$ and hence shows that its K-theory space is contractible. We analyze this contraction carefully and show that compatible shifts give compatible contractions.

The theorems from Chapter 5 have a "defect" at K_0 as the K-theory we take there is connective. We do not have a direct definition of *non-connective* K-theory of a category with cofibrations and weak equivalences, so we define $\mathbb{K}^{-\infty}(w\mathcal{C}^G(X, R))$ as the spectrum with nth space $K(w\mathcal{C}^G(X \times \mathbb{R}^n, R))$ in Section 6.1. We show, using the results from Chapter 5, the following theorems. (They are restated in Section 6.1.)

Theorem. *Let R be a simplicial ring and X a control space. There is a non-connective algebraic K-theory spectrum*

$$\mathbb{K}^{-\infty}(w\mathcal{C}^G(X, R)).$$

Its structure maps are isomorphism on π_k for $k \geq 1$, so it is almost an Ω-spectrum. It agrees on the ith stable homotopy group π_i, $i \geq 1$ with the connective K-theory $K(w\mathcal{C}^G(X, R))$.

Theorem. *Let X be a control space and $Y \subseteq X$ a "good" closed subspace. Denote by gw the germwise equivalences on X away from Y. There is a homotopy fiber sequence of spectra*

$$\mathbb{K}^{-\infty}(w\mathcal{C}^G(Y, R)) \to \mathbb{K}^{-\infty}(w\mathcal{C}^G(X, R)) \to \mathbb{K}^{-\infty}(gw\mathcal{C}^G(X, R)).$$

This homotopy fiber sequence is our generalization of the homotopy fiber sequence obtained from a "Karoubi filtration" of additive categories in [CP97].

Theorem. *Assume X has a flasque shift. Then $\mathbb{K}^{-\infty}(w\mathcal{C}^G(X,R))$ is contractible.*

Theorem. *A weak equivalence of simplicial rings $R \to S$ induces a weak equivalence of spectra*

$$\mathbb{K}^{-\infty}(w\mathcal{C}^G(X,R)) \to \mathbb{K}^{-\infty}(w\mathcal{C}^G(X,S)).$$

We further prove a coarse Mayer-Vietoris theorem. These theorems are versions for non-connective K-theory of the theorems from Chapter 5 which were mentioned above. They do not have the "defect" at K_0 any more. Chapter 6 is devoted to their proofs, Section 6.1 contains a complete summary of our results on non-connective K-theory including the precise statements of the Theorems above.

Following the idea from [BFJR04] we construct in Chapter 7 for a simplicial ring R a homology theory with coefficients in $\mathbf{K}_R$, the G-equivariant non-connective algebraic K-theory spectrum of R. We calculate that it has the right coefficients and compare it to the connective K-theory of R in Section 7.5. In Section 7.6 we finally construct the assembly map and state the Farrell-Jones Conjecture for a simplicial ring. We prove Theorem C from above as Lemma 7.33, which reduces the conjecture to the vanishing of the algebraic K-theory of the control space $(E_{\mathcal{F}}G)^{cc}$. It is the same control space which appears in [BLR08], up to an insignificant change, see Remark 7.34. As we believe the methods from there apply, we state this vanishing as a conjecture.

Appendix A contains a brief summary on simplicial sets and simplicial rings. Appendix B contains a review of Waldhausen's algebraic K-theory of spaces, which is the construction of algebraic K-theory we use in this thesis. Appendix C investigates mapping telescopes in the simplicial setting, this is a tool we need in the proof of Lemma 5.10.

Relations to simplicial modules. If we take G the trivial group and the one-point control space, all our constructions yield constructions in the category of finite-dimensional cellular simplicial R-modules. We strongly suppose that all of the results are well-known for (uncontrolled) cellular simplicial R-modules, but there does not seem to be a canonical reference available. There is an interesting technical point: We do not have homotopy groups available in the category $\mathcal{C}^G(X,R)$, hence we are forced to do all proofs without referring to homotopy groups. Thus a byproduct of our constructions we get proofs of foundational homotopy theoretic facts about cellular simplicial R-modules without using homotopy groups. Notable are the gluing lemma for homotopy equivalences (Lemma 3.8) and the Extension Axiom. Further our proof that a weak equivalence of simplicial rings induces an equivalence on K-theory uses only the $\mathcal{S}_\bullet$-construction. All the proofs in the literature the author is aware of use the plus-construction description of K-theory of simplicial rings, a tool which does not seem to be available in our category. (Note however that the plus-construction style proofs give the stronger result that an n-connected map of simplicial rings gives an $(n+1)$-connected map on algebraic K-theory. This kind of result does not seem to be readily available in our setting.) Our results also give a construction for a non-connective algebraic K-theory spectrum for any simplicial ring.

Remarks for the knowledgeable reader. We give some extra explanations for readers familiar with [BLR08]. Although the constructions and theorems of this thesis follow closely the ideas of [BFJR04] and [BLR08] we construct everything from scratch. We need to modify some definitions and constructions; partly to streamline some constructions, partly to correct some small mistakes. We elaborate on some points in the order they are treated here.

In the definition of a control space (Definition 1.6) we require the diagonal to be contained in any control condition $E \in \mathcal{E}$. This assures $Y \subseteq Y^E$ for every Y and implies $E \cup E' \subseteq E \circ E'$ for two control conditions E, E' (Remark 1.10).

The definition of *locally finite* differs between [BFJR04] and [BLR08], but the latter refers to the former for the constructions and theorems. We want to use the latter definition, so compared to [BFJR04] we need some extra conditions when we look at subspaces of a control space. These extra conditions will be satisfied in all our applications. We introduce the notion of a *froper* subspace (Definition 5.3). This is a technical condition we need on the subspaces of control spaces which occur in the homotopy fiber sequence and the coarse Mayer-Vietoris theorem. Corresponding assumptions should have been made in [BLR08].

Further it is convenient to introduce the notion of a *set over a control space X* and in particular of a *locally finite set* (Definition 3.17). This allows to hide the complexity of a locally finite controlled module behind the simpler notion of a locally finite set, which makes some proofs easier to digest.

The homotopy fiber sequence with germs of Theorem 6.4 is a precise analog of the homotopy fiber sequence one gets for a Karoubi filtration of an additive category. Each choice of germ support sets on a control space X gives such a Karoubi filtration of the categories of [BLR08]. There are Karoubi filtrations which do not arise from such a situation, but it seems that no such ones are used in [BLR08]. In that sense our Theorem 6.4 generalizes the article [CP97] for all our applications.

We make the condition of the existence of an Eilenberg swindle of Proposition 4.4 of [BFJR04] into an axiomatic definition and call it a *flasque shift* (Definition 5.22). We further discuss compatibility of the induced contractions to actually get a map of spectra.

The summary in Section 6.1 covers the same range of tools which are repeated in Section 3 of [BLR08], in particular we reprove all of it in our setting.

The definition of a G-equivariant homology theory in [BFJR04] contains some inconsistencies, we tried to give a more consistent definition in Chapter 7, cf. Remark 7.2.

Notations and conventions

A spectrum is always a spectrum of topological spaces in the sense of Bousfield-Friedlander [BF78], i.e. a sequence $X_i, i \in \mathbb{N}$, of topological spaces with maps $\Sigma X_i \to X_{i+1}$. An Ω-spectrum is a spectrum whose adjoint structure maps $\Omega X_i \to X_{i+1}$ are weak equivalences. We assume familiarity with the homotopy theory of spectra.

We also freely use the language of simplicial sets, simplicial rings and simplicial modules, see [GJ99] or Appendix A for a brief review.

The third main tool we use is Waldhausen's algebraic K-theory of spaces [Wal85], we give a very brief summary of the results we need in Appendix B.

We sometimes use the property that for a diagram in a category

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{I}"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{II}"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

the whole diagram I + II is a pushout if I and II are pushouts and II is a pushout if I and I + II are pushouts. The dual version is proved in [Bor94a, I.2.5.9], the third possible implication does not hold in general.

The set of natural numbers $\mathbb{N}$ contains zero. All rings have a unit.

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1. Definitions

1.1. Simplicial Modules

We give a concise summary of the theory of simplicial modules we need. See Appendix A for a bit more details.

Let $\Delta = \{[n] \mid n \in \mathbb{N}\}$ be the category of finite ordinals and order-preserving morphisms. Let $\mathcal{R}ings$ be the category of rings with unit and let $c\mathcal{R}ings$ the category of commutative rings with unit. Let Ab be the category of abelian groups.

Definition 1.1. *A simplicial abelian group is a functor* $\Delta^{\mathrm{op}} \to Ab$. *A simplicial ring* R *is a functor* $\Delta^{\mathrm{op}} \to \mathcal{R}ings$. *A simplicial ring* R *is called commutative if it factors over* $c\mathcal{R}ings$.

Definition 1.2. *A simplicial left module over* R *is a simplicial abelian group* M *together with a bilinear map* $\mu\colon R \times M \to M$ *of simplicial abelian groups, the left multiplication satisfying the usual associativity and unitality diagrams. A simplicial right* R*-module is a simplicial abelian group* M *together with a bilinear map* $\mu\colon M \times R \to M$, *the right multiplication, which satisfies the corresponding dual diagrams for associativity and unitality.*

Alternative characterizations of simplicial R-modules are given in Appendix A.2. If M is a simplicial left R-module it has the structure of a simplicial right R^{op}-module, where R^{op} is the opposite ring. So usually we only treat left modules. If R is commutative the notion of simplicial left and right modules over R agree. The category of simplicial left R-modules has all limits and colimits and they are formed in simplicial abelian groups. We denote the coproduct of R-modules M^k by $\coprod_k M^k$ and do not use the sum notation.

Let M be a right R-module, N be a left R-module, then the tensor product $M \otimes_R N$ is defined. It is a simplicial abelian group. It gives a bifunctor $M, N \mapsto M \otimes_R N$ which is a left adjoint in each variable, so it commutes with colimits. Each simplicial abelian group is a left and right $\mathbb{Z}$-module, where $\mathbb{Z}$ is considered as simplicial ring, and each left R-module is a R-$\mathbb{Z}$-bimodule, i.e. it has a right $\mathbb{Z}$-multiplication compatible with the left R-multiplication. If M is an R-S-bimodule and N a left S-module then $M \otimes_S N$ is a left R-module.

For A a simplicial set $\mathbb{Z}[A]$ is the free simplicial abelian group on A ([GJ99, p. 4]). For M a simplicial left R-module define $M[A]$ as the simplicial left R-module $M \otimes_{\mathbb{Z}} \mathbb{Z}[A]$. For M, N simplicial left R-modules define $\mathrm{HOM}_R(M, N)$ as the simplicial abelian group $[n] \mapsto \mathrm{Hom}_R(M[\Delta^n], N)$. If M is an R-S-bimodule $\mathrm{HOM}_R(M, N)$

inherits a *left* S-multiplication. The functor $M \otimes_R -$ is left adjoint to $\mathrm{HOM}_R(M, -)$. Both $M \mapsto M[A]$ and $A \mapsto M[A]$ commute with colimits.

The simplicial left R-module M arises *by attaching an n-dimensional R-cell* from M' if there is a map $R[\partial\Delta^n] \to M'$ such that M is isomorphic to the pushout

$$\begin{tikzcd}
R[\partial\Delta^n] \arrow[r] \arrow[d] & M' \arrow[d] \\
R[\Delta^n] \arrow[r] & M' \cup_{R[\partial\Delta^n]} R[\Delta^n]
\end{tikzcd}$$

of R-modules. There is an adjunction (i.e. a natural bijection)

$$\mathrm{Hom}_R(M[A], N) \rightleftarrows \mathrm{Hom}_{sSet}(A, \mathrm{HOM}_R(M, N)). \qquad (1)$$

It follows that one can always attach cells of the same dimension simultaneously and lower dimensional cells can be attached first. We use the terms n-dimensional R-cell and n-cell interchangeably. (See also Section 1.3 for some more details on this adjunction.)

Definition 1.3. *A map $f\colon N \to M$ of simplicial R-modules is said to be a cellular inclusion (or cellular) if M arises from N by attaching cells and f is the inclusion of N. A simplicial left R-module M is called* cellular *if the map $* \to M$ from the trivial R-module to M is a cellular map. We write cellular inclusions as $N \rightarrowtail M$.*

If $N \rightarrowtail M$ is a cellular inclusion we can write M as colimit of the sequence of cellular inclusions

$$M_{-1} \rightarrowtail M_0 \rightarrowtail M_1 \rightarrowtail \cdots \rightarrowtail M_i \rightarrowtail \cdots$$

with $M_{-1} = N$ and M_i is the pushout in

$$\begin{tikzcd}
\coprod_k R[\partial\Delta^i] \arrow[r, "\coprod_k \partial e_k"] \arrow[d, tail] & M_{i-1} \arrow[d, tail] \\
\coprod_k R[\Delta^i] \arrow[r, "\coprod_i e_k"] & M_i
\end{tikzcd} . \qquad (2)$$

The maps $\partial e_k\colon R[\partial\Delta^i] \to M_{i-1} \rightarrowtail M$ are called the *attaching maps* and the maps $e_k\colon R[\Delta^i] \to M_i \rightarrowtail M$ are called the *i-cells of M*. If M is cellular then the M_i are uniquely determined and M_i is called the *i-skeleton* of M. A choice of the above pushouts for each i is called a *cellular structure* on M. It determines M up to isomorphism. Such a choice is almost never unique. From the cellular structure we get the n-cells $R[\Delta^n] \to M$, which on the other hand determine the cellular structure. For a cellular simplicial R-module we always choose a fixed cellular structure, i.e. the module comes with a choice of cells. We say that M is *of dimension n* or *of finite dimension* if it has only cells of dimension smaller or equal than n. We say M is *finite* if it has only finitely many cells.

Lemma 1.4. *Let $A \to B$ be an inclusion of simplicial sets. Let M be a cellular simplicial R-module. Then $M[A] \to M[B]$ is a cellular inclusion.*

Proof. We claim that $M[\partial\Delta^n] \to M[\Delta^n]$ is a cellular inclusion. This follows because $R[\Delta^m \times \Delta^n]$ arises from $R[\partial\Delta^m \times \partial\Delta^n]$ by attaching $(n \cdot m)$- and $(n \cdot m - 1)$-cells. As B arises from A by attaching cells the claim follows. $\square$

It follows that for a cellular module M all modules $M[A]$ are cellular.

Remark 1.5. Table 1.1 shows how the simplicial case specializes to the case of discrete R-modules. Recall that a simplicial set is called *discrete* if the only non-degenerate simplices have dimension 0. A discrete simplicial ring is essentially the same as an ordinary ring known from algebra, so we call these *discrete rings* if we want to distinguish them from simplicial rings.

simplicial R-modules, R simplicial ring	discrete R-modules, R discrete ring
cellular module	free module
cellular structure	choice of a basis
cellular inclusion	direct summand with free complement
$M[A]$ (A a simplicial set)	$\bigoplus_{a\in A} M$ (A a set)
$\coprod_I R[\Delta^n]$	$\bigoplus_I R$
of finite dimension	—
finite	finite dimensional

Table 1.1.: How the notions for simplicial rings specialize to discrete rings.

The simplicial notions specialize to the algebraic notions if R is discrete and we only attach 0-cells. So in some sense the algebraic theory is contained in the simplicial theory.

1.2. Control

It is hard to motivate the notion of control before actually giving the definition. Maybe the most elementary non-trivial situation where "control phenomena" arise is the following: Given A and B topological spaces and let $H\colon A \times I \to B$ be a homotopy of maps $A \to B$. In some sense H is moving a lot of points around in B, namely H gives paths from $H(a, 0)$ to $H(a, 1)$ in B for all $a \in A$. We want to talk about when H "does not move the points too far away from each other, but in a uniform way". The simplest case would be to give a metric on B and require that the distance between $H(a, 0)$ and $H(a, t)$ for all $t \in [0, 1]$ is bounded by a global constant α. The following constructions give a framework for doing this kind of arguments. See the introduction of [BLR08] for references for "controlled topology". Note that the given example is only for motivation, we want to apply control to simplicial modules instead.

The following is adapted from [BFJR04, 2.3ff.].

Definition 1.6. *Let X be a topological Hausdorff space. A morphism control structure on X consists of a set $\mathcal{E}$ of subsets E of $X \times X$ (i.e. relations on X), called the morphism control conditions. We require:*

(i) For $E, E' \in \mathcal{E}$ there is an $\overline{E} \in \mathcal{E}$ such that $E \circ E' \subseteq \overline{E}$ where "$\circ$" is the composition of relations.

(ii) For $E, E' \in \mathcal{E}$ there is an $E'' \in \mathcal{E}$ such that $E \cup E' \subseteq E''$.

(iii) Each $E \in \mathcal{E}$ is symmetric, i.e. $(x, y) \in E \Leftrightarrow (y, x) \in E$.

(iv) The diagonal $\Delta \subseteq X \times X$ is a subset of each $E \in \mathcal{E}$.

For convenience we usually assume that the diagonal Δ is itself in $\mathcal{E}$. To keep the notation simpler we also often assume $E \circ E' \in \mathcal{E}$, cf. Remark 1.10.

Remark 1.7. We require X to be a topological space, but this is only used later to define the notion of locally finiteness in Section 3.4. It is convenient to include it in the definition from the beginning. The Hausdorff property is used only in the proof of Lemma 3.22 in Section 3.4. If one wants to ignore the topology one can assume that X has the discrete topology. There is relatively few interaction between the control structure and the topology, the most notably one being the notion of a *proper control space*. On the other hand the definition of *continuous control* starts with a topological space Z and defines a control structure on $Z \times [1, \infty)$, thus in this case the control space comes with a distinguished topology. The notions of a proper control space and continuous control are defined later. The topology can be ignored until Section 3.4.

For $U \subseteq X$ a subset and $E \in \mathcal{E}$ we define the *E-thickening* of U in X as

$$U^E = \{x \in X \mid \exists y \in U \colon (x, y) \in E\}.$$

An arbitrary relation $\alpha \in X \times X$ is called *E-controlled* for $E \in \mathcal{E}$ if $\alpha \subseteq E$. It is called *controlled* if there is an $E \in \mathcal{E}$ such that it is E-controlled. An example of a relation is the graph of a (not necessarily continuous) map $f \colon X \to X$, so f is called (E-)controlled if the graph of f is (E-)controlled. Note that if f is E-controlled then the image of $U \subseteq X$ under f is contained in U^E.

Definition 1.8. *Given X and a morphism control structure $\mathcal{E}$ on X. An object support structure on $(X, \mathcal{E})$ is a set $\mathcal{F}$ of subsets F of X, called the object support conditions. We require:*

(i) For $F, F' \in \mathcal{F}$ there is an $F'' \in \mathcal{F}$ such that $F \cup F' \subseteq F''$.

(ii) For $F \in \mathcal{F}$ and $E \in \mathcal{E}$ there is an $F''' \in \mathcal{F}$ such that $F^E \subseteq F'''$.

We call $(X, \mathcal{E}, \mathcal{F})$ a *control space*. There are cases where we are not interested in object support conditions $\mathcal{F}$ so we leave them out, which is the same as setting $\mathcal{F} = \{X\}$.

The control space $(X, \mathcal{E}, \mathcal{F})$ is called *proper* if for each $K \subseteq X$ compact, for $E \in \mathcal{E}$ and $F \in \mathcal{F}$ there is a $K' \subseteq X$ compact such that $(F \cap K)^E \cap F \subseteq K'$.

A map of control spaces from $(X_1, \mathcal{E}_1, \mathcal{F}_1)$ to $(X_2, \mathcal{E}_2, \mathcal{F}_2)$ is a (not necessary continuous) map $f\colon X_1 \to X_2$ such that for each $E_1 \in \mathcal{E}_1$ and $F_1 \in \mathcal{F}_1$ there are $E_2 \in \mathcal{E}_2$ and $F_2 \in \mathcal{F}_2$ with $(f \times f)(E_1) \subseteq E_2$ and $f(F_1) \subseteq F_2$.

Examples 1.9. (cf. [BFJR04, Section 2.3]) As said, setting $\mathcal{F} := \{X\}$ has the same effect as leaving out the object support conditions, so we abbreviate $(X, \mathcal{E}, \{X\})$ as $(X, \mathcal{E})$.

(i) (metric control). Let X have a metric d. Then

$$\mathcal{E}_d = \{E \mid \text{there is an } \alpha \text{ such that } E = \{(x, y) \mid d(x, y) \leq \alpha\}\}$$

is a morphism control structure on X.

(ii) (continuous control). Let Z be a topological space and $[1, \infty)$ the half-open interval with closure $[1, \infty]$. Define a morphism control structure $\mathcal{E}_{cc}$ on $X := Z \times [1, \infty)$ as follows. E is in $\mathcal{E}_{cc}$ if it is symmetric and

  (a) For every $x \in Z$ and each neighborhood U of $x \times \infty$ in $Z \times [1, \infty]$ there is a neighborhood $V \subseteq U$ of $x \times \infty$ in $Z \times [1, \infty]$ such that $E \cap ((X \setminus U) \times V) = \varnothing$.

  (b) $p_{[1,\infty)} \times p_{[1,\infty)}(E) \in \mathcal{E}_d([1, \infty))$, where d is the standard euclidean metric on $[1, \infty)$ and $p_{[1,\infty)}$ is the projection to $[1, \infty)$.

(iii) (products). If $(X_1, \mathcal{E}_1, \mathcal{F}_1)$ and $(X_2, \mathcal{E}_2, \mathcal{F}_2)$ are control spaces, then $(X_1 \times X_2, \mathcal{E}_1 \times \mathcal{E}_2, \mathcal{F}_1 \times \mathcal{F}_2)$ is one, where by a misuse of notation $\mathcal{E}_1 \times \mathcal{E}_2 := \{E_1 \times E_2 \mid E_1 \in \mathcal{E}_1, E_2 \in \mathcal{E}_2\}$ and $\mathcal{F}_1 \times \mathcal{F}_2 := \{F_1 \times F_2 \mid F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$.

(iv) (pullbacks). If $(X_2, \mathcal{E}_2, \mathcal{F}_2)$ is a control space and $f\colon X_1 \to X_2$ an arbitrary map, then $(X_1, f^{-1}(\mathcal{E}_2), f^{-1}(\mathcal{F}_2))$ is one, where

$$f^{-1}(\mathcal{E}_2) := \{E' \mid \exists E \in \mathcal{E}_2 \colon (x, y) \in E' \text{ if and only if } (f(x), f(y)) \in E\}$$

and similar $f^{-1}(\mathcal{F}_2) := \{f^{-1}(F) \mid F \in \mathcal{F}_2\}$.

(v) (intersection). If $(X, \mathcal{E}_1, \mathcal{F}_1)$ and $(X, \mathcal{E}_2, \mathcal{F}_2)$ are control spaces, then $(X, \mathcal{E}_1 \cap \mathcal{E}_2, \mathcal{F}_1 \cap \mathcal{F}_2)$ is one where we misuse the notation and set $\mathcal{E}_1 \cap \mathcal{E}_2 := \{E_1 \cap E_2 \mid E_1 \in \mathcal{E}_1, E_2 \in \mathcal{E}_2\}$ and $\mathcal{F}_1 \cap \mathcal{F}_2 := \{F_1 \cap F_2 \mid F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$.

(vi) (uncontrolled). $(X, \{X \times X\}, \{X\})$ is a control space which imposes no control condition at all.

(vii) (compact support). Let set $F \subseteq X$ be in $\mathcal{F}_c$ if it is compact. These are the compact object support conditions. They are object support conditions for $(X, \mathcal{E}_d)$ where X is a proper metric space (i.e. closed balls are compact) or for the continuous control conditions $\mathcal{E}_{cc}$ on $Z \times [1, \infty)$.

Remark 1.10. Note that as we require morphism control conditions to be symmetric we have $E \circ E' = E' \circ E$ for $E, E' \in \mathcal{E}$. Note further that $\Delta \subset E$ implies $E \subseteq E \circ E'$ and $Y \subseteq Y^E$. So in particular $E \cup E' \subseteq E \circ E'$, so Definition 1.6(iv) and (i) imply 1.6(ii). We leave (ii) in the definition for the analogy to [BFJR04, 2.3].

In many examples we have $E \circ E' \in \mathcal{E}$, but that does not need to hold always, a counterexample is at the end of Section 1.3.

If $\overline{\mathcal{E}} \subseteq \mathcal{E}$ is cofinal, i.e. for each $E \in \mathcal{E}$ there is an $\overline{E} \in \overline{\mathcal{E}}$ with $E \subseteq \overline{E}$, then in all our applications $\overline{\mathcal{E}}$ and $\mathcal{E}$ give equivalent control structures, which means that the categories $\mathcal{C}(X, R, \mathcal{E}, \mathcal{F})$ and $\mathcal{C}(X, R, \overline{\mathcal{E}}, \mathcal{F})$ defined below are equivalent (or even equal). Hence we can always pass to a cofinal subset of $\mathcal{E}$, or in the other direction assume that every E' for which there is an $E \in \mathcal{E}$ with $\Delta \subseteq E' \subseteq E$ is also in $\mathcal{E}$. This is sometimes convenient to have and we will use this freely. Assuming $E \circ E' \in \mathcal{E}$ for $E, E' \in \mathcal{E}$ is an example of that.

1.3. Controlled simplicial modules

From now on all modules are simplicial left R-modules for R a simplicial ring.

Let M be a cellular simplicial R-module. We always assume that M has a chosen cellular structure. This determines the n-cells of M which are by definition maps $R[\Delta^n] \to M$. By the adjunction (1) a map $\alpha\colon R[\Delta^n] \to M$ of simplicial R-modules is the same as a map $\widetilde{\alpha}\colon \Delta^n \to M$ of simplicial sets, where in turn this map is determined by the image $\widetilde{\alpha}(\mathrm{id}_{[n]})$ of the generating simplex of Δ^n. So an n-dimensional R-cell of M gives an element in M_n which determines the cell. In particular the cellular structure on M gives a subset of elements in $\coprod_{n \in \mathbb{N}} M_n$; we denote this subset by $\diamond_R M$ and call it the *cells of* M, imitating [Wei02, II.6.1]. For a cell $e \in \diamond_R M$ we usually denote its associated *characteristic map* $R[\Delta^n] \to M$ by $\overline{e}$. Up to isomorphism, $\diamond_R M$ and the boundaries of its elements determine M.

Remark 1.11 (Motivation). We want to construct algebraic K-Theory following Waldhausen. For this one needs a suitable category of “cofibrant” objects. For a discrete ring R these are the free or the projective modules. For a simplicial ring R a suitable category is the category of cellular R-modules.

In the setting of [BFJR04] a “module over X” for a discrete ring S can be considered as an S-module M together with a direct sum decomposition $M \cong \bigoplus_{x \in X} M^x$. Each subset $U \subseteq X$ determines a direct summand $\bigoplus_{x \in U} M^x$ of M. A map $f\colon M \to N$ between such S-modules over X has associated some “control data”, namely $(a, b) \in X \times X$ is in the “support of f” if and only if

$$M^a \to \bigoplus_{z \in X} M^z \xrightarrow{f} \bigoplus_{y \in X} M^y \to M^b$$

is not the zero map. We then require that the support of f is contained in some $E \in \mathcal{E}$. While for free or projective S-modules each short exact sequence splits, not every short exact sequence of cellular simplicial R-modules (exactness defined degreewise) splits, as simplicial modules have additionally structure maps, in particular face maps δ_i^*, $\delta_i\colon [n-1] \rightarrowtail [n]$. Also some of the "geometry" of X should play a role when considering the cellular simplicial modules over X, so it is not enough to define a cellular simplicial module over X to be a direct sum of cellular simplicial modules $(M_n^x)_n$ for each $x \in X$, we want to have face maps which have a "distance". Hence the face maps δ_i^* also have to be "controlled" and moreover all structure maps should be. The way to require this is the following.

A cellular R-module M is "generated" by the cells $\diamond_R M$. The way to make M an E-controlled module over X for $E \in \mathcal{E}$ is to choose for each cell $e \in \diamond_R M$ an element $\kappa_R(e) \in X$. Then one considers the map of R-modules $\overline{e}\colon R[\Delta^n] \to M$ and require that $\overline{e}$ hits only elements in M which are "over $y \in X$" with $(\kappa_R(e), y) \in E$. Note that a general element in M is a linear combination of (degeneracies of) cells, so it does not "live over one $y \in X$"; the general strategy is to say that if $m = n + e$ in M then the support of m is contained in the union of the support of n and e. We make this precise now.

Definition 1.12. *Let $(X, \mathcal{E}, \mathcal{F})$ be a control space. A general R-module over X is a cellular R-module M together with a map $\kappa_R\colon \diamond_R M \to X$.*

A boundary of a simplex in $\diamond_R M$ does not need to be contained in $\diamond_R M$. Define the set of simplices of M as $\diamond M := \coprod_{n\in\mathbb{N}} M_n$. Each element $a \in M_m \subseteq \diamond M$ has a unique representation as

$$a = \sum_{e_i \in I(a)} r_i \cdot \sigma_i^* e_i \tag{3}$$

with $I(a) \subseteq \diamond_R M$ finite, $r_i \in R_m$, $r_i \neq 0$, $\sigma_i\colon [m] \twoheadrightarrow [n_i]$ surjective and $e_i \in \diamond_R M \cap M_{n_i}$. We can interpret I as a map $\diamond M \to \diamond_R M$. We extend $\kappa_R\colon \diamond_R M \to X$ to a map $\kappa\colon \diamond M \to \mathcal{P}(X)$ by setting

$$\kappa(a) := \kappa_R(I(a)) = \{\kappa_R(e_i) \mid i \in I(a)\}.$$

For $\sigma\colon [m] \twoheadrightarrow [n]$ surjective and $e \in \diamond_R M \cap M_n$ we have $\kappa(\sigma^*(e)) = \kappa(e) = \{\kappa_R(e)\}$. This means that degeneracies of cells are automatically "controlled". This is not true for boundaries of a cell, so we have to require it: Let e be an n-cell of M and $\delta\colon [l] \rightarrowtail [n]$ be injective. This gives a relation $\mathcal{R}(e,\delta) := \kappa_R(\{e\}) \times \kappa_R(I(\delta^*(e))) \subseteq X \times X$ on X. We get a bigger relation $\mathcal{R}(e)$ by taking the union over all injective maps $\delta\colon [l] \rightarrowtail [n]$ and all l, so set

$$\mathcal{R}(e) := \bigcup_{l,\, \delta:[l]\rightarrowtail[n]} \mathcal{R}(e,\delta) = \kappa_R(\{e\}) \times \kappa_R\Bigg(\bigcup_{l,\, \delta:[l]\rightarrowtail[n]} I\big(\delta^*(e)\big)\Bigg).$$

Then $\mathcal{R}(e) \subseteq X \times X$ measures the "distance" of e and its boundaries over the control space X.

Definition 1.13. *Let $(X, \mathcal{E}, \mathcal{F})$ be a control space. A controlled R-module (M, κ_R) over X is a general R-module (M, κ_R) over X such that there are $E \in \mathcal{E}$ and $F \in \mathcal{F}$ with* *(i) (boundary control):*

$$\bigcup_{e \in \diamond_R M} \mathcal{R}(e) \subseteq E$$

(ii) (support):

$$\kappa_R(\diamond_R M) \subseteq F.$$

We say (M, κ_R) is E-controlled and has support in F. We often leave κ_R understood.

Let (M, κ_R), (M', κ'_R) be general R-modules over X and $f \colon M \to M'$ a map of simplicial R-modules. For each $e \in \diamond_R M$ we get a relation $\mathcal{R}_f(e)$ by setting $x \sim y$ if $x = \kappa_R(e)$ and $y \in \kappa'(f(e))$.

Definition 1.14. *Let $(X, \mathcal{E}, \mathcal{F})$ be a control space. Let (M, κ_R), (M', κ'_R) be controlled R-modules over X and $f \colon M \to M'$ be a map of simplicial R-modules. Let $E \in \mathcal{E}$.*

The map f is E-controlled if

$$\bigcup_{e \in \diamond_R M} \mathcal{R}_f(e) \subseteq E.$$

The map f is controlled if there is an $E \in \mathcal{E}$ such that f is E-controlled.

Remark 1.15. If $f \colon M \to M'$ is E-controlled and $g \colon M' \to M''$ is E'-controlled, then $g \circ f$ is $E' \circ E$-controlled. If $f_1, f_2 \colon M \to M'$ are E_1-, resp. E_2-controlled and $E_1 \cup E_2 \subseteq E_3$, then $f_1 + f_2$ is E_3-controlled. A map $f \colon M \to M'$ which is Δ-controlled is a map "over X", i.e. it maps cells over $x \in X$ to a linear combination of cells over x.

Definition 1.13 of control for a cellular simplicial R-module only works well if the module in question is finite-dimensional, i.e. it has only cells up to a certain dimension, see Section 1.1. Hence we require finite-dimensionality for all modules we consider. We make the following definition.

Definition 1.16. *Let $(X, \mathcal{E}, \mathcal{F})$ be a control space. The finite-dimensional controlled R-modules over X together with the controlled morphisms form a category which we denote by*

$$\mathcal{C}(X, R, \mathcal{E}, \mathcal{F}).$$

A map $\vartheta \colon (X, \mathcal{E}, \mathcal{F}) \to (X', \mathcal{E}', \mathcal{F}')$ of control spaces induces a functor ϑ_* from $\mathcal{C}(X, R, \mathcal{E}, \mathcal{F})$ to $\mathcal{C}(X', R, \mathcal{E}', \mathcal{F}')$ which sends (M, κ_R) to $(M, \vartheta \circ \kappa_R)$.

For the following we fix a control space $(X, \mathcal{E}, \mathcal{F})$. We abbreviate $\mathcal{C}(X, R, \mathcal{E}, \mathcal{F})$ by $\mathcal{C}(X, R)$, by $\mathcal{C}(X)$ or simply by $\mathcal{C}$. Let $(M, \kappa_R) \in \mathcal{C}(X, R)$ be a controlled R-module, let A be a finite-dimensional simplicial set. The map $p \colon A \to *$ induces a map $p_* \colon M[A] \to M$ of simplicial R-modules which maps cells to cells, i.e. $\diamond_R M[A]$ to $\diamond_R M$. Therefore $(M[A], \kappa_R \circ p_*)$ is a controlled R-module and the map p_* is Δ-controlled. More generally for every map $p \colon A \to B$ of simplicial sets we get a map $M[p]\colon M[A] \to M[B]$ which is Δ-controlled if $M[A]$ and $M[B]$ have the just defined control maps.

Let $\underline{R}$ be R considered as left R-module. Define for $x \in X$ the control map $\kappa^x_{\underline{R}}\colon \diamond_R \underline{R} \to X$, ($\diamond_R \underline{R} = \{1_R\}$), by $\kappa^x_{\underline{R}}(1_R) := x$. This gives a controlled R-module $(\underline{R}, \kappa^x_{\underline{R}})$. This also gives controlled R-modules $(R[\Delta^n], \kappa^x_{\underline{R}} \circ p_*)$, which we denote by $\underline{R[\Delta^n]}_x$ for the next construction. In general we write them as $R[\Delta^n]$ and leave the control map and the x understood. This works for more general simplicial sets than Δ^n, we will use it e.g. for $\partial\Delta^n$ below.

Definition 1.17. *A Δ-controlled map of controlled R-modules over X is called a cellular inclusion if it is one after forgetting the control.*

The next characterization is clear but often helpful.

Lemma 1.18. *A map $M \to N$ is a cellular inclusion of controlled R-modules if and only if it induces an injective map $\diamond_R M \to \diamond_R N$ and is Δ-controlled.*

Note that inducing a map $\diamond_R M \to \diamond_R N$ is a strong condition, it means that cells are mapped to cells and strongly depends on the chosen cell structure.

If $M \to N$ is a cellular inclusion of controlled R-modules it follows that M arises from N by attaching controlled cells. Here N is said to arise from N' by attaching a controlled n-cell over $x \in X$ if there is a pushout diagram

$$\begin{tikzcd}
\underline{R[\partial\Delta^n]}_x \arrow[r, "\partial e"] \arrow[d, tail] & N' \arrow[d, tail] \\
\underline{R[\Delta^n]}_x \arrow[r, "e"] & N
\end{tikzcd}$$

in $\mathcal{C}(X)$ where the vertical maps are Δ-controlled, i.e. induced by $\partial\Delta^n \to \Delta^n$. It is convenient to assume that $x = \kappa_R(e(\mathrm{id}_{[n]}))$, so that $\underline{R[\Delta^n]}$ is concentrated over $\kappa_R(e)$, $e := e(\mathrm{id}_{[n]})$ and the boundary control $R(e)$ of e in B is the same as the morphism control of ∂e. By definition all modules in $\mathcal{C}(X)$ are cellular and for $A \rightarrowtail B$ an inclusion of simplicial sets the map $M[A] \to M[B]$ is a cellular inclusion in $\mathcal{C}$ (using Lemma 1.4).

Remark 1.19. The category $\mathcal{C}(X, R, \mathcal{E}, \mathcal{F})$ has finite coproducts. If M_1 is an E_1-controlled module with support on F_1 and M_2 is an E_2-controlled modules with support on F_2 and $E_1 \cup E_2 \subseteq E''$ as well as $F_1 \cup F_2 \subseteq F''$, then $M_1 \amalg M_2$ is an E''-controlled module with support on F''. In the context of categories with cofibrations and weak equivalences the twofold coproduct is usually denoted by $M_1 \vee M_2$, a reminiscent of the one-point union, so we will also use that notation. Certain larger coproducts exist in $\mathcal{C}(X, R, \mathcal{E}, \mathcal{F})$, e.g. for a fixed module M and any index set I the I-fold coproduct $\coprod_I M$ exists. This only becomes important when we need the mapping telescopes from Appendix C in Lemma 5.10.

Remark 1.20. (Isomorphic objects) Let M and N be in $\mathcal{C}(X, R, \mathcal{E}, \mathcal{F})$ and let M be E-controlled with support on F. If $f\colon M \to N$ and $g\colon N \to M$ are inverse isomorphisms with f and g being E'-controlled, then N is $E' \circ E \circ E'$-controlled and has support on $F^{E'}$.

Note that if M, N are isomorphic as simplicial modules they need not to be controlled isomorphic, even if one of the isomorphisms is controlled. This is already wrong for discrete R-modules with R a discrete ring. This implies in particular that the choice of a cellular structure *does* matter.

Example 1.21. A counterexample is the following: Let $X = \mathbb{N}$ the discrete space of the natural numbers with the standard euclidean metric. We get the morphism control conditions $\mathcal{E}_d$. Consider the module $M := \bigoplus_{i \in \mathbb{N}} R\{e_i\}$. One basis of this module is $b_i^1 := e_i, i \in \mathbb{N}$. Another one is $b_i^2 := e_i - e_{i-1}, i \in \mathbb{N}$ with $e_{-1} := 0$. So we get two structures of a controlled module on M by mapping the ith element of each basis to i. Call the modules (M_1, κ_R^1) and (M_2, κ_R^2). The identity is an (uncontrolled) isomorphism of M_1 and M_2. The map $\mathrm{id}_* \colon (M_2, \kappa_R^2) \to (M_1, \kappa_R^1)$ is controlled, as $b_i^2 = b_i^1 - b_{i-1}^1$. However, in the inverse direction $b_i^1 = \sum_{k=0}^{i} b_k^2$, so the map $\mathrm{id}_* \colon (M_1, \kappa_R^1) \to (M_2, \kappa_R^2)$ is *not controlled*!

Remark 1.22. Note that albeit $\mathcal{C}(X, R, \mathcal{E}, \mathcal{F})$ is an additive category, the K-theory defined by the split inclusions and isomorphisms (as used in [BLR08]) is not the right one. We have to take homotopies into account. We define suitable structures of a category with cofibrations and weak equivalences on $\mathcal{C}(X, R, \mathcal{E}, \mathcal{F})$ in Chapter 3 which allows us to use Waldhausen's construction of algebraic K-theory.

If X is a point or more generally the control space is $(X, \{X \times X\}, \{X\})$ the category $\mathcal{C}(X, R)$ is equivalent to the category of finite-dimensional cellular simplicial R-modules.

Let $(X, R, \mathcal{E}, \mathcal{F})$ be a control space. According to our definition of $\mathcal{C}(X, R, \mathcal{E}, \mathcal{F})$ each module has support on some $F \in \mathcal{F}$ and for two modules M, N there is an F' such that both have support on F'. So we can replace a morphism control condition E by the control conditions $E \cap (F \times F)$ for all $F \in \mathcal{F}$ without changing the category $\mathcal{C}(X, R, \mathcal{E}, \mathcal{F})$. Stated formally we set $\overline{\mathcal{E}} := \{E \cap F \times F \mid E \in \mathcal{E}, F \in \mathcal{F}\}$ and get

$$\mathcal{C}(X, R, \mathcal{E}, \mathcal{F}) = \mathcal{C}(X, R, \overline{\mathcal{E}}, \mathcal{F}).$$

We therefore can require a weaker condition of maps of control spaces, as in [BFJR04, 3.3 (ii)]. There the morphism condition on a map $f \colon (X, \mathcal{E}, \mathcal{F}) \to (Y, \mathcal{E}', \mathcal{F}')$ reads as follows

> "[...] for $E \in \mathcal{E}$ and every $F \in \mathcal{F}$ there exists an $E' \in \mathcal{E}'$ with $(f \times f)(E \cap F \times F) \subseteq E'$."

This corresponds to replacing $\mathcal{E}$ by $\overline{\mathcal{E}}$ as above and then using the usual condition $(f \times f)(E) \subseteq E'$. One readily checks that $\overline{\mathcal{E}}$ is still a morphism control structure in the sense of Definition 1.6, cf. also Remark 1.10. We will come across maps $X \to Y$ which give maps of control spaces from $(X, \overline{\mathcal{E}_X}, \mathcal{F}_X)$ to $(Y, \mathcal{E}_Y, \mathcal{F}_Y)$ but not from $(X, \mathcal{E}_X, \mathcal{F}_X)$ to $(Y, \mathcal{E}_Y, \mathcal{F}_Y)$.

1.4. The "Fundamental Lemma"

Let R be a simplicial ring. Let M, N be simplicial left R-modules and let A be a simplicial set. Let $\mathrm{Hom}_R(M, N)$ be the set of R-module homomorphisms from M to N and let $\mathrm{HOM}_R(M, N)$ be the simplicial mapping space of homomorphisms from M to N (cf. Section 1.1). There is an adjunction (cf. (1))

$$\mathrm{Hom}_R(M[A], N) \rightleftarrows \mathrm{Hom}_{sSet}(A, \mathrm{HOM}_R(M, N)). \tag{4}$$

It sends a map $f \colon M[A] \to N$ to the map $A \mapsto \mathrm{HOM}_R(M, N)$ which assigns to a n-simplex $a \in A$ the map $M[\Delta^n] \xrightarrow{\overline{a}_*} M[A] \xrightarrow{f} N$, where $\overline{a} \colon \Delta^n \to A$ is the characteristic map of a. In the other direction $M[A]$ is a colimit of $M[\Delta^n]_a := M[\Delta^n]$ for $a \in A$ an n-simplex, so the maps $M[\Delta^n]_a \to N$ glue to a map $M[A] \to N$. The bijection is natural in M, N and A. (See [GJ99, p. 158/Prop. 5.1 on p. 20] for the case of simplicial abelian groups.)

There is a controlled version of this adjunction. It is not a full-fledged genuine adjunction, but it is enough for what we need. Let $(X, \mathcal{E}, \mathcal{F})$ be a control space, let $M, N \in \mathcal{C}(X, R, \mathcal{E}, \mathcal{F})$ be controlled modules over X and let A be a simplicial set. Let $E \in \mathcal{E}$ be a morphism control condition.

Definition 1.23. *Define* $\mathrm{Hom}_R^E(M, N)$ *as the set of* E*-controlled maps from* M *to* N*, i.e. the subset of maps* $f \in \mathrm{Hom}_R(M, N)$ *such that* f *is* E*-controlled.*

Define $\mathrm{HOM}_R^E(M, N)$ *as the spaces of* E*-controlled maps from* M *to* N*, i.e. the subspace of elements* $f \colon M[\Delta^n] \to N$ *in* $\mathrm{HOM}_R(M, N)_n$ *for all* n *such that* f *is an* E*-controlled map. As the boundaries and degeneracies of* f *are again* E*-controlled this is a well-defined simplicial subset of* $\mathrm{HOM}_R(M, N)$*.*

Lemma 1.24 (Fundamental Lemma, 1st part)**.** *For* $M, N \in \mathcal{C}(X, R, \mathcal{E}, \mathcal{F})$*, a simplicial set* A *and* $E \in \mathcal{E}$ *there is a bijection*

$$\mathrm{Hom}_R^E(M[A], N) \xrightarrow{\cong} \mathrm{Hom}_{sSet}(A, \mathrm{HOM}_R^E(M, N))$$

which is induced by the adjunction (4).

A priori this bijection is only natural for Δ-controlled maps. We discuss naturality after the proof.

Proof. The map $M[A] \to N$ is E-controlled if and only if each map $M[\Delta^n]_a \to N = M[\Delta^n] \xrightarrow{\overline{a}_*} M[A] \to N$ is E-controlled, as $M[\Delta^n] \xrightarrow{\overline{a}_*} M[A]$ is Δ-controlled. □

For each map f in $\mathcal{C}$ there is some $E \in \mathcal{E}$ such that f is E-controlled, so the set of homomorphisms from M to N in $\mathcal{C}$ is the colimit of $\mathrm{Hom}_R^E(M, N)$ over $E \in \mathcal{E}$. As $\mathcal{E}$ is filtered by condition (ii) of Definition 1.6, this is a filtered colimit of sets and even a union in our case. As the colimit does not depend on $\mathcal{F}$ we denote it by $\mathrm{Hom}_R^{\mathcal{E}}(M, N)$. Similar define $\mathrm{HOM}_R^{\mathcal{E}}(M, N)$ as the filtered colimit (or as the union) of $\mathrm{HOM}_R^E(M, N)$.

In general $\mathrm{Hom}_{sSet}(A,-)$ does *not* commute with filtered colimits. However if A is a *finite* simplicial set it does. So we get for A finite a bijection

$$\mathrm{Hom}_R^{\mathcal{E}}(M[A],N) \xrightarrow{\cong} \mathrm{Hom}_{sSet}(A, \mathrm{HOM}_R^{\mathcal{E}}(M,N)).$$

An E_N-controlled map $N \to N'$ induces maps $\mathrm{Hom}_R^E(M,N) \to \mathrm{Hom}_R^{E_N \circ E}(M,N')$ and $\mathrm{HOM}_R^E(M,N) \to \mathrm{HOM}_R^{E_N \circ E}(M,N')$, and therefore maps

$$\mathrm{Hom}_R^{\mathcal{E}}(M,N) \to \mathrm{Hom}_R^{\mathcal{E}}(M,N') \quad \text{and} \quad \mathrm{HOM}_R^{\mathcal{E}}(M,N) \to \mathrm{HOM}_R^{\mathcal{E}}(M,N').$$

Similar an E_M-controlled map $M' \to M$ induces maps

$$\mathrm{Hom}_R^E(M[A],N) \to \mathrm{Hom}_R^{E \circ E_M}(M'[A],N) \quad \text{and}$$
$$\mathrm{HOM}_R^E(M[A],N) \to \mathrm{HOM}_R^{E \circ E_M}(M'[A],N)$$

and therefore maps

$$\mathrm{Hom}_R^{\mathcal{E}}(M[A],N) \to \mathrm{Hom}_R^{\mathcal{E}}(M'[A],N) \ \text{ and } \ \mathrm{HOM}_R^{\mathcal{E}}(M[A],N) \to \mathrm{HOM}_R^{\mathcal{E}}(M'[A],N).$$

Corollary 1.25 (Fundamental Lemma, 2nd part). *If A is a finite simplicial set and $M, N \in \mathcal{C}(X,R,\mathcal{E},\mathcal{F})$ we have a bijection*

$$\mathrm{Hom}_R^{\mathcal{E}}(M[A],N) \xrightarrow{\cong} \mathrm{Hom}_{sSet}(A, \mathrm{HOM}_R^{\mathcal{E}}(M,N))$$

which is natural in A, M and N.

Remark 1.26. This is not quite an adjunction, but it is one "for all practical purposes", in particular for our practical purposes. The problem is that $\mathrm{HOM}_R^{\mathcal{E}}(M,N)$ is not a finite simplicial set, so the set of homomorphisms on the right-hand side is not a Hom-functor of the category of finite simplicial sets. For an adjunction one would need e.g. to be able to apply it to $\mathrm{HOM}_R^{\mathcal{E}}(M,N)$ for A, but the bijection does not hold for this in general non-finite simplicial set.

Actually for practical applications the first part of the Fundamental Lemma is more useful as it has an explicit control condition. However the second version includes the naturality which makes the similarity to an adjunction more explicit.

1.5. Equivariant controlled modules

We need equivariant versions of the notions of Sections 1.1 to 1.4. So for the following let G be an arbitrary discrete group. Most of the definitions are straightforward generalizations. We are mainly interested in free actions with regard to our applications, cf. the discussion in [BFJR04, Section 3] and Section 7.1. All our notions here specialize to the notions of Sections 1.1 to 1.4 if we let G be the trivial group $\{1\}$.

Definition 1.27. *Let M be a simplicial R-module. An action of G on M is a group homomorphism $\rho\colon G \to \mathrm{Aut}_R(M)$ from G to the automorphism group of M. (This is the same as a map $G \to \mathrm{Hom}_R(M,M)$ which takes group multiplication to composition.)*

An action of G on M gives an action on each M_n. The action is called *cell-permuting* if it induces an action on the chosen cell structure $\diamond_R M$. An action is *free* if it is cell-permuting and the action on $\diamond_R M$ is free. A free action makes each M_n into a free $R_n[G]$-module.

Let M, N be simplicial R-modules which have G-actions ρ_M and ρ_N respectively. A map $f\colon M \to N$ of simplicial R-modules is called *G-equivariant* if for each $g \in G$ the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\rho_M(g)"'] & N \arrow[d, "\rho_N(g)"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

commutes. A G-equivariant map $i\colon L \to M$ is called a *cellular inclusion* if it is one after forgetting the G-action. If the action is cell-permuting it follows that M arises from L by attaching G-cells, i.e. cells of the form $R[\Delta^n][G/H]$, H a subgroup of G. If the G-action on M (and hence on L) is free M arises by attaching free G-cells $\bigoplus R[\Delta^n][G/1]$. (See [DL98] for a discussion of the analogous case of G-CW-complexes.) The module $R[\Delta^n][G/H]$ inherits its G-action from the G-set G/H, which is interpreted as a discrete simplicial set with a G-action. Equivalently $R[\Delta^n][G/H]$ is the coproduct $\coprod_{g \in G/H} R[\Delta^n]$ which inherits the obvious G-action by the permutation of factors.

If M is a simplicial R-module with G-action and A a simplicial set then $M[A]$ has a G-action by the functoriality of $-[A]$. Denote by $\mathrm{Hom}_R(M,N)^G$ the set of G-equivariant homomorphisms from M to N. It is a subset of $\mathrm{Hom}_R(M,N)$. Denote similarly by $\mathrm{HOM}_R(M,N)^G$ the subspace of $\mathrm{HOM}_R(M,N)$ of G-equivariant homomorphisms. The adjunction (1) gives an adjunction

$$\mathrm{Hom}_R(M[A],N)^G \rightleftarrows \mathrm{Hom}_{sSet}(A, \mathrm{HOM}_R(M,N)^G). \tag{5}$$

(See below for a more detailed discussion of this in the controlled context.)

Definition 1.28. *A G-equivariant control space is a control space $(X, \mathcal{E}, \mathcal{F})$ such that X has a G-action and the morphism control conditions $E \in \mathcal{E}$ and the object support conditions $F \in \mathcal{F}$ are G-invariant, i.e. $gE = E$ with diagonal action and $gF = F$ for all E, F and all $g \in G$.*

A *free* G-equivariant control space is a G-equivariant control space $(X, \mathcal{E}, \mathcal{F})$ such that the action on X is free. A map of G-equivariant control spaces $(X, \mathcal{E}_X, \mathcal{F}_X) \to (Y, \mathcal{E}_Y, \mathcal{F}_Y)$ is a map of control spaces which is also G-equivariant.

The control space $(X, \mathcal{E}, \mathcal{F})$ is called *G-proper* if for each G-compact set $K \subseteq X$, for $E \in \mathcal{E}$ and $F \in \mathcal{F}$ there is a G-compact set $K' \subseteq X$ such that $(F \cap K)^E \cap F \subseteq K'$.

Examples 1.29. These are from [BFJR04, Definitions 2.7,2.9; Section 3.1,3.2].

(i) (G-equivariant continuous control). Let Z be a topological space with continuous G-action. Define the equivariant continuous morphism control structure

$\mathcal{E}_{Gcc}(Z)$ on $X := Z \times [1, \infty)$ as follows. E is in $\mathcal{E}_{Gcc}(Z)$ if it is symmetric, invariant under the G-operation and

(a) For every $x \in Z$ and each G_x-invariant neighborhood U of (x, ∞) in $Z \times [1, \infty]$ there is a G_x-invariant neighborhood $V \subseteq U$ of (x, ∞) in $Z \times [1, \infty]$ such that $E \cap ((X \smallsetminus U) \times V) = \varnothing$.

(b) $p_{[1,\infty)} \times p_{[1,\infty)}(E) \in \mathcal{E}_d([1, \infty))$, where d is the standard euclidean metric on $[1, \infty)$ and $p_{[1,\infty)}$ is the projection to $[1, \infty)$.

(ii) (G-compact support). Let X be a G-space. Define the G-compact object support conditions $\mathcal{F}_{Gc}$ as follows. A set $F \subseteq X$ is in $\mathcal{F}_{Gc}$ if it is G-compact, i.e. of the form GK for some compact set $K \subseteq X$. These are object support conditions for the control space $(Z \times [1, \infty), \mathcal{E}_{Gcc})$ from the previous item.

(iii) (Resolutions). A *resolution* of the G-space Z is a free G-space $\overline{Z}$ together with an equivariant continuous map $p\colon \overline{Z} \to X$ such that for every G-compact set $GK \subseteq Z$ there is a G-compact set $G\overline{K} \subseteq \overline{Z}$ with $p(G\overline{K}) = GK$. Further the G-action on $\overline{Z}$ should be properly discontinuous and $G \backslash \overline{Z}$ should be Hausdorff. The *standard resolution* is $Z \times G \to Z$. (It is a resolution for Z Hausdorff.)

Definition 1.30. *Let $(X, \mathcal{E}, \mathcal{F})$ be a free G-equivariant control space. A controlled simplicial R-module with G-action over X is a controlled module (M, κ_R) over X such that M has a cell-permuting G-action and $\kappa_R\colon \diamond_R M \to X$ is G-equivariant. It follows that the G-action on M is free.*

A morphism of controlled simplicial R-modules with G-action over X from (M, κ_R) to (N, κ_R) is a G-equivariant morphism $M \to N$ which is controlled over X.

Denote the category of finite-dimensional controlled simplicial R-modules over X with free G-action and its morphisms as

$$\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F}).$$

We abbreviate it by $\mathcal{C}^G(X, R)$, $\mathcal{C}^G(X)$ or $\mathcal{C}^G$.

All further definitions of Section 1.3 transfer to $\mathcal{C}^G$.

Remark 1.31. A cellular simplicial R-module with free cell-permuting G-action is the same as a free cellular $R[G]$-module, where $R[G]$ is the (simplicial) group ring. However, a controlled simplicial R-module with G-action *is not* the same as a controlled $R[G]$-module. The reason is that we want to “distribute” the G-action over X, i.e. it should be “horizontally” while the R-multiplication should be “vertically”, i.e. agnostic of the X.

The adjunction (4)

$$\mathrm{Hom}_R(M[A], N) \rightleftarrows \mathrm{Hom}_{sSet}(A, \mathrm{HOM}_R(M, N))$$

restricts to the equivariant adjunction (5)

$$\mathrm{Hom}_R(M[A], N)^G \rightleftarrows \mathrm{Hom}_{sSet}(A, \mathrm{HOM}_R(M, N)^G)$$

as follows. For a G-equivariant map $M[A] \to N$ and all $a \in A$ the induced maps $M[\Delta^n]_a \to M[A] \to N$ are also G-equivariant and vice versa those equivariant maps glue to an equivariant map $M[A] \to N$.

This gives the corresponding equivariant controlled versions of the Fundamental Lemma 1.24 and 1.25. For $E \in \mathcal{E}$ define $\mathrm{Hom}_R^E(M, N)^G$ and $\mathrm{HOM}_R^E(M, N)^G$ as the subset resp. subspace of G-equivariant maps in $\mathrm{Hom}_R^E(M, N)$ and $\mathrm{HOM}_R^E(M, N)$.

Lemma 1.32 (Equivariant Fundamental Lemma, 1st part)**.** *For M and N in $\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})$, a simplicial set A and $E \in \mathcal{E}$ there is a bijection*

$$\mathrm{Hom}_R^E(M[A], N)^G \xrightarrow{\cong} \mathrm{Hom}_{sSet}(A, \mathrm{HOM}_R^E(M, N)^G)$$

which is induced by the adjunction (5).

Corollary 1.33 (Equivariant Fundamental Lemma, 2nd part)**.** *If A is a finite simplicial set and $M, N \in \mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})$ we have a bijection*

$$\mathrm{Hom}_R^{\mathcal{E}}(M[A], N)^G \xrightarrow{\cong} \mathrm{Hom}_{sSet}(A, \mathrm{HOM}_R^{\mathcal{E}}(M, N)^G)$$

which is natural in A, M and N.

Here we use the obvious definitions of $\mathrm{Hom}_R^{\mathcal{E}}(M[A], N)^G$ and $\mathrm{HOM}_R^{\mathcal{E}}(M, N)^G$ as the subset and subspace of G-equivariant maps. (Alternatively we could define it as the colimit over E.)

Proof of Lemma 1.32 and Corollary 1.33. The bijections are restrictions of the bijections of Lemma 1.24 and Corollary 1.25 by (5), so the control condition holds. The naturality of Corollary 1.33 follows from the naturality of Corollary 1.25. □

Remark 1.34 (Removing equivariance). Denote by $\underline{R[G/1]}$ a cellular simplicial controlled R-module $(\coprod_{g \in G/1} \underline{R}_g, \kappa_R)$ over X with the obvious free G-action, where $\underline{R}_g$ denotes a 1-dimensional free R-module. It is determined the value of κ_R on the cell $(1_R)_e$ for $e \in G$. If we forget the control we have an adjunction

$$\mathrm{HOM}_{R[G]}(\underline{R[G/1]}, N) \rightleftarrows \mathrm{HOM}_R(\underline{R}, N).$$

For the controlled modules this gives natural isomorphisms

$$\mathrm{HOM}_R^E(\underline{R[G/1]}, N)^G \cong \mathrm{HOM}_R^E(\underline{R}, N)$$

and

$$\mathrm{HOM}_R^{\mathcal{E}}(\underline{R[G/1]}, N)^G \cong \mathrm{HOM}_R^{\mathcal{E}}(\underline{R}, N)$$

where on the right-hand side we forget the G-action on N and consider N as a non-equivariant controlled cellular R-module. This simplifies the right-hand side of the Fundamental Lemma in the case $M = \underline{R[G/1]}$.

Of course all the remarks of Section 1.4 apply.

Remark 1.35. Let us sketch the prospective use of the lemma. We will be concerned with problems where the simplicial set A changes. A common situation is the inclusion $\Lambda_i^n \to \Delta^n$ of a horn into the n-simplex which induces a map $M[\Lambda_i^n] \to M[\Delta^n]$. We then want to know if we could extend a map $M[\Lambda_i^n] \to N$ to a map on $M[\Delta^n]$.

The Fundamental Lemma translates this problem into a situation of simplicial maps $\Lambda_i^n \to Q$ with $Q := \mathrm{HOM}_R(M, N)$ where we can analyze (and solve) it. The crucial point is that we do not need to care about control any more, the Fundamental Lemma does this for us. Further the Equivariant Fundamental Lemma now even takes care about the equivariance, so from that point of view this situation is not more complicated than the non-equivariant one. These are the reasons why we call this two facts “Fundamental Lemma” as they hide a lot of complications for us.

2. Structures on $\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})$

We want to define the algebraic K-Theory of $\mathcal{C}(X, R, \mathcal{E}, \mathcal{F})$ and $\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})$. The established way to do this is to use Waldhausen's $\mathcal{S}_\bullet$-construction which takes as input a *category with cofibrations and weak equivalences* and produces an infinite loop space or equivalently a connective spectrum ([Wal85]), see Appendix B for a brief overview. The following discusses the ingredients we need to produce such a structure on $\mathcal{C}^G$, the actual result is Proposition 3.3 in Section 3.1. We assume first that $\mathcal{F} = \{X\}$, i.e. that we do not have any object support conditions. For non-trivial object support conditions there are only two extra things to check which are discussed in Section 3.3.

The first step is that we show that $\mathcal{C}^G$ is a *category with cofibrations* (cf. B.1). This is established in Section 2.2. Then we define cylinders in Section 2.3, which are used to define homotopies in Section 2.4. The homotopy equivalences are the first choice of the weak equivalences for $\mathcal{C}^G(X, R)$ (the second are the germwise weak equivalences which we define in Chapter 4). To prove the axioms of a category of weak equivalences we have to do some intermediate steps. We prove that the mapping cylinder of a homotopy equivalence is a deformation retract of the source (Section 2.6), and then that the pushout of cofibration which is a homotopy equivalence is again a homotopy equivalence (Section 2.7). We finish this chapter with a proof of the Extension Axiom for homotopy equivalences.

Remark 2.1. A lot of the proofs in the following sections are *formal* in the sense that they do not use specific properties of $\mathcal{C}^G$ but merely some basic Lemmas about $\mathcal{C}^G$. Some of these Lemmas are e.g. the existence of pushouts along cofibrations (Lemma 2.6), and the horn-filling and the homotopy extension properties (Lemmas 2.21 and 2.29). However, it does not seem worthwhile to try to make this formality precise by giving an axiomatic framework as things will not get easier to understand.

One consequence of this formality is that quite a few proofs look exactly like proofs for the corresponding statements for uncontrolled cellular R-modules. Of course our results specialize to the uncontrolled case, but the converse is not true in general. The main obstacle is that we do not have homotopy groups available in the controlled setting, which make things considerably easier in the uncontrolled case, as a map between uncontrolled cellular R-modules is a homotopy equivalence if and only if it induces an isomorphism on homotopy groups of the underlying simplicial sets. The author suspects that the uncontrolled cases of the results of this chapter are well-known, but there does not seem to be a standard reference for this.

Another point we have to pay attention to is the following. When we deal with an infinite number of cells or an infinite number of steps, e.g. for an induction, we have to take care of the control conditions. In particular $\mathcal{C}^G$ does not contain all infinite colimits, for in general the set $\bigcup_{i=0}^{\infty} E_i, E_i \in \mathcal{E}$ is not longer controlled. This is the main reason we assume our modules to be of finite dimension. If we do only finitely many steps we can often do the proof without mentioning control conditions at all, and we will do so.

2.1. Pushouts along cellular inclusions and cofibrations

The categories $\mathcal{C}$ and $\mathcal{C}^G$ do not have all pushouts, e.g. for $R := \mathbb{Z}$ the pushout of $* \leftarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$ is $\mathbb{Z}/2$ and not free. But pushouts along cellular inclusions exist even canonically.

Lemma 2.2 (Pushouts along cellular inclusions)**.** *Let $(A, \kappa_R^A) \to (B, \kappa_R^B)$ be a cellular inclusion in $\mathcal{C}^G$, let $f\colon (A, \kappa_R^A) \to (C, \kappa_R^C)$ be any controlled map in $\mathcal{C}^G$. Then the pushout $D := C \cup_A B$,*

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, "f"] & B \arrow[d] \\
C \arrow[r, tail] & D
\end{tikzcd}$$

of simplicial R-modules has a canonical structure of an object (D, κ_R^D) in $\mathcal{C}^G$. Further $(C, \kappa_R^C) \to (D, \kappa_R^D)$ is a cellular inclusion.

Hence $\mathcal{C}^G$ has canonical pushouts along cellular inclusions.

Remark 2.3. It is important that the pushouts are *canonical* to get functorial mapping cylinders later. Being "canonical" should mean that for each diagram there is a *preferred choice* of the pushout, only depending on the diagram.

Proof. We first prove this for $\mathcal{C}$, i.e. $G = \{1\}$. Clearly D exists as a finite-dimensional simplicial R-module, as the category of finite-dimensional simplicial R-modules is cocomplete. It suffices to show that $C \to D$ is a cellular inclusion.

