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The Liftability of Elliptic Surfaces

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Zusammenfassung

Das Thema dieser Arbeit ist die Liftbarkeit elliptischer Flächen. Ein Schema X über einer einem Körper $k \supset \mathbb{F}_p$ heißt liftbar nach Characteristik 0 wenn ein lokaler Ring $(R, \mathfrak{m})$ existiert mit $R/\mathfrak{m} \simeq k$ und $\mathbb{Z} \subset R$, sowie ein flaches Schema $\mathcal{X}$ über R mit $\mathcal{X} \otimes_R k \simeq X$. Eine Fläche heißt elliptisch, wenn sie eine elliptische Faserung besitzt, d.h. einen Morphismus auf eine Kurve, dessen generische Faser eine glatte Kurve vom Geschlecht 1 ist. Elliptische Flächen existieren im Überfluss, denn viele Fläche von Kodaira-Dimension kleiner gleich 0 sind elliptisch, und jede Fläche von Kodaira-Dimension 1 besitzt eine elliptische oder quasi-elliptische Faserung.

Zur Untersuchung der Liftbarkeit elliptischer Faserungen ziehen wir die Modultheorie elliptischer Kurven heran. Im ersten Kapitel der vorliegenden Arbeit studieren wir Deformationen und Liftungen glatter elliptischer Faserungen. Es stellt sich heraus, dass die Deformationstheorie dieser Objekte so gut kontrollierbar ist, dass wir in der Lage sind, nicht liftende Beispiele elliptische Faserungen über Körpern der Characterisik 2 und 3 zu konstruieren. Es handelt sich um die ersten bekannten Beispiele. Auch konstruieren wir eine Klasse elliptischer Faserungen deren Liftbarkeit äquivalent zu einer offenen Vermutung von Oort ist. Das unterstreicht die Komplexität des Liftungsproblems für elliptische Faserungen. Als weitere Anwedung klassifizieren wir die Deformationen bielliptischer Flächen und zeigen deren Liftbarkeit.

Im zweiten Kapitel beschäftigen wir uns mit semistabilen elliptischen Faserungen. Mittels der Modultheorie für verallgemeinerte elliptische Kurven, entwickelt von Deligne und Rapoport und erweitert durch Conrad, können wir zeigen, dass jede semistabile elliptische Faserung mit Schnitt und separabler modularer Invariante nach Characteristik 0 lifted.

Das dritte Kapitel handelt von elliptischen Faserungen die bestimmte Zahmheitseigenschaften erfüllen. Nach Ausschluss von Charakterisik 2 und 3 zeigen wir, dass jede Jacobische elliptische Faserung mit zahmer modularer Invariante lifted. Für nicht jacobische Faserungen gilt ein vergleichbares Resultat, falls ein Multischnitt vom Grade prim zu p existiert.

Als Fazit erhalten wir, dass obgleich nicht liftende elliptische Faserungen existieren, dieses Verhalten nicht das typische ist. Ist p im Verhältniss zu gewissen arithmetischen Invarianten der fraglichen Faserung hinreichend groß, so gilt Liftbarkeit.

Summary

The topic of this work is the liftability of elliptic surfaces. A scheme X over a field $k \supset \mathbb{F}_p$ is called *liftable to characteristic zero*, if there is a local ring $(R, \mathfrak{m})$ with $R/\mathfrak{m} \simeq k$ and $\mathbb{Z} \subset R$, as well as a flat scheme $\mathcal{X}$ over R, such that $\mathcal{X} \otimes_R k \simeq X$. A surface is called *elliptic* if it has an elliptic fibration, i.e. a morphism to a curve, such that the generic fibre is a smooth genus-1 curve. Elliptic surfaces exist in abundance because they are common in Kodaira dimension less than one and every surface in Kodaira dimension one has a unique elliptic or quasi-elliptic fibration, given by the canonical bundle.

To investigate the liftability of elliptic fibrations, we make extensive use of the moduli theory of elliptic curves. In the first Chapter of this work, we study deformations and liftings of smooth elliptic fibrations. It turns out that we can control their deformations fairly well, which allowes us to give examples of non-liftable elliptic fibrations of Kodaira dimension one over fields of characteristic two and three. Those are the first examples currently known. We also construct a class of elliptic surfaces whose liftability is equivalent to an open conjecture of Oort. This illustrates the complexity of the lifting problem for elliptic surfaces. To give a further application, we classify deformations of bielliptic surfaces and show that they are liftable.

In the second Chapter we are concerned with semistable elliptic fibrations. Using the moduli theory of generalized elliptic curves developed by Deligne and Rapoport and extended by Conrad, we can show that every semistable elliptic fibration, possesing a section and having a separable modular invariant, is liftable to characterstic zero.

The third chapter deals with elliptic fibrations satisfying certain tameness properties. Excluding characteristic two and three, we prove that a Jacobian elliptic fibration with tame modular invariant is liftable in the category of algebraic spaces. For non-Jacobian fibrations we have a similar result, given the existence of a multisection of degree prime to p.

As a conclusion, we can say that, although non-liftable elliptic fibrations do exist, this is not the typical behaviour. Given that p is sufficiently large in comparison to certain arithmetic invariants of the surface in question, liftability is seen to hold true.

Introduction

Drei Orangen, zwei Zitronen:-
Bald nicht mehr verborgene Gleichung,
Formeln, die die Luft bewohnen,
Algebra der reifen Früchte! [1]

The question of liftability arises naturally in the modern formulation of algebraic geometry due to Grothendieck, because algebraic geometry can be done not just over fields, but over more general bases such as arbitrary rings. Now let R be a local ring with residue field k. Given a scheme X over k, we can ask for a *lifting* of X over R, this is a *flat* R-scheme $\mathcal{X}$, with $\mathcal{X} \otimes_R k \simeq X$.

In other words, a lifting of X is a family over R which contains the given X as a special fibre. If the ring R is a k-algebra, this question is trivial, because we can always take a product family. However, the question becomes non-trivial, if R is a ring of mixed characteristic, saying k is of positive characteristic and the fraction field of R is of characteristic zero. In that case we speak of a lifting of X to characteristic zero.

We are going to study liftability for surfaces having a fibration onto a curve such that the generic fibre is a smooth curve of genus one. This question was posed by Katsura and Ueno in [KU85]. Our results are twofold: We construct non-liftable elliptic surfaces of Kodaira dimension one over fields of characteristic two and three, showing that liftability does not hold in general. For general p we construct a special class of elliptic fibrations, whose liftability would imply an open conjecture of F. Oort. This illustrates the complexity of the problem.

On the other hand, we establish a series of affirmative lifting results for certain classes of elliptic surfaces.

This work is organized in three Chapters. The overall principle of its organization is derived from its mathematical objects of study: Beginning with the most special class of elliptic fibrations, namely the smooth ones, we work towards higher degrees of generality in the following Chapters.

In the *first Chapter* we study smooth elliptic fibrations. Smooth elliptic fibrations over proper bases can be interpreted as elliptic fibre bundles, i.e. étale locally over the base they become constant fibrations. We classify deformations of Jacobian elliptic fibre bundles (those possesing sections), and prove that an arbitrary elliptic fibre bundle has a formal lifting if and only if its Jacobian does so (Theorem 1.6.5). In the Kodaira dimension one case, we get:

Theorem (1.7.1)**.** *If X/C is an elliptic fibre bundle of Kodaira dimension one, then its unique elliptic fibration extends in a unique way to every deformation.*

[1] This is the first stanza of the poem "Drei Orangen, zwei Zironen" by Karl Krolow. A word by word english translation reads: "Three oranges, two lemons:- / Equation, soon no longer concealed, / Formulas, inhabiting the air, / Algebra of ripe fruit!"

This result has two important applications: To see that a Jacobian elliptic fibre bundle of Kodaira dimension one is not liftable, it is enough to show that it is not liftable as a Jacobian elliptic fibre bundle. This allowes to prove:

Theorem (1.7.5)**.** *There exist elliptic fibre bundles in characteristic two and three, that do not lift to characteristic zero.*

Another result connects the liftability of certain elliptic fibrations to the Oort conjecture predicting that given a curve C of higher genus, and a cyclic subgroup G of $\mathrm{Aut}(C)$, then the pair (C, G) can be lifted. Our result is:

Theorem (1.7.6)**.** *Choose an elliptic curve E and let G act on E by translation. Then the quotient $X = (E \times C)/G$ is liftable if and only if the pair (C, G) is liftable.*

As a further application of the theory, we treat the case of bielliptic surfaces. This is of interest since for small p one encounters phenomena which do not appear when considering the same class of surfaces in characteristic zero. For example, there are bielliptic surfaces with obstructed deformations. This was already observed by W. Lang in [Lan95]. There is also the possibility of deforming a Jacobian bielliptic surface into a non-Jacobian one, which can never happen in characteristic zero. We obtain a classification of deformations of bielliptic surfaces:

Theorem (1.8.10)**.** *If X is a bielliptic surface, then both elliptic fibration extend under deformations. In other words: Every deformation of a bielliptic surface is bielliptic.*

In particular, we show that every bielliptic surface has a projective lifting to characterisitc zero (Corollary 1.8.9).

In the *second Chapter* we are concerned with semistable elliptic fibrations. Jacobian semistable elliptic fibrations can be equipped with group structures. In that case we speak of *generalized elliptic curves*. These objects are the subject of a powerful moduli theory developed in [DR73]. Directly from this theory, we get:

Theorem (Proposition 2.4.1 and Corollary 2.4.2)**.** *Let E/C be a generalized elliptic curve. Suppose that for some integer $n \geq 3$ and prime to p, we have $E[n]$ is a constant group scheme isomorphic to $(\mathbb{Z}/n\mathbb{Z})^2$, and that the modular invariant of E is separable. Then E lifts to characteristic zero.*

The result is proven just by lifting the separable map from C into the moduli space $\mathcal{M}_{\Gamma(n)}$. The separability assumption on the modular invariant is indeed necessary for the existence of liftings in the category of generalized elliptic curves, as we prove in Theorem 2.4.4.

We are now looking for a way to obtain lifting results which are independent of the presence of a level structure. The level structure was needed to make the associated moduli space representable. However, we can do with weaker assumptions: For example, suppose we are mapping into a moduli stack which is a neutral gerbe under an étale structure group over its coarse moduli space.

This is the case for the moduli stack of generalized elliptic curves with geometrically irreducible fibres $\mathcal{M}_1$ if we restrict to the non-special locus of the coarse moduli space $\mathbb{P}^1$, which is given by removing the points $j = 0$ and $j = 1728$ (see Proposition 2.2.3). The structure group here is just $\mathbb{Z}/2\mathbb{Z}$. The usefulness of gerbes with étale structure groups in this context stems form the fact that sections of gerbes lift uniquely. We are hence reduced to lift the morphism into the coarse moduli space.

The reason why we have to exclude $j = 0$ and $j = 1728$ is that at these points the rank of the inertia of $\mathcal{M}_1$ jumps up. The difficult part is to handle those special points in order to obtain a global lifting. This is achieved by a combination of deformation theory arguments, formal patching methods, and descent arguments. For an outline of the strategy see the beginning of Section 6. The main theorem we obtain in this fashion is:

Theorem (2.6.7)**.** *Let E/C be a generalized elliptic curve, over a proper smooth curve C/k. Assume that the modular invariant of E is separable and that the total space E/k is smooth. Then there exists a projective lifting $\mathcal{E}/\mathcal{C}$ of E/C over the ring of Witt vectors.*

We can also give a version which holds for semistable fibration in general, without assuming the existence of a section. This is Theorem 2.7.4.

In the *third Chapter* we generalize the results of Chapter 2 to some classes of non-semistable fibrations. First, we treat the Jacobian case. Given a Jacobian elliptic fibration E/C we can find a separable covering $C' \to C$, such that the minimal regular model of the base change $E \times_C C'$ is semistable. Our strategy is to lift this model and afterwards to obtain a lifting of E itself by descent.

In order to do so, we rely on the theory of deformations of tame coverings. This forces us to assume throughout the Chapter, that $p \geq 5$ holds, otherwise the modular invariant of a semistable fibration will never be a tame covering.

The descent procedure, which is the central part of the proof, involves taking quotients by finite group actions. This has the consequence, that we do not construct a lifting of E itself, but rather of a birational model of E, which has only rational double point singularities.

At this point the theory of simultaneous resolution comes in. In the algebraic category, it was developed by M. Artin in [Art74]. It allowes us to resolve the singularities of the lifting constructed above. The theorem we obtain in this way reads:

Theorem (3.4.3)**.** *Let $E \to C$ be a Jacobian elliptic fibration with tame modular invariant. Then there exists a Jacobian lifting $\widetilde{\mathcal{E}}/\mathcal{C}$ in the category of algebraic spaces over some finite flat extension of the ring of Witt vectors.*

Now, let X/C be an elliptic fibration, which is not assumed to posses a section. The generic fibre X_K, over the fraction field K of C, is a smooth genus-1 curve. We can associate a Jacobian elliptic fibration by forming the minimal regular model E/C of the Jacobian E_K of X_K. We have that X_K is an E_K-torsor which corresponds to a cohomology class $[X_K] \in H^1(\mathrm{Gal}(K^s/K), E_K)$.

To construct a lifting, we need a tameness assumption on the modular invariant of the Jacobian as before, and additionally we need that X/C has a multisection of degree m prime to p. The latter will allow us to describe X by some cocycle taking values in $E_K[m]$. From this a lifting argument can be derived because such a cocycle is easily seen to lift.

With techniques similar to the Jacobian case we prove now:

Theorem (3.4.4)**.** *Let X/C be an elliptic fibration which is minimal and has regular total space. Let E_K the Jacobian of the generic fibre, and let E/C be the minimal regular model of E_K thereof. Assume the modular invariant $j_0 \colon C \to \mathbb{P}^1_k$ is tame, and that X/C posseses a multisection of degree prime to p. Then there exists a lifting $\mathcal{X}/\mathcal{C}$ of X/C in the category of algebraic spaces over a finite flat extension of the ring of Witt vectors.*

Background and motivation

There are several motivations to study liftability. Since algebraic geometry can be done over fields of arbitrary characterstics, the question arises, to what extend geometry over fields of positive characteristic differs from geometry over the complex numbers. One way to approach this question is via liftability.

If one is rather interested in a specific scheme X over a field $k \supset \mathbb{F}_p$, it often makes sense to construct a lifting to characteristic zero, where additional techniques from complex geometry become available. Important invariants, for example Betti numbers, are constant in flat families which are proper and therefore can be read off a specific lifting.

If it turns out that a given X is not only liftable, but also is liftable over the ring of Witt vectors $W_\infty(k)$ (see below), it can be said that X behaves in some respect like a characterstic zero scheme. For example, we have the following theorem by Deligne and Illusie:

Theorem ([DI87]). *Let X/k be a smooth and proper scheme of dimension $< p$. Assume that X is liftable over $W_\infty(k)$. Then the spectral sequence of Hodge to de Rham*

$$E_1^{ij} = H^j(X, \Omega^i_{X/k}) \Rightarrow H^*_{DR}(X)$$

degenerates at E_1. Furthermore, the Kodaira vanishing theorem holds for X.

Following this motivation, we continue with an overview on what is known concerning liftability. Let $k \supset \mathbb{F}_p$ be a perfect field. The first step is to choose a lifting of k itself. There is a canonical choice, namely the ring of Witt vectors $W_\infty(k)$ of k. Abstractly, we can say that $W_\infty(k)$ is, up to isomorphism, the unique complete discrete valuation ring whose maximal ideal is generated by p and whose residue field $W_\infty(k)/(p)$ is k. For example, we have $W_\infty(\mathbb{F}_p) = \mathbb{Z}_p$, the p-adic numbers. For an explicit construction of $W_\infty(k)$ see [Ser79, II 6]. By completness, we can write $W_\infty(k)$ as an inverse limit over its truncations $W_n(k) = W_\infty(k)/(p^n)$.

Now, let X be a smooth and projective k-scheme. The one-dimensional case can be solved entirely with deformation theoretic methods. Namely, assuming X is a curve, we find

$$H^2(X, \Theta_X) = 0,$$

thus infinitesimal deformations of X are unobstructed. Given a deformation $\mathcal{X}_n$ over $W_n(k)$ we can lift it to $\mathcal{X}_{n+1}$ over $W_{n+1}(k)$. Continuing in this fashion, we get an adic $W_\infty(k)$-formal scheme (see 1.2 in Chapter 1). To obtain a lifting in the category of "ordinary" schemes, one applies Grothendieck's Algebraization Theorem (Theorem 1.1.3). To this end, we have to lift an ample line bundle, but again, the obstruction to lifting a line bundle sits inside

$$H^2(X, \mathscr{O}_X) = 0.$$

The case of curves, which was developed in [SGA 1], should be kept in mind as a model case for the deformation theoretic approach toward the lifting problem. The interesting point here is, that the use of explicit equations for X is completely avoided.

For a projective X the "naive" approach of lifting the defining equations comes to mind. In fact, this does work if X is globally a complete intersection [Liu03, Proposition 3.1]. In general however, this is not practical, because the scheme obtained by lifting the equations is not necessarily flat.

Note that the objects we are primary interested in, namely surfaces of Kodaira dimension one, are never complete intersections.

Let us sum up what is know in the case of surfaces. We follow the classification according to Kodaira. Bomberi and Mumford have established in a series of articles, that the Kodaira classification holds, *cum granum salis*, independently of the characteristic.

Every surface of Kodaira dimension $-\infty$ is liftable: For $\mathbb{P}^2$, this is clear. For a ruled surface X, this follows by interpreting it as a projective bundle over a base curve, and then applying deformation arguments. Algebraization is always possible, because we have $H^2(X, \mathscr{O}_X) = 0$, meaning we can lift an ample line bundle.

For X of Kodaira dimension zero, the treatment is split according to the classes of special surfaces.

- Let X be a $K3$-surface. There is a result by Rudakov and Safarevich [RŠ76], showing that it has no global vector fields. By duality, this implies $H^2(X, \Theta_X) = 0$. This settles the existence of formal liftings. The existence of projective liftings was shown by Deligne in [Del81]. The method is to study the (obstructed) deformation functor of X along with an ample line bundle.
- The liftability of abelian surfaces is a special case of more general results for abelian varieties. Those result are due to Mumford and Oort (see [Oor79]).
- In the class of Enriques surfaces, liftability is non-trivial over fields of characteristic 2. This was treated by Liedtke in [Lie10].
- The case of bielliptic surfaces is integrated in this work (see Chapter 1 section 8). It is also true that every quasi-bielliptic surface lifts, but we do not treat that case here.

For surfaces of general type, there are liftability results as well as examples of non-liftable surfaces. It is instructive to recall how the counterexamples are realized. A surface X of general type over $\mathbb{C}$ satisfies the Bogomolov-Miyaoka-Yau inequality, which implies:

$$K_X^2/\chi(\mathscr{O}_X) \leq 9$$

Given a surface over a field $k \supset \mathbb{F}_q$ which violates this inequality, one immediately concludes that it is non-liftable because it involves only numerical invariants which are preserved under deformations. Examples of surfaces, violating Bogomolov-Miyaoka-Yau, have been constructed by Szpiro, Hirzebruch and others (see [Lie09, Section 7] for an overview).

The remaining case in this overview are surfaces of Kodaira dimension one. This is the topic of this work. Recall that every surface of this class has a unique elliptic or quasi-elliptic fibration. A fibration is called quasi-elliptic if the generic fibre is a cuspidal curve of arithmetic genus one.

In comparison to the theory of surfaces of general type, numerical invariants do not play a central role in Kodaira dimension one. Namely, one always has $K_X^2 = 0$ and if X is elliptic and not quasi-elliptic, then $\chi(\mathscr{O}_X) \geq 0$ holds. The last fact was used by Raynaud: In [Ray78] he constructed quasi-elliptic surfaces with $\chi(\mathscr{O}_X) < 0$, which are therefore non-liftable.

In this work we will only consider the case of elliptic surfaces. This class encompasses the class of Kodaira dimension one surfaces and interacts with the special classes of surfaces of Kodaira dimension less than one. Because of the vastness of this class, it seems unlikely that every elliptic surface is liftable. However, giving a counterexamples is not easy because one cannot rely on numerical invariants as in the general type case. The existence of non-liftable elliptic surfaces for small p ranks among the central results of this work.

As regards affirmative liftings results, there are some partial results of Seiler on Weierstrass fibrations for $p \geq 5$ ([Sei88]). In this work, we give a series of lifting results, both for Jacobian and non-Jacobian fibrations. In order to do so we make assumptions like the separability or tameness of the modular invariant (in the Jacobian case) or the existence of a multisection of order prime to p (in the non-Jacobian case). Those assumptions are satisfied by a reasonably broad class. Therefore it can be said that liftability is a rather common behaviour for elliptic surfaces. It is not clear to the author if there are liftability results for surfaces of general type, allowing a comparable statement.

Notations and conventions

Let k be an algebraically closed field. By an *elliptic fibration* we mean a morphism of k schemes: $f\colon X \to C$ where C is a proper, smooth and connected curve over k, and f is proper, flat with 1-dimensional fibers, inducing $\mathscr{O}_C \simeq f_*\mathscr{O}_X$ and with the property that the generic fiber X_η is a smooth genus-1 curve.

By a *minimal regular model* of an elliptic fibration $X \to C$, we mean an elliptic fibration $\widetilde{X}/C$ such that *(a)* we have an isomorphism of generic fibers $X_\eta \simeq \widetilde{X}_\eta$ and *(b)* the total space $\widetilde{X}$ is regular and there are no (-1)-curves in the fibers of $\widetilde{X} \to C$. Minimal regular models of elliptic fibrations exist and are unique. Note that $\widetilde{X}$ only depends on X_η. Therefore we sometimes speak of the minimal regular model of a genus-1 curve over η.

An elliptic fibration is called *Jacobian*, if it possesses a section lying in the open subset of X where $X \to C$ is smooth.

CHAPTER 1

Smooth elliptic fibrations and elliptic fibre bundles

The topic of this chapter is the study of the deformation theory of elliptic fibrations of the simplest kind: so called *elliptic fibre bundles*.

1. Preliminaries: Deformation theory and thickened schemes

We will start with recalling some basic facts and definitions from deformation theory, following Schlessinger's fundamental paper [Sch68]. Let k be an algebraically closed field of characterisitc $p > 0$, and let W denote the ring of infinite Witt vectors over k.

Let $\mathcal{A}\mathrm{lg}_W$ denote the category of artinian local W-algebras with residue field k. Typical objects of $\mathcal{A}\mathrm{lg}_W$ are the rings of Witt vectors of finite length $W_n(k) = W_\infty(k)/(p^{n+1})$. However, since k is a W-algebra, also $k[t]/(t^2)$ is in $\mathcal{A}\mathrm{lg}_W$. So we see that choosing to work over W allows us to treat liftings (i.e. deformations in the arithmetic direction) and k-equivariant deformations uniformly.

Let X be a scheme over k. By a deformation of X over $\Lambda \in \mathcal{A}\mathrm{lg}_W$ we mean a couple $(\mathcal{X}, \epsilon)$ where $\mathcal{X}$ is a *flat* scheme over Λ and ϵ is an isomorphism

$$\epsilon\colon \mathcal{X} \otimes_R k \xrightarrow{\sim} X.$$

A morphism of deformations $(\mathcal{X}_1, \epsilon_1)$ and $(\mathcal{X}_2, \epsilon_2)$ is a Λ-morphism $\varphi\colon \mathcal{X}_1 \to \mathcal{X}_2$ such that $(\varphi \otimes_\Lambda k) \circ \epsilon_1 = \epsilon_2$. In particular, the reduction of φ is an isomorphism, and hence by flatness, φ is an isomorphism.

Thickened schemes (which we denote with curly letters) like $\mathcal{X}/\Lambda$ will play an important role in this work. Due to the lack of a reference we include the following lemma, which illustrates the importance of flatness:

1.1.1. Lemma. *Let $f\colon \mathcal{X} \to \mathrm{Spec}(\Lambda)$ be a flat scheme over some Λ in $\mathcal{A}\mathrm{lg}_W$. Denote its reduction by X. Assume that $H^0(X, \mathscr{O}_X) = k$. Then we have*

$$H^0(\mathcal{X}, \mathscr{O}_X) = \Lambda.$$

Moreover, the statement remains true if we exchange Λ by R and assume X/R to be proper.

PROOF. We define a functor T on Λ-modules as follows

$$M \mapsto H^0(\mathcal{X}, f^*(M)).$$

By flatness, T is left exact. We claim that the natural map $M \to T(M)$ is an isomorphism for every finite length module M over Λ. A module of length one is isomorphic to k. For k the claim is true since $k \simeq H^0(X, \mathscr{O}_X) \simeq H^0(\mathcal{X}, \mathscr{O}_\mathcal{X} \otimes k)$.

We are going to prove the statement by induction: Let M be a Λ-module of finite length. We have a exact sequence

$$0 \to M_1 \to M \to k \to 0. \qquad (1.1.1.1)$$

Here, M_1 is a Λ-module of length $l(M) - 1$. We apply the functor T to (1.1.1.1):

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & k \\ & & \downarrow & & \downarrow & & \downarrow \wr \\ 0 & \longrightarrow & T(M_1) & \longrightarrow & T(M) & \longrightarrow & T(K) \end{array}$$

The map $M_1 \to T(M_1)$ is an isomorphism by induction hypothesis. Thus, by the five-lemma, we see that the middle map is an isomorphism. Finally, we obtain the statement by considering $\Lambda \to T(\Lambda)$.

The statement for X/R follows from the first part by the way of the Theorem of Formal Functions. □

1.1. Lifting of étale coverings. A key problem that will appear in Sections 7 and 8 is of the following form: Given a deformation $\mathcal{X}$ of some scheme X, what properties and additional structures carry over to $\mathcal{X}$? One example for such properties are étale coverings.

1.1.2. Theorem (Theorem 5.5 and Theorem 8.3 [SGA 1])**.** *Let $\mathcal{S}$ be a scheme with a closed subscheme S_0 having the same topological space as $\mathcal{S}$ itself. Then the functor*

$$\mathcal{X} \mapsto \mathcal{X} \times_{\mathcal{S}} S_0$$

form the category of étale $\mathcal{S}$-schemes to the category of étale S_0-schemes is an equivalence of categories.

This theorem can be seen as a geometric form of Hensel's Lemma from commutative algebra. We note two special cases: The categories of étale Galois covering of $\mathcal{X}$ and X_0 are equivalent, and so are the categories of finite étale group schemes. Recall that an étale covering $S' \to S$ is called Galois with group G if G acts on S' as an S-scheme and we have an isomorphism

$$G \times S' \simeq S' \times_S S' \text{ given by } (\sigma, x) \mapsto (\sigma(x), x).$$

1.2. Grothendieck's Algebraization Theorem. Let X be a k-scheme, and let R be a complete local ring with $R/m = k$. The problem of constructing a lifting of X over R can be subdivided into two subproblems: The first one is the construction of a lifting $\mathcal{X} = \varinjlim X_n$ in the category of formal schemes. The second one is to algebraize the formal lifting. By this, we mean the construction of an R-scheme $\overline{\mathcal{X}}$ with the property that the completion $\varinjlim_n (\overline{\mathcal{X}} \otimes R/m^{n+1})$ is isomorphic to $\mathcal{X}$.

To accomplish the first step, we can use techniques from deformation theory. In particular, in the case where X is smooth, there is a cohomological condition, namely $H^2(X, \Theta_X) = 0$, which is sufficient for the existence of a formal lifting.

As for second step, this is a different kind of problem. It is not clear what a direct approach would look like. Indeed, the problem of deciding whether a given formal deformation can be algebrized is a subtle one. It corresponds to convergence problems which come up in the deformation theory of complex manifolds. Fortunately, there is the following theorem of Grothendieck:

1.1.3. Theorem ([EGA III.1] Theorem 5.4.5)**.** *Let $\mathcal{X}$ be a proper, adic* Spf(R)*-formal scheme. Let $\mathscr{L}$ be an invertible sheaf on $\mathcal{X}$ with the property that the restriction $\mathscr{L} \otimes_R k$ is ample on X. Then $\mathcal{X}$ is algebraizable.*

This theorem essentially translates the algebraization problem into a deformation theory problem: We have to find a formal lifting, together with a lifting of a line bundle L over X, which is ample.

We recall the notion of an adic Spf(R)-formal scheme (see [EGA I, 10.12]). This is the type of formal scheme which arises by "piling up" infinitesimal deformation over the truncations of R. More generally, a morphism of locally noetherian formal schemes $f\colon \mathcal{X} \to \mathcal{Y}$ is called adic, if it exists an ideal of definition $\mathcal{J}$ of $\mathcal{Y}$ such that $\mathcal{I} = f^*(\mathcal{J})\mathscr{O}_{\mathcal{X}}$ is an ideal of definition of $\mathcal{X}$. We also say $\mathcal{X}$ is an adic $\mathcal{Y}$-formal scheme.

Setting $\mathcal{X}_n = (\mathcal{X}, \mathscr{O}_{\mathcal{X}}/\mathcal{I}^{n+1})$ and $\mathcal{Y}_n = (\mathcal{Y}, \mathscr{O}_{\mathcal{Y}}/\mathcal{J}^{n+1})$ we obtain, for $m \leq n$, a cartesian diagram:

$$\begin{array}{ccc} X_m & \longrightarrow & X_n \\ \downarrow{\scriptstyle f_m} & \square & \downarrow{\scriptstyle f_n} \\ Y_m & \longrightarrow & Y_n \end{array}$$

An inductive system of such cartesian diagrams is called adic Y_n-inductive system. The category of those inductive systems is equivalent to the category of adic $\mathcal{Y}$-formal schemes.

2. Elliptic fibre bundles: Definition and examples

1.2.1. Definition. Let S be a scheme over some ring R. An R-morphism $X \to S$ is called *elliptic fibre bundles* if étale locally on S, it is isomorphic to a trivial family of elliptic curves. More precisely: For every closed point $x \in S$ there exists an étale neighborhood $U \to S$ and an elliptic curve E over R such that

$$U \times_S X \simeq U \times_{\mathrm{Spec}(R)} E.$$

We observe that an elliptic fibre bundle $X \to S$ is a proper and smooth morphism. Later on, we will see that every *smooth* elliptic fibration over a proper base is in fact an elliptic fibre bundle. The notion of elliptic fibre bundles should not be confused with so called *iso-trivial* elliptic fibration, because an iso-trivial fibration is not necessarily smooth.

1.2.2. *Example* Let k be field, and let C be an elliptic curve over k, such that C has a non-trivial 2-torsion point $c \in C$. Given another elliptic curve E over k we can make the following construction:

On the product $E \times_k C$ we have a diagonal action of $G = \mathbb{Z}/2\mathbb{Z}$ namely:

$$(x, y) \mapsto (-x, y + c).$$

This action is free because it is free on the second factor. Thus the quotient $X = (E \times C)/G$ will be a smooth surface. The second projection is equivariant for the G-action, thus it descends to $f\colon X \to C/G$. We sum up everything in a diagram:

$$\begin{array}{ccc} E \times C & \longrightarrow & C \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & C/G \end{array} \qquad (1.2.2.1)$$

We observe that (1.2.2.1) is in fact cartesian, because $X \times_{C/G} C$ is a $\mathbb{Z}/2\mathbb{Z}$-torsor over X just like $E \times C$, and the natural map $E \times C \to X \times_{C/G} C$ is equivariant.

From (1.2.2.1) if follows at once that $X \to C/G$ is an elliptic fibre bundle: We have one global covering, namely $C \to C/G$ such that the pullback of X splits into a product.

What makes it interesting to study deformations of such straight forward objects? If the construction given above is carried out over a field of characteristic two, then the surface X is denote as *Igusa-surface*. Many specific characteristic p

phenomena can be demonstrated with this example. Igusa himself was the first to observe this in his classic paper [Igu55]. He calculated the invariants

$$h^1(X, \mathscr{O}_X) = 2 \text{ and } b_2(X) = 2.$$

This is impossible for a complex algebraic surface. The modern interpretation of this phenomenon is that X has a non-reduced Picard scheme. Namely $b_2(X)/2$ can be interpreted as the dimension of the Albanese variety of X, which equals the dimension of the reduction of the Picard scheme, whereas $h^1(X, \mathscr{O}_X)$ can be interpreted as the tangent space at the identity of the Picard scheme.

We will see later on, how the deformation theory of the Igusa surface differs from its well behaved relatives over fields of odd characteristics.

Another motivation is the existence of non-liftable elliptic fibre bundles over fields of characteristic two and three. In fact, once we understand the deformation theory of elliptic fibre bundles well enough, it will be easy to show that the examples given in Section 5 are in fact non-liftable surfaces.

At the end of Section 4 we will give a classification of elliptic fibre bundles over smooth and proper curves.

3. Elliptic curves and their moduli

The main tool of this section will be the theory of moduli of elliptic curves as it is explained in "Arithmetic moduli of elliptic curves" by Katz and Mazur [KM85]. We recall the definition of an elliptic curve over a base scheme S: An elliptic curve E/S is a proper smooth curve with geometrically connected fibers of genus one, with a given section $\epsilon \colon S \to E$. There exists a unique commutative group scheme structure with zero section ϵ on J/S (see [KM85, Theorem 2.1.2]).

We define the *moduli stack of elliptic curves* $\mathcal{M}_1$ to be the following category fibered over the category of all schemes: Objects of $\mathcal{M}_1$ are elliptic curves E/S over arbitrary base schemes S. Morphism in $\mathcal{M}_1$ are given by cartesian diagram

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & \square & \downarrow \\ S' & \longrightarrow & S \end{array} \tag{1.3.0.2}$$

such that the induced morphism $E' \to E \times_S S'$ is an isomorphism of elliptic curves over S'. This fibered category is in fact a Deligne-Mumford stack.

The stack $\mathcal{M}_1$ cannot be a scheme, because any elliptic curve has a non-trivial automorphism: namely the involution. By fixing additional data on elliptic curves, we can get rid of those automorphisms. In this chapter, we will make use of the following fundamental construction:

Let $\mathcal{M}_{\Gamma(n)}$ be the stack over the category of schemes, whose objects are pairs $(E/S, \gamma)$ where γ is an isomorphism $E[n] \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^2$. We also speak of $(E/S, \gamma)$ as an elliptic curve with a *full level-n structure*. Morphism are defined as in $\mathcal{M}_1$, with the restriction that they have to be compatible with the level structures.

Note that there are no elliptic curves with full level-n structures over a scheme S if n is not invertible in $\mathscr{O}_S$.

To determine the automorphism group of a pair (E, γ) we use the following theorem, which is of independent use for us:

1.3.1. Theorem ([KM85] Corollary 2.7.3)**.** *Let $\omega \colon E \to E$ be an automorphisms of an elliptic curve E over a connected base S. Let $d \geq 4$ be an integer and let $G \subset E$ be a subgroup scheme which is locally free over S of order d. If ω induces* *the identity morphism of G and $d > 4$ then $\omega = \mathrm{Id}$. If $d = 4$ then $G = E[2]$ and $\omega = -\mathrm{Id}$.*

Recall that the order of $E[n]$ is n^2. Thus we get that the objects of $\mathcal{M}_{\Gamma(n)}$ have no non-trivial automorphisms, given $n \geq 3$. Once we have seen that there are no automorphisms, it makes sense to inquire about representability. In fact there is the following theorem:

1.3.2. Theorem ([KM85] Corollary 4.7.2)**.** *Given $n \geq 3$, the moduli stack of elliptic curves with full level-n structures $\mathcal{M}_{\Gamma(n)}$ is representable by a smooth affine curve over $\mathbb{Z}[1/N]$.*

Let A be some ring. To use geometric arguments in combination with moduli theory, it will be useful for us to work in the category $\mathcal{M}_1/A$ instead of $\mathcal{M}_1$. The objects in $\mathcal{M}_1/A$ are elliptic curves over A-schemes, and the morphisms are cartesian squares where the bottom arrows are A-linear. The category $\mathcal{M}_{\Gamma(n)}/A$ is defined analogously. Let X be the scheme representing $\mathcal{M}_{\Gamma(n)}$. Then $\mathcal{M}_{\Gamma(n)}/A$ is represented by $X \otimes A$ (see [KM85, 4.13]).

Abusing notation, we shall write $\mathcal{M}_{\Gamma(n)}$ for X.

4. Deformations of Jacobian elliptic fibre bundles

A *Jacobian* elliptic fibre bundle is a pair $(J/S, \epsilon)$ where $J \to S$ is an elliptic fibre bundle and ϵ is a section of $J \to S$. We can consider $(J/S, \epsilon)$ as an elliptic curve over S. In particular, there exists a unique commutative group scheme structure on J/S. However, it should be noted, that not every elliptic curve is an elliptic fibre bundle: For E/S to be fibre bundle, it is a necessary condition that the modular invariant of E/S is constant.

Let k be an algebraically closed field of characteristic $p > 0$, and denote by W the ring of Witt vectors of k. We are going to work over $\Lambda \in \mathcal{A}\mathrm{lg}_W$, i.e. a local artinian W-algebra with residue field k.

1.4.1. Proposition. *Let $\mathcal{S}$ be a proper flat Λ scheme such that $\mathcal{S} \otimes_\Lambda k$ is integral. Let $\mathcal{J}/\mathcal{S}$ be an elliptic curve over a base scheme. For some integer $n \geq 3$ which is prime to p, assume that the n-torsion subgroup scheme of $\mathcal{J}$ is split i.e; there is an isomorphism*

$$(\mathbb{Z}/n\mathbb{Z})^2 \simeq \mathcal{J}[n].$$

Then there exists an elliptic curve $\mathcal{E}$ over Λ such that $\mathcal{J}$ is isomorphic to $\mathcal{E} \times_\Lambda \mathcal{S}$.

Proof. We can choose a level-n-structure on $\mathcal{J}/\mathcal{S}$. By Theorem 1.3.2 we obtain a morphism $c\colon \mathcal{S} \to \mathcal{M}_{\Gamma(n)}$ such that $\mathcal{J} \simeq c^*(\mathcal{E}^{univ})$ where $\mathcal{E}^{univ}$ is the universal family of the moduli problem; i.e. $\mathcal{E}^{univ}$ is an elliptic curve with a level-n structure, but the level structure will no longer be relevant for us.

Again by Theorem 1.3.2 we know that $\mathcal{M}_{\Gamma(n)}$ is affine. Thus c factors over the affine hull of $\mathcal{S}$, namely $\mathrm{Spec}(H^0(\mathcal{S}, \mathscr{O}_\mathcal{S}))$. This is just $\mathrm{Spec}(\Lambda)$ by Lemma 1.1.1. Therefore $\mathcal{J}$ is just the pullback of an elliptic curve $\mathcal{E}$ over Λ. □

In particular, we see that $\mathcal{J}/\mathcal{S}$ is an elliptic fibre bundle under the assumptions of Proposition 1.4.1. This can be generalized, because for an arbitray elliptic curve $\mathcal{J}/\mathcal{S}$, and an interger n prime to p, we always have that $\mathcal{J}[n]$ is a finite and étale group scheme over $\mathcal{S}$, so there exists an étale Galois covering $\mathcal{S}' \to \mathcal{S}$, such that $\mathcal{J}[n] \times_\mathcal{S} \mathcal{S}' \simeq (\mathbb{Z}/n\mathbb{Z})^2$.

1.4.2. Proposition. *Let $\mathcal{S}$ be a proper flat Λ-scheme, such that the special fibre $\mathcal{S} \otimes_\Lambda k$ is regular. Let $\mathcal{J}/\mathcal{S}$ be an elliptic fibre bundle, and let $\mathcal{S}' \to \mathcal{S}$ be a finite* *étale Galois covering with group G such that $\mathcal{J}[n]\times_{\mathcal{S}}\mathcal{S}' \simeq (\mathbb{Z}/n\mathbb{Z})^2$ for some $n \geq 3$. Then*

$$\mathcal{J} \simeq (\mathcal{E} \times_\Lambda \mathcal{S}')/G,$$

where $\mathcal{E}$ is an elliptic over Λ, and the action is the diagonal action given by the Galois action on $\mathcal{S}'$ and by a homomorphism $G \to \mathrm{Aut}(\mathcal{E})$ on the left factor.

Proof. The scheme $\mathcal{S}'$ is connected, and because $\mathcal{S}\otimes_\Lambda k$ is regular, so is $\mathcal{S}'\otimes_\Lambda k$. In particular, $\mathcal{S}'\otimes_\Lambda k$ is integral. Hence the elliptic curve

$$\mathcal{J}\times_{\mathcal{S}}\mathcal{S}' \to \mathcal{S}'$$

satisfies the assumptions of Proposition 1.4.1. Thus there exists an elliptic curve $\mathcal{E}$ over Λ and an isomorphism $\mathcal{J}\times_{\mathcal{S}}\mathcal{S}' \simeq \mathcal{E}\times_\Lambda \mathcal{S}'$. In other words, we know that $\mathcal{J}$ and $\mathcal{E}\times_\Lambda\mathcal{S}'$ are twists of each other, becoming isomorphic under the base change $\mathcal{S}'\to\mathcal{S}$.

Twists of the fibration $\mathcal{E}\times_\Lambda\mathcal{S}$ are classified up to isomorphism by the Galois cohomology set $H^1(G, A(\mathcal{S}'))$, where A is the group scheme $\mathrm{Aut}(\mathcal{E}\times_\Lambda\mathcal{S})$ and we consider its $\mathcal{S}'$-valued points as Galois modul under G.

We claim that the Galois action on A is trivial: We have a closed immersion $A\subset \mathrm{Aut}(\mathcal{E}[n]\times_\Lambda\mathcal{S})$ by rigidity (Theorem 1.3.1). However, since Λ is a strict henselian ring, we find that $\mathcal{E}[n]\times_\Lambda\mathcal{S}$ is the constant group scheme $(\mathbb{Z}/n\mathbb{Z})^2$ on $\mathcal{S}$ which in turn implies that $\mathrm{Aut}(\mathcal{E}[n]\times_\Lambda\mathcal{S})$ is the constant group scheme $\mathrm{GL}(2,\mathbb{Z}/n\mathbb{Z})$ on $\mathcal{S}$.

Finite étale group schemes over $\mathcal{S}$ correspond to finite abstract groups with a continuous $\pi_1(\mathcal{S})$-action. We saw that A can be embedded into a group scheme with trivial $\pi_1(\mathcal{S})$-action, hence the action on A has to be trivial as well. The action of G on A is an induced action of a finite quotient $\pi_1(\mathcal{S}) \twoheadrightarrow G$, and therefore trivial as well. Thus

$$H^1(G, \mathrm{Aut}(\mathcal{E}\times_\Lambda\mathcal{S})(\mathcal{S}')) \simeq \mathrm{Hom}(G, \mathrm{Aut}(\mathcal{E}\times_\Lambda\mathcal{S})(\mathcal{S}')).$$

For a homomorphism ρ in the above group, the corresponding twist looks like $(\mathcal{E}\times_\Lambda\mathcal{S}')/G$, where the action of $\sigma\in G$ is given by

$$(x,y)\mapsto(\rho(\sigma)(x),\sigma y).$$ □

Now, let $S\simeq\mathcal{S}\otimes_R k$ denote the reduction of $\mathcal{S}$. Given an elliptic curve E/S we can use the above results to give a necessary and sufficient criterion for the existence of Jacobian liftings:

1.4.3. Corollary. *Let J be an elliptic curve over S, given by $(E\times_k S')/G$ for some étale Galois covering $S'\to S$ with group G (Proposition 1.4.2). Denote the action of G on E by ρ_0. Then there exists a lifting $\mathcal{J}\to\mathcal{S}$ if and only if there exists a lifting $\mathcal{E}$ of E over Λ together with an extension of the action ρ_0.*

Proof. To show sufficiency is easy. The covering $S'\to S$ lifts uniquely to $\mathcal{S}'\to\mathcal{S}$ which is again Galois with group G. If a lifting $\mathcal{E}$ of E with the prescribed properties exists, simply put $\mathcal{J} = (\mathcal{E}\times_\Lambda\mathcal{S}')/G$. This quotient will exist in the category of schemes because G is finite.

In order to show necessity, assume that we have a lifting $\mathcal{J}\to\mathcal{S}$. Like before, we also have the unique lifting $\mathcal{S}'\to\mathcal{S}$ of the Galois covering. Observe that $\mathcal{J}[N]\times_{\mathcal{S}}\mathcal{S}'$ is split, since $\mathcal{J}[N]$ is a finite and étale group scheme and the reduction is split by assumption. Using Proposition 1.4.2, we find that $\mathcal{J}\simeq(\mathcal{E}\times_\Lambda\mathcal{S}')/G$, where the action of G on $\mathcal{E}$ is denoted by ρ. We claim that ρ lifts the action ρ_0:

Consider the induced action of ρ on $\mathcal{E}[N]$ for some integer N. The categories of étale group schemes over k and R are equivalent, hence ρ is determined by its action on the reduction $E[N]$.

For $N \geq 3$ we know that the group homomorphisms, given by restricting the automorphism group of an elliptic scheme to its N-torsion is injective [KM85, Corollary 2.7.2]. However the isomorphism type of $J[N]$ allows to read of the action of G on $J[N]$, for it is given by a class in

$$H^1(G, \mathrm{Aut}(J[N])(S')) \simeq \mathrm{Hom}(G, \mathrm{Aut}(E[N])(S'))$$

and the element of the latter group which corresponds to $J[N]$ is just ρ_0. Hence the restriction of ρ to the reduction has to be ρ_0. □

The result of this section can be aplied to give a rough classification of elliptic fibre bundles over smooth and proper curves over k. Non-Jacobian elliptic fibre bundles are closely related to the Jacobian ones: Let X/C be an elliptic fibre bundle. We have that the relative Picard functor $\mathrm{Pic}_{X/C}$ is representable, and that its identity component is a Jacobian elliptic fibre bundle over J/C. This bundle is called the *Jacobian* of X. Certain properties of X can be derived from those of J. The following proposition makes exemplary use of this:

1.4.4. Proposition. *The linebundle $\mathscr{L} \simeq R^1 f_* \mathscr{O}_X$ associated to an elliptic fibre bundle $f\colon X \to C$ is a torsion line bundle.*

Proof. Let $g\colon J \to C$ be the Jacobian of X, and dentote by $\epsilon\colon C \to J$ the zero section. Since $J = \mathrm{Pic}^0(X/C)$ we have that

$$\mathscr{L} \simeq \mathrm{Lie}(J/C) \simeq \epsilon^* \Theta_{J/C}$$

(see [LLR04, Proposition 1.3]). Thus it suffices to show that $\Theta_{J/C}$ is a torsion line bundle. By Proposition 1.4.2, we know that J has an étale Galois covering of group G fitting into the diagram

$$\begin{array}{ccc} E \times C' & \xrightarrow{q} & J \\ \downarrow & & \downarrow \\ C' & \longrightarrow & C \end{array}$$

We write J' for $E \times C'$ Since q is étale, it follows $q^*\Theta_{J/C} \simeq \Theta_{J'/C'} \simeq \mathscr{O}_{J'}$. Now, we aplly the norm map $N\colon \mathrm{Pic}(J') \to \mathrm{Pic}(J)$ (see [EGA II, 6.5]), associated to the morphism q:

$$\mathscr{O}_J \simeq N(q^*\Theta_{J'/C'}) \simeq \Theta_{J/C}^{\otimes d}$$

where d is the degree of q. This proves the claim. □

Let X/C be an elliptic fibre bundle. It is a consequence of the above, and of the canonical bundle formula

$$\omega_X \simeq f^*(\mathscr{L}^{-1} \otimes \omega_C)$$

that the Kodaira dimension of an elliptic fibre bundle is uniquely determind by its base curve. This allows the following classification, with respect to the base curve:

- $C \simeq \mathbb{P}^1_k$: X is of the form $\mathbb{P}^1_k \times E$ for an elliptic curve E over k.
- $g(C) = 1$: X is either an abelian surfaces or a bielliptic surface.
- $g(C) \geq 2$: X is an elliptic surface of Kodaira dimension 1.

In this work, we will study deformations of bielliptic surfaces and of elliptic fibre bundles of Kodaira dimension 1. One of the results is that these classes are stable under deformations. The case with $C \simeq \mathbb{P}^1_k$ is trivial, and deformations of abelian surfaces are well understood.

5. Non-liftable elliptic fibre bundles

We postpone the development of the general theory at this point to give some specific examples of Jacobian elliptic bundles that do not have a lifting to characteristic zero.

Characteristic three. For the first example, let k be an algebraically closed field of characteristic three, and let E be an elliptic curve over k with j-invariant 0. By [Sil09, Appendix A, Proposition 1.2] the automorphism group G of E is a semidirect product $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$ where $\mathbb{Z}/4\mathbb{Z}$ acts on $\mathbb{Z}/3\mathbb{Z}$ in the unique non trivial way.

As we shall see later on, there exists a smooth and proper curve C over k such that there is a surjection $\pi_1(C) \twoheadrightarrow G$. Denote by $C' \to C$ the associated finite and étale Galois cover. Now we set

$$J = (E \times_k C')/G,$$

where the action of G on E is the action of the automorphism group.

1.5.1. Lemma. *Let the characterstic of k be three, and let Λ be in $\mathcal{A}\mathrm{lg}_W$. If for an elliptic curve $\mathcal{E}$ over Λ the order of the automorphism group of $\mathcal{E}$ is greater than six, it follows $3 \cdot \Lambda = 0$.*

Proof. Assume by contradiction, that the order of $\mathrm{Aut}_0(\mathcal{E})$ is greater than six. Since two is a unit, there is a Weierstraß equation for $\mathcal{E}$ of the following form:

$$y^2 = x^3 + a_2x^2 + a_4x + a_6$$

Admissible transformations look like $x \mapsto u^2x + r$ and $y \mapsto u^3y + u^2sx + t$. The specific form of the equation implies $t = 0$ and $s = 0$. Standart arguments show that either $u^4 = 1$ or $u^6 = 1$. Thus, an automorphism group of order greater than six would have to contain an element of the form $x \mapsto x + r$.

We get an equation $a_2 = a_2 + 3r$, which implies $3r = 0$. But r has to be a unit, for otherwise the reduction map would not be injective on the automorphism group. Thus $3 = 0$ follows. □

Now we get as a direct consequence of Corollary 1.4.3:

1.5.2. Proposition. *The elliptic bundle J can be lifted (as Jacobian fibration) only over rings in which $3 = 0$ holds.*

Characteristic two. Now assume k is a field of characteristic two. Given an elliptic curve E over k with j-invariant 0, the group of automorphisms will be a semidirect product $G = Q \rtimes \mathbb{Z}/3\mathbb{Z}$, where Q is the quaternion group. Similarly to Lemma 1.5.1, one shows that neither G nor Q can lift to rings with $2 \neq 0$. Now assume the existence of two curves C_G and C_Q together with étale Galois covers $C'_G \to C_G$ of group G and $C'_Q \to C_Q$ of group Q respectively.