As $A \rightarrowtail B$ is cellular, B has a filtration

$$A = B_{-1} \rightarrowtail B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_i \rightarrowtail \cdots$$

and each B_i is a pushout

$$\begin{tikzcd}
\coprod_{J_i} R[\partial\Delta^i] \arrow[r, tail] \arrow[d] & \coprod_{J_i} R[\Delta^i] \arrow[d] \\
B_{i-1} \arrow[r, tail] & B_i
\end{tikzcd}.$$

Set inductively D_i as the pushout

$$\begin{tikzcd}
B_{i-1} \arrow[r, tail] \arrow[d] & B_i \arrow[d] \\
D_{i-1} \arrow[r, tail] & D_i
\end{tikzcd}$$

control conditions we have	
A, B	E_B-controlled
C	E_C-controlled
f	E_f-controlled
g_C, g_B	E-controlled

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, "f"] & B \arrow[d, "\widetilde{f}"] \arrow[ddr, bend left, "g_B"] & \\
C \arrow[r, tail] \arrow[drr, bend right, "g_C"'] & D \arrow[dr, dashed, "g"] & \\
 & & T
\end{tikzcd}$$

control conditions we get	
D	$E_C \cup E_f \circ E_B$-controlled
$\widetilde{f}$	E_f-controlled
g	E-controlled

Table 2.1.: Control conditions on pushouts along cellular inclusions in $\mathcal{C}^G$.

with $D_{-1} := C$. Then the D_i give a filtration

$$C = D_{-1} \rightarrowtail D_0 \rightarrowtail D_1 \rightarrowtail \cdots \rightarrowtail D_i \rightarrowtail \cdots$$

of D and D_i is the pushout

$$\begin{tikzcd}
\coprod_{J_i} R[\partial\Delta^i] \arrow[r, tail] \arrow[d] & \coprod_{J_i} R[\Delta^i] \arrow[d] \\
D_{i-1} \arrow[r, tail] & D_i
\end{tikzcd}.$$

So $C \to D$ is cellular, thus D is cellular. The cells of D are indexed by the cells of C and the cells of B which are not in A, so $\diamond_R D \cong \diamond_R C \cup (\diamond_R B \smallsetminus \diamond_R A)$ and we *specify* this isomorphism to get a canonical model for D, call it φ in the following. (Recall that $\diamond_R D$ and boundary data determine D.)

Define κ_R^D as

$$\kappa_R^D(e) := \begin{cases} \kappa_R^C(e) & \text{if } \varphi(e) \in \diamond_R C, \\ \kappa_R^B(e) & \text{if } \varphi(e) \in \diamond_R B \smallsetminus \diamond_R A. \end{cases}$$

Let $f\colon A \to B$ be E_f-controlled, let B be E_B-controlled and C be E_C-controlled. Then the boundary control for $e \in \diamond_R D$ with $\varphi(e) \in \diamond_R B$ is $E_f \circ E_B$ and the boundary control for $e \in \diamond_R D$ with $\varphi(e) \in \diamond_R C$ is E_C. So (D, κ_R^D) is an $E_C \cup E_f \circ E_B$-controlled module. Further $B \to D$ is E_f-controlled. If B and C have object support on $F \in \mathcal{F}$ then D has object support on F.

If $B \to T$, $C \to T$ are compatible E-controlled maps, the induced map $D \to T$ is also E-controlled. This shows that the pushout D in simplicial modules gives the pushout (D, κ_R^D) in $\mathcal{C}$.

If A, B, C have a free G-action then D has a free G-action, e.g. because $\varphi\colon \diamond_R D \cong \diamond_R C \cup (\diamond_R B \smallsetminus \diamond_R A)$ is an isomorphism of free G-sets. Hence (D, κ_R^D) is also a pushout in $\mathcal{C}^G$. □

Remark 2.4. For reference the control conditions for a pushout along a cellular inclusion $\mathcal{C}^G$ are recorded in Table 2.1.

Definition 2.5. *A cofibration in $\mathcal{C}^G$ is a map $A \to B$ which is isomorphic to a cellular inclusion, i.e. there is a diagram in $\mathcal{C}^G$*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\cong"] & B \arrow[d, "\cong"] \\
A' \arrow[r, tail] & B'
\end{tikzcd} \tag{6}$$

where $A \to A'$ and $B \to B'$ are isomorphisms and $A' \rightarrowtail B'$ is a cellular inclusion. We denote cofibrations again by "$\rightarrowtail$".

One way to state the definition is to say that B arises from A by attaching cells *up to isomorphism.* This implies in particular that the cellular structure of A and B might not be compatible.

Lemma 2.2 remains true for cofibrations except that pushouts might no longer be canonical:

Lemma 2.6 (Pushouts along cofibrations)**.** *Let $A \rightarrowtail B$ be a cofibration in $\mathcal{C}^G$ and $A \to C$ any controlled map. Then the pushout D of $C \leftarrow A \rightarrowtail B$ exists in $\mathcal{C}^G$ and the map $C \to D$ is a cofibration which can be chosen to be a cellular inclusion.*

Proof. Choose a cellular inclusion $A' \rightarrowtail B'$ isomorphic to $A \rightarrowtail B$. Diagram (6) is also a pushout diagram so we set $D := C \cup_{A'} B'$ using Lemma 2.2 and get the big diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\cong"] & B \arrow[d, "\cong"] \\
A' \arrow[r, tail] \arrow[d] & B' \arrow[d] \\
C \arrow[r, tail] & D
\end{tikzcd} .$$

Both smaller squares are pushouts and therefore the outer one is the desired pushout in $\mathcal{C}^G$. Note that pushouts might not be canonical as one might not be able to choose the cellular inclusion $A' \rightarrowtail B'$ canonically depending on $A \rightarrowtail B$. ☐

2.2. $\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})$ is a category with cofibrations

Let co$\mathcal{C}^G$ be the class of cofibrations in $\mathcal{C}^G$.

Lemma 2.7. co$\mathcal{C}^G$ *is a subcategory of $\mathcal{C}^G$.*

Proof. Identities are cofibrations. We have to prove that composition of two cofibrations is a cofibration.

Composition of cellular inclusions is a cellular inclusion. Assume we have cofibrations $A \rightarrowtail B$, $B \rightarrowtail C$. Then $A \rightarrowtail B$ is isomorphic to a cellular inclusion

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, "\cong"] & B \arrow[d, "\cong"] \\
A' \arrow[r, tail] & B'
\end{tikzcd}$$

and the pushout C' in

$$\begin{tikzcd} B \arrow[r, tail] \arrow[d, "\cong"] & C \arrow[d, "\cong"] \\ B' \arrow[r, tail] & C' \end{tikzcd}$$

can be chosen by Lemma 2.6 such that $B' \to C'$ is a cellular inclusion. So $A \to C$ is isomorphic to the composition of cellular inclusions

$$A' \rightarrowtail B' \rightarrowtail C'$$

and hence it is a cofibration. □

Lemma 2.8. *The category $\mathcal{C}^G$ together with the class* co$\mathcal{C}$ *is a category with cofibrations in the sense of B.1/[Wal85].*

Proof. We have to check four axioms.

(0) co$\mathcal{C}^G$ is a subcategory.

(i) Isomorphisms are in co$\mathcal{C}^G$.

(ii) All maps $* \to M$ are in co$\mathcal{C}^G$.

(iii) Cofibrations admit cobase change.

Items (ii) and (i) are satisfied by definition. Item (iii) holds by Lemma 2.6 and (0) by Lemma 2.7. □

Remark 2.9. This is the time to make an important remark about our choice of cofibrations. In general retracts of maps in co$\mathcal{C}^G$ are no longer cofibrations! Otherwise the pushout along a cofibration might not yield again a cellular simplicial R-module, so we would have to include retracts of cellular simplicial R-modules in our definition of $\mathcal{C}^G$, but we had trouble to find a good notion of control for these kind of R-modules. In the world of discrete R-modules this means that we are only looking at free modules and not at projective ones because we need properties which depend on a basis of our modules which are hard to get for projective modules. However, our category is big enough to have *homotopy* retracts, which we will have reason to discuss later.

A counterexample for R a discrete ring is the following. Let B be an R-module which is stably free but not free, so $B \oplus R^n \cong R^m$. Then cellular inclusions are inclusions of direct summands with free complement, compare with Table 1.1 on page 21. Assume $m > n$ and choose an inclusion $a\colon R^n \to R^m$ such that the complement is free. Thus a is a cellular inclusion. Then the inclusion into the second summand $0 \oplus \mathrm{id}\colon R^n \to B \oplus R^n$ is a retract of the cellular inclusion $a \oplus \mathrm{id}_{R^n}$,

$$\begin{tikzcd}
R^n \arrow[r, "0\oplus\mathrm{id}"] \arrow[d, "i_1"] & B \oplus R^n \arrow[d] \\
R^n \oplus R^n \arrow[r, "a\oplus\mathrm{id}"] \arrow[d, "\mathrm{pr}_1"] & R^n \oplus B \oplus R^n \arrow[d] \\
R^n \arrow[r, "0\oplus\mathrm{id}"] & B \oplus R^n
\end{tikzcd} .$$

But the complement of $0 \oplus \mathrm{id}$ is B and not free. So if we would require retracts of cofibrations to be cofibrations then $0 \oplus \mathrm{id}$ must be a cofibration. It would follow, as

$$\begin{tikzcd}
R^n \arrow[r, tail] \arrow[d] & B \oplus R^n \arrow[d] \\
* \arrow[r, tail] & B
\end{tikzcd}$$

is a pushout along a cofibration, that B must be in our category, although B is not free.

2.3. A cylinder functor

For the next constructions we need the notion of a *mapping cylinder* for any map $f\colon A \to B$ in $\mathcal{C}^G$. In the context of categories with cofibrations a *Cylinder Functor* (B.3/[Wal85, 1.6]) serves this purpose.

Remark 2.10. Note that albeit [Wal85] requires a category with cofibrations *and* weak equivalences we actually do not need the weak equivalences in the definition of a Cylinder Functor. However the weak equivalences are needed in the statement of the Cylinder Axiom, so we will only mention the axiom here briefly in 2.18 and defer the discussion of it.

We first recall the definition of a Cylinder Functor.

Recollection 2.11 (Cylinder Functor, B.3/[Wal85, 1.6]). Let $\mathcal{C}$ be a category with cofibrations. A *Cylinder Functor* is a functor which takes a map $f\colon A \to B$ in $\mathrm{Ar}\mathcal{C}$ to a diagram

$$\begin{tikzcd}
A \arrow[r, "\iota_0"] \arrow[dr, "f"'] & T(f) \arrow[d, "p"] & B \arrow[l, "\iota_1"'] \arrow[dl, "\mathrm{id}"] \\
& B &
\end{tikzcd} .$$

Here ι_0 is called the *front inclusion*, ι_1 is called the *back inclusion* and p is called the *projection.* Further the following two axioms should be satisfied.

(i) (Cyl 1) Front and back inclusion assemble to an exact functor

$$\mathrm{Ar}\mathcal{C} \longrightarrow \mathcal{F}_1\mathcal{C}$$
$$f \mapsto \big(\iota_0 \vee \iota_1\colon A \vee B \rightarrowtail T(f)\big).$$

(ii) (Cyl 2) $T(* \to A) = A$ for every $A \in \mathcal{C}$ and the projection and the back inclusion are the identity map on A.

Here $\mathrm{Ar}\mathcal{C}$ is the arrow category of $\mathcal{C}$ and $\mathcal{F}_1\mathcal{C}$ is the full subcategory of $\mathrm{Ar}\mathcal{C}$ with objects the cofibrations. Both can be made into categories with cofibrations, with the cofibrations of $\mathcal{F}_1\mathcal{C}$ being slightly non-obvious. See Appendix B and [Wal85, 1.1] for the precise definitions. We use the notion "$A \vee B$" from [Wal85] for the coproduct of A and B.

Definition 2.12. *Let $f\colon A \to B$ be a map in $\mathcal{C}^G$. Define $T(f)$ as the canonical pushout in*

$$\begin{tikzcd}
A[1] \arrow[r, tail] \arrow[d, "f"'] & A[\Delta^1] \arrow[d] \\
B \arrow[r] & T(f)
\end{tikzcd} \quad . \tag{7}$$

Proposition 2.13. *The assignment $f \mapsto T(f)$ gives a Cylinder Functor on $\mathcal{C}^G$.*

There is a lot to check. We will first construct the required data and show the functoriality. Then we check the axioms. The whole proof occupies almost the rest of this section and we state the steps as lemmas.

Lemma 2.14. *T gives a functor from $\mathrm{Ar}\mathcal{C}^G$ into diagrams in $\mathcal{C}^G$, taking $f\colon A \to B$ to a commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "\iota_0"] \arrow[dr, "f"'] & T(f) \arrow[d, "p"] & B \arrow[l, "\iota_1"'] \arrow[dl, "\mathrm{id}"] \\
& B &
\end{tikzcd} \quad . \tag{8}$$

Proof. First $A \mapsto A[\Delta^1]$ is a functor and as $A[1] \rightarrowtail A[\Delta^1]$ is a cellular inclusion the pushout in (7) is functorial as by Lemma 2.2 there is a canonical choice of $T(f)$. Therefore $T(f)$ depends functorially on f.

The map ι_0 is given by $A[0] \to A[\Delta^1] \to T(f)$, ι_1 is given by $B \to T(f)$ in (7). The maps $\mathrm{id}\colon B \to B$ and $A[\Delta^1] \xrightarrow{\mathrm{pr}} A \xrightarrow{f} B$ induce the map p:

$$\begin{tikzcd}
A[1] \arrow[r, tail] \arrow[d, "f"'] & A[\Delta^1] \arrow[d] \arrow[dr, bend left, "\mathrm{pr}"] & & . \\
B \arrow[r] \arrow[drr, bend right, "\mathrm{id}"'] & T(f) \arrow[dr, dashed, "p"] & A \arrow[d, "f"] & \\
& & B &
\end{tikzcd}$$

This gives diagram (8). One checks that

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
A' \arrow[r, "f'"] & B'
\end{tikzcd}$$

gives a map of diagrams (8). Hence we made T into a functor. ☐

Lemma 2.15 (Cyl 2)**.** *$T(* \to A) = A$ for $A \in \mathcal{C}^G$ and p and ι_1 are the identity.*

Proof. We have $*[\Delta^1] = *$, so $T(* \to A)$ is the pushout of $* \leftarrow * \to A$, which *is* A by inspection of the proof of Lemma 2.2. (Of course a canonical isomorphism would suffice, for the proof as well as for the definition of (Cyl 2).) ☐

We split the proof of (Cyl 1) into two parts which will be the next two lemmas.

Lemma 2.16. *Front and back inclusion give a functor $\mathrm{Ar}\mathcal{C}^G \to \mathcal{F}_1\mathcal{C}^G$,*

$$f \mapsto (A \vee B \rightarrowtail T(f)).$$

Proof. The only thing to show is that $A \vee B \to T(f)$ is a cellular inclusion. Consider the diagram

$$\begin{tikzcd}
* \arrow[r, tail] \arrow[d, tail] \arrow[rd, phantom, "\mathrm{I}"] & A \arrow[d, tail] & . \\
A \arrow[r, tail] \arrow[d, "f"'] \arrow[rd, phantom, "\mathrm{II}"] & A \vee A \arrow[r, tail] \arrow[d] \arrow[rd, phantom, "\mathrm{III}"] & A[\Delta^1] \arrow[d] \\
B \arrow[r, tail] & A \vee B \arrow[r] & T(f)
\end{tikzcd}$$

Here $A \vee A \rightarrowtail A[\Delta^1]$ is the cellular inclusion $\iota_0 \vee \iota_1 \colon A[0] \vee A[1] = A[0 \amalg 1] \rightarrowtail A[\Delta^1]$. We claim that every possible square is a pushout along a cellular inclusion. I is a pushout square by definition, as well as I + II. It follows that II is one. Further II + III is a pushout square by definition of $T(f)$, so III is one. Hence the lower map $A \vee B \to T(f)$ is a cellular inclusion by Lemma 2.2. ☐

The last and most difficult part is to check the exactness of the functor $\mathrm{Ar}\mathcal{C}^G \to \mathcal{F}_1\mathcal{C}^G$.

Lemma 2.17. *The functor of Lemma 2.16 is exact.*

Let us briefly recall the cofibrations in $\mathrm{Ar}\mathcal{C}$ and $\mathcal{F}_1\mathcal{C}^G$, cf. also Appendix B. For notation let

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
A' \arrow[r, "f'"] & B'
\end{tikzcd} \tag{9}$$

be a map in $\mathrm{Ar}\mathcal{C}^G$ from $A \to B$ to $A' \to B'$. It is a cofibration in $\mathrm{Ar}\mathcal{C}^G$ if both vertical maps are cofibrations. The category $\mathcal{F}_1\mathcal{C}^G$ is the full subcategory of $\mathrm{Ar}\mathcal{C}^G$ with objects being the cofibrations in $\mathcal{C}^G$. Hence if f and f' are cofibrations the diagram also shows a map in $\mathcal{F}_1\mathcal{C}^G$. It is a cofibration in $\mathcal{F}_1\mathcal{C}^G$ if $A \to A'$ and $A' \cup_A B \to B'$ are cofibrations in $\mathcal{C}^G$. See [Wal85, Lemma 1.1.1] for details and a proof that the composition of cofibrations in $\mathcal{F}_1\mathcal{C}^G$ is again a cofibration.

Proof. Both $A \vee B$ and $T(f)$ are colimits, hence commute with pushouts, which are formed pointwise in $\mathrm{Ar}\mathcal{C}^G$ and $\mathcal{F}_1\mathcal{C}^G$. The functor respects the zero object $* \to *$. So we only have to show that it maps cofibrations to cofibrations. More concretely we have to show that for a map (9) which is a cofibration in $\mathrm{Ar}\mathcal{C}^G$ the maps $A \vee B \to A' \vee B'$ and $(A' \vee B') \cup_{A \vee B} T(f) \to T(f')$ are cofibrations in $\mathcal{C}^G$. As functors respect isomorphisms we can assume that all cofibrations are cellular inclusions.

So assume we have a diagram (9) where the vertical maps are cellular inclusions. We can factor (9) into

$$\begin{tikzcd}
A \arrow[r, equal, "\mathrm{id}"] \arrow[d, "f"] & A \arrow[r] \arrow[d, "f^*"] & A' \arrow[d, "f'"] \\
B \arrow[r] & B' \arrow[r, equal, "\mathrm{id}"] & B'
\end{tikzcd}$$

(where we write the maps horizontally for presentational reasons). If suffices to check each map individually. As $B \rightarrowtail B'$ is a cellular inclusion and $A \vee B'$ is the pushout of $B' \leftarrowtail B \to A \vee B$ the map $A \vee B \to A \vee B'$ is a cellular inclusion by Lemma 2.2. By the same reason $A \vee B' \to A' \vee B'$ is a cellular inclusion.

Recalling that by Definition 2.12 $T(f)$ is the pushout $B \cup_f A[\Delta^1]$

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & A[\Delta^1] \arrow[d] \\
B \arrow[r] & T(f)
\end{tikzcd}$$

we see that $T(f^*)$ is the pushout

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & T(f) \arrow[d] \\
B' \arrow[r] & T(f^*)
\end{tikzcd}$$

and hence (by "canceling A" by a similar pushout argument as in the proof of Lemma 2.16) it is the pushout

$$\begin{tikzcd}
A \vee B \arrow[r] \arrow[d] & T(f) \arrow[d] \\
A \vee B' \arrow[r] & T(f^*)
\end{tikzcd}$$

so the map $(A \vee B') \cup_{A \vee B} T(f) \to T(f^*)$ is the identity and therefore a cellular inclusion. Using the canceling argument for B we can write the other map

$$(A' \vee B') \cup_{A \vee B'} T(f^*) \to T(f')$$

as

$$A' \cup_{A[0]} A[\Delta^1] \cup_{f^*} B' \to A'[\Delta^1] \cup_{f'} B'. \tag{10}$$

Here the first object is a cylinder where we glued in spaces at both sides. But because $A \rightarrowtail A'$ is a cellular inclusion so is $A'[0] \cup_{A[0]} A[\Delta^1] \cup_{A[1]} A'[1] \rightarrowtail A'[\Delta^1]$. We have the commutative diagram

$$\begin{tikzcd}
A[1] \arrow[r, tail] \arrow[d, tail] \arrow[dd, tail, bend right=60, "f^*"'] & A'[0] \cup_{A[0]} A[\Delta^1] \arrow[d, tail] & \\
A'[1] \arrow[r, tail] \arrow[d, "f'"'] & A'[0] \cup_{A[0]} A[\Delta^1] \cup_{A[1]} A'[1] \arrow[r, tail] \arrow[d] & A'[\Delta^1] \arrow[d] \\
B' \arrow[r, tail] & A'[0] \cup_{A[0]} A[\Delta^1] \cup_{f^*} B' \arrow[r] & A'[\Delta^1] \cup_{f'} B'
\end{tikzcd}$$

where every square and in particular the lower right one is a pushout (by the same reasoning as in the proof of Lemma 2.16). The lower right horizontal map is the map (10). Hence using Lemma 2.2 one last time it follows that the map (10) is a cellular inclusion. □

The Cylinder Functor T satisfies the Cylinder Axiom with respect to the homotopy equivalences. We defer the proof until we actually define what a homotopy equivalence is. (The definition involves the Cylinder Functor.) We state it here for completeness.

Lemma 2.18 (Cylinder Axiom)**.** *T satisfies the Cylinder Axiom, i.e. $T(f) \to B$ is a homotopy equivalence for all $f\colon A \to B$ in $\mathcal{C}$..*

Proof. This will be proved in 2.33. □

Remark 2.19. Compared to the topological case the mapping cylinders we get are *reduced* as $*[\Delta^1] = *$. This is necessary for the Cylinder Functor to be exact. In particular T respects *cofiber sequences* (cf. 2.43, see the proof of Lemma 2.48 for an application of this property).

2.4. Homotopies, horn-filling and the homotopy extension property

We can apply the Cylinder Functor to $\mathrm{id}_A\colon A \to A$ and get the module $T(\mathrm{id}_A) = A[\Delta^1]$ together with maps $\iota_i\colon A[i] \to A[\Delta^1]$, $i = 0, 1$, and $p\colon A[\Delta^1] \to A$. We call $A[\Delta^1]$ a *cylinder* for A and use it to define the notion of homotopy in $\mathcal{C}^G$.

Definition 2.20. *Let $A, B \in \mathcal{C}^G$ and let f, g be maps $A \to B$. A homotopy from f to g is a map $H\colon A[\Delta^1] \to B$ such that $H \circ \iota_0\colon A[0] \to A[\Delta^1] \to B$ is equal to f and $H \circ \iota_1\colon A[1] \to A[\Delta^1] \to B$ is equal to g. Then f and g are said to be homotopic, written $f \simeq g$.*

We sometimes abbreviate $H \circ \iota_0$ as H_0, etc. If $C \rightarrowtail A$ is a cellular inclusion (or more generally any map) we say that a homotopy $H\colon A[\Delta^1] \to B$ is *relative to* C or *constant* or *trivial* on C if the induced composed homotopy $C[\Delta^1] \to A[\Delta^1] \to B$ factors over $p\colon C[\Delta^1] \to C$, i.e. it can be written as the composition $C[\Delta^1] \to C \to B$.

To actually show something about homotopies we need the horn-filling property, which we discuss next.

Let Δ^n be the standard n-simplex. The *ith horn* Λ^n_i is the simplicial subset of Δ^n generated by all but the ith face of Δ^n. Recall that a simplicial set K is called *fibrant* or *Kan* if for all n and i every map $\Lambda^n_i \to K$ can be extended to a map $\Delta^n \to K$. This process is called "horn-filling", so K is also said to have the "horn-filling" property or the Kan Extension Property (after [Kan57]). Kan sets are homotopically nice, e.g. if K and L are Kan, being homotopic is an equivalence relation on maps $L \to K$, and such a map is an π_*-isomorphism if and only if it is a homotopy equivalence. See the [GJ99, I.3] for the basic definitions of horns and horn-filling in the case of simplicial sets and a proof that any simplicial group has the Kan Extension Property.

We need the analogous property in the category $\mathcal{C}^G$. The following lemma and its proof is a prototype for a lot of arguments that will follow. Its key ingredient is the Fundamental Lemma 1.32 and this is in fact the reason why we call 1.32 the "Fundamental Lemma".

Lemma 2.21 (Horn-filling)**.** *Let $\Lambda^n_i \subseteq \Delta^n$ be a horn. Given objects M, P from $\mathcal{C}^G$. Then any E-controlled map $M[\Lambda^n_i] \to P$ can be extended to an E-controlled map $M[\Delta^n] \to P$. We say that we can fill the horn $M[\Lambda^n_i]$ over P.*

Proof. We want a dashed map in

$$\begin{tikzcd}
M[\Lambda^n_i] \arrow[r] \arrow[d] & P \\
M[\Delta^n] \arrow[ur, dashed] &
\end{tikzcd},$$

where $M[\Lambda^n_i] \to M[\Delta^n]$ is Δ-controlled, i.e. induced by $\Lambda^n_i \to \Delta^n$. By the Fundamental Lemma 1.32 this is equivalent to the existence of a dashed map (of simplicial sets) in

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d] & \mathrm{HOM}^E_R(M, P)^G \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}.$$

But $\mathrm{HOM}^E_R(M, P)^G$ is a simplicial (abelian) group, hence it has the horn-filling property for simplicial sets. So the dashed lift exists. □

In the following we use the notions "Kan Extension Property" and "horn-filling" interchangeably. We need a relative version of horn-filling.

Lemma 2.22 (Relative horn-filling)**.** *Let $M, P \in \mathcal{C}^G$. Let A be a cellular submodule of M, let $\Lambda^n_i \subseteq \Delta^n$ be a horn. Any controlled maps $A[\Delta^n] \to P$ and $M[\Lambda^n_i] \to P$ which agree on $A[\Lambda^n_i]$ can be extended to a controlled map $M[\Delta^n] \to P$.*

Remark 2.23. Checking the proof shows that the extension is only $E' \circ E^n$-controlled, with M being E-controlled, E' the control of the maps to P and n the dimension of M. This is not as good as in the previous Lemma, but sufficient for our applications.

Proof of 2.22. First assume that M arises from A by attaching only one k-cell and that $G = \{1\}$. We prove that

$$\begin{tikzcd}
R[\Delta^k \times \Lambda^n_i \cup \partial\Delta^k \times \Delta^n] \arrow[r] \arrow[d] & M[\Lambda^n_i] \cup_{A[\Lambda^n_i]} A[\Delta^n] \arrow[d] \\
R[\Delta^k \times \Delta^n] \arrow[r] & M[\Delta^n]
\end{tikzcd}$$

is a pushout in $\mathcal{C}^G$. First M is the pushout

$$\begin{tikzcd}
R[\partial\Delta^k] \arrow[r] \arrow[d] & A \arrow[d] \\
R[\Delta^k] \arrow[r] & M
\end{tikzcd} .$$

As adjoining a simplicial set commutes with colimits we get pushouts

$$\begin{tikzcd}
R[\partial\Delta^k][\Delta^n] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(A)}"] & A[\Delta^n] \arrow[d] \\
R[\Delta^k][\Delta^n] \arrow[r] & M[\Delta^n]
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
R[\partial\Delta^k][\Lambda^n_i] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(B)}"] & A[\Lambda^n_i] \arrow[d] \\
R[\Delta^k][\Lambda^n_i] \arrow[r] & M[\Lambda^n_i]
\end{tikzcd} .$$

Further $R[\Delta^k \times \Lambda^n_i \cup \partial\Delta^k \times \Delta^n]$ and $M[\Lambda^n_i] \cup_{A[\Lambda^n_i]} A[\Delta^n]$ are the pushouts

$$\begin{tikzcd}
R[\partial\Delta^k][\Lambda^n_i] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(C)}"] & R[\Delta^k][\Lambda^n_i] \arrow[d] \\
R[\partial\Delta^k][\Delta^n] \arrow[r] & R[\Delta^k \times \Lambda^n_i \cup \partial\Delta^k \times \Delta^n]
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
A[\Lambda^n_i] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(D)}"] & M[\Lambda^n_i] \arrow[d] \\
A[\Delta^n] \arrow[r] & M[\Lambda^n_i] \cup_{A[\Lambda^n_i]} A[\Delta^n]
\end{tikzcd} .$$

We have a larger diagram

$$\begin{tikzcd}
R[\partial\Delta^k][\Delta^n] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(E)}"] & R[\Delta^k \times \Lambda^n_i \cup \partial\Delta^k \times \Delta^n] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(F)}"] & R[\Delta^k][\Delta^n] \arrow[d] \\
A[\Delta^n] \arrow[r] & M[\Lambda^n_i] \cup_{A[\Lambda^n_i]} A[\Delta^n] \arrow[r] & M[\Delta^n]
\end{tikzcd} .$$

We want that the square (F) is a pushout. This now follows from the rules for composing pushouts: We first show that (E) is a pushout. We write (B) + (D) for the "composition" (stacking) of the diagrams (B) and (D) along the only common map, etc. We have that (B) + (D) is a pushout which is the same diagram as (C) + (E). As (C) is a pushout it follows that (E) is one. Then (E) + (F) is the pushout (A), therefore (F) is one.

We want to find a dashed lift in

$$\begin{tikzcd}
R[\Delta^k \times \Lambda^n_i \cup \partial\Delta^k \times \Delta^n] \arrow[r] \arrow[d] & M[\Lambda^n_i] \cup_{A[\Lambda^n_i]} A[\Delta^n] \arrow[r] \arrow[d] & P \\
R[\Delta^k \times \Delta^n] \arrow[r] & M[\Delta^n] \arrow[ur, dashed] &
\end{tikzcd} , \qquad (11)$$

but as the square is a pushout it suffices to find a lift $R[\Delta^k \times \Delta^n] \to P$. For this it suffices by the Fundamental Lemma 1.33 to find a lift in the diagram of simplicial sets

$$\begin{tikzcd}
\Delta^k \times \Lambda^n_i \cup \partial\Delta^k \times \Delta^n \arrow[r] \arrow[d] & \mathrm{HOM}^{\mathcal{E}}_R(\underline{R}, P) \\
\Delta^k \times \Delta^n \arrow[ur, dashed] &
\end{tikzcd} .$$

Such a lift exists as the vertical inclusion arises by repeated horn-filling (cf. [GJ99, p. 18/19]) and $\mathrm{HOM}^{\mathcal{E}}_R(\underline{R}, P)$ is an abelian group and hence Kan.

If $G \neq \{1\}$ we have to find a lift $R[G/1][\Delta^k \times \Delta^n]$, but we can first find the lift for $R[\{1\}/1][\Delta^k \times \Delta^n]$ and then extend equivariantly. Alternatively the whole argument works for R replaced with $R[G/1]$ and the equivariance of the Fundamental Lemma.

If we want to attach more than one cell we have to take care of the control conditions. Assume M is E-controlled. We can arrange Diagram (11) such that the left horizontal maps are E-controlled (and the left vertical map is Δ-controlled). If the right horizontal map to P is E'-controlled, then by the first part of the Fundamental Lemma 1.32 the lift is $E' \circ E$-controlled.

Assume that M arises from A by attaching only cells of dimension k. As they can be attached individually we can take the lift for each cell, each of which is $E' \circ E$-controlled, and glue it to a common lift, which therefore is again $E' \circ E$-controlled. The general case follows by induction and the finite-dimensionality of M. □

Remark 2.24. This is one of the places where we need the finite-dimensionality of our modules, at least for the proof of the lemma.

Corollary 2.25. *Let M, N be in $\mathcal{C}^G$. Being homotopic is an equivalence relation on $\mathrm{Hom}_{\mathcal{C}^G}(M, N)$.*

For the proof of the corollary and following proofs it is convenient to have a diagram language available. We provide a digression to describe the language we will use. So the reader can skip to Definition 2.27 if the following proof of Corollary 2.25 is clear.

We need a quick definition: For each map $f\colon A \to B$ we have the *constant* or *trivial homotopy* $\mathrm{Tr}_f\colon A[\Delta^1] \xrightarrow{p} A \xrightarrow{f} B$ from f to f.

Proof of Corollary 2.25 using notation from the digression below. Let f, g and h be maps $A \to B$ and H, $G\colon A[\Delta^1] \to B$ be homotopies such that $H_0 = f$, $H_1 = g = G_0$ and $G_1 = h$. We use that by Lemma 2.21 we can fill horns like $A[\Lambda^2_1] \to B$.

The map f is homotopic to itself via the trivial homotopy Tr_f,

$$\begin{tikzcd} f \arrow[r, "\mathrm{Tr}"] & f \end{tikzcd}\,.$$

This shows reflexivity. From H and G we get a homotopy F from f to h by filling the horn

$$\begin{tikzcd} & h \\ f \arrow[r, "H"] & g \arrow[u, "G"] \end{tikzcd} \quad \text{to} \quad \begin{tikzcd} & h \\ f \arrow[r, "H"] \arrow[ur, "F"] & g \arrow[u, "G"] \end{tikzcd}\,.$$

This proves transitivity. For H we get a homotopy $\overline{H}$ from g to f by filling a horn

$$\begin{tikzcd} & f \\ f \arrow[r, "H"] \arrow[ur, "\mathrm{Tr}"] & g \end{tikzcd} \quad \text{to} \quad \begin{tikzcd} & f \\ f \arrow[r, "H"] \arrow[ur, "\mathrm{Tr}"] & g \arrow[u, "\overline{H}"] \end{tikzcd}\,.$$

This shows symmetry and finishes the proof. ☐

The same proof, using relative horn-filling, shows:

Corollary 2.26. *Let M, N be in $\mathcal{C}^G$, let $A \subseteq M$ be a cellular submodule. Being homotopic relative A is an equivalence relation on* $\mathrm{Hom}_{\mathcal{C}^G}(M, N)$. ☐

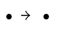
2.4.1. Digression: Describing maps by diagrams

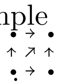
In the following we will often have to describe maps like $A[\Delta^1 \times \Delta^1] \to B$ or $A[\Delta^1 \cup_{\Delta^0} \Delta^1] \to B$. Here we introduce some notation we will use. We tried to make it intuitive so that one can read off the map directly from a single diagram.

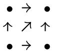
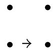
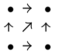
If one recognizes the simplicial sets which are meant by pictures like

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd}\,, \quad \begin{tikzcd} \bullet \arrow[r] & \bullet \\ \bullet \arrow[r] \arrow[u] \arrow[ur] & \bullet \arrow[u] \end{tikzcd}\,, \quad \begin{tikzcd} \bullet & \bullet \\ \bullet \arrow[r] \arrow[u] & \bullet \end{tikzcd}\,, \quad \begin{tikzcd} \bullet & \bullet \\ \bullet & \bullet \arrow[u] \end{tikzcd}\,, \quad \text{or} \quad \begin{tikzcd} \bullet \arrow[r] & \bullet \\ \bullet \arrow[r] \arrow[u] \arrow[ur] & \bullet \end{tikzcd}$$

one can skip the next two paragraphs in this section, but one should have a look at the third paragraph, where we discuss subsets of $\Delta^1 \times \Delta^1$.

There are some simplicial sets which come from (abstract) simplicial complexes and hence can be described combinatorially and even by pictures. Recall that an (abstract) simplicial complex consists of a set of vertices which is partially ordered, together with $(k+1)$-element subsets for each $k \geq 0$ on which the induced ordering is total and which are called the k-simplices (cf. [Hat02, p. 107]). The easiest example (except the point) is probably the simplicial interval Δ^1. We draw the corresponding simplicial complex as  , which shows the two 0-simplices (vertices) and the one 1-simplex (arrow). The direction of the arrow determines the order of the vertices, it goes from the smaller one to the bigger one. We will call the smaller vertex the 0th vertex and the bigger one the 1st vertex. Note that when we talk about the boundary, the notation is different: One gets the 0th boundary by *leaving out the 0th simplex*, hence the 0th boundary is the 1st vertex!

A more complicated example of a simplicial complex would be the model for $\Delta^1 \times \Delta^1$ which we draw as  . It consists of four 0-simplices (vertices), five 1-simplices (arrows) and two 2-simplices (triangles enclosed by arrows). One 2-simplex consists of the three upper left vertices and the other one consists of the three lower right vertices. Note that the arrows are the most important part, the dots are merely there to make the pictures easier to read. If three of the arrows enclose a triangle this should always mean that the corresponding 2-simplex is in the simplicial complex.

We make an important exception when we want to draw subsets of $\Delta^1 \times \Delta^1 =$  . If we draw  , this should denote the simplicial complex generated by the lower horizontal 1-simplex. This includes the lower vertices as endpoints, but it should not include the upper vertices. They are just drawn to make it unambiguous which subset of  we mean.

As a simplicial complex uniquely determines a simplicial set, each of the pictures also defines a simplicial set. We are only interested in the simplicial sets and not in the simplicial complexes they depict. (The reader is invited to look at the pictures four paragraphs earlier to recognize the simplicial sets they define.)

We are interested in controlled simplicial R-modules of the form $A[\Delta^1 \times \Delta^1]$, for A any controlled simplicial R-module, and want to describe maps out of it. Recall first, that each simplicial set is the "union of its simplices", or more precisely the colimit of standard simplices indexed over the simplex category (cf. [GJ99, p. 6/7 and Lemma 2.1]), and we can index over the non-degenerated simplices only.

Therefore the simplicial R-module $A[\Delta^1 \times \Delta^1]$ is a quotient of

$$A[\Delta^2] \quad \amalg \quad A[\Delta^2] \quad \amalg \quad \coprod^5 A[\Delta^1] \quad \amalg \quad \coprod^4 A[\Delta^0].$$

Hence to give a map $A[\Delta^1 \times \Delta^1] \to B$ we only have to give *compatible* maps

$$A[\Delta^2] \to B \qquad A[\Delta^2] \to B \qquad \coprod^5 A[\Delta^1] \to B \qquad \coprod^4 A[\Delta^0] \to B. \tag{12}$$

For the induced map $A[\Delta^1 \times \Delta^1] \to B$ to be controlled it suffices that both maps $A[\Delta^2] \to B$ are controlled as $A[\Delta^1 \times \Delta^1]$ is the pushout along a cellular inclusion in

$$\begin{tikzcd}
A[\Delta^1] \amalg A[\Delta^1] \arrow[r, tail] \arrow[d] & A[\Delta^2] \amalg A[\Delta^2] \arrow[d] \\
A[\Delta^1] \arrow[r, tail] & A[\Delta^1 \times \Delta^1]
\end{tikzcd}.$$

The last two groups of maps in (12) have to be controlled because compatibility implies they are restrictions of the first two maps. We will develop a graphical language to specify these maps now.

Of course the maps $A[\Delta^2] \to B$ determine the other ones, so we only need to give these. However we will only write down the maps on the boundary, as these will be more important for us. This makes it easier to draw the pictures and the map on the 2-simplex itself will be specified in the text. The pictures we will draw look like the following:

$$\begin{tikzcd}
 & r . \\
p \arrow[ur, "\beta"] \arrow[r, "\gamma"'] & q \arrow[u, "\alpha"']
\end{tikzcd}$$

This should denote a map $A[\Delta^2] \to B$ as follows. Restricted to the 0th, 1st or 2nd boundary it is α, β or $\gamma\colon A[\Delta^1] \to B$, respectively. Restricted to the 0th, 1st or 2nd vertex it is p, q, or $r\colon A[\Delta^0] = A \to B$, respectively. Sometimes we will draw the decoration on the arrow, sometimes beside it if it improves the readability.

More examples are drawn below. The left picture shows a homotopy from α to β, the middle shows a horn $A[\Lambda^2_0] \to B$, and the right the map $A[\Delta^2] \to B$ which arises by filling the horn in the middle.

$$\begin{tikzcd}
\alpha \arrow[r, "H" description] & \beta
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \alpha \\
\alpha \arrow[ur, "\mathrm{Tr}" description] \arrow[r, "H" description] & \beta
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \alpha \\
\alpha \arrow[ur, "\mathrm{Tr}" description] \arrow[r, "H" description] & \beta \arrow[u, "\overline{H}" description]
\end{tikzcd}$$

The right diagram shows how we proved the symmetry of the relation "homotopic" above. We sometimes call $\overline{H}$ the "inverse homotopy" to H.

Sometimes we leave out the decorations for vertices, as they are uniquely determined by the decorations on the arrows, and draw dots instead. All this works for more complicated simplicial sets as long as we can draw diagrams for them.

This ends the digression.

The next goal is to show that cofibrations have the homotopy extension property.

Definition 2.27. *Let A, B, P be in $\mathcal{C}^G$. A map $A \to B$ has the homotopy extension property (HEP) if for all maps*

$$A[\Delta^1] \to P, \qquad B[0] \to P$$

which coincide when precomposed with $A[0] \to A[\Delta^1]$ resp. $A[0] \to B[0]$ there is a map $B[\Delta^1] \to P$ extending both, i.e. the lift in

$$\begin{tikzcd}
A[\Delta^1] \cup_{A[0]} B[0] \arrow[r] \arrow[d] & P \\
B[\Delta^1] \arrow[ur, dashed] &
\end{tikzcd} \tag{13}$$

exists.

Remark 2.28. The pushout $A[\Delta^1] \cup_{A[0]} B[0]$ exists by Lemma 2.2. Of course the homotopy extension property for arbitrary P and f is equivalent to the special case where $P = A[\Delta^1] \cup_{A[0]} B[0]$ and $f = \mathrm{id}$.

Lemma 2.29. *Cofibrations have the homotopy extension property.*

Proof. It suffices to prove the Lemma for cellular inclusions, because if f is isomorphic to f' then f has the HEP if and only if f' has the HEP. Then it is the relative horn-filling property for $\Lambda^1_0 \subseteq \Delta^1$ of Lemma 2.22. $\square$

2.5. Homotopy equivalences

Being homotopic is an equivalence relation on controlled maps $A \to B$ in $\mathcal{C}^G$ by Corollary 2.25, so we do not have to care if our homotopies go from 0 to 1 or vice versa and further can concatenate homotopies. We will use this freely if needed.

There is an obvious notion of homotopy equivalences.

Definition 2.30. *A map $f\colon A \to B$ is a homotopy equivalence in $\mathcal{C}^G$ if there is a map $g\colon B \to A$, its homotopy inverse, such that $f \circ g$ is homotopic to id_B and $g \circ f$ is homotopic to id_A.*

If for $A, B \in \mathcal{C}^G$ there is such a homotopy equivalence, then A and B are called to be homotopy equivalent. We write $A \simeq B$ for homotopy equivalent modules.

As this is a very important definition in this thesis let us unwrap the definition a bit. Recall that the objects A and B in $\mathcal{C}^G = \mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})$ are cellular simplicial R-modules with a free cell-permuting action of G and an equivariant control map to X. Then $f\colon A \to B$ being a (controlled) homotopy equivalence means that there is an $E \in \mathcal{E}$ and an E-controlled map $g\colon B \to A$ as well as E-controlled homotopies $H^A\colon A[\Delta^1] \to A$ from $g \circ f$ to id_A and $H^B\colon B[\Delta^1] \to B$ from $f \circ g$ to id_B. Note that the maps f and g and in particular the homotopies are required to be G-equivariant where $A[\Delta^1]$ and $B[\Delta^1]$ inherit the G-action from A and B respectively.

In the following we will not explicitly mention the control conditions $E \in \mathcal{C}$, as all maps in $\mathcal{C}^G$ are by definition E-controlled for some $E \in \mathcal{E}$. Note however, that some of our constructions increase the control conditions, i.e. they result in maps which are only E'-controlled with $E \subseteq E'$. An example for this is the pushout-construction of Lemma 2.2. Sometimes it is important that the control conditions are not increased and we will explicitly mention this in that cases. An example for that is the horn-filling property of Lemma 2.21 above.

The homotopy equivalences in $\mathcal{C}^G$ will be our category of *weak equivalences* for the category with cofibrations (cf. B.2). We now prove the *Saturation Axiom* as well as the *Cylinder Axiom* for homotopy equivalences in $\mathcal{C}^G$.

Recall that the saturation axiom (see Appendix B/[Wal85]) for weak equivalences means that if for maps $f\colon A \to B$, $g\colon B \to C$ two of the maps f, g, and $g \circ f$ are weak equivalences then so is the third one. (This is sometimes also called the “2-of-3”-property.)

Lemma 2.31 (Saturation Axiom)**.** *The homotopy equivalences in $\mathcal{C}^G$ satisfy the 2-of-3 property.*

Proof. Given maps f, g, and h with $g \circ f = h$,

$$\begin{tikzcd}
& \bullet \arrow[dr, "g"] & . \\
\bullet \arrow[ur, "f"] \arrow[rr, "h"'] & & \bullet
\end{tikzcd}$$

We prove only one case, the other two are similar. Note that for $\beta \simeq \beta'$ we have $\alpha \circ \beta \circ \gamma \simeq \alpha \circ \beta' \circ \gamma$ and that homotopy inverses are unique up to homotopy, as for two homotopy inverses $\overline{f}$, $\overline{f}'$ of f we have $\overline{f} \simeq \overline{f} \circ f \circ \overline{f}' \simeq \overline{f}'$.

Assume that g and h have homotopy inverses $\overline{g}$ and $\overline{h}$. Set

$$\overline{f} := \overline{h} \circ g.$$

Then $\overline{f}$ is a homotopy inverse for f as

$$\begin{aligned}
f \circ \overline{f} = f \circ \overline{h} \circ g &\simeq \overline{g} \circ g \circ f \circ \overline{h} \circ g \\
&= \overline{g} \circ h \circ \overline{h} \circ g \\
&\simeq \overline{g} \circ g \\
&\simeq \mathrm{id}
\end{aligned}$$

and

$$\overline{f} \circ f = \overline{h} \circ g \circ f = \overline{h} \circ h \simeq \mathrm{id}\ .$$

□

Note that the construction of homotopy inverses by this lemma gives a control condition for the inverse and the homotopies which are usually much bigger than the original control conditions involved before.

For the next result we need the notion of a (controlled) *deformation retraction* in $\mathcal{C}^G$. This is the expected notion, but we define it nonetheless.

Definition 2.32 (Deformation retraction)**.** *Let $i\colon A \rightarrowtail M$ be a cellular inclusion in $\mathcal{C}^G$, i.e. we can consider A as a submodule of M. A is a deformation retract of M if there is a map $r\colon M \to A$ such that $r \circ i$ is id_A and $i \circ r$ is homotopic to id_M relative A.*

The map i is called the inclusion and r is called the retraction or deformation retraction.

In particular each deformation retraction is a homotopy equivalence and if A is a deformation retract of M then in particular the inclusion $A \to M$ is a homotopy equivalence. (But being a deformation retraction is of course a much stronger property, as we will see later.)

For the Cylinder Axiom recall that the Cylinder Functor T of Proposition 2.13 gives in particular a retraction $p\colon T(f) \to B$ for every map $f\colon A \to B$.

Lemma 2.33 (Cylinder Axiom)**.** *The map $p\colon T(f) \to B$ is a deformation retraction. Therefore T satisfies the Cylinder Axiom (Lemma 2.18/[Wal85, 1.6]).*

Proof. We only have to prove that $\iota_1 \circ p\colon T(f) \to B \to T(f)$ is homotopic relative B to $\mathrm{id}_{T(f)}$. Recall that $T(f)$ is defined as the pushout of $B \leftarrow A[1] \to A[\Delta^1]$. We see that $\iota_1 \circ p$ is induced by $p_1\colon A[\Delta^1] \to A[1] \to A[\Delta^1]$. The following diagram of pushouts shows the situation

$$\begin{tikzcd}
B \arrow[d] & A[1] \arrow[l] \arrow[r] \arrow[d] & A[\Delta^1] \arrow[d] & T(f) . \arrow[d, "p"] \\
B \arrow[d] & A[1] \arrow[l] \arrow[r] \arrow[d] & A[1] \arrow[d] & B \arrow[d, "i"] \\
B & A[1] \arrow[l] \arrow[r] & A[\Delta^1] & T(f)
\end{tikzcd}$$

As $-[\Delta^1]$ commutes with pushouts we only have to give a homotopy H from $\mathrm{id}_{A[\Delta^1]}$ to p_1 which is relative to $A[1]$, i.e. a map $A[\Delta^1 \times \Delta^1] \to A[\Delta^1]$ with

$$\begin{aligned}
&A[\Delta^1 \times 0] \to A[\Delta^1 \times \Delta^1] \to A[\Delta^1] = \mathrm{id} \quad \text{and} \\
&A[\Delta^1 \times 1] \to A[\Delta^1 \times \Delta^1] \to A[\Delta^1] = p_1 \ .
\end{aligned}$$

But there is a well-known map $\widehat{H}\colon \Delta^1 \times \Delta^1 \to \Delta^1$ of simplicial sets inducing such a map.

(That map can, similar to our notation above, be described as follows. Denoting the simplices of Δ^1 as $0 \to 1$ we could just write $\widehat{H}$ down as

$$\begin{tikzcd}
1 \arrow[r] & 1 , \\
0 \arrow[u] \arrow[r] \arrow[ur] & 1 \arrow[u]
\end{tikzcd}$$

cf. [Lam68, I.5.4] for the “dual case”.)

Thus the homotopy H which is induced by $\widehat{H}$ is a homotopy relative to $A[1]$ which induces the desired homotopy. ☐

2.6. Homotopy equivalences and mapping cylinders

The Cylinder Functor T from Lemma 2.13 gives for each map $f\colon A \to B$ in $\mathcal{C}^G$ a mapping cylinder, i.e. a factorization of f as $p \circ \iota_0\colon A \to T(f) \to B$ where ι_0 is a cellular inclusion. We have just proved in Lemma 2.33 that p is homotopy equivalence, so by the Saturation Axiom 2.31 it follows that f is homotopy equivalence if and only if ι_0 is one. However, much more is true.

Proposition 2.34. *Let $f\colon A \to B$ be a homotopy equivalence in $\mathcal{C}^G$. Then A is a deformation retract of $T(f)$ via the inclusion ι_0.*

The proof takes several steps and occupies the rest of this section. It is an adaption of the corresponding proof for topological spaces which the author learned from F. Waldhausen [WalAT, pp. 140ff.].

Let $f\colon A \to B$ be the controlled homotopy equivalence from above. Let us collect and name the data we have: We have maps

$$f\colon A \to B, \qquad g\colon B \to A$$

and homotopies

$$H^A\colon A[\Delta^1] \to A, \qquad H^B\colon B[\Delta^1] \to B$$

from $g \circ f$ to id_A, resp. from $f \circ g$ to id_B. Assume all of these are controlled by $E \in \mathcal{E}$.

We will construct:

(i) A retraction $T(f) \xrightarrow{r} A$.

(ii) A homotopy $\iota_0 \circ r \simeq \mathrm{id}_{T(f)}$.

(iii) A better homotopy $\iota_0 \circ r \simeq \mathrm{id}_{T(f)}$ which is relative to A.

All these maps will satisfy control conditions. These three steps suffice to prove Proposition 2.34, so proof will be complete after proving the last step in Lemma 2.39.

For the first step we prove a general lemma first.

Lemma 2.35. *Suppose we have a map $f\colon A \to B$. Let $g\colon B \to P$ be a map such that $g \circ f$ is homotopic via $H\colon A[\Delta^1] \to P$ to a map $h\colon A \to P$. Suppose all maps are E-controlled. Then there exists an E-controlled map*

$$T(f) \to P$$

such that composition with the front inclusion $A \to T(f) \to P$ is equal to h and the composition with the back inclusion $B \to T(f) \to P$ is equal to g.

Remark 2.36. The lemma is useful when converting a homotopy commutative diagram as on the left below into the strict commutative one on the right below,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & P \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[r, "\iota_0"] \arrow[dr, "h"'] & T(f) \arrow[d] \\ & P \end{tikzcd},$$

where $T(f) \simeq B$. It is stated here such that we can refer to it later.

Proof. The mapping cylinder $T(f)$ was defined as the pushout $B \cup_{A[1]} A[\Delta^1]$. To construct the desired map it therefore suffices to use the strict commutative diagram

$$\begin{tikzcd}
A[\Delta^1] \arrow[dr, "H"] & \\
A[1] \arrow[u, tail] \arrow[r, "g\circ f"] \arrow[d, "f"'] & P \\
B \arrow[ur, "g"'] &
\end{tikzcd}$$

to get an induced map $T(f) \to P$ which satisfies all the desired properties, in particular it is E-controlled, which can be looked up in Table 2.1 on page 37. ☐

Lemma 2.37. *There is a retraction $r\colon T(f) \to A$ for the inclusion $\iota_0\colon A \rightarrowtail T(f)$. The composition with the back inclusion $B \xrightarrow{\iota_1} T(f) \xrightarrow{r} A$ is equal to g.*

Proof. To construct the retraction we apply Lemma 2.35 with $P := A$, $H := H^A$ and $f := f, g := g$. Let r be the map given by the Lemma. ☐

Now we have to construct a homotopy from $T(f) \xrightarrow{r} A \to T(f)$ to the identity. This might not be relative to A, but we will correct this later.

Lemma 2.38. *The map $T(f) \xrightarrow{r} A \xrightarrow{\iota_0} T(f)$ is homotopic to the identity.*

Proof. Recall from Lemma 2.33 that $T(f) \to B$ is a deformation retraction, i.e. the composition $T(f) \xrightarrow{p} B \xrightarrow{\iota_1} T(f)$ is a homotopic to the identity. So we pre- and postcompose $T(f) \to A \to T(f)$ with $T(f) \to B \to T(f)$ and get a map which is homotopic to it. This can be written as

$$\begin{tikzcd}
T(f) \arrow[r, "r"] & A \arrow[r, "\iota_0"] \arrow[dr, "f"'] & T(f) \arrow[d, "p"] \\
B \arrow[u, "\iota_1"] \arrow[ur, "g"'] & & B \arrow[d, "\iota_1"] \\
T(f) \arrow[u, "p"] & & T(f)
\end{tikzcd}$$

with compositions identified as f and g. But $f \circ g$ is homotopic to id_B via H^B by assumption. So we are left with

$$\begin{tikzcd}
B \arrow[r, "\mathrm{id}"] & B \arrow[d, "\iota_1"] \\
T(f) \arrow[u, "p"] & T(f)
\end{tikzcd}$$

which is homotopic to $\mathrm{id}_{T(f)}$ again by Lemma 2.33. Being homotopic is an equivalence relation by Corollary 2.25 so $\iota_0 \circ r$ is homotopic to $\mathrm{id}_{T(f)}$. ☐

Let $s := \iota_0 \circ r \colon T(f) \to A \to T(f)$ and let H be the homotopy from $\mathrm{id}_{T(f)}$ to s we get by Lemma 2.38. We have $s \circ \iota_0 = \iota_0$ as well as $\mathrm{id}_{T(f)} \circ \iota_0 = \iota_0$ so on the endpoints H is relative to the cellular inclusion $\iota_0 \colon A \to T(f)$. We want to make the whole homotopy relative to A, i.e. $A[\Delta^1] \xrightarrow{\iota_0[\Delta 1]} T(f)[\Delta^1] \xrightarrow{H} T(f)$ should be equal to $A[\Delta^1] \xrightarrow{p} A \xrightarrow{\iota_0} T(f)$.

Lemma 2.39. *Let s be the map $T(f) \xrightarrow{r} A \xrightarrow{\iota_0} T(f)$. There is a homotopy relative A from the identity on $T(f)$ to s.*

Proof. (We use the diagram notation of Digression 2.4.1.) A is a retract of $T(f)$ and $\iota_0 \colon A \to T(f)$ has the homotopy extension property. We will use this homotopy extension property to construct a certain map $T(f)[\Delta^1 \times \Delta^1] \to T(f)$ which restricted to $1 \times \Delta^1$ will be the desired homotopy from $\mathrm{id}_{T(f)}$ to s relative to A.

Note that s is an idempotent, i.e. $s^2 = s$. We use the notation from above. The proof will proceed as follows. We will prescribe the map $T(f)[\Delta^1 \times \Delta^1] \to T(f)$ on the subspace $A[\Delta^1 \times \Delta^1] \rightarrowtail T(f)[\Delta^1 \times \Delta^1]$ and on the top, bottom and left part of $\Delta^1 \times \Delta^1 = \begin{tikzcd}\bullet \arrow[r] & \bullet \\ \bullet \arrow[u] \arrow[r] \arrow[ur] & \bullet \arrow[u]\end{tikzcd}$, i.e. on $T(f)[\begin{tikzcd}\bullet \arrow[r] & \bullet \\ \bullet \arrow[u] \arrow[r] & \bullet\end{tikzcd}]$. Then we check that the two maps are compatible. This will give a map

$$T(f)[\begin{tikzcd}\bullet \arrow[r] & \bullet \\ \bullet \arrow[u] \arrow[r] & \bullet\end{tikzcd}] \cup A[\begin{tikzcd}\bullet \arrow[r] & \bullet \\ \bullet \arrow[u] \arrow[r] \arrow[ur] & \bullet \arrow[u]\end{tikzcd}] \to T(f)$$

which can be extended by the homotopy extension property to the desired map $T(f)[\begin{tikzcd}\bullet \arrow[r] & \bullet \\ \bullet \arrow[u] \arrow[r] \arrow[ur] & \bullet \arrow[u]\end{tikzcd}] \to T(f)$.

Both maps will be constructed from the same map, which we describe first. Horn-filling gives for any map $T(f)[\begin{tikzcd}\bullet & \bullet \\ \bullet \arrow[u] \arrow[ur] & \end{tikzcd}] \to T(f)$ a map $T(f)[\begin{tikzcd}\bullet \arrow[r] & \bullet \\ \bullet \arrow[u] \arrow[ur] & \end{tikzcd}] \to T(f)$, in particular we get for the first diagram below the second one, where $\overline{H}$ is the "inverse homotopy". Extending this as in the third diagram below gives a map $G \colon T(f)[\Delta^1 \times \Delta^1] \to T(f)$.

$$\begin{tikzcd}\bullet & \bullet \\ \bullet \arrow[u, "H"] \arrow[ur, "\mathrm{Tr}"] & \end{tikzcd} \qquad \begin{tikzcd}\bullet \arrow[r, dotted, "\overline{H}"] & \bullet \\ \bullet \arrow[u, "H"] \arrow[ur, "\mathrm{Tr}"] & \end{tikzcd} \qquad \begin{tikzcd}\bullet \arrow[r, dotted, "\overline{H}"] & \bullet \\ \bullet \arrow[u, "H"] \arrow[ur, "\mathrm{Tr}"] \arrow[r, "\mathrm{Tr}"] & \bullet \arrow[u, "\mathrm{Tr}"]\end{tikzcd}$$

Define the map $A[\Delta^1 \times \Delta^1] \to T(f)$ as the restriction of G to $A[\Delta^1 \times \Delta^1]$. Define the map $T(f)[\begin{tikzcd}\bullet \arrow[r] & \bullet \\ \bullet \arrow[u] \arrow[r] & \bullet\end{tikzcd}] \to T(f)$ as

$$\begin{tikzcd}\bullet \arrow[r, "\overline{H}\circ s"] & \bullet \\ \bullet \arrow[u, "H"] \arrow[r, "\mathrm{Tr}"] & \bullet\end{tikzcd}$$

so on the $\begin{tikzcd}\bullet & \bullet \\ \bullet \arrow[u] \arrow[r] & \bullet\end{tikzcd}$-part it is the restriction of G, but on the upper part $\begin{tikzcd}\bullet \arrow[r] & \bullet \\ \bullet & \bullet\end{tikzcd}$ we replace the homotopy $\overline{H}$ by $\overline{H} \circ s$. This replacement is crucial for the proof.

We check that these maps are compatible. First $\overline{H}$ is a homotopy from s to id, hence $\overline{H} \circ s$ is a homotopy from s^2 to s; but $s^2 = s$ so it agrees with H on the upper left vertex. Second, restricted to A the map s is the inclusion $\iota_0\colon A \to T(f)$, hence $\overline{H} \circ s \circ \iota_0 = \overline{H} \circ \iota_0$. So this glues to a map

$$T(f)[\ \begin{smallmatrix} \bullet & \rightarrow & \bullet \\ \uparrow & & \\ \bullet & \rightarrow & \bullet \end{smallmatrix}\ ] \cup A[\ \begin{smallmatrix} \bullet & \rightarrow & \bullet \\ \uparrow & \nearrow & \uparrow \\ \bullet & \rightarrow & \bullet \end{smallmatrix}\ ] \to T(f).$$

This can be interpreted as a map $T(f)[0 \times \Delta^1] \to T(f)$ together with a homotopy on the submodule $T(f)[0 \times \{0,1\}] \cup A[0 \times \Delta^1]$. So using the homotopy extension property of Lemma 2.29 we get map $T(f)[\Delta^1 \times \Delta^1] \to T(f)$. This map in turn defines a homotopy when restricting along $T(f)[1 \times \Delta^1] \to T(f)[\Delta^1 \times \Delta^1]$ (which is $T(f)[\ \begin{smallmatrix} \bullet & & \bullet \\ & & \uparrow \\ \bullet & & \bullet \end{smallmatrix}\ ] \to T(f)[\ \begin{smallmatrix} \bullet & \rightarrow & \bullet \\ \uparrow & \nearrow & \uparrow \\ \bullet & \rightarrow & \bullet \end{smallmatrix}\ ]$). This homotopy starts at the identity, ends at the map s and is the constant homotopy on A. Hence it is the desired homotopy. ☐

This proves Proposition 2.34.

2.7. Pushouts of homotopy equivalences which are cofibrations

Lemma 2.40. *Let*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

be a pushout diagram in $\mathcal{C}^G$ where $A \to C$ is a cofibration and a homotopy equivalence. Then $B \to D$ is a homotopy equivalence.

Remark 2.41. This is a key result on the way to prove the Gluing Lemma for homotopy equivalences. Almost exactly the same proof works if we assume that $A \to B$ is a cofibration instead of $A \to C$, as the proof of part (ii) of Lemma 2.42 below is symmetric in B and C.

Proof. We can factor $f\colon A \to C$ into $A \rightarrowtail T(f) \to C$. Taking the pushouts along the cellular inclusion $A \rightarrowtail T(f)$ and along the cofibration $A \rightarrowtail C$ gives a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ T(f) \arrow[r] \arrow[d] & Q \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

and the induced map $Q \to D$ completes the lower square to a pushout square.

The following Lemma shows that both maps $B \to Q$ and $Q \to D$ are homotopy equivalences, so their composition $B \to D$ is one. ☐

Lemma 2.42. *In the situation above the following holds.*

(i) The map $B \to Q$ is a deformation retraction.
(This uses that $A \to C$ is a homotopy equivalence.)

(ii) The map $Q \to D$ is a homotopy equivalence.
(This uses that $A \to C$ is a cofibration.)

Proof of part (i). By Proposition 2.34 A is a deformation retract of $T(f)$ via the cellular inclusion $A \rightarrowtail T(f)$. Therefore we have a retraction $r\colon T(f) \to A$ and a homotopy $H\colon T(f)[\Delta^1] \to T(f)$ from $\iota_0 \circ r$ to $\mathrm{id}_{T(f)}$ which is relative to A. This induces a retraction $Q \to B$ via

$$\begin{tikzcd}
B \arrow[r, "\mathrm{id}"] & B \\
A \arrow[u] \arrow[r, "\mathrm{id}"] \arrow[d, "\iota_0"'] & A \arrow[u] \arrow[d] \\
T(f) \arrow[r, "r"] & A
\end{tikzcd}$$

and a homotopy of $Q \to B \to Q$ to id_Q via

$$\begin{tikzcd}
B[\Delta^1] \arrow[r, "\mathrm{pr}"] & B \\
A[\Delta^1] \arrow[u] \arrow[r, "\mathrm{pr}"] \arrow[d] & A \arrow[u] \arrow[d] \\
T(f)[\Delta^1] \arrow[r, "H"] & T(f)
\end{tikzcd} .$$

As H is relative to A this diagram commutes and gives a map on the pushout with the desired properties. (Here we used that $-[\Delta^1]$ commutes with colimits.)

Hence $Q \to B$ is a deformation retraction. ☐

Proof of part (ii). Written out $Q \to D$ is the map

$$B \cup_{A[1]} A[\Delta^1] \cup_{A[0]} C \to B \cup_A C$$

induced by $A[\Delta^1] \to A$. We have to construct a homotopy inverse for this map. We will construct a homotopy equivalence $A[\Delta^1] \cup_{A[0]} C \to C$ which is relative to $\iota_1^A\colon A[1] \rightarrowtail A[\Delta^1] \cup_{A[0]} C$, resp. to $j_A\colon A \rightarrowtail C$, hence glues along $A[1] \to B$ to the desired homotopy equivalence

$$B \cup_{A[1]} A[\Delta^1] \cup_{A[0]} C \xrightarrow{\simeq} B \cup_A C \ ,$$

as $-[\Delta^1]$ commutes with pushouts. We therefore have to construct for

$$e\colon A[\Delta^1]_{A[0]} C \to C$$

(induced by $A[\Delta^1] \to A$) maps

$$g\colon C \to A[\Delta^1] \cup_{A[0]} C$$

and homotopies

$$\begin{gathered} H\colon C[\Delta^1] \to C \\ G\colon \big(A[\Delta^1] \cup_{A[0]} C\big)[\Delta^1] \to A[\Delta^1] \cup_{A[0]} C \end{gathered}$$

with the properties

$$\begin{aligned} H_0 &= \mathrm{id} & H_1 &= e \circ g \\ G_0 &= \mathrm{id} & G_1 &= g \circ e \\ G \circ \iota_1^A[\Delta^1] &= \iota_1^A & H \circ j_A[\Delta^1] &= j_A \\ g \circ j_A &= \iota_1^A & e \circ \iota_1^A &= j_A\ . \end{aligned}$$

Using the homotopy extension property of the cofibration $j_A\colon A \rightarrowtail C$ (Lemma 2.29) there is a retraction $R\colon C[\Delta^1] \to A[\Delta^1] \cup_{A[0]} C$. Define g as the composition

$$C \xrightarrow{\iota_1^C} C[\Delta^1] \xrightarrow{R} C \cup_{A[0]} A[\Delta^1].$$

We get $g \circ j_A = \iota_1^A$.

Define H as the composition $e \circ R\colon C[\Delta^1] \to C \cup_{A[0]} A[\Delta^1] \to C$. One checks that H is a homotopy from id_C to $e \circ g$ relative to A.