This gives rise to two Jacobian elliptic fibre bundles: $J_G \simeq (C'_G \times E)/G$ and $J_Q \simeq (C'_Q \times E)/Q$. Again by Corollary 1.4.3 it follows:

1.5.3. Proposition. *The elliptic bundles J_G and J_Q can be lifted (as Jacobian fibrations) only over rings in which $2 = 0$ holds.*

To finish this discussion, we have to establish the existence of curves with specific étale Galois coverings.

To this end, we use a powerful theory which is developed in [PS00]. First we fix some group theoretic invariants. Let G be a finite group with the property that the maximal p-Sylow subgroup P is normal. We set $H = G/P$. Then one can write G as a semidirect product $P \rtimes H$.

We denote by $\mathcal{P}$ the maximal elementary abelian quotient of P, and consider it as a $\mathbb{F}_p$-vector space, which is possible since it is a p-torsion group. Let $Z(H)$

be the set of irreducible characters with values in k, and let V_χ be an irreducible k-representation of H with character χ. On P, we have an H action coming from the structure of the semidirect product. This induces an H-representation on $\mathcal{P}$. Since H is of order prime to p, this representation is semisimple, and we write

$$\mathcal{P} \otimes_{\mathbb{F}_p} k \simeq \oplus V_\chi^{m_\chi}.$$

The m_χ are thus numerical invariants of the group G.

Theorem (Theorem 7.4 [PS00]). *Let G be a group having a normal p-Sylow subgroup P. Suppose $H = G/P$ is abelian. Then there exists a curve of genus $g \geq 2$ having an étale Galois covering with group G if the minimal number of generators of H is less or equal than $2g$, and $m_\chi \leq g - 1$ holds for every $\chi \in Z(H)$.*

In the characteristic three example we had $G = \mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$. The minimal number of generators of $H = \mathbb{Z}/4\mathbb{Z}$ is one, the representation of H on $\mathcal{P}$ is obviously irreducible and given by the sign involution. Thus the assumption of the theorem are satisfied for some curve of genus 2.

In the characteristic two examples we also have that the maximal p-Sylow group is normal. Thus for g sufficiently large, we will find curves with the required coverings.

Later on, we will prove that J, J_G and J_Q are non-liftable even if we drop any additional assumptions on the liftings. This is done in two steps: First, we prove that a lifting in the category of elliptic fibre bundles exists if and only if a Jacobian lifting exists (this is the content of Section 6). The final step is to see that *every* lifting of an elliptic fibre bundle of Kodaira dimension one is an elliptic fibre bundle. This will be done in Section 7.

6. Deformations of elliptic torsors

We start with some general theory on deformations of torsors under smooth commutative group schemes. This mainly rephrases [SGA III.2, Remarque 9.1.9].

We use the same notations as in the previous sections, i.e. k is an algebraically closed field, W its ring of Witt vectors, and $\mathcal{A}\mathrm{lg}_W$ is the category of artinian local W-algebras with residue field k. We fix a small extension of algebras in $\mathcal{A}\mathrm{lg}_W$

$$0 \to I \to \Lambda \to \Lambda_0 \to 0.$$

Let $\mathcal{S}$ be a flat Λ-scheme and set $\mathcal{S}_0 = \mathcal{S} \otimes_\Lambda \Lambda_0$. We have a closed immersion $i\colon \mathcal{S}_0 \to \mathcal{S}$. Let $\mathcal{G}$ be a smooth commutative $\mathcal{S}$-group scheme and set $\mathcal{G}_0 = \mathcal{G} \times_{\mathcal{S}} \mathcal{S}_0$.

For a group functor $\mathcal{F}$ on the category of $\mathcal{S}_0$ schemes, we defined the pushforward functor $i_*\mathcal{F}$ on $\mathcal{S}$-scheme by sending a $\mathcal{S}$-scheme $\mathcal{T}$ to $\mathcal{F}(\mathcal{T} \times_{\mathcal{S}} \mathcal{S}_0)$. There is a natural specialization map $s\colon \mathcal{G} \to i_*\mathcal{G}_0$ of group functors. To investigate its kernel, we introduce a coherent sheaf on $\mathcal{S}_0$:

$$\mathscr{L} = \mathrm{Lie}(\mathcal{G}_0/\mathcal{S}_0) \otimes_{\mathscr{O}_{\mathcal{S}_0}} I.$$

We have a sequence of group functors on $\mathcal{S}$ namely

$$0 \to i_*\mathscr{L} \to \mathcal{G} \xrightarrow{s} i_*\mathcal{G}_0 \to 0, \qquad (1.6.0.1)$$

whose exactness follows from the smoothness of $\mathcal{G}_0$, as can be seen affine locally. Taking étale cohomology of (1.6.0.1), we obtain the fundamental long exact sequence

$$0 \to i_*\mathcal{G}_0(\mathcal{S})/s(\mathcal{G}(\mathcal{S})) \to H^1(\mathcal{S}, i_*\mathscr{L}) \to H^1(\mathcal{S}, \mathcal{G}) \xrightarrow{s} \to H^1(\mathcal{S}, i_*\mathcal{G}_0) \to H^2(\mathcal{S}, i_*\mathscr{L}). \qquad (1.6.0.2)$$

The sheaves $\mathcal{L}$ and $i_*\mathcal{L}$ are coherent modules. We find

$$H^i(\mathcal{S}, i_*\mathscr{L}) \simeq H^i(\mathcal{S}_0, \mathscr{L}) \simeq H^i_{zar}(\mathcal{S}, \mathrm{Lie}(G_0/S_0)) \otimes I.$$

Moreover, we claim that the group $H^1(\mathcal{S}, i_*\mathcal{G}_0)$ is isomorphic to $H^1(\mathcal{S}_0, \mathcal{G}_0)$: By the Leray spectral sequence we get an exact sequence of étale cohomology groups

$$0 \to H^1(\mathcal{S}, i_*\mathcal{G}_0) \to H^1(\mathcal{S}_0, \mathcal{G}_0) \to H^0(\mathcal{S}_0, R^1 i_*\mathcal{G}_0),$$

with vanishing last term: It is enough to show that $(R^1 i_*\mathcal{G}_0)_x = 0$ for every closed point x of $\mathcal{S}$; i.e. of $\mathcal{S}_0$. By [Mil80, Theorem 1.15] it follows that $(R^1 i_*\mathcal{G}_0)_x \simeq H^1(\mathrm{Spec}(\mathscr{O}^\wedge_{\mathcal{S}_0,x}), \mathcal{G}_0)$ and the last group vanishes since $\mathrm{Spec}(\mathscr{O}^\wedge_{\mathcal{S}_0,x})$ has only the trivial étale covering. Here, we make use of the assumption that k is algebraically closed.

In our situation, this means the following: Let J/C be a Jacobian elliptic fibre bundle over a curve C over k. Let $\mathcal{J}_0/\mathcal{C}_0$ be a Jacobian lifting of J/C over $\Lambda_0 \in \mathcal{A}\mathrm{lg}_W$. As above, we fix a small extension

$$0 \to I \to \Lambda \to \Lambda_0 \to 0.$$

1.6.1. Proposition. *Let $\mathcal{J} \to \mathcal{C}$ be a Jacobian lifting of $\mathcal{J}_0/\mathcal{C}_0$ to Λ. Then every $\mathcal{J}_0$-torsor $\mathcal{X}_0$ over $\mathcal{C}_0$ lifts to a $\mathcal{J}$-torsor over $\mathcal{C}$. Furthermore, let m be an integer prime to p. Then the restriction map*

$$H^1(\mathcal{C}, \mathcal{J})[m] \xrightarrow{\sim} H^1(\mathcal{C}_0, \mathcal{J}_0)[m]$$

is bijective. This means that liftings of torsors are unique up to p-torsion.

Proof. From the sequence (1.6.0.2) we know that the obstruction to lifting the cohomology class associated to $\mathcal{X}_0$ lies inside $H^2(\mathcal{C}_0, \mathrm{Lie}(\mathcal{J}_0/\mathcal{C}_0)) \otimes I$. Note that since $\mathrm{Lie}(\mathcal{J}_0/\mathcal{C}_0)$ is a coherent $\mathscr{O}_{\mathcal{C}_0}$-modul, we can compute its cohomology with respect to the Zariski topology. Since $\mathcal{C}_0$ is one-dimensional, this group is zero.

Once we have lifted the cohomology class, we have to answer the question, whether it is associated to a representable $\mathcal{J}$-torsor. By Lemma 1.6.3 below, this will be the case if it is torsion. We claim that $H^1(\mathcal{C}, \mathcal{J})$ is torsion: Since $H^1(C, J)$ is torsion, it is enough to show that $H^1(\mathcal{C}_0, \mathrm{Lie}(\mathcal{J}_0/\mathcal{C}_0)) \otimes I$ is torsion, then the assertion will follow by induction. But the former group is a Λ-module, and Λ itself is annihilated by some power of p.

The second statement follows now directly by taking m-torsion in (1.6.0.2). □

1.6.2. *Remark* In the case where the base is zero dimensional, one recovers the well known fact that the Tate-Šafarevič group of an elliptic curve over a complete local ring with algebraically closed residue field is zero, since the first cohomology of the Lie algebra vanishes.

The following lemma by Raynaud concludes the proof of Proposition 1.6.1 and will be useful for us in other situations as well:

1.6.3. Lemma ([Ray70] Lemme XIII)**.** *Let A be an projective abelian scheme over some base scheme S. Let $[X] \in H^1(S, A)$ be a m-torsion cohomology class and let X be its representing algebraic space. Then we have a canonical finite morphism $X \to A$. In particular, X is a projective scheme over S.*

For a proof of a generalized version, see Lemma 2.7.2 below. We want to rephrase Proposition 1.6.1 in the language of deformation functors. For that purpose, we define two deformation functors associated to an elliptic bundle $X \to C$ over k.

1.6.4. Definition. By a deformation of X over some $\Lambda \in \mathcal{A}\mathrm{lg}_W$, we mean a pair $(\mathcal{X}, \epsilon)$, where $\mathcal{X}$ is a flat scheme over $\mathrm{Spec}(\Lambda)$ and ϵ is an isomorphisms $\epsilon\colon \mathcal{X} \otimes_\Lambda k \simeq X$. Let $\mathcal{D}\mathrm{ef}_X\colon \mathcal{A}\mathrm{lg}_W \to (Sets)$ denote the functor, which sends $\Lambda \in \mathcal{A}\mathrm{lg}_W$ to the set of isomorphism classes of deformations of X/C.

By a *deformation of a fibration* X/C, we understand a deformation $(\mathcal{X}, \epsilon)$ of X together with a map $\mathcal{X} \to \mathcal{C}$, such that the isomorphism ϵ is in fact an isomorphism of C schemes. The functor of deformation of X as fibration is denoted by $\mathcal{F}\mathrm{ib}_{X/C}\colon \mathcal{A}\mathrm{lg}_W \to (Sets)$.

Two deformations of $\mathcal{X}/\mathcal{C}$ and $\mathcal{X}'/\mathcal{C}$ are called *isomorphic* if there exists an isomorphisms of deformations, which is also an isomorphisms of $\mathcal{C}$-schemes.

Let $(J/C, e_0)$ be a Jacobian elliptic fibre bundle. We define the functor $\mathcal{J}\mathrm{ac}_{J/C}$ by sending $\Lambda \in \mathcal{A}\mathrm{lg}_W$ to a pairs $(\mathcal{J}/\mathcal{C}, e)$, where $\mathcal{J}/\mathcal{C}$ is a deformation of J/C and e is a lift of e_0. For an element $(\mathcal{J}, e)$ of $\mathcal{J}\mathrm{ac}_{J/C}$ notice that a different choice of e leads to an isomorphic element of $\mathcal{J}\mathrm{ac}_{J/C}$. Thus we view $\mathcal{J}\mathrm{ac}_{J/C}$ as a subfunctor of $\mathcal{F}\mathrm{ib}_{J/C}$; i.e. the subfunctor of those deformations admitting a section.

We get a natural map $\mathcal{F}\mathrm{ib}_{X/C} \to \mathcal{J}\mathrm{ac}_{J/C}$ as follows: For a deformation $\mathcal{X}/\mathcal{C}$ (not necessarily having a section) we consider the zero component of its Picard scheme. Since k is of characterstic p, we can always lift an appropriate p-th power of a relativly ample line bundle of $X \to C$. Therefore the representabilty of $\mathrm{Pic}_{X/C}$ follows from:

Theorem (Theorem 4.8 [Kle05])**.** *Let Z be a projective and flat S-scheme, having integral geometric fibres. Then $Pic_{Z/S}$ is representable by a separated S-scheme.*

Since $\mathcal{X}/\mathcal{C}$ is an elliptic fibre bundle, the Picard scheme will be smooth, and the zero component $\mathrm{Pic}^0_{\mathcal{X}/\mathcal{C}}$ is a smooth elliptic scheme over $\mathcal{C}$. Now, we have that $\mathcal{X}/\mathcal{C}$ is in a natural way a torsor under $\mathrm{Pic}^0_{\mathcal{X}/\mathcal{C}}$ coming from the isomorphism

$$\mathrm{Pic}^1_{\mathcal{X}/\mathcal{C}} \simeq \mathcal{X}/\mathcal{C}.$$

We define the natural map $\mathcal{F}\mathrm{ib}_{X/C} \to \mathcal{J}\mathrm{ac}_{J/C}$ by sending the fibration $\mathcal{X}/\mathcal{C}$ to $\mathrm{Pic}^0_{\mathcal{X}/\mathcal{C}}$. In this language, Proposition 1.6.1 now becomes the first part of our main theorem:

1.6.5. Theorem. *The map of functors $\mathcal{F}\mathrm{ib}_{X/C} \to \mathcal{J}\mathrm{ac}_{J/C}$ is formally smooth and moreover*

$$\dim(\mathcal{F}\mathrm{ib}_{X/C}(k[\epsilon])) = \dim(\mathcal{J}\mathrm{ac}_{X/C}(k[\epsilon])) + h^1(C, \mathrm{Lie}(J/C)).$$

Proof. Recall that $\mathcal{F}\mathrm{ib}_{X/C} \to \mathcal{J}\mathrm{ac}_{J/C}$ is formally smooth if for a surjection $\Lambda \twoheadrightarrow \Lambda_0$ in $\mathcal{A}\mathrm{lg}_W$ the induced map

$$\mathcal{F}\mathrm{ib}_{X/C}(\Lambda) \to \mathcal{F}\mathrm{ib}_{J/C}(\Lambda_0) \times_{\mathcal{J}\mathrm{ac}_{J/C}(\Lambda_0)} \mathcal{J}\mathrm{ac}_{J/C}(\Lambda)$$

is surjective. By induction it suffices to verify this for small extensions. However, this follows directly from Proposition 1.6.1 applied to every element $\mathcal{J}$ of $\mathcal{J}\mathrm{ac}_{J/C}(\Lambda)$ with reduction $\mathcal{J}_0$ over Λ_0 and a $\mathcal{J}_0$ torsor $\mathcal{X}_0$.

To prove the statement about the tangent space dimensions, first note that $\mathcal{F}\mathrm{ib}_{X/C}$ fullfills the Schlesinger criteria and carries therefore a vector space structure on its tangent space. We are going to determine the kernel of the following linear map

$$\mathcal{F}\mathrm{ib}_{X/C}(k[\epsilon]) \to \mathcal{J}\mathrm{ac}_{J/C}(k[\epsilon]),$$

which consists of torsors under $J \otimes k[\epsilon]$. To determine this group, we use again (1.6.0.2). The first term vanishes, since every section $C \to J$ lifts to the trivial deformation. Hence the kernel is given by $H^1(C, \mathrm{Lie}(J)) \otimes I$. □

6.1. Algebraization. So far we only studied the infinitesimal deformation theory of elliptic fibre bundles. Let X/C be an elliptic fibre bundle. We saw that X/C has unobstructed deformations, if and only if the Jacobian J/C of X/C does so. In particular, it is possible to construct a formal lifting, by "piling up" infinitesimal deformations. With regard to lifting questions, we are not done yet, because we require a lifting to be a scheme, and not just a formal one.

If we just study Jacobian deformations, this question is trivial, because a formal deformation $\mathcal{J}/\mathcal{C}$ with zero-section $e\colon \mathcal{C} \to \mathcal{J}$ has a natural polarization, namely the line bundle $\mathscr{L}$ given by the subscheme $e(\mathcal{J})$. Because the fibers of $\mathcal{J}$ are irreducible, $\mathscr{L}$ is $\mathcal{C}$-ample. Adding a fibre class to $\mathscr{L}$, we obtain an ample line bundle. Thus by Theorem 1.1.3 we conclude that $\mathcal{X}$ can be algebraized.

1.6.6. Proposition. *Let X/C be an elliptic fibre bundle, with Jacobian J/C. Let $\mathcal{J}/\mathcal{C}$ be a lifting of J/C. Assume that the cohomology class $[X] \in H^1(C, J)$ associated to X is m-torsion, with m prime to p. Then there exists a projective lifting $\mathcal{X}/\mathcal{C}$ of X/C over R.*

Proof. Set $\mathcal{J}_n = \mathcal{J} \otimes_R R/m^{n+1}$ and $\mathcal{C}_n = \mathcal{C} \otimes_R R/m^{n+1}$. In Proposition 1.6.1, we saw that there is an isomorphism

$$H^1(\mathcal{C}_n, \mathcal{J}_n)[m] \xrightarrow{\sim} H^1(\mathcal{C}_0, \mathcal{J}_0)[m].$$

Thus we find a lifting $\mathcal{X}_n$ of X over R/m^{n+1} such that $[\mathcal{X}_n] \in H^1(\mathcal{C}_n, \mathcal{J}_n)[m]$ for every n.

Since $[X]$ is m-torsion, we can apply Lemma 1.6.3 to get a canonical finite map $\phi_n\colon \mathcal{X}_n \to \mathcal{J}_n$. These maps fit together to a map of formal schemes

$$\phi\colon \varinjlim_n \mathcal{X}_n \to \varinjlim_n \mathcal{J}_n.$$

Let $\mathscr{L}$ be a linebundle on $\varinjlim_n \mathcal{J}_n$ with the property that the restriction to the reduction is ample. Then $\phi^*\mathscr{L}$ will be ample on $\mathcal{X}_0$ because ϕ_0 is finite. Hence we can apply Grothendieck's algebraization theorem Theorem 1.1.3 and conclude that $\mathcal{X}$ has a projective algebraization. ☐

7. Elliptic fibre bundles of Kodaira dimension one

We saw that an elliptic fibre bundle E/C has Kodaira dimension 1, if and only if $g(C) \geq 2$. It is a general fact form the theory of elliptic surfaces, that on a surface of Kodaira dimension 1, there exists exactly one elliptic fibration, given by a suitable power of the canonical bundle. We generalize this fact to deformations, by showing that the unique fibration lifts to an arbitrary deformation of the total space and that this lifting is unique. In other words:

1.7.1. Theorem. *For an elliptic bundle $f\colon X \to C$ of Kodaira dimension one, the forgetful map of deformation functors $\mathcal{F}\mathrm{ib}_{X/C} \to \mathcal{D}\mathrm{ef}_X$ is an isomorphism.*

We first show injectivity:

1.7.2. Proposition. *Let $f\colon \mathcal{X} \to \mathcal{C}$ be a deformation over $\Lambda \in \mathcal{A}\mathrm{lg}_W$ of an elliptic fibre bundle $f\colon X \to C$. Then f is defined by a suitable power of the canoical sheaf $\omega_{\mathcal{X}/\Lambda}$. In particular, f is unique.*

Proof. By smoothness of f, we get an exact sequence

$$0 \to f^*\Omega^1_{\mathcal{C}/\Lambda} \to \Omega^1_{\mathcal{X}/\Lambda} \to \Omega^1_{\mathcal{X}/\mathcal{C}} \to 0.$$

The outer terms are invertible sheaves. It follows $\omega_{\mathcal{X}/\Lambda} = f^*\Omega^1_{\mathcal{C}/\Lambda} \otimes \Omega^1_{\mathcal{X}/\mathcal{C}}$.

We claim that $\Omega^1_{\mathcal{X}/\mathcal{C}}$ is a torsion element in $\mathrm{Pic}(\mathcal{X})$. Let $f_0\colon X \to C$ denote the reduction of f. We saw (Proposition 1.4.4) that $\mathscr{L} = R^1 f_{0*}\mathscr{O}_X$ is a torsion element of $\mathrm{Pic}(C)$. The same is true for $\Omega^1_{X/C}$ because $(f_0^*\mathscr{L})^\wedge \simeq \Omega^1_{X/C}$. We conclude that $\Omega^1_{\mathcal{X}/\mathcal{C}}$ is of finite order by induction on the length of Λ: The tangent space of the functor of deformation of invertible sheaves is $H^1(X, \mathscr{O}_X)$, which is a p-torsion group, since k is of characteristic p. So let n denote the order of $\Omega^1_{\mathcal{X}/\mathcal{C}}$. This implies

$$\omega_{\mathcal{X}/\Lambda}^{\otimes mn} \simeq f^*(\Omega^1_{\mathcal{C}/\Lambda})^{\otimes mn}.$$

Since $\Omega^1_{\mathcal{C}/\Lambda}$ is ample on $\mathcal{C}$, we find that $\omega_{\mathcal{X}/\Lambda}$ is semiample on $\mathcal{X}$. Furthermore, $\mathrm{Proj}(\mathrm{Sym}(\omega_{\mathcal{X}/\Lambda}^{\otimes n}))$ is isomorphic to $\mathrm{Proj}(\mathrm{Sym}(\Omega^1_{\mathcal{C}/\Lambda})^{\otimes n}) \simeq \mathcal{C}$. This follows because $H^0(\mathcal{X}, f^*(\Omega^1_{\mathcal{C}/\Lambda})^{\otimes mn}) \simeq H^0(\mathcal{C}, (\Omega^1_{\mathcal{C}/\Lambda})^{\otimes mn})$ by the projection formula.

Denote the map $\mathcal{X} \to \mathcal{C}$ given by the canonical sheaf by f_{can}. On the reduction we find $f \otimes_\Lambda k = f_0 = f_{can} \otimes_\Lambda k$. However, liftings of f_0 are unique. It is enough to prove this for first order deformations. The tangent space of the functor of deformations of $f_0 \colon X \to C$ is $H^0(X, f_0^*\Theta_C)$. However, this vector space is trivial since the dual $f_0^*\Omega^1_C$ of $f^*\Theta_C$ has non-zero global sections, because $g(C) \geq 2$. □

To prove the surjectivity of $\mathcal{F}\mathrm{ib}_{X/C} \to \mathcal{D}\mathrm{ef}_X$ we need an estimate for $h^1(X, \Theta_X)$:

1.7.3. Lemma. *Denote by $g \geq 2$ the genus of C, and set $\mathscr{L} = R^1 f_*\mathscr{O}_X$. We get*

$$h^1(X, \Theta_X) \leq g - 1 + h^0(C, \mathscr{L}) + h^0(C, \mathscr{L}^2) + 3g - 3. \tag{1.7.3.1}$$

If X is Jacobian, we get equality in (1.7.3.1).

Proof. Since f is smooth, we have an exact sequence

$$0 \to \Theta_{X/C} \to \Theta_X \to f^*\Theta_C \to 0. \tag{1.7.3.2}$$

This gives rise to an exact sequence of cohomology groups

$$H^1(X, \Theta_{X/C}) \to H^1(X, \Theta_X) \to H^1(X, f^*\Theta_C).$$

Thus $h^1(X, \Theta_X) \leq h^1(X, f^*\Theta_C) + h^1(X, \Theta_{X/C})$. We claim that $\Theta_{X/C}$ is isomorphic to $f^*\mathscr{L}$: This follows from the canonical bundle formula

$$\omega_X \simeq f^*(\mathscr{L}^{-1} \otimes \omega_C)$$

and from the expression

$$(\Theta_{X/C})^{-1} \simeq \omega_{X/C} \simeq \omega_X \otimes (f^*\omega_C)^{-1}.$$

To compute $h^1(X, \Theta_{X/C})$ we use the Leray spectral sequence and the projection formula ($f_*\Theta_{X/C} \simeq f_*f^*\mathscr{L} \simeq \mathscr{L}$) to obtain

$$0 \to H^1(C, \mathscr{L}) \to H^1(X, \Theta_{X/C}) \to H^0(C, \underbrace{R^1 f_* f^*\mathscr{L}}_{\simeq \mathscr{L}^{\otimes 2}}) \to 0.$$

By Riemann-Roch we get $h^1(C, \mathscr{L}) = g - 1 + h^0(C, \mathscr{L})$. Thus

$$h^1(X, \Theta_{X/C}) = g - 1 + h^0(C, \mathscr{L}) + h^0(C, \mathscr{L}^{\otimes 2}).$$

For $h^1(X, f^*\Theta_C)$ we obtain with the same approach and using $\Theta_C \simeq f_*f^*\Theta_C$ the following sequence

$$0 \to H^1(C, \Theta_C) \to H^1(X, f^*\Theta_C) \to H^0(C, \mathscr{L} \otimes \Theta_C) \to 0.$$

Since $g > 1$, the last term vanishes and we get $h^1(X, f^*\Theta_C) = 3g - 3$.