For the other composition consider the commutative diagram

$$\begin{tikzcd}
C \cup_{A[0]} A[\Delta^1] \arrow[rr, "e"] \arrow[dr, "j"'] & & C \arrow[rr, "g"] \arrow[dr, "\iota_1^C"'] & & C \cup_{A[0]} A[\Delta^1] \\
& C[\Delta^1] \arrow[ur, "\mathrm{pr}"] \arrow[rr, dashed] & & C[\Delta^1] \arrow[ur, "R"'] &
\end{tikzcd}$$

where dashed map is the projection to $C[1]$. It is homotopic relative $C[1]$ to the identity. This gives a homotopy G from the composition $g \circ e$ to the identity, using that R is a retraction for j. One checks that G is relative to $A[1]$.

This shows that e is a homotopy equivalence and therefore makes $Q \to D$ into one. $\square$

2.8. The Extension Axiom

Our next goal is to prove the so called Extension Axiom for the homotopy equivalences in $\mathcal{C}^G$. We first recall its definition.

Recollection 2.43 (cf. Appendix B/[Wal85, 1.2]). Let $\mathcal{C}$ be a category with cofibrations. A *cofiber sequence* in $\mathcal{C}$ is a sequence $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{C}$ where $A \rightarrowtail B$ is a cofibration and $B \twoheadrightarrow C$ is isomorphic to the map $B \twoheadrightarrow B/A := B \cup_A *$ in the pushout

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d] & B \arrow[d] & . \\
* \arrow[r, tail] & B/A &
\end{tikzcd}$$

A subcategory $w\mathcal{C}$ of weak equivalences of $\mathcal{C}$ satisfies the *Extension Axiom* if for each map of cofiber sequences

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, "f_A"] & B \arrow[r, two heads] \arrow[d, "f_B"] & C \arrow[d, "f_C"] \\
A' \arrow[r, tail] & B' \arrow[r, two heads] & C'
\end{tikzcd}$$

where f_A and f_C are weak equivalences the map f_B is a weak equivalence. Sometimes B (resp. f_B) is called an *extension* of A by C (resp. of f_A by f_C).

We show that the homotopy equivalences in $\mathcal{C}^G$ satisfy the Extension Axiom. We first need a relative homotopy lifting property.

Lemma 2.44 (Relative homotopy lifting property)**.** *Let $A \rightarrowtail B$ be a cellular inclusion in $\mathcal{C}^G$. Let $U \rightarrowtail P$ also be a cellular inclusion in $\mathcal{C}^G$ and let $P \twoheadrightarrow Q := P/U$ the quotient map. Then $A \to B$ has the relative homotopy lifting property with respect to $P \to Q$. This means, that given a solid commutative diagram of controlled maps*

$$\begin{tikzcd}
A[\Delta^1] \cup B[0] \arrow[r] \arrow[d] & P \arrow[d] \\
B[\Delta^1] \arrow[r] \arrow[ur, dashed] & Q
\end{tikzcd} \tag{14}$$

then the dashed lift exists.

Remark 2.45. This generalizes the fact that cellular inclusions have the homotopy extension property, which is the case $Q = *$ and which is proved in Lemma 2.29. The proof here is similar but we need an additionally lemma (Lemma 2.46 below), which we show after the proof.

The map $P \twoheadrightarrow Q$ is a Δ-controlled map in $\mathcal{C}^G$ and it is a Kan fibration after forgetting the control, as it is a homomorphism of simplicial abelian groups. We give an explicit reference for this in Lemma 2.47.

Proof. Assume both horizontal maps in (14) are E'-controlled and that B is E_B-controlled.

We first assume that A arises from B by attaching only one free G-cell. So we have to solve the lifting problem for the outer diagram

$$\begin{tikzcd}
R[G/1][\partial\Delta^n \times \Delta^1 \cup \Delta^n \times 0] \arrow[r] \arrow[d, tail] & A[\Delta^1] \cup B[0] \arrow[r] \arrow[d, tail] & P \arrow[d] \\
R[G/1][\Delta^n \times \Delta^1] \arrow[r] & B[\Delta^1] \arrow[r] & Q
\end{tikzcd},$$

where the left horizontal maps are E_B-controlled and the leftmost vertical map is Δ-controlled. We need to find a map $R[G/1][\Delta^n \times \Delta^1] \to P$ which fits into the outer diagram. We can apply the Fundamental Lemma 1.32 as the left vertical map is Δ-controlled, i.e. induced by a map on simplicial sets. Note that the composition of both horizontal maps is $E := E_B \circ E'$-controlled.

By the Fundamental Lemma 1.32 the map $R[G/1][\partial\Delta^n \times \Delta^1 \cup \Delta^n \times 0] \to P$ which is E-controlled and G-equivariant corresponds to a map of simplicial sets $\partial\Delta^n \times \Delta^1 \cup \Delta^n \times 0 \to \mathrm{HOM}^E(R[G/1], P)^G$ and $R[G/1][\Delta^n \times \Delta^1] \to Q$ corresponds to $\Delta^n \times \Delta^1 \to \mathrm{HOM}^E(R[G/1], Q)^G$. As $P \twoheadrightarrow Q$ is Δ-controlled it induces a map

$$\mathrm{HOM}^E(R[G/1], P)^G \to \mathrm{HOM}^E(R[G/1], Q)^G$$

where the HOM-spaces *have the same control condition.* So the naturality of the Fundamental Lemma 1.32 gives the diagram

$$\begin{tikzcd}
\partial\Delta^n \times \Delta^1 \cup \Delta^n \times 0 \arrow[r] \arrow[d] & \mathrm{HOM}^E(R[G/1], P)^G \arrow[d] \\
\Delta^n \times \Delta^1 \arrow[r] & \mathrm{HOM}^E(R[G/1], Q)^G
\end{tikzcd}. \tag{15}$$

A lift there provides the desired lift. For such a lift to exist it would be sufficient that the right map is a Kan fibration of simplicial sets, but in general it is not even surjective. But we can find a lift if we extend the control condition on the upper space.

First we can replace $\mathrm{HOM}^E(R[G/1], P)^G$ by $\mathrm{HOM}^E(R, P)$ by Remark 1.34 to get rid of G. Then Lemma 2.46 below provides the desired lift and shows that it is $E_P \circ E$-controlled. This proves the Lemma if A arises from B by attaching only one cell.

For the general case of $A \rightarrowtail B$ a cellular inclusion note that cells of the same degree can be attached independently. Then the procedure above produces inductively for each n a lift $B_n[\Delta^1] \to P$ which has the *fixed* control condition $((E_P)^{n+1} \circ E)$, where B_n is the submodule of B consisting of A and all cells of B up to dimension n. As B is finite-dimensional this finishes the proof. □

For the proof above we needed the following lemma.

Lemma 2.46. *Given the situation of Lemma 2.44, in particular let* $p\colon P \twoheadrightarrow Q$ *be the quotient map as above. Given the diagram*

$$\begin{tikzcd}
\partial\Delta^n \times \Delta^1 \cup \Delta^n \times 0 \arrow[r] \arrow[d] & \mathrm{HOM}_R^E(R,P) \arrow[d] \\
\Delta^n \times \Delta^1 \arrow[r] & \mathrm{HOM}_R^E(R,Q)
\end{tikzcd}$$

and let P *be an* E_P*-controlled module. Then there is a lift of the lower map to a map*

$$\Delta^n \times \Delta^1 \to \mathrm{HOM}_R^{E_P \circ E}(R,P)$$

making the diagram

$$\begin{tikzcd}
\partial\Delta^n \times \Delta^1 \cup \Delta^n \times 0 \arrow[r] \arrow[d] & \mathrm{HOM}_R^{E_P \circ E}(R,P) \arrow[d] \\
\Delta^n \times \Delta^1 \arrow[r] \arrow[ur] & \mathrm{HOM}_R^{E_P \circ E}(R,Q)
\end{tikzcd}$$

commutative.

Proof. The problem is that $\mathrm{HOM}_R^E(R,P) \to \mathrm{HOM}_R^E(R,Q)$ is not surjective in general and hence no Kan fibration.

As $\mathrm{HOM}(R,P) = P$ and $\mathrm{HOM}^E(R,P) \subseteq \mathrm{HOM}^{E_P \circ E}(R,P) \subseteq \mathrm{HOM}(R,P)$ there is a commutative diagram

$$\begin{tikzcd}
\mathrm{HOM}^E(R,P) \arrow[r, "\subseteq"] \arrow[d] & \mathrm{HOM}^{E_P \circ E}(R,P) \arrow[r, "\subseteq"] \arrow[d, "p_2"] & P \arrow[d, "p"] \\
\mathrm{HOM}^E(R,Q) \arrow[r, "\subseteq"] & \mathrm{HOM}^{E_P \circ E}(R,Q) \arrow[r, "\subseteq"] & Q
\end{tikzcd} .$$

The map p_2 is surjective onto its image $\mathrm{Im}(p_2)$ in $\mathrm{HOM}^{E_P \circ E}(R,Q)$ hence it is a Kan fibration by Lemma 2.47 below, as it is a homomorphism of simplicial abelian groups.

We show that $\mathrm{HOM}_R^E(R,Q) \subseteq \mathrm{Im}(p_2)$. Let $\alpha\colon R[\Delta^n] \to Q$ be an element in $\mathrm{HOM}_R^E(R,Q)_n$, let e be the generator of $R[\Delta^n]$. So α is E-controlled and $R[\Delta^n]$ is concentrated over $\kappa(e)$. We have $\alpha(e) = \sum_i r_i \cdot \sigma_i^* e_i^Q$ with $r_i \in R_n$, $r_i \neq 0$, e_i^Q a cell in Q of dimension $\leq n$ and $(\kappa_R(e), \kappa(e_i^Q)) \in E$. As p is a quotient by a cellular inclusion there is for each e_i^Q a (unique) $e_i^P \in P$ with $p(e_i^P) = e_i^Q$, as cells in Q correspond to cells in P which are not in U. As p is Δ-controlled it follows that $\kappa(e_i^P) = \kappa(p(e_i^P)) = \kappa(e_i^Q)$, hence $(\kappa(e), \kappa(e_i^P)) \in E$. As P is E_P-controlled each e_i^P determines an E_P-controlled map $R[\Delta^{k_i}]_i \to P$ where $R[\Delta^{k_i}]_i$ is concentrated over $\kappa(e_i^P)$.

So let $R[\Delta^n]_{i'}$ be concentrated over $\kappa(e)$ such that the composition $R[\Delta^n]_{i'} \xrightarrow{(\sigma_i)_*} R[\Delta^{k_i}]_i \to P$ is $E_P \circ E$-controlled. Hence it is a map in $\mathrm{HOM}^{E_P \circ E}(R,P)$. It follows that

$$\alpha(e) = \sum_i r_i \cdot (\sigma_i)_* p_2(e_i^P) = p_2\Big(\sum_i r_i \cdot (\sigma_i)_* e_i^P\Big),$$

so $\alpha(e) \in \mathrm{Im}(p_2)$. As this is true for all $\alpha \in \mathrm{HOM}^E(R,Q)$ we have $\mathrm{HOM}^E(R,Q) \subseteq \mathrm{Im}(p_2)$. So we have a diagram

$$\begin{tikzcd}
\partial\Delta^n \times \Delta^1 \cup \Delta^n \times 0 \arrow[r] \arrow[d] & \mathrm{HOM}_R^E(R,P) \arrow[r, "\subseteq"] \arrow[d] & \mathrm{HOM}_R^{E_P \circ E}(R,P) \arrow[d, "p_2"] \\
\Delta^n \times \Delta^1 \arrow[r] & \mathrm{HOM}_R^E(R,Q) \arrow[r, "\subseteq"] & \mathrm{Im}(p_2)
\end{tikzcd}$$

where p_2 is a Kan fibration of simplicial sets. As the left vertical map arises by filling horns there is a lift $\Delta^n \times \Delta^1 \to \mathrm{HOM}_R^{E_P \circ E}(R,P)$ which has the desired properties. □

The next lemma just states the well-known fact that a surjective map of simplicial abelian groups is a Kan fibration. It is stated and proved here as we referred to that fact.

Lemma 2.47. *Any surjective map $B \twoheadrightarrow C$ of simplicial abelian groups is a Kan fibration.*

Consequently for a cellular inclusion of simplicial R-modules $A \rightarrowtail B$, the map $B \twoheadrightarrow B/A$ is a Kan fibration of simplicial sets.

Proof. The kernel of $B \twoheadrightarrow C$ (which is formed degreewise) is a simplicial subgroup A of B. Each A_n acts freely on B_n by multiplication and C_n is the quotient of this action. Now [GJ99, Corollary V.2.7, p. 263] shows that the quotient map is a Kan fibration. □

Recall that for E-controlled maps $f, g \colon A \to B$ in $\mathcal{C}^G$ the maps $f+g$ and $-f$ are again E-controlled in $\mathcal{C}^G$. We turn to the Extension Axiom.

Lemma 2.48 (Extension axiom)**.** *Let*

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, "\sim"] & B \arrow[r, two heads] \arrow[d] & C \arrow[d, "\sim"] \\
A' \arrow[r, tail] & B' \arrow[r, two heads] & C'
\end{tikzcd}$$

be a map of cofiber sequences in $\mathcal{C}^G$. Assume that $A \to A'$ and $C \to C'$ are homotopy equivalences. Then $B \to B'$ is a homotopy equivalence.

Proof. We can factor the vertical maps functorially by using the Cylinder Functor from Proposition 2.13. As a Cylinder Functor is exact it respects the cofiber sequences (cf. Remark 2.19). We get a diagram

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, "\sim"] & B \arrow[r, two heads] \arrow[d] & C \arrow[d, "\sim"] & . \\
T_A \arrow[r, tail] \arrow[d, "\sim"] & T_B \arrow[r, two heads] \arrow[d, "\sim"] & T_C \arrow[d, "\sim"] & \\
A' \arrow[r, tail] & B' \arrow[r, two heads] & C' &
\end{tikzcd}$$

By Proposition 2.34 A and C are deformation retracts of T_A and T_C, respectively, with the inclusions being the left and the right vertical upper maps. What remains to be shown is that the vertical upper middle map is a homotopy equivalence. This is proved in Lemma 2.49 below, where it is shown that B is a deformation retract of T_B. □

Lemma 2.49. *Assume we have a cofiber sequence* $A \rightarrowtail B \twoheadrightarrow \overline{B}$ *in* $\mathcal{C}^G$ *where* $\overline{B} = B/A$ *for brevity. Suppose we have a diagram*

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, tail] & B \arrow[r, two heads] \arrow[d, tail] & \overline{B} \arrow[d, tail] \\
T_A \arrow[r, tail] & T_B \arrow[r, two heads] & T_{\overline{B}}
\end{tikzcd}$$

in $\mathcal{C}^G$ *where the horizontal lines are cofiber sequences and the vertical arrows are cellular inclusions. Suppose that* A *and* $\overline{B}$ *are deformation retracts of* T_A *and* $T_{\overline{B}}$ *with inclusions the left and right vertical maps. Then* B *is a deformation retract of* T_B *with inclusion the middle vertical map.*

Remark 2.50. Of course we are interested in a situation where the vertical maps are inclusions into the mapping cylinder. When we assume that $T_A \rightarrowtail T_B \twoheadrightarrow T_{\overline{B}}$ is a cofibration sequence this means that $T_{\overline{B}}$ comes with an isomorphism to $\overline{T_B} = T_B/T_A$ which is compatible with the projection from T_B.

Lemma 2.47 and therefore Lemma 2.44 applies to the maps $B \to \overline{B}$ and $T_B \to T_{\overline{B}}$, so we have the relative homotopy lifting property with respect to these maps.

We prove a slightly stronger statement than Lemma 2.49:

Lemma 2.51. *Assume that we are in the situation of Lemma 2.49. Let* D_0 *be a cellular submodule of* D*. Then each controlled map* $(D, D_0) \to (T_B, B)$ *of pairs in* $\mathcal{C}^G$ *is controlled homotopic relative* D_0 *to a map into* B*.*

Proof of Lemma 2.49 using 2.51. By Lemma 2.51 the map $\mathrm{id}\colon (T_B, B) \to (T_B, B)$ is controlled homotopic relative B to a map $T_B \to B$. This is the desired deformation retraction. □

The proof of Lemma 2.51 may look a little bit complicated. The following "toy situation" is easier: Assume all objects are abelian groups and we want to prove that the middle map is surjective if the outer ones are. We prove the "toy fact" first, the actual proof will follow this proof closely. The reader is advised to try the proof oneself first, as it is really easy, and to look it up only for the notation.

Remark 2.52 (Proof of the toy situation). Assume we have a commutative diagram of abelian groups

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "f_B"'] & \overline{B} \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & \overline{B}'
\end{tikzcd}$$

where the horizontal lines are short exact sequences. Assume that the outer maps are surjective, then the middle map is surjective.

Let α be an element in B'. We will denote the constructed elements by consecutive Greek letters and denote projections to the quotient by a bar. So $\overline{\alpha}$ is an element in $\overline{B}'$. As $\overline{B} \to \overline{B}'$ is surjective there is an element β in $\overline{B}$ which maps to $\overline{\alpha}$. As $B \to \overline{B}$ is surjective there is an element γ in B which maps to $\beta \in \overline{B}$. The elements $f_B(\gamma)$ and α do not need to be equal in B', but they become equal when projected to $\overline{B}'$, so $\alpha - f_B(\gamma)$ factors through $A' \rightarrowtail B'$. As $A \to A'$ is surjective there is an element δ in A which maps to $\alpha - f_B(\gamma)$ in B'. Hence, considered in B, $f_B(\delta + \gamma)$ equals α.

Proof of Lemma 2.51. Let $\alpha\colon (D, D_0) \to (T_B, B)$ be a controlled map. This gives a map $\overline{\alpha}$ into $(\overline{B}, T_{\overline{B}})$. As $\overline{B}$ is a deformation retract of $T_{\overline{B}}$ we get a homotopy $\overline{H}\colon D[\Delta^1] \to T_{\overline{B}}$ from $\overline{\alpha}$ to a map into $\overline{B}$ which is constant on D_0. It comes from the deformation of $(\overline{B}, T_{\overline{B}})$ precomposed with α. Lemma 2.44 applies to the map $T_B \to T_{\overline{B}}$. So we get a lift H of $\overline{H}$, relative to α and D_0.

$$\begin{tikzcd}
D_0[\Delta^1] \cup D[0] \arrow[r, "\alpha"] \arrow[d] & T_B \arrow[d, two heads] \\
D[\Delta^1] \arrow[r, "\overline{H}"] \arrow[ur, dashed, "H"] & T_{\overline{B}}
\end{tikzcd}$$

This is a homotopy from α to a better map, call it $\beta\colon D \to T_B$. However, β might not yet factor through B in which case the lemma would follow. But composition with $T_B \to T_{\overline{B}}$ gives a map $\overline{\beta}$ to $T_{\overline{B}}$ which factors through $\overline{B}$.

$$\begin{tikzcd}
 & B \arrow[r, two heads] \arrow[d] & \overline{B} \arrow[d] \\
D \arrow[r, "\beta"'] \arrow[urr, bend left, "\overline{\beta}"'] & T_B \arrow[r, two heads] & T_{\overline{B}}
\end{tikzcd}$$

Using Lemma 2.44 again this time for $B \twoheadrightarrow \overline{B}$ and the constant homotopy of $\overline{\beta}$ in $\overline{B}$ we get some lift of $\overline{\beta}$ to B, call it γ.

$$\begin{tikzcd}
* \arrow[r] \arrow[d, tail] & B \arrow[r, two heads] & \overline{B} \\
D \arrow[ur, dashed, "\gamma"] \arrow[urr, bend right, "\overline{\beta}"'] & &
\end{tikzcd}$$

It follows that the difference $\beta - \gamma\colon D \to T_B$ is zero when composed with $T_B \to T_{\overline{B}}$. Hence it factors through T_A. As the restrictions of β and γ to D_0 both lie in B the restriction of $\beta - \gamma$ to D_0 factors through A. So $\beta - \gamma$ gives a map $(D, D_0) \to (T_A, A)$. We can show the situation by the following commuting diagrams.

$$\begin{tikzcd}
T_A \arrow[r, tail] & T_B \arrow[r, two heads] & T_{\overline{B}}\,, \\
 & D \arrow[ul, dashed, "\beta-\gamma"] \arrow[u, "\beta-\gamma"'] \arrow[ur, "0"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, tail] & B \arrow[r, two heads] & \overline{B}\,, \\
 & D_0 \arrow[ul, dashed, "\beta-\gamma"] \arrow[u, "\beta-\gamma"'] \arrow[ur, "0"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r] & T_A \\
D_0 \arrow[r] \arrow[u, dashed, "\beta-\gamma"] & D \arrow[u, dashed, "\beta-\gamma"]
\end{tikzcd}$$

Hence, as $A \to T_A$ is a deformation retraction, there is a homotopy G relative to D_0 of $\beta - \gamma$ to a map into A. It comes from the deformation of (A, T_A) precomposed with $\beta - \gamma$. Call the resulting map $\delta\colon D \to A$. Via the inclusion $(T_A, A) \to (T_B, B)$ the map G can be viewed as a homotopy to T_B with:

$$\begin{array}{ll} G\colon & D[\Delta^1] \to T_B \\ G_{|0} & = \beta - \gamma \\ G_{|1} & = \delta \\ G_{|D_0[\Delta^1]} & = \beta - \gamma_{|D_0} \end{array} .$$

Therefore $G + \gamma\colon D[\Delta^1] \to T_B$ is a homotopy from β to $\delta + \gamma$, where δ and γ factor through B so the sum also factors through B. Furthermore the homotopy is constant on D_0. Concatenating the two homotopies H and G thus gives a homotopy relative D_0 from α to a map into B. This is what we wanted to show.

Note that all maps above are in fact in $\mathcal{C}^G$, because maps in $\mathcal{C}^G$ form an abelian group and being homotopic relative a subspace is an equivalence relation in $\mathcal{C}^G$ by Corollary 2.26. $\square$

3. Algebraic K-theory of categories of controlled modules

To define the algebraic K-Theory of category $\mathcal{C}^G$ following Waldhausen we have to put on $\mathcal{C}^G$ the structure of a *category with cofibrations and weak equivalences* (see [Wal85]/Appendix B). We do that in Section 3.1, which relies on the results established in Chapter 2. The resulting category with cofibrations and weak equivalences is called $\mathcal{C}^G_a$, where the "a" should stand for "all" objects. That is, if we refer to $\mathcal{C}^G_a$ we always refer to $\mathcal{C}^G$ together with the particular classes of cofibrations and weak equivalences. We define the weak equivalences to be the homotopy equivalences and denote them by $w\mathcal{C}^G$. If we refer to $\mathcal{C}^G_a$ with the homotopy equivalences as weak equivalences we often just write $w\mathcal{C}^G_a$. Note however that $\mathcal{C}^G_a$ is "too big" in the sense that it has infinite sums and hence an Eilenberg-swindle, therefore its algebraic K-Theory vanishes. But it contains subcategories satisfying certain finiteness conditions whose K-Theory is interesting and which inherit the structure of a category with cofibrations and weak equivalences. They are discussed in the following sections. First we discuss object support conditions, then three different finiteness conditions, the finite, homotopy finite and homotopy finitely dominated modules, denoted by $\mathcal{C}^G_f$, $\mathcal{C}^G_{hf}$ and $\mathcal{C}^G_{hfd}$. In the last section we discuss and compare the algebraic K-theory of these categories. We also show that $\mathcal{C}^G(G/1, R)$ and $\mathcal{C}(\mathrm{pt}, R[G])$ are equivalent categories and that they have equivalent algebraic K-theory. We further prove that a weak equivalence of simplicial rings induces a weak equivalence on the algebraic K-theory of the categories of controlled modules over a control space.

For the whole chapter G is a discrete group and all control spaces as well as all modules are G-equivariant and free. We usually leave this understood.

3.1. $\mathcal{C}^G_a$ as a category with cofibrations and weak equivalences

Recollection 3.1. Let R be a simplicial ring, let $(X, \mathcal{E}, \mathcal{F})$ be a control space with morphism control conditions $\mathcal{E}$ and object support conditions $\mathcal{F}$. Let G be a (discrete) group. Recall that $\mathcal{C}^G = \mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})$ is the category of finite-dimensional controlled simplicial R-modules with G-action. Recall that a cofibration is a map which is isomorphic to a cellular inclusion. Recall that a homotopy is map $M[\Delta^1] \to N$ and that a homotopy equivalence is a map which has a homotopy inverse.

Definition 3.2. *Denote the category of cofibrations in $\mathcal{C}^G$ by $\mathrm{co}\mathcal{C}^G$. Define the weak equivalences in $\mathcal{C}^G$ to be the homotopy equivalences and denote the subcategory of them* *by $w\mathcal{C}^G$. Define $\mathcal{C}^G_a(X, R, \mathcal{E}, \mathcal{F})$ to be $\mathcal{C}^G$ together with these two subcategories. (Here the "a" should stand for "all objects"). We abbreviate $\mathcal{C}^G_a(X, R, \mathcal{E}, \mathcal{F})$ by $\mathcal{C}^G_a(X, R)$ or $\mathcal{C}^G_a$.*

We first assume that $\mathcal{F} = \{X\}$, i.e. that we have no object support conditions, as we did in the last chapter. The case of general $\mathcal{F}$ will be considered in Section 3.3, as its treatment is parallel to the treatment of the finiteness conditions.

Proposition 3.3. *$\mathcal{C}^G_a(X, R, \mathcal{E})$ is a category with cofibrations and weak equivalences. It has a Cylinder Functor which satisfies the Cylinder Axiom and its weak equivalences satisfy the Saturation and the Extension Axiom.*

Let us recall the definition of a category with cofibrations and weak equivalences in the sense of [Wal85, 1.2].

Recollection 3.4 (Category of weak equivalences [Wal85, 1.2]/B.2). Let $\mathcal{C}$ be a category with cofibrations (cf. Definition B.1). A *category of weak equivalences* in $\mathcal{C}$ is a subcategory $w\mathcal{C}$ satisfying the following axioms.

(i) $w\mathcal{C}$ contains all isomorphisms of $\mathcal{C}$.

(ii) (Gluing lemma). If we have the diagram

$$\begin{tikzcd}
B \arrow[d] & A \arrow[l, tail] \arrow[r] \arrow[d] & C \arrow[d] \\
B' & A' \arrow[l, tail] \arrow[r] & C'
\end{tikzcd}$$

with $A \rightarrowtail B$ and $A' \rightarrowtail B'$ cofibrations and all three vertical arrows are in $w\mathcal{C}$, then the induced map

$$B \cup_A C \to B' \cup_{A'} C'$$

on the pushouts is also in $w\mathcal{C}$.

We formulate the steps in the proof of Proposition 3.3 as lemmas.

Lemma 3.5. *$\mathcal{C}^G_a$ is a category with cofibrations.*

Proof. This is Lemma 2.8. ☐

Lemma 3.6. *The category $\mathcal{C}^G_a$ has a Cylinder Functor which fulfills the Cylinder Axiom.*

Proof. The existence of a Cylinder Functor is Proposition 2.13. The Cylinder Axiom is Lemma 2.33. ☐

The next step is to show that the homotopy equivalences fulfill the axioms of a category of weak equivalences. The class of homotopy equivalences contains all isomorphisms and it is a subcategory by the Saturation Axiom (Lemma 2.31). It remains to prove the gluing lemma. Here we use a tool from [GJ99], the notion of a *category of cofibrant objects.* This is the only place where we need the notion, so we are brief and refer to [GJ99, p. 122] for details.

Recollection 3.7 (Category of cofibrant objects). A *category of cofibrant objects* is a category $\mathcal{D}$ which satisfies the following axioms.

(0) The category contains all finite coproducts.

(i) The 2-of-3 property holds for weak equivalences.

(ii) The composition of cofibrations is a cofibration, isomorphisms are cofibrations.

(iii) Pushout diagrams of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & B \arrow[d, "i_*"] \\ C \arrow[r] & D \end{tikzcd}$$

exist when i is a cofibration. In this case i_* is a cofibration which is additionally a weak equivalence if i is one.

(iv) For each object there is a cylinder object.

(v) For each X the unique map $* \to X$ from the initial object is a cofibration.

The notion of a cylinder object in [GJ99, p. 123] is slightly different then our notion, but if the Cylinder Axiom 2.33 holds our Cylinder Functor applied to the identity yields a cylinder object in the sense of [GJ99, p. 123].

Lemma 3.8. *The category $\mathcal{C}_a^G$ fulfills the axioms of a category of cofibrant objects. Hence it fulfills the gluing lemma.*

Proof of Lemma 3.8. The proof that a category of cofibrant objects has a gluing lemma is Lemma II.8.8 in [GJ99, p. 127]. It remains to check the axioms of a category of cofibrant objects.

As $\mathcal{C}_a^G$ is a category with cofibrations by Lemma 2.8 it has finite coproducts, the composition of two cofibrations is a cofibration as well as isomorphisms are cofibrations, the unique map from the initial object is a cofibration and pushouts of cofibrations are cofibrations.

Further the weak equivalences satisfy the 2-of-3 axiom by the Saturation Axiom (Lemma 2.31). Pushouts of cofibrations which are homotopy equivalences are cofibrations which are homotopy equivalences by Lemma 2.40. Further we have a Cylinder Functor which satisfies the Cylinder Axiom. This gives a cylinder object for each object. This shows all axioms and hence implies the Gluing Lemma for $\mathcal{C}_a^G$. □

Lemma 3.8 gives a direct corollary:

Corollary 3.9. *The category $\mathcal{C}_a^G$ has the structure of a category with cofibrations and weak equivalences.*

Lemma 3.10. *The class $w\mathcal{C}_a$ satisfies the saturation axiom.*

Proof. This is already part of the structure of a category with cofibrations which was proved in Lemma 3.8. The actual proof is Lemma 2.31. □

Lemma 3.11. *The homotopy equivalences in $\mathcal{C}_a^G$ satisfy the Extension Axiom.*

Proof. This was proved in Lemma 2.48. □

This finishes the proof of Proposition 3.3. Next we consider full subcategories of $\mathcal{C}_a^G$.

3.2. Subcategories of $\mathcal{C}_a^G$

Let us recall the data we have for $\mathcal{C}_a^G$. We have a simplicial ring R and a control space $(X, \mathcal{E})$. (We did not impose object support conditions so far.) Below, E always denotes an arbitrary, not fixed, element of $\mathcal{E}$. We defined $\mathcal{C}_a^G = \mathcal{C}_a^G(X, R, \mathcal{E})$ as the category with cofibrations and weak equivalences with the following data.

$\operatorname{Obj} \mathcal{C}_a^G$	E-controlled cellular R-modules
$\operatorname{Mor} \mathcal{C}_a^G$	E-controlled maps
$\mathrm{co}\mathcal{C}_a^G$	maps isomorphic to cellular inclusions
$w\mathcal{C}_a^G$	homotopy equivalences

This gives the structure of a category with cofibrations and weak equivalences on $\mathcal{C}_a^G$, where "a" stands for "all". The category is not of interest itself as it has an Eilenberg-swindle. But all the other categories we consider are subcategories of this one (with possibly other weak equivalences). So this category will save us a lot of work in proving the axioms for the other ones.

We will now impose restrictions on the objects we consider. This means, we single out a subset of objects $\operatorname{Obj} \mathcal{C}_?^G$ of $\operatorname{Obj} \mathcal{C}_a^G$ and consider the full subcategory $\mathcal{C}_?^G$ generated by it. We define the cofibrations and weak equivalences simply by restriction to this subcategory. As a table:

$\operatorname{Obj} \mathcal{C}_?^G$	a subset of $\mathcal{C}_a^G$ specified by ?
$\operatorname{Mor} \mathcal{C}_?^G$	all morphisms in $\operatorname{Mor} \mathcal{C}_a^G$ between objects of $\operatorname{Obj} \mathcal{C}_?^G$.
$\mathrm{co}\mathcal{C}_?^G$	$\mathrm{co}\mathcal{C}_a^G \cap \operatorname{Mor} \mathcal{C}_?^G$
$w\mathcal{C}_?^G$	$w\mathcal{C}_a^G \cap \operatorname{Mor} \mathcal{C}_?^G$

There is an easy criterion when $\mathcal{C}_?^G$ is again a category with cofibrations and weak equivalences.

Lemma 3.12. *Let $A \to C$ be a map and $A \rightarrowtail B$ a cofibration in $\mathcal{C}_?^G$. If the pushout*

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d] & B \arrow[d] \\
C \arrow[r] & C \cup_A B
\end{tikzcd}$$

in $\mathcal{C}_a^G$ is also contained in $\mathcal{C}_?^G$, then $\mathcal{C}_?^G$ is a category with cofibrations and weak equivalences. If additionally for each $A \in \mathcal{C}_?^G$ the cylinder $A[\Delta^1]$ is again in $\mathcal{C}_?^G$, then the Cylinder Functor T of $\mathcal{C}_a^G$ restricts to $\mathcal{C}_?^G$, so in particular $\mathcal{C}_?^G$ has a Cylinder Functor which satisfies the Cylinder Axiom. Further $\mathcal{C}_?^G$ satisfies the Saturation Axiom and the Extension Axiom.

Proof. The first part is clear, as this is the cobase change, which is the only axiom which requires new objects. With $A[\Delta^1]$ also $T(f) = A[\Delta^1] \cup_{A[1]} C$ is in $\mathcal{C}_?^G$, which shows the second part. The validity of the three axioms is completely clear. $\square$

Remark 3.13. Note that if $A[\Delta^1]$ is contained in $\mathcal{C}_?^G$ for each A in $\mathcal{C}_?^G$, then the class $w\mathcal{C}_?^G$ of weak equivalences in $\mathcal{C}_?^G$ agrees with the homotopy equivalences in $\mathcal{C}_?^G$.

The lemma reduces the number of conditions we have to check to two. We state them again because we will refer to them several times in the following.

Conditions 3.14. *The conditions of Lemma 3.12 are:*

(C1) For $C \leftarrow A \rightarrowtail B$ in $\mathcal{C}_?^G$ the pushout is in $\mathcal{C}_?^G$.

(C2) For A in $\mathcal{C}_?^G$, $A[\Delta^1]$ is in $\mathcal{C}_?^G$.

3.3. Object support conditions

So far we have assumed that there are no object control conditions. We impose them now. Let $(X, \mathcal{E}, \mathcal{F})$ be a control space. Let (M, κ) be a controlled module over $(X, \mathcal{E}, \mathcal{F})$. Recall that M has support on $F \in \mathcal{F}$ if $\kappa_R(\diamond_R M) \subseteq F$. Note that $\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})$ is the full subcategory of $\mathcal{C}^G(X, R, \mathcal{E}, \{X\})$ spanned by modules with support in some $F \in \mathcal{F}$.

Define $\mathcal{C}_s^G(X, R, \mathcal{E}, \mathcal{F})$ by restricting $\mathcal{C}_a^G$ to the objects with support on some $F \in \mathcal{F}$. We abbreviate $\mathcal{C}_s^G(X, R, \mathcal{E}, \mathcal{F})$ by $\mathcal{C}_s^G$. We have:

$$\operatorname{Obj} \mathcal{C}_s^G := \{M \in \mathcal{C}_a^G \mid \operatorname{supp} M \subseteq F, F \in \mathcal{F}\} \ .$$

Lemma 3.15. *$\mathcal{C}_s^G$ is a category with cofibrations and weak equivalences. It has a Cylinder Functor satisfying the Cylinder Axiom and the class of weak equivalences satisfy the Extension and the Saturation Axiom.*

Proof. We only have to check (C1) and (C2) from 3.14. Assume we have a cofibration $A \rightarrowtail B$ and a map $A \to C$ with $\operatorname{supp}(A) \subseteq F_A$, $\operatorname{supp}(B) \subseteq F_B$, $\operatorname{supp}(C) \subseteq F_C$, with $F_A, F_B, F_C \in \mathcal{F}$. Let D be the pushout $C \cup_A B$ in $\mathcal{C}_a^G$. If $A \rightarrowtail B$ is a cellular inclusion, then by the construction of Lemma 2.2 $\operatorname{supp}(D) \subseteq F_B \cup F_C \in \mathcal{F}$. Let be $A \rightarrowtail B$ be E-isomorphic to the cellular inclusion $A' \rightarrowtail B'$. Then $\operatorname{supp}(B') \subseteq {F_B}^E$, hence $\operatorname{supp}(D) \subseteq F_C \cup {F_B}^E \in \mathcal{F}$. This shows (C1).

For A, $\operatorname{supp}(A[\Delta^1]) = \operatorname{supp}(A) \subseteq F_A \in \mathcal{F}$. This shows (C2). $\square$

The category $\mathcal{C}_s^G$ is still too big, but it is equally convenient to have as the category $\mathcal{C}_a^G$. For the following, if we have object support conditions on X, we (re-)define $\mathcal{C}_a^G(X, R, \mathcal{E}, \mathcal{F})$ as $\mathcal{C}_s^G(X, R, \mathcal{E}, \mathcal{F})$, this is consistent for $\mathcal{F} = \{X\}$. If we want to restrict further to fewer objects, i.e. require extra conditions, we still only have to check conditions (C1) and (C2) of 3.14, so Lemma 3.12 still holds with this definition of $\mathcal{C}_a^G$. We will assume that we have object support conditions from now on and continue to construct further subcategories of $\mathcal{C}_a^G$.

3.4. Finite objects

Next we discuss finiteness conditions. These will give categories which have interesting algebraic K-theory. It is convenient to introduce the notion of a *set over a control space* $(X, \mathcal{E}, \mathcal{F})$ and define when such a set over X is *locally finite.*

Definition 3.16. *Let $(X, \mathcal{E}, \mathcal{F})$ be a control space. A set over X is a set L together with a map $\kappa\colon L \to X$ such that its image $\kappa(L)$ is contained in an $F \in \mathcal{F}$. Let (L, κ), (L', κ') be sets over X. A map $f\colon L \to L'$ is a* map of sets over X *if the relation $\{(\kappa(l), \kappa' \circ f(l)) \mid l \in L\}$ is a subset of an $E \in \mathcal{E}$.*

If G is not trivial then we assume that L has a free G-action and that κ is G-equivariant. Further all maps of sets over X are supposed to be G-equivariant.

For a controlled module (M, κ_R) we get the set $(\diamond_R M, \kappa_R)$ over X. This is our prime example. Note that if (M, κ_R^1), (M, κ_R^2) are controlled modules over X such that $(\diamond_R M, \kappa_R^1)$ and $(\diamond_R M, \kappa_R^2)$ are isomorphic as sets over X then (M, κ_R^1) and (M, κ_R^2) are controlled isomorphic.

Definition 3.17. *A set (L, κ) over the control space $(X, \mathcal{E}, \mathcal{F})$ is locally finite if for each point $x \in X$ there is a neighborhood U_x of x such that $\kappa^{-1}(U_x) \subseteq L$ is finite.*

This notion of local finiteness gives rise to the notion of finiteness of a controlled module.

Definition 3.18. *A module $(M, \kappa_R) \in \mathcal{C}^G$ is called finite if it is locally finite, that is $(\diamond_R M, \kappa_R)$ is a locally finite set over X. Remember that by definition all objects of $\mathcal{C}^G$ are finite-dimensional.*

Definition 3.19. *Define $\mathcal{C}_f^G = \mathcal{C}_f^G(X, R, \mathcal{E}, \mathcal{F})$ by restricting $\mathcal{C}_a^G$ to the finite modules. Stated as a table*

$$\operatorname{Obj} \mathcal{C}_f^G := \{M \in \mathcal{C}_a^G \mid M \textit{ is finite } \} \ .$$

Lemma 3.20. *$\mathcal{C}_f^G$ is a category with cofibrations and weak equivalences. It has a Cylinder Functor satisfying the Cylinder Axiom and the class of weak equivalences satisfies the Extension and the Saturation Axiom.*

We need some preparations before we prove this. The next two lemmas discuss how we can change the control map of a controlled module.

Remark 3.21. Note that modules isomorphic to finite modules do not need to be finite again, if the control space is not “good”. Take as example $\mathbb{R} \setminus \{0\}$ with metric control and as module $(\bigoplus_{n\in\mathbb{N}} \underline{R} \cdot e_n, \kappa)$ with $\kappa(e_n) = 1/n$. This is finite (as 0 is not in $\mathbb{R} \setminus \{0\}$) and isomorphic to $(\bigoplus_{n\in\mathbb{N}} \underline{R} \cdot e_n, \kappa')$ with $\kappa'(e_n) = 1$, which is not finite. If the control space is proper (cf. Sections 1.2 and 1.5) then modules isomorphic to finite modules are again finite, see also Remark 3.24.

Lemma 3.22. *Let $f' \colon A \to B$ be a cofibration in $\mathcal{C}^G_f$. Then it is isomorphic to a cellular inclusion in $\mathcal{C}^G_f$.*

Proof. By definition f' is only isomorphic to a cellular inclusion in $\mathcal{C}^G_a$, which does not need to be in $\mathcal{C}^G_f$ by Remark 3.21.

By Lemma 2.6 the pushout of f' along id_A can be chosen as

$$\begin{tikzcd}
A \arrow[r, "f'"] \arrow[d, "\mathrm{id}_A"] & B \arrow[d] \\
A \arrow[r, "f"] & D
\end{tikzcd}$$

such that f is a cellular inclusion. Then D is isomorphic to the finite module B, but need not be finite itself. We show in the next lemma that D can indeed be made into a finite module. That lemma finishes the proof. ☐

Lemma 3.23. *Let $f \colon A \to D$ be a cellular inclusion in $\mathcal{C}^G$ such that A is finite and (D, κ^D) is isomorphic to a finite module (B, κ^B). Then there is a control map $\overline{\kappa}^D \colon \diamond_R D \to X$ such that $(D, \overline{\kappa})$ is a finite module which is isomorphic to (D, κ^D).*

It follows that $A \to (D, \overline{\kappa})$ is isomorphic to a cellular inclusion in $\mathcal{C}^G_f$.

Proof. We do two steps to improve $\kappa_0 := \kappa^D$. For the proof we have to improve the set $(\diamond_R D, \kappa_0)$ over X and make it into a locally finite one. Hence we will define maps $\kappa_1, \kappa_2 \colon \diamond_R D \to X$ which are controlled isomorphic to κ_0 and “improve” κ_0, in particular κ_2 is a locally finite over X. Than we can take $\overline{\kappa} := \kappa_2$. We prove the non-equivariant case first, i.e. assume $G = \{e\}$.

First we define κ_1 such that its image is contained in the image of κ^B. As (B, κ^B) and (D, κ_0) are controlled isomorphic there is an $E \in \mathcal{E}$ such that for each $e \in \diamond_R D$ there is an $x(e) \in \operatorname{Im} \kappa^B \subseteq X$ such that $(\kappa_0(e), x(e)) \in E$. Set $\kappa_1(e) := x(e)$.

As $\kappa^B \colon \diamond_R B \to X$ is a locally finite set over X its image has no accumulation points, i.e. for each point $x \in X$ there is a neighborhood U_x containing only finitely many points of the image of κ^B. Thus the same is true for the image of κ_1.

Set

$$T := \{x \in X \mid \kappa_1^{-1}(x) \text{ is infinite}\}.$$

As X is Hausdorff we have that (D, κ_1) is finite if and only if T is empty. So T are the “trouble points”. We change κ_1 on $\kappa_1^{-1}(T)$. Fix a degree n. In the following we want to ignore the part in D_n and B_n coming from cells of lower dimension, i.e. from degeneracies of such cells. Thus we assume that the $(n-1)$-skeleton of B and D

as simplicial R-module is trivial by replacing D with $D' := D/\mathrm{sk}_{n-1}D$ and B with $B' := \mathrm{sk}_{n-1}B$, i.e. we collapse all cells of dimension $< n$ to a point. Then D' is a cellular R-module with no cells of dimension $< n$ and the k-cells for $k \geq n$ are in one-to-one correspondence to the k-cells of D, similar for B'. The new modules D' and B' are still isomorphic controlled modules (with control map defined by restriction of the ones from D and B). Their degree n parts D'_n and B'_n are the free R_n-module on their n-cells. We do not change the name of the control maps.

Set for $x \in T$

$$TD_n^x := \{e \mid e \in (\diamond_R D')_n, \kappa_1(e) = x\}.$$

In words these are the cells of D' in degree n over $x \in T$ (here $(\diamond_R D')_n := \diamond_R D' \cap D'_n$). TD_n^x is a basis of the direct summand concentrated over x of the R_n-module D'_n. As x is in T this module might is not finitely generated, so we have to change the control map on it. (In fact, for each x there is an n' such that this module is not finitely generated, but it might not be for the n in question. To keep the notation simpler we do not exclude that case, but we could.)

We now use that (D', κ_1) is still E'-isomorphic to the finite module (B', κ^B) for some $E' \in \mathcal{E}$.

Let $\langle TD_n^x \rangle$ be the free direct summand of the free R_n-module D'_n with basis TD_n^x. There is an isomorphism $\theta\colon B'_n \to D'_n$ of R_n-modules, so in particular θ is surjective. It is E'-controlled.

This gives a map

$$\varphi_n\colon (\diamond_R B')_n \to \mathcal{P}_f\Big(\bigcup_{x \in X} TD_n^x\Big)$$

from the n-cells of B' to the finite subsets of $\bigcup_{x \in X} TD_n^x$ by writing $\theta(e)$, for $e \in (\diamond_R B')_n$, as a linear combination of the basis $(\diamond_R D')_n$ of D'_n and setting $q \in \varphi_n(e) \subseteq TD_n^x$ if q occurs as a summand. (Note that not every element in $(\diamond_R D')_n$ is in TD_n^x, so $\varphi_n(e)$ might be the empty set.) It is important that φ_n assigns to each e a *finite* set.

The projection to $\langle TD_n^x \rangle$ composed with θ gives a map $\theta_x\colon B'_n \to \langle TD_n^x \rangle$ which is surjective. By the control condition θ_x is still surjective if restricted to the submodule of B'_n (freely) generated by the cells over $\{x\}^{E'}$. Thus if $q \in TD_n^x$ occurs as a summand of $\theta(e)$ then $\kappa^B(e)$ is in $\{x\}^{E'}$ and each such q occurs as a summand for some e. Hence there is a "partial section"

$$i_n^x\colon TD_n^x \to (\diamond_R B')_n$$

to φ_n such that $\big(\kappa_1(d), \kappa^B(i_n^x(d))\big) \in E'$ for $d \in TD_n^x$. ("Partial section" means the union of $\varphi_n \circ i_n^x(d)$ over all $d \in TD_n^x$ contains TD_n^x.)

We can do the construction for all n up to the dimension of B and D. (We only need the data φ_n and i_n^x for all n and $x \in T$, which are maps of sets, and use that $(\diamond_R B')_n = (\diamond_R B)_n$.) Therefore these give maps of (subsets of) $\diamond_R B$ and $\diamond_R D$.)

Define $\kappa_2\colon \diamond_R D \to X$ by

$$\kappa_2(d) = \begin{cases} \kappa^B(i_n^x(d)), & \text{if } \kappa_1(d) = x \in T,\ d \in (\diamond_R D)_n, \\ \kappa_1(d), & \text{else.} \end{cases}$$

Then $(\kappa_2(d), \kappa_1(d)) \in E'$, hence (D, κ_1) and (D, κ_2) are controlled isomorphic. We claim that (D, κ_2) is finite.

We only have to show that for each $y \in X$ there are only finitely many d with $\kappa_2(d) = y$. So let $d \in \kappa_2^{-1}(y)$. We show that d lies in a finite union of finite sets which only depend on y.

If $z := \kappa_1(d) \notin T$ then $\kappa_2(d) = \kappa_1(d)$ so $d \in \kappa_1^{-1}(z)$ which by definition of T is a finite set. If $\kappa_1(d) = x \in T$ then there is an $e_B \in (\diamond_R B)_n$ such that $e_B = i^n_x(d)$ and $\kappa^B(e_B) = y$. There are only finitely e_B with $\kappa^B(e_B) = y$ hence it suffices to show that $\bigcup_{x\in X}(i^x_n)^{-1}(e_B)$ is finite for each e_B. But as i^x_n is a "partial section" of φ_n the set $\bigcup_{x\in X}(i^x_n)^{-1}(e_B)$ is contained in $\varphi(e_B)$, which is a finite set. Hence each d with $\kappa_2(d) = y$ is contained in a finite union of finite sets and therefore $\kappa_2^{-1}(y)$ is finite. This proves the finiteness of (D, κ_2) and therefore the first part for G trivial.

If G is not trivial first note that κ_1 can be chosen to be G-equivariant by simply doing the choice on a set of representatives. Further θ and φ_n are G-equivariant. As G acts freely on X we can just define i^x_n only for x in a set of representatives of G-orbits and extend equivariantly. Then κ_2 is equivariant, as κ^B, κ_1 and the collection of i^x_n are. This shows the equivariant case.

Now the map $A \to (D, \overline{\kappa})$ induces an injective map $\diamond_R A \to \diamond_R D$. Redefining $\overline{\kappa}$ such that the inclusion is a Δ-controlled map of sets over X finishes the proof (cf. Lemma 1.18). ☐

Remark 3.24. The proof of the Lemma implies the following for the control space X if T was not empty: There are points $x \in X$ and $E \in \mathcal{E}$ such that $\{x\}^E$ is not contained in a compact subset. Namely i^x_n must hit infinitely many cells of B over points in $\{x\}^E$, but B is locally finite. In particular X is not a proper control space in the sense of Section 1.2.

The main point of the Lemma is that B and D might have different cellular structures, see Remark 1.20 why one has to be careful in such cases.

Proof of Lemma 3.20. Again we only have to check the conditions 3.14. We prove (C1) first. If $A \rightarrowtail B$ is a cellular inclusion in $\mathcal{C}^G_f$, i.e. A and B are finite objects, the pushout along any map $A \to C$ exists in $\mathcal{C}^G_f$, as the canonical pushout provided by Lemma 2.2 lies in $\mathcal{C}^G_f$. (The set over X of the pushout is a union of the set $(\diamond_R C, \kappa_R)$ over X and a subset of $(\diamond_R B, \kappa_R)$.)

So let $A \rightarrowtail B$ be a cofibration in $\mathcal{C}^G_f$. This means it is isomorphic to a cellular inclusion $A' \rightarrowtail B'$ in $\mathcal{C}^G_a$. By Lemma 3.22 it is also isomorphic to a cellular inclusion in $\mathcal{C}^G_f$. So we get a diagram

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, "\cong"] & B \arrow[d, "\cong"] \\
A \arrow[r, tail] & D
\end{tikzcd}$$

where the lower row is a cellular inclusion in $\mathcal{C}^G_f$. Hence pushouts along cofibrations exists in $\mathcal{C}^G_f$ which shows (C1).

For (C2) note that if A is finite $A[\Delta^1]$ is again finite. This follows as $R[\Delta^n][\Delta^1] = R[\Delta^n \times \Delta^1]$ has only finitely many R-cells, so for each $e \in A$ a cell with $\kappa_R(e) = x \in X$ the module $A[\Delta^1]$ contains only finitely many extra cells, each of which is again over x. This proves the lemma. ☐

We do not have functoriality of $\mathcal{C}_f^G$ for all maps of control spaces any more, but we have the following obvious criterion.

Lemma 3.25. *Let* $\varphi\colon (X, \mathcal{E}_X, \mathcal{F}_X) \to (Y, \mathcal{E}_Y, \mathcal{E}_Y)$ *be a map of control spaces which maps locally finite sets over* X *to locally finite sets over* Y*. Then* φ *induces a functor* $\mathcal{C}_f^G(X, R, \mathcal{E}_X, \mathcal{F}_X) \to \mathcal{C}_f^G(Y, R, \mathcal{E}_Y, \mathcal{F}_Y)$*.* ☐

Remark 3.26. Note that inclusions of subspaces do not map locally finite sets to locally finite sets in general. A counterexample is the inclusion $\mathbb{R} \smallsetminus \{0\} \to \mathbb{R}$, cf. Remark 3.21. However closed inclusions do map locally finite sets to locally finite set and hence do induce a functor of categories of controlled modules.,

3.5. Homotopy finite objects

We call a controlled module $M \in \mathcal{C}_a^G$ *homotopy finite* if there is a finite module $M' \in \mathcal{C}_f^G$ and a homotopy equivalence $M' \xrightarrow{\sim} M$ in $\mathcal{C}_a^G$. We define $\mathcal{C}_{hf}^G = \mathcal{C}_{hf}^G(X, R, \mathcal{E}, \mathcal{F})$ by restricting to the homotopy finite objects. Stated again:

$$\operatorname{Obj} \mathcal{C}_{hf}^G := \left\{ M \in \mathcal{C}_a^G \mid M \text{ is homotopy finite } \right\} .$$

Lemma 3.27. $\mathcal{C}_{hf}^G$ *is a category with cofibrations and weak equivalences. It has a Cylinder Functor satisfying the Cylinder Axiom and the class of weak equivalences satisfies the Extension and the Saturation Axiom.*

Proof. The proof is formal. We check (C2) of 3.14 first. Let A, B be homotopy finite objects and $A \to B$ a map. Then $B[\Delta^1]$ and even $T(A \to B)$ are weakly equivalent to B by Lemma 2.33 and hence homotopy finite.

To check (C1), we use that we know (C1) holds for finite objects and that the gluing lemma holds. We replace the pushout diagram

$$\begin{tikzcd} C & A \arrow[l] \arrow[r, tail] & B \end{tikzcd} ,$$

with A, B, C homotopy finite, step by step by a diagram with finite objects. So assume that there are finite objects A', B', C' weakly equivalent to A, B, C. Note that we have inverses for weak equivalences, which we will use freely. Below we denote mapping cylinders by M_A, M_B, etc. and cofibrations by $\rightarrowtail$.

We get a chain of maps of diagrams. In the following the arrows marked with $\overset{\bullet}{\to}$ are defined by composition. The first step is

$$\begin{tikzcd} C \arrow[d, equal] & A \arrow[l] \arrow[r, tail] & B \\ C & A' \arrow[l, "\bullet"'] \arrow[r, tail] \arrow[u, "\sim"] & M_B \arrow[u, "\sim"] \end{tikzcd} ,$$

where A' is finite and M_B is the mapping cylinder of $A' \to A \rightarrowtail B$, which still is homotopy finite. Next we get a map

$$\begin{tikzcd}
C \arrow[d, "\sim"'] & A' \arrow[l] \arrow[r, tail] \arrow[d, equal] & M_B \arrow[d, equal] \\
C' & A' \arrow[l, "\bullet"'] \arrow[r, tail] & M_B
\end{tikzcd}$$

by C being homotopy finite. Then take

$$\begin{tikzcd}
C' & A' \arrow[l] \arrow[r, tail] \arrow[d, equal] & M_B \arrow[d, equal] \\
M_{C'} \arrow[u, "\sim"] & A' \arrow[l, tail] \arrow[r, tail] & M_B
\end{tikzcd}$$

with $M_{C'}$ being the cylinder of $A' \to C'$ which is finite as A' and C' are finite. Finally we get a map

$$\begin{tikzcd}
M_{C'} \arrow[d, equal] & A' \arrow[l, tail] \arrow[r, tail] \arrow[d, equal] & M_B \arrow[d, "\sim"'] \\
M_{C'} & A' \arrow[l, tail] \arrow[r, "\bullet"] & B'
\end{tikzcd}$$

as M_B is weakly equivalent to B'. Using the gluing lemma four times gives that $C \cup_A B$ is weakly equivalent to the finite object $M_{C'} \cup_{A'} B'$. □

A map of control spaces $(X, \mathcal{E}_X, \mathcal{F}_X) \to (Y, \mathcal{E}_Y, \mathcal{E}_Y)$ induces a functor

$$\mathcal{C}^G_{hf}(X, R, \mathcal{E}_X, \mathcal{F}_X) \to \mathcal{C}^G_{hf}(Y, R, \mathcal{E}_Y, \mathcal{F}_Y)$$

if it maps finite sets over X to finite sets over Y, cf. Lemma 3.25.

3.6. Homotopy finitely dominated objects

We call an object $M \in \mathcal{C}^G_a$ *homotopy finitely dominated* if it is a (strict) retract of a homotopy finite object. We define $\mathcal{C}^G_{hfd} = \mathcal{C}^G_{hfd}(X, R, \mathcal{E}, \mathcal{F})$ by restricting to the homotopy finitely dominated objects. Stated again:

$$\operatorname{Obj} \mathcal{C}^G_{hfd} := \{ M \in \mathcal{C}^G_a \mid M \text{ is retract of a homotopy finite object } \} \ .$$

Let us recall some definitions first.

Definition 3.28. *Let M, M' be objects in $\mathcal{C}^G_a$.*

(i) M is called a retract of M' *if there are maps $i \colon M \to M'$, $r \colon M' \to M$ such that $r \circ i = \mathrm{id}_M$.*

(ii) M is called a homotopy retract of M'*, or* dominated by M'*, if there are maps $i \colon M \to M'$, $r \colon M' \to M$ and a homotopy $H \colon M[\Delta^1] \to M$ from $r \circ i$ to id_M.*

We first need a lemma.

Lemma 3.29. *Let $A \in \mathcal{C}_a^G$. Then the following are equivalent.*

(i) A is a homotopy retract of a finite module A'.

(ii) A is a retract of a homotopy finite module A''.

(iii) A is a homotopy retract of a homotopy finite module A'''.

Proof. Clearly (i) ⇒ (iii) and (ii) ⇒ (iii) hold. We show (iii) ⇒ (i) first.

As A''' is homotopy finite, there is a finite module B and maps $f \colon A''' \to B$, $g \colon B \to A'''$ such that $g \circ f \simeq \mathrm{id}_{A'''}$, so A is a homotopy retract of B via $A \xrightarrow{i} A''' \xrightarrow{f} B$ and $B \xrightarrow{g} A''' \xrightarrow{r} B$.

Now we show (i) ⇒ (ii). We have maps $i \colon A \to A'$, $r \colon A' \to A$ with $r \circ i \simeq \mathrm{id}_A$. Using Lemma 2.35 we can make the homotopy commutative diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[dr, "\mathrm{id}"'] & A' \arrow[d, "r"] \\
& A
\end{tikzcd}$$

into a strict commutative one, namely

$$\begin{tikzcd}
A \arrow[r] \arrow[dr, "\mathrm{id}"'] & T(i) \arrow[d] \\
& A
\end{tikzcd} .$$

Hence A is a retract of $T(i)$ and as $T(i) \xrightarrow{\sim} A'$ is a homotopy equivalence, $T(i)$ is homotopy finite. □

Lemma 3.30. *$\mathcal{C}_{hfd}^G$ is a category with cofibrations and weak equivalences. It has a Cylinder satisfying the Cylinder Axiom Functor and the class of weak equivalences satisfies the Extension and the Saturation Axiom.*

Proof. Again we only show (C1) and (C2) from 3.14. Assume that A, B, C are retracts of homotopy finite objects A', B', C'. Note that we can make the coretraction into a cofibration by replacing A' with the mapping cylinder of $A \to A'$, so we will assume that the coretractions i_A, i_B, i_C are actually cofibrations.

We want to show that $C \cup_A B$ is a retract of a homotopy finite object. We reduce this to the case where A is homotopy finite. Consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, tail] & B \arrow[d, equal] \\
A' \arrow[r, "\bullet"] \arrow[d] & B \arrow[d, equal] \\
A \arrow[r, tail] & B
\end{tikzcd} .$$

(As before $\xrightarrow{\bullet}$ denotes a map defined by composition.) We can factor the horizontal maps into cofibrations simultaneously using the Cylinder Functor. We obtain

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, tail] & M \arrow[r, "\sim"] \arrow[d] & B \arrow[d, equal] \\
A' \arrow[r, tail] \arrow[d] & \overline{M} \arrow[r, "\sim"] \arrow[d] & B \arrow[d, equal] \\
A \arrow[r, tail] & M \arrow[r, "\sim"] & B
\end{tikzcd} .$$

There and in all following diagrams the composition of the vertical arrows is always the identity, which holds in the diagram above by the functoriality of the Cylinder Functor. By the gluing lemma $C \cup_A M$ is weakly equivalent to $C \cup_A B$. Then the diagram, extended by C,

$$\begin{tikzcd}
C \arrow[d, equal] & A \arrow[l] \arrow[r, tail] \arrow[d, tail] & M \arrow[d] \\
C \arrow[d, equal] & A' \arrow[l, "\bullet"'] \arrow[r, tail] \arrow[d] & \overline{M} \arrow[d] \\
C & A \arrow[l] \arrow[r, tail] & M
\end{tikzcd} ,$$

shows that $C \cup_A M$ is a retract of $C \cup_{A'} \overline{M}$. We are done if we show that $C \cup_{A'} \overline{M}$ is finitely dominated. As M is homotopy equivalent to the homotopy finitely dominated object B, Lemma 3.29 shows that $\overline{M}$ is again a retract of a homotopy finite module M'.

Now we can use that we have coretractions $C \rightarrowtail C'$, $\overline{M} \rightarrowtail M'$ with C', M' homotopy finite objects, which are also cofibrations. This gives a commuting retraction diagram

$$\begin{tikzcd}
C \arrow[d] & A' \arrow[l] \arrow[r, tail] \arrow[d, equal] & \overline{M} \arrow[d, tail] \\
C' \arrow[d] & A' \arrow[l, "\bullet"'] \arrow[r, tail, "\bullet"] \arrow[d, equal] & M' \arrow[d] \\
C & A' \arrow[l] \arrow[r, tail] & \overline{M}
\end{tikzcd} ,$$

where we want to emphasize, that the map $A' \rightarrowtail M'$, defined by composition, is a cofibration. Thus $C \cup_{A'} \overline{M}$ is a retract of $C' \cup_{A'} M'$, which is homotopy finite, as being a pushout of homotopy finite objects along a cofibration.

For (C2), $A[\Delta^1]$ is dominated by $A'[\Delta^1]$. $\square$

Again a map of control spaces $(X, \mathcal{E}_X, \mathcal{F}_X) \to (Y, \mathcal{E}_Y, \mathcal{E}_Y)$ induces a functor

$$\mathcal{C}^G_{hfd}(X, R, \mathcal{E}_X, \mathcal{F}_X) \to \mathcal{C}^G_{hfd}(Y, R, \mathcal{E}_Y, \mathcal{F}_Y)$$

if it maps finite sets over X to finite sets over Y, cf. Lemma 3.25.

3.7. Connective algebraic K-theory of controlled modules

The categories $\mathcal{C}_f^G$, $\mathcal{C}_{hf}^G$ and $\mathcal{C}_{hfd}^G$ are all categories with cofibrations and weak equivalences, so we can use Waldhausen's $\mathcal{S}_\bullet$-construction from [Wal85] to produce an algebraic K-Theory spectrum $K(\mathcal{C}_?^G)$ and therefore also the corresponding infinite loop space. Define $K_n(\mathcal{C}_?^G)$ for $n \geq 0$ as the nth homotopy group $\pi_n K(\mathcal{C}_?^G)$. This algebraic K-Theory spectrum is always connective so we do not assign any name to its negative homotopy groups. Later we will define a non-connective algebraic K-theory spectrum which might have negative K-groups. It may differ in K_0 with the K-groups defined here.

Remark 3.31. There is a slight set-theoretical problem, as $\mathcal{C}_a^G$ is not a small category according to our definition but it needs be one to apply the K-theory construction. However, we take the usual approach (see e.g. [Wal85, Remark before 2.1.1]) and fix a suitable large set-theoretical small category of simplicial R-modules to begin with. Then all the categories we consider are again small. (We could get such a category by fixing a large cardinal and require all elements to lie in it.)

We will assume such a choice from now on.

All three categories from Section 3.4 to 3.6 are sufficiently small so they have interesting algebraic K-Theory. We make the definition explicit.

Definition 3.32 (Algebraic K-theory of categories of controlled modules)**.** *Let G be a group, $(X, \mathcal{E}, \mathcal{F})$ be a free G-equivariant control space. Let R be a simplicial ring. Define the algebraic K-theory space of the category with cofibrations and weak equivalences $\mathcal{C}_f^G(X, R, \mathcal{E}, \mathcal{F})$ as the infinite loop space*

$$K(w\mathcal{C}_f^G(X, R, \mathcal{E}, \mathcal{F}))$$

where K is Waldhausen's algebraic K-theory of spaces [Wal85]. We define similar the algebraic K-theory of $\mathcal{C}_{hf}^G$ and $\mathcal{C}_{hfd}^G$.

We first compare the different notions of finiteness.

Proposition 3.33. *Let $(X, \mathcal{E}, \mathcal{F})$ be a control space and R a simplicial ring.*

(i) The inclusion $\mathcal{C}_f^G(X, R, \mathcal{E}, \mathcal{F}) \to \mathcal{C}_{hf}^G(X, R, \mathcal{E}, \mathcal{F})$ is exact and induces a homotopy equivalence on K-Theory.

(ii) The inclusion $\mathcal{C}_{hf}^G(X, R, \mathcal{E}, \mathcal{F}) \to \mathcal{C}_{hfd}^G(X, R, \mathcal{E}, \mathcal{F})$ is exact and induces an isomorphism on K_n for $n \geq 1$ and an injection $K_0(\mathcal{C}_{hf}^G) \to K_0(\mathcal{C}_{hfd}^G)$.