In the Jacobian case let $s \colon C \to X$ denote the section. The natural map $f^*\Omega^1_C \to \Omega^1_X$ has a global splitting given locally by $d(g) \mapsto d(s^*g) \otimes 1$. Dualizing yields a splitting of (1.7.3.2). □

Now, we are set up to show the surjectivity of the inclusion $\mathcal{F}\mathrm{ib}_{X/C} \to \mathcal{D}\mathrm{ef}_X$.

1.7.4. Proposition. *Let Λ be an object of $\mathcal{A}\mathrm{lg}_W$. Every deformation $\mathcal{X} \in \mathcal{D}\mathrm{ef}_X(\Lambda)$ of the total space of X admits a lifting of the fibration on X; in other words $\mathcal{X} \in \mathcal{F}\mathrm{ib}_{X/C}(\Lambda)$.*

PROOF. Denote by J the Jacobian of X. By Proposition 1.4.2, we know that there is an étale Galois covering $C' \to C$ with group G such that $J' = J \times_C C' = E \times_k C'$, for some elliptic curve E over k. Since forming Pic^0 commutes with base change, the Jacobian associated to the fibration $X' = X \times_C C'/C'$ will be J'. We denote by $\mathcal{X}' \to \mathcal{X}$ the unique lifting of $X' \to X$.

We claim that $\mathcal{X}'$ admits an elliptic fibration. To see this, we show that the deformation functors $\mathcal{F}\mathrm{ib}_{X'/C'}$ and $\mathcal{D}\mathrm{ef}_{J'}$ are isomorphic.

The functor of Jacobian deformations of J' is unobstructed (see Corollary 1.4.3), so we conclude by Proposition 1.6.1, that $\mathcal{F}\mathrm{ib}_{X'/C'}$ is unobstructed as well. It remains to show that

$$h^1(X', \Theta_{X'}) = \dim(\mathcal{F}\mathrm{ib}_{X'/C'}(k[\epsilon])).$$

Let g denote the genus of C. We have $h^1(X', \Theta_{X'}) \leq 4g - 2$ by Lemma 1.7.3. As for $\mathcal{F}\mathrm{ib}_{X'/C'}(k[\epsilon])$, we have $(3g-3)+1$ dimensions coming from the functor of Jacobian deformations of J': Namely $3g-3$ from the deformations of the C', and one dimension coming from E. The first cohomology of the Lie algebra $\mathscr{O}_{C'}$ of J' gives g additional dimensions.

Now, we come back to our deformation $\mathcal{X}$ of X. By Proposition 1.7.2 the fibration $g\colon \mathcal{X}' \to \mathcal{C}'$ is defined in terms of the canonical bundle of $\mathcal{X}'$. The action of G an $\mathcal{X}'$ induces an action on the canoical model $\mathcal{C}'$ of $\mathcal{X}'$. For $\sigma \in G$ we get a diagram:

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\sigma} & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathcal{C}' & \xrightarrow{\sigma} & \mathcal{C}' \end{array}$$

This implies that $g\colon \mathcal{X}' \to \mathcal{C}'$ descents to a fibration $f\colon \mathcal{X} \to \mathcal{C}$ on $\mathcal{X}$. □

We are going to present some applications of the Theorem 1.7.1. Recall that in Section 4, we constructed a Jacobian elliptic fibre bundle J over some curve of genus two over a field of characteristic 3, which was shown to be non-liftable as Jacobain elliptic fibre bundle. We also gave two examples denoted by J_G and J_Q, showing the same behaviour in characteristic two.

1.7.5. Theorem. *The elliptic fibre bundles J (in characteristic three) and J_G, J_Q (in characteristic two) do not admit a formal lifting to characteristic zero.*

PROOF. We already saw that J, J_G and J_Q cannot be lifted as Jacobian elliptic fibre bundles (Proposition 1.5.2 and Proposition 1.5.3). From Proposition 1.6.1, it follows that the same is true for liftings which are not Jacobian but admit an elliptic fibration. Finally, observe that the base curves on both cases are of genus $g \geq 2$, which implies via canonical bundle formula, that the Kodaira dimensions of J, J_G and J_Q are 1. Now by Theorem 1.7.1, we get that every deformation is elliptic. □

One can make an interesting remark here. Recall the following conjecture:

Conjecture (F. Oort, 1985). *Let k be a field of characteristic p, and let C be a smooth, proper and connected curve of genus at least 2. Let G be a cyclic subgroup of* $\mathrm{Aut}(C)$. *Then there exists a lifting to characterisitc zero of the pair* (C, G).

The conjecture is known to hold if the order of G is not divided by p^3 ([GM98]). Given a curve C and a cyclic subgroup $G \subset \mathrm{Aut}(C)$, we construct an elliptic surface $X \to B = C/G$, which has a formal lifting to characteristic zero if and only if the pair (C, G) is liftable.

Let E be an ordinary elliptic curve over k. An action of G on E is given by choosing a point of order $\mathrm{ord}(G)$. We set $X = (E \times C)/G$, where we divide out by the diagonal action. Note that the action of G on the product is free. The surface X will have an elliptic fibration coming from the projection $E \times C \to C$. However, this fibration will in general not define an elliptic fibre bundle, because the fixed points of the G action on C will give rise to multiple fibers.

1.7.6. Theorem. *The surface X is formally liftable to characteristic zero if and only if there exists a lifting $\mathcal{C}$ of C together with a lifting of G.*

Proof. The "if"-part is clear. Assume there exists a lifting $\mathcal{X}$ of X over some local artinian ring Λ with residue field k. Since the covering $X' = E \times C \to X$ given by the quotient map is étale, we find a lifting $\mathcal{X}' \to \mathcal{X}$ which is again Galois with group G.

Now, X' is an elliptic fibre bundle of Kodaira dimension one, hence by Proposition 1.7.4 we know that $\mathcal{X}'$ comes with its canonical fibration $\mathcal{X} \to \mathcal{C}$ lifting $X \to C$. Since $\mathcal{C}$ is the canonical model of $\mathcal{X}$, we get an induced G action on $\mathcal{C}$, coming from the G-action on $\mathcal{X}$. Since the automorphism scheme of a curve of higher genus is unramified, we conclude that the G action on $\mathcal{C}$ is faithful. $\square$

8. Bielliptic surfaces

Let X be minimal smooth surface over k, of Kodaira dimension zero and with invariants $b_1 = b_2 = 2$. Directly from the invariants, we get that the Albanese of X is an elliptic curve. The associated map $f\colon X \to \mathrm{Alb}(X)$ is either a smooth elliptic fibration (see [BM77, Proposition 5]) or a quasi-elliptic fibration. In the elliptic case, we call X a bielliptic surface. In view of our classification of elliptic fibre bundles we can say that a bielliptic surface is an elliptic fibre bundle over an elliptic curve, which is *not* an abelian surface.

Based on the methods we have developed so far, we will explain how to retrieve the classification of bielliptic surfaces over arbitrary ground fields, as given in [BM77]. Let X/C be a bielliptic surface, and let $J = \mathrm{Pic}^0_{X/C}$ denote its Jacobian. We know there is an étale Galois covering $C' \to C$ such hat $J \times_C C'$ splits into a product of elliptic curves. Since Pic^0 commutes with base change, this just means that the Jacobian J' of $X' = X \times_C C'$ is a product $E \times C'$ of elliptic curves, and hence an abelian surface. The same is true for X': by [CD89, Corollary 5.3.5] the Betti numbers of J' and X' coincide, hence the X' is abelian by the Enriques classification.

By Lemma 1.6.3 we obtain a finite C'-morphism $\varphi\colon X' \to J'$. We can choose the zero section of X' such that φ becomes an isogeny of abelian surfaces. Denote its kernel by K. There exists an integer n such that $K \subset X'[n]$. So we get a factorization of $X' \xrightarrow{[n]} X'$ into

$$X \to J' \to X.$$

Now consider the composition $E \times C = J' \to X'$. We denote the image of C in X' by C''. This one-dimensional abelian subvariety of X' gives a multisection section of $X' \to C'$, thus $X' \times_{C'} C'' \simeq E \times C''$.

To sum up the discussion, we have established a diagram:

$$\begin{array}{ccccc} E \times C'' & \longrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ C'' & \longrightarrow & C' & \longrightarrow & C \end{array}$$

in which both squares are cartesian, and the arrows in the bottom line are morphisms of elliptic curves. Let G denote the kernel of $C'' \to C$. We get a G-action ρ on $E \times C''$ such that $X = (E \times C'')/G$. We want to separate the action ρ of

G on both factors. To that purpose, denote the action of G on C'' by $\rho_{C''}$. The composition $\rho\circ(\mathrm{Id}\times(\rho_{C''})^{-1})$ is C''-linear, thus induces an action of G on E, which we denote by ρ_E. We can assume that ρ_E is faithful, otherwise we divide out by its kernel. Since $(\mathrm{Id}\times\rho_{C''})$ is translation, this cannot be the case for $(\rho_E\times\mathrm{Id})$ because in that case the quotient were an abelian surface.

To obtain an explicit classification, it remains to understand the image of ρ_E. It has a non-trivial subgroup Γ fixing the zero section, which is of the form $\mathbb{Z}/d\mathbb{Z}$, where $d\in\{2,3,4,6\}$ as follows from the classification of automorphisms of elliptic curves, and from the fact that G, and hence Γ, is abelian. There is a complementary subgroup A, which acts trivially on Pic^0_E. This is the translation part. In consequence we get that $\rho_E(G)$ is the direct product $A\cdot\Gamma$.

This group being abelian puts a severe restriction on A: It has to be fixed by the generator ω of Γ. With this information one assembles the following well-known list of bielliptic surfaces:

(a) We have the Jacobian surface $X_a=(E\times C)/(\mathbb{Z}/2\mathbb{Z})$, where the $\mathbb{Z}/2\mathbb{Z}$-action is given by

$$(x,y)\mapsto(-x,x+a)$$

for point of a of order 2. Note that if $p=2$, this forces C to be ordinary.

There is one non-trivial element in the Tate-Safarevich group of X_a: Namely, $X_{a1}=(E\times C')/(\mathbb{Z}/2\mathbb{Z})^2$ with action

$$(x,y)\mapsto(-x,x+a)\quad\text{and}\quad(x,y)\mapsto(x+b,x+c)$$

where a is like before, c is another point of order 2 of C, and b is a point of order 2 in E. Note that the existence of two non-trivial 2-torsion point excludes $p=2$.

However, there is a characteristic two version of this: We have $X_{a2}=(E\times C)/(\mu_2\cdot\mathbb{Z}/2\mathbb{Z})$ where μ_2 acts by translation on both factors and $\mathbb{Z}/2\mathbb{Z}$ acts as in case a.

(b) Let E be an elliptic curve with $j(E)=0$. Then we have an automorphism ω of E of order 3. We construct the Jacobian surface $X_b=(E\times C)/(\mathbb{Z}/3\mathbb{Z})$ by using the action:

$$(x,y)\mapsto(\omega(x),y+a),$$

where a is a 3-torsion point of C. Again, C is required to be ordinary. If $p\neq3$, the fixed point set of ω consists of exactly one non-trivial 3-torsion point $b\in E$. This allowes us to construct $X_{b1}=(E\times C)/(\mathbb{Z}/3\mathbb{Z})^2$ where the action is

$$(x,y)\mapsto(\omega(x),y+a)\quad\text{and}\quad(x,y)\mapsto(x+b,y+c)$$

where a is like before, c is another point of order 3 of C, and b is a point of order 3 in E.

There is no characteristic three version of this construction: Since $j(E)=0$ forces E to be supersingular, we have no 3-torsion points. The automorphism ω fixes the subscheme α_3 (consider the (trivial) action on $\mathrm{Lie}(E)$). However, there is no α_3 action on C, since C must be ordinary.

(c) Let E be an elliptic curve with $j(E)=1728$. Then there exists an automorphism ω of E of order 4. We construct the Jacobian surface $X_c=(E\times C)/(\mathbb{Z}/4\mathbb{Z})$ by using the action:

$$(x,y)\mapsto(\omega(x),y+a),$$

where a is a 4-torsion point of C, and C is has to be ordinary if $p=2$. The fixed point set of ω consists of exactly one non-trivial 2-torsion point $b\in$

E, given $p \neq 2$. This allowes us to construct $X_{b1} = (E\times C)/(\mathbb{Z}/4\mathbb{Z}\cdot\mathbb{Z}/3\mathbb{Z})$ where the action is

$$(x, y) \mapsto (\omega(x), y + a) \text{ and } (x, y) \mapsto (x + b, y + c)$$

where a is like before, c is a point of order 2 of C, and b is a point of order 2 in E.

The same arguments as in the former case show, that there is no characteristic three version of this construction.

(d) Let again $j(E) = 0$. We have an automorphism $-\omega$ of E of order 6. The associated surfaces $X_d = (E \times C)/(\mathbb{Z}/6\mathbb{Z})$ is obtained from the action

$$(x, y) \mapsto (\omega(x), y + a),$$

where a is a torsion point of order 6 of C. We claim that $-\omega$ no non-trivial fixed points: To give a fix point x of $-\omega$ satisfies $\omega(x) = -x$. However, x cannot be a 2-torsion point, thus the above relation implies that the orbit of x under ω has only to elements, contradiction.

We have reobtained the classification in [BM77]. The next step is to study deformations of bielliptic surfaces. To do this, we need to calculate some basic invariance.

1.8.1. Proposition. *Let $f\colon X \to C$ be a Jacobian bielliptic surface. Write*

$$X = (E \times C')/(\mathbb{Z}/d\mathbb{Z})$$

as in the classification. If d is not a power of p and $d \neq 2$ we have

$$h^0(\Theta_X) = 1, \qquad h^1(\Theta_X) = 1, \qquad h^2(\Theta_X) = 0, \qquad h^1(C, \mathrm{Lie}(X/C)) = 0.$$

If $d = 2$ and $p \neq 2$ we get

$$h^0(\Theta_X) = 1, \qquad h^1(\Theta_X) = 2, \qquad h^2(\Theta_X) = 1, \qquad h^1(C, \mathrm{Lie}(X/C)) = 0.$$

Whereas if d is a power of p it holds

$$h^0(\Theta_X) = 2, \qquad h^1(\Theta_X) = 4, \qquad h^2(\Theta_X) = 2, \qquad h^1(C, \mathrm{Lie}(X/C)) = 1.$$

Let Y/C be a non-Jacobian bielliptic surface with Jacobian X/C. Then we have

$$h^i(Y, \Theta_Y) = h^i(X, \Theta_X).$$

Proof. Since f is smooth, we have an exact sequence

$$0 \to \Theta_{X/C} \to \Theta_X \to f^*\Theta_C \to 0. \tag{1.8.1.1}$$

Because the action of $\mathbb{Z}/d\mathbb{Z}$ on $E \times C'$ is diagonal action, we see that (1.8.1.1) is split, by comparing the sequence with its pullback via the quotient map. Therefore Θ_X decomposes as

$$\Theta_X \simeq \Theta_{X/C} \oplus f^*\Theta_C. \tag{1.8.1.2}$$

Since C is an elliptic curve, we find $f^*\Theta_C \simeq \mathscr{O}_X$. As for $\Theta_{X/C}$, it will be a torsion line bundle of order ℓ equal to the order of the induced action of $\mathbb{Z}/d\mathbb{Z}$ on Θ_E. To see this, note that $\mathbb{Z}/d\mathbb{Z}$ acts trivially on $\Theta_E^{\otimes \ell}$ because the induced action is by roots of unity. Thus a section of $\Theta_E^{\otimes \ell}$ will descend to a section of $\Theta_{X/C}^{\otimes \ell}$. We claim that $\mathrm{ord}(\Theta_{X/C}) = \ell$ where $d = \ell p^n$ with ℓ prime to p. This is seen as follows:

The action on Θ_E is determined by the action on the k-vector space $\mathrm{Lie}(E)$. If d is a power of p, this action has to be trivial, since $\mathrm{Hom}(\mathbb{Z}/p^n\mathbb{Z}, \mathbb{G}_m) = 0$. To determine the action in general, note that we have a subgroupscheme $H \subset E$ of height one, such that $\mathrm{Lie}(H) = \mathrm{Lie}(H)$. In fact, the total space of H is given by $\mathrm{Spec}(\mathscr{O}_E/\mathfrak{m}_{0,E}^p)$ and H is isomorphic either to μ_p or to α_p. Because of height one, the map given by the Lie functor $\mathrm{Aut}(H) \to \mathrm{Aut}(\mathrm{Lie}(H))$ is injective. In fact, it will be an isomorphism if we restrict to maps of p-Lie algebras (see [Mum70, Section 14]).

The group scheme H is of rank p, so if $p > 4$ we get $\operatorname{ord}(\omega) = \operatorname{ord}(\omega|_H)$ by rigidity [KM85, Corollary 2.7.3]. If $p = 3$ and $d = 2$, we know that ω will act on $\operatorname{Lie}(E)$ as involution. If $d = 4$ the same argument applies to ω^2.

If $p = 2$ and $d = 3$ we have $H = \alpha_2$ since $j(E) = 0$. Assume ω induces the identity on H. Then the associated trace map $\operatorname{tr}_\omega = \operatorname{Id} + \omega + \omega^2$ would give multiplication by 3, which is an isomorphisms on H. However, we know that tr_ω is the zero map on E (see Lemma 1.8.5 below).

Now, it is easy to calculate the invariants. Denote by $\epsilon\colon C \to X$ the zero section of X. We have $\mathscr{L} = \operatorname{Lie}(X) = \epsilon^*\Theta_{X/C}$, and since $f^*\operatorname{Lie}(X) \simeq \Theta_{X/C}$ it follows that

$$\operatorname{ord}(\operatorname{Lie}(X)) = \operatorname{ord}(\Theta_{X/C}).$$

The statement about the cohomology of $\operatorname{Lie}(X)$ follows, since it is a line bundle of degree zero and therefore

$$h^1(C, \operatorname{Lie}(X)) = h^0(C, \operatorname{Lie}(X)).$$

However, the last term is not zero if and only if $\operatorname{Lie}(X)$ is trivial. We also get that

$$h^0(X, \Theta_X) = h^0(X, \Theta_{X/C}) + h^0(X, f^*\Theta_C) = h^0(X, \operatorname{Lie}(X)) + 1.$$

To compute $h^1(X, \Theta_X)$ we treat both summands in (1.8.1.2) separately. We have $\Theta_{X/C} \simeq f^*R^1f_*\mathscr{O}_X \simeq f^*\operatorname{Lie}(X)$. By the projection formula and the Leray spectral sequence we get

$$0 \to H^1(C, \mathscr{L}) \to H^1(X, \Theta_{X/C}) \to H^0(C, \underbrace{R^1f_*f^*\mathscr{L}}_{\simeq \mathscr{L}^{\otimes 2}}) \to 0.$$

Thus we have:

$$h^1(X, \Theta_{X/C}) = \begin{cases} 0 & \text{if } \ell > 2 \\ 1 & \text{if } \ell = 2 \\ 2 & \text{if } \ell = 1 \end{cases}$$

For $h^1(X, f^*\Theta_C)$ we obtain similarly:

$$0 \to H^1(C, \Theta_C) \to H^1(X, f^*\Theta_C) \to H^0(C, \mathscr{L} \otimes \Theta_C) \to 0$$

Because $\Theta_C \simeq \mathscr{O}_C$ we find $h^1(X, f^*\Theta_C) = 2$ if $\mathscr{L}$ is trivial and $h^1(X, f^*\Theta_C) = 1$ otherwise.

This proves the statement about $h^1(X, \Theta_X)$. To compute $h^2(X, \Theta_X)$ we just observe that $\chi(\Theta_X) = 0$, because $\chi(\Theta_{E\times C'}) = 0$.

The statement about a non-Jacobian bielliptic surface $g\colon Y \to C$ with Jacobian X/C follows from the expression

$$R^1g_*\mathscr{O}_Y \simeq \operatorname{Lie}(X/C) \simeq \epsilon^*\Theta_{J/C}$$

and $g^*(\mathscr{L}) = \Theta_{Y/C}$. This was already used in the proof of Proposition 1.4.4. □

8.1. Deformations of abelian varities. To classify deformation of bielliptic surfaces, we are going to study deformation of their abelian cover, the advantage being that there exists a well developed theory for deformation of abelian varieties.

Most noteably, we will use the Serre-Tate Theorem, which allowes to understand deformations of abelian varieties in more explicit terms through deformations of p-divisible groups. To state that result, we repeat some basic definitions and facts: Let p be a prime number, and let S be a scheme. A sheaf of groups for the *fppf*-topology is called a *p-divisible group*, if G is p-divisible and p-primary, i.e.

$$G = \varinjlim G[p^n]$$

and the groups $G[p^n]$ are finite flat group scheme over S (see [Gro74] where p-divisible groups go by the name "Barsotti-Tate groups"). The main examples which we are in fact interested in are p-divisible groups associated with abelian schemes. For an abelian S-scheme A, we set

$$A[p^\infty] = \varinjlim A[p^n].$$

The deformation theory of abelian schemes is controlled by their p-divisible groups. To be precise, let R be a ring in which $p^N = 0$. For a nilpotent ideal $I \subset R$ we define the category $\mathcal{T}$ of triples:

$$(A, \mathcal{G}, \epsilon)$$

where A is an abelian scheme over R/I, $\mathcal{G}$ is a p-divisible group over R and ϵ an isomorphisms $\mathcal{G} \otimes_R R/I \simeq A[p^\infty]$. Now we have the theorem of Serre and Tate:

1.8.2. Theorem (Theorem 1.2.1 [Kat81])**.** *There is an equivalence between $\mathcal{T}$ and the category of abelian schemes over R given by*

$$\mathcal{A} \mapsto (\mathcal{A} \otimes_R R/I, \mathcal{A}[p^\infty], \text{ natural } \epsilon).$$

We will use the following statement to understand the lifting behavior of morphisms of the latter:

1.8.3. Lemma (Lemma 1.1.3 [Kat81])**.** *Let $\mathcal{G}$ and $\mathcal{H}$ be p-divisible groups over R. Assume $I^{\nu+1} = 0$. Let G and H denote their restrictions to* $\mathrm{Spec}(R/I)$*. Then the following holds:*

(i) *The groups* $\mathrm{Hom}_R(\mathcal{G}, \mathcal{H})$ *and* $\mathrm{Hom}_{R/I}(G, H)$ *have no p-torsion.*
(ii) *The reduction map* $\mathrm{Hom}_R(\mathcal{G}, \mathcal{H}) \to \mathrm{Hom}_{R/I}(G, H)$ *is injective.*
(iii) *For any homomorphism $f\colon G \to H$ there exists a unique homomorphism ϕ_ν lifting $[p^\nu] \circ f$.*
(iv) *In order for f to lift to a homomorphism $f\colon \mathcal{G} \to \mathcal{H}$, it is necessary and sufficient for the homomorphism ϕ_ν to annihilate $\mathcal{G}[p^\nu]$.*

We want to apply Theorem 1.8.2 to the case of an ordinary elliptic curve E over an algebraically closed field k. We have

$$E[p^\infty] = \mu_{p^n} \oplus \mathbb{Q}_p/\mathbb{Z}_p.$$

For an arbitrary deformation $\mathcal{E}$ over Λ, where $\Lambda \in \mathcal{A}\mathrm{lg}_W$, the splitting of $E[p^\infty]$ does not lift, but we get an extension

$$0 \to \mu_{p^\infty} \to E[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p \to 0. \tag{1.8.3.1}$$

From Theorem 1.8.2 it follows that the deformation $\mathcal{E}$ is uniquley determined by the class of the extension (1.8.3.1) in $\mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty})$. This group is isomorphic to the multiplicative group $1 + \mathfrak{m} \subset \Lambda^\times$. The element $q \in 1 + \mathfrak{m}$ associated to a deformation $\mathcal{E}$ is called the *Serre-Tate parameter* of $\mathcal{E}$. Using this parameter, we can express the universal family as $\mathcal{E}^{univ} \to \mathrm{Spf}(W[[q-1]])$, such that for an $W[[q-1]]$-algebra Λ, the pullback $\mathcal{E} \otimes_{W[[q-1]]} \Lambda$ has the extension structure determined by the image of q in Λ (See [KM85, 8.9]).

8.2. The versal families. Let $X = (E \times F)/G$ over k be a Jacobian bielliptic surface, with fibration $X \to C$. First, we study the deformation functor $\mathcal{J}\mathrm{ac}_{X/C}$.

By Proposition 1.4.2 we know the structure of Jacobian deformations of X. They will be of the form $(\mathcal{E} \times \mathcal{F})/\Gamma$. Here, $\mathcal{E}$ is a deformation of E extending the automorphism ω, and we are going to denote the deformation functor of such pairs by (E, ω). Likewise, $\mathcal{F}$ is a deformation of F with a torsion point lifting the point of F which appears in the definition of the action of Γ, and we denote the deformation functor of such a pair by (F, c).

The functor $\mathcal{J}\mathrm{ac}_{X/C}$ is isomorphic to the product of the deformation functors

$$(E,\omega)\times(F,c).$$

To write down a versal family for $\mathcal{J}\mathrm{ac}_{X/C}$, we treat the problem separately for both factors.