Remark 3.34. In view of the theorem one can consider $\mathcal{C}_{hfd}^G$ as an "idempotent completion" of $\mathcal{C}_{hf}^G$. We prove in Corollary C.5 in Appendix C that idempotents and certain homotopy idempotents split in $\mathcal{C}_{hfd}^G$. The author does not know if every homotopy idempotent splits in $\mathcal{C}_{hfd}^G$ but suspects that it does. Hence it is not clear that $K_0(\mathcal{C}_{hfd}^G)$ is the "correct" group from this point of view. However, this does not matter much as we construct a non-connective delooping later anyway, whose K_0 will be the "correct" group.

Proof. To prove (i) we use Waldhausen's Approximation Theorem B.7 and apply it to the inclusion functor. See B.6 for a recollection of the assumptions we have to check. A map is a homotopy equivalence in $\mathcal{C}_f^G$ if and only if it is one in $\mathcal{C}_{hf}^G$, so (App 2) is satisfied.

So given $A \in \mathcal{C}_f^G$ and $B \in \mathcal{C}_{hf}^G$ and let $f \colon A \to B$ be a map. For B there is by definition a $B_f \in \mathcal{C}_f^G$ homotopy equivalent to B, i.e. there are maps $g \colon B_f \to B$ and $\overline{g} \colon B \to B_f$ with both compositions being homotopic to the identity. Define a map $j \colon A \to B_f$ as $j := \overline{g} \circ f$. Then $g \circ j$ is homotopic to f, so Lemma 2.35 gives for the homotopy commutative diagram on the left below the strict commutative diagram on the right:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] & B \\
B_f \arrow[ur, "g"'] &
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\iota_0"'] & B \\
T(j) \arrow[ur, "H"'] &
\end{tikzcd}$$

with $T(j) \in \mathcal{C}_f^G$, $A \to T(j)$ a cofibration and a map $B_f \to T(j)$ which is a homotopy equivalence. As there is a commutative diagram

$$\begin{tikzcd}
B_f \arrow[r, "\sim"] \arrow[d, "\sim"'] & B \\
T(j) \arrow[ur, "H"'] &
\end{tikzcd}$$

the Saturation Axiom (Lemma 2.31) shows that H is a homotopy equivalence. This shows (i).

For (ii) we use Exercise 1.10.2 from [TT90]. This is the following cofinality result (in notation from there):

> Let $\mathbf{A}$ and $\mathbf{B}$ be Waldhausen categories. Suppose $\mathbf{A}$ is a full subcategory of $\mathbf{B}$ closed under extensions, that $w(\mathbf{A}) = \mathbf{A} \cap w(\mathbf{B})$, and that a map in $\mathbf{A}$ is a cofibration in $\mathbf{A}$ iff it is a cofibration in $\mathbf{B}$ with quotient isomorphic to an object of $\mathbf{A}$. Suppose that $\mathbf{B}$ has mapping cylinders satisfying the cylinder axiom, and that $\mathbf{A}$ is closed under them. Suppose finally that $\mathbf{A}$ is cofinal in $\mathbf{B}$ in that for all B in $\mathbf{B}$ there is B' in $\mathbf{B}$ such that $B \cup B'$ is isomorphic to an object of $\mathbf{A}$.
>
> Then $K(\mathbf{A}) \to K(\mathbf{B}) \to$ "$K_0(\mathbf{B})/K_0(\mathbf{A})$" is a homotopy fibre sequence.

The sequence $K(\mathbf{A}) \to K(\mathbf{B}) \to$ "$K_0(\mathbf{B})/K_0(\mathbf{A})$" can either be interpreted as a homotopy fiber sequence of infinite loop spaces or of spectra. We use it for infinite loop-spaces, then the last term is simply the discrete group $K_0(\mathbf{B})/K_0(\mathbf{A})$. (In the spectra-version it would be the Eilenberg-MacLane spectrum $\mathbb{H}M$ with $M := K_0(\mathbf{B})/K_0(\mathbf{A})$, which is characterized by $\pi_0\mathbb{H}M = M$ and $\pi_n\mathbb{H}M = 0$ for $n \neq 0$ up to weak equivalence of spectra.)

We have to check the conditions. Most of them are clear or shown in the previous sections, in particular a map $A \to A'$ in $\mathcal{C}_{hf}^G$ which is a cofibration in $\mathcal{C}_{hfd}^G$ is a cofibration in $\mathcal{C}^G_{hf}$, and therefore its quotient is again in $\mathcal{C}^G_{hf}$. We first show the cofinality and then that $\mathcal{C}^G_{hf}$ is closed under extensions in $\mathcal{C}^G_{hfd}$.

For $B \in \mathcal{C}^G_{hfd}$ there is an $A \in \mathcal{C}^G_{hf}$ such that B is a retract of A, i.e. there are maps $r\colon A \to B$, $i\colon B \to A$ such that $r \circ i = \mathrm{id}_B$. By replacing A with $T(i)$ we can assume that i is a cofibration, hence there is a cofiber sequence

$$B \overset{i}{\rightarrowtail} A \overset{p}{\twoheadrightarrow} C := A/B.$$

The retraction $r\colon A \to B$ and the map $* \to C$ give a map $A \to B \vee C$, and $* \to B$ and $A \to C$ give another one. The sum of these maps makes the diagram

$$\begin{tikzcd}
B \arrow[r, tail] \arrow[d, "="'] & A \arrow[r, two heads] \arrow[d] & C \arrow[d, "="'] \\
B \arrow[r, tail] & B \vee C \arrow[r, two heads] & C
\end{tikzcd}$$

commutative and both rows are cofiber sequences. By the Extension Axiom 2.48 the map $A \to B \vee C$ is a homotopy equivalence, hence $B \vee C \in \mathcal{C}^G_{hf}$.

Next we need to show that $\mathcal{C}^G_{hf}$ is closed under extensions in $\mathcal{C}^G_{hfd}$. So let

$$A \rightarrowtail B \twoheadrightarrow C$$

be a cofiber sequence in $\mathcal{C}^G_{hfd}$ with $A, C \in \mathcal{C}^G_{hf}$ and B ("the extension of A by C") in $\mathcal{C}^G_{hfd}$. As $\mathcal{C}^G_{hf}$ is cofinal there is a $B' \in \mathcal{C}^G_{hfd}$ such that $B \vee B' \in \mathcal{C}^G_{hf}$. Then

$$A \rightarrowtail B \vee B' \twoheadrightarrow C \vee B'$$

is a cofiber sequence with $A, B \vee B' \in \mathcal{C}^G_{hf}$, hence the quotient $C \vee B'$ is in $\mathcal{C}^G_{hf}$ by the gluing lemma. Similar but easier we get cofiber sequences

$$C \rightarrowtail C \vee B' \twoheadrightarrow B'$$

showing $B' \in \mathcal{C}^G_{hf}$ and

$$B' \rightarrowtail B \vee B' \twoheadrightarrow B$$

showing $B \in \mathcal{C}^G_{hf}$, what we wanted to show. Part (ii) of the proposition follows. □

If X is a point the category $\mathcal{C}(X, R)$ is just the category of simplicial R-modules. We have a similar result in the presence of G.

Lemma 3.35. *Let R be a simplicial ring and G a (discrete) group. Then the categories $\mathcal{C}^G(G/1, R, \{G \times G\}, \{G\})$ and $\mathcal{C}(\mathrm{pt.}, R[G], \{\mathrm{pt.}\}, \{\mathrm{pt.}\})$ are equivalent and both are equivalent to the category of finite-dimensional cellular simplicial $R[G]$-modules.*

The equivalences respect the finiteness conditions f, hf and hfd.

Proof. Let $(M, \kappa_R) \in \mathcal{C}^G(G/1, R)$. First we can forget κ_R as each two controlled modules (M, κ_R), (M, κ'_R) are isomorphic. Then M is a finite-dimensional cellular simplicial R-module with a cell-permuting G-action. This is the same as a finite-dimensional cellular simplicial $R[G]$-module. G-orbits in $\diamond_R M$ are in bijection to cells (elements) in $\diamond_{R[G]} M$. This proves the first part.

If M is a finite R-module over $G/1$ (with discrete topology) then $\diamond_R M$ contains only finitely many G-orbits, hence $\diamond_{R[G]} M$ is finite. Homotopies and retracts in $\mathcal{C}_a^G(G/1, R)$ and $\mathcal{C}_a(\text{pt.}, R[G])$ correspond to each other. This shows the lemma. □

Corollary 3.36. *The algebraic K-Theory of $\mathcal{C}^G_{hfd}(G/1, R, \{G \times G\}, \{G\})$ is homotopy equivalent to $K(R[G])$, the algebraic K-theory of the simplicial ring $R[G]$.*

Proof. The definition of algebraic K-Theory of simplicial rings we use is in [Wal85, 2.3]. As $\mathcal{C}^G_{hfd}(G/1, R)$ is equivalent to the category of finite $R[G]$-modules the corollary follows. □

Remark 3.37. As always one has to be careful about the "correct" definition of K_0. The definition in the corollary yields the "correct" K_0, whereas $\mathcal{C}^G_f$ and $\mathcal{C}^G_{hf}$ would only give the part of K_0 corresponding to the image of $K_0(\mathbb{Z})$. See also the Remark after [Wal85, Thm. 2.3.2].

We finish with an observation about a change of the rings.

Theorem 3.38. *Let $f \colon R \to S$ be map of simplicial rings which is a weak equivalence. Then f induces a map $\mathcal{C}^G_?(X, R) \to \mathcal{C}^G_?(X, S)$ which is an equivalence on algebraic K-Theory. (Here ? can be f, hf or hfd.)*

The proof will take the rest of this section. We do some preparations first. If M is a cellular R-module then $S \otimes_R M$ is a cellular S-module and we get a natural bijection $\diamond_R M \cong \diamond_S(S \otimes_R M)$ which makes $S \otimes_R M$ into a controlled S-module. This construction respects all finiteness conditions and cofibrations, so we get an exact functor $S \otimes_R - \colon \mathcal{C}^G_?(X, R) \to \mathcal{C}^G_?(X, S)$. We make two observations first before we prove Theorem 3.38.

Lemma 3.39. *Let $R \to S$ be a weak equivalence of simplicial rings and P a cellular (uncontrolled) R-module. Let $\eta \colon P \to \operatorname{res} S \otimes_R P$ be the unit of the adjunction between the induction $S \otimes_R -$ and the restriction res_R. Then η is a weak equivalence of simplicial R-modules and in particular a homotopy equivalence of simplicial sets.*

Proof. This follows from the gluing lemma and induction over the dimension of P. We get a pushout-diagram

$$\begin{tikzcd}
\coprod R[\Delta^n] \arrow[d, "\simeq"] & \coprod R[\partial\Delta^n] \arrow[l] \arrow[r] \arrow[d, "\simeq"] & P_{n-1} \arrow[d, "\simeq"] \\
\coprod S[\Delta^n] & \coprod S[\partial\Delta^n] \arrow[l] \arrow[r] & S \otimes_R P_{n-1}
\end{tikzcd}$$

where the vertical maps are weak equivalences of simplical R-modules, hence by the gluing lemma for simplicial R-modules (cf. [GJ99, II.8.12;III.2.14]) the pushout $P_n \to S \otimes_R P_n$ is a weak equivalence. As simplicial abelian groups are fibrant as simplicial sets the weak equivalence is a homotopy equivalence of simplicial sets. ☐

Lemma 3.40. *Let $M, P \in \mathcal{C}_a^G(X,R)$. For any map*

$$g\colon S \otimes_R M \to S \otimes_R P$$

in $\mathcal{C}_a^G(X,S)$ there is a map $f\colon M \to P$ in $\mathcal{C}_a^G(X,R)$ such that g is homotopic to $S \otimes_R f$ in $\mathcal{C}_a^G(X,S)$. If $A \rightarrowtail M$ is a cellular inclusion in $\mathcal{C}_a^G(X,R)$ and $f'\colon A \to P$ a map such that

$$\begin{tikzcd} S \otimes_R A \arrow[r, "S\otimes_R f'"] \arrow[d, tail] & S \otimes_R P \arrow[d, "\mathrm{id}"] \\ S \otimes_R M \arrow[r, "g"] & S \otimes_R P \end{tikzcd} \tag{16}$$

commutes, then g is homotopic to $S \otimes_R f$ relative to $S \otimes_R A$.

Proof. We do induction over the dimension of the cells attached to A. So assume that M is the pushout

$$\begin{tikzcd} \coprod R[\partial\Delta^n] \arrow[r] \arrow[d] & A \arrow[d] \\ \coprod R[\Delta^n] \arrow[r] & M \end{tikzcd}$$

and we have maps $f'\colon A \to P$, $g\colon S \otimes_R M \to S \otimes_R P$ such that Diagram (16) commutes. We have to construct a map $f\colon M \to P$ and a homotopy from g to $S \otimes_R f$ relative to $S \otimes_R A$. It suffices to construct this for each cell individually. So choose one cell $e\colon R[\Delta^n] \to M$ attached to A and let $x := \kappa_R(e)$. As M is controlled there is an $E \in \mathcal{E}$ and cellular submodules $A' \subseteq A$, $P' \subseteq P$ with support on $\{x\}^E$ such that the attaching map of e factors as $R[\partial\Delta^n] \to A' \rightarrowtail A$ and the restriction g' of g to $A' \cup_{R[\partial\Delta^n]} R[\Delta^n]$ factors over P'. In other words we get a commutative diagram

$$\begin{tikzcd} S \otimes_R A' \arrow[r] \arrow[d, tail] & S \otimes_R P' \arrow[d, "\mathrm{id}"] \\ S \otimes_R (A' \cup_{R[\partial\Delta^n]} R[\Delta^n]) \arrow[r, "g'"] & S \otimes_R P' \end{tikzcd}$$

in $\mathcal{C}_a^G(X,S)$ and it suffices to construct a map $f\colon A' \cup_{R[\partial\Delta^n]} R[\Delta^n] \to P'$ and homotopy of g' to $S \otimes_R f$. We use the adjunction of $S \otimes_R -$ and the restriction res along $R \to S$ to obtain the diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d, tail] & P' \arrow[d, "\eta"] \\ A' \cup_{R[\partial\Delta^n]} R[\Delta^n] \arrow[r, "\widehat{g'}"] & \operatorname{res} S \otimes_R P' \end{tikzcd} .$$

Here $\widehat{g'}$ is the adjoint to g' and similar η is the unit of the adjunction. Using the adjunction of $R[-]$ and the forgetful functor to simplicial sets we can simplify further to the diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, tail] & P' \arrow[d, "\eta"] \\
\Delta^n \arrow[r] & \operatorname{res} S \otimes_R P'
\end{tikzcd} .$$

of simplicial sets. It suffices to construct a lift up to homotopy there. Now as $R \to S$ is a weak equivalence the unit $\eta\colon P' \to \operatorname{res} S \otimes_R P'$ is a weak equivalence of simplicial sets and a homotopy equivalence by Lemma 3.39. If follows that in the last diagram there exists a lift up to homotopy. This lift gives a map

$$f\colon A' \cup_{R[\partial\Delta^n]} R[\Delta^n] \to P'$$

such that $\eta \circ f$ is homotopic relative A to $\widehat{g'}$. As P' has support on $\{x\}^E$ the map f is always at least E^2-controlled. This proves the lemma for a single cell.

For all cells together note that for each cell in the same dimension we get the same control condition E, hence the constructed lift has on each cell the same control condition, so the homotopy and the lift glue together. Then we use the homotopy extension property to get a map $S \otimes_R M \to S \otimes_R P$ homotopic to g which relative to A and the n-skeleton of M. Induction and the finite-dimensionality of M finishes the proof. □

Proof of Theorem 3.38. We apply the Approximation Theorem B.7 to the functor $F := S \otimes_R -\colon \mathcal{C}^G_?(X,R) \to \mathcal{C}^G_?(X,R)$. We prove (App 1) first. Let $\alpha\colon M \to M'$ be a map in $\mathcal{C}^G_?(X,R)$ such that $S \otimes_R \alpha$ is a homotopy equivalence in $\mathcal{C}^G_?(X,S)$. By Lemma 3.40 there is a map $\beta'\colon M' \to M$ such that the homotopy inverse $\beta\colon S \otimes_R M' \to S \otimes_R M$ of $S \otimes_R \alpha$ in $\mathcal{C}^G_?(X,S)$ is homotopic to $S \otimes_R \beta'$. Hence there is a homotopy $H\colon S \otimes_R M[\Delta^1] \to S \otimes_R M$ from $S \otimes_R \mathrm{id}_R$ to $S \otimes_R (\beta' \circ \alpha)$ in $\mathcal{C}^G_?(X,S)$ which is homotopic relative to $M[\partial\Delta^1]$ to a homotopy $S \otimes_R H'$ where H' is a homotopy from id_R to $\beta' \circ \alpha$, using Lemma 3.40 again. Vice versa for $\alpha \circ \beta'$, so α is also a homotopy equivalence in $\mathcal{C}^G_?(X,R)$.

For (App 2) take $M \in \mathcal{C}^G_?(X,R)$ and $N \in \mathcal{C}^G_?(X,S)$ and a map $S \otimes_R M \to N$. Assume that it is a cellular inclusion by taking the mapping cylinder. We show that N is homotopy equivalent relative $S \otimes_R M$ to a module $S \otimes_R \overline{M}$, with $\overline{M} \in \mathcal{C}^G_?(X,R)$.

We proceed by induction over the dimension of cells of N which are not in $S \otimes_R M$. Hence assume we have an R-module $\overline{M^n}$, an S-module N^n homotopy equivalent to N relative to $S \otimes_R M$, and a cellular inclusion $S \otimes_R \overline{M^n} \to N^n$ such that the S-cells of N^n which are not in $S \otimes_R \overline{M^n}$ are of dimension $\geq n+1$. We only need to show that there is an R-module $\overline{M^{n+1}}$ such that the union of the $n+1$-skeleton N^n_{n+1} of N^n with $S \otimes_R \overline{M^n}$ is homotopy equivalent to $S \otimes_R \overline{M^{n+1}}$ relative to $S \otimes_R \overline{M^n}$.

Then we can take the pushout

$$\begin{tikzcd}
N^n_{n+1} \arrow[r, tail] \arrow[d] & N^n \arrow[d] \\
S \otimes_R \overline{M^{n+1}} \arrow[r, tail] & N^{n+1}
\end{tikzcd}$$

to get a module N^{n+1} homotopy equivalent relative $S \otimes_R \overline{M^n}$ to N^n which contains $S \otimes_R \overline{M^{n+1}}$ as a cellular submodule. This gives the induction step, the induction terminates after finitely many steps as N and hence each N^n is finite-dimensional.

By assumption N^n_{n+1} is the pushout of

$$\begin{tikzcd}
S[\coprod \Delta^{n+1}] & S[\coprod \partial\Delta^{n+1}] \arrow[l] \arrow[r, "\varphi^{n+1}"] & S \otimes_R \overline{M^n}\ .
\end{tikzcd}$$

where φ^{n+1} is the attaching map for the cells. By Lemma 3.40 there is a map $\psi^{n+1}\colon R[\coprod \partial\Delta^{n+1}] \to \overline{M^n}$ such that $S \otimes_R \psi^{n+1}$ is homotopic to a φ^{n+1}. Call the homotopy H^{n+1}. Applying the gluing lemma to the diagram (where all vertical maps are homotopy equivalences)

$$\begin{tikzcd}
S[\coprod \Delta^{n+1}] \arrow[d] & S[\coprod \partial\Delta^{n+1}] \arrow[l] \arrow[r, "\varphi^{n+1}"] \arrow[d] & S \otimes_R \overline{M^n} \arrow[d] \\
S[\coprod \Delta^{n+1}][\Delta^1] & S[\coprod \partial\Delta^{n+1}][\Delta^1] \arrow[l] \arrow[r, "H^{n+1}"] & S \otimes_R \overline{M^n}[\Delta^1] \\
S \otimes_R R[\coprod \Delta^{n+1}] \arrow[u] & S \otimes_R R[\coprod \partial\Delta^{n+1}] \arrow[l] \arrow[r, "S\otimes\psi^{n+1}"] \arrow[u] & S \otimes_R \overline{M^n} \arrow[u]
\end{tikzcd}$$

shows that the pushout of the first row is homotopy equivalent to the pushout of the last row. (This is a simplicial version of the topological fact that homotopic attaching maps yield homotopy equivalent CW-complexes.) In the last row $S \otimes_R -$ commutes with the pushout, define $\overline{M^{n+1}}$ as the pushout of

$$\begin{tikzcd}
R[\coprod \Delta^{n+1}] & R[\coprod \partial\Delta^{n+1}] \arrow[l] \arrow[r, "\psi^{n+1}"] & \overline{M^n}\ .
\end{tikzcd}$$

The resulting map $S \otimes_R \overline{M^{n+1}} \to N^n_{n+1}$ is the desired homotopy equivalence.

Summarizing we get for $m := \dim N$ a diagram

$$\begin{tikzcd}
S \otimes_R M \arrow[r] \arrow[d] & N \arrow[d, "\simeq"] \\
S \otimes_R \overline{M^m} \arrow[r, "\cong"] & N^m
\end{tikzcd}$$

which we can make into the desired diagramm

$$\begin{tikzcd}
S \otimes_R M \arrow[r] \arrow[d, "S\otimes -"'] & N . \\
S \otimes_R \overline{M} \arrow[ur, "\sim"'] &
\end{tikzcd}$$

using a homotopy inverse for the left map and defining $\overline{M}$ as the mapping cylinder of $M \to \overline{M^m}$ to make the diagram strictly commutative. This proves (App 2). The theorem follows by the Approximation Theorem B.7. $\square$

This finishes the discussion of the different structures of categories with cofibrations. Next we will discuss another class of weak equivalences.

4. Germs

Germwise weak equivalences give a new class of weak equivalences on $\mathcal{C}^G(X,R,\mathcal{E},\mathcal{F})$ larger than the class of homotopy equivalences. It is convenient to introduce a whole "category of germs".

We motivate the notion of germs in Section 4.2. We show in Section 4.1 that for $M \in \mathcal{C}^G(X,R,\mathcal{E},\mathcal{F})$ a controlled module and $U \subseteq X$ a subset of the control space $(X,\mathcal{E},\mathcal{F})$ we can "restrict" M to U. This allows us to define germs in Section 4.3 and show in the following sections that the germwise weak equivalences give a category of weak equivalences on $\mathcal{C}^G(X,R,\mathcal{E},\mathcal{F})$ which satisfies the Gluing Lemma, the Extension Axiom and the Saturation Axiom.

4.1. Modules with support on subsets

Lemma/Definition 4.1. *Let (M,κ_R) be an E-controlled module in $\mathcal{C}^G(X,R,\mathcal{E},\mathcal{F})$. Let U be a G-invariant subset of X.*

Then there is an $E' \in \mathcal{E}$ and a cellular submodule (M_U,κ) of (M,κ) with

- *(i)* $\kappa^{-1}(U) \cap \diamond_R M_U = \kappa^{-1}(U)$
- *(ii)* $\operatorname{supp} M_U \subseteq U^{E'}$.

Here E' only depends on E and the dimension of M.

Remark 4.2. One could say that M_U is the "restriction of M to U". The first condition ensures that "over U" there is no difference between M and M_U whereas the second conditions implies in particular that M_U is controlled isomorphic to a module over U.

Recall that all modules in $\mathcal{C}^G$ are required to be finite-dimensional. This is strongly linked to this definition and the definition of control for a module in Section 1.3: as 4.1 would not be true for every infinite-dimensional controlled module with the definition of control given in Section 1.3. The proof below will show that E' can be chosen to be E^n, for n the dimension of M.

Proof. We define M_U by specifying a subset of the generating cells of M. Let n be the dimension of M. Let C_n be set of cells e_i^n of M of top dimension n with $\kappa(e_i^n) \in U$. Define C_l, $l < n$ by descending induction.

For all boundaries $\delta^*(e_i^{l+1})$ of cells $e_i^{l+1} \in C_{l+1}$ we have a unique representation (cf. (3) in Section 1.3)

$$\delta^*(e_i^{l+1}) = \sum_{e_j^l \in I(\delta^*(e_i^{l+1}))} r_j \cdot \sigma_j^* e_j^l.$$

Let I_U be the subset of $e \in (\diamond_R M)_l$ with $\kappa_R(e) \in U$. Then define

$$C_l := I_U \cup \bigcup_{e_i^{l+1} \in C_{l+1}} I(\delta^*(e_i^{l+1})).$$

Then $\bigcup_{i=0}^n C_i =: \diamond_R M_U$ generates the desired submodule M_U of M. The G-invariance is automatic. As the boundaries are controlled by E, M_U has support on U^{E^n} with $E^n := E \circ \overset{n}{\cdots} \circ E \in \mathcal{E}$. $\square$

4.2. Motivation and controlled preliminaries

So what should a "germ" be? It is an already established notion in the form of "germs at infinity", see [BFJR04] for the case of discrete controlled modules and [CPV98] for a slightly different version in the setting of controlled topological spaces. We generalize the "infinity". The basic idea is that for a control space $(X, \mathcal{E})$, a subspace $Y \subseteq X$, and the category $\mathcal{C}^G_{hfd}(X, R, \mathcal{E})$ we want to ignore "anything over Y systematically". In particular modules with support on Y should be equivalent to the trivial module and hence maps factoring over such a module should be equivalent to the zero map. The categorical properties require us to remove more. For example there are objects which are isomorphic to ones having support on Y but itself have only support on Y^E for some $E \in \mathcal{E}$, we want them to be equivalent to the zero module as well. The technical tool to do this is to introduce a new category of weak equivalences, the germwise weak equivalences. Let us elaborate on this a bit before providing the formal definition.

Assume we have a control space $(X, \mathcal{E})$ and a subspace Y of X. An instructive example for the following is $(\mathbb{R}_{\geq 0}, \mathcal{E}_d)$, the positive real numbers with metric control, with subspace $\{0\}$. We use the following observation: If we have a module M on X and we are only interested in the part "away from Y", we can consider the modules $M_{X \smallsetminus Y^{E_i}}, E_i \in \mathcal{E}$, cf. Definition 4.1. We want maps from M to be equivalent if the agree on $M_{X \smallsetminus Y^{E_i}}$, in particular $M_{X \smallsetminus Y^{E_i}} \rightarrowtail M$ will be a germwise weak equivalence. (Note that for $\{0\} \subset (\mathbb{R}_{\leq 0}, \mathcal{E}_d)$ the sets $X \smallsetminus Y^{E_i}$ are of the form (α, ∞), $\alpha \in \mathbb{R}_{\geq 0}$.)

We axiomatize the properties we need from the sets $X \smallsetminus Y^{E_i}$ below in Definition 4.4 as *germ support sets* which hopefully makes the arguments more transparent. The sets $X \smallsetminus Y^{E_i}$ are our only example for them. Note that object support conditions $\mathcal{F}$ do not play any role in these discussion, so we will not mention them in this chapter any more.

Remark 4.3. The aim of this theory is to play the role of a Karoubi filtration of an additive category from [CP97], which gives a homotopy fiber sequence in the algebraic K-theory of these additive categories. In particular we want the precise analogue of Proposition 4.2 of [BFJR04], which says that an inclusion of control spaces gives a Karoubi filtration, and hence a fiber sequence. The problem why we cannot apply that theory is the following: For *discrete* R-modules M in $\mathcal{C}^G(X)$ and $Y \subseteq X$ a G-equivariant subspace we can decompose M as $M = M_Y \oplus M_{X \smallsetminus Y}$, but this decomposition not possible for a general simplicial R-module. Therefore we do not have "complements" for simplicial modules, and further we have to take homotopy equivalences into account. Readers familiar with the notion of a Karoubi filtration from [CP97] should view the category $\mathcal{C}^G$ together with the germwise weak equivalences $gw\mathcal{C}^G$ as an analogue of the Karoubi quotient $\mathcal{U}/\mathcal{A}$ from [CP97, Def. 3.3], for $\mathcal{U}$ an $\mathcal{A}$-filtered additive category.

Definition 4.4. *Let $(X, \mathcal{E})$ be a G-equivariant control space. A set of germ support sets is a collection $\mathcal{U}$ of G-invariant subsets U_i of X such that*

(i) For $U_1, U_2 \in \mathcal{U}$ there is a $U_3 \in \mathcal{U}$ with $U_3 \subseteq U_1 \cap U_2$.

(ii) For $U_1 \in \mathcal{U}$ and $E \in \mathcal{E}$ there is a $U_4 \in \mathcal{U}$ with ${U_4}^E \subseteq U_1$.

We can draw some immediate conclusions.

Lemma 4.5. *Suppose we have a set of germ support sets $\mathcal{U}$ on $(X, \mathcal{E})$.*

(i) For each $U_i \in \mathcal{U}$ the restricted module M_{U_i} from 4.1 exists.

(ii) For $U_1, U_2 \in \mathcal{U}$ there is a $U_3 \in \mathcal{U}$ and cellular inclusions $M_{U_3} \rightarrowtail M_{U_1}$ and $M_{U_3} \rightarrowtail M_{U_2}$.

(iii) For $M \in \mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})$, U_1 and an E-controlled map $f\colon A \to M$ there is a $U_4 \subseteq U_1 \subseteq X$ such that the restricted map $f\colon A_{U_4} \to M$ factors through M_{U_1},

$$\begin{tikzcd} A_{U_4} \arrow[r, dotted] \arrow[dr, "f"'] & M_{U_1} \arrow[d] \\ & M \end{tikzcd} ,$$

which implies in particular that for any map $f\colon A \to B$ and any U_1 there is a U_4 such that f restricts to

$$\begin{tikzcd} A_{U_4} \arrow[r, "f"] \arrow[d] & B_{U_1} \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Proof. The first claim follows from 4.1, the second and third are then easy to check. ☐

Remark 4.6. The definition of "germ support sets" is made such that Lemma 4.5 is true (and a bit more which we will need later in a proof). The first point of the definition requires that we can always make two germ support sets smaller to get a common germ support set, whereas the second point is that we can always restrict quite far, getting a certain "margin" to the original set.

Like for object support conditions and morphism control conditions we can alway pass to a cofinal subset of germ support sets, this time with respect to the descending order "$\supseteq$". This means that if $\mathcal{U} \subseteq \mathcal{U}'$ are germ support sets such that for each $U' \in \mathcal{U}'$ there is a $U \in \mathcal{U}$ with $U \subseteq U'$ then $\mathcal{U}$ and $\mathcal{U}'$ are equivalent for all purposes. Note that the axioms for germ support sets are dual to the ones for object support conditions, i.e. $\{X \setminus U \mid U \in \mathcal{U}\}$ is a set of object support conditions. This is no coincidence, but in applications where the concepts are used simultaneously they are "orthogonal", i.e. the specific germ support sets and object support conditions are chosen independently and are not dual to each other.

Lemma 4.7. *Let $(X, \mathcal{E})$ be a free G-equivariant control space and Y a G-invariant subspace of X. The collection of sets $\{U_i\} := \{X \setminus Y^{E_i} \mid E_i \in \mathcal{E}\}$ is a set of germ support sets.*

Proof. As Y and each E_i is G-invariant the G-invariance of U_i follows.

Given $U_1 = X \setminus Y^{E_1}$ and $U_2 = X \setminus Y^{E_2}$ then there is a E_3 with $E_1 \cup E_2 \subseteq E_3$. It follows for $U_3 := X \setminus Y^{E_3}$ that $Y^{E_1} \cup Y^{E_2} \subseteq Y^{E_3}$ and hence

$$U_1 \cup U_2 = (X \setminus Y^{E_1}) \cap (X \setminus Y^{E_2}) = X \setminus (Y^{E_1} \cup Y^{E_2}) \supseteq X \setminus Y^{E_3} = U_3.$$

This proves the first part.

For the second part let $U_1 = X \setminus Y^{E_1}$ and given an $E \in \mathcal{E}$. We prove that for $U_4 := X \setminus Y^{E \circ E_1}$ we have ${U_4}^E \subseteq U_1$. So take $x \in {U_4}^E$. Then there is a $(x, y) \in E$ with $y \notin Y^{E \circ E_1}$. It follows $x \notin Y^{E_1}$. Hence $x \in U_1$. $\square$

Definition 4.8. *We call the germ support sets $\mathcal{U}_Y := \{U_i\} = \{X \setminus Y^{E_i} \mid E_i \in \mathcal{E}\}$ from Lemma 4.7 the germs support sets away from Y.*

4.3. The category of germs

For the category of controlled simplicial R-modules $\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})$ and a collection of germ support sets $\mathcal{U} = \{U_i\}$ satisfying Definition 4.4 we define the *category of germs* at $\mathcal{U}$. This category will be denoted by $\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})^g$ where the little "g" refers to the chosen collection $\mathcal{U}$ which usually will be left understood.

Definition 4.9. *Let $(X, \mathcal{E}, \mathcal{F})$ be a control space and $\mathcal{U} = \{U_i\}$ a set of germ support sets on X. Consider the category $\mathcal{C}^G = \mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})$. A partial map from A to B is a map $A_U \to B$ for some $U \in \mathcal{U}$. For $U_1 \subseteq U$, $U, U_1 \in \mathcal{U}$ and a partial map $f\colon A_U \to B$ we call the composition $A_{U_1} \to A_U \to B$ the restriction of f to U_1. It is again a partial map.*

Two partial maps $f_{1,2}\colon A_{U_{1,2}} \to B$ are equivalent if there is a $U_3 \subseteq U_1 \cap U_2$ such that the restrictions to U_3 are equal, i.e.

$$A_{U_3} \rightarrowtail A_{U_1} \xrightarrow{f_1} B \qquad \text{equals} \qquad A_{U_3} \rightarrowtail A_{U_2} \xrightarrow{f_2} B.$$

Note that a partial map $A_U \to B$ does not need to be a restriction of a map $A \to B$, but every map $A \to B$ gives a partial map $A_U \to B$ by restriction.

Lemma 4.10. *Let f, f' be partial maps from A to B, let g, g' be partial maps from B to C. Assume that f is equivalent to f' and g is equivalent to g'. Then there is a $\overline{U}$ such that composition of the partial maps $g \circ f\colon A_{\overline{U}} \to C$ is defined. Further $g \circ f$ and $g' \circ f'$ are equivalent as partial maps.*

Proof. Assume we have the following maps:

$$\begin{array}{ll} f\colon A_{U_1} \to B & \qquad g\colon B_{U_3} \to C \\ f'\colon A_{U_2} \to B & \qquad g'\colon B_{U_4} \to C. \end{array}$$

Then by Lemma 4.5 (iii) there are $\overline{U}_1$ and $\overline{U}_2$ such that f and f' restrict to

$$f\colon A_{\overline{U}_1} \to B_{U_3} \qquad\qquad f'\colon A_{\overline{U}_2} \to B_{U_4},$$

hence we can compose.

Further there is a $\overline{U}_3 \subseteq U_3 \cap U_4$ such that $g = g'\colon B_{\overline{U}_3} \to C$ and a $\overline{\overline{U}}_1 \subseteq \overline{U}_1 \cap \overline{U}_2$ such that $f = f'\colon A_{\overline{\overline{U}}_1} \to B$. Therefore there is a $V \subseteq \overline{\overline{U}}_1$ such that we get the equality of the restrictions

$$g \circ f = g' \circ f'\colon A_V \to B_{\overline{U}_3} \to C. \qquad \square$$

Definition 4.11. *We call an equivalence class of partial maps from A to B a germ. We usually write $f\colon A \to B$ for a germ from A to B by a misuse of notation.*

Remark 4.12. Note that a more precise notation for a germ from A to B would be the notation $[f\colon A_U \to B]$, but this is inconvenient to use.

Definition 4.13. *Let $(X, \mathcal{E}, \mathcal{F})$ be a control space and $\mathcal{U}$ a set of germ support conditions. Define the category of germs of $\mathcal{C}^G$ as the category with the same objects as $\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})$ and with morphisms the germs from A to B. The composition is given by Lemma 4.10, leaving $\mathcal{U}$ understood.*

Denote the category of germs by $\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})^g$ or abbreviated by $\mathcal{C}^{Gg}$. Here and in the following the little "g" always refers to the collection $\mathcal{U}$.

We have an obvious functor $\mathcal{C}^G \to \mathcal{C}^{Gg}$ by which we can regard morphisms and objects in $\mathcal{C}$ as lying in $\mathcal{C}^g$. It take a map $A \to B$ to a partial map $A_U \to B$ for $U \in \mathcal{U}$ and then takes equivalence classes. The functor is surjective on objects by definition. This gives for each simplicial set K and $M \in \mathcal{C}^{Gg}$ an object $M[K]$ in $\mathcal{C}^{Gg}$. So in particular we get the object $A[\Delta^1]$. It will serve as a cylinder.

Definition 4.14. *A homotopy from f to g in $\mathcal{C}^{Gg}$ is germ $H\colon A[\Delta^1] \to B$ such that $H \circ \iota_0$ is f and $H \circ \iota_1$ is g.*

The following lemma shows that the notion of homotopy is unambiguous.

Lemma 4.15. *Restriction commutes with adjoining a simplicial set, e.g. $A_U[\Delta^1]$ is $A[\Delta^1]_U$.*

Proof. This follows by inspection. ☐

We note an obvious but important fact.

Lemma 4.16. *For any module A and any subset U_i the canonical inclusion $A_{U_i} \rightarrowtail A$ is an isomorphism in $\mathcal{C}^{Gg}$.* ☐

Remark 4.17. The interplay between the categories $\mathcal{C}^G$ and $\mathcal{C}^{Gg}$ is important, so let us give a few remarks about it. A germ from A to B in $\mathcal{C}^{Gg}$ is represented by a partial map $A_U \to B$ in $\mathcal{C}^G$, this means, using Lemma 4.5(iii), that we can transport finite diagrams without cycles in $\mathcal{C}^{Gg}$ back to $\mathcal{C}^G$ and investigate them there. For example if we want to show that a diagram commutes in $\mathcal{C}^{Gg}$ we can look at the corresponding diagram of partial maps in $\mathcal{C}^G$ and check its commutativity. Note however, that if it does not commute in $\mathcal{C}^G$ it still might commute in $\mathcal{C}^{Gg}$ as some restriction might commute.

So although the functor $\mathcal{C}^G \to \mathcal{C}^{Gg}$ is neither injective nor surjective on morphisms from A to B in general, we still can and must use the source category $\mathcal{C}^G$ to examine the target category $\mathcal{C}^{Gg}$. We will use this strategy in the case of pushout-diagrams.

4.4. Germwise weak equivalences

As before we fix some set of germ support sets $\mathcal{U}$ and consider germs g with respect to $\mathcal{U}$. The notion of homotopy in $\mathcal{C}^{Gg}$ yields the notion of homotopy equivalence. Hence we can make the following definition.

Definition 4.18. *A map $A \to B$ in $\mathcal{C}^G$ is called a germwise weak equivalence or gw-equivalence if it becomes a homotopy equivalence in $\mathcal{C}^{Gg}$.*

We will show that an analogue of the Gluing Lemma and the Extension Axiom hold in $\mathcal{C}^{Gg}$, which will provide us with these axioms for the *gw*-equivalences. We will not prove that $\mathcal{C}^{Gg}$ is a category with cofibrations and weak equivalences as we do not need that. In particular we refrain from defining cofibrations there. The author does not know, if $\mathcal{C}^{Gg}$ can be made into a category with cofibrations and weak equivalences.

We first need a very helpful technical lemma.

Lemma 4.19. *Let*

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d] & B \arrow[d] \\
C \arrow[r, tail] & Q
\end{tikzcd}$$

be a pushout along a cofibration in $\mathcal{C}^G$, then it is a pushout in $\mathcal{C}^{Gg}$.

As before we can assume that $A \rightarrowtail B$ is a cellular inclusion. We prove first that if we restrict the diagram it does not matter.

Lemma 4.20. *Let $C \leftarrow A \rightarrow B$ be a diagram in $\mathcal{C}^G$ with pushout Q where $A \rightarrowtail B$ is a cellular inclusion. Let $C_{U_2} \leftarrow A_{U_1} \rightarrowtail B_{U_3}$ be a restriction of the diagram, $U_i \in \mathcal{U}$, i.e. the diagram*

$$\begin{tikzcd}
C_{U_2} \arrow[d] & A_{U_1} \arrow[l] \arrow[r, tail] \arrow[d] & B_{U_3} \arrow[d] \\
C & A \arrow[l] \arrow[r, tail] & B
\end{tikzcd}$$

is strictly commutative in $\mathcal{C}^G$. (It follows that $A_{U_1} \rightarrowtail B_{U_3}$ is a cellular inclusion.) Let Q^U be the pushout of the restricted diagram in $\mathcal{C}^G$.

Then the induced map between the pushouts $Q^U \rightarrow Q$ is an isomorphism in $\mathcal{C}^{Gg}$.

Remark 4.21. In general $Q^U \rightarrow Q$ will not be injective in $\mathcal{C}^G$. E.g. if A_{U_1} is "small" and C a point the map $B_{U_3} \rightarrow Q^U$ might not collapse some cells of B_{U_3} which are collapsed by $B \rightarrow Q$.

Proof. We take the canonical model for Q and Q^U from Lemma 2.2 and therefore know precisely how they are constructed. In particular we have $\diamond_R Q \cong \diamond_R C \cup (\diamond_R B \setminus \diamond_R A)$ and $\diamond_R Q^U \cong \diamond_R C_{U_2} \cup (\diamond_R B_{U_3} \setminus \diamond_R A_{U_1})$ and the control maps are the ones induced by the control maps of C and B. We will denote all control maps by κ as there is no ambiguity. We choose a $\overline{U}$ such that ${Q^U}_{\overline{U}}$ has support contained in $U_1 \cap U_2 \cap U_3$. Such a $\overline{U}$ exists by Lemma 4.5.

We show that ${Q^U}_{\overline{U}} \rightarrowtail Q$ is a cellular inclusion such that over $\overline{U}$ the cells of ${Q^U}_{\overline{U}}$ and Q are in bijection. Let $e_1 \in \diamond_R Q^U$ be a cell which comes neither from $\diamond_R C$ nor from $\diamond_R B \setminus \diamond_R A$. The it must be from $\diamond_R B_{U_3} \setminus \diamond_R A_{U_1}$, hence it follows that it comes from a cell $e_2 \in \diamond_R A \setminus \diamond_R A_{U_1}$. Therefore $\kappa(e_2) \notin U_1$, in particular $e_1 \notin {Q^U}_{\overline{U}}$. So ${Q^U}_{\overline{U}}$ is a cellular submodule of Q.

So let now e_1 be a cell in $\diamond_R Q$ with $\kappa(e_1) \in \overline{U}$. Then there is an e_2 which is either in $\diamond_R C$ or in $\diamond_R B \setminus \diamond_R A$ which maps to e_1. In both cases $\kappa(e_2) = \kappa(e_1) \in \overline{U}$, hence $e_2 \in \diamond_R C_{U_1}$ or $e_2 \in \diamond_R B_{U_3}$, so it provides an element e_3 in $\diamond_R {Q^U}_{\overline{U}}$ mapping to e_1 in $\diamond_R Q$. This shows all cells of Q which lie over $\overline{U}$ are already in ${Q^U}_{\overline{U}}$.

It follows that the composition ${Q^U}_{\overline{U}} \rightarrowtail Q^U \rightarrowtail Q$ is an isomorphism in $\mathcal{C}^g$. As the first map is one, so then is the second, which proves the lemma. □

Proof of Lemma 4.19. We assume $A \rightarrowtail B$ is a cellular inclusion. We now have to check the universal property. Denote $B \rightarrow Q$ by i_B and $C \rightarrow Q$ by i_C.

Assume that T is an object in $\mathcal{C}^{Gg}$ and let $f\colon B \rightarrow T$, $g\colon C \rightarrow T$ be two germs which agree when precomposed with $A \rightarrow B$. This means that there are U_1, U_2, $U_3 \in \mathcal{U}$ such that the maps

$$A_{U_1} \rightarrow B_{U_3} \xrightarrow{f} T$$
$$A_{U_1} \rightarrow C_{U_2} \xrightarrow{g} T$$

exist and agree in $\mathcal{C}^G$. If we take the pushout Q^U of $C_{U_2} \leftarrow A_{U_1} \rightarrow B_{U_2}$ in $\mathcal{C}^G$ we get a (unique) induced map $Q^U \rightarrow T$ (in $\mathcal{C}^G$) and by the last lemma the canonical map $Q^U \to Q$ is an isomorphism in $\mathcal{C}^{Gg}$ so this gives a germ $t\colon Q \to T$ compatible with the germs $B \to T$ and $C \to T$. This gives the existence.

Assume there is another germ $t'\colon Q \to T$ compatible with f and g. By taking representing partial maps of $t' \circ i_B$ and $t' \circ i_C$ we get a solid commutative diagram

$$\begin{tikzcd}
A_{V_1} \arrow[r] \arrow[d] & B_{V_3} \arrow[d, dashed] \arrow[ddr, bend left, "t'\circ i_B"] & \\
C_{V_2} \arrow[r, dashed] \arrow[drr, bend right, "t'\circ i_C"'] & Q^V \arrow[dr, dashed] & \\
 & & T
\end{tikzcd} . \tag{17}$$

We get a unique induced map $Q^V \to T$ on the pushout which represents t', i.e. $Q^V \to T$ represents the same germ as $Q^V \xrightarrow{\cong_g} Q \xrightarrow{t'} T$. But all partial maps in the solid diagram (17) represent the same germs which also induce the map $t'\colon Q^U \to T$. It follows that the outer diagram

$$\begin{tikzcd}
Q^U \arrow[d, "\cong"'] \arrow[dr, "t"] & \\
Q \arrow[r, "t", "t'"'] & T \\
Q^V \arrow[u, "\cong"] \arrow[ur, "t'"'] &
\end{tikzcd}$$

commutes in $\mathcal{C}^{Gg}$ and this implies $t = t'$. This shows the uniqueness. ☐

Lemma 4.22. *Homotopy of germs in $\mathcal{C}^{Gg}$ is an equivalence relation.*

Proof. Reflexivity is clear using Lemma 4.15. Symmetry follows by the existence of a “twist” map $A[\Delta^1] \to A[\Delta^1]$ by the Kan Extension property. Transitivity follows as soon as we show that homotopies can be concatenated.

By the horn-filling property we have a map $A[\Delta^1] \to A[\Delta^1 \cup_{\Delta^0} \Delta^1]$. We only need to show that two homotopies $A[\Delta^1] \to B$ can be glued to a map $A[\Delta^1 \cup_{\Delta^0} \Delta^1] \to B$ if the end of the first is the same germ as the start of the second. But the last lemma showed the source of this is a pushout in $\mathcal{C}^{Gg}$ so this can be done. ☐

The lemma implies that the homotopy equivalences in $\mathcal{C}^{Gg}$ behave in the same way as one is used to. In particular we have the following lemma.

Lemma 4.23. *The gw-equivalences fulfill the Saturation Axiom.*

Proof. The gw-equivalences are the homotopy equivalences in $\mathcal{C}^{Gg}$, so one can regard the question there. But the elaborated proof we gave for the Saturation Axiom 2.31 in $\mathcal{C}^G$ unsurprisingly also works for the homotopy equivalences in $\mathcal{C}^{Gg}$. ☐

4.5. $\mathcal{C}^G$ as a category with cofibrations and germwise weak equivalences

The goal of this section is to show that the germwise weak equivalences also satisfy the axioms of a category of weak equivalences in $\mathcal{C}^G$ (cf. B.2 / Recollection 3.4). The main point is to prove the Gluing Lemma. That requires the same steps as in the proof of the Gluing Lemma for the homotopy equivalences which we did in Section 3.1. Fortunately big parts of that proof there can be reused, so we follow the same strategy. After that we also show that the Extension Axiom holds for the germwise weak equivalences, which is parallel to Section 2.8.

We state the crucial lemma first, then deduce the existence of a structure of a category of germwise weak equivalences on $\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})$ before we prove that lemma afterwards. We follow the same strategy we used to show Proposition 3.3, the proofs are parallel to those of Section 3.1 and Sections 2.4 to 2.8. Therefore we sometimes only remark the differences to the proofs from there, in particular all statements about $\mathcal{C}^G$ from there which do not involve weak equivalences apply here and we will use them without further remarks.

Lemma 4.24. *Given a pushout diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "\sim_{gw}"', "f"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

in $\mathcal{C}^G$ where $A \to C$ is a cofibration and a gw-equivalence. Then $B \to D$ is a gw-equivalence.

Proof. Deferred after Remark 4.28. ☐

Remark 4.25. Of course it also follows that $B \to D$ is a cofibration as $\mathcal{C}^G$ is a category with cofibrations, which we proved already.

Lemma 4.26. *The category $\mathcal{C}^G = \mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})$ together with the cofibrations and gw-equivalences as weak equivalences is a category of cofibrant objects in the sense of Remark 3.7.*

Proof. From the axioms for a category of cofibrant objects (see Remark 3.7 for a list) we only need to show a part of 3.7(iii): The pushout of an acyclic cofibration is an acyclic cofibration, but this is Lemma 4.24. The remaining points either do not involve weak equivalences or are already shown: The Saturation Axiom is Lemma 4.23 and the Cylinder Axiom is implied by $w\mathcal{C} \subseteq gw\mathcal{C}$. Hence Lemma 4.24 implies this lemma. ☐

Lemma 4.27. *The category $\mathcal{C}^G = \mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})$ together with the cofibrations and gw-equivalences as weak equivalences is a category with cofibrations and weak equivalences. It has a Cylinder Functor which satisfies the Cylinder Axiom and its weak equivalences satisfy the Saturation Axiom.*

Proof. This is implied by Lemma 4.26 which in particular implies the Gluing Lemma for gw-equivalences, as categories of cofibrant objects satisfy the Gluing Lemma by [GJ99, II.8.8]. ☐

Remark 4.28. We denote this category as $gw\mathcal{C}_a^G$. It also satisfies the Extension Axiom which we will prove in Lemma 4.34.

We start the proof of Lemma 4.24. Basically we have to redo Sections 2.4 to 2.7 for germwise homotopy equivalences. However some proofs transfer to our situation, so we only sketch the steps and describe the differences.

We first need a version of the homotopy extension property. We repeat that there is a functor $\mathcal{C}^G \to \mathcal{C}^{Gg}$ by which we can interpret "real" maps as germs. Hence it makes sense for example to say that two maps in $\mathcal{C}^G$ are "germwise homotopic", this should mean there is a homotopy in $\mathcal{C}^{Gg}$ between the maps regarded as germs. We abbreviate this and say they are g-homotopic, and the same for other notions.

Lemma 4.29 (g-HEP)**.** *Let $A \rightarrowtail C$ be a cofibration in $\mathcal{C}^G$, let Q be in $\mathcal{C}^G$, and let $C \to Q$ be a germ (g-map) in $\mathcal{C}^{Gg}$. Assume there is a g-homotopy $H\colon A[\Delta^1] \to Q$ in $\mathcal{C}^{Gg}$ starting at $H_0\colon A \rightarrowtail C \to Q$. Then H can be extended to a g-homotopy $\overline{H}\colon C[\Delta^1] \to Q$ in $\mathcal{C}^{Gg}$.*

Proof. We have to find a dashed arrow in the diagram

$$\begin{tikzcd}
A[\Delta^1] \cup C \arrow[r] \arrow[d] & Q \\
C[\Delta^1] \arrow[ur, dashed] &
\end{tikzcd}$$

in $\mathcal{C}^{Gg}$. We can assume that $A \to C$ is a cellular inclusion.

The germ $C \to Q$ is represented by a partial map $C_{U_1} \to Q$, so H_0 is represented by the partial map $A_{U_1} \to C_{U_1} \to Q$ where $A_{U_1} \to C_{U_1}$ is a cellular inclusion. Let the g-homotopy H be represented by $A[\Delta^1]_{U_2} \to Q$. There is a $U_3 \subseteq U_1 \cap U_2$ such that $(A[\Delta^1] \cup C)_{U_3} = A_{U_3}[\Delta^1] \cup C_{U_3}$ and the germ $A[\Delta^1] \cup C \to Q$ is represented by the pushout $A_{U_3}[\Delta^1] \cup C_{U_3} \to Q$ in $\mathcal{C}^G$. The map l in the diagram

$$\begin{tikzcd}
A_{U_3}[\Delta^1] \cup C_{U_3} \arrow[r] \arrow[d] & Q \\
C_{U_3}[\Delta^1] \arrow[ur, dashed, "l"'] &
\end{tikzcd}$$

exists in $\mathcal{C}^G$ by the homotopy extension property from Lemma 2.29. It is a diagram of partial maps for the desired diagram of germs above, so in particular $\overline{H} := l$ represents a solution for the extension problem. ☐

Analogous we have a horn-filling property.

Lemma 4.30. *Every horn can be filled in $\mathcal{C}^{Gg}$.*

Proof. Given a germ $A[\Lambda^n_i] \to B$ in $\mathcal{C}^{Gg}$. This is represented by a partial map $A[\Lambda^n_i]_U \to B$ and $A[\Lambda^n_i]_U$ is $A_U[\Lambda^n_i]$ by Lemma 4.15. But $A_U[\Lambda^n_i]$ can be filled in $\mathcal{C}^G$, so there is a map $A_U[\Delta^n] \to B$ which extends the horn, which is a partial map $A[\Delta^n]_U \to B$. ☐

The notion of a deformation retraction (see Definition 2.32) makes sense in $\mathcal{C}^{Gg}$. We need it in a case where the inclusion exists already in $\mathcal{C}^G$. So the data we have for A being a deformation retract of M in $\mathcal{C}^{Gg}$ (in our application) are an inclusion $i\colon A \to M$ in $\mathcal{C}^G$, a germ $r\colon M \to A$ such that $r \circ i = \mathrm{id}_A$ in $\mathcal{C}^{Gg}$, and a g-homotopy $M[\Delta^1] \to M$ from $i \circ r$ to id_M in $\mathcal{C}^{Gg}$. As Definition:

Definition 4.31. *If $i\colon A \to M$ is a map in $\mathcal{C}^G$ and $r\colon M \to A$ a map in $\mathcal{C}^{Gg}$ such that A is a deformation retract of M in $\mathcal{C}^{Gg}$ with inclusion i and retraction r then A is called a g-deformation retract of M and r is called a g-deformation retraction.*

Remark 4.32. Note that the notion of a g-deformation retract is only useful for discussions about $\mathcal{C}^G$ if at least one of maps "inclusion" or "retraction" is in $\mathcal{C}^G$. In our cases this will always be the inclusion.

As before we denote the Cylinder Functor (or mapping cylinder) applied to a map $f\colon A \to B$ in $\mathcal{C}^G$ by $T(f)$.

Lemma 4.33. *Let $f\colon A \to B$ be a gw-equivalence in $\mathcal{C}^G$. Then A is a g-deformation retract of $T(f)$ via the inclusion ι_0.*

Proof. We do the same three steps as in the proof of Proposition 2.34. The first step is to get the g-retraction $r\colon T(f) \to A$. But the proof of Lemma 2.37 works in $\mathcal{C}^{Gg}$. We only need that T is also a Cylinder Functor in $\mathcal{C}^{Gg}$, which follows as we have cylinders and pushout by Lemmas 4.15 and 4.19. Hence we get a germ r with $r \circ i = \mathrm{id}_A$ as germs.

The next step is to construct a g-homotopy $i \circ r \simeq \mathrm{id}_{T(f)}$. But the proof of Lemma 2.38 applies verbatim. We just have to interpret all maps and homotopies as maps in $\mathcal{C}^{Gg}$. So we get a homotopy $H\colon T(f)[\Delta^1] \to T(f)$ from $i \circ r$ to $\mathrm{id}_T(f)$ in $\mathcal{C}^{Gg}$.

So the last step is to make that homotopy relative to (i.e. constant on) A. Again we can apply the proof of Lemma 2.39 verbatim, interpreting it in $\mathcal{C}^{Gg}$. (It was written in that way.) Just note that all the constructions we do there are available: To construct maps from $M(f)[\begin{smallmatrix} \bullet & \to & \bullet \\ \uparrow & \nearrow & \uparrow \\ \bullet & \to & \bullet \end{smallmatrix}]$ etc. we need that they are pushouts and arise by adjoining the simplicial set, which is provided by Lemmas 4.19 and 4.15. The needed homotopy extension property and the horn-filling are provided by Lemmas 4.29 and 4.30. We get the desired relative homotopy. This finishes the proof. ☐

Finally we can prove Lemma 4.24.

Proof of Lemma 4.24. We again follow the same strategy as in the non-germ case, namely in the proof of Lemma 2.40. We factor f as $A \to T(f) \to C$ and consider the two pushouts

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
T(f) \arrow[r] \arrow[d] & Q \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

in $\mathcal{C}^G$. If we show that $B \to Q$ and $Q \to D$ are gw-equivalences then we are done. But $Q \to D$ is even a w-equivalence by Lemma 2.42(ii), which only uses that $A \rightarrowtail C$ is a cofibration.

For the other part Lemma 2.42(i) applies again verbatim if we interpret it as being in $\mathcal{C}^{Gg}$. ☐

The next step is to prove the Extension Axiom for the gw-equivalences in $\mathcal{C}^G$. Again we use the same strategy as in before. We will be brief. See Section 2.8 for the original proof.

Lemma 4.34. *The category $gw\mathcal{C}^G$ of weak equivalences satisfies the Extension Axiom.*

Proof. This is the same proof as for Lemma 2.48, once we prove the analogue of Lemma 2.49 below. ☐

The analogue of Lemma 2.49 is of course (note the "g"-deformation retracts):

Lemma 4.35. *Assume we have a cofiber sequence $A \rightarrowtail B \twoheadrightarrow \overline{B}$ in $\mathcal{C}^G$ where $\overline{B} = B/A$ for brevity. Suppose we have a diagram*

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, tail] & B \arrow[r, two heads] \arrow[d, tail] & \overline{B} \arrow[d, tail] \\
T_A \arrow[r, tail] & T_B \arrow[r, two heads] & T_{\overline{B}}
\end{tikzcd}$$

in $\mathcal{C}^G$ where the horizontal lines are cofiber sequences and the vertical arrows are cellular inclusions. Suppose that A and $\overline{B}$ are g-deformation retracts of T_A and $T_{\overline{B}}$ with inclusions the left and right vertical maps. Then B is a g-deformation retract of T_B with inclusion the middle vertical map.

Again there is a slightly stronger statement which is more convenient to prove. Its formulation is actually a bit more "germ-like" than Lemma 2.51.

Note that if we have a cellular inclusion $D_0 \rightarrowtail D$ and a germ support set $U \in \mathcal{U}$ then we have a cellular inclusion $D_{0U} \rightarrowtail D_U$. We write $(D, D_0)_U$ for such a situation, similar as we write (D, D_0) for the pair.

Lemma 4.36. *Assume we are in the situation of Lemma 4.35. Then for each controlled map* $(D, D_0) \to (T_B, B)$ *there is a germ support set* U *such that the partial map* $(D, D_0)_U \to (T_B, B)$ *is controlled homotopic relative* $(D_0)_U$ *to a map into* B.

It is clear Lemma 4.36 implies Lemma 4.35. We describe how to interpret the proof of Lemma 2.51 to get a proof for Lemma 4.36.

Interpretion of the Proof of 2.51. The proof is nearly the same as for the original Lemma 2.51, which was written with this application in mind. We describe the differences.

The difference in the prerequisite is that the two deformation retractions are now *germwise* deformation retractions. This means we have only partial homotopies $T_A[\Delta^1]_U \to T_A$ and $T_{\overline{B}}[\Delta^1]_V \to T_{\overline{B}}$ from the inclusions to maps into the respective submodules. (A *partial homotopy* is just a partial map from $A[\Delta^1]$ to B.) There are two situations in the proof where we need them, in both cases we construct a homotopy of maps from D to T_A by composing a map $D \to T_A$ with the homotopy $T_A[\Delta^1] \to T_A$ (resp. to $T_{\overline{B}}$).

But by Lemma 4.5 respective Lemma 4.10 we can always restrict further such that we can compose the map from D with the homotopy. This means we can follow the proof as written and when we get to one of the situations where we have to compose with a partial homotopy, we replace D by $D_{U'}$ for a suitable germ control set $U' \in \mathcal{U}$, get a real homotopy of maps from $D_{U'}$, and continue with the proof with $D_{U'}$ instead of D. After doing this twice we get a map $D_{U''} \to B$ which is homotopic relative to $(D_0)_{U''}$ to our original map. This provides the proof. □

To summarize we proved the following Theorem. Denote by $gw\mathcal{C}^G_a(X, R, \mathcal{E}, \mathcal{F})$ the category $\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})$ together with the usual cofibrations $\mathrm{co}\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})$ and the germwise weak equivalences gw as weak equivalences.

Theorem 4.37. $gw\mathcal{C}^G_a(X, R, \mathcal{E}, \mathcal{F})$ *is a category with cofibrations and weak equivalences. It has a Cylinder Functor which satisfies the Cylinder Axiom and its weak equivalences satisfy the Saturation and the Extension Axiom.*

The discussion of Chapter 3 about the different categories with cofibrations and weak equivalences given by finiteness conditions is orthogonal to, hence unaffected by, the choice of the weak equivalences. With the obvious notations we get the corollary.

Corollary 4.38. *The three categories* $gw\mathcal{C}^G_f(X, R, \mathcal{E}, \mathcal{F})$, $gw\mathcal{C}^G_{hf}(X, R, \mathcal{E}, \mathcal{F})$ *and* $gw\mathcal{C}^G_{hfd}(X, R, \mathcal{E}, \mathcal{F})$ *are categories with cofibrations and weak equivalences. They all have a Cylinder Functor which satisfies the Cylinder Axiom and their weak equivalences satisfy the Saturation and the Extension Axiom.*

5. Connective algebraic K-theory of $\mathcal{C}_f^G$

Let G be a group, $(X, \mathcal{E}, \mathcal{F})$ a G-equivariant control space and R a simplicial ring.

In Chapter 2 and 3 we constructed a category with cofibrations and weak equivalences $\mathcal{C}_f^G(X, R, \mathcal{E}, \mathcal{F})$, the finite controlled simplicial R-modules over $(X, \mathcal{E}, \mathcal{F})$ with weak equivalences $w\mathcal{C}_f^G$ the homotopy equivalences, and took its (connective) algebraic K-theory. Here we prove three basic theorems relating the K-theory of these for different control spaces. These are the following, omitting some technical conditions: First, if $Y \subseteq X$ is a G-invariant closed subspace then there is a homotopy fiber sequence relating $K(w\mathcal{C}_f^G(Y))$ and $K(w\mathcal{C}_f^G(X))$. Second, we prove a "coarse Mayer-Vietoris" theorem for control spaces, which gives a homotopy fiber square relating the K-theory of control spaces X, A, B and $A \cap B$, when $A \cup B = X$ and (X, A, B) forms an "excisive triple". Third we show that if X has a "flasque shift", then its algebraic K-theory vanishes.

Remark 5.1. The three theorems correspond to the tools of Section 4 of [BFJR04]. The homotopy fiber sequence we construct corresponds to the homotopy fiber sequence of a Karoubi filtration ([CP97]) arising from an inclusion of control spaces ([BFJR04, Prop. 4.2]), the coarse Mayer-Vietoris of course corresponds to the coarse Mayer-Vietoris ([BFJR04, Prop. 4.3]) and the "flasque shift" is an axiomatization of the conditions of [BFJR04, Prop. 4.4] which produces an Eilenberg-Swindle.

There are two notable differences between [BFJR04] and our treatment. First and minor, we are still dealing with connective algebraic K-theory, as there is no direct definition of non-connective algebraic K-theory for a general category with cofibrations and weak equivalences. This is solved in the next chapter where we define non-connective algebraic K-theory using the tools of this chapter and prove the corresponding three theorems for non-connective algebraic K-theory. Second, our definition of (local) finiteness, which is taken from [BLR08], differs slightly but in a crucial way from the definition in [BFJR04], so we have to carry an extra condition all along. This new condition on a subspace $Y \subseteq X$, which we call *froper* (see Def. 5.3), is satisfied in all the applications we discuss.

5.1. The homotopy fiber sequence with germs

Let G be a group and R be a simplicial ring. Let $(X, \mathcal{E}, \mathcal{F})$ be a G-equivariant control space, $Y \subseteq X$ a G-invariant subspace and let $\mathcal{U}_Y := \{X \setminus Y^{E_i} \mid E_i \in \mathcal{E}\}$ be the set of germs support sets away from Y (cf. Definition 4.8). By Lemma 3.20 the category $\mathcal{C}_f^G(X, R, \mathcal{E}, \mathcal{F})$ has the structure of a category with cofibrations and weak equivalences $w\mathcal{C}_f^G$, which are the homotopy equivalences. It further has a

Cylinder Functor satisfying the Cylinder Axiom and the weak equivalences satisfy the Saturation Axiom and the Extension Axiom.

It has a second class of weak equivalences, the germwise weak equivalences, denoted by $gw\mathcal{C}^G_f$, which depend on the set of germ support conditions $\mathcal{U}_Y$. Corollary 4.38 shows that $\mathcal{C}^G_f(X, R, \mathcal{E}, \mathcal{F})$ together with $gw\mathcal{C}^G_f$ also has the structure of a category with cofibrations and weak equivalences which has a Cylinder Functor satisfying the Cylinder Axiom and the class of weak equivalences satisfy the Saturation Axiom and the Extension Axiom.

We have the inclusion $w\mathcal{C}^G_f \subseteq gw\mathcal{C}^G_f$ of the two kinds of weak equivalences.

Lemma 5.2. *Waldhausen's generic fibration theorem B.5 gives a homotopy fiber sequence*

$$K(w\mathcal{C}^G_f(X, R, \mathcal{E}, \mathcal{F})^{gw}) \to K(w\mathcal{C}^G_f(X, R, \mathcal{E}, \mathcal{F})) \to K(gw\mathcal{C}^G_f(X, R, \mathcal{E}, \mathcal{F})) \tag{18}$$

of algebraic K-theory spaces.

Proof. One checks directly that all assumptions of B.5 are satisfied. □

In the Lemma $\mathcal{C}^G_f(X)^{gw}$ is the full subcategory of $w\mathcal{C}^G_f(X)$ with objects M such that $* \to M$ is a gw-equivalence (and with $w\mathcal{C}^G_f(X) \cap \mathcal{C}^G_f(X)^{gw}$ as its weak equivalences). We can describe its K-theory in terms of the K-theory of $\mathcal{C}^G(Y)$ if Y is a *froper* subspace of X. We define and discuss this notion now.

Definition 5.3 (Froper subspace)**.** *Let $(X, \mathcal{E}, \mathcal{F})$ be a G-equivariant control space and let Y be a G-invariant subspace of X. We say that Y is* froper *if it is closed in X and if for each $E \in \mathcal{E}$ there exists a G-equivariant map $f_E \colon Y^E \to Y$ which is controlled, is the identity on Y and maps locally finite sets over X with support in Y^E to locally finite sets over Y.*

Remark 5.4. See Definition 3.17 to recall the definition of a locally finite set over X. We need to check the locally finiteness with respect to X, as Y^E might not be closed. Note that as Y is closed a set over Y is a locally finite set over X if and only if it is a locally finite set over Y.

We have a range of examples where the conditions are always satisfied.

Lemma 5.5. *Let $(X, \mathcal{E}, \mathcal{F})$ be a free G-equivariant control space. If X is proper, locally compact and G acts properly discontinuous on X then each G-invariant closed subspace $Y \subseteq X$ is froper.*

Note that as the action of G on X is free, G acts properly discontinuous if and only if for each x in X there is a neighborhood U of x such that $gU \cap U = \varnothing$ for $g \in G$, $g \neq 1_G$.

Proof. Assume first that G is the trivial group. Take any E-controlled map $f_E \colon Y^E \to Y$. Let (L, κ) be a locally finite set over X with support in Y^E, i.e. $\kappa(L) \subseteq Y^E \subseteq X$. We show that $(L, f_E \circ \kappa)$ is locally finite over Y.

There is an $F_0 \in \mathcal{F}$ with $\kappa(L) \subseteq F_0$, hence $f_E \circ \kappa(L) \subseteq F_0^E$. So pick an $F \in \mathcal{F}$ with $F_0^E \subseteq F$. We get $\kappa(L) \subseteq F$, $f_E \circ \kappa(L) \subseteq F$. Take $x \in Y$ and a compact neighborhood U of x in X. The preimage $f_E^{-1}(U \cap F)$ is contained in $(U \cap F)^E$. As X is proper there is a compact set U' containing $(U \cap F)^E \cap F$. Hence $\kappa^{-1}(f_E^{-1}(U)) = \kappa^{-1}(f_E^{-1}(U \cap F)) \subseteq \kappa^{-1}(U' \cap F) = \kappa^{-1}(U')$ is a finite subset of L.

If G is not trivial, we can choose f_E to be G-equivariant. Then each of Y, E, F_0 and F is G-invariant. Further from U we get a G-compact set GU and by the G-properness of X we have that $(L, f_E \circ \kappa)$ is G-finite on GU, i.e. $(f_E \circ \kappa)^{-1}(GU)$ contains only finitely many G-orbits. As G acts properly discontinuous and free there is a neighborhood V of x in X containing only one representative of any G-orbit, hence $(f_E \circ \kappa)^{-1}(V)$ is finite. □

Lemma 5.6. *Let $i\colon Y \to X$ be the inclusion of a froper subspace, denote by g the germs away from Y. There is a functor*

$$w\mathcal{C}_f^G(Y, R, i^{-1}\mathcal{E}, i^{-1}\mathcal{F}) \to w\mathcal{C}_f^G(X, R, \mathcal{E}, \mathcal{F})^{gw}.$$

After applying algebraic K-theory it induces an isomorphism on π_n for $n \geq 1$ and an injection on π_0.

Remark 5.7. Lemmas 5.2 and 5.6 say that we almost have a homotopy fiber sequence

$$K(w\mathcal{C}_f^G(Y, R, i^{-1}\mathcal{E}, i^{-1}\mathcal{F})) \to K(w\mathcal{C}_f^G(X, R, \mathcal{E}, \mathcal{F})) \to K(gw\mathcal{C}_f^G(X, R, \mathcal{E}, \mathcal{F}))$$

but there is a “defect” on π_0 of the first term. It would be an actual homotopy fiber sequence if we replace f (finite) with hfd (homotopy finitely dominated) but we need the “finite” version in the next section.

Definition 5.8. *We say that a functor is coconnected if it is like in the lemma, i.e. it induces an injective map on π_0 and bijective maps on π_n for $n \geq 1$.*

Proof of Lemma 5.6. The proof takes the rest of this section. We do a reduction first. Let $i\colon Y \subseteq X$ be the inclusion. We define a functor $F\colon w\mathcal{C}_a^G(Y) \to w\mathcal{C}_a^G(X)^{gw}$ as follows.

A controlled module (M, κ_R) over Y is mapped to the controlled module $(M, i \circ \kappa_R)$ in $\mathcal{C}_a^G(X)$. A morphism $A \to B$ defines a morphism $F(A) \to F(B)$. The control conditions are satisfied because for each $E_Y \in i^{-1}\mathcal{E}$ there is an $E \in \mathcal{E}$ with $E_Y \subseteq E$ and similar for each $F_Y \in i^{-1}\mathcal{F}$ there is an $F \in \mathcal{F}$ with $F_Y \subseteq F$, by definition of $i^{-1}\mathcal{E}$ and $i^{-1}\mathcal{F}$. For each $M \in \mathcal{C}_a^G(Y)$ the map $* \to F(M)$ is a germwise equivalence as $F(M)_{X \smallsetminus Y} = *$, hence $(M, i \circ \kappa_R)$ is in $\mathcal{C}_a^G(X)^{gw}$.

Clearly F maps cofibrations to cofibrations and homotopy equivalences to homotopy equivalences, so F is an *exact* functor $\mathcal{C}_a^G(Y) \to \mathcal{C}_a^G(X)^{gw}$. It also respects the finiteness conditions. Namely as Y is froper it is closed in X, so F maps finite modules to finite modules. As F respects homotopies it takes homotopy finite modules to homotopy finite modules and homotopy finitely dominated modules to homotopy finitely dominated modules. This shows the existence of the functor.

Therefore there is a commutative square

$$\begin{tikzcd}
w\mathcal{C}^G_f(Y,R,i^{-1}\mathcal{E},i^{-1}\mathcal{F}) \arrow[r, "(1)"] \arrow[d] & w\mathcal{C}^G_{hfd}(Y,R,i^{-1}\mathcal{E},i^{-1}\mathcal{F}) . \arrow[d, "(3)"] \\
w\mathcal{C}^G_f(X,R,\mathcal{E},\mathcal{F})^{gw} \arrow[r, "(2)"] & w\mathcal{C}^G_{hfd}(X,R,\mathcal{E},\mathcal{F})^{gw}
\end{tikzcd} \tag{19}$$

In Proposition 3.33 we proved that (1) is coconnected. We show that the functor (2) is coconnected in Lemma 5.9 below. We further show that (3) induces a weak equivalence on algebraic K-theory in Lemma 5.10 below. Then the claim follows. ☐

Lemma 5.9. *Let $(X,\mathcal{E},\mathcal{F})$ be a control space and $\mathcal{U}=\{U_i\}$ an arbitrary set of germ support sets. The inclusion*

$$\mathcal{C}^G_f(X,R,\mathcal{E},\mathcal{F})^{gw} \to \mathcal{C}^G_{hfd}(X,R,\mathcal{E},\mathcal{F})^{gw}$$

is coconnected.

Proof. The proof follows the proof of Proposition 3.33 closely and uses the results which are established in that proof. The inclusion factors as

$$\mathcal{C}^G_f(X)^{gw} \to \mathcal{C}^G_{hf}(X)^{gw} \to \mathcal{C}^G_{hfd}(X)^{gw}.$$

The Approximation Theorem B.7 applies to the first functor like in Proposition 3.33 (i) with the additional fact that if $* \to M$ is a gw-equivalence and M is homotopy equivalent to N then $* \to N$ is a gw-equivalence.

To the second functor we can apply the same cofinality result from [TT90] as in the proof of 3.33 (ii). See there for the statement and the strategy. We also use that we established there that $\mathcal{C}^G_{hf}(X)$ is cofinal in $\mathcal{C}^G_{hfd}(X)$.

Hence we only have to prove that $\mathcal{C}^G_{hf}(X)^{gw}$ is cofinal and closed under extensions in $\mathcal{C}^G_{hfd}(X)^{gw}$. Pick an $A \in \mathcal{C}^G_{hfd}(X)^{gw}$. We show that there is a $B' \in \mathcal{C}^G_{hfd}(X)^{gw}$ such that $A \vee B' \in \mathcal{C}^G_{hf}(X)^{gw}$, that is $* \to A \vee B'$ is a gw-equivalence and there is a homotopy equivalence $A \vee B' \to D$ where D is finite.

We have that $* \to A$ is a gw-equivalence and A is homotopy finitely dominated. It was shown in the proof of 3.33(ii) that there is a $B \in \mathcal{C}^G_{hfd}(X)$ such that $A \vee B$ is homotopy finite, i.e. there is a finite D' and a homotopy equivalence $f\colon D' \to A \vee B$. The map $* \to D'$ does not need to be a gw-equivalence. But we can change B and D' simultaneously such that it becomes one.

So choose a $U_A \in \mathcal{U}$ such that $A_{U_A} \to A$ is homotopic to the trivial map and choose a $U \subseteq U_A$ and a submodule $D'_U \subseteq D'$ such that $\operatorname{supp}(f(D'_U)) \subseteq U_A$. Note that D'_U is again *finite*. The cone on $D'_U \subseteq D'$ gives a finite module $C_U D'$ such that $* \to C_U D'$ is a gw-equivalence. The map f gives a map $D'_U \to A \vee B$ and the cone $C_U(A \vee B)$ on f is homotopy equivalent to $C_U D'$ hence still homotopy finite and germwise trivial. (The cone of a map $f\colon M \to N$ is $T(f)/M$, the pushout of $* \leftarrow M \xrightarrow{\iota_0} T(f)$.)

The map $D'_U \to A \vee B$ gives two maps $f_A\colon D'_U \to A$ and $f_B\colon D'_U \to B$ and vice versa, as $A \vee B$ is also the categorical product of A and B. As f_A maps D_U into A_{U_A} it is homotopic to the trivial map, hence $D'_U \to A \vee B$ is homotopic to $f_B\colon D'_U \to B \to A \vee B$. Therefore $C_U(A \vee B)$ is homotopy equivalent to $A \vee C(f_B)$. Here $C(f_B)$ is the cone of f_B, it is still homotopy finitely dominated and $* \to C(f_B)$ is a gw-equivalence. This shows that $A \vee C(f_B)$ is homotopy equivalent to $C_U D'$, hence homotopy finite, and $* \to A \vee C(f_B)$ is a gw-equivalence, hence we get the desired cofinality with $B' := C(f_{B'})$.

The extension property is easier. We know from Proposition 3.33 that if $A \rightarrowtail B \twoheadrightarrow C$ is a cofiber sequence in $\mathcal{C}^G_{hfd}(X)^{gw}$ with A and C in $\mathcal{C}^G_{hf}(X)^{gw}$ then B is in $\mathcal{C}^G_{hf}(X)$. But as the Extension Axiom holds for gw-equivalences it follows from the map of cofiber sequences

$$\begin{tikzcd}
* \arrow[r, tail] \arrow[d, "\sim_{gw}"] & * \arrow[r, two heads] \arrow[d] & * \arrow[d, "\sim_{gw}"] \\
A \arrow[r, tail] & B \arrow[r, two heads] & C
\end{tikzcd}$$

that $* \to B$ is also a gw-equivalence. ☐

Next we prove that the map (3) in Diagram (19) induces a weak equivalence on K-theory which needs the results of Appendix C about mapping telescopes. The next lemma only works for $? = hfd$. It is the main reason why we consider homotopy finitely dominated modules. Recall that the germs are the germs away from Y.

Lemma 5.10. *The functor $F\colon w\mathcal{C}^G_{hfd}(Y) \to w\mathcal{C}^G_{hfd}(X)^{gw}$ satisfies the Approximation Property B.6. Hence it induces an equivalence on algebraic K-Theory.*

Proof. We already know that $w\mathcal{C}^G(X)$ and $w\mathcal{C}^G(Y)$ satisfy the Saturation Axiom and $\mathcal{C}^G(Y)$ satisfies the Cylinder Axiom, hence once we know that F satisfies the Approximation Property B.6, Waldhausen's Approximation Theorem B.7 applies and shows that F induces an equivalence on algebraic K-theory.

(App 1) is easy: As F is fully faithful and $F(A[\Delta^1]) = F(A)[\Delta^1]$ it reflects weak equivalences, i.e. if $F(\alpha)$ is a weak equivalence, then α is one.

(App 2) is more complicated. We follow a strategy we adapted from the proof of [CPV98, 2.12].

Given A in $\mathcal{C}^G_{hfd}(Y)$ and a map $\alpha\colon F(A) \to M$ in $\mathcal{C}^G_{hfd}(X)^{gw}$. We have to show that there is an A' together with a map $A \to A'$ in $\mathcal{C}^G_{hfd}(Y)$ such that the diagram

$$\begin{tikzcd}
F(A) \arrow[r] \arrow[d] & M \\
F(A') \arrow[ur] &
\end{tikzcd}$$

commutes and $F(A') \to M$ is a homotopy equivalence.

We can assume that $\alpha\colon F(A) \to M$ is a cellular inclusion by replacing M with the mapping cylinder $T(\alpha)$ as α factors as $F(A) \rightarrowtail T(\alpha) \xrightarrow{\sim} M$. Therefore we consider $F(A)$ as a submodule of M.

From the assumptions we have that $* \to M$ is a gw-equivalence. This means there is a germ support set $U = X \smallsetminus Y^E$ such that $M_U \to M$ is homotopic to the trivial map, let H be such a homotopy. Lemma 5.11 below shows that we can choose U such that $M_{U \cup Y} = M_U \amalg M_Y$, that is "$U$ is far away from Y". In particular we have $F(A) \subseteq M_Y$. By combining H with the constant homotopy Tr on M_Y we get a homotopy

$$H \amalg \mathrm{Tr}\colon (M_U \amalg M_Y)[\Delta^1] \to M.$$

At 0 it is the inclusion, so it is compatible with the identity $M[0] \xrightarrow{\mathrm{id}} M$ there. The homotopy extension property 2.29 then gives a homotopy from id_M to a map $\mu\colon M \to M$ which extends this homotopy. The homotopy is relative to $F(A)$ and μ is zero on M_U.

This means μ has support on $X \smallsetminus U$. By definition $X \smallsetminus U = Y^E$ for an $E \in \mathcal{E}$. Let E_μ be a control condition satisfied by μ. Then μ factors through $M_{Y^{E \circ E_\mu}}$, so we get the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\mu"] \arrow[dr, dashed, "p"] & M \\
M_{Y^E} \arrow[u, tail] \arrow[r, dashed] & M_{Y^{E \circ E_\mu}} \arrow[u, tail, "i"']
\end{tikzcd}$$

where we define p as the factorization of μ over $M_{Y^{E \circ E_\mu}}$ and i as the inclusion $M_{Y^{E \circ E_\mu}} \to M$.

From p and i we get two maps. First we get back μ as $\mu = i \circ p$. So this composition is homotopic relative $F(A)$ to the identity. And second we get the map $\eta := p \circ i$, which is a homotopy idempotent via $\eta^2 = p \circ i \circ p \circ i = p \circ \mu \circ i$ using the homotopy $p \circ H \circ i$. This is also relative to $F(A)$, i.e. we have commutative diagrams

$$\begin{tikzcd}
F(A) \arrow[d] \arrow[dr] & \\
M_{Y^{E \circ E_\mu}} \arrow[r, "\eta"] & M_{Y^{E \circ E_\mu}}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
F(A)[\Delta^1] \arrow[r, "\mathrm{Pr}"] \arrow[d, "{\mathrm{inc}[\Delta^1]}"] & F(A) \arrow[d, "\mathrm{inc}"] \\
M_{Y^{E \circ E_\mu}}[\Delta^1] \arrow[r, "p \circ H \circ i"] & M_{Y^{E \circ E_\mu}}
\end{tikzcd} .$$

Let $\kappa\colon \diamond_R M_{Y^{E \circ E_y}} \to X$ be the control map of $M_{Y^{E \circ E_y}}$, assume $\operatorname{Im} \kappa \subseteq Y^{E'}$. As $i\colon Y \subseteq X$ is froper there is a controlled map $f_{E'}\colon Y^{E'} \to Y$ and therefore $\overline{\kappa} := f_{E'} \circ \kappa\colon \diamond_R M_{Y^{E \circ E_y}} \to Y$ is a control map such that $i \circ \overline{\kappa}$ and κ are controlled isomorphic. This makes $(M_{Y^{E \circ E_y}}, \overline{\kappa})$ into an object in $\mathcal{C}^G_a(Y)$ such that $F((M_{Y^{E \circ E_y}}, \overline{\kappa}))$ and $(M_{Y^{E \circ E_y}}, \kappa)$ are controlled isomorphic. Define $\overline{\eta}$ via

$$\begin{tikzcd}
(M_{Y^{E \circ E_y}}, \overline{\kappa}) \arrow[r, "\overline{\eta}"] \arrow[d, "\cong"] & (M_{Y^{E \circ E_y}}, \overline{\kappa}) \arrow[d, "\cong"] \\
M_{Y^{E \circ E_\mu}} \arrow[r, "\eta"] & M_{Y^{E \circ E_\mu}}
\end{tikzcd} .$$

(Here we identified objects of $\mathcal{C}^G(Y)$ with their images under the fully faithful functor F.) Then $\overline{\eta}$ is a homotopy idempotent in $\mathcal{C}^G_a(Y)$.

Now we use Appendix C on mapping telescopes. We summarized the all results we need in Proposition C.4. By Proposition C.4 there is for the homotopy idempotent $\overline{\eta}$ a module $\mathrm{Tel}(\overline{\eta})$ in $\mathcal{C}^G_a(Y)$ satisfying the properties we need below. We show there is a chain of homotopy equivalences

$$F(\mathrm{Tel}(\overline{\eta})) \xrightarrow{\cong} \mathrm{Tel}(\eta) \xrightarrow{\simeq} \mathrm{Tel}(\mu) \xrightarrow{\simeq} \mathrm{Tel}(\mathrm{id}_M) \xrightarrow{\simeq} M. \tag{20}$$

Then Lemma 5.12 below shows that $\mathrm{Tel}(\overline{\eta})$ is in $\mathcal{C}^G_{hfd}(Y)$.

First $F(\mathrm{Tel}(\overline{\eta})) \cong \mathrm{Tel}(F(\overline{\eta}))$ so we can completely work in $\mathcal{C}^G_a(X)$. As the homotopy idempotents η and $F(\overline{\eta})$ are isomorphic, Proposition C.4 (ii) shows that there is an isomorphism $\mathrm{Tel}(F(\overline{\eta})) \to \mathrm{Tel}(\eta)$.

For η and μ we have two strict commutative diagrams

$$\begin{tikzcd}
M \arrow[r, "\mu"] \arrow[d, "p"] & M \arrow[d, "p"] & , & M_{Y^{E\circ E_\mu}} \arrow[r, "\eta"] \arrow[d, "i"] & M_{Y^{E\circ E_\mu}} \arrow[d, "i"] \\
M_{Y^{E\circ E_\mu}} \arrow[r, "\eta"] & M_{Y^{E\circ E_\mu}} & & M \arrow[r, "\mu"] & M
\end{tikzcd}$$

which give maps $p_*\colon \mathrm{Tel}(\mu) \to \mathrm{Tel}(\eta)$ and $i_*\colon \mathrm{Tel}(\eta) \to \mathrm{Tel}(\mu)$ by Proposition C.4 (ii) which further shows that the composition $p_* \circ i_*$ is

$$(p \circ i)_* = \eta_*\colon \mathrm{Tel}(\eta) \to \mathrm{Tel}(\eta)$$

and $i_* \circ p_*$ is

$$\mu_*\colon \mathrm{Tel}^1(\mu) \to \mathrm{Tel}(\mu)$$

As μ and η are coherent by Lemma C.2 both maps are homotopic to the identity (on the corresponding telescope) by Proposition C.4 (vi). So $i_*\colon \mathrm{Tel}(\eta) \to \mathrm{Tel}(\mu)$ is a homotopy equivalence.

Now $\mu\colon M \to M$ is homotopic to id_M hence Proposition C.4 (iii) gives a homotopy equivalence $\mathrm{Tel}(\mu) \to \mathrm{Tel}(\mathrm{id}_M)$. By Proposition C.4 (iv) there is a homotopy equivalence $\mathrm{Tel}(\mathrm{id}_M) \to M$.

We claim that all maps are relative to $F(A)$. This follows from the following diagram where the lower vertical maps are the inclusions ι from Proposition C.4 (i).

$$\begin{tikzcd}
F(A) \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] & & & & \\
F(B) \arrow[r] \arrow[d] & M_{Y^{E\circ E_\mu}} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M \arrow[d] \arrow[dr] & \\
F(\mathrm{Tel}^1(\overline{\eta})) \arrow[r] & \mathrm{Tel}^1(\eta) \arrow[r] & \mathrm{Tel}^1(\mu) \arrow[r] & \mathrm{Tel}^1(\mathrm{id}_M) \arrow[r] & M
\end{tikzcd}$$

It is commutative by Proposition C.4 (v). Hence we get with $A' := \mathrm{Tel}(\overline{\eta})$ a strict commutative diagram

$$\begin{tikzcd}
F(A) \arrow[r] \arrow[d] & M \\
F(A') \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

as desired. This proves the Approximation Property. □

There are two lemmas left we used above have to prove.

Lemma 5.11. *There is a germ support set U away from Y such that $M_{Y\cup U} = M_Y \amalg M_U$ is a coproduct of controlled modules over X.*

Proof. If M is an E-controlled module there is by Definition 4.1 an E' only depending on E and M such that for each $V \subseteq X$ we have $\operatorname{supp} M_V \subseteq V^{E'}$. Also there is an E_j such that $(Y^{E'})^{E'} \subseteq Y^{E_j}$ or equivalently $(X \smallsetminus Y^{E_j})^{E'}$ is a subset of X disjoint to $Y^{E'}$. Set $U := X \smallsetminus Y^{E_j}$. It follows that $\operatorname{supp} M_U$ is disjoint from $\operatorname{supp} M_Y$, and so the lemma follows. □

Lemma 5.12. *Let $i\colon Y \subseteq X$ be a froper subspace of the control space $(X, \mathcal{E}, \mathcal{F})$. If A is a finite module in $\mathcal{C}^G_f(X)$ which has support on Y^E for some $E \in \mathcal{E}$ then A is isomorphic to a finite module with support on Y.*

Let further $(M, \kappa_R) \in \mathcal{C}^G_a(Y, R, i^{-1}\mathcal{E}, i^{-1}\mathcal{F})$ and assume that M is homotopy finitely dominated on X, i.e. $(M, i \circ \kappa_R) \in \mathcal{C}^G_{hfd}(X)$. Then M is homotopy finitely dominated on Y, i.e. $M \in \mathcal{C}^G_{hfd}(Y)$.

Proof. By the assumption that Y is froper, hence there is an E-controlled map $\omega\colon Y^E \to Y$ which is the identity on Y and take the locally finite set $(\diamond_R A, \kappa_R)$ to the locally finite set $(\diamond_R A, \omega \circ \kappa_R)$. This makes $A' := (A, \omega \circ \kappa_R)$ into a locally finite module over Y such that A and A' are E-controlled isomorphic.

For M there is by assumption a finite module B in $\mathcal{C}^G_f(X)$, maps $M \xrightarrow{j} B \xrightarrow{r} M$ and a homotopy from $r \circ j$ to id_M. If the support of j and r is in E then r is zero outside $U := Y^E$, j factors as $M \xrightarrow{j'} B_U \to B$ and $r \circ j$ agrees with $r' \circ j'\colon M \to B_U \to M$, r' being the obvious restriction. Then $r' \circ j'$ is still homotopic to id and B_U is a finite module with support on some $Y^{E'}$. Hence by the first part it is isomorphic to a finite module with support on Y and M is dominated by this module. □

Remark 5.13. Although Lemma 5.12 looks quite innocent, it is the main place where we need the assumption that Y is froper. In particular it seems that the result is not true without a suitable assumption on Y. In [BFJR04] the assumption that the control space X is proper ensured this, but as the definition of “locally finite” was changed in [BLR08] it seems not to be clear if this is still sufficient. (By Lemma 5.5 it would be sufficient to require X to be proper, locally compact and having a properly discontinuous free G-action, but this might be too strong for our applications.)

Let us summarize the main result of this section as a Lemma for further references.

Lemma 5.14. *For $(X, \mathcal{E}, \mathcal{F})$ a G-equivariant control space, $i\colon Y \subseteq X$ a froper subspace and the germs away from Y. There is a homotopy fiber sequence of spaces*

$$K(w\mathcal{C}_f^G(X, R, \mathcal{E}, \mathcal{F})^{gw}) \to K(w\mathcal{C}_f^G(X, R, \mathcal{E}, \mathcal{F})) \to K(gw\mathcal{C}_f^G(X, R, \mathcal{E}, \mathcal{F})).$$

Further there is a map

$$K(w\mathcal{C}_f^G(Y, R, i^{-1}\mathcal{E}, i^{-1}\mathcal{F})) \to K(w\mathcal{C}_f^G(X, R, \mathcal{E}, \mathcal{F})^{gw})$$

which is an isomorphism on π_i for $i \geq 1$.

5.2. Coarse Mayer-Vietoris

The "coarse Mayer-Vietoris" is an G-equivariant, "coarse" version of the Mayer-Vietoris principle. It gives a homotopy pullback square of spaces for a triple (X, A, B) of control spaces satisfying certain conditions. In this section we discuss the version for connective K-theory.

Definition 5.15 (Coarsely excisive, cf. [BFJR04, Prop. 4.3])**.** *Let $(X, \mathcal{E}, \mathcal{F})$ be a G-equivariant control space and let $A, B \subseteq X$ be G-invariant subspaces with $A \cup B = X$. Then A and B as well as $A \cap B$ inherit G-equivariant control structures. We call the triple (X, A, B) coarsely excisive if for every $E \in \mathcal{E}$ and $F \in \mathcal{F}$ there is an $E' \in \mathcal{E}$ and an $F' \in \mathcal{F}$ such that*

$$(A \cap F)^E \cap (B \cap F)^E \subseteq (A \cap B \cap F')^{E'} \cap F'.$$

Recall the definition of a froper subspace from Definition 5.3.

Lemma 5.16 (Coarse Mayer-Vietoris, cf. [BFJR04, Prop. 4.3])**.** *Let $(X, \mathcal{E}, \mathcal{F})$ be a free G-equivariant control space and let A and B be closed subspaces of X such that $A \cup B = X$, assume that A is froper in X. Suppose that the triple (X, A, B) is coarsely excisive.*

Choose two sets of germs support conditions: Let $\mathcal{U}_A$ be the germ support conditions on A away from $A \cap B$, denote the germs by g. Let $\mathcal{U}_X$ be the germ support conditions on X away from B, denote the germs by g'.

Then the diagram

$$\begin{tikzcd}
w\mathcal{C}_f^G(A, i_A^{-1}\mathcal{E}, i_A^{-1}\mathcal{F})^{gw} \arrow[r] \arrow[d] & w\mathcal{C}_f^G(A, i_A^{-1}\mathcal{E}, i_A^{-1}\mathcal{F}) \arrow[d] \\
w\mathcal{C}_f^G(A \cup B, \mathcal{E}, \mathcal{F})^{g'w} \arrow[r] & w\mathcal{C}_f^G(A \cup B, \mathcal{E}, \mathcal{F})
\end{tikzcd}$$

is a homotopy pullback of spaces after applying algebraic K-theory, where i_A is the obvious inclusion.

Remark 5.17. In view of Lemma 5.6 we could say that

$$\begin{tikzcd}
w\mathcal{C}_f^G(A \cap B, j^{-1}\mathcal{E}, j^{-1}\mathcal{F}) \arrow[r] \arrow[d] & w\mathcal{C}_f^G(A, i_A^{-1}\mathcal{E}, i_A^{-1}\mathcal{F}) \arrow[d] \\
w\mathcal{C}_f^G(B, i_B^{-1}\mathcal{E}, i_B^{-1}\mathcal{F}) \arrow[r] & w\mathcal{C}_f^G(A \cup B, \mathcal{E}, \mathcal{F})
\end{tikzcd}$$

is *almost* a homotopy pullback on K-theory except at π_0, where j, i_A and i_B are the obvious inclusions. We will correct this in the next chapter where we define non-connective algebraic K-theory.

We write $A \cup B$ instead of X for emphasis.

Proof. The vertical functors exist because A is closed in $A \cup B$. Note that both kinds of germ support sets are subsets of A. For each $V \in \mathcal{U}_A$ there is an $U \in \mathcal{U}_X$ with $U \subseteq V$. Therefore the diagram extends to a map of homotopy fiber sequences

$$\begin{tikzcd}
w\mathcal{C}_f^G(A)^{gw} \arrow[r] \arrow[d] & w\mathcal{C}_f^G(A) \arrow[r] \arrow[d] & gw\mathcal{C}_f^G(A) \arrow[d] \\
w\mathcal{C}_f^G(A \cup B)^{g'w} \arrow[r] & w\mathcal{C}_f^G(A \cup B) \arrow[r] & g'w\mathcal{C}_f^G(A \cup B)
\end{tikzcd} \tag{21}$$

and we are done if we show that $F\colon gw\mathcal{C}_f^G(A) \to g'w\mathcal{C}_f^G(A \cup B)$ induces an equivalence on K-theory. We apply the Approximation Theorem B.7.

First we show (App 1). Let $f\colon M_0 \to M_1$ be a map in $gw\mathcal{C}_f^G(A)$ such that the map $F(f)\colon F(M_0) \to F(M_1)$ is a $g'w$-equivalence. We have to show f is a gw-equivalence, this uses that (X, A, B) is coarsely excisive. Assume $F \in \mathcal{F}$ contains the support of M_0 and M_1. We have to show that for $U \in \mathcal{U}_X$ there is a $V \in \mathcal{U}_A$ such that $U \cap F \supseteq V \cap F$. Assume $U = X \smallsetminus (B^E)$, as (X, A, B) is coarsely excisive there is an $E' \in \mathcal{E}$ and an $F' \in \mathcal{F}$ such that

$$(A \cap F)^E \cap (B \cap F)^E \subseteq (A \cap B \cap F')^{E'} \cap F'$$

Define V as $A \smallsetminus (A \cap B)^{E'}$. A calculation shows $U \cap F \supseteq V \cap F$ which shows (App 1).

For (App 2) take $M \in \mathcal{C}_f^G(A)$ and a map $F(M) \to N$. As M has support on A there is an E such that the map factors as $F(M) \to N_{A^E} \to N$. Now N_{A^E} is isomorphic to an object N' of $\mathcal{C}_f^G(A)$ by Lemma 5.12 as A is froper in $A \cup B$ and N is finite. Further the resulting map $N' \to N_{A^E} \to N$ is a $g'w$-equivalence. This shows (App 2). ☐

Remark 5.18. This proof is the reason why we use the finite modules. In particular it is not true in general, that for a homotopy finite module M the module M_U is again homotopy finite, the same for homotopy finitely dominated, hence the Approximation Property for the rightmost vertical map in (21) does not follow.

5.3. Flasque shift and Eilenberg swindle

There is a very helpful vanishing criterion: If a category $\mathcal{C}$ with cofibrations and weak equivalence has a *flasque functor* (also called an *Eilenberg swindle*) then its algebraic K-theory vanishes. If we are careful we can describe the contraction of the algebraic K-theory space and hence discuss when such contractions are compatible.

Definition 5.19. *Let $\mathcal{C}$ be a category with cofibrations and weak equivalences. An exact functor $T\colon \mathcal{C} \to \mathcal{C}$ is called* flasque functor *or an* Eilenberg functor *if there is a natural weak equivalence of functors*

$$\varepsilon\colon T \xrightarrow{\sim} T \vee \mathrm{Id}_C \,.$$

Lemma 5.20 (Eilenberg swindle)**.** *Let $\mathcal{C}, \mathcal{D}$ be categories with cofibrations and weak equivalences and let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor.*

Assume that $\mathcal{C}, \mathcal{D}$ have flasque functors $T_\mathcal{C}$ and $T_\mathcal{D}$. Then the algebraic K-theory of $\mathcal{C}$ and $\mathcal{D}$ vanishes, i.e. there is a pointed homotopy

$$K(\mathcal{C}) \wedge I_+ \to K(\mathcal{C})$$

from the trivial map to the identity, and similar for $\mathcal{D}$.

Assume further that $T_\mathcal{C}$, $T_\mathcal{D}$ are compatible with F, i.e. the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "T_\mathcal{C}"] & \mathcal{D} \arrow[d, "T_\mathcal{D}"] \\
\mathcal{C} \arrow[r, "F"] & \mathcal{D}
\end{tikzcd}$$

commutes and the natural transformations

$$F \circ T_\mathcal{C} \xrightarrow{F \circ \varepsilon_\mathcal{C}} F \circ (T_\mathcal{C} \vee \mathrm{Id}_\mathcal{C}) \quad \textit{and} \quad T_\mathcal{D} \circ F \xrightarrow{\varepsilon_\mathcal{D} \circ F} (T_\mathcal{D} \vee \mathrm{Id}_\mathcal{D}) \circ F$$

agree. Then the nullhomotopies for $K(\mathcal{C})$ and $K(\mathcal{D})$ are compatible in the sense that the diagram

$$\begin{tikzcd}
K(\mathcal{C}) \wedge I_+ \arrow[r, "K(F) \wedge I_+"] \arrow[d] & K(\mathcal{D}) \wedge I_+ \arrow[d] \\
K(\mathcal{C}) \arrow[r, "K(F)"] & K(\mathcal{D})
\end{tikzcd}$$

commutes.

Proof. For $\mathcal{C}$, the natural equivalence $\varepsilon := \varepsilon_\mathcal{C}$ induces a (pointed) homotopy $e = e_\mathcal{C}$ from $K(T)$ to $K(T \vee \mathrm{Id}_\mathcal{C})$,

$$e\colon K(T) \simeq K(T \vee \mathrm{Id}_\mathcal{C})$$

by [Wal85, 1.3.1]. As $K(\mathcal{C})$ is an infinite loop space it has a homotopy associative multiplication $\mu\colon K(\mathcal{C}) \vee K(\mathcal{C}) \to K(\mathcal{C})$ with a homotopy inverse. Composing $K(T)$

with the homotopy inverse we get a functor $K(T)^{-1}\colon K(\mathcal{C}) \to K(\mathcal{C})$. We can take the coproduct of this functor with the homotopy from above and get a homotopy (of maps)

$$K(T)^{-1} \vee K(T) \simeq K(T)^{-1} \vee \big(K(T) \vee \mathrm{Id}_{\mathcal{C}}\big). \tag{22}$$

Here "$\simeq$" means there is a homotopy from the left map to the right map. The multiplication and the homotopy inverse in $K(\mathcal{C})$ give a homotopy from the trivial map to the first term above:

$$* \simeq K(T)^{-1} \vee K(T).$$

For the last term in (22) the homotopy associativity and then the multiplication and the homotopy inverse give homotopies

$$K(T)^{-1} \vee \big(K(T) \vee \mathrm{Id}_{\mathcal{C}}\big) \simeq \big(K(T)^{-1} \vee K(T)\big) \vee \mathrm{Id}_{\mathcal{C}} \simeq * \vee \mathrm{Id}_{\mathcal{C}} \xrightarrow{\cong} \mathrm{Id}_{\mathcal{C}}$$

which shows the first claim.

The exact functor F gives a map of infinite loop spaces, hence in particular $K(F)$ is compatible with the multiplication, the inverse and all corresponding homotopies. The only thing we have to check is that the homotopy e is compatible with $K(F)$, i.e. $K(F) \circ K(e_{\mathcal{C}}) = K(e_{\mathcal{D}}) \circ K(F)$. But that follows from $F \circ e_{\mathcal{C}} = e_{\mathcal{D}} \circ F$. This shows the second claim. □

If our control space $(X, \mathcal{E}, \mathcal{F})$ has a certain self-map, then the category $\mathcal{C}^G(X)$ (and all its variations) has a flasque functor.

Lemma 5.21 (Eilenberg swindle for control spaces)**.** *Assume there is a G-equivariant self map $s\colon X \to X$ with the following properties:*

(i) s maps locally finite sets over X to locally finite sets over X.

(ii) For each $x \in X$ there is a neighbourhood U and an n such that $(s^n)^{-1}(U)$ is empty.

(iii) For every $E \in \mathcal{E}$ there exists an E' with

$$\bigcup_{n\geq 1} (s \times s)^n(E) \subseteq E'.$$

(iv) For every $F \in \mathcal{F}$ there exists an $F' \in \mathcal{F}$ with $\bigcup_{n\geq 1} s^n(F) \subseteq F'$.

(v) For every $F \in \mathcal{F}$ the set $\{(x, s(x)) \mid x \in F\}$ is contained in some $E \in \mathcal{E}$.

Then s induces a flasque functor T on $w\mathcal{C}^G_f(X, R, \mathcal{E}, \mathcal{F})$.

Definition 5.22. *We call such an s a flasque shift on X.*

Remark 5.23. The remark of [BFJR04, Prop. 4.4] applies: In many applications the last three properties follow because for every $E \in \mathcal{E}$ one has $(s \times s)(E) \subseteq E$ and for every $F \in \mathcal{F}$ one has $s(F) \subseteq F$.

Lemma 5.21 is our generalization of Proposition 4.4 of [BFJR04]. Compared to [BFJR04, Prop. 4.4] (v) was simplified in view of (iv), and in (iii) we replaced $E \cap F \times F$ by E in view of the Remark at the end of Section 1.3.

Note that the Lemma might not be true for a different class of weak equivalences. One needs (roughly) that a certain countable coproduct of weak equivalences is again one. This is not true for germwise weak equivalences if the shift goes "in the direction" of the germs. Otherwise $gw\mathcal{C}_f(\mathbb{R}_+)$, germs away from 0, which is used, below would be trivial.

Proof. The map s induces an exact functor $s_*\colon \mathcal{C}^G_a(X, R, \mathcal{E}, \mathcal{F}) \to \mathcal{C}^G_a(X, R, \mathcal{E}, \mathcal{F})$ via $(M, \kappa_R) \mapsto (M, s \circ \kappa_R)$. It respects the control conditions because of (iii) and (iv) for $n = 1$. There is an obvious natural isomorphism $\varepsilon\colon s_* \to \mathrm{Id}$, $\varepsilon(M, \kappa_R) = (M, s \circ \kappa_R) \xrightarrow{\text{"Id"}} (M, \kappa_R)$. It is controlled because of (v).

Using the coproduct in $\mathcal{C}^G_a(X, R, \mathcal{E}, \mathcal{F})$ we can form

$$T := \coprod_{n \geq 1}^{\infty} {s_*}^n \colon (M, \kappa_R) \longmapsto \Big(\coprod_{n\geq 1}^{\infty} M, \coprod_{n \geq 1}^{\infty} (s^n \circ \kappa_R) \Big)$$

We claim that this is our flasque functor. For $M \in \mathcal{C}^G_a(X)$ the properties (iii) and (iv) are precisely the conditions needed that $T(M)$ is still controlled and has a valid object support. Property (iii) also implies this for maps, hence T gives indeed a functor $\mathcal{C}^G_a(X) \to \mathcal{C}^G_a(X)$. Taking $T \vee \mathrm{Id}$ is the same as $\coprod_{n\geq 0}^{\infty} {s_*}^n$, hence $s_* \circ (\mathrm{Id} \vee T) \cong T$ and the natural transformation $s_* \to \mathrm{Id}$ gives a natural isomorphism $T \xrightarrow{\cong} T \vee \mathrm{Id}$. It is clear that T maps cofibrations to cofibrations and one checks that it also maps homotopy equivalences to homotopy equivalences. This shows T is a flasque functor on $w\mathcal{C}^G_a(X)$, we now show that it respects the finiteness conditions. This uses properties (i) and (ii).

So let M be a finite module. Consider the locally finite set $(\diamond_R M, \kappa_R)$. Pick $x \in X$, a neighborhood U of x and an n such that $(s^n)^{-1}(U)$ is empty. As s maps locally finite sets to locally finite sets also s^i does so. Hence $\Big(\coprod_{i\geq 1}(s^i \circ \kappa_R)\Big)^{-1}(U)$ is equal to $\left(\coprod_{i=1}^{n}(s^i \circ \kappa_R)\right)^{-1}(U)$ which is finite union of locally finite sets and hence again locally finite. Then T also respects the homotopy finite and the homotopy finitely dominated modules. $\square$

If we have compatible flasque shifts we get a comparison map.

Lemma 5.24. *Let $f\colon X \to Y$ be a map of control spaces and let $s_X\colon X \to X$ and*

$s_Y\colon Y \to Y$ be flasque shifts such that the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "s_X"'] & Y \arrow[d, "s_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

commutes. Then the flasque functors T_X and T_Y induced by s_X and s_Y are compatible in the sense of Lemma 5.20.

Proof. As T_X is defined as $\coprod_{n=1}^{\infty} {s_*}^n$ and f_* commutes with s and coproducts we have $f_* \circ T_X = T_Y \circ f_*$ and hence are left to check that the natural transformations

$$f_* \circ \varepsilon_X\colon f_* \circ T_X \to f_* \circ (T_X \vee \mathrm{Id})$$

and

$$\varepsilon_Y \circ f_*\colon T_Y \circ f_* \to (T_Y \vee \mathrm{Id}) \circ f_*$$

agree. But ε_X is induced by the natural transformation $\delta_X = \text{“ Id ”}\colon {s_X}_* \to \mathrm{Id}$ via $\varepsilon_X = \delta_X \circ T_X$ and similar ε_Y. Hence the commutativity $f_* \circ \delta_X = \delta_Y \circ f_*$ shows the lemma. ☐

Corollary 5.25. *For f, s_X and s_Y as in the lemma we get compatible contractions of $K(\mathcal{C}_f^G(X))$ and $K(\mathcal{C}_f^G(Y))$.* ☐

Example 5.26. Consider $\mathbb{R}_+$ with metric control. It has a flasque shift via $x \mapsto x + 1$. Hence $K(w\mathcal{C}_f(\mathbb{R}_+, R, \mathcal{E}_d))$ is contractible. More generally for $(X, \mathcal{E}, \mathcal{F})$ a G-equivariant control space we get the G-equivariant control space $(X \times \mathbb{R}_+, \mathcal{E} \times \mathcal{E}_d, \mathcal{F} \times \{\mathbb{R}_+\})$ whose algebraic K-theory $K(w\mathcal{C}_f^G(X \times \mathbb{R}_+, R))$ is contractible.

If Y is another control space and $f\colon X \to Y$ a map of control spaces then

$$f_*\colon K(w\mathcal{C}_f^G(X \times \mathbb{R}_+, R)) \to K(w\mathcal{C}_f^G(Y \times \mathbb{R}_+, R))$$

is compatible with the contractions. This gets important in the next chapter when we construct deloopings.

6. Non-connective algebraic K-theory

Our definition for the algebraic K-theory of the (G-equivariant) control space $(X, \mathcal{E}, \mathcal{F})$ with coefficients in the simplicial ring R is, unwrapping the definition (cf. Section 3.7/Appendix B/[Wal85]),

$$K(\mathcal{C}_f^G(X, R, \mathcal{E}, \mathcal{F})) := \Omega|\mathcal{S}_\bullet w\mathcal{C}_f^G(X, R, \mathcal{E}, \mathcal{F})| \ .$$

This is a topological space. But according to [Wal85] it has the structure of an infinite loop space, in other words it is the zeroth space of a connective Ω-spectrum. (Recall "connective" means the spectrum has no negative homotopy groups.) Such a spectrum is sometimes called a *delooping* of the space, hence the space above has a connective delooping. It is well-known that for purposes of the Farrell-Jones Conjecture one needs a *non-connective* delooping, i.e. a spectrum which has negative homotopy groups (cf. [LR05]). For R a discrete ring these negative homotopy groups of the non-connective algebraic K-theory spectrum for R are the negative algebraic K-groups of R first defined by Bass [Bas68].

The purpose of this chapter is to define such a non-connective delooping of the algebraic K-theory space of a control space and to discuss the corresponding homotopy fiber sequence, Mayer-Vietoris-theorem and the Eilenberg swindle in that case. These are the non-connective versions of the three basic theorems of the last section.

Unfortunately we have no definition of non-connective algebraic K-theory of a general category with cofibrations and weak equivalences as we have for additive categories by work of Pedersen and Weibel ([PW85]). Hence we are forced to do the delooping by manipulations of the control space. This is inspired by their ideas.

Let us summarize the results we want to prove in this chapter. As always X is a (G-equivariant) control space. Further $\mathbb{K}^{-\infty}$ denotes the non-connective algebraic K-theory we are about to construct. Roughly speaking $\mathbb{K}^{-\infty}$ is a functor which can be applied to categories of the form $w\mathcal{C}_f^G(X, R, \mathcal{E}, \mathcal{F})$.

(i) For $A \subseteq X$ a suitable subspace the sequence

$$w\mathcal{C}_f^G(A, R) \to w\mathcal{C}_f^G(X, R) \to gw\mathcal{C}_f^G(X, R)$$

induces a homotopy fiber sequence on $\mathbb{K}^{-\infty}$-spectra.

(ii) For $A \cup B = X$ suitable subspaces the square

$$\begin{tikzcd}
\mathcal{C}_f^G(A \cap B, R) \arrow[r] \arrow[d] & \mathcal{C}_f^G(A, R) \arrow[d] \\
\mathcal{C}_f^G(B, R) \arrow[r] & \mathcal{C}_f^G(A \cup B, R)
\end{tikzcd}$$

gives homotopy pullback of spectra after applying $\mathbb{K}^{-\infty}$.

The conditions on the subspaces $A, B \subseteq X$ are nearly the same as in Lemma 5.6 and Lemma 5.16. (We need two extra froper conditions for $A \cap B \subseteq B$ and $B \subseteq X$.) The results would not be true as stated without non-connective K-theory. In some sense this solves the "K_0-problem" noted before.

6.1. Overview and the theorems

In [PW85] and [PW89] Pedersen and Weibel defined the non-connective K-theory of an additive category via a geometric construction which in our language uses the control space $\mathbb{Z}^n$ or $\mathbb{R}^n$ with metric control. Roughly speaking for R a ring the space $K(w\mathcal{C}_f(\mathbb{R}^n, R, \mathcal{E}_d, \{\mathbb{R}^n\}))$ is a non-connective n-fold delooping of $K(R)$. We take that as the motivation for the following definition.

Lemma/Definition 6.1. *Let $(X, \mathcal{E}, \mathcal{F})$ be a G-equivariant control space, R a simplicial ring, G a discrete group. Define the nth term of the non-connective algebraic K-theory spectrum $\mathbb{K}^{-\infty}(w\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F}))$ as*

$$K(w\mathcal{C}_f^G(X \times \mathbb{R}^n, R, \mathcal{E} \times \mathcal{E}_d, \mathcal{F} \times \{\mathbb{R}^n\})).$$

There are structure maps

$$K(w\mathcal{C}_f^G(X \times \mathbb{R}^n, R, \mathcal{E} \times \mathcal{E}_d, \mathcal{F} \times \{\mathbb{R}^n\})) \to \Omega K(w\mathcal{C}_f^G(X \times \mathbb{R}^{n+1}, R, \mathcal{E} \times \mathcal{E}_d, \mathcal{F} \times \{\mathbb{R}^{n+1}\}))$$

which induce an isomorphism on π_i for $i \geq 1$ and an injection on π_0.

Remark 6.2. For non-connective K-theory the spectra for finite, homotopy finite and homotopy finitely dominated modules are all stably equivalent, hence the difference does not matter. We choose the finite version to work with, as it is a bit easier to handle in the proof.

Remark 6.3. Recall that $\mathcal{E}$ '×' $\mathcal{E}_d$ and $\mathcal{F}$ '×' $\{\mathbb{R}^n\}$ is a misuse of notation. Also recall Definition 5.3 of a froper subspace.

We have the following theorems.

Theorem 6.4. *Let $(X, \mathcal{E}, \mathcal{F})$ be a control space and $i\colon Y \subseteq X$ a froper subspace. Let gw be the germwise weak equivalences on X away from Y. Then*

$$\mathbb{K}^{-\infty}(w\mathcal{C}^G(Y, i^{-1}\mathcal{E}, i^{-1}\mathcal{F})) \to \mathbb{K}^{-\infty}(w\mathcal{C}^G(X, \mathcal{E}, \mathcal{F})) \to \mathbb{K}^{-\infty}(gw\mathcal{C}^G(X, \mathcal{E}, \mathcal{F}))$$

is a homotopy fiber sequence of spectra.

Theorem 6.5 (Coarse Mayer-Vietoris for non-connective K-Theory)**.** *Let $(X, \mathcal{E}, \mathcal{F})$ be a free G-equivariant control space and let A and B be froper subspaces of X such that $A \cup B = X$, assume further that $A \cap B$ is froper in A. Suppose that the triple (X, A, B) is coarsely excisive (Definition 5.15).*

Then the diagram

$$\begin{tikzcd}
\mathbb{K}^{-\infty}(w\mathcal{C}^G(A\cap B, j^{-1}\mathcal{E}, j^{-1}\mathcal{F})) \arrow[r] \arrow[d] & \mathbb{K}^{-\infty}(w\mathcal{C}^G(A, i_A^{-1}\mathcal{E}, i_A^{-1}\mathcal{F})) \arrow[d] \\
\mathbb{K}^{-\infty}(w\mathcal{C}^G(B, i_B^{-1}\mathcal{E}, i_B^{-1}\mathcal{F})) \arrow[r] & \mathbb{K}^{-\infty}(w\mathcal{C}^G(A\cup B, \mathcal{E}, \mathcal{F}))
\end{tikzcd}$$

is a homotopy pullback of spectra, where i_A, i_B, j are the obvious inclusions.

Theorem 6.6 (Flasque shift)**.** *Assume $(X, \mathcal{E}, \mathcal{F})$ is a free G-equivariant control space which has a flasque shift. Then $\mathbb{K}^{-\infty}(w\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F}))$ is contractible.*

Theorem 6.7 (Change of rings)**.** *Let $f\colon R \to S$ be a map of simplicial rings. It induces a map*

$$\mathbb{K}^{-\infty}(w\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})) \to \mathbb{K}^{-\infty}(w\mathcal{C}^G(X, S, \mathcal{E}, \mathcal{F}))$$

which is a weak equivalence if f is a weak equivalence of simplicial rings.

The proofs rely on the corresponding Lemmas for the connective K-theory together with some tricks using the Eilenberg swindle. They are given in the next sections. Having defined $\mathbb{K}^{-\infty}$ for categories of controlled modules we can make the following definition for a simplicial ring R.

Definition 6.8 (Non-connective algebraic K-theory spectrum)**.** *Let R be a simplicial ring. Define the non-connective algebraic K-theory spectrum of R as*

$$\mathbb{K}^{-\infty}(R) := \mathbb{K}^{-\infty}(w\mathcal{C}(\mathrm{pt}, R))$$

6.2. The non-connective K-theory spectrum $\mathbb{K}^{-\infty}$

In this section we prove Lemma 6.1. We need an auxiliary lemma first which we will often use implicitely.

Lemma 6.9. *Let $(X, \mathcal{E}_X)$, $(Y, \mathcal{E}_Y)$ be a control spaces. Recall that then $(X \times Y, \mathcal{E}_X \times \mathcal{E}_Y)$ is a control space.*

(i) If $\mathcal{E} = \mathcal{E}_X \times \mathcal{E}_Y$ then $(A \times B)^E = A^{E_X} \times B^{E_Y}$ for $E_X \in \mathcal{E}_X$, $E_Y \in \mathcal{E}_Y$ and $E = E_X \times E_Y \in \mathcal{E}$.

(ii) If $A \subseteq X$ is froper and $B \subseteq Y$ is froper than $A \times B \subseteq X \times Y$ is froper.

Proof. The first part is clear by definition. For the second part use that A, B are froper to choose maps $f_A\colon A^{E_X} \to A$, $f_B\colon B^{E_Y} \to B$. Then $f_A \times f_B$ is controlled, the identity on $A \times B$ and maps locally finite sets to locally finite sets. ☐

We prove the statement about the structure maps of $\mathbb{K}^{-\infty}(w\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F}))$.

Proof of 6.1. Let $\mathbb{R}^n_+$ be $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$ and $\mathbb{R}^n_-$ be $\mathbb{R}_{\leq 0} \times \mathbb{R}^{n-1}$ both with metric control. Note that their intersection is $0 \times \mathbb{R}^{n-1} \cong \mathbb{R}^{n-1}$. We then have a commutative diagram

$$\begin{tikzcd}
K(w\mathcal{C}^G_f(X \times \mathbb{R}^{n-1})) \arrow[r] \arrow[d] & K(w\mathcal{C}^G_f(X \times \mathbb{R}^n_+)) \arrow[d] \\
K(w\mathcal{C}^G_f(X \times \mathbb{R}^n_-)) \arrow[r] & K(w\mathcal{C}^G_f(X \times \mathbb{R}^n))
\end{tikzcd} . \tag{23}$$

By Example 5.26 we have a flasque shift on $\mathbb{R}^n_\pm$ and hence on $X \times \mathbb{R}^n_\pm$, so the upper right and lower left corners are contractible. Hence we get a map

$$K(w\mathcal{C}^G_f(X \times \mathbb{R}^{n-1})) \to \Omega K(w\mathcal{C}^G_f(X \times \mathbb{R}^n)) \tag{24}$$

from the upper left term to the homotopy pullback of the rest. We discuss its connectivity.

The triple $(\mathbb{R}^n, \mathbb{R}^n_+, \mathbb{R}^n_-)$ is froper and coarsely excisive, as one readily checks. It follows that $(X \times \mathbb{R}^n, X \times \mathbb{R}^n_+, X \times \mathbb{R}^n_-)$ is also froper and coarsely excisive. Lemma 5.16 then gives the following homotopy pullback of spaces where the germs are away from $\mathbb{R}^n_- \subset \mathbb{R}^n$ (and where we neglect the control conditions for the moment):

$$\begin{tikzcd}
K(w\mathcal{C}^G_f(X \times \mathbb{R}^n_+)^{gw}) \arrow[r] \arrow[d] & K(w\mathcal{C}^G_f(X \times \mathbb{R}^n_+)) \arrow[d] \\
K(w\mathcal{C}^G_f(X \times \mathbb{R}^n)^{gw}) \arrow[r] & K(w\mathcal{C}^G_f(X \times \mathbb{R}^n))
\end{tikzcd} . \tag{25}$$

By Lemma 5.6 there is a square

$$\begin{tikzcd}
K(w\mathcal{C}^G_f(X \times \mathbb{R}^{n-1})) \arrow[r, "(1)"] \arrow[d] & K(w\mathcal{C}^G_f(X \times \mathbb{R}^n_+)^{gw}) \arrow[d] \\
K(w\mathcal{C}^G_f(X \times \mathbb{R}^n_-)) \arrow[r, "(2)"] & K(w\mathcal{C}^G_f(X \times \mathbb{R}^n)^{gw})
\end{tikzcd} \tag{26}$$

where (1) and (2) are coconnected (cf. Definition 5.8). We can take the diagram (26) + (25), which is diagram (23) above. As (25) is a homotopy pullback it follows that the homotopy pullback of (26) is homotopy equivalent to $\Omega K(w\mathcal{C}^G_f(X \times \mathbb{R}^n))$. Hence we get the diagram

$$\begin{tikzcd}
K(w\mathcal{C}^G_f(X \times \mathbb{R}^{n-1})) \arrow[rrd, bend left, "\gamma"] \arrow[rd, dashed, "\beta"] \arrow[rdd, bend right] & & \\
& \Omega K(w\mathcal{C}^G_f(X \times \mathbb{R}^n)) \arrow[r, "\alpha"] \arrow[d] & K(w\mathcal{C}^G_f(X \times \mathbb{R}^n_+)^{gw}) \arrow[d] \\
& K(w\mathcal{C}^G_f(X \times \mathbb{R}^n_-)) \arrow[r, "\alpha'"] & K(w\mathcal{C}^G_f(X \times \mathbb{R}^n)^{gw})
\end{tikzcd}$$

where the map α is an isomorphism on π_i for $i \geq 1$ as α is the homotopy pullback of α', which has the same property. The same is true for γ which is the map (1) above, hence also for β, but β is the map of (24). We further see that γ is injective on π_0, hence the same is true for β.

We neglected the control conditions. E.g. for the upper right space Lemma 5.16 needs them to be $i_*^{-1}(\mathcal{E} \times \mathcal{E}_d)$, where i_* is the map induced by the inclusion $\mathbb{R}^n_+ \subset \mathbb{R}^n$ and $\mathcal{E}_d$ the metric control. But $i^{-1}\mathcal{E}_d$ *is* the metric control on $\mathbb{R}^n_+$. Hence $(\mathbb{R}^n_+, \mathcal{E}_d)$ and $(\mathbb{R}^n_+, i^{-1}\mathcal{E}_d)$ are the same control spaces. The same is true for all the other spaces occurring. Therefore the control structures in the statement agree with the ones we have by Lemma 5.16. ☐

Lemma 6.10. *Let f be a map $(X, \mathcal{E}_X, \mathcal{F}_X) \to (Y, \mathcal{E}_Y, \mathcal{F}_Y)$ of equivariant control spaces. Assume f maps locally finite set to locally finite sets (cf. Definition 3.17).*

Then f induces a map $\mathbb{K}^{-\infty}(f)$ on the non-connective K-theory spectra,

$$\mathbb{K}^{-\infty}(f)\colon \mathbb{K}^{-\infty}(\mathcal{C}^G_f(X, R, \mathcal{E}_X, \mathcal{F}_X)) \to \mathbb{K}^{-\infty}(\mathcal{C}^G_f(Y, R, \mathcal{E}_Y, \mathcal{F}_Y)).$$

Proof. The assumption implies that f maps finite modules to finite modules, hence f gives a map $\mathcal{C}^G_f(X, R, \mathcal{E}_X, \mathcal{F}_X) \to \mathcal{C}^G_f(Y, R, \mathcal{E}_Y, \mathcal{F}_Y)$. We show that $f \times \mathbb{R}^n$ also maps locally finite sets to locally finite sets, the proof for $\mathbb{R}^n_+$, $\mathbb{R}^n_-$ and $\mathbb{R}^{n-1}$ is the same.

Clearly $f \times \mathbb{R}^n\colon (X \times \mathbb{R}^n, \mathcal{E}_X \times \mathcal{E}_d, \mathcal{F}_X \times \{\mathbb{R}^n\}) \to (Y \times \mathbb{R}^n, \mathcal{E}_Y \times \mathcal{E}_d, \mathcal{F}_Y \times \{\mathbb{R}^n\})$ is a map of control spaces. Let (L, κ) be a locally finite set over $X \times \mathbb{R}^n$, we have to show that $(L, (f \times \mathbb{R}^n) \circ \kappa)$ is locally finite over $Y \times \mathbb{R}^n$. Let (y, t) be a point in $Y \times \mathbb{R}^n$. Choose a compact neighborhood K of t in $\mathbb{R}^n$. We can restrict L to $X \times K$ and it is still a locally finite set, call it L_K. Using the projection $p_X\colon X \times K \to X$ it becomes a locally finite set over X by Lemma 6.11 below. Hence over Y the set $(L_K, p_Y \circ (f \times \mathbb{R}^n) \circ \kappa)$ becomes a locally finite set. So there is a neighborhood U of y such that $(p_Y \circ (f \times \mathbb{R}^n) \circ \kappa)^{-1}(U)$ is finite. Then $U \times K$ is a neighborhood over which $(L, (f \times \mathbb{R}^n) \circ \kappa)$ is finite.

It follows that we also get induced maps

$$\mathcal{C}^G_f(X \times \mathbb{R}^n, R, \mathcal{E}_X \times \mathcal{E}_d, \mathcal{F}_X \times \{\mathbb{R}^n\}) \to \mathcal{C}^G_f(Y \times \mathbb{R}^n, R, \mathcal{E}_Y \times \mathcal{E}_d, \mathcal{F}_Y \times \{\mathbb{R}^n\})$$

and the same for $\mathbb{R}^n_+$, $\mathbb{R}^n_-$, $\mathbb{R}^{n-1}$, hence maps of the commutative squares like (23) which defines the structure map (24). By Example 5.26 the contractions on the corners are compatible, hence the induced structure map is natural with respect to f.

Thus the collection of $K(f \times \mathbb{R}^n)$ give the desired map of spectra. ☐

We have to provide a lemma we just used.

Lemma 6.11. *Let $(X, \mathcal{E}, \mathcal{F})$ and $(K, \mathcal{E}_K, \mathcal{F}_K)$ be control spaces with K being compact as a topological space. Then the projection $p\colon (X \times K, \mathcal{E} \times \mathcal{E}_K, \mathcal{F} \times \mathcal{F}_K) \to (X, \mathcal{E}, \mathcal{F})$ maps locally finite sets to locally finite sets.*

Proof. Clearly p is a map of control spaces. Let (L, κ) be a locally finite set over $X \times K$. We show that $(L, p \circ \kappa)$ is locally finite. Choose for each point $w \in X \times K$ a neighborhood U_w such that $\kappa^{-1}(U_w)$ is finite. Let x be a point in X, then there is a neighborhood U of x such that $p^{-1}(U) = U \times K$ is covered by finitely many U_w, as K is compact. Hence $(p \circ \kappa)^{-1}(U) \subseteq \bigcup_w \kappa^{-1}(U_w)$ is also finite. □

6.3. Non-connective algebraic K-theory for germwise equivalences

Let $\mathcal{U}$ be a set of germ support conditions on $(X, \mathcal{E}, \mathcal{F})$ and denote the germs by g. We need a non-connective delooping of the spaces $K(gw\mathcal{C}_f^G(X, R, \mathcal{E}, \mathcal{F}))$ and $K(w\mathcal{C}_f^G(X, R, \mathcal{E}, \mathcal{F})^{gw})$. For this we need theorems corresponding to the results of Chapter 5 for germs. We sketch the results in this section. The construction is largely parallel to Definition 6.1. If $Y \subseteq X$ is froper we further show that the non-connective deloopings of $K(w\mathcal{C}_f^G(X, R, \mathcal{E}, \mathcal{F})^{gw})$ and $K(w\mathcal{C}_f^G(Y, R, i^{-1}\mathcal{E}, i^{-1}\mathcal{F}))$ are equivalent.

Lemma/Definition 6.12. *Let G be a discrete group, $(X, \mathcal{E}, \mathcal{F})$ be a G-equivariant control space and R a simplicial ring. Let $\mathcal{U}$ be a set of germ support conditions and denote the germwise weak equivalences by gw.*

Define the nth space of the non-connective algebraic K-theory spectrum of the category $gw\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})$ as

$$K(gw\mathcal{C}_f^G(X \times \mathbb{R}^n, R, \mathcal{E} \times \mathcal{E}_d, \mathcal{F} \times \{\mathbb{R}^n\}).$$

Denote the spectrum by $\mathbb{K}^{-\infty}(gw\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F}))$. As before there are structure maps

$$\mathbb{K}^{-\infty}(gw\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F}))_n \to \Omega\mathbb{K}^{-\infty}(gw\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F}))_{n-1}$$

which induce an isomorphism on π_i for $i \geq 1$.

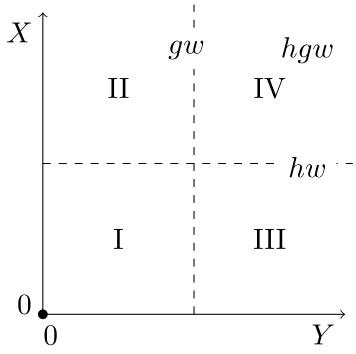
We sketch the results to prove the lemma. First we need a more general statement. We already showed that for a control space X and germ support conditions on it we get a homotopy fiber sequence. If we have two control spaces with germ support conditions on both of them we can combine them to get three different germ support conditions on the product control space, and correspondingly some more homotopy fiber sequences. This is summarized in the following lemma.

Lemma 6.13. *Let X and Y be control spaces and $\mathcal{U}_X$ and $\mathcal{U}_Y$ be germ support conditions on X, resp. Y. Then we get three canonical germ support conditions on $X \times Y$, namely $\mathcal{U}_X \times \{Y\}$, $\{X\} \times \mathcal{U}_Y$ and $\mathcal{U}_X \times \mathcal{U}_Y$ (with the usual misuse of notation). Call the corresponding germwise weak equivalences gw, hw and hgw. We have inclusions $w \subseteq gw \subseteq hgw$ and $w \subseteq hw \subseteq hgw$, where w are the homotopy* *equivalences. We get a diagram where each row and each column is a homotopy fiber sequences after applying K-theory:*

$$\begin{tikzcd}
(w\mathcal{C}^{hw})^{gw} \arrow[r] \arrow[d] & w\mathcal{C}^{gw} \arrow[r] \arrow[d] & hw\mathcal{C}^{gw} . \arrow[d] \\
w\mathcal{C}^{hw} \arrow[r] \arrow[d] & w\mathcal{C} \arrow[r] \arrow[d] & hw\mathcal{C} \arrow[d] \\
gw\mathcal{C}^{hw} \arrow[r] & gw\mathcal{C} \arrow[r] & hgw\mathcal{C}
\end{tikzcd}$$

(Of course $\mathcal{C}$ is an abbreviation for $\mathcal{C}^G_f(X \times Y, R, \mathcal{E}_X \times \mathcal{E}_Y, \mathcal{F}_X \times \mathcal{F}_Y)$.)