Deforming elliptic curves with automorphisms. Let $(\mathcal{E}^{univ},\omega)\to\mathrm{Spec}(R)$ be the universal deformation of E along with its automorphism ω. This functor is indeed pro-representable: if a lifting of ω exists for a given deformation of E, then it is unique.

If $\mathrm{ord}(\omega)=2$, then ω is the involution which obviously extends to any deformation of E. Hence in that case R is just the deformation ring of E, which is isomorphic to $W[[t]]$.

If $\mathrm{ord}(\omega)>2$ then the j-invariant of $\mathcal{E}$ is either 0 or 1728. If the order is prime to p, there are no obstruction against lifting (E,ω), thus $R=W$.

We treat the remaining cases. First, assume $p=2$ and $d=\mathrm{ord}(\omega)=4$. We know from [JLR09, Lemma 1.1] that there is no elliptic curve over W with j-invariant 1728 and good reduction. This means we have to pass to a ramified extension of W. We will work over $R=W[i]$ where i is a primitive forth root of unity. The following curve $\mathcal{E}_2$ is taken from [JLR09, §2.A]

$$y^2+(-i+1)xy-iy=x^3-ix^2.$$

It has $j=1728$ and Discriminant $\Delta=11-2i$, and is therefore of good reduction.

For $p=3$ and $d=3$, again by [JLR09], there will be no elliptic curve over W with j-invariant 0 and good reduction. So let $R=W[\pi]$, where $\pi^2=3$. Consider the elliptic curve $\mathcal{E}_3$ given by the Weierstraß equation

$$y^2=x^3+\pi x^2+x,$$

whose j-invariant is 0 and whose discriminant is $\Delta=-16$. In particular, it has good reduction.

In both cases ($p=2$ or 3), the curve $\mathcal{E}_p$ has an automorphism of order four or three respectively, since on the generic fibre, automorphisms are given by the action of certain roots of unity, and we have chosen the base rings in such a way that they contain the necessary roots. An automorphism of the generic fibre extends to the entire family, and its order will not change after passing to the reduction, as can be seen by considering an étale torsion subscheme of sufficiently high order.

We claim that the elliptic curves over the rings constructed above are the universal families for the deformation problem (E,ω). This follows from the fact that the respective base rings are the smallest possible extensions of W over which the deformation problem can be solved, and from the fact that an elliptic curve over a local complete ring with algebriacally closed residue field is determined by its modular invariant (see Proposition 2.2.1 below.)

Deforming elliptic curves with torsion points. Now we treat the second factor. If p does not divide d, then a d-torsion point lifts uniquely to any deformation of F. Therefore (F,c) is pro-represented by $W[[t]]$. Note that the parameter t can be identified with the modular invariant $j(F)$ if $j(F)(j(F)-1728)$ is non-zero.

Assume now, that p does divide d. By the above, we can assume $d=p^n$. Since F is ordinary, we can write

$$\mathcal{F}^{univ}\to\mathrm{Spf}(W[[q-1]]) \qquad (1.8.3.2)$$

where q is the Serre-Tate parameter. Let Λ be in $\mathcal{A}\mathrm{lg}_W$. For a deformation $\mathcal{F}=\mathcal{F}^{univ}\otimes_{W[[q-1]]}\Lambda$, we have that

$$0\to\mu_{p^n}\to\mathcal{F}[p^n]\to\mathbb{Z}/p^n\mathbb{Z}\to 0$$

is split, if and only if the image of q in Λ has a p^n-th root. This follows from the explicit description of the group scheme $F^{univ}[p^\infty]$ as given in [KM85, 8.7]. We conclude that $W[[q-1]][\sqrt[p^n]{q}]$ is a versal hull of the functor (F, c).

1.8.4. Proposition. *The functor $\mathcal{J}\mathrm{ac}_{X/C}$ has versal hull R, where R is given in the table below:*

	$p=2$	$p=3$	$p>3$
$d=2$	$W[[t_E]]\otimes W[[q-1]][\sqrt[2]{q}]$	$W[[t_E]]\otimes W[[t_F]]$	$W[[t_E]]\otimes W[[t_F]]$
$d=3$	$W[[t_E]]\otimes W$	$W[\pi]\otimes W[[q-1]][\sqrt[3]{q}]$	$W\otimes W[[t_F]]$
$d=4$	$W[i]\otimes W[[q-1]][\sqrt[4]{q}]$	$W\otimes W[[t_F]]$	$W\otimes W[[t_F]]$
$d=6$	$W\otimes W[[q-1]][\sqrt[2]{q}]$	$W[\pi]\otimes W[[q-1]][\sqrt[3]{q}]$	$W\otimes W[[t_F]]$

It is easy to read off and interpret the dimension of the tangent space of the deformation functor. For example, in the case where $p=3$ and $d=3$ we have $\dim(\mathrm{Hom}(W[\pi]\otimes W[\sqrt[3]{j_E}], k[\epsilon])) = 3$. The parameter j_F gives one dimension, and the rest is due to relations, coming from obstructions. As explained before, $h^1(X, \Theta_X) = 4$ holds in this case, so there still is one dimension missing.

To account for this missing dimension, we have to study all deformations of X, not just the Jacobian ones. This is settled by Theorem 1.6.5. Observe that in all the cases it holds

$$h^1(X, \Theta_X) - \dim(\mathcal{J}\mathrm{ac}_{X/C}(k[\epsilon])) = h^1(C, \mathrm{Lie}(X)).$$

Therefore $\dim(\mathcal{F}\mathrm{ib}(k[\epsilon])) = h^1(X, \Theta_X)$, and its makes sense to ask if the absolute deformation functor of X is isomorphic to $\mathcal{F}\mathrm{ib}_{X/C}$. In the next section, we will see that this is indeed the case.

8.3. Classification of deformations. The most important step to classify deformations of bielliptic surfaces is to show that for a bielliptic surface X/C over k the functors $\mathcal{F}\mathrm{ib}_{X/C}$ and $\mathcal{D}\mathrm{ef}_X$ are isomorphic.

Denote by $J \to C$ the Jacobian of $X \to C$. If d is not a power of p, the claim follows already, since in that case $\mathcal{J}\mathrm{ac}_{J/C}$ is unobstructed and has the right tangent dimension. Hence we get a chain of isomorphisms

$$\mathcal{J}\mathrm{ac}_{X/C} \simeq \mathcal{F}\mathrm{ib}_{X/C} \simeq \mathcal{D}\mathrm{ef}_X .$$

In the case where d is a power of p, we have to use a more refined strategy. The key idea is again to use the étale covering of X. It will turn out that this is more difficult than in the Kodaira dimension one case because the étale cover of X is an elliptic abelian surface A and not every deformation of the covering admits a fibration. In particular, A has more deformations than X.

In the course of the proof, we will use the following lemma:

1.8.5. Lemma. *Let $\mathcal{E}$ be an elliptic scheme over a base scheme $\mathcal{S}$. Let Ω be an automorphism of $\mathcal{E}$ of order d. Then the trace map*

$$\mathrm{tr}_\Omega = \mathrm{Id} + \Omega + \cdots + \Omega^{d-1}$$

gives the zero map on $\mathcal{E}$.

Proof. We can prove the statement fibrewise. So assume that S is the spectrum of a field. Now, for any $\mathcal{S}$-scheme T and a T-valued point $x \in \mathcal{E}_{\mathcal{T}}(\mathcal{T})$ we find that

$$\mathrm{tr}_\Omega(x) = \mathrm{tr}_\Omega(\Omega(x)).$$

In other words: The orbits of Ω are contained in the fibres of tr_Ω. This means in particular, that tr_Ω factors over the quotient scheme $\mathcal{E}/(\mathbb{Z}/d\mathbb{Z})$, where the action is given by Ω. However, since Ω fixes the zero-section, this quotient is isomorphic to $\mathbb{P}^1_{\mathcal{S}}$. The conclusion follows, because there is no non-constant map $\mathbb{P}^1_{\mathcal{S}} \to \mathcal{E}$. □

1.8.6. Proposition. *Let $f\colon X \to C$ be a Jacobian bielliptic surface over k. Then f extends to any deformation $\mathcal{X}$ of X over $\Lambda \in \mathcal{A}lg_R$.*

Proof. We have an étale Galois cover $A = E \times F$ of X. The Galois group is isomorphic to $\Gamma = \mathbb{Z}/d\mathbb{Z}$. The action on E is by an automorphism ω, fixing the zero section. For a deformation $\mathcal{X}$ of X, we get a diagram

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{A} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{X} \end{array}$$

where the right hand column is the unique lifting of the left. By [MFK94, Theorem 6.14] we can give $\mathcal{A}$ a structure of an abelian scheme, extending the group structure on A. Now the strategy is as follows: First, we show that $\mathcal{A}$ has an automorphism Ω lifting $\mathrm{Id} \times \omega$. Then we study the action of Ω on the p-divisible group $\mathcal{A}[p^\infty]$, and use the trace map defined by Ω to lift the projection $E[p^\infty] \times F[p^\infty] \to F[p^\infty]$. This lifting will descend to the desired lifting of the fibration f on X.

Denote the image of $1 \in \Gamma$ in $\mathrm{Aut}(\mathcal{A})$ by σ. Note that σ does not necessarily fix the zero section. We study its action on $\mathcal{A}$: Set $c = \sigma(0) \in \mathcal{A}(\mathrm{Spec}(\Lambda))$ and denote by t_{-c} the morphism given by translation the $-c$. We set

$$\Omega = \sigma \circ t_{-c} \ \text{ and } \ \Omega' = t_{-c} \circ \sigma.$$

Both maps fix the zero section of $\mathcal{A}$ and are therefore group automorphisms of $\mathcal{A}$. Furthermore, they lift the automorphism $\mathrm{Id} \times \omega$ of $E \times F$, which implies $\Omega = \Omega'$ since the lift of an automorphism is unique.

This means that Ω and t_c commute, and since σ and Ω are of order d, we get that c is a torsion point of order d, which lifts the action of Γ by translation on F.

To proceed with the proof, we pass to the category of p-divisible groups, as explained in Theorem 1.8.2. Our aim is to lift the second projection $\mathrm{pr}_2\colon E[p^\infty] \times F[p^\infty] \to F[p^\infty]$. We know there exists some integer N such that there exists a unique lift ϕ_N of $[N] \circ \mathrm{pr}_2$ (Lemma 1.8.3). We compare ϕ_N with the trace tr_Ω defined by Ω. The restriction $\overline{\mathrm{tr}_\Omega}$ of tr_Ω to $A[p^\infty]$ gives the map

$$[d] \circ \mathrm{pr}_2\colon A[p^\infty] \to F[p^\infty] = \mathrm{Im}(\overline{\mathrm{tr}_\Omega})$$

because tr_Ω is multiplication by d on the factor $F[p^\infty]$ and the zero map on the factor $E[p^\infty]$ (see Lemma 1.8.5). Now, we get that $[d] \circ \phi_N$ is a lift of $[N] \circ \overline{\mathrm{tr}_\Omega}$. Again, because an endomorphisms has at most one lifting, it follows

$$[d] \circ \phi_N = [N] \circ \mathrm{tr}_\Omega\,.$$

Factoring out by $\mathcal{A}[N]$, we see that tr_Ω is a lift of $[d] \circ [\mathrm{pr}_2]$. It remains to show that $\mathcal{A}[d]$ lies in the kernel of tr_Ω.

To see this, we consider the exact sequence of finite flat group schemes

$$0 \to \mathcal{A}[d]^0 \to \mathcal{A}[d] \to \mathcal{A}[d]^{et} \to 0.$$

We first show $\mathrm{tr}_\Omega(\mathcal{A}[d]^0) = 0$. Again, we have an exact sequence

$$0 \to \mathcal{A}[d]^{mult} \to \mathcal{A}[d]^0 \to \mathcal{A}[d]^{bi} \to 0.$$

The outer groups denote the multiplicative part and the biinfinitesimal part respectively. The category of multiplicative groups schemes is dual to the category of étale group schemes via Cartier duality - thus endomorphisms lift uniquely, and we get $\mathrm{tr}_\Omega(\mathcal{A}[d]^{mult}) = 0$. Now, we consider the sequence of p-divisible groups

$$0 \to \mathcal{A}[p^\infty]^{mult} \to \mathcal{A}[p^\infty]^0 \to \mathcal{A}[p^\infty]^{bi} \to 0.$$

Since Ω maps $\mathcal{A}[p^\infty]^{mult}$ into itself, we get an induced action of Ω on $\mathcal{A}[p^\infty]^{bi}$, and in particular, tr_Ω descends to $\mathcal{A}[p^\infty]^{bi}$. If this group is non-trivial, it is a lift of $E[p^\infty]$ on which tr_Ω is zero. Again by uniqueness of lifts, we get $\mathrm{tr}_\Omega(\mathcal{A}[d]^{bi}) = 0$.

We saw that $\mathrm{tr}_\Omega(\mathcal{A}[d]^0) = 0$ and it remains to show $\mathrm{tr}_\Omega(\mathcal{A}[k]^{et}) = 0$. However, this is clear, since we deal with étale group schemes. We conclude that pr_2 extends to $\mathcal{X}$. □

So far, we have treated only Jacobian bielliptic surfaces. But the non-Jacobian cases are mostly trivial. Consulting the list of bielliptic surfaces in section 8, we see that the Tate-Safarevich group are trivial if the associated Jacobians have obstructed deformations, except in one case in charactertic two.

This is the surface $X = X_{a2} = (E \times F)/(\mu_2 \cdot \mathbb{Z}/2\mathbb{Z})$. The non-split abelian surface $A = (E \times F)/\mu_2$ is an étale $\mathbb{Z}/2\mathbb{Z}$-cover of X.

Now let $\mathcal{X}$ be a deformation of X. Once more we have a diagram:

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{A} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{X} \end{array}$$

We claim that $\mathcal{A}$ admits a lifting of the elliptic fibration $f\colon A \to F/\mu_2$: We have an exact sequence

$$0 \to \mathcal{A}[p^\infty]^{tor} \to \mathcal{A}[p^\infty] \to \mathcal{A}[p^\infty]^{et} \to 0. \qquad (1.8.6.1)$$

The morphism $A[p^\infty]^{et} \to F[p^\infty]^{et}$ induced by f lifts uniquely to

$$\varphi\colon \mathcal{A}[p^\infty]^{et} \to \mathcal{F}[p^\infty]^{et}$$

since we are dealing with étale group schemes. Denote by $\mathcal{B}$ the p-divisible group obtained by pushout of (1.8.6.1) along φ. We still have $\mathcal{A}[p^\infty]^{tor} \subset \mathcal{B}$ and inside $\mathcal{A}[p^\infty]^{tor}$ we have the kernel of the unique lift of $A^{tor} \to F[p^\infty]^{tor}$. Dividing out $\mathcal{B}$ by that kernel we obtain a lifting of f.

As in the proof of Theorem 1.8.6, we see that f descends to $\mathcal{X}$. Therefore $\mathcal{X}$ is elliptic. To sum up, we have the following theorem:

1.8.7. Theorem. *Every deformation $\mathcal{X}$ of a bielliptic surface X induces a lifting of the elliptic fibration $X \to C = \mathrm{Alb}(X)$.*

Next, we show that a versal deformation of a bielliptic surface is algebraizable.

1.8.8. Proposition. *Let X be a bielliptic surface over k. Denote by $\mathcal{X}^{vers} \to \mathrm{Spf}(R)$ a formal versal family of $\mathcal{D}\mathrm{ef}_X$. Then there exists a projective scheme $\overline{\mathcal{X}}$ over R, such that $\mathcal{X}^{vers}$ is the completion of $\overline{\mathcal{X}}$ with respect to the special fibre.*

Proof. For an arbitrary deformation $\mathcal{X}$ of X, denote by $\mathcal{A} \to \mathcal{X}$ the unique lifting of the abelian covering $A \to X$. In the proof of Proposition 1.8.6 we saw that the abelian scheme $\mathcal{A}$ has an automorphism Ω lifting the automorphism $\omega \times \mathrm{Id}$ of $E \times F$.

The automorphism of $\mathcal{X}$, obtained from Ω by descent, will again be denoted by Ω. Now Ω is a $\mathcal{C}$-automorphism of $\mathcal{X}$; i.e. its action is confined to the fibres of the fibration.

We claim that the fixed locus of Ω is flat over $\mathcal{C}$: Every closed point $x \in \mathcal{C}$ has an étale neighborhood $\mathcal{U} \to \mathcal{C}$, such that the the pullback

$$\mathcal{X}_\mathcal{U} = \mathcal{X} \times_\mathcal{C} \mathcal{U}$$

can be given the structure of an abelian scheme, in such a way that the base change of Ω to $\mathcal{X}_\mathcal{U}$ becomes a group automorphism. We consider the endomorphism $\Omega - \mathrm{Id}$

of $\mathcal{X}_{\mathcal{U}}$. It is a surjective map of abelian schemes, and therefore flat by [MFK94, Lemma 6.12]. In particular, its kernel, i.e the fix locus of Ω, is flat over $\mathcal{U}$.

Thus we found a relative Cartier divisor $\mathcal{Z}$ of $\mathcal{X} \to \mathcal{C}$. Its degree can be computed on the reduction. It equals the order of the subgroup scheme of E fixed by ω. In particular, it is positive, which means that $\mathcal{Z}$ is a relatively ample divisor for $\mathcal{X} \to \mathcal{C}$.

Now denote by $\mathcal{X}^{vers} \to \mathrm{Spf}(R)$ a versal family of $\mathcal{F}\mathrm{ib}_{X/C}$. It is a formal scheme over the hull of the deformation functor $\mathcal{F}\mathrm{ib}_{X/C}$, therefore admitting an elliptic fibration $F\colon \mathcal{X}^{vers} \to \mathcal{C}$ lifting $X \to C$. Denote by $\mathfrak{m}$ the maximal ideal of R , and set $X_n = \mathcal{X}^{vers} \otimes_R R/\mathfrak{m}^{n+1}$.

The construction of $\mathcal{Z}$ gives rise to a compatible system of relatively ample line bundles $\mathscr{O}_{X_n}(Z_n)$. Tensoring with the line bundle coming from the divisor of a fibre of $\mathcal{X}^{vers} \to \mathcal{C}$, we obtain a system of ample line bundles $\mathscr{H}_n$. Thus by Grothendieck's Algebraization Theorem (Theorem 1.1.3), we conclude that $\mathcal{X}^{vers}$ is the completion of some projective scheme $\overline{\mathcal{X}^{vers}}$ over $\mathrm{Spec}(R)$. □

1.8.9. Corollary. *Every bielliptic surface over k lifts projectively to characterisitc zero.*

Proof. The existence of liftings of Jacobian bielliptic surfaces follows from the explicit description of their deformation rings (Proposition 1.8.4). By Proposition 1.6.1 this implies the existence of formal liftings for *any* bielliptic surface. Those formal liftings are in turn algebraizable by Proposition 1.8.8. □

Proposition 1.8.8 helps us to answer another natural question: X is called bielliptic because it has two transversal elliptic fibrations: The smooth one, denoted by f, coming from the projection $E \times F \to F$ and a second one, denoted by g, with base curve $\mathbb{P}^1_k$ coming from $E \times F \to E$. We saw that the first fibration is preserved under deformation, but what about the second one?

1.8.10. Proposition. *Let X be a bielliptic surface, then every deformation $\mathcal{X}$ of X extends both elliptic fibration.*

Proof. We are going to show that the versal deformation $\mathcal{X}^{vers} \to \mathrm{Spf}(R)$ admits an extension of g, so that the claim follows by versality.

Denote by K the fraction field of R. We can use surface theory to analyze the generic fibre $\overline{\mathcal{X}}_K$ of the algebraization $\overline{\mathcal{X}}$ of $\mathcal{X}^{vers}$. Denote by $\mathscr{L} = \mathscr{O}_{\overline{\mathcal{X}}}(\mathcal{Z})$ the line bundle associated to the divisor $\mathcal{Z}$, constructed in the proof of Proposition 1.8.8. The canonical bundle of $\overline{\mathcal{X}}_K$ has self-intersection number 0. It follows that the line bundle $\mathscr{L}_K^{\otimes m}$, gives rise to an elliptic fibration $g'\colon \overline{\mathcal{X}}_K \to \mathbb{P}^1_K$, if we choose m sufficiently big [Bǎd01, Theorem 7.11].

Since $\overline{\mathcal{X}}$ is proper and normal, we can extend g' to a rational map $g'\colon \overline{\mathcal{X}} \to \mathbb{P}^1_R$ which is defined on a non-empty open subset intersecting the special fibre. Now, there are sections $s_1, s_2\colon \mathrm{Spec}(R) \to \mathbb{P}^1_R$ whose associated closed subschemes are disjoint and who do lie inside the image of g'. Taking the closures of the inverse images of those sections under g', we get two divisors $\mathcal{G}_1$ and $\mathcal{G}_2$ in $\overline{\mathcal{X}}$ who have disjoint specializations on a non empty open subset of the special fibre (namely the locus where g' is defined).

We claim that their reductions G_1 and G_2 are irreducible (and hence disjoint): The group of divisors of X modulo numerical equivalence is generated by two classes F and G, where F is a fibre class of f and G is one of g. The intersection numbers are

$$F \cdot F = 0, \qquad F \cdot G > 0, \qquad G \cdot G = 0.$$

In particular, there are no effective divisors on X with negative self-intersection. It follows that the specialization of a curve of canonical type is again of canonical type. However, every curve of canonical type on X is irreducible, hence the claim follows.

Considering the global sections of $\mathscr{L}^{\otimes m}$ associated to the effective divisors $\mathcal{G}_1$ and $\mathcal{G}_2$, we find that $\mathscr{L}^{\otimes m}$ is globally generated. It follows that the map given by $\mathscr{L}^{\otimes m}$ is in fact a morphism, lifting g. □

We illustrate the theorem by looking at a special case: Let k be of characteristic three, and let X denote the Jacobian bielliptic surface of index $d = 3$ over k. What does the fibre $\mathcal{X}_\eta$ of the versal family of X over the generic point of the base look like? The smooth fibration with elliptic base curve does not have a section. The three sections which appear when we basechange with the algebraic closure of η do not descend to $\mathcal{X}_\eta$. Instead, we have a multi-section of degree three.

There is an explicit construction of a bielliptic surface with $d = 3$ over $\mathbb{Q}$, which shows the same behaviour. It was given in [BS03] as a counterexamples to the Hasse principle which cannot be explained by the Manin obstruction.

CHAPTER 2

Liftings of semistable elliptic fibrations

1. Semistable genus-1 curves and generalized elliptic curves

Among elliptic fibrations, the semistable ones play a central role, because any elliptic fibration has a semistable model after a suitable base change. If we consider Jacobian semistable elliptic fibrations possesing a group structure, we have a powerfull moduli theory at our disposal. It was developed by Deligne and Rapoport [DR73]. This theory is the main ingredients of this Section.

We start with recalling basic facts and definitions from [DR73] (see also the first Section of [Con07]). The concept of a semistable elliptic fibration is a special case of:

2.1.1. Definition. Let $p\colon C \to S$ be proper, flat morphism of finite presentation and of relative dimension one. Then C/S is called a semistable genus-1 curve if every geometric fibre of C is either a smooth connected curve of genus 1 or a Néron n-gon.

In [DR73] semistable curves are called *stable* (see [DR73, II Definition 1.4]). We recall the definition of a Néron n-gon: The standart Néron n-gon is the curve over $\mathrm{Spec}(\mathbb{Z})$ which is obtained by taking a disjoint union of $\mathbb{P}^1_{\mathbb{Z}}$'s, indexed by $\mathbb{Z}/n\mathbb{Z}$, and gluing the zero section of the i'th copy to the infinity section of $i+1$'th copy. A (Néron) 1-gon over an arbitrary scheme S is the pullback of the standart n-gon along $S \to \mathrm{Spec}(\mathbb{Z})$. For example, the 1-gon is just the rational nodal curve. Notice, that the arithmetic genus of a semistable genus-1 curve is in fact 1.

We recall the definition of the *graph of irreducible components* $\Gamma(C)$ of a reduced curve C over an algebraically closed field (see [DR73, I 3.5]). The set of vertices consists of the set $'\Gamma^0$ of irreducible components of C and the set $''\Gamma^0$ of singularities of C. For a point $p \in C(k)$ we call the preimages of p under the normalization map $\widetilde{C} \to C$ the *branches* of C at p. The set of edges of C is defined to be the set of couples (p, b) where $p \in C(k)$ is a singular point, and b is a branch at p. Note that we have a bijection between components of C and components of $\widetilde{C}$. Using this, we say that an edge (p, b) joints $p \in {''\Gamma^0}$ with a component in $'\Gamma^0$ if b lies on the corresponding component of $\widetilde{C}$.

If C is a Néron n-gon with $n > 1$, then $\Gamma(C)$ is a cycle consisting of $2n$ edges and $2n$ vertices.

By [DR73, II Proposition 1.6.] we know that a semistable genus-1 curve $f\colon C \to S$ satisfies $\mathscr{O}_S = f_*\mathscr{O}_C$ universally. Therefore, we see that a semistable genus-1 curve which is generically smooth and lives over a base which is a proper, smooth and connected curve over some field k, is indeed an elliptic fibration.