Proof (Sketch). The homotopy fiber sequences result from the generic Fibration Theorem B.5. There are only two things to check. The first is that $(\mathcal{C}^{hw})^{gw} \cong (\mathcal{C}^{gw})^{hw}$ which is clear. The second is that $gw\mathcal{C}^{hgw} \simeq gw\mathcal{C}^{hw}$ on K-theory. If $X = Y = \mathbb{R}_+$ and the germs are away from zero this is the following picture.



A module in $\mathcal{C}^{hgw}$ is nullhomotopic on (IV) (i.e. there is a germ support set from $\mathcal{U}_X \times \mathcal{U}_Y$ such that the restricted module is contractible), a module in $\mathcal{C}^{hw}$ is nullhomotopic on (IV) + (III), so we have an inclusion. But gw-equivalences ignore (I) + (III), so each module in $\mathcal{C}^{hgw}$ is gw-equivalent to one which is also trivial on (III).

For the general case and a more precise proof we use the Approximation Theorem B.7. We want to approximate a map $A \to B$, $A \in \mathcal{C}^{hw}$, $B \in \mathcal{C}^{hgw}$. B can be restricted to a module on $U_X \times Y \cup X \times (Y \smallsetminus U_Y)$ (being (I) + (II) + (IV) above) for suitable germ support sets $U_X \subseteq X$ and $U_Y \subseteq Y$ and the restriction is a gw-equivalence, as well as the restricted module is in $\mathcal{C}^{hw}$. By the homotopy extension property the identity on A is homotopic to a map which is zero on some $X \times U'_Y$, hence maps into the restricted submodule of B. Thus we get the second approximation property up to homotopy which we can rectify as done before. □

Proof of 6.12 (Sketch). The proof goes analogous to the proof of Lemma 6.1, but one needs to establish analogous results for the lemmas used first. We sketch these here.

Denote the germs on $X \times \mathbb{R}^n_+$ away from $X \times 0 \times \mathbb{R}^{n-1}$ by hw and the germs on $X \times \mathbb{R}^n$ away from $X \times \mathbb{R}^n_-$ by $h'w$. For the germwise weak equivalences gw and hw we can apply Lemma 6.13, as well as we can apply it to gw and $h'w$. We we get a diagram

$$\begin{tikzcd}
gw\mathcal{C}^G_f(X \times \mathbb{R}^n_+)^{hw} \arrow[r] \arrow[d] & gw\mathcal{C}^G_f(X \times \mathbb{R}^n_+) \arrow[r] \arrow[d] & hgw\mathcal{C}^G_f(X \times \mathbb{R}^n_+) \arrow[d] . \\
gw\mathcal{C}^G_f(X \times \mathbb{R}^n)^{h'w} \arrow[r] & gw\mathcal{C}^G_f(X \times \mathbb{R}^n) \arrow[r] & h'gw\mathcal{C}^G_f(X \times \mathbb{R}^n)
\end{tikzcd}$$

where the horizontal lines are homotopy fiber sequences after applying algebraic K-theory. The proof of Lemma 5.16 (coarse Mayer-Vietoris) shows that the rightmost vertical map satisfies the Approximation Property B.6. It applies for the germwise equivalences hgw and $h'gw$ and the subspace $X \times \mathbb{R}^n_+ \subset X \times \mathbb{R}^n$, as we are in exactly the same situation of comparing two germwise weak equivalences. Hence the rightmost map induces an equivalence on K-theory. Next we identify the leftmost column.

We have to show that $gw\mathcal{C}^G_f(X \times \mathbb{R}^{n-1}) \to gw\mathcal{C}^G_f(X \times \mathbb{R}^n_+)^{hw}$ is coconnected, but we can do the same proof as for the corresponding Lemma 5.6 for homotopy equivalences. Lemma 5.6 uses a lot of other results and we comment briefly on how they transfer to gw-equivalences. The main tool we use during the proof of Lemma 5.6 is the Approximation Theorem B.7. To apply it to a functor F we have to check conditions (App 1) and (App 2) of the Approximation Property B.6 for F. While (App 1) is often easy to show, (App 2) requires more work as we have to construct for any map $F(A) \to B$ a map $A \to A'$ and a gw-equivalence $F(A') \to B$. However, the proofs in Section 5.1 construct such an A' and a map $F(A') \to B$ which is a homotopy equivalence, i.e. a w-equivalence. As $w \subseteq gw$ this map is also a gw-equivalence, so the proofs in Section 5.1 also show (App 2) for the case of gw-equivalences.

First Proposition 3.33 (comparison between the finiteness condition f, hf and hfd) is used, whose proof applies verbatim for gw-equivalences, as (App 1) is always clear. The next is Lemma 5.9, which is the analogue of Proposition 3.33 for $\mathcal{C}^{Ggw}$ instead of $\mathcal{C}^G$. It also holds but note that the gw-equivalences there are the hw-equivalences above. The most delicate part is Lemma 5.10, but the proof there shows that for each module M in $\mathcal{C}^G_{hfd}(X \times \mathbb{R}^n_+)^{hw}$ any map $N \to M$ factors as a map $N \to M'$ followed by a w-equivalence $M' \to M$, where N and M' are in $\mathcal{C}^G_{hfd}(X \times 0)$. So the crucial map is also a gw-equivalence and Lemma 5.10 holds for them.

The last thing is to check the Eilenberg-swindle on $gw\mathcal{C}^G_f(X \times \mathbb{R}^n_+)$. Everything from Definition 5.19 on generalizes to gw-equivalences, with the exception that we have to check that the flasque functor T we construct is actually exact (cf. Remark 5.23), where the point is of course that it maps gw-equivalences to gw-equivalences. But this holds as the shift $s\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$, $t \mapsto t+1$ is "orthogonal" to the germs g; that is if f is a gw-equivalence which comes with some (three) germ support sets for the germwise inverses and homotopies, then $s(f)$ is a gw-equivalence which we can choose to have the *same germ support sets*! This implies that $T(f) = \coprod_n s^n(f)$ is again a gw-equivalence.

Now everything works exactly like in the proof of Lemma 6.1. ☐

Remark 6.14. Note that because the construction of $\mathbb{K}^{-\infty}(gw\mathcal{C}^G(X))$ is analogous to the construction of $\mathbb{K}^{-\infty}(w\mathcal{C}^G(X))$ we get a natural map of spectra from the latter to the former.

We also need a non-connective spectrum for $w\mathcal{C}^{gw}$. Our proof of the coarse Mayer-Vietoris theorem does not seem to work for the germwise-trivial objects, so we omit the connectivity assumptions on the structure maps.

Lemma/Definition 6.15. *Let $(X, \mathcal{E}, \mathcal{F})$ be a G-equivariant control space and $i\colon Y \subseteq X$ a froper subspace. Denote by g the germs away from Y. There is a non-connective algebraic K-theory spectrum $\mathbb{K}^{-\infty}(w\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})^{gw})$, defined in degree n as*

$$K(w\mathcal{C}^G_f(X \times \mathbb{R}^n, R, \mathcal{E} \times \mathcal{E}_d, \mathcal{F} \times \{\mathbb{R}^n\})^{gw}).$$

There further is a stable equivalence of spectra

$$\mathbb{K}^{-\infty}(w\mathcal{C}^G(Y, R, i^{-1}\mathcal{E}, i^{-1}\mathcal{F})) \to \mathbb{K}^{-\infty}(w\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F})^{gw}).$$

Note that a priori we do not make any connectivity assumptions about the structure maps of the spectrum. However the proof of the second claim shows that the structure maps of $\mathbb{K}^{-\infty}(w\mathcal{C}^G(X)^{gw})$ are isomorphisms on π_i for $i \geq 1$.

Proof. As before we have a diagram

$$\begin{tikzcd}
w\mathcal{C}^G_f(X \times \mathbb{R}^{n-1})^{gw} \arrow[r] \arrow[d] & w\mathcal{C}^G_f(X \times \mathbb{R}^n_+)^{gw} \arrow[d] \\
w\mathcal{C}^G_f(X \times \mathbb{R}^n_-)^{gw} \arrow[r] & w\mathcal{C}^G_f(X \times \mathbb{R}^n)^{gw}
\end{tikzcd}.$$

There is a flasque shift on $\mathbb{R}^n_\pm$ which respects the gw-equivalences as noted in the last proof, hence the upper right and lower left corners have an Eilenberg swindle and vanishing K-theory. This gives the structure maps of the spectrum.

Lemma 5.6 provides a functor $w\mathcal{C}^G_f(Y \times \mathbb{R}^n) \to w\mathcal{C}^G_f(X \times \mathbb{R}^n)^{gw}$ which is an isomorphism on π_i for $i \geq 1$. The functor is compatible with the flasque shifts, so we get the desired stable equivalence of spectra $\mathbb{K}^{-\infty}(w\mathcal{C}^G(Y)) \to \mathbb{K}^{-\infty}(w\mathcal{C}^G(X)^{gw})$. (It follows that the structure maps of $\mathbb{K}^{-\infty}(w\mathcal{C}^G(X)^{gw})$ are an isomorphism on π_i for $i \geq 1$.) ☐

Note that this third version of the non-connective K-theory spectrum has also the expected functoriality properties.

6.4. The homotopy fiber sequence and coarse Mayer-Vietoris for $\mathbb{K}^{-\infty}$

We can now prove Theorem 6.4 which gives the homotopy fiber sequence for non-connective algebraic K-theory.

Proof of Theorem 6.4. We know by the connective homotopy fiber sequence of Lemma 5.14 that

$$K(w\mathcal{C}^G_f(X\times\mathbb{R}^n,R,\mathcal{E},\mathcal{F})^{gw})\to K(w\mathcal{C}^G_f(X\times\mathbb{R}^n,R,\mathcal{E},\mathcal{F}))\to K(gw\mathcal{C}^G_f(X\times\mathbb{R}^n,R,\mathcal{E},\mathcal{F}))$$

is a homotopy fiber sequence for all n. Therefore we get a homotopy fiber sequence on non-connective K-theory spectra

$$\mathbb{K}^{-\infty}(w\mathcal{C}^G(X)^{gw})\to\mathbb{K}^{-\infty}(w\mathcal{C}^G(X))\to\mathbb{K}^{-\infty}(gw\mathcal{C}^G(X)).$$

By Lemma 6.15 the map $\mathbb{K}^{-\infty}(w\mathcal{C}^G(Y))\to\mathbb{K}^{-\infty}(w\mathcal{C}^G(X)^{gw})$ is a stable equivalence of spectra, hence the result follows. ☐

Remark 6.16. We want to give a bit more details why we get the fiber sequence of spectra. A fibration of spectra is defined to be a degreewise fibration. We can factor the rightmost horizontal map into a (degreewise) fibration and a degreewise cofibration which is also a degreewise weak equivalence. We get a (solid) diagram

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, dashed] & \bullet \arrow[r] \arrow[d, tail, "\sim"] & \bullet \arrow[d, "\mathrm{id}"] \\
\bullet \arrow[r] & \bullet \arrow[r, two heads] & \bullet
\end{tikzcd}$$

where the upper row is the original row. Taking the fiber of the lower row (which is the homotopy fiber of both rows) gives a map from the leftmost upper spectrum to the (homotopy) fiber.

Degreewise this is a replacement by a fibration sequence; as degreewise the upper row was a homotopy fiber sequence, the left vertical map is degreewise a weak equivalence, hence also a weak (=stable) equivalence of spectra. Hence the upper row is weakly equivalent to a fiber sequence and hence a homotopy fiber sequence.

The proof of the non-connective coarse Mayer-Vietoris Theorem 6.5 proceeds similarly.

Proof of Theorem 6.5 (Sketch). Using Theorem 6.4 we can proceed as in the proof of the non-connective case (Lemma 5.16), namely we can extend the diagram to the right to

$$\begin{tikzcd}
\mathbb{K}^{-\infty}(w\mathcal{C}^G(A\cap B,j^{-1}\mathcal{E},j^{-1}\mathcal{F})) \arrow[r] \arrow[d] & \mathbb{K}^{-\infty}(w\mathcal{C}^G(A,i_A^{-1}\mathcal{E},i_A^{-1}\mathcal{F})) \arrow[r] \arrow[d] & \mathbb{K}^{-\infty}(gw\mathcal{C}^G(A,i_A^{-1}\mathcal{E},i_A^{-1}\mathcal{F})) \arrow[d] \\
\mathbb{K}^{-\infty}(w\mathcal{C}^G(B,i_B^{-1}\mathcal{E},i_B^{-1}\mathcal{F})) \arrow[r] & \mathbb{K}^{-\infty}(w\mathcal{C}^G(A\cup B,\mathcal{E},\mathcal{F})) \arrow[r] & \mathbb{K}^{-\infty}(gw\mathcal{C}^G(A\cup B,\mathcal{E},\mathcal{F}))
\end{tikzcd}$$

and show that the rightmost map gives a stable equivalence of spectra, which is the argument of Lemma 5.16 applied degreewise. ☐

The two results we just proved are precisely the analoga of Proposition 4.2 (iii) (Coarse Pair/Karoubi fiber sequence) and Proposition 4.3 (Coarse Mayer-Vietoris) of [BFJR04]. These are the fundamental results about the algebraic K-theory and in particular the coarse Mayer-Vietoris is the crucial ingredient for the construction of the equivariant homology theory of [BFJR04, Section 5]. Thus we can prove the corresponding results now in our setting.

The proofs of 6.6 and 6.7 are now easy. We state them here for completeness.

Proof of 6.6. If X has a flasque shift, then the space in each degree of the spectrum $\mathbb{K}^{-\infty}(w\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F}))$ is contractible, hence the spectrum itself is contractible. ☐

Proof of 6.7. The map $f\colon R \to S$ induces maps $\mathcal{C}^G_f(X \times \mathbb{R}^n, R) \to \mathcal{C}^G_f(X \times \mathbb{R}^n, S)$ which by Theorem 3.38 induces an equivalence on connective algebraic K-theory if f is a weak equivalence of rings. The maps are compatible with the contractions on $\mathbb{R}^n_\pm$ and hence induce a map of spectra. It is an weak equivalence of spectra if f is a weak equivalence of rings. ☐

7. An equivariant homology theory

We define a functor $h^G(-, \mathbf{K}_R)$ from G-CW-complexes to spectra and show that it is an unreduced generalized G-equivariant homology theory with coefficients in $\mathbf{K}_R$, the non-connective G-equivariant algebraic K-theory spectrum of the simplicial ring R. Here $\mathbf{K}_R$ is a functor from the orbit category $\operatorname{Or} G$—which consists of G-spaces G/H and G-equivariant maps—to spectra, and evaluated on G/H it gives a spectrum weakly equivalent to $\mathbb{K}^{-\infty}(R[H])$.

Recall that a G-CW-complex is a CW-complex with G-action such that the image of an open cell under the action of $g \in G$ is either disjoint to the cell or fixed pointwise. Equivalently a G-CW-complex is a space arising by attaching G-cells in increasing dimension, where a G-cell of dimension n is a G-space of the form $(G/H \times D^n, G/H \times \partial D^n)$ with $H \subseteq G$ a subgroup of G. (A convenient way to look at G-CW-complexes is to consider them as spaces under the orbit category $\operatorname{Or} G$ of G, but we will not need this viewpoint and refer to [DL98] for details.) We now define and discuss what we mean by a G-equivariant homology theory.

Definition 7.1 (Unreduced G-equivariant homology theory). *Let G be a group. An unreduced G-equivariant homology theory is a functor $h^G(-)$ from G-CW-complexes to spectra such that:*

(0) (Functoriality) If $f\colon X \to Y$ is a G-equivariant map between G-CW-complexes then there is a map

$$h^G(f)\colon h^G(X) \to h^G(Y)$$

of spectra. (This is mentioned so that we can refer to it later.)

(i) (Homotopy Invariance) If f is an G-equivariant homotopy equivalence then $h^G(f)$ is an equivalence of spectra.

(ii) (Mayer-Vietoris) A homotopy pushout square of G-CW-complexes induces a homotopy pullback square of spectra.

(iii) (Compact support) Let $\{X_i \mid i \in I\}$ be the G-finite (or equivalently G-compact) subcomplexes of X. Then the natural map

$$\operatorname*{hocolim}_{i \in I} h^G(X_i) \to h^G(X)$$

is an equivalence of spectra.

Remark 7.2. This definition follows closely the definition in the beginning of Section 5 of [BFJR04] where an “equivalent” definition is given, too. However this “equivalent” definition actually uses a notion of reduced homology theory (without saying so) and the reformulation seems to be slightly weaker for uncountable G-CW-complexes. The last issue is related to the trick we discuss now.

The above axioms (0) to (ii) correspond to the Eilenberg-Steenrod axioms. Axiom (iii) is sometimes called the *direct limit axiom* (e.g. in [Hat02, 4.F]) and goes back to Adams [Ada71]. However, it is also mentioned in [Spa89, p. 203] (which was published in 1966!) as the *axiom of compact support*, hence our name. It implies Milnor’s Additivity Axiom [Mil62] which is more commonly required, hence we prove the stronger axiom. (Note that according to [Hat02, 4.F/p.455] the stronger axiom is needed to provide representability by a spectrum in the non-equivariant case.)

The proof of (i) and (ii) is easier if we restrict to finite G-CW-complexes (which are the G-compact ones). Then we invoke a general trick that taking (iii) as a definition we could extend any homology theory to arbitrary G-CW-complexes. As (iii) will hold for us, the Mayer-Vietoris axiom is automatically satisfied for all G-CW-complexes.

We define $h^G(-, \mathbf{K}_R)$ in Section 7.1. Section 7.2 shows that the functor restricted to finite G-CW-complexes satisfies (0) to (ii) of Definition 7.1 ((iii) is trivially true), i.e. it is a homology theory on finite G-CW-complexes. Section 7.3 shows how to extend any homology theory on finite G-CW-complexes to a homology theory on all G-CW-complexes by requiring 7.1(iii) to hold, and uses this to show that $h^G(-, \mathbf{K}_R)$ is a G-equivariant homology theory in the sense of Definition 7.1. We briefly remark how one gets from our definition to a corresponding reduced homology theory and how to obtain a homology theory for pairs in Section 7.4.

We define $\mathbf{K}_R$, the G-equivariant algebraic K-theory spectrum of R, as the G-equivariant spectrum $G/H \mapsto h^G(G/H, \mathbf{K}_R)$ in Section 7.5. Then it is of course trivially true that $\mathbf{K}_R$ are the coefficients of $h^G(-, \mathbf{K}_R)$, but we have no other definition at hand. We calculate $h^G(G/H, \mathbf{K}_R) \simeq \mathbb{K}^{-\infty}(R[H])$ to justify the name. Finally we show there is a weak a map of spectra $K(R) \to \mathbb{K}^{-\infty}(R)$ from the connective algebraic K-theory to the non-connective one and this map is an isomorphism on stable homotopy groups π_i for $i \geq 1$. Last we discuss the assembly map for $h^G(-, \mathbf{K}_R)$ in Section 7.6.

7.1. A G-equivariant homology theory with coefficients $\mathbf{K}_R$

Recall that in the previous section we defined for a free G-equivariant control space $(X, \mathcal{E}, \mathcal{F})$ and a simplicial ring R the non-connective spectrum $\mathbb{K}^{-\infty}(w\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F}))$. For a set of germ support conditions on X denoted by g we have the corresponding spectrum $\mathbb{K}^{-\infty}(gw\mathcal{C}^G(X, R, \mathcal{E}, \mathcal{F}))$.

Let X be a G-CW-complex. We get a G-equivariant control space by taking the G-equivariant continuous control on $X \times [1, \infty)$ (see Example 1.29), denote it by $\mathcal{E}_{Gcc}(X)$. The control space is in general not free. But we can take the *standard resolution* (cf. Example 1.29 (iii) and [BFJR04, 3.1]) which is the space $X \times G \times [1, \infty)$

with the projection $p\colon X \times G \times [1, \infty) \to X \times [1, \infty)$. We get a free G-equivariant control space by pulling back the G-equivariant control condition from $X \times [1, \infty)$ and taking the G-equivariant compact object support conditions pulled back from $X \times G$ (cf Ex.1.29 and Ex.1.9). Thus we make the following definition.

Definition 7.3. *For X a G-CW-complex define the free G-equivariant control space as*

$$X^{cc} := (X \times G \times [1, \infty), p^{-1}\mathcal{E}_{Gcc}(X), p_{X\times G}^{-1}\mathcal{F}_{Gc})$$

Note that G acts free and properly discontinuous on X^{cc}.

Lemma 7.4. *For $f\colon X \to Y$ a map of G-CW-complexes we get a map $f^{cc}\colon X^{cc} \to Y^{cc}$ of free G-equivariant control spaces. The map takes locally finite sets to locally finite sets.*

Proof. Let $f^{cc}\colon X \times G \times [1, \infty) \to Y \times G \times [1, \infty)$ be the map induced by $f\colon X \to Y$. We have to check that it gives a map of control spaces and that it takes locally finite sets to locally finite sets.

The part for the control spaces is proven in Proposition 3.2 of [BFJR04] if we use that taking the standard resolution is functorial. For the proposition to apply we have to invoke the "trick" at the end of Section 1.3 and replace $\mathcal{E}$ by "$\mathcal{E} \cap (\mathcal{F} \times \mathcal{F})$" (misuse of notation), which gives equivalent control spaces, i.e. the same category $\mathcal{C}^G(X)$. We hide this change of the control structure as it is irrelevant for us.

A locally finite set (M, κ) over X^{cc} has compact support in $X \times G$-direction, hence restricted to $X \times G \times [n, n+2]$ it is even finite. Therefore the set $(M, f \circ \kappa)$ over Y^{cc} is always finite on $Y \times G \times [n, n+2]$ and therefore in particular locally finite. □

There is a canonical inclusion $X \times G \times 0 \to X^{cc}$ of control spaces which gives us germ support conditions on X^{cc}. These are the "germs at infinity", denote them by g_∞.

Definition 7.5. *Define $h^G(-, \mathbf{K}_R)$ as the functor from G-CW-complexes to spectra*

$$X \longmapsto \Omega\mathbb{K}^{-\infty}(g_\infty w\mathcal{C}^G(X^{cc})).$$

Remark 7.6. The looping is just a degree shift to get $h^G(G/G, \mathbf{K}_R) \simeq \mathbb{K}^{-\infty}(R[G])$. Usually we suppress it for the following proofs, although it has the nice property that it makes $h^G(-, \mathbf{K}_R)$ into an Ω-spectrum.

We now can state our theorem.

Theorem 7.7. *$h^G(-, \mathbf{K}_R)$ is a G-equivariant homology theory with coefficients $G/H \mapsto \mathbb{K}^{-\infty}(R[H])$.*

This will be proven as Corollary 7.21 in Section 7.3.

7.2. Homotopy invariance and Mayer-Vietoris property

We prove that $h^G(-, \mathbf{K}_R)$ is a homology theory if restricted to finite G-complexes. We want to apply the theorems from Chapter 6.

To apply the coarse Mayer-Vietoris principle (6.5) we will need the next result. It is an adaption of Proposition 5.3 of [BFJR04] which we reprove here as we need the details of the proof. For the Mayer-Vietoris property of $h^G(-, \mathbf{K}_R)$ we will apply the following proposition with both A and B compact, but for the homotopy invariance we need a non-compact B. Note that Proposition 5.3 of [BFJR04] does not hold true for $A \cap B = \varnothing$ as stated, we give the full correct statement here. (However, the proof from [BFJR04] applies to both situations.) The result is used to show that the triple (X^{cc}, A^{cc}, B^{cc}) is coarsely excisive (Definition 5.15). We also need an addendum which ensures that the subspaces are froper (Definition 5.3).

Lemma 7.8 (Adaption of [BFJR04, Prop. 5.3])**.** *Let X be a G-CW-complex and let $A, B \subseteq X$ be two G-invariant closed subsets with A being G-compact and $A \cup B = X$. Assume that there exists a G-invariant open neighborhood U of $A \cap B$ in X such that U is homeomorphic to $(A \cap B) \times (-1, 1)$, where G acts trivially on $(-1, 1)$. Moreover, $U \cap A$ should correspond to $(A \cap B) \times [0, 1)$ and $U \cap B$ to $(A \cap B) \times (-1, 0]$.*

Let E be in $\mathcal{E}_{Gcc}(X)$. Assume $(A \cap B)$ is not empty, then there exists an $E' \in \mathcal{E}_{Gcc}(X)$ such that

$$(A \times [1, \infty))^E \cap (B \times [1, \infty))^E \subseteq ((A \cap B) \times [1, \infty))^{E'}$$

If $A \cap B = \varnothing$ then there is a t such that

$$(A \times [1, \infty))^E \cap (B \times [1, \infty))^E \cap X \times [t, \infty) = \varnothing.$$

Remark 7.9. Note that this Lemma implies the same result for the control spaces obtained by resolutions, as we have no object support conditions imposed so far.

Lemma 7.10 (Addendum)**.** *Given the situation above. Consider the control space X^{cc} which was defined as*

$$(X \times G \times [1, \infty), \mathcal{E}_{Gcc}(X \times G), p^{-1}_{X \times G} \mathcal{F}_{Gc}).$$

Then the subspace $(A \cap B) \times G \times [1, \infty)$ is froper in $A \times G \times [1, \infty)$ as well as $B \times [1, \infty) \times G$ is froper in $X \times G \times [1, \infty)$.

Proof of Lemma 7.8. We follow closely the proof of [BFJR04] and give some more details.

Let $Z_A := (A \times [1, \infty))^E \smallsetminus (A \times [1, \infty))$. For $0 < \varepsilon < 1$ let U_ε be the neighborhood of $A \cap B$ corresponding to $(A \cap B) \times [-\varepsilon, \varepsilon]$. The continuous control condition implies that for every $x \in A \times \infty$ and for the G_x-invariant neighborhood $U_\varepsilon \times [1, \infty) \cup A \times [1, \infty)$ there is a G_x-invariant neighborhood V such that $V^E \subseteq U_\varepsilon \times [1, \infty) \cup A \times [1, \infty)$. We can shrink V to a neighborhood $V' \times [t^x_\varepsilon, \infty)$. As A is G-compact it is covered by finitely many orbits of such neighborhoods and therefore there is a $V'' \supset A$ and t^A_ε such that $(V'' \times [t^A_\varepsilon, \infty))^E \subseteq U_\varepsilon \times [1, \infty) \cup A \times [1, \infty)$. It follows that

$$Z_A \cap \big(X \times [t^A_\varepsilon, \infty)\big) \subseteq U_\varepsilon \times [1, \infty).$$

We have the analogous result for $Z_B := (B \times [1, \infty))^E \smallsetminus (B \times [1, \infty))$. Assume that E has the metric control condition α, i.e. for $((x,t),(x',t')) \in E$ we have $|t - t'| \leq \alpha$. Let U°_ε be the neighborhood of $A \cap B$ corresponding to $(A \cap B) \times (-\varepsilon, \varepsilon)$. Then for $0 < \varepsilon < 1$ the continuous control condition implies that for every x in the compact G-invariant subspace $A \smallsetminus U^\circ_\varepsilon$ and its G_x-invariant neighborhood $(A \smallsetminus U_{\varepsilon/2}) \times [1, \infty)$ there is a G_x-invariant neighborhood V such that $V^E \subseteq (A \smallsetminus U_{\varepsilon/2}) \times [1, \infty)$. As $A \smallsetminus U^\circ_\varepsilon$ is G-compact there is a t^B_ε such that

$$\big((A \smallsetminus U^\circ_\varepsilon) \times [t^B_\varepsilon, \infty)\big)^E \subseteq (A \smallsetminus U_{\varepsilon/2}) \times [1, \infty)$$

As B in contained in the complement of $(A \smallsetminus U_{\varepsilon/2})$ and E is symmetric it follows that

$$(B \times [t^B_\varepsilon + \alpha, \infty))^E \subseteq (X \smallsetminus (A \smallsetminus U^\circ_\varepsilon)) \times [t^B_\varepsilon, \infty) = (U_\varepsilon \cup B) \times [t^B_\varepsilon, \infty)$$

where we used that due to the metric control $B \times [t^B_\varepsilon + \alpha, \infty)^E \subseteq X \times [t^B_\varepsilon, \infty)$. In particular we have

$$Z_B \cap \big(X \times [t^B_\varepsilon + \alpha, \infty)\big) \subseteq U_\varepsilon \times [1, \infty).$$

Set $t_\varepsilon := \max(t^A_\varepsilon, t^B_\varepsilon + \alpha)$, so $(Z_A \cup Z_B) \cap (X \times [t_\varepsilon, \infty)) \subseteq U_\varepsilon \times [1, \infty)$. If $A \cap B = \varnothing$ it follows that $(Z_A \cup Z_B) \cap (X \times [t_\varepsilon, \infty))$ is empty and hence

$$(A \times [1, \infty))^E \cap (B \times [1, \infty))^E \cap X \times [t, \infty) = \varnothing.$$

Else define for $t > t_{\frac{1}{2}}$ the function

$$\varepsilon(t) := \min\{\varepsilon \mid (Z_A \cup Z_B) \cap X \times [t, \infty) \subseteq U_\varepsilon \times [1, \infty)\}.$$

Note that $\varepsilon(t_\varepsilon) \leq \varepsilon$ and $\varepsilon(t) \leq \varepsilon(t')$ for $t \geq t'$ and $\varepsilon(t)$ goes to zero for t going to infinity.

For $z \in A \cap B$ let $L_\varepsilon(z) \subseteq U$ correspond to $\{z\} \times (-\varepsilon, \varepsilon)$. Set

$$E' := \Big\{ \big((x,t),(y,t)\big) \quad \begin{array}{cl} | & t < t_{\frac{1}{2}} \text{ or } x = y \\ \text{or} & x \in A \cap B \text{ and } y \in L_{\varepsilon(t)}(x) \\ \text{or} & y \in A \cap B \text{ and } x \in L_{\varepsilon(t)}(y) \Big\}. \end{array}$$

One sees that E' is symmetric and G-invariant. One checks that it is a continuous control set. It satisfies our claim. Figure 7.1 shows a picture of the situation. □

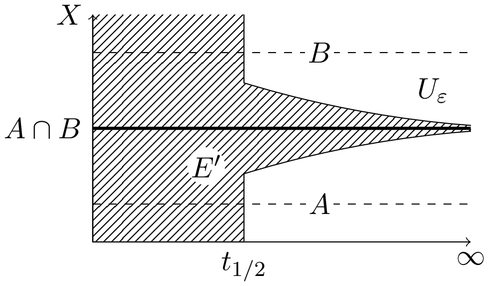


Figure 7.1.: A sketch of E'

Proof of the Addendum. We treat the case $B \subseteq X$. We have to construct a controlled retraction $(B \times G \times [1,\infty))^E \to B \times G \times [1,\infty)$ which maps locally finite sets to locally finite sets.

We use the notation and results from the previous proof. It suffices to define a map $f\colon Z_B \to B \times [1,\infty)$ and extend it G-equivariantly. If $(x,t) \in Z_B \cap X \times [t_{1/2},\infty)$ then it is contained in $U_{1/2} \times [t_{1/2},\infty)$. Hence $x \in U_{1/2}$ has a parametrization as $(x',s) \in A \cap B \times [-1/2, 1/2]$, so we can write (x,t) as (x',s,t). Then we map (x',s,t) to $(x',0,t)$. Else $t \leq t_{1/2}$ and we choose some y in B and map (x,t) to (y,t). This gives a map $f\colon Z_B \to B \times [1,\infty)$.

This map f is E'-controlled for the E' constructed to E in the previous proof. It gives a retraction $f \times G\colon (B \times G \times [1,\infty))^E \to B \times G \times [1,\infty)$ which is G-equivariant and E'-controlled. Let (M,κ) be a locally finite set over X^{cc} with support in B^E. We have to show that $M' := (f \times G)_*(M)$ is a again a locally finite set.

Using that G acts properly discontinuous and free on $B \times G$ we find for $x \in B \times G \times [1,\infty)$ a neighborhood $V \subseteq B \times G \times [i,i+2]$ with $gV \cap V = \varnothing$ for $g \in G \setminus \{e\}$. As M has G-compact support in $B \times G$-direction it is finite on V. As $V' := (f \times G)^{-1}(V)$ has also the property that $gV' \cap V' = \varnothing$ for $g \in G \setminus \{e\}$ and is again contained in $B \times G \times [i,i+2]$ it follows that M is finite on V', and therefore M' is finite on V.

This shows that $f \times G$ maps locally finite sets to locally finite sets. (Note that the lemma is only true for the resolutions, as there might be no retraction without taking resolutions, e.g. if G acts freely on $A \cap B$ but not on A.) ☐

We can now prove the homotopy invariance. This is a slight variation of Proposition 5.6 of [BFJR04], as there seem to be some typos in the proof.

Lemma 7.11. *The functor $h^G(-,\mathbf{K}_R)$ is homotopy invariant.*

Proof. It suffices to show that the canonical map $X \times I \to X$ induces an equivalence of spectra.

For $i = 0,1$ let $X_i := [i,1] \times X$ and $Z_i := [i,\infty) \times X$. We get four control spaces X_i^{cc} and Z_i^{cc}. Let $Z_i^{cc'}$ be the control space Z_i^{cc} with the different object support condition $p^{-1}_{X\times G}\mathcal{F}_{Gc}$, i.e. we do not require compact support in $[i,\infty)$-direction. We want to check that the triple $(Z_0^{cc'}, Z_1^{cc'}, X_0^{cc})$ satisfies the conditions of the coarse Mayer-Vietoris (Theorem 6.5). Lemma 7.8 shows that the triple $(Z_0^{cc'}, Z_1^{cc'}, X_0^{cc})$ is coarsely excisive and the addendum shows $X_1^{cc} = X_0^{cc} \cap Z_1^{cc'}$ is froper in X_0^{cc}. One checks further that $Z_1^{cc'}$ and X_0^{cc} are froper in $Z_0^{cc'}$. Thus the assumptions of the coarse Mayer-Vietoris (Theorem 6.5) are satisfied, which therefore provides a homotopy fiber square of spectra

$$\begin{tikzcd}
\mathbb{K}^{-\infty}(w\mathcal{C}^G(X_1^{cc})) \arrow[r] \arrow[d] & \mathbb{K}^{-\infty}(w\mathcal{C}^G(X_0^{cc})) \arrow[d] \\
\mathbb{K}^{-\infty}(w\mathcal{C}^G(Z_1^{cc'})) \arrow[r] & \mathbb{K}^{-\infty}(w\mathcal{C}^G(Z_0^{cc'}))
\end{tikzcd} .$$

Both lower spaces have a flasque shift by Lemma 7.12 below and hence are trivial. It follows that the upper map is a homotopy equivalence.

However, we want to have the result for germwise equivalences away from 1 in the continuous control direction. It suffices to prove it for the control spaces $\iota\colon X_i \times G \times 1 \subseteq X_i^{cc}$, then we get that in the map of homotopy fiber sequences

$$\begin{tikzcd}
\mathbb{K}^{-\infty}(w\mathcal{C}^G(X_i \times G \times 1, \iota^{-1}\mathcal{E}, \iota^{-1}\mathcal{F})) \arrow[r] \arrow[d] & \mathbb{K}^{-\infty}(w\mathcal{C}^G(X_i^{cc})) \arrow[r] \arrow[d] & \mathbb{K}^{-\infty}(g_\infty w\mathcal{C}^G(X_i^{cc})) \arrow[d] \\
\mathbb{K}^{-\infty}(w\mathcal{C}^G(X_0 \times G \times 1, \iota^{-1}\mathcal{E}, \iota^{-1}\mathcal{F})) \arrow[r] & \mathbb{K}^{-\infty}(w\mathcal{C}^G(X_0^{cc})) \arrow[r] & \mathbb{K}^{-\infty}(g_\infty w\mathcal{C}^G(X_0^{cc}))
\end{tikzcd}$$

the first two vertical maps are equivalences of spectra and hence the third vertical arrow is also an equivalence of spectra, which by definition is the map $h^G(X_0, \mathbf{K}_R) \to h^G(X_1, \mathbf{K}_R)$.

If we pull back the control conditions of X_i^{cc} to $X_i \times G \times 1$ we get no control conditions at all, as X_i is compact. It follows that $X_1 \times G \times 1 \to X_0 \times G \times 1$ induces a homotopy equivalence on the spectra, which is the left map in the diagram above. □

We still have to provide a flasque shift:

Lemma 7.12. *There is a flasque shift on*

$$Z_i^{cc'} = \big([i,\infty) \times X \times G \times [1,\infty), \mathcal{E}_{Gcc}([i,\infty) \times X \times G), \mathcal{F}_{Gc}(X \times G)\big).$$

It may be given by

$$\begin{array}{rccc} \varphi\colon & [i,\infty) \times X \times G \times [1,\infty) & \to & [i,\infty) \times X \times G \times [1,\infty) \\ & (s,x,\gamma,r) & \mapsto & (s+\frac{1}{r}, x, \gamma, r). \end{array}$$

Proof. We can assume $X = \mathrm{pt}$ and $G = \{e\}$, as φ is constant in these directions and the continuous control on the space is the intersection of the pullbacks of the continuous control conditions on $[i,\infty)\times[1,\infty)$, $X \times [1,\infty)$, and $G \times [1,\infty)$; the same is true for the object support conditions. Therefore, on $[i,\infty) \times [1,\infty)$, we have no equivariance, no extra metric control conditions (except the one in $[1,\infty)$-direction) and no object support conditions. The shift reduces to $\varphi(s,r) \mapsto (s+1/r,r)$.

We recall how the continuous control conditions on the space $[i,\infty)\times[1,\infty)$ look like (cf. Example 1.9(ii)). We say that a set $E \subseteq \big([i,\infty)\times[1,\infty)\big)^2$ is a continuous control condition if it satisfies the following two conditions. First we have the original continuous control condition which we can rephrase as: For each point $(x,\infty)\in[i,\infty)\times[1,\infty]$ and each neighborhood U of (x,∞) in $[i,\infty)\times[1,\infty]$ there is a neighborhood V of (x,∞) in $[i,\infty)\times[1,\infty]$ such that V^E is contained in U. We denote this condition by (C). Then there is an extra metric control condition saying that for E there is an α such that for each $((x,r),(x',r'))\in E$ we have $|r-r'|\leq\alpha$. We denote this condition by (M).

To prove that φ is a flasque shift we have to check the five conditions (i)–(v) of 5.21. First φ is a homeomorphism onto its image, this shows that it maps locally finite sets to locally finite sets, hence (i). Now pick an $x=(s,r)$ and $\varepsilon<1$. Then $(s^n)^{-1}([s-\varepsilon,s+\varepsilon]\times[r-\varepsilon,r+\varepsilon])$ is empty for $n>(s+\varepsilon)(r+\varepsilon)$, which shows (ii). We have no object support conditions, so (iv) is satisfied.

For (v) it suffices to show that $D:=\{(x,\varphi(x))\}=\{((s,r),(s+\frac{1}{r},r))\}$ is a valid continuous control condition, i.e. satisfies (C) and (M). Pick an $s_0\in[i,\infty)$ and a neighborhood U of (s_0,∞) in $[i,\infty)\times[1,\infty]$. We can assume that

$$U=[s_0-\varepsilon,s_0+\varepsilon]\times[l,\infty].$$

We set as attempt $V:=[s_0-\frac{\varepsilon}{2},s_0+\frac{\varepsilon}{2}]\times[J,\infty]$ and try to determine J. We have $V^D=\{(s+\frac{1}{r},r)\mid(s,r)\in V\}$. If $J\geq l$ we get $V^D\subseteq[i,\infty)\times[J,\infty)$, so we only need to check the first coordinate. For this we have for $r\geq 2/\varepsilon$

$$\begin{aligned}\left|s+\frac{1}{r}-s_0\right| &\leq |s-s_0|+\frac{1}{r}\\ &\leq \frac{\varepsilon}{2}+\frac{1}{r}\leq\varepsilon,\end{aligned}$$

hence with $J:=\max(l,\frac{2}{\varepsilon})$ we have $V^D\subseteq U$. This shows (C). Condition (M) is obviously true. Hence property (v) holds.

Condition (iii) is left, which is more intricate. For each E satisfying (C) and (M) we have to show that $D:=\bigcup_n(s\times s)^n(E)$ also satisfies (C) and (M). Again choose an s_0 and a neighborhood $U:=[s_0-\varepsilon,s_0+\varepsilon]\times[l,\infty]$. We have to find a K such that for $V:=[s_0-\varepsilon/2,s_0+\varepsilon/2]\times[K,\infty]$ we have $V^D\subseteq U$. Assume that E satisfies (M) with metric control condition α. Then D satisfies (M) with the same metric control condition. Thus if we assume $K\geq l+\alpha$ we have $V^D\subseteq X\times[l,\infty)$ and hence can ignore that direction.

Pairs in D have the form $(s+n/r,r),(s'+n/r',r')$ with $((s,r),(s',r'))\in E$. So we have to find a K such that for each n and each pair $((s,r),(s',r'))\in E$ with $|s+\frac{n}{r}-s_0|\leq\frac{\varepsilon}{2}$ and $r\geq K$ we have

$$\left|s'+\frac{n}{r'}-s_0\right|\leq\varepsilon.$$

It suffices to show that

$$\left|s + \frac{n}{r} - s' - \frac{n}{r'}\right| \le \frac{\varepsilon}{2}.$$

For $\omega > 0$ set $U^q_\omega := [q - \omega/2, q + \omega/2] \times [1, \infty]$. We can consider the cover $\{U^q_\omega \mid q \in [i, s_0 + \varepsilon]\}$ of $[i, s_0 + \varepsilon] \times [1, \infty]$. By the continuous control condition we get for each U^q_ω a neighborhood V^q_ω of (q, ∞) such that $(V^q_\omega)^E \subseteq U^q_\omega$. Without loss of generality V^q_ω has the form $I^q_\omega \times [r^q_\omega, \infty]$, I^q_ω some interval. As $[i, s_0 + \varepsilon]$ is compact, finitely many I^q_ω cover it. Take r_ω as the maximum of the finitely many r^q_ω for which $I^q_\omega \times [r^q_\omega, \infty]$ covers $[i, s_0 + \varepsilon] \times \{\infty\}$. Then for each $r \ge r_\omega$ and $s \le s_0 + \varepsilon$ we have that $((s, r), (s', r'))$ in E implies $|s - s'| \le \omega$.

Set $\omega := \varepsilon/4$ and get r_ω. Pick $(s + n/r, r) \in V$. We then have $s + \frac{n}{r} \le s_0 + \varepsilon$. We further (can) assume $r \ge r_\omega$. Now have the estimates together to show that

$$V^D = ([s_0 - \varepsilon/2, s_0 + \varepsilon/2] \times [K, \infty])^D \subseteq [s_0 - \varepsilon, s_0 + \varepsilon] \times [l, \infty] = U,$$

if we set K large enough.

For $((s + n/r, r), (s' + n/r', r'))$ in D we can estimate

$$\begin{aligned}\left|s + \frac{n}{r} - s' - \frac{n}{r'}\right| &\le |s - s'| + n \cdot \left|\frac{1}{r} - \frac{1}{r'}\right| \\ &\le \frac{\varepsilon}{4} + n \cdot \frac{|r' - r|}{rr'}.\end{aligned}$$

The metric control condition provides $|r' - r| \le \alpha$. Further from $s + \frac{n}{r} \le s_0 + \varepsilon$ we get $n \le (s_0 - s + \varepsilon) \cdot r$, hence $n \le 2\varepsilon \cdot r$. We can estimate further

$$\begin{aligned}\frac{\varepsilon}{4} + n \cdot \frac{|r' - r|}{rr'} &\le \frac{\varepsilon}{4} + 2\varepsilon \cdot r \cdot \frac{\alpha}{rr'} \\ &\le \frac{\varepsilon}{4} + 2\varepsilon \frac{\alpha}{r'}\end{aligned}$$

and hence for $r' \ge 4\alpha$ this is bounded by $\varepsilon/2$. Setting $K := \max(r_{\varepsilon/4}, 4\alpha, j) + \alpha$ we get

$$\Big([s_0 - \frac{\varepsilon}{2}, s_0 + \frac{\varepsilon}{2}] \times [K, \infty]\Big)^E \subseteq U.$$

This shows the property (C) and hence finishes the proof. $\square$

Remark 7.13. The difference to the proof of [BFJR04, Prop. 5.6] is, besides that we give more details, that the conditions on $Z^{cc'}_i$ are different. We explicitly do not require extra metric control in the $[i, \infty)$-direction and no compact support in that direction, as this would make Lemma 7.12 wrong.

Lemma 7.14 (Mayer-Vietoris). *Assume we have a homotopy pushout of finite G-CW-complexes*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X_3 \end{tikzcd} .$$

Then

$$\begin{tikzcd}
h^G(X_0, \mathbf{K}_R) \arrow[r] \arrow[d] & h^G(X_1, \mathbf{K}_R) \arrow[d] \\
h^G(X_2, \mathbf{K}_R) \arrow[r] & h^G(X_3, \mathbf{K}_R)
\end{tikzcd}$$

is a homotopy pullback square of spectra.

Proof. We assume that $X_0 \neq \varnothing$. The case $X_3 = X_2 \amalg X_1$ seems to need an extra treatment. We do it in Lemma 7.15 below.

By the homotopy invariance we can replace X by the double mapping cylinder of $X_1 \leftarrow X_0 \rightarrow X_2$ and thus are in the situation of Lemma 7.8. Together with its Addendum it shows the assumptions of Theorem 6.5 (coarse Mayer-Vietoris). Applying it to $X_2^{cc} \leftarrow X_0^{cc} \rightarrow X_1^{cc}$ yields that the diagram

$$\begin{tikzcd}
\mathbb{K}^{-\infty}(w\mathcal{C}^G(X_0^{cc})) \arrow[r] \arrow[d] & \mathbb{K}^{-\infty}(w\mathcal{C}^G(X_1^{cc})) \arrow[d] \\
\mathbb{K}^{-\infty}(w\mathcal{C}^G(X_2^{cc})) \arrow[r] & \mathbb{K}^{-\infty}(w\mathcal{C}^G(X_3^{cc}))
\end{tikzcd} \tag{A}$$

is a homotopy pushout of spectra. The pullback of the control conditions along $\iota\colon X_i \times 1 \to X_i^{cc}$ gives no control conditions at all. As all spaces are non-empty the categories $\mathcal{C}_f^G(X_i \times G \times 1, R)$ are all equivalent and hence in the diagram

$$\begin{tikzcd}
\mathbb{K}^{-\infty}(w\mathcal{C}^G(X_0 \times G \times 1)) \arrow[r] \arrow[d] & \mathbb{K}^{-\infty}(w\mathcal{C}^G(X_1 \times G \times 1)) \arrow[d] \\
\mathbb{K}^{-\infty}(w\mathcal{C}^G(X_2 \times G \times 1)) \arrow[r] & \mathbb{K}^{-\infty}(w\mathcal{C}^G(X_3 \times G \times 1))
\end{tikzcd} \tag{B}$$

all maps are equivalences of spectra, in particular it is a homotopy pushout square. We have obtained for $i = 0, \ldots, 3$ homotopy fiber sequences

$$\mathbb{K}^{-\infty}(w\mathcal{C}^G(X_i \times G \times 1, \iota^{-1}\mathcal{E}, \iota^{-1}\mathcal{F})) \to \mathbb{K}^{-\infty}(w\mathcal{C}^G(X_i^{cc})) \to \mathbb{K}^{-\infty}(g_\infty w\mathcal{C}^G(X_i^{cc})).$$

The third terms form the square

$$\begin{tikzcd}
h^G(X_0, \mathbf{K}_R) \arrow[r] \arrow[d] & h^G(X_1, \mathbf{K}_R) \arrow[d] \\
h^G(X_2, \mathbf{K}_R) \arrow[r] & h^G(X_3, \mathbf{K}_R)
\end{tikzcd} . \tag{C}$$

So we get a sequence of the squares of spectra which is a homotopy fiber sequence at each corner

$$\begin{tikzcd}
\bullet \arrow[rr] \arrow[dd] & & \bullet \arrow[dd] & & \bullet \arrow[rr] \arrow[dd] & & \bullet \arrow[dd] & & \bullet \arrow[rr] \arrow[dd] & & \bullet \arrow[dd] \\
& \text{(A)} & & \longrightarrow & & \text{(B)} & & \longrightarrow & & \text{(C)} & \\
\bullet \arrow[rr] & & \bullet & & \bullet \arrow[rr] & & \bullet & & \bullet \arrow[rr] & & \bullet
\end{tikzcd}$$

where (A) and (B) are homotopy pushout squares. It follows that (C) is a homotopy pushout square. ☐

We still need to show the case $A \cap B = \varnothing$, i.e. $A \amalg B = X$. This seems to be more complicated.

Lemma 7.15. *Let $X = A \amalg B$ the disjoint union of G-CW-complexes. Then $h^G(A \amalg B, \mathbf{K}_R) \cong h^G(A, \mathbf{K}_R) \vee h^G(A, \mathbf{K}_R)$.*

Proof. The proof goes pretty deep into the definitions. We do a few reductions first.

Note first that if the control space Y is the disjoint union of two control spaces Y_1, Y_2 and we take as control conditions also the disjoint unions, then the category $\mathcal{C}_f^G(Y, R)$ is equivalent to $\mathcal{C}_f^G(Y_1, R) \times \mathcal{C}_f^G(Y_2, R)$. (Note that "disjoint union of control spaces" just means that for each $y_i \in Y_i$ the pair (y_1, y_2) is not in any $E \in \mathcal{E}$.) The product extends to the categories with cofibrations and weak equivalences and, as Waldhausen's K-theory respects finite products of categories, to the K-theory. This works for finite products with $\mathbb{R}^n$. Hence, as for spectra product and coproduct agree up to homotopy, we get the result for our non-connective K-theory.

In our situation consider the control spaces $(A \amalg B)^{cc}$ and $A^{cc} \amalg B^{cc}$. They differ only in the control conditions and there is a map from the latter to the former. So if we show that the Approximation Theorem applies to

$$F\colon g_\infty w\mathcal{C}_f^G(A^{cc} \amalg B^{cc}) \to g_\infty w\mathcal{C}_f^G((A \amalg B)^{cc}),$$

with the germs at infinity, we are done. Lemma 7.8 shows for $A \cap B = \varnothing$ that for each $E \in \mathcal{E}_{Gcc}$ there is a t such that for all $a \in A \times [t, \infty)$ and $b \in B \times [t, \infty)$ the pair (a, b) is not in E. This means, that "near infinity $(A \amalg B)^{cc}$ is a disjoint union". We now make that statement into a proof. Abbreviate $g_\infty w\mathcal{C}_f^G(A^{cc} \amalg B^{cc})$ by $\mathcal{D}$ and $g_\infty w\mathcal{C}_f^G((A \amalg B)^{cc})$ by $\mathcal{D}'$, so we have to show that $F\colon \mathcal{D} \to \mathcal{D}'$ satisfies the two conditions of the Approximation property.

For (App 1) let $f\colon M \to M'$ be a map in $\mathcal{D}$. Assume that $F(f)$ is a germwise equivalence in $\mathcal{D}'$ with all associated data controlled by E. By Lemma 7.8 there is a t such that $E \cap (X \times [t, \infty))^2$ splits as $E^A \amalg E^B$. Hence there is a t such that if we restrict the germwise inverse and the homotopies to $X \times [t, \infty)$ they come from $\mathcal{D}$. This shows f is a $g_\infty w$-equivalence in $\mathcal{D}$.

Let M be an object in $\mathcal{C}_f^G(A^{cc} \amalg B^{cc})$. It is a controlled module over $A \amalg B$ with no "crossing" differentials (i.e. differentials from A to B or vice versa). Let P be in $\mathcal{C}_f^G((A \amalg B)^{cc})$ and let $f\colon F(M) \to P$. Recall that we have to find a module P' in

$\mathcal{C}^G_f(A^{cc} \amalg B^{cc})$ and maps such that the diagram

$$\begin{tikzcd}
F(M) \arrow[r] \arrow[d] & P \\
P' \arrow[ur, "\sim_{g_\infty w}"'] &
\end{tikzcd}$$

commutes and the lower diagonal map is a $g_\infty w$-equivalence. We can assume that $M \to P$ is an inclusion by replacing it by the mapping cylinder. Use again Lemma 7.8 to choose a t such that $P_{X \times [t,\infty)}$ has no "crossing" differentials. $P_{X\times[t,\infty)} \to P$ is a $g_\infty w$-equivalence. If we enlarge it to $P' := F(M) \cup P_{[t,\infty)}$ the map $P' \to P$ is still a $g_\infty w$-equivalence and P' has no "crossing differentials", hence comes from $\mathcal{C}^G_f(A^{cc} \amalg B^{cc})$. This shows the (App 2). □

Thus $h^G(-, \mathbf{K}_R)$ is a homology theory on G-finite G-CW-complexes. We will calculate its coefficients in Section 7.5.

7.3. Direct Limit Axiom and extension to arbitrary G-CW-complexes

We now extend our discussion from the previous section to arbitrary G-CW-complexes. In the following *finite* always means G-finite.

Let h^G be a functor from finite G-CW-complexes to spectra which satisfies the axioms (0) to (iii) from above. (Note that (0) just demands the functoriality and in (iii) the indexing category of the homotopy colimit is finite in which case it has a terminal object and hence (iii) is always true, cf. Remark7.17 below.)

Lemma/Definition 7.16. *Define a functor $\widehat{h}^G$ from all G-CW-complexes to spectra by*

$$\widehat{h}^G(X) := \operatorname*{hocolim}_{X_i \subseteq X \text{ finite}} h^G(X_i)$$

Then $\widehat{h}^G$ extends h^G in the sense that for X a finite G-CW-complex the canonical map $\widehat{h}^G(X) \to h^G(X)$ is an equivalence of spectra.

Further $\widehat{h}^G$ is a G-equivariant homology theory which satisfies the compact support axiom (iii).

Remark 7.17. Recall that we have a functorial definition of the *homotopy colimit* of a functor $F\colon \mathcal{C} \to$ (Spectra) by setting $\operatorname{hocolim}_{c\in\mathcal{C}} F(c)$ as the realization of the simplicial object

$$[n] \mapsto \bigvee_{c_0 \to \cdots \to c_n \in \mathcal{C}} F(c_0)$$

with the obvious structure maps (cf. [GJ99, Example IV.1.8,p. 199]). It has the property that for an objectwise weak equivalence $F(c) \xrightarrow{\simeq} F'(c)$ it gives an equivalence hocolim $F \to$ hocolim F'. Further if $\mathcal{C}' \subseteq \mathcal{C}$ is cofinal then the map from $\operatorname{hocolim}_{\mathcal{C}'} F$ to $\operatorname{hocolim}_{\mathcal{C}} F$ is an equivalence. There also is a canonical map $\operatorname{hocolim}_{\mathcal{C}} F \to \operatorname{colim}_{\mathcal{C}} F$ which does not need to be an equivalence in general.

Proof of 7.16. Let $\{X_i\}$ be the system of finite subcomplexes of X. If X is finite then it is a terminal object in $\{X_i\}$, hence

$$\operatorname{hocolim} h^G(X_i) \simeq h^G(X).$$

To prove homotopy invariance we show that the projection $p\colon X \times I \to X$ induces an equivalence of spectra. Note that $\{X_i \times I \subseteq X \times I\}$, with $X_i \subseteq X$ finite, is a cofinal subsystem of the finite subcomplexes of $X \times I$. Hence p gives a map

$$\operatorname{hocolim} h^G(X_i \times I) \to \operatorname{hocolim} h^G(X_i)$$

which is pointwise a weak equivalence, hence gives a weak equivalence on the homotopy colimit (cf. Remark 7.17). This shows the homotopy invariance.

For the Mayer-Vietoris Axiom note that it suffices to prove the case when X is the union of subcomplexes A and B. Then each finite subcomplex X_i of X gives a pair of finite subcomplexes of A and B whose union is X_i. Hence if we index over the finite subcomplexes of X we get diagrams

$$\begin{tikzcd}
A_i \cap B_i \arrow[r] \arrow[d] & A_i \arrow[d] \\
B_i \arrow[r] & X_i
\end{tikzcd} \tag{27}$$

of finite G-CW-complexes. Hence $\widehat{h}^G(-)$ of the diagram

$$\begin{tikzcd}
A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\
B \arrow[r] & X
\end{tikzcd} \tag{28}$$

is the homotopy colimit diagram of the homotopy pushout diagrams (solid)

$$\begin{tikzcd}
h^G(A_i \cap B_i) \arrow[r] \arrow[d] & h^G(A_i) \arrow[r, dashed] \arrow[d] & \bullet \arrow[d, dashed, "\simeq"] \\
h^G(B_i) \arrow[r] & h^G(X_i) \arrow[r, dashed] & \bullet
\end{tikzcd}$$

We can extend the diagrams functorially to the right as shown, with the rows being homotopy cofiber sequences. Then the right horizontal map is an equivalence as the solid diagram is a homotopy pushout. If we take the homotopy colimit over i we get a similar diagram where the rows are still homotopy cofiber sequences and the rightmost vertical map is an equivalence. Hence

$$\begin{tikzcd}
\widehat{h}^G(A \cap B) \arrow[r] \arrow[d] & \widehat{h}^G(A) \arrow[d] \\
\widehat{h}^G(B) \arrow[r] & \widehat{h}^G(X)
\end{tikzcd}$$

is a homotopy pushout. (We used that (27) and (28) are also *homotopy* pushouts of G-CW-complexes.) This shows the Mayer-Vietoris property.

The compact support axiom is now easy. For $X_i \subseteq X$ the finite subcomplexes we have that $\widehat{h}^G(X_i) \to h^G(X_i)$ is an equivalence, hence

$$\operatorname{hocolim} \widehat{h}^G(X_i) \to \operatorname{hocolim} h^G(X_i) \to \widehat{h}^G(X)$$

is one.

This proves the lemma. ☐

Our compact support axiom states that $h^G(X)$ is equivalent to the homotopy colimit over the finite subcomplexes of X. For $h^G(X, \mathbf{K}_R)$ we can replace the homotopy colimit by the actual colimit as we show now. This was already done in [BFJR04], so one can view the following lemma as a justification for that.

Lemma 7.18. *Given X and $X_i \subseteq X$ a cofinal filtered system of subcomplexes of X. If $h^G(X_i)$ consists of CW-complexes in each degree for each X_i and the induced maps $h^G(X_i) \to h^G(X_j)$ are cellular inclusions in each degree then*

$$\operatorname{hocolim} h^G(X_i) \to \operatorname{colim} h^G(X_i)$$

is an equivalence.

Remark 7.19. This is not automatically clear as a cofibration of spectra is more than a cofibration in each degree.

That $X_i \subseteq X$ is a cofinal filtered system means that besides it is filtered, i.e. for X_1, X_2 there is an X_3 with $X_1 \subseteq X_3$, $X_2 \subseteq X_3$, we also have $\bigcup_i X_i = X$. The lemma applies to our situation of $h^G(-, \mathbf{K}_R)$ defined above and explains why the "directed unions" axiom of [BFJR04] holds, although it is stated as $\operatorname{colim}_{X_i} h^G(X_i, \mathbf{K}_R) \to h^G(X, \mathbf{K}_R)$ (in our notation) being a weak equivalence. The assumptions of the lemma hold because an inclusion of subcomplexes $X_i \subseteq X_j$ gives an inclusion of control spaces $X_i^{cc} \subseteq X_j^{cc}$ and then an inclusion of categories, which further gives an inclusion of the (bi-)simplicial sets involved in the $\mathcal{S}_\bullet$-construction. Then realization respects inclusions and taking the loop space does not change the argument.

Proof. The colimit of spectra is formed by taking the colimit degreewise. We show that $\operatorname{hocolim} h^G(X_i) \to \operatorname{colim} h^G(X_i)$ is a π_*-isomorphism. An element in $\pi_n \operatorname{colim} h^G(X)$ is represented by a map $f \colon S^{n+k} \to h^G(X)_k$ for some $k > 0 - n$. As

$\operatorname{colim} h^G(X_i)_n$ is the colimit of cellular inclusions over a filtered set it follows that there is an X_j such that f factors as $S^{n+k} \to h^G(X_j)_k \to \operatorname{colim} h^G(X_i)_k$, as S^{n+k} is compact. Hence f factors also as $S^{n+k} \to (\operatorname{hocolim} h^G(X_i))_k \to \operatorname{colim} h^G(X_i)_k$. This shows surjectivity.

Assume that $f \in \pi_n \operatorname{hocolim} h^G(X_i)$ becomes zero in $\operatorname{colim} h^G(X_i)$. Then there is a k' such that $S^{n+k'} \to \operatorname{colim} h^G(X_i)_{k'}$ is nullhomotopic, i.e. extends to a disk $D^{n+k'+1} \to \operatorname{colim} h^G(X_i)_{k'}$. As $D^{n+k'+1}$ is again compact this shows that f is zero in $\pi_n \operatorname{hocolim} h^G(X_i)$, hence it shows the injectivity. □

In Section 7.1 we have defined a functor $h^G(-, \mathbf{K}_R)$ from all G-CW-complexes to spectra. We just showed that if we use the results from Section 7.2 we could restrict it to the finite G-CW-complexes and the extend it again to all to get a G-equivariant homology theory. We now show that the compact support axiom (iii) holds for $h^G(-, \mathbf{K}_R)$. This implies that it is itself a homology theory.

Lemma 7.20. *The functor $h^G(-, \mathbf{K}_R)$ satisfies the compact support axiom.*

Corollary 7.21. *For all G-CW-complexes X we have an equivalence*

$$\widehat{h}^G(X, \mathbf{K}_R) \xrightarrow{\simeq} h^G(X, \mathbf{K}_R).$$

Hence $h^G(-, \mathbf{K}_R)$ is itself a homology on all G-CW-complexes.

Proof of Corollary 7.21. We have $\widehat{h}^G(X) = \operatorname{hocolim} h^G(X_i) \xrightarrow{\simeq} h^G(X)$. □

Proof of 7.20. This follows closely the proof of the corresponding Proposition 5.5 of [BFJR04]. We have to show that for X a G-CW-complex and $X_i \subseteq X$ the finite subcomplexes we have that

$$\operatorname*{hocolim}_{X_i \subseteq X} h^G(X_i, \mathbf{K}_R) \to h^G(X, \mathbf{K}_R)$$

is an equivalence of spectra. Note that the set of finite subcomplexes of X is *filtered*, that is, each two finite subcomplexes are contained in a common finite subcomplex.

We get that X^{cc} is the filtered union of the X_i^{cc}. Because of the object support condition which requires compact object support of the modules in the X-direction we have that the category $\mathcal{C}_f^G(X^{cc})$ is the filtered union of $\mathcal{C}_f^G(X_i^{cc})$, as each object support set lies in one X_i^{cc}. The rest is an argument about Waldhausen's algebraic K-theory.

Recall that in degree 0 the spectrum $h^G(X, \mathbf{K}_R)$ was defined as $K(g_\infty w\mathcal{C}_f^G(X^{cc}))$, where $K(w\mathcal{C})$ is defined as $\Omega|\mathcal{S}_\bullet w\mathcal{C}|$. Here $\mathcal{S}_\bullet$ is Waldhausen's construction [Wal85] which takes a category with cofibrations and weak equivalences to a bisimplicial set. One checks that it takes an inclusion of categories to an inclusion of bisimplicial sets. Geometric realization respects this, so we have a homeomorphism

$$\operatorname*{colim}_i |\mathcal{S}_\bullet w\mathcal{C}_i| \xrightarrow{\simeq} |\mathcal{S}_\bullet w\mathcal{C}| \ .$$

Now as S^1 is compact (and $\Omega|\mathcal{S}_\bullet w\mathcal{C}|$ is the space of maps $S^1 \to |\mathcal{S}_\bullet w\mathcal{C}|$), as well as the structure map in the colimit are cellular inclusions, we see that $\operatorname{colim} \Omega|\mathcal{S}_\bullet w\mathcal{C}_i| \to \Omega|\mathcal{S}_\bullet w\mathcal{C}|$ is a homeomorphism.

For the degree n of $h^G(X, \mathbf{K}_R)$ we get the same result with the slightly modified control space $(X \times [1, \infty) \times \mathbb{R}^n)$. Hence the map

$$\operatorname*{colim}_{X_i \subseteq X} h^G(X_i, \mathbf{K}_R) \to h^G(X, \mathbf{K}_R)$$

is an equivalence. That we can replace the colimit by a homotopy colimit follows by Lemma 7.18. ☐

7.4. Reduced and unreduced theories

Let us briefly discuss the notion of a *reduced G-homology theory*, in particular let us notice a few things which are different from the non-equivariant case. It is well-known (see e.g. [Hat02, p. 161]) that for the non-equivariant case the notion of a reduced homology theory and an unreduced homology theory are essentially equivalent. For the G-equivariant case the same is true but we have to be more careful from where our functors start.