Semistable genus-1 curves are special, because they can carry group structures:

2.1.2. Definition. A generalized elliptic curve is a triple $(E/S, +, \epsilon)$ where E/S is a semistable genus-1 curve, $+\colon E^{sm} \times_S E \to E$ is an S-morphism and $\epsilon\colon S \to E^{sm}(S)$ is a section such that:

- "+" restricts to a commutative groups scheme structure on the smooth locus E^{sm}.
- "+" is an action of E^{sm} on E such that on singular geometric fibers the translation action by each rational point in the smooth locus induces a rotation on the graph of irreducible components.

If the base S is regular and one-dimensional, and the total space of C is regular, we can always equip E^{sm}/S with a group structure. This follows from the theory of Néron models. However, it is important to point out that this is no longer true if we allow higher dimensional or non-reduced bases.

2.1.3. *Example* There exists a Jacobian elliptic fibration over $E/\mathbb{P}^1_k$, with regular total space, which has exactly five singular fibers: Four of those are nodal curves, and one is an 8-gon. This is one of the rational elliptic surfaces which can be found in the big list [Mir90]. Its modular invariant j_0 is separable and of degree 12.

We use the methods developed later in this work to construct a specific deformation of E. Let R be a complete discrete valuation ring, with fraction field K. Denote by W the Weierstraß-model of E. By Corollary 2.6.2 below, it exists a lifting $\mathcal{W}/\mathbb{P}^1_R$ of W over R. Furthermore, we can choose the lifting such that the modular invariant of $\mathcal{W}_K = \mathcal{W} \otimes_R K$ has only single poles. It follows that $\mathcal{W}_K$ is an elliptic surface with 12 singular fibers, each of them a rational nodal curve.

After an extension of $R' \supset R$, it is possible to find a minimal simultaneous resolution $\mathcal{E} \to \mathcal{W}$ (see Proposition 3.4.5). The surface $\mathcal{E} \otimes_R k$ is isomorphic to E. Hence we have lifted the elliptic surface E. However, we cannot lift the group law on $E/\mathbb{P}^1_k$: For assume that $\mathcal{E}/\mathbb{P}^1_R$ were a generalized elliptic curve. Then we know that the subgroup scheme $\mathcal{E}[8]$ is flat [DR73, II 1.18]. Zariski's Main Theorem implies that for a quasi-finite and flat map, the fiber rank cannot grow under specialization. This leads to a contradiction if we consider one of the nodal fibers, specializing into the 8-gon. Note that if F is a 1-gon, $\operatorname{ord}(F[8]) = 8$, and if F is a 8-gon, $\operatorname{ord}(F[8]) = 8^2$.

One consequence of this discussion is that a reasonable theory of Néron models over bases of higher dimension cannot exist. Another consequence is that a deformation of a generalized elliptic curve will be a semistable curve (see [DR73, I Proposition 1.5]) but in general we lose the group structure, even in the case where we have a lifting of the zero section.

If we restrict ourselves to semistable curves with geometrically integral fibers, the situation is much better. We will call a generalized elliptic curve E/S a *Weierstraß curve*, if E has geometrically irreducible fibers. Every semistable genus-1 curve J/S with geometrically irreducible fibers and a section lying in the smooth part is in fact a Weierstraß curve. Namely, there exists a unique group law on J/S, depending only on the choice of the section (see [DR73, II Proposition 2.7])

Given a Weierstraß curve E over S, we denote by $\epsilon\colon S \to E^{sm}$ the zero section. We define an invertible sheaf on S by

$$\omega = \epsilon^*\Omega^1_{E/S}.$$

The name Weierstraß curve stems from the fact that a Weierstraß curve E/S can be embedded into a two dimensional projective bundle over $P \to S$. Inside P we have that E is given as the vanishing locus of a cubic equation, the so called Weierstraß equation.

Following [Tat75, §1] we can define two sections

$$c_4 \in H^0(S, \omega^{\otimes 4}) \text{ and } \Delta \in H^0(S, \omega^{\otimes 12})$$

depending on the coefficients of a Weierstraß equation for E. The ratio $j = c_4^3/\Delta$ defines a section of ω, and depends only on the isomorphism type of E. Because we assume E to have no fibres of additive type, c_4 and Δ will not vanish simultaneously. Thus we get a well defined section of $\mathbb{P}^1_S$ over S, given by the basis $1, j$ of $H^0(S, \omega)$. The induced map $S \to \mathbb{P}^1_{\mathbb{Z}} \times S \to \mathbb{P}^1_{\mathbb{Z}}$ is called the *modular invariant* of E, and denoted by $j(E)$.

Given an arbitrary semistable genus-1 curve E/S with a section lying in E^{sm} we can reduce to the Weierstraß case, using the following:

2.1.4. Theorem ([DR73] IV Proposition 1.3)**.** *Let E/S be a semi-stable genus-1 curve over a scheme S. Let $H \subset E^{sm}$ be an effective divisor, lying in E^{sm}; i.e. a Cartier divisor. Assume that $H \to S$ is flat.*

Then there exists an unique semistable genus-1 curve E^c over S, and a S-morphism $c\colon E \to E^c$, such that fibrewise over geometric points the base, c is given by contraction fibre of E which do not intersect H.

For the application we have in mind, just choose H to be the closed subscheme given by the section. The Weierstraß curve E^c obtained by contraction in that way is sometimes called the *Weierstraß model* of E/C. Using E^c, we can define the modular invariant for an arbitrary generalized elliptic curve.

2. Modular invariants and isomorphism types over complete bases

It is well know that an elliptic curve over an algebraically closed field is uniquely determind by its modular invariant. We want to understand to what kind of base schemes this fact generalizes.

2.2.1. Proposition. *Let R be a complete local noetherian ring with algebraically closed residue field k. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ denote two elliptic curves over R with identical modular invariant j. Then $\mathcal{E}_1$ and $\mathcal{E}_2$ are isomorphic.*

Proof. The reductions of $\mathcal{E}_1$ and $\mathcal{E}_2$ are isomorphic, because k is algebraically closed. So we find an elliptic curve E over k with isomorphisms

$$\epsilon_1\colon E \simeq \mathcal{E} \otimes_R k \quad\text{and}\quad \epsilon_2\colon E \simeq \mathcal{E}_2 \otimes_R k.$$

The couples $(\mathcal{E}_1, \epsilon_1)$ and $(\mathcal{E}_2, \epsilon_2)$ are now consider as deformations of E over R.

We write $A = W(k)[[T]]$ for the universal deformation ring of E. Let j_0 denote the modular invariant of E. Denote by $\mathscr{O}^{\wedge}_{\mathbb{P}^1_{\mathbb{Z}}, j_0}$ the completion of the local ring of the coarse moduli space $\mathbb{P}^1_{\mathbb{Z}}$ at the point $j = j_0$. Since there exists a universal family over A we get a map $\operatorname{Spec}(A) \to \operatorname{Spec}(\mathscr{O}^{\wedge}_{\mathbb{P}^1_{\mathbb{Z}}, j_0})$.

Following [KM85, Proposition 8.2.3] we can describe this map as quotient map

$$\operatorname{Spec}(A) \to \operatorname{Spec}(A)/\operatorname{Aut}(E) \simeq \operatorname{Spec}(\mathscr{O}^{\wedge}_{\mathbb{P}^1_{\mathbb{Z}}, j_0}).$$

Here, the action of $\operatorname{Aut}(E)$ on $\operatorname{Spec}(A)$ is defined pointwise: A homomorphism $\operatorname{Spec}(\Lambda) \to \operatorname{Spec}(A)$ is just a deformation $(\mathcal{E}, \epsilon)$ of E over $\operatorname{Spec}(\Lambda)$. On this data we act with $\operatorname{Aut}(E)$ by moving the isomorphism ϵ on the reduction. Coming back to R, the two deformations give us morphisms

$$\begin{array}{ccc} \operatorname{Spec}(R) & \rightrightarrows & \operatorname{Spec}(A) \\ & \searrow & \downarrow j \\ & & \operatorname{Spec}(\mathscr{O}^{\wedge}_{\mathbb{P}^1_R, j_0}) \end{array}$$

By what we saw above, we get that the two morphism are permutated by the $\operatorname{Aut}(E)$ action. This action, however, does not change the isomorphism type of the pullback. $\square$

Again from the "Formulaire" [Tat75] we take the following description of the isomorphisms functor of two Weierstraß curves:

2.2.2. Proposition ([Tat75] 5.3)**.** *Let E and E' be two Weierstraß curves over a scheme S.*

(i) $\mathrm{Isom}(E, E')$ *is representable, finite and unramified over S.*
(ii) *If $j_E = j_{E'}$, then* $\mathrm{Isom}(E, E') \to S$ *is surjective.*
(iii) *If $j_E = j_{E'}$ and the values* 0 *or* 1728 *are never assumed, then* $\mathrm{Isom}(E, E')$ *is étale of rank 2 over S.*

Note that the last statement in the proposition says that $\mathrm{Isom}(E, E')$ is a $\mathbb{Z}/2\mathbb{Z}$-torsor in that case. In the non-special case, Proposition 2.2.1 follows directly from this.

Let $\mathcal{M}_1$ be the moduli stack of Weierstraß curves. Following [DR73, VI], the individual modular invariants defined above give a morphisms $\mathcal{M}_1 \to \mathbb{P}^1_{\mathbb{Z}}$, which factors over the coarse moduli space M_1 of $\mathcal{M}_1$. This factorization is in fact an isomorphism. Hence we get a section j of $\mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1)$, such that for every curve E we have that $j(E)$ is the pullback of j.

Let U be an open subset of $M_1 \simeq \mathbb{P}^1_{\mathbb{Z}}$. We denote by $\mathcal{M}_1|_U$ the full subcategory of $\mathcal{M}_1$ consisting of those curves E/S, whose modular invariants factor through U. This means we have a diagram:

$$\begin{array}{ccc} S & \longrightarrow & \mathcal{M}_1 \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathbb{P}^1_{\mathbb{Z}} \end{array}$$

Evidently, the coarse moduli space of $\mathcal{M}_1|_U$ is just U. We define a stack $\mathcal{M}_1/U$ over the category of U-schemes as follows: For a scheme S with U-structure $f \colon S \to U$, let $(\mathcal{M}_1/U)(S)$ be the category of Weierstraß curves E/S satisfying $j(E) = f$. In symbols, $\mathcal{M}_1/U$ is just the fibre product of stacks $\mathcal{M}_1 \times_{\mathbb{P}^1_{\mathbb{Z}}} U$. We seek to describe this stack in the case where U does not contain special values.

2.2.3. Proposition. *Let $j \in \mathscr{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1)$ denote the modular invariant given by the universal family. Let $U \subset \mathbb{P}^1_R$ be an open subset where $j(j-1728)$ is invertible. Then $\mathcal{M}_1/U \to U$ is a neutral $\mathbb{Z}/2\mathbb{Z}$-gerbe for the étale topology.*

Proof. To show that $\mathcal{M}_1/U \to U$ is a gerbe, we have to check the following two axioms:

(i) *Non-emptiness.* There exists an étale covering $\{U_\alpha \to U\}$ of U such that for every U_α holds $(\mathcal{M}_1/U)(U_\alpha) \neq \emptyset$.
(ii) *Transitivity.* For an U-scheme S and two objects E_1 and E_2 of $(\mathcal{M}_1/U)(V)$ there is an étale neighborhood S' of every point $x \in S$ such that the pullbacks E_1' and E_2' to S' become isomorphic.

A gerbe is neutral if it satisfies not only (i) but rather has a global section. A global section of $\mathcal{M}_1/U \to U$ can be given by the wellknown family

$$y^2 + xy = x^3 - \frac{36}{j-1728}x - \frac{1}{j-1728}$$

with modular invariant j.

Transitivity follows form the properties of the isomorphism functor: For two Weierstraß curves E_1 and $E_2 \in (\mathcal{M}_1/U)(S)$ we know that $\mathrm{Isom}_S(E_1, E_2)$ is a $\mathbb{Z}/2\mathbb{Z}$-torsor. Thus, there exists a étale covering $S' \to S$ such that $\mathrm{Isom}_S(E_1, E_2)$ is split. □

Applying the description of neutral gerbes as classifying stacks [LMB00, Lemme 3.21], to $\mathcal{M}_1/U \to U$, we see that once we have fixed a map $j\colon S \to U$, giving a Weierstraß curve over S with modular invariant j is equivalent to giving a $\mathbb{Z}/2\mathbb{Z}$-torsor over S. This presentation depends of course on the choice of the global section.

While Proposition 2.2.1 classifies smooth elliptic curves over complete rings with algebraically closed residue field by the means of their modular invariants, we will also need an analog of this, for non-smooth generalized elliptic curves. This is provided by the theory of Tate curves.

The construction of Tate curves, following Raynaud, is given in [DR73, VII 1]. We review it briefly: Given a scheme S and a section $t \in H^0(S, \mathscr{O}_S)$, one can define a scheme $\overline{\mathscr{G}}^t_m$ over S. Restricting $\overline{\mathscr{G}}^t_m$ to $S[1/t]$ it becomes isomorphic to $\mathbb{G}_m$. The restriction to the closed subscheme given by (t) is an infinite chain of $\mathbb{P}^1$'s glued together at infinity and zero. The smooth locus $\mathscr{G}^t_m \subset \overline{\mathscr{G}}^t_m$ can be equipped with the structure of a commutative algebraic group, extending the group scheme $\mathbb{G}_m$ over $S[1/t]$. Its components over $t = 0$ can be indexed with $\mathbb{Z}$, and are denoted by T_k. Here, T_0 is to be the zero component.

The group scheme structure itself extends to an action:

$$\mathscr{G}^t_m \times \overline{\mathscr{G}}^t_m \to \overline{\mathscr{G}}^t_m.$$

Now assume t to be nilpotent, and let q be a section of $\mathscr{G}^t_m$, which does not specialize into the identity component of $\mathscr{G}^t_m$. Then we get a $\mathbb{Z}$-action on $\overline{\mathscr{G}}^t_m$ by multiplying with $g^i, i \in \mathbb{Z}$, and the quotient $\overline{\mathscr{G}}^t_m/q^{\mathbb{Z}}$ exists in the category of schemes [DR73, VII Proposition 1.6].

This quotient is a generalized elliptic curve. Working over a ring A which is complete with respect to some ideal (t), we can construct a curve $\overline{\mathscr{G}}^t_m/q^{\mathbb{Z}}$ as inductive limit of the analogous construction over the truncations $A/(q^n)$. More precisely we have:

2.2.4. Proposition (VII Corollaire 2.6)**.** *Let A be a complete local noetherian ring. Let t be an element in the maximal ideal. Then*

$$\overline{\mathscr{G}}^t_m/(t^k)^{\mathbb{Z}}$$

is a generalized elliptic curve over A, with the property that the non-smooth locus is defined by t, and that the special fiber is a k-gon.

Furthermore, given a generalized elliptic curve E over A with the same properties, there exists $u \in A^\times$ such that

$$E \simeq \overline{\mathscr{G}}^t_m/(ut^k)^{\mathbb{Z}}.$$

3. Moduli of generalized elliptic curves with level structures

We fix a positive integer n. We want to define level structures just as in the smooth case. In order to do so, we have to understand the n-torsion subscheme $E[n]$ of a generalized elliptic curve E/C. It is defined as the kernel of

$$[n]\colon E^{sm} \to E^{sm}.$$

By the fibrewise criterion for flatness [EGA IV.3, 11.3.11] we get that $E[n]$ is flat. If furthermore n is invertible in $\mathscr{O}_S$, the morphism $[n]$ is fibrewise unramified, hence étale. In general $E[n]$ will not be finite, because the rank can drop under specialization at singular fibers. However, if the rank of $E[n]$ at the singular fibers is sufficiently large, we get following:

2.3.1. Proposition ([DR73] II Corollaire 1.20)**.** *Let E/S denote a generalized elliptic curve. Assume that every geometric fibre of E is either smooth or an m-gon, with $n|m$. Then, $E[n]$ is a finite flat groups scheme of order n^2.*

Let E/S be a generalized elliptic curve. A Cartier divisor $D \subset E$ is called S-ample if $\mathscr{O}_E(D)$ is relatively ample for the map $E \to S$. This implies that D intersects every irreducible component of every geometric fibre of E/S.

2.3.2. Definition. Let E/S be a generalized elliptic curve. A $\Gamma(n)$-*structure* on E/S is an isomorphism of S-group schemes

$$\alpha \colon E[n] \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^2,$$

such that the image of α is S-ample.

A $\Gamma_1(n)$-*structure* on E/S is a fixed torsion point of exact order n, or equivalently on embedding of S-group schemes

$$\beta \colon \mathbb{Z}/n\mathbb{Z} \hookrightarrow E[n]$$

such that the image of β is S-ample.

A $\Gamma_0(n)$-*structure* on E/S is a S-subgroupscheme

$$A \subset E[n]$$

which is locally for the étale topology isomorphic to $\mathbb{Z}/n\mathbb{Z}$, such that the closed subscheme associated to A is S-ample.

Now we can define the associated stacks:

2.3.3. Definition. For $\Gamma \in \{\Gamma(n), \Gamma_1(n), \Gamma_0(n)\}$ we define $\mathcal{M}_\Gamma$ to be the stack of pairs $(E/S, \gamma)$ where E/S is a generalized elliptic curve over a $\mathbb{Z}[1/n]$-scheme S, and γ is a Γ-structure.

Note that $\mathcal{M}_\Gamma$ is indeed a *fpqc*-stack over $\mathbb{Z}$, because the required ampleness of the level structures gives us an ample line bundle, compatible with descent data.

We will sum up some facts about the representability of the moduli stacks defined above. If we restrict ourself to $\mathbb{Z}[1/n]$-schemes, that is consider the stacks $\mathcal{M}_\Gamma \otimes \mathbb{Z}[1/n]$ for the various level structures, we know form [DR73] that all of them are Deligne-Mumford stacks.

A Deligne-Mumford stack is an algebraic space, if and only if its inertia group is trivial, i.e. if the parametrized objects have only trivial automorphisms. This is indeed the case for $\mathcal{M}_{\Gamma(n)}$ if $n \geq 3$, and for $\mathcal{M}_{\Gamma_1(n)}$ if $n \geq 4$. Whereas, for $\mathcal{M}_{\Gamma_0(n)}$, this is never true for the very reason that the involution maps a subgroup into itself.

In [DR73] it is shown that the stacks $\mathcal{M}_{\Gamma(n)} \otimes \mathbb{Z}[1/n], \mathcal{M}_{\Gamma_1(n)} \otimes \mathbb{Z}[1/n]$ and $\mathcal{M}_{\Gamma_0(n)} \otimes \mathbb{Z}[1/n]$ are proper, smooth and 1-dimensional over $\mathrm{Spec}(\mathbb{Z}[1/n])$ (see [DR73, III Theorem 3.4]). This can used to show that for $\mathcal{M}_{\Gamma(n)}$ and $\mathcal{M}_{\Gamma_1(n)}$ the notions of "being representable as an algebraic space" and "being representable as a scheme" coincide.

Thus, for adequately chosen n, the functors $\mathcal{M}_{\Gamma(n)} \otimes \mathbb{Z}[1/n]$ and $\mathcal{M}_{\Gamma_1(n)} \otimes \mathbb{Z}[1/n]$ are representable by smooth and proper curves over $\mathrm{Spec}(\mathbb{Z}[1/n])$. In that case, we will write $\mathcal{M}_{\Gamma(n)}$ and $\mathcal{M}_{\Gamma_1(n)}$ for the fine moduli schemes.

4. Level structures and liftability

Let R be a complete noetherian ring with maximal ideal $\mathfrak{m}$ and algebraically closed residue field $R/\mathfrak{m} \simeq k$. For the rest of this Chapter, we will be concerned with lifting a generalized elliptic curve $E/C/k$ to a generalized elliptic curve $\mathcal{E}/\mathcal{C}/R$. The moduli theory for curves with level-n structures gives the following trivial lifting result:

2.4.1. Proposition. *Let $(E/C, \gamma_0)$ be a pair consisting of a generalized elliptic curve over a proper smooth base curve C/k. and a Γ-structure γ on E, where $\Gamma \in \{\Gamma(n \geq 3), \Gamma_1(n \geq 4)\}$. Denote by $c_0 \colon C \to \mathcal{M}_\Gamma \otimes k$ the classifying morphism.*

Then a liftings $(\mathcal{E}/\mathcal{C}, \gamma)$ over R of the pair $(E/C, \gamma_0)$ corresponds to a lifting $c \colon \mathcal{C} \to \mathcal{M}_\Gamma \otimes R$ of c_0.

2.4.2. Corollary. *Let $(E/C, \gamma_0)$ be a pair like in Proposition 2.4.1. Assume, that the modular invariant $j_0 \colon C \to \mathbb{P}^1_k$ is a separable morphism. Then there exists a lifting $(\mathcal{E}/\mathcal{C}, \gamma)$.*

Proof. Let $c_0 \colon C \to \mathcal{M}_\Gamma \otimes k$ denote the morphism given by the pair $(E/C, \gamma_0)$. We have a triangle:

$$\begin{array}{ccc} C & \xrightarrow{c_0} & \mathcal{M}_\Gamma \otimes k \\ & {\scriptstyle j_0} \searrow & \downarrow \\ & & \mathbb{P}^1_k \end{array}$$

Since j_0 is separable, it follows that c_0 is separable as well. We can always lift a separable covering, for example by the means of formal patching (see Proposition 2.5.8 below). So, let $c \colon \mathcal{C} \to \mathcal{M}_\Gamma \otimes R$ be a lifting of c_0. The pullback of the universal family along c is the desired lifting of E. □

We can also use Proposition 2.4.1 in the opposite way: Namely to produce examples of non-liftable generalized elliptic curves. The idea is as follows: Choose $\Gamma \in \{\Gamma(n), \Gamma_1(n), \Gamma_0(n)\}$ for some n with the property that the genus of the components of the coarse moduli space M_Γ is at least 2. Formulas for the genus of M_Γ can be found for example in [Hus04, 11. §3]. The condition $g(M_\Gamma) \geq 2$ will be satisfied for sufficiently large n.

Let $(E/C, \gamma)$ be a generalized elliptic curve with Γ-structure γ, such that the morphism into the coarse moduli space $c \colon C \to M_\Gamma \otimes k$ is purely inseparable. Denote the component of $M_\Gamma \otimes k$ into which C is mapped by Y.

The curves C and Y have equal genus. Thus it is a consequence of Hurwitz's formula, that c is not liftable: A lifting of c would be a special case of a deformation of a purely inseparable map into a separable one. However, the genus of curves and the degree of morphisms are invariant under deformation, and Hurwitz's formula shows that there is now separable map between curves of equal genus, given that genus is greater or equal to 2. So we have as a trivial consequence of Proposition 2.4.1:

2.4.3. Lemma. *For $(E/C, \gamma_0)$ as above, there does not exists a lifting to a pair $(\mathcal{E}/\mathcal{C}, \gamma)$ if the fraction field of R contains $\mathbb{Q}$.*

The Γ-structures we are dealing with are defined in terms of the étale subgroup scheme $E[n]$. It is therefore natural to expect that an infinitesimal deformation of a generalized elliptic curve induces a lifting of a given Γ-structure. Exemplarily we prove for $\Gamma = \Gamma_0(n)$ the following statement:

2.4.4. Theorem. *Let n be choosen such that $g(M_{\Gamma_0(n)} \otimes k) \geq 2$. Let $(E/C, \gamma)$ be a generalized elliptic curve with a $\Gamma_0(n)$-structure $A \subset E$, such that classifying morphism $C \to M_{\Gamma_0(n)}$ is purley inseparable. Then there does not exist a lifting of E within the category of generalized elliptic curves, if the fraction field of R contains $\mathbb{Q}$.*

Proof. Assume a lifting $\mathcal{E}/\mathcal{C}$ exists. We have to lift the subscheme $A \subset E[n]$ given by the level strucutre. To that purpose we have to work in the category of formal schemes. So denote by $\widehat{\mathcal{E}} \to \widehat{\mathcal{C}}$ the completion of $\mathcal{E}/\mathcal{C}$ over $\mathrm{Spf}(R)$.

There exists a unique lifting $\mathcal{A}/\widehat{\mathcal{C}}$ of A. The group scheme $\hat{\mathcal{E}}[n]$ is étale, so we have bijection of morphisms

$$\mathrm{Hom}_{\widehat{\mathcal{C}}}(\mathcal{A}, \widehat{\mathcal{E}}[n]) \to \mathrm{Hom}_C(A, E[n])$$

thus we can lift the map $A \hookrightarrow E[n]$. By uniqueness, the lifting will be compatible with the group structure of $E[n]$. Since its kernel is finite, it has to vanish by Nakayama.

However, $\mathcal{A}$ itself is finite, thus we can algebraize the inclusion $\mathcal{A} \hookrightarrow \hat{\mathcal{E}}$ and obtain $A \subset \mathcal{E}$. By construction, this inclusion factors over $\mathcal{E}[n]$. Now, we are just in the situation of Lemma 2.4.3, and obtain a contradiction. ☐

It is tempting to try to weaken the assumption in Theorem 2.4.4 on the hypothetical lifting $\mathcal{E}/\mathcal{C}$ from "generalized elliptic" to "semistable genus 1", because being semistable is an open condition, whereas "generalized elliptic" is not (see Example 2.1.3). However, the presence of a group structure is essential for the argument. This problem could be avoided, if the fibration E/C could be chosen to be Weierstraß, because in this situation we always have a natural group structure.

This is not possible, because one cannot put a $\Gamma_0(n)$-structure on a Weierstraß fibration E/C: Given such a datum, we could turn into a $\Gamma_1(n)$-structure after an étale base change $C' \to C$. However, the universal family over $\mathcal{M}_{\Gamma_1(n)}$ is never Weierstraß, given $n > 1$, because there are n-gons which can by given a $\Gamma_1(n)$-structure: Just take a n-torsion section, generating the component group. Those n-gons arise as pullbacks form the universal family. Also note, that $\mathcal{M}_{\Gamma_1(n)}$ is connected, since the determinant of the associated modular group is trivial.

5. Liftings of generalized elliptic curves with separable modular invariants

We saw in Corollary 2.4.2 that lifting a generalized elliptic curve E/C is easy, if the modular invariant is separable and if one has a sufficiently strong level structure on E. In this Section, we will demonstrate that one can weaken the assumption on the level structure by using descent theory.

For technical reasons, we have to work in the formal catgory. This is why we need the following definition:

2.5.1. Definition. Let $\mathcal{S} = \varinjlim \mathcal{S}_n$ be a formal scheme. An *adic generalized elliptic curve* over $\mathcal{S}$ is an adic $\mathcal{S}$-formal scheme, such that every truncation $\mathcal{E}_n \to \mathcal{S}_n$ is a generalized elliptic curve.

Note that giving an adic generalized elliptic curve $f\colon \mathcal{E} \to \mathcal{S}$ is equivalent to giving a inductive S_n-adic system of generalized elliptic curves:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathcal{E}_{n-1} & \longrightarrow & \mathcal{E}_n & \longrightarrow & \mathcal{E}_{n+1} & \longrightarrow & \cdots \\ & & {\scriptstyle f_{n-1}}\downarrow & \square & \downarrow{\scriptstyle f_n} & \square & \downarrow{\scriptstyle f_{n+1}} & & \\ \cdots & \longrightarrow & \mathcal{S}_{n-1} & \longrightarrow & \mathcal{S}_n & \longrightarrow & \mathcal{E}_{n+1} & \longrightarrow & \cdots \end{array}$$

Typically, we will encounter adic generalized elliptic curves in the following situation: The base scheme $\mathcal{S}$ itself will be an adic $\mathrm{Spf}(R)$-formal scheme. Given an adic generalized elliptic curve $\mathcal{E}/\mathcal{S}$, its total space $\mathcal{E}$ is an adic $\mathrm{Spf}(R)$-formal scheme.

Then the cartesian diagrams in the above system are morphism of objects of the moduli stack $\mathcal{M} \otimes R$ of elliptic curves over R-schemes. Whence we get an inductive system of morphisms $c_n \colon \mathcal{S}_n \to \mathcal{M} \otimes R/\mathfrak{m}^{n+1}$, or equivalently a morphism

$$c \colon S \to \widehat{\mathcal{M}} = \varinjlim \mathcal{M} \otimes R/\mathfrak{m}^{n+1}.$$

This observation carries over to the case where we consider an inductive S_n-adic system of generalized elliptic curves with some Γ-structure and exchange $\mathcal{M}$ by $\mathcal{M}_\Gamma$.

Before we begin with the construction of specific liftings, we first prove a uniqueness result. This is based on a generalization of Proposition 2.2.2 to the non-Weierstraß case, which was given by Conrad:

2.5.2. Theorem ([Con07] 3.1.2)**.** *Let E/S and E'/S be generalized elliptic curves over some base scheme S. The functor $\mathrm{Isom}_S(E, E')$ classifying isomorphisms of generalized elliptic curves over S-schemes is represented by a quasi-finite and separated S-scheme of finite presentation. In particular, it is quasi-affine over S.*

2.5.3. Proposition. *Let E and E' be two generalized elliptic curves over some irreducible one-dimensional basescheme S. Assume that the modular invariants of E and E are identical and non-constant, and denote their common modular invariant by $j \colon S \to \mathbb{P}^1_{\mathbb{Z}}$. Assume further, that over a geometric point $x \in C$, where E and E' are singular, the number of components if E_x and E'_x coincide.*

Then there exists a unique closed subscheme $T \subset \mathrm{Isom}_S(E, E')$ of rank 2, which is étale over S.

Proof. Following Theorem 2.5.2 we have that the functor $\mathrm{Isom}_S(E, E')$ is represented by a quasi-finite and separated S-scheme X of finite presentation. Over the open subset $\mathcal{U} \subset \mathcal{S}$ where $j(j - 1728)$ and $1/j$ are invertible we know from Proposition 2.2.2 that $X \times_S \mathcal{U} \to \mathcal{U}$ is étale of rank 2.

In order to construct T, we first show that if it exists, then it is unique and its formation commutes with flat base change.

Let $i \colon T \hookrightarrow X$ and $i' \colon T' \hookrightarrow X$ be two closed subschemes of X with the above properties. From the separateness of $X \to S$, we get that T and T' share the same topological space.

To prove equality of T and T' we can work locally on S and X: Let $\mathrm{Spec}(A) \to \mathcal{S}$ be the localization in some closed point. Let $\mathrm{Spec}(B)$ be an affine subset of $X \times_S \mathrm{Spec}(A)$. Denote by $I, I' \subset B$ the ideals given by the pullbacks of T and T'. Consider the exact sequence

$$0 \to J \to B/(I \cap I') \to B/I \to 0 \qquad (2.5.3.1)$$

Let $m \subset A$ denote the maximal ideal. We are going to show that $J = 0$. Tensoring with A/m, we get an exact sequence:

$$\mathrm{Tor}_1^A(B/I, A/m) \to J \otimes A/m \to B/(I \cap I') \otimes A/m \to B/I \otimes A/m \to 0.$$

By assumption on T, we know that B/I is A-flat and hence the leftmost term vanishes. Combining this with Nakayama, we see that it is enough to show that we have equality of T and T' in every fibre over every closed point of S.

Given a fibre X_s, we have that T_x and T'_x are reduced subschemes sharing the same topological space, hence they are equal.

Now, it makes sense to speak of T as *the* finite and étale subscheme of X of rank 2. We are going to show that its formation commutes with flat base change over S. Let $S' \to S$ be a flat morphism. Then $U' = U \times_S S'$ is a non-empty open subset of S', over which $X' = X \times_S S'$ is étale and of rank 2. With these properties, one can prove uniqueness of T as before.

The existence of T is still not established at this point. However, the uniqueness allowes us to construct T locally in the $fpqc$-toplogy. In particular, we can assume S to be affine.

Let Z be the complement of U. Denote by $\mathcal{Z}$ the completion of S along Z. We consider the $fpqc$-morphism $\mathcal{Z} \coprod U \to S$. Over U, the existence of T is trivial. Let x be a point in Z and set $S_x = \mathrm{Spec}(\mathscr{O}^{\wedge}_{S,x})$.

At first, we treat the case where $j(j-1728)$ is contained in the maximal ideal of $\mathscr{O}^{\wedge}_{S,x}$. The fibers $E_x = E \times_S S_x$ and $E'_x = E' \times_S S_x$ are smooth. It follows from Proposition 2.2.1 that there is an isomorphism $\varphi\colon E_x \to E'_x$. The two sections $\pm\varphi$ give rise to an étale finite subscheme of $\mathrm{Isom}_{S_x}(E_x, E'_x)$ of rank 2.

Now assume that $1/j$ lies in the maximal ideal of $\mathscr{O}^{\wedge}_{S,x}$, so that E'_x and E_x are non-smooth. We denote by $(t) \subset \mathscr{O}^{\wedge}_{S,x}$ the ideal which identifies their common non-smooth locus (which is determind by the modular invariant). Let n be the common number of irreducible components of E_x and E'_x over the residue field of the special point of $\mathscr{O}^{\wedge}_{S,x}$. We can assume that the residue field is algebraically closed. By Proposition 2.2.4 there do exist elements $u_1, u_2 \in A^\times$, such that

$$\mathcal{E}_x \simeq \mathscr{G}^t_m/(ut^k)^{\mathbb{Z}} \text{ and } \mathcal{E}'_x \simeq \mathscr{G}^t_m/(u't^k)^{\mathbb{Z}}.$$

However, the elements u and u' are determined by the modular invariant and therefore equal. Having proven that E_x and E'_x are isomorphic we argue as in the smooth case. □

2.5.4. Corollary. *Let $\mathcal{S}$ be a one-dimensional noetherian formal adic formal* $\mathrm{Spf}(R)$*-scheme. Set $S = \mathcal{S} \otimes_R k$. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two generalized elliptic curves over $\mathcal{S}$. Assume we have an isomorphism $\varphi_0\colon E_1 \to E_2$. Assume furthermore, that the modular invariants of $\mathcal{E}_1$ and $\mathcal{E}_2$ are identical and non-constant, and denote their common modular invariant by $j\colon \mathcal{S} \to \hat{\mathbb{P}}^1_R$. Then there exists a unique lifting $\varphi\colon \mathcal{E} \to \mathcal{E}'$ of φ_0.*

Proof. The objects $\mathcal{E}$, $\mathcal{E}'$ and $\mathcal{S}$ live in the category of adic formal schemes over $\mathrm{Spf}(R)$. Let $\mathcal{E}_n = E \otimes_R R/\mathfrak{m}^{n+1}, \mathcal{E}'_n = \mathcal{E}' \otimes_R R/\mathfrak{m}^{n+1}$ and $\mathcal{S}_n = \mathcal{S} \otimes_R R/\mathfrak{m}^{n+1}$. It is equivalent to work with the inductive systems $(\mathcal{E}_n \to \mathcal{S}_n)$ and $(\mathcal{E}'_n \to \mathcal{S}_n)$. In particular, it suffices to construct a unique lifting of φ_0 to some $\varphi_n\colon \mathcal{E}_n \to \mathcal{E}'_n$ for every n.

By Proposition 2.5.3 it exists a unique étale subscheme T of rank 2 inside $\mathrm{Isom}_{\mathcal{S}_n}(\mathcal{E}_n, \mathcal{E}'_n)$. By uniqueness, we know that φ_0 gives a section $S \to T \otimes_R k$. Now by étaleness of T, the claim follows. □

Liftings of covers and formal patching. When we deal with generalized elliptic curves, we will encounter morphisms with wild ramification. The modular invariant $C \to \mathbb{P}^1_k$ of E/C can be wildly ramified in general. Even worse, if $p|6$ it is always wildly ramified: One can see this either using a direct calculation with the formula for j, or by representing the completion of the local ring of the coarse moduli space at $j = 0$ as a quotient of the universal deformation ring, as in Chapter 1 Proposition 2.2.1.

We give the latter argument: Let E_0/k be an elliptic curve with modular invariant $j_0 = 0$, and denote by A its universal deformation ring. Then $\mathscr{O}^{\wedge}_{\mathbb{P}^1_k, j=0}$ is the ring of invariants A^G where $G = \mathrm{Aut}_0(E_0)/\mathbb{Z}/2\mathbb{Z}$. The action is trivial on the special fibre, so we get an inertia group of order 6 of $p = 3$, and one of order 12 if $p = 2$.

If we want to construct liftings of generalized elliptic curves, by using liftings of their modular invariants, this discussion shows, that we have to use techniques which allow to construct and controll liftings of wildly ramified coverings. This can be accomplished by the so called theory of *formal patching*. The starting point is the following descent result:

2.5.5. Theorem (Ferrand-Raynaud (see [Har03] Theorem 3.1.6). *Let V be an affine scheme, let Z be a closed subset of V, and let $U = V - Z$ be the complement. Let $\mathcal{Z}$ be the completion of V at Z. Also let $\mathcal{Z}^\circ = \mathcal{Z} \times_V U$. Then the base change functor*

$$\mathcal{C}oh(V) \to \mathcal{C}oh(\mathcal{Z}) \times_{\mathcal{C}oh(\mathcal{Z}^\circ)} \mathcal{C}oh(U) \tag{2.5.5.1}$$

is an equivalence of categories.

The objects of the fibre product of categories on the right hand side of (2.5.5.1) are by definition triples $(\mathscr{F}, \mathscr{G}, \alpha)$ where $\mathscr{F}$ and $\mathscr{G}$ are in $\mathcal{C}oh(\mathcal{Z})$ or $\mathcal{C}oh(V^\circ)$ respectively and

$$\alpha\colon \mathscr{F}|_{\mathcal{Z}^\circ} \xrightarrow{\sim} \mathscr{G}|_{\mathcal{Z}^\circ}$$

is an isomorphism over the intersection. This makes clear why Theorem 2.5.5 is called a patching result: We are gluing $\mathscr{F}$ and $\mathscr{G}$ along the intersection $\mathcal{Z}^\circ$ by using the isomorphism α of their pullbacks as gluing isomorphism.

There is an adapted version of Theorem 2.5.5 for one-dimensional schemes, avoiding the affiness assumption on V:

2.5.6. Theorem ([Pri00] 3.2). *Let V be a noetherian one-dimensional scheme. Let $Z \subset V$ be a finite set of closed points, $U = V - Z$ the complement. For a closed point $x \in V$ we set $V_x = \mathrm{Spec}(\mathscr{O}^\wedge_{V,x})$. Set $\mathcal{Z}^\circ = \mathcal{Z} \times_V U$.*

Then the base change functor

$$\mathcal{C}oh(X) \to \mathcal{C}oh(\mathcal{Z}) \times_{\mathcal{C}oh(\mathcal{Z}^\circ)} \mathcal{C}oh(U)$$

is an equivalence of categories.

A typical situation, in which 2.5.6 can be applied is when V is a curve over the spectrum of a local artinian R-algebra. For our purpose, it is practical to use a version which works entirely in the formal category. In order to do that, we fix some notations:

Let $\mathcal{V}$ be a smooth 1-dimensional adic $\mathrm{Spf}(R)$-formal scheme, with reduction $V = \mathcal{V} \otimes_R k$. For a closed point $x \in V = \mathcal{V} \otimes_R k$ we define the formal germ of $\mathcal{V}$ at x to be

$$\mathcal{V}_x := \mathrm{Spf}(\mathscr{O}^\wedge_{\mathcal{V},x}).$$

The ring $\mathscr{O}^\wedge_{\mathcal{V},x}$ is isomorphic to $R[[T]]$, for it is a lifting of the regular ring $k[[t]]$ over R. Let $U \subset V$ denote the complement of x and let $\mathcal{U} \subset \mathcal{V}$ be the formal open subscheme with reduction U. We want to determine $\mathcal{V}^\circ_x = \mathcal{V}_x \times_{\mathcal{V}} \mathcal{U} \subset \mathcal{V}_x$.

The ring of functions $\mathscr{O}_{\mathcal{V}^\circ_x}$ of this formal open subscheme is given by

$$\varprojlim_n \mathrm{Frac}(\mathscr{O}_{\mathcal{V}_x} \otimes R/\mathfrak{m}^{n+1})$$

This is an inverse limit of local artinian algebras, given as the total quotient rings of the truncations of $\mathscr{O}_{\mathcal{V}^\circ_x}$. One can think of $\mathcal{V}^\circ_x$ as a formal punctured neighborhood of x.

If R is a discrete valuation ring, $\mathscr{O}_{\mathcal{V}^\circ_x}$ is isomorphic to the discrete valuation ring $R[[t]]\{t^{-1}\}$. This is the subring of the Laurent series ring $R((t))$, formed by series $\sum_{i\in\mathbb{Z}} a_i t^i$, satisfying the convergence condition $\lim_{i\to-\infty} |a_i| = 0$. Here, $|\ |$ is the absolute value associated to the valuation on R.

2.5.7. Theorem (R. Pries). *Let $\mathcal{V}$ be a smooth formal curve over* $\mathrm{Spf}(R)$*, let C be the special fibre, let $W = \{x_1, \dots x_n\}$ be a set of closed points in C. Denote by $U \subset C$ the complement of W, and let $\mathcal{U}$ be a formal open subscheme of $\mathcal{C}$ whose* *special fiber is U. Let $\mathcal{C}_{x_1}, \dots \mathcal{C}_{x_m}$ be the formal germs of $\mathcal{C}$ in x_i, and denote by ${}^\circ\mathcal{C}_{x_1}, \dots, {}^\circ\mathcal{C}_{x_m}$ their formal punctured neighborhoods. The the base change functor*

$$\mathcal{C}oh(\mathcal{C}) \to \mathcal{C}oh(\coprod \mathcal{C}_{x_i}) \times_{\mathcal{C}oh(\coprod {}^\circ\mathcal{C}_{x_i})} \mathcal{C}oh(\mathcal{U})$$

is an equivalence of categories.

This is a slight variation of [Pri00, Theorem 3.4] where it is stated for proper curves. However, the properness is only used to apply Grothendieck's formal existence theorem to pass from coherent schemes over a formal scheme to coherent schemes over the algebraization. At this point, we do not require this step.

To proof Theorem 2.5.7 we reduce every occurring formal scheme modulo $\mathfrak{m}^n$, to find ourselves in the situation of Theorem 2.5.6. Taking the limit over n, we obtain the statement.

So far, we have only stated theorems dealing with the category of coherent schemes. However, this includes the case of coverings and even more, for an equivalence between categories of coherent sheaves induces an equivalence on the categories of finite algebras (i.e. coverings), finite groups schemes, torsors, Galois coverings and so on.

To demonstrate the usefullness of Theorem 2.5.7 we state an existence result for liftings of separable coverings of curves.

2.5.8. Proposition ([Saï04] Proposition 1.6)**.** *Let $\mathcal{C}$ be a proper and smooth formal curve over $\mathrm{Spf}(R)$ with reduction C and let $W = \{x_1, \dots, x_n\}$ be a set of closed points in C. Let $f_0 \colon C' \to C$ be a finite separable (Galois) cover (of group G) whose branch locus is contained in W. Denote by $\mathcal{C}_{x_1}, \dots, \mathcal{C}_{x_n}$ the formal germs of $\mathcal{C}$ at the points in W. We write $C_{x_i} = \mathcal{C}_{x_i} \otimes_R k$ for the reductions.*

Assume that for each $1 \leq i \leq n$ there exists a (Galois) cover

$$F_i \colon \mathcal{C}'_i \to \mathcal{C}_{x_i} \text{ lifting } C' \times_C C_{x_i} \to C_{x_i}.$$

Then there exists a (Galois) cover (of group G), unique up to isomorphism, which lifts f_k and which is isomorphic to the covers F_i when restricted to ${}^\circ\mathcal{C}_{x_i}$.

The proof of Proposition 2.5.8 uses Theorem 2.5.7 to glue the local lifting f_k with the one lifting $F_{\mathcal{U}}$ which exists generically over $\mathcal{U}$ by étalness of the restriction of f_0. The crucial point is that the restrictions of F_i to the punctured neighborhood $\mathcal{C}^\circ_{x_i}$ is also étale, and therefore isomorphic to the restriction of $F_{\mathcal{U}}$.

Consequences of Proposition 2.5.8 are that liftings of separable maps always exists (because there are no local obstructions) and that the problem of lifting a Galois covering *to a Galois covering* is reduced to a purely local one.

To finish this discussion, we mention Grothendieck's formal GAGA theorem (see for example [Ill05, Theorem 4.2]):

2.5.9. Theorem. *Let $\mathcal{X}$ be a proper noetherian scheme, separated and of finite type over $\mathrm{Spec}(R)$. Let $\widehat{\mathcal{X}}/\mathrm{Spf}(R)$ be the formal scheme, obtained by completing X along its special fibre over R. Then the completion functor $\mathcal{C}oh(\mathcal{X}) \to \mathcal{C}oh(\widehat{\mathcal{X}})$ is an equivalence of categories.*

6. The constructions

Before we start out with the actual constructions, a few comments on the method we use seem to be in order. Like with most interesting mathematical problems, there is no canonical method for the construction of liftings of generalized elliptic curves. Our method has two technical advantages:

(i) The use of Weierstraß equations is avoided.
(ii) The proofs apply uniformly for every residue characteristic.

Point (i) and point (ii) are tightly connected: Using Weierstraß equations, one might achieve some results given $p \geq 5$, for small p however, the situation becomes unmanageable.

Instead of Weierstraß equations, the key data in our approach is the modular invariant. In Corollary 2.5.4 we saw, that a lifting of a given generalized elliptic curve is up to isomorphism determind by its modular invariant. This uniqueness makes it possible to patch together local liftings, which is a ubiquitous technique in deformation theory.

However, the situation for generalized elliptic curves is very different from, say, vector bundles. The problem of lifting a generalized elliptic curve, in such a way that the lifting realizes a prescribed modular invariant, does not become easier if we work affine locally.

Rather, one has to distinguish two cases: There is the nice case, where the modular invariant is non-special and the singular fibres are irreducible. Here, we obtain the desired lifting simply by using that the modular stack of such objects is a neutral $\mathbb{Z}/2\mathbb{Z}$-gerbe (Proposition 2.2.3). There is also the hard case, where special values appear or non-irreducible fibers. Both phenomenons have in common, that they lead to “jumps” in the automorphism groups. In that cases, we construct “micro” local liftings with prescribed modular invariants. By “micro” local we mean that our construction takes place over completions of the base at closed points (i.e. *fpqc*-locally).

The smooth but special case is handled by deformation theory, and for the non-smooth case we entirely rely on the magic of Tate curves.

Once we have constructed the local liftings, we have to glue. It might be possible to apply *fpqc*-descent directly, but this would force us to work with formal schemes which are not adic over the base scheme.

Instead, we circumvent this problem by using the formal patching result explained before in the following way: The situation becomes nicer once we can choose a level structure on the local liftings. This is possible after a finite and flat basechange of the whole situation. Now we get maps into some moduli space, which can be glued using formal patching for finite morphism.

The upshot is that gluing is possible after basechange. The rest is descent theory. We use Corollary 2.5.4 to lift the descent data which is given on the reduction.

Our main application of formal patching techniques is this:

2.6.1. Theorem. *Let $E \to C$ be a Weierstraß curve with separable modular invariant $j_0\colon C \to \mathbb{P}^1_k$. Then there exists a lifting $\mathcal{E} \to \mathcal{C}$ over R.*

Proof. Let $\mathcal{U} \subset \hat{\mathbb{P}}^1_R$ be the formal open subset where $j(j-1728)$ is invertible. (This includes also $j = \infty$). Set $U = \mathcal{U} \otimes_R k$.

Let $V = C \times_{\mathbb{P}^1_k} U$ be the preimage of U in C. Let $j_{\mathcal{V}}\colon \mathcal{V} \to \mathcal{U}$ be a formal lifting of $j_0|_V$. By definition of V, we have that $E|_V$ is a Weierstraß curve with non-special modular invariant. In this situation, we obtain a lifting $\mathcal{E}_{\mathcal{V}}/\mathcal{V}$ with modular invariant $j_{\mathcal{V}}$ by Proposition 2.2.3.

Our task is now to complete $\mathcal{E}|_{\mathcal{V}}$ to a lifting $\mathcal{E}/\mathcal{C}$ of E/C. Denote by $s \in \mathbb{P}^1_k$ a special point, i.e. a point where $j = 0$ or $j = 1728$. (In case there is more than one special point, one has to repeat the argument below twice.)

Let $\mathcal{S} = \mathrm{Spf}(\mathscr{O}^{\wedge}_{\widehat{\mathbb{P}^1_R},s})$ be the formal spectrum of the complete local of ring of $\widehat{\mathbb{P}^1_R}$ at s. Let $\{y_1, \dots, y_n\}$ be the points in C mapping to s. We are going to lift every single morphism

$$C_i = \mathrm{Spec}(\mathscr{O}^{\wedge}_{C,y_i}) \xrightarrow{j(E_i)} \mathcal{S} \otimes_R k$$

given by the modular invariant of the restriction E_i of E to C_i. This is achieved by lifting the E_i itself. The ring $R[[T]]$ is a lifting of $\mathscr{O}^{\wedge}_{C,y_i} \simeq k[[t]]$ and it is complete for the $\mathfrak{m}$-adic topology. We claim that over $\operatorname{Spf}(R[[T]]) = \varinjlim \operatorname{Spec}(R[[T]]/\mathfrak{m}^n)$, it exists a formal lifting $\mathcal{F}_i \to \operatorname{Spf}(R[[T]])$ of E_i: Since $E_i \to C_i$ is smooth, this follows because the obstruction to lifting a nilpotent thickening of order $n-1$ over $R[[T]]/(\mathfrak{m}^n)$ to one over $R[[T]]/(\mathfrak{m}^{n+1})$ lies inside

$$H^2(E_i, \Theta_{E_i/\operatorname{Spec}(k[[t]])}) \otimes_k (\mathfrak{m}^n)/(\mathfrak{m}^{n+1}) = 0.$$

Denote by $\mathcal{C}_i = \operatorname{Spf}(R[[T]]) \xrightarrow{j(\mathcal{F}_i)} \widehat{\mathbb{P}^1_R}$ the modular invariant of $\mathcal{F}_i$. Hence we have a lifting of the covering

$$\coprod C_i \xrightarrow{\coprod j(E_i)} \mathcal{S} \otimes_R k \text{ given by } \coprod \mathcal{C}_i \xrightarrow{\coprod j(\mathcal{F}_i)} \mathcal{S}.$$

Invoking formal patching (Theorem 2.5.7), there exists a covering $j\colon \mathcal{C} \to \widehat{\mathbb{P}^1_R}$ of formal schemes such that the restriction $\mathcal{C} \times_{\widehat{\mathbb{P}^1_R}} \mathcal{U} \to \mathcal{U}$ is isomorphic to $\mathcal{V} \to \mathcal{U}$ and $\mathcal{C} \times_{\widehat{\mathbb{P}^1_R}} \mathcal{S} \to \mathcal{S}$ is isomorphic to $\coprod \mathcal{C}_i \to \mathcal{S}$. We will make explicit use of this isomorphism

$$\begin{array}{ccc} \mathcal{C} \times_{\widehat{\mathbb{P}^1_R}} \mathcal{S} & \xrightarrow{\alpha} & \coprod \mathcal{C}_i \\ & \searrow \quad \swarrow_{\coprod j(\mathcal{F}_i)} & \\ & \mathcal{S} & \end{array}$$

by pulling back $\coprod \mathcal{F}_i$ to $\coprod \alpha^* \mathcal{F}_i$ which gives an elliptic curve over $\mathcal{C} \times_{\widehat{\mathbb{P}^1_R}} \mathcal{S}$ with modular invariant coinciding with the restriction of $j_{\mathcal{V}}$. We set $\mathcal{E}_i = \alpha^* \mathcal{F}_i$.