Above we defined an unreduced homology theory. It starts from the unpointed G-CW-complexes and has the property that evaluated on the point it is not trivial. A reduced homology starts from *pointed* G-CW-complexes and is trivial on the point. Pointed means that X comes with a map $* \to X$ from the one-point space with trivial G-action, we usually assume that it goes to the 0-skeleton. From a reduced homology we get an unreduced by evaluating on X_+, X together with a disjoint basepoint. In the other way we get a reduced homology theory from an unreduced one by taking the kernel (or fiber) of the map $h^G(X) \to h^G(*)$. Assuming that X has a basepoint gives a canonical section to this map.

Via this construction we can extend the above homology theory to pairs. First take the associated reduced homology theory and then set $h^G(X, A)$ to be the reduced theory evaluated on $X \cup_A CA$, the mapping cone of the inclusion $A \to X$. Note that if $A = \varnothing$ we get $X \cup_A CA = X_+$, hence we get the original theory back.

7.5. Coefficients and comparison to algebraic K-theory of simplicial rings

As a byproduct of our construction we get a definition for the algebraic K-theory spectrum over the orbit category $\operatorname{Or} G$.

Definition 7.22. *Let R be a simplicial ring, G a group. Define the* $\operatorname{Or} G$*-spectrum* $\mathbf{K}_R$ *via* $G/H \mapsto h^G(G/H, \mathbf{K}_R)$.

We show now that this has the "right" coefficients, i.e. we claim that this is the right notion of a functor $\mathbf{K}_R$ from the orbit category. However, we have no alternative definition, so the only thing we will show is the following lemma, which calculates the coefficients.

Lemma 7.23. *For H a subgroup of G we have an equivalence of spectra*

$$h^G(G/H, \mathbf{K}_R) \simeq \mathbb{K}^{-\infty}(R[H]).$$

Here the right-hand side is the non-connective algebraic K-theory of the simplicial group ring $R[H]$. We defined it in Definition 6.8 as the non-connective K-theory of the category $\mathcal{C}(\mathrm{pt}, R[H])$, so it is a spectrum which has in degree n the space $K(\mathcal{C}_f(\mathbb{R}^n, R[H], \mathcal{E}_d))$. We split the proof into several lemmas.

Remark 7.24. We recall a notation we already used in Section 1.3. If (M, κ_R) is a cellular module over the control space X we denote by M_x the part of the module which "lives" over $x \in X$. More precisely this is the graded subgroup of M which is generated by $\kappa_R^{-1}(x)$. Note that it does not inherit differentials, nor it is a module over a ring. (It is merely a sequence of abelian groups where the ith item is an R_i-module.) As a graded abelian group we have $M \cong \bigoplus_{x \in X} M_x$, so this notion is useful if we want to change κ_R. Usually we treat the cases degreewise and hence assume that $\underline{M}_x$ is an R-module where now R is a discrete ring. We also denote the 1-dimensional free R-module by $\underline{R}$ to distinguish it from the ring itself.

Lemma 7.25. *For any group H there is an equivalence of categories*

$$\varphi\colon \mathcal{C}_f^H(H, R) \xrightarrow{\cong} \mathcal{C}_f(*, R[H]).$$

Hence it induces an equivalence on non-connective algebraic K-theory.

Proof. Let (M, κ_R) be a module in $\mathcal{C}_f^H(H, R)$. As it arises by attaching cells, it suffices to describe the case $M = \bigoplus_{h \in H} \underline{R}_h$. In that case $M \cong R[H]$ where the right-hand side has an H-operation coming from the $R[H]$-module structure which is compatible with the H-operation on the left-hand side. This gives a module M' in $\mathcal{C}_f(*, R[H])$.

For the other direction a cellular $R[H]$-module comes (by our definition) with a chosen basis $\{e_i\}$. We get a cellular R-module by choosing the basis $\{h \cdot e_i\}$ and forgetting the H-action. We have to define κ_R, but we can simply set $\kappa_R(e_i) := e \in H$ and extend equivariantly. Thus the categories are equivalent.

The equivalence respects the notion of homotopy and hence homotopy equivalences. This gives us the equivalence on K-theory spaces.

For the control spaces $\mathbb{R}^n \times H$ and $\mathbb{R}^n$ we get the analogous maps by simply leaving the $\mathbb{R}^n$-direction untouched. Hence we get the desired equivalence of the non-connective K-theory spectra. □

Remark 7.26. We make a notation explicit which we already used sometimes. From Example 1.9 we know a lot of control conditions, but in particular we know how we can pull back morphism control conditions along a projection. So if $p\colon X \times Y \to Y$ is a projection and Y has a morphism control structure $\mathcal{E}$ we abbreviate the pullback of this structure along p by $\mathcal{E}^Y$. If $\mathcal{E}$ is for example the continuous control structure on $Y \times [1,\infty)$ we abbreviate the pullback along p by $\mathcal{E}^Y_{cc}$ instead of writing $(p\times[1,\infty))^{-1}\mathcal{E}_{cc}(Y\times[1,\infty))$ and similar for metric control $\mathcal{E}_d$. In general a superscript denotes the space from which the morphism control condition is pulled back.

Let H be a subgroup of G. Take first the control space $(H,\{H\times H\},\{H\})$, the free H-equivariant control space with no control conditions at all. Then let $(G/H,\mathcal{E}_\Delta)$ be the G-equivariant control space with $\mathcal{E}_\Delta = \{\Delta\}$, Δ the diagonal, i.e. controlled maps are the maps over G/H. Take the standard resolution of this control space to get the free G-equivariant control space $(G/H \times G, \mathcal{E}^{G/H}_\Delta, \mathcal{F}_{Gc})$, where we assume additionally G-compact object support.

Lemma 7.27. *There is a functor*

$$\Phi\colon \mathcal{C}^H_f(H,R,\{H\times H\},\{H\}) \to \mathcal{C}^G_f(G/H\times G, R, \mathcal{E}^{G/H}_\Delta, \mathcal{F}_{Gc})$$

which induces an equivalence of categories and hence an equivalence on algebraic K-theory.

Proof. Let (M,κ_R) be a module in the source of Φ. Recall that we denote the cells of M by $\diamond_R M$ and that κ_R is a map $\diamond_R M \to H$. As $\mathbb{Z}[G]$ is a right $\mathbb{Z}[H]$-module, define

$$\Phi(M) := \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M.$$

Then $\Phi(M)$ is again a cellular R-module and $\diamond_R\Phi(M) \cong G\times_H(\diamond_R M)$ as (free) G-sets. Define $\kappa'_R\colon \diamond_R\Phi(M) \to G/H\times G$ by extending $(e,m)\mapsto([e],\kappa_R(m))$ G-equivariantly. The control condition in the target can be described as the G-equivariant extension of the rule $(([e],g),([x],g'))\in E$ if and only if $[x]=[e]$. Hence Φ takes controlled modules to controlled modules and controlled maps to controlled maps. Thus Φ is a functor. Its image consists of exactly the modules which are concentrated over the G-orbit of $([e],e)$.

We have to show that each module (N,κ_R) in the target is isomorphic to a module of this form. Choose a set $g_i\in G$ of representatives of G/H, then the set $([e],g_i)$ is a system of representatives of the G-orbits of $G/H\times G$. As (N,κ_R) has G-compact support it lies in only finitely many of the G-orbits of $([e],g_i)$. Let e^j_i be the cells over $([e],g_i)$, i.e. $\{e^j_i\}_j = \kappa_R^{-1}(([e],g_i))$. Then define κ'_R as $\kappa'_R(e^j_i) := ([e],e)$ and extend equivariantly. Note that (N,κ_R) and (N,κ'_R) are controlled isomorphic and (N,κ'_R) lies in the image of Φ, an inverse is given by the restriction of (N,κ'_R) to $[e]\times H$, as no boundary maps from cells over $([e],g)$ to cells over $([x],g')$ are allowed for $[x]\neq[e]$. This shows the equivalence of the categories. The part about the algebraic K-theory follows similar like for the preceding lemma. □

Lemma 7.28. *Let $H\subseteq G$ be a subgroup, let $\mathcal{E}^{G/H}_\Delta$ be the set of control conditions on $G/H\times G$ which is the pullback of the discrete control condition on G/H. Let*

$\mathcal{E}_d^{[1,\infty)}$ be the pullback of the metric control condition on $[1,\infty)$. Take germs to be the germs at infinity on $G/H \times G \times [1,\infty)$. Then there is an equivalence of spectra

$$\begin{aligned}\Omega\,\mathbb{K}^{-\infty}\big(g_\infty w\mathcal{C}^G(G/H \times G \times [1,\infty), R, \mathcal{E}_\Delta^{G/H} \cap \mathcal{E}_d^{[1,\infty)}, \mathcal{F}_{Gc})\big) \longrightarrow \\ \mathbb{K}^{-\infty}\big(w\mathcal{C}^G(G/H \times G, R, \mathcal{E}_\Delta^{G/H}, \mathcal{F}_{Gc})\big)\end{aligned}$$

Proof. It suffices to prove that the middle term in the fiber sequence for germs (6.4)

$$w\mathcal{C}^G\big(G/H \times G \times [1,\infty), R, \mathcal{E}_\Delta^{G/H} \cap \mathcal{E}_d^{[1,\infty)}, \mathcal{F}_{Gc}\big)$$

has an Eilenberg-swindle. But the map $(\gamma, g, t) \mapsto (\gamma, g, t+1)$ is a flasque shift (5.22). ☐

Lemma 7.29. *There is a functor*

$$\begin{aligned}g_\infty w\mathcal{C}_f^G(G/H \times G \times [1,\infty), R, \mathcal{E}_\Delta^{G/H} \cap \mathcal{E}_d^{[1,\infty)}, \mathcal{F}_{Gc}^{G/H\times G}) \longrightarrow \\ g_\infty w\mathcal{C}_f^G(G/H \times G \times [1,\infty), R, \mathcal{E}_{Gcc}^{G/H} \cap \mathcal{E}_d^{[1,\infty)}, \mathcal{F}_{Gc}^{G/H\times G})\end{aligned}$$

which induces an equivalence on algebraic K-theory. (Where again the germs are taken at infinity.)

Proof. It is a simple check that $\mathcal{E}_\Delta^{G/H} \cap \mathcal{E}_d^{[1,\infty)} \subseteq \mathcal{E}_{Gcc}^{G/H} \cap \mathcal{E}_d^{[1,\infty)}$, hence the functor exists.

The main idea is the same as in the proof of (M-V). Due to the discreteness of the space $G/H \times G$ and the G-compact support conditions we find for each morphism control condition E a t such that $E \cap [t,\infty)^2$ is the diagonal. The rest is provided by an application of the approximation property. We note again that an extra factor of $\mathbb{R}^n$ in the control spaces does not change the arguments, hence we get an equivalence on non-connective algebraic K-theory. ☐

Summarizing:

Proof of Lemma 7.23. By the preceding lemmas we get a chain of equivalences of spectra

$$\begin{aligned}\mathbb{K}^{-\infty}(R[H]) &\longleftarrow \mathbb{K}^{\infty}\big(w\mathcal{C}^H(H,R)\big) \\ &\longrightarrow \mathbb{K}^{\infty}\big(w\mathcal{C}^G(G/H \times G, R, \mathcal{E}_\Delta^{G/H}, \mathcal{F}_{Gc})\big) \\ &\longleftarrow \Omega\,\mathbb{K}^{-\infty}\Big(g_\infty w\mathcal{C}^G\big(G/H \times G \times [1,\infty), R, \mathcal{E}_\Delta^{G/H} \cap \mathcal{E}_d^{[1,\infty)}, \mathcal{F}_{Gc}^{G/H\times G}\big)\Big) \\ &\longrightarrow \Omega\,\mathbb{K}^{-\infty}\Big(g_\infty w\mathcal{C}^G\big(G/H \times G \times [1,\infty), R, \mathcal{E}_{Gcc}^{G/H} \cap \mathcal{E}_d^{[1,\infty)}, \mathcal{F}_{Gc}^{G/H\times G}\big)\Big) \\ &= h^G(G/H, \mathbf{K}_R).\end{aligned}$$

That shows the lemma. ☐

One last thing is to compare this to the usual algebraic K-theory of simplicial rings.

Lemma 7.30. *For any simplicial ring R there is a (weak) map of spectra*

$$K(R) \to \mathbb{K}^{-\infty}(R)$$

from the connective algebraic K-theory spectrum to the non-connective one (cf. [Wal85]) which is an isomorphism on π_i for $i \geq 1$. (A weak map means there is a chain of maps from the left to the right but the maps in the wrong direction are weak equivalences.)

Proof. The zeroth term of $\mathbb{K}^{-\infty}(R)$ is $K(R)$. As the source is an Ω-spectrum by [Wal85, 1.5.3] and the structure maps on the target give isomorphisms on π_i for $i \geq 1$ the result follows once we constructed the maps.

For the map we use the idea of the "up or across"-Lemma from [Fie77]. Recall that $K(R)$ is an Ω-spectrum with ith term equal to $\Omega|w\mathcal{S}_\bullet{}^i R\text{-Mod}|$, i.e. the i-fold iterated $\mathcal{S}_\bullet$-construction. Our definition of $\mathbb{K}^{-\infty}(R)$ was $\Omega|w\mathcal{S}_\bullet\mathcal{C}_f(\mathbb{R}^j, R, \mathcal{E}_d)|$. But we can also iterate the $\mathcal{S}_\bullet$-construction there and get a bispectrum

$$(i,j) \mapsto \Omega|w\mathcal{S}_\bullet{}^i\mathcal{C}_f(\mathbb{R}^j, R, \mathcal{E}_d)|.$$

The structure maps of this bispectrum $X_{i,j}$ are weak equivalences in the i-direction and it contains both other spectra. Namely $X_{i,0}$ agrees with the spectrum $K(R)$ whereas $X_{0,j}$ agrees with the spectrum $\mathbb{K}^{-\infty}(R)$. Choosing inverses for the homotopy equivalences $X_{i,j} \to \Omega X_{i+1,j}$ we get the desired maps (first up to homotopy) as

$$X_{i,0} \to \Omega X_{i,1} \to \cdots \to \Omega^i X_{i,i} \xleftarrow{\sim} \Omega^{i-1} X_{i,i-1} \xleftarrow{\sim} \cdots \xleftarrow{\sim} X_{i,0}.$$

A diagram chase shows that they are compatible with the structure maps, up to a sign. We get the strict maps in the usual way, e.g. if we assume that the structure maps of $K(R)$ are closed inclusions with the homotopy extension property. From that we can choose the inverses compatibly. The author is not sure if $K(R)$ already has this property, but certainly it can be arranged up to a weak equivalence (in the wrong direction, unfortunately). ☐

Remark 7.31. Note that it also follows that the possible difference at K_0 can be corrected by simply taking the "correct" definition of the *space* $K(R)$ in degree zero. Namely assume that $K_0(R)$ is K_0 not of the finite, but of the homotopy finitely dominated modules (cf. [Wal85, 2.3.2]). Define $\mathbb{K}^{-\infty}(R)$ not by the finite modules over $\mathbb{R}^n$ but by the homotopy finitely dominated modules over $\mathbb{R}^n$, which give an equivalent spectrum as remarked before. Then we get indeed a map which is also an isomorphism on π_0.

7.6. The assembly map

A family $\mathcal{F}$ of subgroups of a group G is a non-empty set of subgroups closed under taking conjugates and subgroups. The *classifying space* $E_{\mathcal{F}}G$ for G and $\mathcal{F}$ is a G-CW-complex whose H-fixed point set is contractible for $H \in \mathcal{F}$ and empty otherwise. It is unique up to G-homotopy equivalence. (See e.g. [DL98] for a construction for any G and $\mathcal{F}$.)

Definition 7.32 (K-Theory Assembly Map with coefficients in a simplicial ring)**.** *Let R be a simplicial ring, G a group and $\mathcal{F}$ a family of subgroups of G. Define the assembly map for $h^G(-, \mathbf{K}_R)$ and the family $\mathcal{F}$ to be the map*

$$h^G(E_{\mathcal{F}}G, \mathbf{K}_R) \to h^G(\mathrm{pt}, \mathbf{K}_R) \tag{29}$$

induced by the G-equivariant map $E_{\mathcal{F}}G \to \mathrm{pt}$.

We have $h^G(\mathrm{pt}, \mathbf{K}_R) \simeq \mathbb{K}^{-\infty}(R[G])$ as shown as part of Lemma 7.23, hence if we take homotopy groups we get for each $n \in \mathbb{Z}$ the map

$$h^G_n(E_{\mathcal{F}}G, \mathbf{K}_R) \to \mathbb{K}^{-\infty}_n(R[H])$$

which is also called the assembly map. (The lower n denotes the nth homotopy group π_n of the corresponding spectra.)

In [BLR08] it is shown that for a discrete ring R, G a word-hyperbolic group in the sense of Gromov and $\mathcal{F} = \mathcal{VC}yc$ the family of virtually cyclic subgroups of G the assembly map is a weak equivalence of spectra. (A group is called *virtually cyclic* if it contains a cyclic subgroup of finite index.) It is conjectured to be a weak equivalence for any discrete ring R, any group G and $\mathcal{F}$ the family of virtually cyclic subgroups of G. This is the so-called Farrell-Jones Conjecture for algebraic K-theory, which is known—together with its variant for L-theory—to imply a plethora of other conjectures, in particular it implies the Borel Conjecture which states that closed aspherical manifolds of dimension ≥ 5 are topologically rigid. See [LR05] for an older overview and the introduction of [BLR08] for a nice summary. The conjecture is known for many groups but the general case seems to be completely open at the time of writing.

We can mimic the approach to the proof of the main theorem of [BLR08] in our setting. Let X be a G-CW-complex. We defined X^{cc} in Definition 7.3 as

$$X^{cc} := (X \times G \times [1, \infty), p^{-1}\mathcal{E}_{Gcc}(X), p^{-1}_{X \times G}\mathcal{F}_{Gc})$$

and the subspace $X \times G \times 1$ inherits the control conditions, namely

$$X^0 := (X \times G \times 1, \{X \times G \times 1\}, p^{-1}_{X \times G}\mathcal{F}_{Gc}) \ .$$

Denote by g_∞ the germ support conditions on X^{cc} away from X^0, i.e. the "germs at infinity".

Define for any simplicial ring R the categories with cofibrations and weak equivalences $\mathcal{T}^G(X)$ ("target"), $\mathcal{O}^G(X)$ ("obstruction") and $\mathcal{D}^G(X)$ ("domain") as

$$\mathcal{T}^G(X) := w\mathcal{C}^G(X^0, R), \quad \mathcal{O}^G(X) := w\mathcal{C}^G(X^{cc}, R), \quad \mathcal{D}^G(X) := g_\infty w\mathcal{C}^G(X^{cc}, R) \quad .$$

By Theorem 6.4 we get a homotopy fiber sequence of spectra

$$\mathbb{K}^{-\infty}\mathcal{T}^G(X) \to \mathbb{K}^{-\infty}\mathcal{O}^G(X) \to \mathbb{K}^{-\infty}\mathcal{D}^G(X) \ .$$

Lemma 7.33. *The assembly map (29) is an isomorphism if and only if the spectrum* $\mathbb{K}^{-\infty}\mathcal{O}^G(E_\mathcal{F}G)$ *is contractible.*

Proof. The map $E_\mathcal{F}G \to \mathrm{pt}$ induces a map of homotopy fiber sequences of spectra

$$\begin{tikzcd}
\mathbb{K}^{-\infty}\mathcal{T}^G(E_\mathcal{F}G) \arrow[r] \arrow[d] & \mathbb{K}^{-\infty}\mathcal{O}^G(E_\mathcal{F}G) \arrow[r] \arrow[d] & \mathbb{K}^{-\infty}\mathcal{D}^G(E_\mathcal{F}G) \ . \arrow[d] \\
\mathbb{K}^{-\infty}\mathcal{T}^G(\mathrm{pt}) \arrow[r] & \mathbb{K}^{-\infty}\mathcal{O}^G(\mathrm{pt}) \arrow[r] & \mathbb{K}^{-\infty}\mathcal{D}^G(\mathrm{pt})
\end{tikzcd}$$

As pt^{cc} has a flasque shift $t \mapsto t+1$ (cf. Definition 5.22) the term $\mathbb{K}^{-\infty}\mathcal{O}^G(\mathrm{pt})$ is contractible. Further $(E_\mathcal{F}G)^0$ and pt^0 are equivalent control spaces, as there is no control condition depending on $E_\mathcal{F}G$, thus the left vertical map is a weak equivalence of spectra. It follows that the assembly map $\mathbb{K}^{-\infty}\mathcal{D}^G(E_\mathcal{F}G) \to \mathbb{K}^{-\infty}\mathcal{D}^G(\mathrm{pt})$ is a weak equivalence of spectra if and only if $\mathbb{K}^{-\infty}\mathcal{O}^G(E_\mathcal{F}G)$ is contractible. ☐

Remark 7.34. Actually the definitions in [BLR08] use an extra metric control condition on the G-factor in X^{cc}, namely they require all maps to be controlled with respect to the word-metric on G. However, as remarked there this is not essential for the definition of $h^G(X, \mathbf{K}_R)$. In particular if we require the extra condition we still get a G-equivariant homology theory with coefficients in the algebraic K-theory of R and therefore equivalent functors. We do not need it for our results so we leave it out, but we could have included it.

We believe that the methods of [BLR08] can be made to work in our situation:

Conjecture. *Let R be a simplicial ring, G a word-hyperbolic group and $\mathcal{F} = \mathcal{VC}yc$ the family of virtually cyclic subgroups of G. Then $\mathbb{K}^{-\infty}\mathcal{O}^G(E_\mathcal{F}G)$ is contractible.*

We have just shown that this implies the Farrell-Jones Conjecture for word-hyperbolic groups with coefficients in a simplicial ring:

Conjecture (Farrell-Jones Conjecture for a simplicial ring)**.** *Let R be a simplicial ring, G a word-hyperbolic group and $\mathcal{VC}yc$ the family of virtually cyclic subgroups of G. The algebraic K-theory assembly map for R, G and $\mathcal{VC}yc$*

$$h^G(E_{\mathcal{VC}yc}G, \mathbf{K}_R) \longrightarrow h^G(\mathrm{pt}, \mathbf{K}_R)$$

is a weak equivalence of spectra.

Appendix

A. Simplicial sets, simplicial abelian groups, and simplicial modules

A.1. A quick review on simplicial methods

We introduce very briefly the basic notions of simplicial sets and of simplicial modules over a simplicial ring. A good general modern reference is [GJ99]. An older but still good German reference is [Lam68]. This section does not contain anything new. We assume familiarity with the notions of category theory (see e.g. [ML98, Bor94a]).

Simplicial methods are a generalization of the idea of combinatorial or geometrical simplicial complexes. In particular we will have the notion of n-simplex and we have face (or "boundary") maps. But to get a good combinatorial notion we also need *degeneracy maps*, roughly these allow us to interpret an n-simplex also as certain $n+1$-simplices. Hence we define the category Δ.

Definition A.1. *Let n be a natural number, including zero. Let $[n]$ be the ordered set $0 < 1 < \cdots < n$ of natural numbers $\{0, \ldots, n\}$. Let Δ be the category with objects the ordered sets $[n]$ for all n and morphisms all monotone (=order preserving) maps.*

The category Δ contains two particular classes of maps. Any injective monotone map $[n-1] \to [n]$ is determined by the $i \in [n]$ which is not in the image. Call this map δ_n^i. Any surjective monotone map $[n+1] \to [n]$ is determined by the $j \in [n]$ which is hit twice. Call this map σ_n^j. The category Δ is generated by the δ_n^i and σ_n^i for all n, $0 \leq i, j \leq n$ with some very explicit relations (see e.g. [GJ99, 1.2] or [Lam68, (1.3)]). We often omit the index n.

Definition A.2. *Let $\mathcal{C}$ be a category. A simplicial object in $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. The simplicial objects in $\mathcal{C}$ together with the natural transformations of functors as morphisms form a category. It is denoted by $s\mathcal{C}$.*

A simplicial object C in $\mathcal{C}$ is hence given by a sequence $C_0, C_1, \ldots$ of objects in $\mathcal{C}$ together with *structure maps* $\alpha^* \colon C_n \to C_m$ for each order preserving map $\alpha \colon [m] \to [n]$. Therefore we sometimes denote C by $C_\bullet$. The maps δ_n^i induce maps $(\delta_n^i)_* =: d_i^n \colon C_n \to C_{n-1}$ which we sometimes call the *face maps* or *boundaries*. The maps σ_n^j induce maps $s_i^n := (\sigma_n^j)_* \colon C_n \to C_{n+1}$ which we sometimes call the *degeneracy maps*. As δ^i, σ^j generate Δ the maps d_i, s_j with its relations determine the structure of $C_\bullet$ as a simplicial object. As $s\mathcal{C}$ is a functor category it possesses all limits and colimits if $\mathcal{C}$ does and they are formed degreewise in $\mathcal{C}$.

An object $X_\bullet$ in *sSet* is called a *simplicial set.* An element $x \in X_n$ is a called an *n-simplex* of $X_\bullet$. An n-simplex x is called *degenerate* if it is in the image of a map s_i^n. The simplices which are not degenerate are called *non-degenerate*, they generate the simplicial set $X_\bullet$ in the sense, that each simplex can be written as $\alpha^*(y)$ for y a non-degenerate simplex. In particular maps out of $X_\bullet$ are determined on the non-degenerate simplices. The *standard n-simplex* is defined as the simplicial set $[m] \mapsto \mathrm{Hom}_\Delta([m],[n])$ and denoted by $\Delta^n_\bullet$. By the Yoneda-Lemma a map $\Delta^n_\bullet \to X_\bullet$ of simplicial sets is determined by a map $\Delta^n_n \supset \{\mathrm{id}_{[n]}\} \to X_n$. We sometimes denote that map by $\overline{x}$ and call it the *characteristic map of x*. Each simplicial set can be written as the colimit of standard simplices over its non-degenerate simplices (cf. [GJ99, I.2.1], [Lam68, II.1.4]).

There is a functor $|-|\colon sSet \to Top$ which produces for a simplicial set X a CW-complex $|X|$, called the *geometric realization of X*. It is determined by $|\Delta^n|$ being the geometric n-simplex (convex hull of the standard basis vectors in $\mathbb{R}^{n+1}$) and compatibility with colimits. A simplicial set is called *discrete* if each n-simplex for $n \geq 1$ is degenerate. The geometric realization of a discrete simplicial set is a discrete topological space. A *point* in X is a 0-simplex in X_0. We denote the points of Δ^n canonically by $0,\dots,n$. If x is a point in X define the nth homotopy group $\pi_n(X,x)$ of X as $\pi_n(|X|,|x|)$ of its geometric realization. A map $f\colon X \to Y$ is called a weak equivalence if either X and Y have no simplices or X is not empty and for each $x \in X$ and each n the induced map $\pi_n(X,x) \to \pi_n(Y,f(x))$ is an isomorphism.

In a precise sense geometric realization induces an equivalence between the homotopy theories of simplicial sets and topological spaces ([GJ99, I.11]).

The *boundary* $\partial\Delta^n$ of the standard n-simplex Δ^n is the simplicial subset of Δ^n generated by $d_i(\mathrm{id}_{[n]})$, $0 \leq i \leq n$. Its geometric realization is homeomorphic to the topological $(n-1)$-sphere. The *kth horn* $\Lambda^n_k \subset \Delta^n$ is the simplicial subset generated by $d_j(\mathrm{id}_{[n]})$, for $j \neq k, 0 \leq j \leq n$. Its geometric realization is homeomorphic to the $(n-1)$-disk.

Recall the limits and colimits of simplicial sets are formed in sets. So the product $A \times B$ of two simplicial sets A, B is $(A \times B) := A_n \times B_n$ with the structure maps acting diagonally. A *homotopy* of maps $f,g\colon X \to Y$ is a map $H\colon X \times \Delta^1 \to Y$ such that the zeroth restriction $H \circ \iota_0\colon X \times 0 \subset X \times \Delta^1 \to Y$ equals f and the first restriction $H \circ \iota_1\colon X \times 1 \subset X \times \Delta^1 \to Y$ equals g. Note that in general being homotopic is not an equivalence relation on maps $X \to Y$.

Definition A.3. *An map $f\colon X \to Y$ of simplicial sets is a Kan fibration if for each solid diagram*

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

the dotted lift exists. If the map $X \to *$ to the one-point simplicial set is a Kan fibration, then X is called Kan, fibrant, *or said to have the* extension property.

A map $f\colon X \to Y$ is a homotopy equivalence if there is a homotopy inverse map $g\colon Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity. Homotopy equivalences are weak equivalences. Here we mostly consider homotopy equivalences between Kan sets, so we do not care about the direction of the homotopies due to the following lemma.

Lemma A.4. *Let Y be Kan. Then*

- *(i) Being homotopic is an equivalence relation on maps $X \to Y$.*
- *(ii) We can define $\pi_n(Y, y)$ as pointed homotopy classes of maps $\partial\Delta^{n+1} \to Y$.*
- *(iii) Inclusions of simplicial sets (defined degreewise) have the homotopy extension property for maps into Y.*
- *(iv) If X and Y are Kan then each weak equivalence $X \to Y$ is a homotopy equivalence.*

A.2. Simplicial abelian groups and simplicial rings

An object in *sAb* is called a *simplicial abelian group*. As there is a forgetful functor *Ab* $\to$ *Set* each simplicial abelian group has an underlying simplicial set. Hence the notions for simplicial sets make sense for simplicial abelian groups. Alternatively a simplicial abelian group could be defined as an abelian group object in simplicial sets with respect to the product. This means a simplicial abelian group A is a simplicial set together with maps $\mu\colon A \times A \to A$, the multiplication, and $\iota\colon A \to A$, the inverse, and unit map $u\colon * \to A$ which satisfy the usual commutative diagrams for associativity, unitality and the inverse (cf. [ML98, III.6] or [Bor94b, 3]). A third equivalent way would be to define a simplicial abelian groups $A_\bullet$ as a sequence of abelian groups $A_0, A_1, \dots$ together with group homomorphisms α^* for all $\alpha \in \Delta$.

Note that as *sAb* is a functor category it contains all (small) limits and colimits and they are formed degreewise as limit and colimits in the category of abelian groups. As the forgetful functor respects limits, in particular the product $A \times B$ of two simplicial abelian groups is formed by the product in sets.

The underlying simplicial set of a simplicial abelian group is Kan. The notion of homotopy between simplicial abelian groups is different from the notion of homotopy of simplicial sets, as $A \times \Delta^1$ does not have a canonical abelian group structure if A has one. If X is a simplicial set we can form the simplicial abelian group $\mathbb{Z}[X]$ which is defined as the composition of the functor $X\colon \Delta^{\mathrm{op}} \to \mathit{Set}$ with the free abelian group functor $\mathbb{Z}[-]\colon \mathit{Set} \to \mathit{Ab}$. It is the left-adjoint to the forgetful functor.

We can form the tensor product $A \otimes_{\mathbb{Z}} B$ of two simplicial abelian groups A and B by taking in degree n the tensor product of groups $A_n \otimes_{\mathbb{Z}} B_n$ and letting the structure maps act diagonally. It makes *sAb* into a closed symmetric monoidal category (see [ML98, VII.7] and [Bor94b, 6.1] for a definition). We therefore can define the simplicial abelian group $A[X] := A \otimes_{\mathbb{Z}} \mathbb{Z}[X]$ for a simplicial abelian group

A and a simplicial set X. In [GJ99] this construction is denoted by $A \otimes X$ (e.g. in II.2), but the author believes our notation is more convenient. A homotopy of maps of simplicial abelian groups is then a map $A[\Delta^1] \to B$.

A *simplicial ring* is an object in $s\mathcal{R}ings$. There are two additional equivalent descriptions. First, a simplicial ring $R_\bullet$ is a sequence of rings $R_0, R_1, \dots$ together with the usual simplicial structure maps α^* which additionally are maps of rings (with unit). The second point of view is that a simplicial ring is a monoid in the symmetric monoidal category of simplicial abelian groups.

A (left) module M over a simplicial ring R is a simplicial abelian group M together with a bilinear map $\mu\colon R \times M \to M$ which satisfies the associativity and unit diagrams similar to the ones for simplicial abelian groups. As the map is bilinear it can equivalently be given by a linear map $R \otimes_{\mathbb{Z}} M \to M$ of simplicial abelian groups. The category of (left) R-modules is in general not a category of simplicial objects (for some category $\mathcal{C}$), so we have only one further equivalent formulation: A simplicial R-module M is a sequence of abelian groups $M_0, M_1, \dots$ such that each M_i is an R_i-module and structure maps α^* for $\alpha \in \Delta$ on M which respect addition and the R-multiplication.

If R is an “ordinary” ring we can make it into a simplicial ring $R_\bullet$ by setting $R_i := R$ and $\alpha^* := \mathrm{id}$ for all $\alpha \in \Delta$. This is discrete as simplicial set. Usually we do not distinguish between R and $R_\bullet$ and call both *discrete rings*. Similar if R is discrete and M is an “ordinary” R-module then we get a simplicial R-module $M_\bullet$ by setting $M_i := M$ and $\alpha^* := \mathrm{id}$. We call such R-modules also *discrete* and do not distinguish between M and $M_\bullet$. Note that if M is a discrete R-module it necessarily implies that R is a discrete ring. To emphasize that R is a simplicial ring and we sometimes say the category of module over R the “simplicial R-modules”.

Again simplicial R-modules have all (small) limits and colimits. They are formed in simplicial abelian groups, hence degreewise in abelian groups. This follows in the same way it follows for discrete R-modules, or, if one likes fancy language, from the fact the simplicial R-modules are modules over a monoid in simplicial abelian groups (see e.g. [Bor94b, 4.3] for the general theory, in particular Propositions 4.3.1 and 4.3.2).

B. On categories with cofibrations and weak equivalences

We recall briefly the results from [Wal85] which we need. A good summary can also be found in Section 2 of [CP97]. An extensive resource is also [TT90]. Most definitions are repeated at the places where they are needed for the first time.

Definition B.1 (Category with Cofibrations [Wal85, 1.1])**.** *A category with cofibrations is a pointed category $\mathcal{C}$ together with a subcategory* $\mathrm{co}\mathcal{C}$*, the cofibrations, such that the following axioms hold.*

(i) *Isomorphisms are in* $\mathrm{co}\mathcal{C}$*.*

(ii) *All maps* $* \to A$ *are in* $\mathrm{co}\mathcal{C}$*.*

(iii) *Cofibrations admit cobase change, i.e. if* $A \rightarrowtail B$ *is a cofibration and* $A \to C$ *any map then the pushout*

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

exists in $\mathcal{C}$ *and the map* $C \to D$ *is again a cofibration.*

Here "$\rightarrowtail$*" denotes a cofibration and "*$*$*" denotes the zero object.*

Definition B.2 (Category of weak equivalences [Wal85, 1.2])**.** *Let* $\mathcal{C}$ *be a category with cofibrations. A category of weak equivalences in* $\mathcal{C}$ *is a subcategory* $w\mathcal{C}$ *satisfying the following axioms.*

(i) $w\mathcal{C}$ *contains all isomorphisms of* $\mathcal{C}$*.*

(ii) *(Gluing lemma). If we have the diagram*

$$\begin{tikzcd}
B \arrow[d] & A \arrow[l, tail] \arrow[r] \arrow[d] & C \arrow[d] \\
B' & A' \arrow[l, tail] \arrow[r] & C'
\end{tikzcd}$$

with $A \rightarrowtail B$ *and* $A' \rightarrowtail B'$ *cofibrations and all three vertical arrows are in* $w\mathcal{C}$*, then the induced map*

$$B \cup_A C \to B' \cup_{A'} C'$$

on the pushouts is also in $w\mathcal{C}$*.*

Weak equivalences are denoted by “$\xrightarrow{\sim}$*”.*

A functor $F\colon \mathcal{A} \to \mathcal{B}$ between categories with cofibrations is *exact* if it takes cofibrations to cofibrations, the zero object to the zero object and respects pushouts along cofibrations. A functor between categories with cofibrations and weak equivalences is *exact* if it additionally takes weak equivalences to weak equivalences.

The weak equivalences satisfy the *Saturation Axiom* if for a, b composable maps in $\mathcal{C}$ and two of a, b and $a \circ b$ are weak equivalences, then so is the third.

If $A \rightarrowtail B$ is a cofibration, denote the pushout along $A \to *$ by A/B. Call $B \twoheadrightarrow A/B$ the *quotient* map. Then $A \to B \to B/A$ is a called a *cofiber sequence*. (Here we deviate slightly from [Wal85] where the term “cofibration sequence” is used.)

The category $\mathcal{C}$ is said to satisfy the *Extension Axiom* if for a map of cofiber sequences

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d] & B \arrow[r, two heads] \arrow[d] & C \arrow[d] \\
A' \arrow[r, tail] & B' \arrow[r, two heads] & C'
\end{tikzcd}$$

in $\mathcal{C}$ the maps $A \to A'$ and $C \to C'$ are weak equivalences, then the map $B \to B'$ is a weak equivalence.

Let $\mathcal{C}$ be a category with cofibrations. Let $\mathrm{Ar}\mathcal{C}$ be the category of arrows in $\mathcal{C}$ with morphisms $(A \to C) \to (B \to D)$ being the commutative squares

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}.$$

A morphism is a called a cofibration in $\mathrm{Ar}\mathcal{C}$ if $A \to B$ and $C \to D$ are cofibrations in $\mathcal{C}$. This makes $\mathrm{Ar}\mathcal{C}$ into a category with cofibrations by [Wal85, 1.1.1]. Let $\mathcal{F}_1\mathcal{C}$ be the full subcategory of $\mathrm{Ar}\mathcal{C}$ with objects being the cofibrations. A cofibration in $\mathcal{F}_1\mathcal{C}$ is a commutative square as above (where than $A \to C$ and $B \to D$ are cofibrations) such that $A \to B$ and the induced map $B \cup_A C \to D$ are cofibrations. This definition makes $\mathcal{F}_1\mathcal{C}$ into a category with cofibrations by [Wal85, 1.1.1].

Definition B.3 (Cylinder Functor [Wal85, 1.6]). *A Cylinder Functor on $\mathcal{C}$ is a functor from $\mathrm{Ar}\mathcal{C}$ to diagrams in $\mathcal{C}$ which takes a map $f\colon A \to B$ to a diagram*

$$\begin{tikzcd}
A \arrow[r, "\iota_0"] \arrow[dr, "f"'] & T(f) \arrow[d, "p"] & B \arrow[l, "\iota_1"'] \arrow[dl, "\mathrm{id}"] \\
 & B &
\end{tikzcd}$$

Here ι_0 is called the front inclusion*, ι_1 is called the* back inclusion *and p is called the* projection*. Further the following two axioms should be satisfied.*

(i) (Cyl 1) Front and back inclusion assemble to an exact functor

$$\begin{gathered}\mathrm{Ar}\mathcal{C} \longrightarrow \mathcal{F}_1\mathcal{C}\\ f \mapsto \big(\iota_0 \vee \iota_1 \colon A \vee B \rightarrowtail T(f)\big).\end{gathered}$$

(ii) (Cyl 2) $T(* \to A) = A$ *for every* $A \in \mathcal{C}$*, further the projection and the back inclusion are the identity map on* A*.*

T *is said to satisfy the Cylinder Axiom if the projection* $T(f) \to B$ *is in* $w\mathcal{C}$ *for every* $f\colon A \to B$ *in* $\mathcal{C}$*.*

We turn to the definition of algebraic K-theory.

Definition B.4 (Algebraic K-theory of a category with cofibrations and weak equivalences [Wal85, 1.3])**.** *For any category with cofibrations and weak equivalences* $\mathcal{C}$ *define the algebraic* K*-theory of* $\mathcal{C}$ *as the space*

$$K(w\mathcal{C}) := \Omega|w\mathcal{S}_\bullet\mathcal{C}|$$

where $\mathcal{S}_\bullet$ *is Waldhausen's construction from* [Wal85, 1.3]*,* w *denotes the class of weak equivalences,* $|-|$ *is the nerve of a simplicial category (i.e. the realization of a bisimplicial set) and* Ω *the loop space.*

We do not discuss the construction as we only need the following theorems. There is a canonical connective delooping of $K(w\mathcal{C})$, so it is an infinite loop space. We could also vary the definition to define $K(w\mathcal{C})$ as the corresponding spectrum. The latter point of view is taken in [TT90]. We regard it as a space here.

Let $\mathcal{C}$ be a category with cofibrations which has two classes of weak equivalences $v\mathcal{C}$ and $w\mathcal{C}$ with $v\mathcal{C} \subseteq w\mathcal{C}$. Denote by $\mathcal{C}^w$ the full subcategory of $\mathcal{C}$ with objects A such that $* \to A$ is in $w\mathcal{C}$. It inherits the structure of a category with cofibrations and weak equivalences.

Theorem B.5 (Fibration Theorem [Wal85, 1.6.4])**.** *If* $\mathcal{C}$ *has a Cylinder Functor and the category of weak equivalences* $w\mathcal{C}$ *satisfies the Cylinder Axiom, the Saturation Axiom and the Extension Axiom then there is a homotopy fiber sequence*

$$K(v\mathcal{C}^w) \to K(v\mathcal{C}) \to K(w\mathcal{C})$$

on algebraic K*-theory spaces.*

The result in Thomason-Trobaugh [TT90, 1.8.2] state this as a homotopy fiber sequence of spectra.

Definition B.6 (Approximation Property [Wal85, 1.6],[TT90, 1.9.1])**.** *Let* $F\colon \mathcal{A} \to \mathcal{B}$ *be an exact functor of categories with cofibrations and weak equivalences.* F *has the Approximation Property if the following two axioms hold.*

(i) (App 1) A map f *in* $\mathcal{A}$ *is a weak equivalence if (and only if) its image* $F(f)$ *in* $\mathcal{B}$ *is a weak equivalence.*

(ii) (App 2) Given any object A in $\mathcal{A}$ and a map $x\colon F(A) \to B$ in $\mathcal{B}$ there exists a map $a\colon A \to A'$ in $\mathcal{A}$ and a weak equivalence $x'\colon F(A') \to B$ in $\mathcal{B}$ such that the triangle

$$\begin{tikzcd}
F(A) \arrow[r, "x"] \arrow[d, "F(a)"'] & B \\
F(A') \arrow[ur, "x'"'] &
\end{tikzcd}$$

commutes.

Theorem B.7 (Approximation Theorem [Wal85, 1.6.7],[TT90, 1.9.1])**.** *Let $\mathcal{A}$, $\mathcal{B}$ be categories with cofibrations and weak equivalences which satisfy the Saturation Axiom. Assume $\mathcal{A}$ has a Cylinder Functor satisfying the Cylinder Axiom. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor with the Approximation Property. Then F induces an equivalence*

$$K(F)\colon K(w\mathcal{A}) \to K(w\mathcal{B})$$

on algebraic K-theory spaces.

We used Thomason-Trobaugh's remark in [TT90, 1.9.1] that we can use a weaker version of the approximation property. In [Wal85] there is the further requirement in (App 2) that a is a cofibration, which we can always be arrange due to the existence of a Cylinder Functor.

C. Homotopy idempotents and mapping telescopes in the simplicial setting

A map $\eta\colon K \to K$ in $\mathcal{C}_a^G$ is called a *homotopy idempotent* if η^2 is homotopic to η. Here we provide the necessary tools we need about homotopy idempotents. The only place where we need this theory is the proof of Lemma 5.10, however this is a crucial step there and the lemma itself is the important step to establish the homotopy fiber sequence of Section 5.1. (We defined the category with cofibrations and weak equivalences $\mathcal{C}_a^G = \mathcal{C}_a^G(X, R, \mathcal{E}, \mathcal{F})$ for a control space $(X, \mathcal{E}, \mathcal{F})$ and a simplicial ring R in Section 3.1.)

Some parts of the following proposition need an extra assumption on the idempotent.

Definition C.1. *A homotopy idempotent $\eta\colon K \to K$ with homotopy H from η^2 to η is called coherent if there is a map $G\colon K[\Delta^1 \times \Delta^1] \to K$ whose restrictions to the boundary look as in the following diagram*

$$\begin{tikzcd}
\bullet \arrow[r, "\eta\circ H"] \arrow[d, "H\circ\eta[\Delta^1]"'] \arrow[dr] & \bullet \arrow[d, "H"] \\
\bullet \arrow[r, "H"'] & \bullet
\end{tikzcd} .$$

We show that coherent homotopy idempotents split up to homotopy in $\mathcal{C}_a^G$, this is proved as Corollary C.5. Note that in the Definition we just the diagram language we described in Digression 2.4.1.

Lemma C.2. *If η arises from a homotopy domination, as in the case of Lemma 5.10, then it is coherent.*

Proof. So assume $\eta = p \circ i$ and $i \circ p$ is homotopic to id via H', $i\colon K \to L$, $p\colon L \to K$. Then the coherence homotopy G can be given by the composition

$$K[I \times I] \cong L[I][I] \xrightarrow{i[I][I]} L[I][I] \xrightarrow{H'[I]} L[I] \xrightarrow{H'} K \xrightarrow{p} K. \qquad \square$$

Remark C.3. The author does not know if every homotopy idempotent in $\mathcal{C}_a^G$ is coherent. For the topogical case it is known that there are unpointed homotopy idempotents of infinite-dimensional CW-complexes which do not split, however every pointed homotopy idempotent as well as every homotopy idempotent of finite-dimensional CW-complexes splits. See [HH82] for the last result and further references.

The results we show in this appendix are summarized in the following Proposition.

Proposition C.4. *Let $\eta\colon K \to K$ be a homotopy idempotent in $\mathcal{C}_a^G(X)$. There is a construction* $\mathrm{Tel}(-)$ *which assigns to any homotopy idempotent η an object* $\mathrm{Tel}(\eta)$ *in $\mathcal{C}_a^G(X)$. It has the following properties.*

(i) *There is a cellular inclusion $\iota\colon K \rightarrowtail \mathrm{Tel}(\eta)$*

(ii) *Let*

$$\begin{tikzcd} A \arrow[r, "\mu"] \arrow[d, "f"] & A \arrow[d, "f"] \\ K \arrow[r, "\eta"] & K \end{tikzcd}$$

be a strict commutative diagram of homotopy idempotents. Then f induces a map $f_*\colon \mathrm{Tel}(\mu) \to \mathrm{Tel}(\eta)$. This is functorial in f. In particular if f is an isomorphism then $\mathrm{Tel}(f)$ is isomorphism.

(iii) *If η, μ are homotopic homotopy idempotents then there is a homotopy equivalence*

$$\mathrm{Tel}(\eta) \xrightarrow{\simeq} \mathrm{Tel}(\mu).$$

(iv) *Consider the telescope $\mathrm{Tel}(\mathrm{id}_K)$ of the homotopy idempotent $\mathrm{id}_K\colon K \to K$. There is a map*

$$\mathrm{Tel}(\mathrm{id}_K) \to K$$

which is a homotopy equivalence.

(v) *All maps in (ii) to (iv) are relative to $\iota\colon K \to \mathrm{Tel}(\eta)$, i.e. they commute with this cellular inclusion.*

(vi) *From (ii) we get for $\mu = \eta = f$ an induced map $\eta_*\colon \mathrm{Tel}(\eta) \to \mathrm{Tel}(\eta)$. This map is a homotopy equivalence. If η is coherent, η_* is homotopic to* id.

(vii) *If η is coherent then there is a map $c\colon \mathrm{Tel}(\eta) \to K$ such that $\iota \circ c$ is homotopic to $\eta_*\colon \mathrm{Tel}(\eta) \to \mathrm{Tel}(\eta)$ and hence to the identity on $\mathrm{Tel}(\eta)$. Therefore $\mathrm{Tel}(\eta)$ is a homotopy retract of K. Further $c \circ \iota$ is homotopic to η itself.*

The proof of this proposition takes the rest of this appendix. A summarizing proof, which lists the lemmas where each part is proved, is given at the end of the appendix.

We can draw a direct corollary which says that coherent homotopy idempotents split up to homotopy in $\mathcal{C}_a^G$.

Corollary C.5. *Let $\eta\colon K \to K$ be a coherent homotopy idempotent in $\mathcal{C}_a^G$. Then there is a $B \in \mathcal{C}_a^G$ such that K is homotopy equivalent to $\mathrm{Tel}(\eta) \vee B$. Moreover under* *this equivalence* η *corresponds to the projection* $\mathrm{pr}\colon \mathrm{Tel}(\eta)\vee B \to \mathrm{Tel}(\eta) \to \mathrm{Tel}(\eta)\vee B$*, i.e. there is a homotopy commutative diagram*

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d, "\eta"] & \mathrm{Tel}(\eta) \vee B \arrow[d, "\mathrm{pr}"] \\
K \arrow[r, "f"] & \mathrm{Tel}(\eta) \vee B
\end{tikzcd}$$

where f *is the homotopy equivalence* $K \xrightarrow{\simeq} \mathrm{Tel}(\eta) \vee B$.

Proof. We know by C.4 (vii) that $\mathrm{Tel}(\eta)$ is a homotopy retract of K. Using Lemma 2.35 we can make the homotopy commutative diagram

$$\begin{tikzcd}
\mathrm{Tel}(\eta) \arrow[r, "c"] \arrow[dr, "\mathrm{id}"'] & K \arrow[d, "\iota"] \\
& \mathrm{Tel}(\eta)
\end{tikzcd} \quad \text{into} \quad \begin{tikzcd}
\mathrm{Tel}(\eta) \arrow[r, tail, "\mathrm{inc}"] \arrow[dr, "\mathrm{id}"'] & T(c) \arrow[d] \\
& \mathrm{Tel}(\eta)
\end{tikzcd},$$

which is a strict commutative diagram, where $T(c)$ is the mapping cylinder of c and inc is a cellular inclusion. Take the cofiber of inc and call it B. The sum of the retraction $T(c) \to \mathrm{Tel}(\eta)$ and the quotient map $T(c) \to B$ gives a map $s\colon T(c) \to \mathrm{Tel}(\eta) \vee B$ (using that $\mathcal{C}^G$ is an additive category and hence twofold products and coproducts agree, cf. the proof of 3.33(ii)). The map makes the diagram of cofiber sequences

$$\begin{tikzcd}
\mathrm{Tel}(\eta) \arrow[r, tail] \arrow[d] & T(c) \arrow[r, two heads] \arrow[d, "s"] & B \arrow[d] \\
\mathrm{Tel}(\eta) \arrow[r, tail] & \mathrm{Tel}(\eta) \vee B \arrow[r, two heads] & B
\end{tikzcd}$$

commutative and the Extension Axiom 2.48 shows that s is a homotopy equivalence. This gives the homotopy equivalence $f\colon K \to T(c) \to \mathrm{Tel}(\eta) \vee B$.

By C.4 (vii) the map $\eta\colon K \to K$ factorizes up to homotopy as $c \circ i\colon K \to \mathrm{Tel}(\eta) \to K$. Hence the upper triangle in

$$\begin{tikzcd}
K \arrow[r, "\eta"] \arrow[d, "\iota"'] & K \arrow[d] & \\
\mathrm{Tel}(\eta) \arrow[r, tail] \arrow[ur, "c"] & T(c) \arrow[r, two heads] & B
\end{tikzcd}$$

is homotopy commutative, whereas the lower one commutes strictly. It follows that $K \xrightarrow{\eta} K \to T(c) \to B$ is homotopic to the zero map.

Further $K \to T(c) \to \mathrm{Tel}(\eta)$ equals ι, hence by adding homotopies the map

$$f \circ \eta\colon K \to K \xrightarrow{\simeq} T(c) \xrightarrow{\simeq} \mathrm{Tel}(\eta) \vee B$$

is homotopic to $K \xrightarrow{\iota} \mathrm{Tel}(\eta) \rightarrowtail \mathrm{Tel}(\eta) \vee B$. As

$$\begin{tikzcd}
K \arrow[r, "\iota"] \arrow[d, "\eta"'] & \mathrm{Tel}(\eta) \vee B \arrow[d, "\eta_* \vee 0_B"] \\
K \arrow[r, "\iota"] & \mathrm{Tel}(\eta) \vee B
\end{tikzcd}$$

is strictly commutative by C.4 (ii) and (v), where 0_B denotes the zero map on B, and as $\eta_* \simeq \mathrm{id}$ by C.4 (vi) it follows that

$$\begin{tikzcd}
K \arrow[r, "\simeq", "f"'] \arrow[d, "\eta"] & \mathrm{Tel}(\eta) \vee B \arrow[d, "\mathrm{id} \vee 0_B"] \\
K \arrow[r, "\simeq", "f"'] & \mathrm{Tel}(\eta) \vee B
\end{tikzcd}$$

is homotopy commutative and the claim follows. □

We would like to give a quick and direct definition of Tel(η) and show Proposition C.4 with this. Unfortunately during the proof of (iii) to (vii) we need at some point that squares which commute only up to homotopy also induce maps on telescopes. However this can be arranged only up to homotopy, which in our case involves invoking the horn-filling property of Section 2.4. But this usually involves choices, which might destroy a functoriality we need in the proof. Our way around this is to invoke the horn-filling property only in the very last step and in between work with "long homotopies", which we will define below. In the category of topological spaces these kind of homotopies are sometimes called "Moore homotopies" and amount to replace the interval $[0, 1]$ involved in the definition of homotopy by intervals $[0, n]$, for $n \in \mathbb{N}$. There they make concatenation of homotopies strictly associative.

If we work simplicially we have altogether two kind of difficulties. First concatenation of homotopies usually involves using the horn-filling property and hence is not canonical or even functorial and second, homotopies have a direction. But still we could formally concatenate homotopies and even find a simplicial set I such that the concatenated homotopies give a map $A[I] \to B$ if each homotopy was a map $A[\Delta^1] \to B$ (which may go "from 1 to 0"). This makes "concatenation" of homotopies strictly associative and we can still invoke horn-filling to get an "ordinary" homotopy back.

The technical tools for this are provided in the next section. In the section after that we discuss the theory of mapping telescopes we need.

Note that we always work in the category $\mathcal{C}_a^G$ here, but the theory seems to work much more general. However the does not seem to exist an established general framework including the category $\mathcal{C}_a^G$ we are interested in, so we refrain from stating the results in that generality.

Remark C.6. For the interested reader let us give a short elaboration on the last remark. The point is that the only general structure $\mathcal{C}_a^G$ has is that of a category with cofibrations and weak equivalences with some additional properties, and there does not seem to be a suitable established notion which implies the properties we need. We list the main properties of $\mathcal{C}_a^G$ which we use.

First the category $\mathcal{C}_a^G$ is a category with cofibrations. It has an operation "adjoining a simplicial set" $\mathcal{C}_a^G \times sSet \to \mathcal{C}_a^G$, which constructs from a module M and a simplicial set A a module $M[A]$ and the "Fundamental Lemma" (Section 1.4) holds, which is a weaker than the property than that $A \mapsto M[A]$ is a left adjoint. We have (relative) horn-filling for this construction (Lemma 2.22), and the Cylinder Functor is constructed from $A[\Delta^1]$. Further the weak equivalences are the homotopy equivalences with respect to this notion of cylinder. This makes $\mathcal{C}_a^G$ into a category with cofibrations and weak equivalences and it satisfies the Saturation Axiom, the Cylinder Axiom and the Extension Axiom.

Further we need for the actual construction of the telescope that for a module $M \in \mathcal{C}_a^G$ we can form the countable coproduct $\coprod_{\mathbb{N}} M$.

We now give a quick definition of $\mathrm{Tel}(\eta)$ and sketch Proposition C.4 (i) and (ii) for the convenience of the reader, before we go to the general theory. The full proof of Proposition C.4 will be given in Section C.2

Definition C.7 (Mapping Telescope, simple version)**.** *Let $f\colon A \to A$ be a map in $\mathcal{C}_a^G$. We defined the mapping cylinder $T(f)$ as the canonical pushout (cf. 2.12)*

$$\begin{tikzcd}
A[1] \arrow[r, tail] \arrow[d, "f"'] & A[\Delta^1] . \arrow[d] \\
A \arrow[r] & T(f)
\end{tikzcd}$$

It comes with two cellular maps $\iota_{0,1}\colon A \to T(f)$, the front and back inclusion. We can form the pushout

$$\begin{tikzcd}
\coprod_{i=0}^{\infty} A \amalg \coprod_{i=1}^{\infty} A \arrow[r, "\iota_1 \amalg \iota_0"] \arrow[d, "c"] & \coprod_{i=0}^{\infty} T(f) , \arrow[d] \\
\coprod_{i=1}^{\infty} A \arrow[r] & \mathrm{Tel}(f)
\end{tikzcd}$$

where ι_1 is the back inclusion of A into $T(f)$, ι_0 the front inclusion, both from the ith summand to the ith summand, and c is the map which maps the ith summand of the first coproduct and the $(i+1)$th summand of the second coproduct to the $(i+1)$th summand of the target.

We call $\mathrm{Tel}(f)$ *the telescope of f.*

Remark C.8. Note that the upper horizontal map is a cellular inclusion, so the pushout exists by Lemma 2.2. We also used that we the countable coproducts above exist in $\mathcal{C}_a^G$. Further note that we can write the pushout as the coequalizer

$$\coprod_{i=0}^{\infty} A \rightrightarrows \coprod_{i=0}^{\infty} T(f) \to \mathrm{Tel}(f),$$

where the two maps are the back inclusions and the front inclusions into the "next" summand.

Proof of Proposition C.4 (i)-(ii). Let $\eta\colon K \to K$ be a homotopy idempotent. Take as inclusion $\iota\colon K \to \mathrm{Tel}(\eta)$ the composition of the front inclusion of K into the first summand $T(\eta)$ together with the quotient map to $\mathrm{Tel}(\eta)$. (This the only one of the front/back inclusions we have not used in the construction of $\mathrm{Tel}(\eta)$.)

For (ii) the map f gives a commutative diagram

$$\begin{tikzcd}
A[\Delta^1] \arrow[d, "{f[\Delta^1]}"'] & A \arrow[l, hook', "\iota_1"'] \arrow[r, "\mu"] \arrow[d, "f"'] & A \arrow[d, "f"'] \\
A[\Delta^1] & A \arrow[l, hook', "\iota_1"'] \arrow[r, "\eta"] & A
\end{tikzcd}$$

whose row-wise pushout gives a map $T(\mu) \to T(\eta)$. This glues together to a map $f_*\colon \mathrm{Tel}(\mu) \to \mathrm{Tel}(\eta)$, which additionally is compatible with ι. □

For the rest we have to talk about "long homotopies".

C.1. Some simplicial tools

For this and the next section we will work in the category with cofibrations and weak equivalences $\mathcal{C}_a^G$ from Section 3.1. See Remark C.6 for a rough list of the properties we will use.

We start by defining what we mean by an *interval* in the category of simplicial sets. This is no common notion there, but it is convenient for us to stress the analogies to the topological setting.

Definition C.9. *Let* $i \in \mathbb{N}$*. The one-point simplicial set* $\underline{i}$*, (*$\underline{i}_k = \{i\}$*) is called a point at* i *or interval of length* 0 *from* i *to* i*.*

A simplicial set $I(i, i+1)$ *together with a bijection of its zero simplices* $l\colon I(i,i+1)_0 \to \{i, i+1\}$ *is called an interval of length* 1 *from* i *to* $(i+1)$ *if* $I(i,i+1)$ *is isomorphic to* Δ^1 *as simplicial set. The map* l *is called the labeling.*

Let $i, j \in \mathbb{N}$*,* $i + 2 \leq j$*. Defined recursively, a simplicial set* $I(i,j)$ *together with a bijection* $l\colon I(i,j)_0 \to \{i, i+1, \dots, j\}$ *is called an interval of length* $(i-j)$ *from* i *to* j *if* $I(i,j)$ *is a pushout*

$$\begin{tikzcd}
\underline{j-1} \arrow[r] \arrow[d] & I(j-1,j) \arrow[d] \\
I(i,j-1) \arrow[r] & I(i,j)
\end{tikzcd} \tag{30}$$

where $\underline{j-1}$ *is a point at* $(j-1)$ *and* $\underline{j-1} \to I(j-1,j)$ *is the map "inclusion of the point at* $(j-1)$*" which is determined by mapping the only zero simplex of* $\underline{j-1}$ *to the zero simplex in* $I(j-1,j)$ *labeled by* $j-1$*, similar for* $\underline{j-1} \to I(i,j-1)$*. Further require the labeling* l *to be compatible with this pushout.*

The standard interval from i *to* $(i+1)$ *is the simplicial set* Δ^1 *together with the labeling* $l(0) = i$*,* $l(1) = i+1$*. The standard interval from* i *to* j *for* $i+2 \leq j$ *is the simplicial set arising from the standard interval from* i *to* $j-1$ *by the pushout (30) with* $I(j-1,j)$ *being the standard interval of length* 1*.*

An interval $I(i,j)$ *from* i *to* j *is called* ordered *if it is isomorphic to the standard interval from* i *to* j *and the isomorphism respects the labeling.*

Remark C.10. We like to draw pictures for intervals, e.g. the picture for the standard interval of length 1 is $0 \to 1$.

The standard interval is an interval, but for given i and j there are 2^{j-i} different simplicial sets up to isomorphism which are intervals from i to j. The four ones for $I(0,2)$ are for example

$$\begin{array}{ll} 0 \to 1 \to 2 & \qquad 0 \to 1 \leftarrow 2 \\ 0 \leftarrow 1 \to 2 & \qquad 0 \leftarrow 1 \leftarrow 2 \end{array}$$

This ambiguity is intentional, as we want to allow all those cases. We will see later that in the presence of the Kan condition this does not matter much. Note that the notion of an ordered interval is unambiguous, the upper left interval above is the only ordered interval of the four examples.

We often just write $I(i,j)$ for an interval from i to j leaving all the other data understood. For $A \in \mathcal{C}_a^G$ we also often abbreviate $A[I(i,j)]$ as $A[i,j]$ and $A[\underline{i}]$ as $A[i]$, slightly misusing notation.

We can extend the definition of an interval to the case $j = \infty$.

Definition C.11. *Define an simplicial set* $I(i,\infty)$ *to be an interval from* i *to* ∞ *if it is the filtered colimit (or union) of intervals* $I(i,j)$ *for* $j \to \infty$*. It is called* ordered *if each of the* $I(i,j)$ *is.*

Remark C.12. We will mostly need this notion for ordered intervals from 0 to ∞. To visualize it, its topological realization can be identified with $\mathbb{R}_{\geq 0}$.

C.1.1. Long homotopies

We defined a homotopy as a map $A[\Delta^1] \to B$ in Section 2.4. However, using the notion of interval we just introduced we can generalize it as follows.

Definition C.13. *Let* $I(0,j)$ *be an interval from* 0 *to* j*. Let* $f_0, f_j \colon A \to B$ *be two maps in* $\mathcal{C}_a^G$*. A (long) homotopy from* f_0 *to* f_j *is a map* $H \colon A[I(0,j)] \to B$ *such that the restriction to* $A[0]$ *is* f_0 *and the restriction to* $A[j]$ *is* f_j*. We say that* H *has the length* j*.*

Example C.14. If $f\colon A \to B$ is a map in $\mathcal{C}_a^G$ and $I(0,i)$ any interval we always have the *constant* on *trivial homotopy* $\mathrm{Tr}\colon A[0,i] \to B$ induced by the map $A[0,i] \to A \to B$. We also define it for $i = 0$ and therefore call the map $\mathrm{Tr}\colon A[0,0] = A[0] = A \xrightarrow{f} B$ the trivial homotopy of length 0.

Remark C.15. Usually we omit the "long". We chose to let the homotopies start at zero, as this will make the definition of concatenation of homotopies below easier.

Of course each homotopy in the usual sense is a long homotopy, but for the converse we need the horn-filling property. Namely, by the Kan property each long homotopy induces a homotopy $A[\Delta^1] \to B$, if $i, j \in \mathbb{N}$. However, this is not unique or functorial, at least a priori. This is the reason we need the generalized notion of an interval. It makes the next construction functorial.

C.1.2. Concatenation of homotopies

If $I(0,i)$ and $I(0,j)$ are intervals we define the concatenation $I(0,i) \square I(0,j)$ as the pushout

$$\begin{tikzcd}
\underline{i} \arrow[r] \arrow[d] & I(i,i+j) \arrow[d] \\
I(0,i) \arrow[r] & I(0,i) \square I(0,j)
\end{tikzcd}$$

where $I(i,i+j)$ is defined as a "relabeling" of $I(0,j)$, namely replace the labeling l of $I(0,j)$ by $l(k) = i+k$. This gives a strictly associative, non-commutative construction for intervals.

Homotopies which agree on the start resp. endpoint can be concatenated. For $H_1\colon A[0,i] \to B$, $H_2\colon A[0,j] \to B$ with $H_{1|A[i]} = H_{2|A[0]}$ define the *concatenation*

$$H_1 \square H_2\colon A[0,i+j] \to B$$

as the map induced by the identification on the pushout $I(0,i) \square I(0,j)$. The concatenation of homotopies is strictly associative.

Remark C.16. Note that concatenation of two homotopies adds the lengths. Note further that the (constant) homotopy of length 0 acts as a neutral element with respect to concatenation of homotopies.

We do not have strict inverses for homotopies, but sometimes we have to consider for $H\colon A[I(0,j)] \to B$ the *inverse homotopy* $\overline{H}$, where $I(0,j)$ is some interval. For this define the *reversed interval* $\overline{I(0,j)}$ of $I(0,j)$ as the same simplicial set with the reverse labeling, namely l is replaced by $\bar{l}(k) := j - l(k)$. Then define the inverse homotopy as the obvious map $\overline{H}\colon A[\overline{I(0,j)}] \to B$.

If $j = 1$ and $I(0,j)$ is an ordered interval we draw the homotopy as $\xrightarrow{H}$ and the inverse homotopy as $\xleftarrow{H}$. (Note that we do not need to use the decoration $\overline{H}$.) The following lemma is a prototype for the arguments used later.

Lemma C.17. *Let $H\colon A[0,1] \to B$ be a homotopy. The concatenation $H \mathbin{\Box} \overline{H}$ is homotopic, relative boundary, to the constant (or trivial) homotopy* $\mathrm{Tr}\colon A[0,2] \to A \to B$.

Proof. (This is also a prototype for the proofs used later. Cf. the Digression 2.4.1 for the diagram language we use.) Assume that $I(0,1)$ is the standard interval, the other case proceeds similar.

We have to give a homotopy $A[0,2][\Delta^1] \to B$. Therefore we have to give maps $A[\Delta^2] \to B$ which fit together as shown in the picture of $[0,2] \times \Delta^1$ below. But this can be done easily, as the 2nd degeneracy map $\Delta^2 \to \Delta^1$ provides a map as in the picture below on the right. The whole map arises by gluing these together.

$$\begin{tikzcd}
\bullet \arrow[r, "\mathrm{Tr}"] \arrow[d, "\mathrm{Tr}"] \arrow[dr, "H"] & \bullet \arrow[d, "H"] & \bullet \arrow[l, "\mathrm{Tr}"'] \arrow[d, "\mathrm{Tr}"] \arrow[dl, "H"'] \\
\bullet \arrow[r, "H"] & \bullet & \bullet \arrow[l, "H"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& \bullet \arrow[dr, "H"] & \\
\bullet \arrow[ur, "\mathrm{Tr}"] \arrow[rr, "H"] & & \bullet
\end{tikzcd}$$

□

Remark C.18. To avoid citing the Kan Extension property we actually gave a homotopy *from* the constant homotopy *to* the given one. This is a trick we will use later on several times. If we a willing to use the Kan Extension property we could proceed as follows to get a homotopy in the "right" direction. Use the Kan property to extend the left cone to the right 2-simplex

$$\begin{tikzcd}
& \bullet \arrow[dr, "H"] & \\
\bullet \arrow[rr, "\mathrm{Tr}"] & & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
& \bullet \arrow[dr, "H"] & \\
\bullet \arrow[ur, "\widetilde{H}"] \arrow[rr, "\mathrm{Tr}"] & & \bullet
\end{tikzcd}
\quad .$$

Then $\widetilde{H}$ is a homotopy inverse to H in the "old sense". This allows to construct a diagram

$$\begin{tikzcd}
\bullet \arrow[r, "\mathrm{Tr}"] & \bullet & \bullet \arrow[l, "\mathrm{Tr}"'] \\
\bullet \arrow[u, "\mathrm{Tr}"] \arrow[ur, "\mathrm{Tr}"] \arrow[r, "H"] & \bullet \arrow[u, "\widetilde{H}"] & \bullet \arrow[l, "H"'] \arrow[ul, "\mathrm{Tr}"'] \arrow[u, "\mathrm{Tr}"']
\end{tikzcd}$$

giving the desired homotopy. Another way to get a homotopy in the right direction is to cite the Kan Extension property on the resulting homotopy of homotopies from above.

Similar diagrams show that $\overline{H} \mathbin{\Box} H$ is homotopic to the trivial homotopy.

The lemma (including the remark) also holds if we allow an interval of length n instead of length 1. To prove this one does induction over n and start building the homotopy from the middle, patching in in each induction step some of four trivial homotopies of the remaining homotopies H' of length 1, like

$$\begin{tikzcd}
\bullet \arrow[r, "H'"] & \bullet \\
\bullet \arrow[u, "\mathrm{Tr}"] \arrow[ur, "H'"] \arrow[r, "H'"] & \bullet \arrow[u, "\mathrm{Tr}"']
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
\bullet & \bullet \arrow[l, "\mathrm{Tr}"'] \\
\bullet \arrow[u, "H'"] & \bullet \arrow[l, "\mathrm{Tr}"'] \arrow[ul, "H'"'] \arrow[u, "H'"']
\end{tikzcd}
.$$

(It is recommended to write down the corresponding homotopy oneself for the case of a homotopy of length 2.)

This trick will work for the more complicated situations later, hence we will tacitly only draw the diagrams for length 1 homotopies in the following.

C.1.3. Intervals of different length are homotopy equivalent

We now prove that for $A \in \mathcal{C}_a^G$ the modules $A[0, i]$, $(i \in \mathbb{N})$, are homotopy equivalent to A. We prove even more that the homotopies are relative to the endpoints. These are the key lemmas which enable us to consider only homotopies of length 1 for our results later. It is crucial for the following that we have the horn-filling property.

We first consider the case of intervals of length 1 and 2.

Lemma C.19. *Let $I(0, 2)$, $I(0, 1)$ be intervals, $A \in \mathcal{C}_a^G$. Then $A[0, 2]$ is homotopy equivalent to $A[0, 1]$, relative endpoints. If there is a projection $I(0, 2) \to I(0, 1)$ respecting the endpoints it induces such a homotopy equivalence.*

Remark C.20. That a homotopy equivalence is relative to the endpoints means, that the restriction of both homotopies relative endpoints gives the constant homotopy there. This allows to glue the homotopy equivalences together later.

Remark C.21. As the lemmas suggests, one can choose any two kinds of the interval $I(0, 1)$, however there may be only one for which a projection map $I(0, 2) \to I(0, 1)$ actually exists.

The lemma is implied by the following more direct statement. Assume that the standard 2-simplex Δ^2 has 0-simplices $0, 1, 2$. We denote by $i \rightharpoonup j$ the 3 non-degenerated 1-simplices of it, e.g. $0 \rightharpoonup 1$ denotes the unique 1-simplex with boundaries 0 and 1. All of them are ordered intervals of length 1. Recall that Λ_i^2 denotes the ith horn of Δ^2, e.g. Λ_1^2 is the simplicial subset of Δ^2 generated by all faces but the 1st one (which is $0 \rightharpoonup 2$).

Lemma C.22. *All the inclusions $A[i \rightharpoonup j] \to A[\Delta^2]$ and $A[\Lambda_i^n] \to A[\Delta^2]$ have deformation retractions. For $A[i \rightharpoonup j] \to A[\Delta^2]$ a deformation retraction can be chosen to be induced by the collapse of a 1-simplex not equal to $i \rightharpoonup j$.*

Proof. We prove only two cases, the other ones are similar.

There is a retraction $\Delta^2 \to 0 \rightharpoonup 2$ taking 1 to 2, and there is a homotopy of simplicial sets from the identity to this retraction, relative to $0 \rightharpoonup 2$. This induces the deformation retraction $A[\Delta^2] \to A[0 \rightharpoonup 2]$. (One could also take 1 to 0 and get a homotopy *from* that retraction *to* the identity.)

For the map $A[\Lambda_1^2] \to A[\Delta^2]$ we get, by horn-filling, a retraction $r \colon A[\Delta^2] \to A[\Lambda_1^2]$. Hence we get a map

$$A[(\Delta^2 \times 1) \cup (\Lambda_1^2 \times \Delta^1) \cup (\Delta^2 \times 0)] \xrightarrow{\mathrm{id} \cup \mathrm{id} \cup r} A[\Delta^2].$$

This can be extended, again by horn-filling, to the desired homotopy

$$A[\Delta^2 \times \Delta^1] \to A[\Delta^2].$$

This gives a deformation retraction $A[\Delta^2] \to A[\Lambda^2_1]$. □

Corollary C.23. *Let Λ^2_i be the ith horn and d_i the ith face of Δ^2. Then $A[d_i]$ and $A[\Lambda^2_i]$ are homotopy equivalent relative the 0-simplices of d_i. The homotopy equivalence can be chosen to be one of the maps $A[\Lambda^2_i] \to A[d_i]$ which induced by collapsing one 1-simplex.*

Proof. For simplicity of notation set $i := 1$. By Lemma C.22 we get a composition of homotopy equivalences relative $0 \amalg 2$

$$A[\Lambda^2_1] \to A[\Delta^2] \to A[0 \rightharpoonup 2]$$

where the last map (and hence the composition) can be chosen to be induced by collapsing a 1-simplex not equal to $0 \rightharpoonup 2$. □

Remark C.24. Note that in general the homotopy we get is *not* canonical, as its construction involved choices during the filling of horns. Horn-filling is also implicitly used for the fact that composition of homotopy equivalences is again a homotopy equivalence, hence involving even more choices.

Corollary C.25. *Let $I(0,1)$ be the standard interval and $I(0,i)$ any interval. Then we have a homotopy equivalence relative endpoints*

$$A[0,1] \simeq A[0,i].$$

Proof of the Corollary and Lemma C.19. Corollary C.23 implies $A[0,2] \simeq A[0,1]$ relative endpoints if there is a projection $I(0,2) \to I(0,1)$. It also implies $A[\to] \simeq A[\leftarrow]$ relative endpoint by the chain $A[\to] \simeq A[\to\leftarrow] \simeq A[\leftarrow]$ of homotopy equivalences relative boundary. (Here $\to$, $\leftarrow$ and $\to\leftarrow$ are pictures denoting the corresponding intervals as explained before.) The corollary follows by induction. □

C.1.4. Homotopies of infinite length

Let $I(0,\infty)$ be an infinite interval which we assume to be ordered for simplicity (cf. Definitions C.11 and C.9). Abbreviate $A[I(0,\infty)]$, $A \in \mathcal{C}^G_a$ suggestively as $A[0,\infty)$. We now prove that $A[0,\infty)$ is homotopy equivalent to A. This needs again the horn-filling property.

We suggestively call a map $A[0,\infty) \to B$ a homotopy *of infinite length.* From such a homotopy we want to get a homotopy of length 1. In general, this is of course impossible. But if the homotopy is "convergent" in the sense below this can be done.

Lemma C.26 (Infinite Homotopy)**.** *Let $H\colon A[0,\infty) \to B$ be a convergent homotopy, this means we assume:*

(i) There is a filtration $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots \subseteq A$ by cellular submodules such that $\bigcup_i A_i = A$.

(ii) For each A_i there is an n_i such that $H_{|A_i[n_i,\infty)}$ is the trivial homotopy Tr. *(This means it is induced by the projection on A_i, or "constant in $[n_i,\infty)$-direction").*

Then there exists a homotopy $G\colon A[\Delta^1] \to B$ with $G_{|A[0]} = H_{|A[0]}$ and $G_{|A_i[1]} = H_{|A_i[n_i]}$.

Remark C.27. Recall that A is a cellular R-module with R-cells $\diamond_R A$. That A_i is a cellular submodule hence means that $A_i \subseteq A$ is an R-submodule and that $\diamond_R A_i \subseteq \diamond_R A$. Recall that the 0-simplices of $I(0,\infty)$ are labeled by the natural numbers so let $I(n,\infty)$ denote the obvious subsimplicial set of $I(0,\infty)$ which is an interval from n to ∞. Then $A_i[n_i]$ and $A_i[n_i,\infty)$ denote obvious cellular submodules of $A[0,\infty)$.

Remark C.28. This lemma is well-known in the topological case. $\widehat{H}_1$ may be called the "limit" of the homotopy H.

Proof. Recall that we assumed $I(0,\infty)$ to be an ordered interval.

We first describe a new simplicial set arising from $I(0,\infty)$, it is sketched in Figure C.1 below. Let $I(0,2)$ be the ordered interval of length 2 from 0 to 2 which is a subsimplicial set of $I(0,\infty)$. It is isomorphic to the horn Λ^2_1, so we can glue in a Δ^2 along this horn, define $\widehat{I(0,2)}$ as the pushout

$$\begin{tikzcd}
\Lambda^2_1 \arrow[r] \arrow[d, "j_2"] & \Delta^2 \arrow[d] \\
I(0,\infty) \arrow[r] & \widehat{I(0,2)}
\end{tikzcd} .$$

It has an extra 1-simplex with boundaries 0 and 2 and arises from $I(0,\infty)$ by horn-filling. Now $\widehat{I(0,2)}$ contains a horn Λ^2_1, namely the one generated by the 1-simplex from 0 to 2 and the one from 2 to 3. We take the pushout along the inclusion of the horn $j_3\colon \Lambda^2_1 \to \widehat{I(0,2)}$ and call it $\widehat{I(0,3)}$:

$$\begin{tikzcd}
\Lambda^2_1 \arrow[r] \arrow[d, "j_3"] & \Delta^2 \arrow[d] \\
\widehat{I(0,2)} \arrow[r] & \widehat{I(0,3)}
\end{tikzcd} .$$

This similar has a new 1-simplex with boundaries 0 and 3. Continue to define $\widehat{I(0,n)}$ by induction. Define $\widehat{I(0,\infty)}$ as the filtered colimit of the $\widehat{I(0,n)}$. There is a canonical inclusion $I(0,\infty) \to \widehat{I(0,\infty)}$. Figure C.1 sketches a picture of $\widehat{I(0,\infty)}$ with $I(0,\infty)$ being the bottom line.

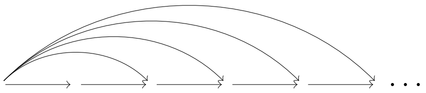


Figure C.1.: A sketch of $\widehat{I(0,\infty)}$.

We have $A\widehat{[0,\infty)} = \bigcup_n A\widehat{[0,n]}$, with $A\widehat{[0,\infty)}$ being an abbreviation for $A[\widehat{I(0,\infty)}]$. As $A\widehat{[0,n]}$ arises from $A\widehat{[0,n-1]}$ by horn-filling we can get from $H\colon A[0,\infty) \to B$ a map $\widehat{H}'\colon A\widehat{[0,\infty)} \to B$ by taking the colimit (which amounts to filling horns countably often).

The idea is that from $\widehat{H}'$ we get for each n a homotopy $A[\Delta^1] \to B$ of length one if we restrict to the simplex from 0 to n, and for increasing n they get better and better. However for this to work we have to be careful when filling the horns in the definition of $\widehat{H}'$. We make this precise now and define a map $\widehat{H}\colon A\widehat{[0,\infty)} \to B$ where we take care of that. We denote the restriction of a map $\widehat{H}'\colon A\widehat{[0,n]} \to B$ or $A\widehat{[0,\infty)} \to B$ to the 1-simplex with boundaries 0 and n of $\widehat{I(0,\infty)}$ by $\widehat{H}'_{|A(0,n)}$.

Note the following: For $i \in \mathbb{N}$ and $n \geq n_i$ we have that $H_{|A_i[n,n+1]}$ is the constant homotopy Tr. Hence for the submodule A_i we can fill the horn $j_{n+1}\colon A_i[\Lambda^2_1] \to A_i\widehat{[0,n]}$ in the following way

$$\begin{tikzcd} {} \arrow[r, bend left, "X"] \arrow[rr, bend left=50, "X"] & {} \arrow[r, "\mathrm{Tr}"] & {} \end{tikzcd}, \tag{31}$$

where X is a homotopy coming from the previous horn-fillings. We now arrange the horn-filling such that these horns are filled in exactly that way.

We can do induction over i. So assume that we have constructed a homotopy $G_i\colon A_i\widehat{[0,\infty)} \to B$ which extends $H\colon A_i[0,\infty) \to B$ and has the property that $G_{i|A(0,n)} = G_{i|A(0,n_i)}$ for all $n \geq n_i$. By the relative horn-filling property 2.22 we can extend G_i to a map $A_{i+1}\widehat{[0,n_{i+1}]} \to B$. From there on we can extend the map further along the constant homotopy as in (31). Hence we get a map $G_{i+1}\colon A_{i+1}\widehat{[0,\infty)}$ with $G_{i+1|A(0,n)} = G_{i+1|A(0,n_{i+1})}$ for all $n \geq n_{i+1}$. This shows the induction step.

Taking the colimit over G_i we get a map $\widehat{H}\colon A\widehat{[0,\infty)} \to B$. We now define $G_{|A_i}\colon A_i[\Delta^1] \to B$ as the restriction of $\widehat{H}$ (or equivalently G_i) to $A_i(0,n_i)$, i.e. to the 1-simplex from 0 to n_i. This is compatible with the inclusion $A_i \to A_{i+1}$ and thus the colimit over i gives the desired homotopy. ☐

Lemma C.29. *The map $A \to A[0,\infty)$ is deformation retraction, so in particular a homotopy equivalence.*

Proof. The map $[0,\infty) \to 0$ induces a retraction $A[0,\infty) \to A$. We have to prove that the composition $A[0,\infty) \to A \to A[0,\infty)$ is homotopic to the identity. We use the Infinite Homotopy Lemma C.26.

Define the convergent homotopy $H\colon A[0,\infty)[0,\infty) \to A[0,\infty)$ as the map induced by the map

$$(i,j) \mapsto \min(i,j)$$

where we use that map $[0,\infty) \times [0,\infty) \to [0,\infty)$ is determined on the 0-simplices. We regard j as the homotopy direction. This map has the following properties:

(i) For $j = 0$ it is the projection to 0, hence the map above.

(ii) For any $j \geq i$ the map $A[0,i][j] \to A[0,i]$ is the identity.

(iii) $\bigcup_i A[0,i]$ is a filtration of $A[0,\infty)$ by cellular modules.

Now the Infinite Homotopy Lemma C.26 applies and hence we get a homotopy $G\colon A[0,\infty)[\Delta^1] \to A[0,\infty)$ from the projection to 0 to the identity. □

C.2. On the mapping telescope

The goal of this section is to prove Proposition C.4. As explained before, we need long homotopies for this, hence we also need long mapping cylinders and long telescopes, which we define now. We need the following definitions only for maps $f\colon A \to A$, hence we state them only for these kind of maps.

Definition C.30. *Let $f\colon A \to A$ be a map in $\mathcal{C}_a^G$ and $I = I(0,i)$ an interval of length $i \geq 1$ (cf. C.9).*

(i) $A[I]$ is called a cylinder for the interval I.

(ii) The pushout

$$\begin{tikzcd} A[i] \arrow[r] \arrow[d, "f"] & A[I] \arrow[d] \\ A \arrow[r] & M^I(f) \end{tikzcd}$$

is called the mapping cylinder of f of length i for the interval I. The map from $A[0]$ to $M^I(f)$ is called the front inclusion and the map from $A[i]$ to $M^I(f)$ is called to back inclusion. The induced map $M^I(f) \to A$ is called the projection.

(iii) The pushout

$$\begin{tikzcd} \coprod_{i=0}^{\infty} A \amalg \coprod_{i=1}^{\infty} A \arrow[r, "\iota_1 \amalg \iota_0"] \arrow[d, "c"] & \coprod_{i=0}^{\infty} M^I(f) \arrow[d] \\ \coprod_{i=1}^{\infty} A \arrow[r] & \mathrm{Tel}^I(f) \end{tikzcd},$$

is called the telescope of f for the interval I. In the diagram ι_1 is the back inclusion of A into $M^I(f)$, ι_0 the front inclusion, both from the ith summand to the ith summand, and c is the map which maps the ith summand of the first *coproduct and the $(i+1)$th summand of the second coproduct to the $(i+1)$th summand of the target.*

The front inclusion into the first mapping cylinder $\iota_0\colon M^I(f)$ (which is not used in the diagram above) gives a map

$$\iota\colon A \to \mathrm{Tel}^I(f)$$

which is a called the front inclusion of the mapping cylinder.

Remark C.31. The telescope consists (as usual) of infinitely many mapping cylinders plugged together to the right. Note that each mapping cylinder has the same interval structure. Note further that we used that the countable coproducts exists in $\mathcal{C}_a^G$.

The definition is an immediate generalization of Definition C.7. The definitions agree if we set $I = I(0,1)$ to be the standard interval of length 1.

Lemma C.32. *Let $f\colon A \to A$ be a map in $\mathcal{C}_a^G$. Let $I = I(0,i)$ be an interval and Δ^1 be the standard interval.*

(i) The mapping cylinders for I and Δ^1 are homotopy equivalent relative to the front and the back inclusion: $M^I(f) \simeq M^{\Delta^1}(f) = T(f)$.

(ii) The mapping telescopes for I and Δ^1 are homotopy equivalent: $\mathrm{Tel}^I(f) \simeq \mathrm{Tel}^{\Delta^1}(f) = \mathrm{Tel}(f)$.

(iii) The telescope $\mathrm{Tel}^I(\mathrm{id}_A)$ of the identity is homotopy equivalent to A, the homotopy equivalence is given by the projection to A.

Each map $I \to \Delta^1$ respecting the endpoints can be chosen to induce the first two homotopy equivalences.

Remark C.33. Note that while we can give the homotopy equivalences quite explicit, the inverse usually will involve a choice, as we used the Kan Extension property to construct it. Hence it is not canonical and not easy to write down.

Proof. Corollary C.25 gives a homotopy equivalence $A[\Delta^1] \simeq A[I]$ relative endpoints, which can assumed to be induced by the projection if there is one. This glues to a homotopy equivalence $M^{\Delta^1}(f) \simeq M^I(f)$ (or use the gluing lemma). This in turn glues to a homotopy equivalence $\mathrm{Tel}^{\Delta^1}(f) \simeq \mathrm{Tel}^I(f)$.

By the first part we can assume the interval I to be ordered. The telescope $\mathrm{Tel}^I(\mathrm{id}_A)$ is then just $A[0,\infty)$. Lemma C.29 provides the homotopy equivalence. ☐

We have the analogue of C.4 (i)-(ii) for long telescopes, which we state briefly.

Lemma C.34. *Let $f\colon A \to A$ be a map in $\mathcal{C}_a^G$ and I an interval.*

(i) There is a cellular inclusion $\iota\colon A = A[0] \rightarrowtail \mathrm{Tel}^I(f)$.

(ii) $\mathrm{Tel}^I(f)$ is a functor from the arrow category of $\mathcal{C}_a^G$.

Proof. The map ι is just the front inclusion. A morphism φ from f to g in $\mathrm{Ar}\mathcal{C}_a^G$ is a commutative square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\varphi"] & A \arrow[d, "\varphi"] \\
B \arrow[r, "g"] & B
\end{tikzcd}$$

which gives a map $M^I(f) \to M^I(g)$ compatible with front/back inclusion and the projection and hence a map $\mathrm{Tel}^I(f) \to \mathrm{Tel}^I(g)$. It is compatible with ι. $\square$

C.2.1. Homotopy commutative squares

We saw that the telescope is functorial for commutative squares of self-maps. There is an important further property. Even for *homotopy commutative* squares we get an induced map on the telescopes. But this is "only up to homotopy" in a sense we will explain now. Take as example a homotopy idempotent $\eta\colon K \to K$. Then the square

$$\begin{tikzcd}
K \arrow[r, "\mathrm{id}_K"] \arrow[d, "\eta"] & K \arrow[d, "\eta"] \\
K \arrow[r, "\eta"] & K
\end{tikzcd} \tag{32}$$

is only homotopy commutative. Clearly if it should induce a map on telescopes the induced map should somehow depend on the homotopy to get an interesting map. We want to allow long homotopies. So let I be an interval and $H\colon K[I] \to K$ a homotopy from $\eta \circ \eta$ to $\eta \circ \mathrm{id}_K$. Take another interval J so we get the telescopes $\mathrm{Tel}^J(\mathrm{id}_K)$ and $\mathrm{Tel}^J(\eta)$. Then the map induced by the homotopy commutative square (32) will only give a map $(\eta, H)_*\colon \mathrm{Tel}^{J \,\square\, I}(\mathrm{id}_K) \to \mathrm{Tel}^J(\eta)$, where $J \,\square\, I$ is the concatenation of intervals from Section C.1.2. By the previous lemma $\mathrm{Tel}^{J \,\square\, I}(\mathrm{id}_K)$ is still homotopy equivalent to $\mathrm{Tel}(\mathrm{id}_K) = \mathrm{Tel}^{\Delta^1}(\mathrm{id}_K)$, but we only get a commutative square

$$\begin{tikzcd}
\mathrm{Tel}^{J \,\square\, I}(\mathrm{id}_K) \arrow[r, "\simeq"] \arrow[d, "(\eta,H)_*"'] & \mathrm{Tel}(\mathrm{id}_K) \arrow[d, "k"] \\
\mathrm{Tel}^J(\eta) \arrow[r, "\simeq"] & \mathrm{Tel}(\eta)
\end{tikzcd}$$

where the horizontal maps are homotopy equivalences, but the right vertical map k is only well-defined up to homotopy because it involves a choice of a homotopy inverse for the upper horizontal equivalence. But such a choice is non-canonical as it is constructed using horn-filling. In particular there does not seem to be a way such that the map k depends functorially on the homotopy commutative square (32).

On the other hand the map $(\eta, H)_*$ on the left will depend "functorially" on η and H if we build in the enlargement of the intervals. This section makes these constructions precise.

Definition C.35. *A square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & A \arrow[d, "a"] \\ B \arrow[r, "g"] & B \end{tikzcd} \tag{33}$$

is homotopy commutative if there is an interval $I = I(0,i)$ *and a specified homotopy* $H^a\colon A[0,i] \to B$ *which goes from* $g \circ a$ *to* $a \circ f$*. This should mean* ${H^a}_{|A[0]} = g \circ a$ *and* ${H^a}_{|A[i]} = a \circ f$*.*

Remark C.36*.* One should view the square as a "map" from f to g in a certain category which includes the homotopies. We refrain from defining this category.

Remark C.37*.* The homotopy of a homotopy commutative square always goes from the lower left corner to the upper right, it is helpful to visualize this as

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & A \arrow[d, "a"] \\ B \arrow[r, "g"] \arrow[ur, Rightarrow, "H"] & B \end{tikzcd}$$

when thinking about the homotopies. The reader is encouraged to do this for the diagrams we will use; we refrain from doing this, to keep the diagrams simpler.

We chose the direction of the homotopy such that it will fit together with our definition of mapping cylinder.

The next observation is the most important one in this section.

Lemma/Definition C.38 (Stacking squares)**.** *Homotopy commutative squares can be composed (stacked). Given two homotopy commutative squares*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & A \arrow[d, "a"] & , & B \arrow[r, "g"] \arrow[d, "b"] & B \arrow[d, "b"] \\ B \arrow[r, "g"] & B & & C \arrow[r, "h"] & C \end{tikzcd} \tag{34}$$

with homotopies H^a, H^b *using intervals* I^a*,* I^b*, then composed (stacked) square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "b\circ a"] & A \arrow[d, "b\circ a"] \\ C \arrow[r, "h"] & C \end{tikzcd}$$

is homotopy commutative with homotopy

$$(H^b \circ a[I^b]) \,\Box (b \circ H^a)\colon A[I^b \,\Box\, I^a] \to C.$$

Proof. Careful checking shows that the given homotopy makes the stacked square homotopy commutative. ☐

The next Proposition is clear, but for it to hold we indeed need homotopies of length greater than 1.

Proposition C.39. *Composition (stacking) of homotopy commutative squares is strictly associative, because concatenation of homotopies is. The length of the homotopies add. If the square is strictly commutative we can and hence will assume that the "homotopy" has length* 0. □

Note that we only "compose in one direction" of the two possible directions in which the square could be "stacked". The reason is simply, that this is the only case we are interested in.

C.2.2. Homotopy commutative squares and mapping cylinders

A homotopy commutative square induces a map on mapping cylinders and hence on telescopes.

Lemma/Definition C.40. *Let I be an interval. A homotopy commutative square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & A \arrow[d, "a"] \\ B \arrow[r, "g"] & B \end{tikzcd} \tag{35}$$

with homotopy $H\colon A[I] \to B$ induces a map called $(H,a)_*\colon M^I(f) \to B$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\iota_0"] \arrow[d, "a"] & M^I(f) \arrow[d] & A \arrow[l, "\iota_1"'] \arrow[d, "a"] \\ B \arrow[r, "g"] & B & B \arrow[l, "\mathrm{id}_B"'] \end{tikzcd} \tag{36}$$

commutes (strictly). Here ι_0 is the front and ι_1 the back inclusion. Each such diagram determines uniquely a homotopy commutative square (35).

If J is another interval, taking the mapping cylinder with J of the left square of (36) gives a map $a[J] \,\square (H,a)_*\colon M^{J \,\square\, I}(f) \to M^J(g)$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\iota_0"] \arrow[d, "a"] & M^{J \,\square\, I}(f) \arrow[d] & A \arrow[l, "\iota_1"'] \arrow[d, "a"] \\ B \arrow[r, "\iota_0"] & M^J(g) & B \arrow[l, "\iota_1"'] \end{tikzcd} \tag{37}$$

commutes. We call the induced maps the cylinder maps of the homotopy commutative diagram, resp. the cylinder maps with respect to J.

Remark C.41. The notation $a[J] \,\square (H,a)_*$ should indicate the map. On the first part it is just $a[J]\colon A[J] \to A[J]$, on the second it is first the homotopy and then the map a.

Proof. The pushout of the strictly commutative diagram

$$\begin{tikzcd}
A[I] \arrow[d, "H"] & A[i] \arrow[l, "\iota_1"'] \arrow[r, "f"] \arrow[d, "a\circ f"] & A \arrow[d, "a"] \\
B & B \arrow[l, "\mathrm{id}_B"'] \arrow[r, "\mathrm{id}_B"] & B
\end{tikzcd}$$

gives the map $(H, a)_*\colon M^I(f) \to B$ and then diagram (36) commutes. Conversely taking the map $A[I] \to M^I(f) \to B$ gets back the homotopy H and the commutativity of (36) shows that H makes the square (35) homotopy commutative.

Taking the mapping cylinder is functorial, even if we do it with J. Noting that for $\iota_0\colon A \to M^I(f)$ we have $M^J(\iota_0) \cong M^{J\,\Box\, I}(f)$ shows the last claim. □

The next definition is visualized in Figure C.2 below. We define a kind of composition of mapping cylinders, but this needs to enlarge the source of the first map. The reason for this will become clearer in the following lemmas.

Definition C.42. *Given maps $f\colon A \to A$, $g\colon B \to B$ and $h\colon C \to C$ as well as $a\colon A \to B$ and $b\colon B \to C$. Assume we have cylinder maps $(H^a, a)_*\colon M^{I^a}(f) \to B$ and $(H^b, b)_*\colon M^{I^b}(g) \to C$ like in Definition C.40 satisfying diagrams like (36). Define the "composition" $(H^a, a)_* \boxdot (H^b, b)_*$ as*

$$M^{I^b\,\Box\, I^a}(f) \xrightarrow{a[I^b]\,\Box(H^a,a)_*} M^{I^b}(g) \xrightarrow{(H^b,b)_*} C$$

More generally let J be another interval. Assume we have cylinder maps with respect to J

$$a[J]\,\Box(H^a, a)_*\colon M^{J\,\Box\, I^a}(f) \to M^J(g) \quad \textit{and} \quad b[J]\,\Box(H^b, b)_*\colon M^{J\,\Box\, I^b}(g) \to M^J(h)$$

Define the "composition" as

$$\Big(a[J]\,\Box(H^a, a)_*\Big) \boxdot \Big(b[J]\,\Box(H^b, b)_*\Big)\colon$$
$$M^{J\,\Box\, I^b\,\Box\, I^a}(f) \xrightarrow{a[J]\,\Box\, a[I^b]\,\Box(H^a,a)_*} M^{J\,\Box\, I^b}(g) \xrightarrow{b[J]\,\Box(H^b,b)_*} M^J(h)$$

The "composition" is compatible with stacking homotopy commutative squares.

Lemma C.43. *Given two homotopy commutative squares*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"] & A \arrow[d, "a"] \\
B \arrow[r, "g"] & B
\end{tikzcd}, \qquad
\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "b"] & B \arrow[d, "b"] \\
C \arrow[r, "h"] & C
\end{tikzcd}$$

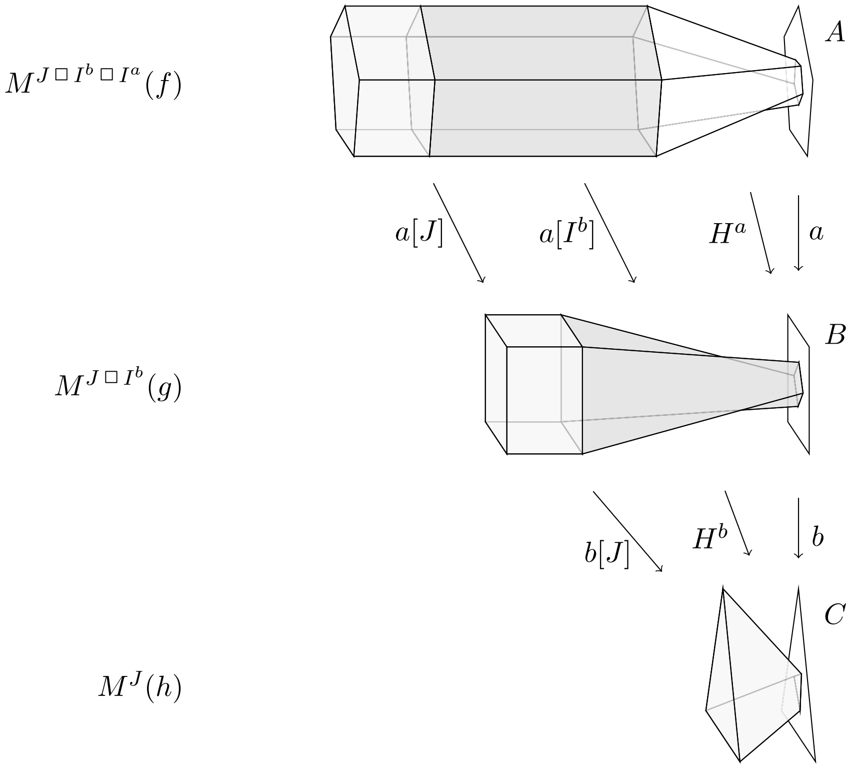


Figure C.2.: Composition of maps of long mapping cylinders. Shows the mapping cylinder construction is "functorial" if performed with long cylinders.

with homotopies $H^a \colon A[I^a] \to B$, $H^b \colon B[I^b] \to C$. *Then the cylinder map of the stacked homotopy commutative square (cf. C.38)*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "b \circ a"] & A \arrow[d, "b \circ a"] \\ C \arrow[r, "h"] & C \end{tikzcd}$$

is equal to the "composition" of the cylinder maps of the individual squares, i.e.

$$\Big((H^b \circ a[I^b]) \square (b \circ H^a), b \circ a\Big)_* = (H^b, b)_* \circ \Big(a[I^b] \square (H^a, a)_*\Big)$$

The same is true for cylinder maps with respect to J.

Proof. We have to check the equality of two maps $M^{J \square I^b \square I^a}(f) \to M^J(h)$. Figure C.2 shows the situation. With its help for the bookkeeping the equality can be checked directly. $\square$

Remark C.44. In some sense the compatibility with composition means that this construction is "functorial", but of course the "composition" of maps of mapping cylinders has a different source than the original maps, so it is not a composition in our category $\mathcal{C}_a^G$. This lemma is the reason why we need long cylinders.

We can directly extend these results to telescopes. For these we always set $J := \Delta^1$, the standard interval of length 1 in the lemmas above.

Definition C.45. *Two homotopy commutative squares*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & A \arrow[d, "a"] & , & B \arrow[r, "g"] \arrow[d, "b"] & B \arrow[d, "b"] \\ B \arrow[r, "g"] & B & & C \arrow[r, "h"] & C \end{tikzcd}$$

with homotopies $H^a\colon A[I^a] \to B$, $H^b\colon B[I^b] \to C$ *induce maps*

$$(H^a, a)_*\colon \operatorname{Tel}^{\Delta^1 \,\square\, I^a}(f) \to \operatorname{Tel}^{\Delta^1}(g),$$
$$(H^b, b)_*\colon \operatorname{Tel}^{\Delta^1 \,\square\, I^b}(g) \to \operatorname{Tel}^{\Delta^1}(h).$$

Define $(H^a, a)_*$ *to be induced by the map* $M^{\Delta^1 \square I^a}(f) \to M^{\Delta^1}(g)$ *from Definition C.40 and similar for* $(H^b, b)_*$.

Define the "composition"

$$(H^b, b)_* \boxdot (H^a, a)_*\colon \operatorname{Tel}^{\Delta^1 \,\square\, I^b \,\square\, I^a}(f) \to \operatorname{Tel}^{\Delta^1}(h)$$

as the map induced by the "composition"

$$M^{\Delta^1 \,\square\, I^b \,\square\, I^a}(f) \to M^{\Delta^1 \,\square\, I^b}(g) \to M^{\Delta^1}(h)$$

from Definition C.42.

Lemma C.46. *The "composition"* $(H^b, b)_* \boxdot (H^a, a)_*$ *is the same map as the map induced by the stacked homotopy commutative square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "b \circ a"] & A \arrow[d, "b \circ a"] & . \\ C \arrow[r, "h"] & C & \end{tikzcd}$$

Proof. This clear in view of Lemma C.43. □

Lemma C.47. *The induced map* $(H, a)_*$ *commutes with the front inclusions* ι, *i.e. the following square commutes.*

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[d, "a"] & \operatorname{Tel}^{\Delta^1 \,\square\, I^a}(f) \arrow[d, "(H,a)_*"] \\ B \arrow[r, "\iota"] & \operatorname{Tel}^{\Delta^1}(g) \end{tikzcd}$$

Proof. This follows e.g. from the commutativity of diagramm (37) in Lemma C.40. □

If we consider a strictly commutative square as a homotopy commutative square with constant homotopy of length 0 then our definition gives the same map as the one from Lemma C.34.

Lemma C.48. *Let I be an interval and consider a homotopy commtative square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & A \arrow[d, "\mathrm{id}"] \\ A \arrow[r, "f"] & A \end{tikzcd}$$

with homotopy the trivial homotopy $\mathrm{Tr}\colon A[I] \to A$ *for the interval* I*. The map* $(\mathrm{Tr}, \mathrm{id})_*\colon \mathrm{Tel}^{\Delta^1 \,\square\, I}(f) \to \mathrm{Tel}^{\Delta^1}(f)$ *is induced by the projection* $\Delta^1 \,\square\, I \to \Delta^1$ *mapping* I *to* $\overline{1} \subseteq \Delta^1$*.*

Proof. This follows by inspection. □

Remark C.49. Note that the problem with cylinders of different lengths might even arise if we would be in the topological setting, as even there composition of homotopies is only associative up to homotopy; but there it is usually hidden by the fact that there is always a canonical homeomorphism $[0,1] \cong [0,i]$ and two such homoemophisms (preserving endpoints) are homotopic.

C.2.3. A homotopy criterion

Assume we have two homotopy commutative squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & A \arrow[d, "a"] \\ B \arrow[r, "g"] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\widetilde{a}"] & A \arrow[d, "\widetilde{a}"] \\ B \arrow[r, "g"] & B \end{tikzcd}$$

with homotopies H^a and $H^{\widetilde{a}}$. By extending the homotopies by trivial homotopies we can assume that they have the same length and are indexed over the same interval I. We want a criterion when the two induced maps $(H^a, a)_*$ and $(H^{\widetilde{a}}, \widetilde{a})_*\colon \mathrm{Tel}^{\Delta^1 \,\square\, I}(f) \to \mathrm{Tel}^{\Delta^1}(g)$ are homotopic.

Lemma C.50. *Assume that in the above situation:*

(i) There is a homotopy $H\colon A[J] \to B$ *from* a *to* $\widetilde{a}$*; and*

(ii) There is "2-homotopy" $G\colon A[I][J] \to B$ *from* H^a *to* $H^{\widetilde{a}}$ *which restricts on* $\underline{0} \times J$ *to* $g \circ H$ *and on* $\underline{i} \times J$ *to* $H \circ f[J]$*.*

Then the two induced maps $(H^a, a)_*, (H^{\widetilde{a}}, \widetilde{a})_*\colon \mathrm{Tel}^{\Delta^1 \,\square\, I}(f) \to \mathrm{Tel}^{\Delta^1}(g)$ *are homotopic. The homotopy is* $(G, H)_*\colon \mathrm{Tel}^{\Delta^1 \,\square\, I}(f)[J] \to \mathrm{Tel}^{\Delta^1}(g)$*.*

To explain what we mean by G being a 2-homotopy it is easiest to draw a picture for $I = J = \Delta^1$. In our standard diagram language for maps from $A[I][J]$ to B (cf. Digression 2.4.1 and Lemma C.17) we could draw a diagram of G as follows:

$$\begin{tikzcd}
g \circ a \arrow[r, "H^a"] \arrow[d, "g\circ H"'] \arrow[dr] & a \circ f \arrow[d, "H\circ f[J]"] \\
g \circ \widetilde{a} \arrow[r, "H^{\widetilde{a}}"'] & \widetilde{a} \circ f
\end{tikzcd} . \tag{38}$$

(To simplify it we usually replace the corners by dots, as they are determined by the map on the edges.) So G can either be viewed as a homotopy from $g \circ H$ to $H \circ f[J]$ or as a homotopy from H^a to $H^{\widetilde{a}}$.

Proof. Interpreting G as homotopy from $g \circ H$ to $H \circ f[J]$ gives a homotopy commutative square

$$\begin{tikzcd}
A[J] \arrow[r, "f[J]"] \arrow[d, "H"] & A[J] \arrow[d, "H"] \\
B \arrow[r, "g"] & B
\end{tikzcd}$$

with homotopy $G\colon (A[J])[I] \to B$. We get an induced map $(G, H)_*\colon \operatorname{Tel}^{\Delta^1 \square I}(f[J]) \to \operatorname{Tel}^{\Delta^1}(g)$. As the telescope is a colimit it commutes with adjoining an interval, hence we can write the domain of the induced map as $\operatorname{Tel}^{\Delta^1 \square I}(f)[J]$. Therefore $(G, H)_*$ is a homotopy, we have to check that it is the desired one.

Assume J is an interval $I(0, j)$ from 0 to j. The strict commuting square

$$\begin{tikzcd}
A[0] \arrow[r, "f"] \arrow[d, "\iota_0"] & A[0] \arrow[d, "\iota_0"] \\
A[J] \arrow[r, "f[J]"] & A[J]
\end{tikzcd}$$

gives a map on telescopes $(\mathrm{Tr}, \iota_0)_*\colon \operatorname{Tel}^{\Delta^1}(f) \to \operatorname{Tel}^{\Delta^1}(f[J])$ which is the same as the inclusion induced by $\underline{0} \to I(0, j) = J$. Lemma C.43 shows that the "composition" $(G, H)_* \boxdot (\mathrm{Tr}, \iota_0)_*$ is the same as the map induced by the homotopy commutative square

$$\begin{tikzcd}
A[0] \arrow[r, "f"] \arrow[d, "H_{|A[0]}"', "=a"] & A[0] \arrow[d, "a"] \\
B \arrow[r, "g"] & B
\end{tikzcd}$$

with homotopy $H^a = G_{|A[I][0]}$ which is the stacking (composition) of the two squares above. It follows that

$$(G, H)_{*|0} = (G, H)_* \boxdot (\mathrm{Tr}, \iota_0)_* = (H^a, a)_*.$$

Similar with the inclusion $\iota_j \colon A[j] \to A[J]$ we get

$$(G,H)_{*|j} = (G,H)_* \boxdot (\mathrm{Tr}, \iota_j)_* = (H^{\widetilde{a}}, \widetilde{a})_*. \qquad \square$$

Remark C.51. Note that we really have a strict equality for the restrictions in the proof, thanks to our careful tracking of homotopies and the functoriality. The author does not know how to prove a lemma like this without using long homotopies and long cylinders. So this lemma is the technical reason why we need long homotopies!

Remark C.52. The lemma we just proved is the key lemma we use in the following. Due to the lemma we only need to show that homotopies like H and G as above exist if we want to show that certain maps on telescopes are homotopic. We usually give G in the form of diagrams like (38).

To make our live easier, we will pretend that all the homotopies we start with have length 1, which allows us to draw nice (and small) diagrams in the proofs. The strategy to extend this to longer homotopies is the same as described after the proof of Lemma C.17 at the end of Section C.1.2. Note that we could not literally assume that the homotopies have length 1, even if we could arrange it! One has to read the proof as if there were longer homotopies to maintain the generality we need.

Remark C.53. If $a = \widetilde{a}$ then H^a and $H^{\widetilde{a}}$ are two different homotopies which make the same square homotopy commutative. Then Lemma C.50 gives a useful criterion when these induce homotopic maps on the telescopes. We will apply the lemma in this situation.

C.2.4. Homotopic maps between telescopes

The next lemma implies Proposition C.4 (iii).

Lemma C.54. *Let $f, g \colon A \to A$ be homotopic maps. Then $\mathrm{Tel}^{\Delta^1}(f)$ and $\mathrm{Tel}^{\Delta^1}(g)$ are homotopy equivalent. The homotopy equivalences are relative to the front inclusions.*

Proof. Let $H \colon A[I] \to A$ be the homotopy from g to f. We get two homotopy commutative squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & A \arrow[d, "\mathrm{id}"] \\ A \arrow[r, "g"] & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\mathrm{id}"] & A \arrow[d, "\mathrm{id}"] \\ A \arrow[r, "f"] & A \end{tikzcd}$$

with homotopies $H \colon A[I] \to A$ and $\overline{H} \colon A[\overline{I}] \to A$ (where the latter is the "inverse" homotopy, cf. Section C.1.2). This gives maps $(H, \mathrm{id})_* \colon \mathrm{Tel}^{\Delta^1 \square I}(f) \to \mathrm{Tel}^{\Delta^1}(g)$ and $(\overline{H}, \mathrm{id})_* \colon \mathrm{Tel}^{\Delta^1 \square \overline{I}}(g) \to \mathrm{Tel}^{\Delta^1}(f)$. By Lemma C.46 the "composition"

$$(\overline{H}, \mathrm{id})_* \boxdot (H, \mathrm{id})_* \colon \mathrm{Tel}^{\Delta^1 \square \overline{I} \square I}(f) \to \mathrm{Tel}^{\Delta^1}(f)$$

is induced by the homotopy commutative square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & A \arrow[d, "\mathrm{id}"] \\
A \arrow[r, "f"] & A
\end{tikzcd}$$

with homotopy $\overline{H} \square H \colon A[\overline{I} \square I] \to A$. By Lemma C.17 it is homotopic relative endpoints to the trivial homotopy, hence Lemma C.50 shows that $(\overline{H}, \mathrm{id})_* \boxdot (H, \mathrm{id})_*$ is homotopic to $(\mathrm{Tr}, \mathrm{id})_*$. By Lemma C.48 $(\mathrm{Tr}, \mathrm{id})_*$ is the map induced by the projection $\Delta^1 \square \overline{I} \square I \to \Delta^1$. The same holds for the other composition.

As $(\mathrm{Tr}, \mathrm{id})_* \colon \mathrm{Tel}^{\Delta^1 \square I}(f) \to \mathrm{Tel}^{\Delta^1}(f)$ is a homotopy equivalence by Lemma C.32 we get two homotopy commutative triangles

$$\begin{tikzcd}
\mathrm{Tel}^{\Delta^1 \square I}(f) \arrow[r, "\simeq"] \arrow[d, "(H{,}\mathrm{id})_*"'] & \mathrm{Tel}^{\Delta^1}(f) \arrow[dl, "\varphi"] \\
\mathrm{Tel}^{\Delta^1}(g) &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{Tel}^{\Delta^1 \square \overline{I}}(g) \arrow[r, "\simeq"] \arrow[d, "(\overline{H}{,}\mathrm{id})_*"'] & \mathrm{Tel}^{\Delta^1}(g) \arrow[dl, "\psi"] \\
\mathrm{Tel}^{\Delta^1}(f) &
\end{tikzcd}$$

where φ is defined using a chosen homotopy inverse of the horizontal map and ψ similarly. We claim both compositions of these maps are homotopic to the identity. Together with these triangles we get a large diagram

$$\begin{tikzcd}
\mathrm{Tel}^{\Delta^1 \square \overline{I} \square I}(f) \arrow[r, "\simeq"] \arrow[d, "(H{,}\mathrm{id})_*"'] & \mathrm{Tel}^{\Delta^1 \square I}(f) \arrow[r, "\simeq"] \arrow[d, "(H{,}\mathrm{id})_*"'] & \mathrm{Tel}^{\Delta^1}(f) \arrow[dl, "\varphi"] \\
\mathrm{Tel}^{\Delta^1 \square \overline{I}}(g) \arrow[r, "\simeq"] \arrow[d, "(\overline{H}{,}\mathrm{id})_*"'] & \mathrm{Tel}^{\Delta^1}(g) \arrow[dl, "\psi"] & \\
\mathrm{Tel}^{\Delta^1}(f) & &
\end{tikzcd}$$

where the square is strictly commutative. The left vertical composition is the "composition" $(\overline{H}, \mathrm{id})_* \boxdot (H, \mathrm{id})_*$ which is homotopic to $(\mathrm{Tr}, \mathrm{id})_*$, which is exactly the composition of the upper horizontal maps. It follows that $\psi \circ \varphi \simeq \mathrm{id}$ and similar for the other composition.

As the homotopy inverses in the definition of ψ and φ can be chosen to respect the front inclusion by Lemma C.32 and all other maps and homotopies are relative to it φ is a homotopy equivalence relative to the front inclusion. This shows the lemma. $\square$

For any map $f \colon A \to A$ and an interval I we not only get a telescope $\mathrm{Tel}^I(f)$ but also a self-map $(\mathrm{Tr}, f)_* \colon \mathrm{Tel}^I(f) \to \mathrm{Tel}^I(f)$. We show that it is a homotopy equivalence. Recall that $\mathrm{Tel}^I(f)$ is a quotient of $\coprod_{n \in \mathbb{N}} M^I(f)$. The map taking the nth component to the $(n+1)$st component is compatible with the quotient, hence induces a map $\mathrm{Tel}^I(f) \to \mathrm{Tel}^I(f)$, which we will call the *shift map* and denote it by sh.

Lemma C.55. *Let I be an interval, $f\colon A \to A$ a self-map. The maps* sh *and* $(\mathrm{Tr}, f)_*$ *from* $\mathrm{Tel}^I(f)$ *to* $\mathrm{Tel}^I(f)$ *are homotopy inverse:*

$$(\mathrm{Tr}, f)_* \circ \mathrm{sh} = \mathrm{sh} \circ (\mathrm{Tr}, f)_* \simeq \mathrm{id}\colon\ \mathrm{Tel}^I(f) \to \mathrm{Tel}^I(f)$$

Proof. For notation let $I = I(0, i)$ be an interval from 0 to i. The first equality is clear. To provide the homotopy it suffices to restrict to a submodule $M^I(f)$ of $\mathrm{Tel}^I(f)$. Then we can restrict the target to the submodule $M^I(f) \cup_A M^I(f)$ of $\mathrm{Tel}^I(f)$ which is the pushout of $M^I \xleftarrow{\iota_1} A[i] \xrightarrow{\iota_0} M^I(f)$, where ι_0, ι_1 are the front resp. back inclusion.

As $M^I(f)$ is the pushout of $A[I] \xleftarrow{\iota_1} A \xrightarrow{f} A$ we get a map $A[I] \to M^I(f)$ which we denote by $\mathrm{id}[I]$ by a slight misuse of notation. Similar we get a map $\varphi\colon A[I \,\square\, I] \to M^I(f) \cup_A M^I(f)$ which can be defined as the map induced by the pushouts of the rows of

$$\begin{tikzcd}
A[I] \arrow[d, "{\mathrm{id}[I]}"] & A[i] \arrow[l] \arrow[r] \arrow[d, "f"] & A[I] \arrow[d, "{\mathrm{id}[I] \circ f[I]}"] \\
M^I(f) & A \arrow[l] \arrow[r] & M^I(f)
\end{tikzcd}.$$

We get two inclusions $j_0, j_1\colon A[I] \to A[I \,\square\, I]$, corresponding to the two inclusions of the interval $I \to I \,\square\, I$. (Recall that $I \,\square\, I$ is the pushout of two copies of I along the endpoint resp. the startpoint.) Forming $\varphi \circ j_0$ gives that $(\varphi \circ j_0)_{|A[0]} = (\varphi \circ j_0)_{|A[i]} \circ f$, hence it glues to a map of mapping cylinders and therefore of telescopes. The same is true for $\varphi \circ j_1$. The induced map of $\varphi \circ j_0$ is the identity, whereas $\varphi \circ j_1$ induces the map $\mathrm{sh} \circ (\mathrm{Tr}, f)_*$. Therefore to show that these maps are homotopic it suffices to produce a homotopy H from j_0 to j_1 with the compatibility $(\varphi \circ H)_{|A[0]} = (\varphi \circ H)_{|A[i]} \circ f$.

For simplicity we only treat the case $I = \Delta^1$. So we have to produce a map $A[\Delta^1 \times \Delta^1] \to A[\Delta^1 \,\square\, \Delta^1]$. Writing $I \,\square\, I$ as $I(0,1) \,\square\, I(1,2)$ and using our usual diagram notation (cf. Digression 2.4.1), the notation of an arrow denoting the target 1-simplex it maps to, we use horn-filling to fill the diagram on the left below symmetrically to the diagram on the right.

$$\begin{tikzcd}
\bullet & \bullet \\
\bullet \arrow[u, "{I(0,1)}"] \arrow[r, "{I(0,1)}"'] & \bullet \arrow[u, "{I(1,2)}"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[r, "{I(1,2)}"] & \bullet \\
\bullet \arrow[u, "{I(0,1)}"] \arrow[r, "{I(0,1)}"'] \arrow[ur] & \bullet \arrow[u, "{I(1,2)}"']
\end{tikzcd}\ . \tag{39}$$

(We first fill the lower right horn and the use the symmetry of the diagram to take the same map for the upper right simplex.) This gives a homotopy H from j_0 to j_1. Restricting to $A[0]$ is the same as restricting to $A[1]$ and "shift" one up, hence one checks that $\varphi \circ H$ has the desired compatibility and glues to the desired homotopy

$$\mathrm{Tel}^I(f)[I] \to \mathrm{Tel}^I(f)$$

from the identity to $\mathrm{sh} \circ (\mathrm{Tr}, f)_*$. If I is a more general interval than Δ^1 the diagram (39) has to be replaced by a similar diagram of I^2, but this can be filled symmetrically in the same way and thus also produces the desired homotopy. $\square$

C.3. Telescopes of homotopy idempotents

Lemma C.56. *Let $\eta\colon K \to K$ be a coherent homotopy idempotent in $\mathcal{C}_a^G$, I an interval. Then the induced map $\eta_* = (\mathrm{Tr}, \eta)_*\colon \mathrm{Tel}^I(\eta) \to \mathrm{Tel}^I(\eta)$ is not only a homotopy equivalence but even homotopic to the identity* id.

Proof. We abbreviate $(\mathrm{Tr}, \eta)_*$ by η_*. We show $\eta_* \circ \eta_* \simeq \eta_*$, then by Lemma C.55 we have $\eta_* \circ \mathrm{sh} \simeq \mathrm{id}$ and therefore $\eta_* \simeq \eta_* \circ \eta_* \circ \mathrm{sh} \simeq \eta_* \circ \mathrm{sh} \simeq \mathrm{id}$ and we are done.

We assume that $H\colon A[I] \to A$ is the homotopy from η^2 to η and that $I = \Delta^1$. (Intervals of longer length work similarly.) As η is coherent we have a diagram

$$\begin{tikzcd}
\bullet \arrow[r, "\eta\circ H"] \arrow[d, "H\circ\eta[I]"'] \arrow[dr, "X"] & \bullet \arrow[d, "H"] \\
\bullet \arrow[r, "H"'] & \bullet
\end{tikzcd} .$$

where X is is just a name to the denote the restriction to the diagonal. Using that diagram we can build the diagram

$$\begin{tikzcd}
\eta^3 \arrow[r, "\mathrm{Tr}"] \arrow[d, "\eta\circ H"'] \arrow[dr, "X"] & \eta^3 \arrow[d, "X"] & \eta^3 \arrow[l, "\mathrm{Tr}"'] \arrow[d, "H\circ\eta[I]"] \arrow[dl, "X"'] \\
\eta^2 \arrow[r, "H"] & \eta & \eta^2 \arrow[l, "H"'] \\
\eta^2 \arrow[r, "\mathrm{Tr}"'] \arrow[u, "\eta\circ\mathrm{Tr}"] \arrow[ur, "H"] & \eta^2 \arrow[u, "H"] & \eta^2 \arrow[l, "\mathrm{Tr}"] \arrow[u, "\mathrm{Tr}\circ\eta[I]"'] \arrow[ul, "H"']
\end{tikzcd}$$

which gives a 2-homotopy G such that Lemma C.50 shows that $(G, H)_*$ is a homotopy from η_*^2 to η_*, but only as maps $\mathrm{Tel}^{\Delta^1 \,\square\, I \,\square\, \overline{I}}(\eta) \to \mathrm{Tel}^{\Delta^1}(\eta)$. But as for Lemma C.54 the triangle for η_*

$$\begin{tikzcd}
\mathrm{Tel}^{\Delta^1 \,\square\, I \,\square\, \overline{I}}(\eta) \arrow[r, "\simeq"] \arrow[d, "\eta_*"'] & \mathrm{Tel}^{\Delta^1}(\eta) \arrow[dl, "\eta_*"] \\
\mathrm{Tel}^{\Delta^1}(\eta) &
\end{tikzcd} ,$$

as well as the one for η_*^2, commutes, here even strictly, and therefore also the maps on the cylinder of the same lengths are homotopic. $\square$

Lemma C.57. *Let $\eta\colon K \to K$ be a coherent homotopy idempotent in $\mathcal{C}_a^G$. Then there is a map $c\colon \mathrm{Tel}^{\Delta^1}(\eta) \to K$ such that the composition $\iota \circ c$ with the inclusion $\iota\colon K \to \mathrm{Tel}^{\Delta^1}(\eta)$ is homotopic to the identity on $\mathrm{Tel}^{\Delta^1}(\eta)$, whereas the other composition $c \circ \iota$ is homotopic to $\eta\colon K \to K$.*

Proof. Let $H\colon A[I] \to A$ be the homotopy from η^2 to η. We get two homotopy commutative squares

$$\begin{tikzcd} K \arrow[r, "\mathrm{id}"] \arrow[d, "\eta"] & K \arrow[d, "\eta"] \\ K \arrow[r, "\eta"] & K \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} K \arrow[r, "\eta"] \arrow[d, "\eta"] & K \arrow[d, "\eta"] \\ K \arrow[r, "\mathrm{id}"] & K \end{tikzcd}$$

with homotopies $H\colon A[I] \to A$ and $\overline{H}\colon A[\,\overline{I}\,] \to A$. Hence we get two induced maps

$$(H,\eta)_*\colon\ \mathrm{Tel}^{\Delta^1 \,\square\, I}(\mathrm{id}_K) \to \mathrm{Tel}^{\Delta^1}(\eta)$$
$$(\overline{H},\eta)_*\colon\ \mathrm{Tel}^{\Delta^1 \,\square\, \overline{I}}(\eta) \to \mathrm{Tel}^{\Delta^1}(\mathrm{id}_K).$$

Consider the "composition" (C.45)

$$(H,\eta)_* \boxdot (\overline{H},\eta)_*\colon\ \mathrm{Tel}^{\Delta^1 \,\square\, I \,\square\, \overline{I}}(\eta) \to \mathrm{Tel}^{\Delta^1}(\eta)$$

which by Lemma C.46 is equal to $(H \circ \eta[I] \,\square\, \eta \circ \overline{H}, \eta^2)_*$. As the homotopy idempotent is coherent (cf. C.1) we have a map $A[I \times I] \to A$ which is on the boundary of I^2:

$$\begin{tikzcd} \bullet \arrow[r, "\eta\circ H"] \arrow[d, "H\circ\eta{[I]}"'] \arrow[dr] & \bullet \arrow[d, "H"] \\ \bullet \arrow[r, "H"'] & \bullet \end{tikzcd} \quad . \tag{40}$$

Thus by pasting two copies of the above square together as shown below we get the 2-homotopy G

$$\begin{tikzcd} \eta^3 \arrow[r, "H\circ\eta{[I]}"] \arrow[d, "\eta\circ H"] & \eta^2 \arrow[d, "H"] & \eta^3 \arrow[l, "\eta\circ H"'] \arrow[d, "H\circ\eta{[I]}"] \\ \eta^2 \arrow[r, "H"] & \eta & \eta^2 \arrow[l, "H"'] \\ \eta^2 \arrow[u, "\eta\circ\mathrm{Tr}"] \arrow[r, "\mathrm{Tr}"] \arrow[ur, "H"] & \eta^2 \arrow[u, "H"] & \eta^2 \arrow[l, "\mathrm{Tr}"'] \arrow[u, "\mathrm{Tr}\circ\eta{[I]}"'] \arrow[ul, "H"'] \end{tikzcd}$$

from $H \circ \eta[I] \,\square\, \eta \circ \overline{H}$ to Tr and by Lemma C.50 $(G,H)_*$ gives a homotopy from the composition $(H,\eta)_* \boxdot (\overline{H},\eta)_*$ to the map $(\mathrm{Tr},\eta)_*$. Similar the other "composition" $(\overline{H},\eta)_* \boxdot (H,\eta)_*$ is homotopic to $(\mathrm{Tr},\eta)_*\colon\ \mathrm{Tel}^{\Delta^1 \,\square\, \overline{I} \,\square\, I}(\mathrm{id}_K) \to \mathrm{Tel}^{\Delta^1}(\mathrm{id}_K)$ using the 2-homotopy

$$\begin{tikzcd} \eta^2 \arrow[d, "H"] & \eta^3 \arrow[l, "\eta\circ H"'] \arrow[r, "H\circ\eta{[I]}"] \arrow[d, "X"] \arrow[dl, "X"'] \arrow[dr, "X"] & \eta^2 \arrow[d, "H"] \\ \eta & \eta \arrow[l, "\mathrm{Tr}"] \arrow[r, "\mathrm{Tr}"'] & \eta \end{tikzcd}$$

where X is the diagonal in diagram (40) and the upper left and right triangles are also from (40).

Now we make $(H, \eta)_*$ and $(\overline{H}, \eta)_*$ into maps of telescopes of the same length as in the proof of Lemma C.54. Define $\widetilde{c}$ and $\widetilde{\iota}$ by choosing a homotopy inverse in the top row of the following diagrams

$$\begin{tikzcd}
\mathrm{Tel}^{\Delta^1 \,\square\, \overline{I}}(\eta) \arrow[r, "\simeq"] \arrow[d, "(\overline{H}{,}\eta)_*"'] & \mathrm{Tel}^{\Delta^1}(\eta) \arrow[dl, "\widetilde{c}"] \\
\mathrm{Tel}^{\Delta^1}(\mathrm{id}_K) &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{Tel}^{\Delta^1 \,\square\, I}(\mathrm{id}_K) \arrow[r, "\simeq"] \arrow[d, "(H{,}\eta)_*"'] & \mathrm{Tel}^{\Delta^1}(\mathrm{id}_K) \arrow[dl, "\widetilde{\iota}"] \\
\mathrm{Tel}^{\Delta^1}(\eta) &
\end{tikzcd} .$$

A similar argument as in the proof of Lemma C.54 using a big triangle shows that $\widetilde{c} \circ \widetilde{\iota}$ is homotopic to $\eta_*\colon \mathrm{Tel}^{\Delta^1}(\mathrm{id}_K) \to \mathrm{Tel}^{\Delta^1}(\mathrm{id}_K)$ and $\widetilde{\iota} \circ \widetilde{c}$ is homotopic to $\eta_*\colon \mathrm{Tel}^{\Delta^1}(\eta) \to \mathrm{Tel}^{\Delta^1}(\eta)$. Lemma C.56 shows that on $\mathrm{Tel}^{\Delta^1}(\eta)$ the map η_* is homotopic to the identity.

Now $\iota_{\mathrm{id}_K}\colon K \to \mathrm{Tel}^{\Delta^1}(\mathrm{id}_K)$ is a homotopy equivalence and even an inclusion for a deformation retraction pr by Lemma C.32. Set $c := \mathrm{pr} \circ \widetilde{c}\colon \mathrm{Tel}^{\Delta^1}(\eta) \to K$, note $\widetilde{\iota} \circ \iota_{\mathrm{id}_K} = \iota_\eta$. Then $\iota \circ c = \widetilde{\iota} \circ \iota_{\mathrm{id}_K} \circ \mathrm{pr} \circ \widetilde{c}$ is homotopic to $\widetilde{\iota} \circ \widetilde{c}$ and hence to $\mathrm{id}_{\mathrm{Tel}^{\Delta^1}(\eta)}$ and $c \circ \iota = \mathrm{pr} \circ \widetilde{c} \circ \widetilde{\iota} \circ \iota_{\mathrm{id}_K}$is homotopic to $\eta\colon K \to K$. This shows the lemma. $\square$

Proof of Proposition C.4. We already proved every single claim. We defined $\mathrm{Tel}(\eta)$ in Definition C.7, provided the inclusion (C.4(i)) and showed the functoriality (C.4(ii)) right after that. The inclusion $\iota\colon K \to \mathrm{Tel}(\mathrm{id}_K)$ is a homotopy equivalence (C.4(iv)) by Lemma C.32 and homotopic maps gives homotopy equivalent telescopes (C.4(iii)) by Lemma C.54. The compatibility with the inclusion (C.4(v)) was noted along the lemmas.

A homotopy idempotent induces a homotopy equivalence on its own telescope by Lemma C.55, and a coherent homotopy idempotent even induces a map homotopic to the identity by Lemma C.56 (C.4(vi)). Finally the retraction up to homotopy to the inclusion (C.4(vii)) is provided in Lemma C.57. $\square$

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