The restrictions $\mathcal{E}_{\mathcal{V}}$ and $\mathcal{E}_i$ to the punctured formal neighborhood $\mathcal{C}_i^\circ$ have thus coinciding modular invariants. We claim that they are in fact isomorphic: The modular invariants of the restrictions are non-special, thus

$$\operatorname{Isom}_{\mathcal{C}_i^\circ}(\mathcal{E}_{\mathcal{V}} \times_{\mathcal{C}} \mathcal{C}_i^\circ, \mathcal{E}_i \times_{\mathcal{C}} \mathcal{C}_i^\circ)$$

is a $\mathbb{Z}/2\mathbb{Z}$-torsor (Proposition 2.2.2). This étale torsor has a splitting over the reduction $\mathcal{C}_i^\circ \otimes_R k$, and this splitting lifts, so we have an isomorphism $\varphi\colon \mathcal{E}_{\mathcal{V}} \times_{\mathcal{C}} \mathcal{C}_i^\circ \to \mathcal{E}_i \times_{\mathcal{C}} \mathcal{C}_i^\circ$.

We want to construct an elliptic curve over $\mathcal{C}$ by gluing $\mathcal{E}_{\mathcal{V}}$ and the $\mathcal{E}_i$. Let $\mathcal{W} \subset \mathcal{C}$ be an open subset such that the restriction $\mathcal{E}_{\mathcal{V} \cap \mathcal{W}} = \mathcal{E}_{\mathcal{V}}|_{\mathcal{V} \cap \mathcal{W}}$ is smooth. Let $n \geq 3$ be an integer prime to p. First of all, we can use φ to glue the n-torsion schemes of $\mathcal{E}_{\mathcal{V} \cap \mathcal{W}}$ and $\mathcal{E}_i$ to a group scheme $\mathcal{G}$ over $\mathcal{C}$ such that $\mathcal{G}|_{\mathcal{V}} \simeq \mathcal{E}_{\mathcal{V}}[n]$ and $\mathcal{G}|_{\mathcal{C}_i} \simeq \mathcal{E}_i[n]$. This is formal patching in the category of group schemes.

We find a finite étale covering $\mathcal{W}' \to \mathcal{W}$ such that the pullback $\mathcal{G} \times_{\mathcal{W}} \mathcal{W}'$ is split. Let $\mathcal{V}' \to \mathcal{V}$ and $\mathcal{C}_i' \to \mathcal{C}_i$ be the étale covering induced by $\mathcal{V}_1' \to \mathcal{C}$. Let $\mathcal{E}_{\mathcal{V}'}$ and $\mathcal{E}_i'$ denote the pullbacks of $\mathcal{E}_{\mathcal{V}}$ and $\mathcal{E}_i$ respectively. Like before, $\mathcal{E}_{\mathcal{V}'}$ and $\mathcal{E}_i'$ are isomorphic on the intersection $\mathcal{W}' \times_{\mathcal{V}'} \mathcal{C}_i'$. By assumption on $\mathcal{G}$, and because taking n-torsion commutes with base change, the n-torsion subschemes of those curves are split. We can form pairs $(\mathcal{E}_{\mathcal{V}'}, \gamma_{\mathcal{V}'})$ and $(\mathcal{E}_i', \gamma_i)$ with compatible $\Gamma(n)$-structures. Denote by $c_{\mathcal{V}'}\colon \mathcal{V}' \to \widehat{\mathcal{M}_{\Gamma(n)}}$ and $c_i\colon \mathcal{C}_i' \to \widehat{\mathcal{M}_{\Gamma(n)}}$ the corresponding morphisms. Evoking formal patching, we obtain $c\colon \mathcal{V}' \to \widehat{\mathcal{M}_{\Gamma(n)}}$. Let $\mathcal{E}_{\mathcal{W}'}'$ be the corresponding elliptic curve over $\mathcal{W}'$. This curve is a gluing of $\mathcal{E}_{\mathcal{V}'}$ and $\mathcal{E}_i'$. We have a diagram:

$$\begin{array}{ccc} \mathcal{W}' & \longrightarrow & \widehat{\mathcal{M}_{\Gamma(n)}} \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \widehat{\mathbb{P}^1_R} \end{array}$$

In this situation, we can apply Proposition 2.6.4 to show that $\mathcal{E}'_{\mathcal{W}'}$ descends to an elliptic curve $\mathcal{E}_{\mathcal{W}}$ over $\mathcal{W}$.

The curves $\mathcal{E}_{\mathcal{W}}$ and $\mathcal{E}_{\mathcal{V}}$ are isomorphic over $\mathcal{V} \cap \mathcal{V}$ and can therefore be glued to a generalized elliptic curve $\mathcal{E}/\mathcal{C}$. This is the formal lifting we wanted to construct. An algebrization of $\mathcal{E}/\mathcal{C}$ exists, because the zero section gives rise to an ample invertible sheaf, thus we can aplly Grothendieck Algebraization 1.1.3. □

The proof of Theorem 2.6.1 allowes to state the following corollary:

2.6.2. Corollary. *Let $E \to C$ be a Weierstraß curve with separable modular invariant $j_0 \colon C \to \mathbb{P}^1_k$. Let $\mathcal{U} \subset \widehat{\mathbb{P}^1_R}$ be the open formal subset where $j(j-1728)$ is invertible. Given a lifting $j_{\mathcal{V}} \colon \mathcal{V} \to \mathcal{U}$ of the restriction j_0 over $\mathcal{U} \otimes_R k$, there is a lifting $\mathcal{E}/\mathcal{C}$ of $E \to C$, such that the modular invariant $\mathcal{C} \to \mathbb{P}^1_R$ of $\mathcal{E}/\mathcal{C}$ extends the j_0.*

We want to generalize Theorem 2.6.1 from Weierstraß curves to arbitrary generalized elliptic curves.

2.6.3. Proposition. *Let E/C be a generalized elliptic curve, with separable modular invariant $j_0 \colon C \to \mathbb{P}^1_k$. Then there exists a formal lifting $\mathcal{E}/\mathcal{C}$ over* $\mathrm{Spf}(R)$.

Proof. As usual, let $\mathcal{U} \subset \widehat{\mathbb{P}}^1_R$ be the ordinary locus, and $U \subset \mathbb{P}^1_k$ its reduction. Let $V = C \times_{\mathbb{P}^1_k} U$ denote the inverse image. We are going to construct a special lifting $j_{\mathcal{V}} \colon \mathcal{V} \to \mathcal{U}$ of j_0 restricted to V. We have

$$C \times_{\mathbb{P}^1_k} \mathrm{Spec}(k[[1/j]]) \simeq \coprod C_i$$

and the covering $C_i = \mathrm{Spec}(k[[t_i]]) \to \mathrm{Spec}(k[[1/j]])$ is given by $1/j \mapsto t_i^{e_i}$ where e_i is the ramification index of j_0 at $t_i = 0$. We lift this to a ring homomorphism $R[[1/j]] \to R[[T_i]]$ by sending $1/j \mapsto T_i^{k_i}$. Here, T_i is a lift of t_i. We denote by $\mathcal{C}_i = \mathrm{Spf}(R[[T_i]]) \to \mathrm{Spf}(R[[1/j]]$ the associated covering. Now, there exists a covering $j_{\mathcal{V}} \colon \mathcal{V} \to \mathcal{U}$ such that $\mathcal{V} \times_{\mathcal{U}} \mathrm{Spf}(R[[1/j]])$ is isomorphic to $\coprod \mathcal{C}_i \to \mathrm{Spf}(R[[1/j]])$.

By Corollary 2.6.2 we find a lifting $\mathcal{E}^c$ of the Weierstraß model of E, with modular invariant $j_{\mathcal{V}}$ over $\mathcal{V}$. Let $\mathcal{W}$ be the open subset where $\mathcal{E}^c \to \mathcal{C}$ is smooth. We denote the restriction of $\mathcal{E}^c$ to $\mathcal{W}$ by $\mathcal{E}_{\mathcal{W}}$. Our objective now is to complete $\mathcal{E}_{\mathcal{W}}$ to a lifting of E.

At first, we construct a suitable generalized elliptic curve over each of the $\mathcal{C}_i$. Let n denote the number of components of the special fibre of $E_i = E \times_C \mathrm{Spec}(k[[t_i]])$. By [DR73, VII Corollaire 2.6] we can represent E_i as Tate curve. More precisely, there exists an element $u_0 \in k[[t_i]]^\times$ such that

$$\mathscr{G}_m^{t_i}/(u_0 t_i^n)^{\mathbb{Z}}.$$

For $u \in R[[T]]^\times$ mapping to u_0 under the specialization map, the curve

$$\mathscr{G}_m^{T_i}/(u T_i^n)^{\mathbb{Z}}$$

is a lifting of E_i. We are looking for a specific lifting, whose modular invariant is the morphism $\mathcal{C}_i \to \mathrm{Spf}(R[[1/j]])$. This morphism is given by the image f of $1/j$ in $\mathscr{O}_{\mathcal{C}_i}$. We claim that there exists a $u \in R[[T_i]]^\times$ specializing to u_0 such that

$$j(\mathscr{G}_m^{T^i}/(uT_i^n)^{\mathbb{Z}})^{-1} = f. \tag{2.6.3.1}$$

We have an expansion of the right hand side ([DR73, VII 2.7]):

$$j(\mathscr{G}_m^{T^i}/(uT_i^n)^{\mathbb{Z}})^{-1} = \pm u T_i^n + \sum_{k=0}^{\infty} d_k (uT_i^n)^k$$

Both sides of equation (2.6.3.1) lie in (T_i^n), thus it is sufficient to solve the equation $\tilde{g}(u) = \tilde{f}$ obtained by cancelling T_i^k. This equation has a solution modulo $\mathfrak{m}$, namely

u_0 and the derivation of $\tilde{g}(u)$ is a unit modulo $\mathfrak{m}$. Using completeness of $R[[T_i]]$ with respect to the ideal $\mathfrak{m} \subset R$, we find a u solving (2.6.3.1) by Hensel's lemma. Denote the generalized elliptic curve given by that u by $\mathcal{E}_i$.

As before, denote by $\mathcal{C}_i^\circ$ the formal tubular neighborhood of the closed point of $\mathcal{C}_i$. The restriction of $\mathcal{E}_i$ to $\mathcal{C}_i^\circ$ is a smooth elliptic curve with non-special j-invariant. Again, it follows that

$$\mathcal{E}_i \times_{\mathcal{C}_i} \mathcal{C}_i^\circ \simeq \mathcal{E}_{\mathcal{W}} \times_{\mathcal{C}} \mathcal{C}_i^\circ,$$

because by construction, the modular invariants of the restrictions coincide.

Now, we choose an open neighborhood $V_i \subset C$ of x_i such that E_i is the only non-singular fiber of $E|_{V_i}$, the section $j_0(j_0 - 1728)$ is invertible over V_i and $E|_{V_i}$ is ordinary. Let $\mathcal{V}_i \subset \mathcal{C}$ be the lifting of V_i. Note that $\mathcal{C}_i \to \mathcal{C}$ factors over $\mathcal{V}_i$.

As in the proof of Theorem 2.6.1, we claim that there exists a finite flat base change $\mathcal{V}_i' \to \mathcal{V}_i$ such that we can put compatible $\Gamma_1(n)$-structures on the pullbacks $\mathcal{E}_{\mathcal{W}'}/\mathcal{W}'$ of $\mathcal{E}|_{\mathcal{V}_i \cap \mathcal{W}}$ and $\mathcal{E}_i'/\mathcal{C}_i'$ of $\mathcal{E}_i$.

This is possible, because we have chosen $\mathcal{V}_i$ such that the n-torsion subschemes of $\mathcal{E}_{\mathcal{W}'}$ and $\mathcal{E}_i'$ are finite and flat. Denote by $\mathcal{A}$ the finite and flat group scheme over $\mathcal{V}_i$, obtained by gluing the particular n-torsion subgroup schemes.

If n is prime to p, one can choose $\mathcal{V}_i' \to \mathcal{V}_i$ splitting $\mathcal{A}$, and then it is clear how to choose a $\Gamma_1(n)$-structure.

So write $n = mp^d$ with m prime to p. By the Chinese Remainder Theorem, choosing a n-torsion section amounts to the same as choosing a m-torsion section and a p^n-torsion section. The m-torsion part can be split after étale pullback. To choose a p^n-torsion section, we make use of the fact that the restriction of $\mathcal{A}$ has a natural extension structure

$$0 \to \mu_{p^n} \to \mathcal{A}|_{V_0} \xrightarrow{r} G \to 0 \tag{2.6.3.2}$$

because $\mathcal{V}_i$ is an inductive limit of schemes with the property that p is a nilpotent element in the structure sheaf. The inverse image $\mathcal{T} = r^{-1}(1)$ of the generator $1 \in G$ is a μ_{p^n}-torsor. Because $\mathcal{A}|_{V_0}$ is killed by p^n, a point $r^{-1}(\mathcal{V}_i)$ gives a splitting of (2.6.3.2). The group scheme G is étale, and locally isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$. Taking $\mathcal{V}_i'$ as a composition of $\mathcal{T} \to \mathcal{S}$ with a suitable étale covering, we get what we want.

Having now formed pairs $(\mathcal{E}_{W'}/\mathcal{W}', \gamma_{\mathcal{W}'})$ and $(\mathcal{E}_i'/\mathcal{C}_i', \gamma_i)$ such that the $\Gamma_1(n)$-structures $\gamma_{\mathcal{W}'}$ and γ_i are compatible with the isomorphisms of $\mathcal{E}_{W'}$ and $\mathcal{E}_i'$ on the intersection $\mathcal{C}_i' \times_{\mathcal{V}_i'} \mathcal{W}'$ we obtain a morphism into the coarse moduli space $c\colon \mathcal{V}_i' \to M_{\Gamma_1(n)} \otimes R$. The stack $\mathcal{M}_{\Gamma_1(n)} \otimes R$ is representable of $n > 2$. For $n \leq 2$, the morphism of stacks $\mathcal{M}_{\Gamma(n)} \to M_{\Gamma(n)}$ is a $\mathbb{Z}/2\mathbb{Z}$-gerbe over the open subset of $M_{\Gamma(n)}$ where $j(j-1728)$ is invertible. In any case, we find a unique lifting $\mathcal{E}'_{\mathcal{V}_i'}$ over $\mathcal{V}_i$. This curve is a gluing of $\mathcal{E}_{\mathcal{W}_i'}$ and $\mathcal{E}_i'$. We have a diagram:

$$\begin{array}{ccc} \mathcal{W}' & \longrightarrow & \mathcal{M}_{\Gamma(n)} \otimes R \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathbb{P}^1_R \end{array}$$

In this situation, we can apply Proposition 2.6.4 to show that $\mathcal{E}'_{\mathcal{W}_i'}$ descends to an elliptic curve $\mathcal{E}_{\mathcal{W}_i}$ over $\mathcal{W}_i$. □

2.6.4. Proposition. *Let $\mathcal{S}$ be a smooth and adic* $\operatorname{Spf}(R)$*-formal scheme. Set $S = \mathcal{S} \otimes_R k$. Let E/S be a generalized elliptic curve. Let $\mathcal{S}' \to \mathcal{S}$ be a finite, flat and adic covering. Let $\mathcal{E}'/\mathcal{S}'$ be an adic generalized elliptic curve such that we have an isomorphism $\mathcal{E}' \otimes_R k \simeq E' = E \times_S S'$, where $S' = \mathcal{S}' \otimes_R k$, and the modular invariant of $\mathcal{E}'$ factors over $\mathcal{S}$.*

Then $\mathcal{E}'$ descents to a formal lifting $\mathcal{E}$ of E over $\mathcal{S}$.

Proof. Set $\mathcal{S}_n = \mathcal{S} \otimes_R R/\mathfrak{m}^{n+1}$, $\mathcal{S}'_n = \mathcal{S}' \otimes_R R/\mathfrak{m}^{n+1}$ and $\mathcal{E}'_n = \mathcal{E}' \otimes_R R/\mathfrak{m}^{n+1}$. Those objects and the morphisms between them from adic inductive systems. Assume we can solve the descent problem for any n, i.e. we have an $\mathcal{E}_n$ with $\mathcal{E}_n \times_{\mathcal{S}_n} \mathcal{S}'_n \simeq \mathcal{E}'_n$. In the adic inductive system $(\mathcal{E}'_n/\mathcal{S}'_n)$, the maps $\mathcal{E}'_m \to \mathcal{E}'_n$, for $m \leq n$, are given by basechange of $\mathcal{E}'_n$ with $\operatorname{Spec}(R/\mathfrak{m}^m) \to \operatorname{Spec}(R/\mathfrak{m}^n)$, and therefore it is clear that they descent to $\mathcal{E}_n$. Thus we get an inductive system $(\mathcal{E}_n/\mathcal{S}_n)$ which is adic. We are therefore reduced to the case of ordinary schemes, and can assume that $\mathcal{S}', \mathcal{S}$ and $\mathcal{E}'$ live over an artinian local ring. The descent argument now works as follows:

By construction, E' has descent data: With the notations

$$S'' = S' \times_S S' \underset{p_2}{\overset{p_1}{\rightrightarrows}} S'$$

this means that we have a descent datum

$$\varphi_0 \colon p_1^* E' \xrightarrow{\sim} p_2^* E'.$$

Our strategy is to lift φ_0 to $\varphi \colon p_1^*\mathcal{E}' \to p_2^*\mathcal{E}'$ and to show consecutively that φ satisfies the cocycle condition. The first step is to observe that the modular invariants of $p_1^*\mathcal{E}'$ and $p_2^*\mathcal{E}'$ are identical. Those morphism however are given by the upper and lower row in the diagram

$$\mathcal{S}' \times_{\mathcal{S}} \mathcal{S}' \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathcal{S}' \to \mathcal{S} \to \mathbb{P}^1_R$$

which is clearly commutative by assumption on the modular invariant of $\mathcal{E}'$. Now Corollary 2.5.4 allowes us to lift φ_0 to φ as desired.

Applying the fact that isomorphisms lift uniquely (the uniqueness part form Corollary 2.5.4), this time to the isomorphism schemes of the $\mathcal{S}'''$-schemes

$$p_{12}^* p_1^* \mathcal{E}' = p_{13}^* p_1^* \mathcal{E}', p_{12}^* p_2^* \mathcal{E}' = p_{23}^* p_1^* \mathcal{E}' \quad \text{and} \quad p_{23}^* p_2^* \mathcal{E}' = p_{13}^* p_2^* \mathcal{E}',$$

we check that φ satisfies the cocycle condition, because φ_0 does so. □

Algebraization. Our strongest result so far (Proposition 2.6.3) shows the existence of formal lifting of generalized elliptic curves with separable modular invariants.

In this section, we are going to study under what circumstances it is possbile to algebraize a formal generalized elliptic curve. For Weierstraß fibrations, algebraization is trivial, because the zero section gives a relatively ample divisor. Thus we have to deal with the lifting behaviour of fibral components. Our treatment is based on the powerful statement:

2.6.5. Proposition (II Proposition 1.15 [DR73]). *Let $p \colon E \to S$ be a generalized elliptic curve. Then there exists a locally finite family of subschemes $(S_n)_{n \geq 1}$ of S which are closed and disjoint, such that*

(i) *$\cup_n S_n$ is the image of the non smooth locus of E in S.*
(ii) *Locally in the $fppf$-topology over S_n, we have that E is isomorphic to the pullback of the standart n-gon over* $\operatorname{Spec}(\mathbb{Z})$*.*

2.6.6. Proposition. *Let $\mathcal{E}/\mathcal{C}$ be an adic generalized elliptic curve, over a smooth and proper formal curve $\mathcal{C}/\operatorname{Spf}(R)$. Assume that the total space $\mathcal{E}$ is smooth over* $\operatorname{Spf}(R)$*, and that $\mathcal{E}/\mathcal{C}$ is generically smooth. Then $\mathcal{E}/\mathcal{C}$ is algebraizable.*

Proof. Denote by E/C the reduction of $\mathcal{E}/\mathcal{C}$. Let $H \subset E$ be the Cartier divisor given as zero section plus a non-singular fibre. Let $F \subset E$ denote the Cartier divisor of all fibre components not intersection the zero section (i.e. from every n-gon we remove the zero component).

We can also view F as the exceptional divisor which comes up when considering E as minimal desingularization of its Weierstraß model. This description allowes to see that $H-F$ is ample on E. This follows also from the fact that the components of F have selfintersection -2.

The latter description globalizes to $\mathcal{E}$: for let $\mathcal{W}/\mathcal{C}$ denote the formal generalized elliptic curve obtained by contracting every fibral component not intersected by the zero section. The non-smooth locus of $\mathcal{W}$ is a closed subscheme, and its inverse image under the contraction map $\mathcal{E} \to \mathcal{W}$ gives a closed subscheme $\mathcal{F}$ of $\mathcal{E}$ which lifts F. We have to show that the ideal $\mathcal{I}_{\mathcal{F}}$ defining $\mathcal{F}$ lifts $\mathscr{O}(-F) = \mathcal{I}_F$ and is locally a principal ideal, i.e. an invertible sheaf. In order to see this, we proof first that $\mathcal{F}$ is R-flat.

Let $S \subset \mathcal{C}$ be the closed subscheme which is obtained as the image of the non-smooth locus of $\mathcal{E}$. We have that S is flat over R, because it is the inverse image of the cuspidal locus $1/j = 0$ in $\widehat{\mathbb{P}^1_R}$ under the modular invariant $\mathcal{C} \to \widehat{\mathbb{P}^1_R}$. However, the modular invariant itself is flat by fibrewise flatness, since its reduction is a non-constant morphism of curves (if S is not empty), and hence flat.

Locally in the $fppf$-toplogy, $\mathcal{E} \times_{\mathcal{C}} S_n$ is isomorphic to the pullback of the standard Néron n-gon over Spec($\mathbb{Z}$) ([DR73, 1.15]). The closed subscheme obtained from the standard Néron n-gon by removing the zero component is clearly flat over Spec($\mathbb{Z}$), hence the assertion follows.

Having shown that $\mathcal{F}$ is flat over R, it follows that the sequence

$$0 \to \mathscr{I}_{\mathcal{F}} \to \mathscr{O}_{\mathcal{X}} \to \mathscr{O}_{\mathcal{F}} \to 0$$

remains exact after restricting to the special fibre (i.e. tensoring over R with k). This can be seen locally on $\mathcal{E}$: Let $\mathcal{U} = \mathrm{Spec}(A)$ some affine open subset. We get

$$0 = \mathrm{Tor}_1^R(\mathscr{O}_{\mathcal{F}\cap\mathcal{U}}, k) \to \mathcal{I}_{\mathcal{F}} \otimes k \to \mathscr{O}_{\mathcal{U}} \otimes k \to \mathscr{O}_{\mathcal{F}\cap\mathcal{U}} \otimes k \to 0.$$

The relation $\mathscr{I}_{\mathcal{F}} \otimes_R k = \mathscr{I}_F$ implies that $\mathscr{I}_{\mathcal{F}}$ is a principle ideal and hence an invertible sheaf, because a lifting of a local generator is a generator by Nakayama. Now, lifting H to $\mathcal{H} \subset \mathcal{E}$ is trivial, thus we can lift $\mathscr{O}(H) \otimes \mathscr{I}_F$, and are done by Grothendieck's Algebraization Theorem. □

The combination of Proposition 2.6.3 and Proposition 2.6.6 gives the main theorem of this section.

2.6.7. Theorem. *Let E/C be a generalized elliptic curve, over a proper smooth curve C/k. Assume that the modular invariant of E is separable and that the total space E/k is smooth.*

Then there exists a projective lifting $\mathcal{E}/\mathcal{C}$ of E/C. □

7. Liftings of principal homegenous spaces

Let E/C be a Jacobian elliptic fibration, with regular total space. We can consider the so called *Tate-Safarevich-group* $H^1_{et}(C, E^{sm})$. Let $i\colon \mathrm{Spec}(K) = \eta \to C$ be the generic point of C. The Leray spectral sequence with repect to i and E^{sm} gives

$$0 \to H^1(C, E^{sm}) \to H^1(\eta, E^{sm}_\eta) \to H^0(S, R^1 i_* E^{sm}_\eta) \to H^2(C, E^{sm}).$$

The group $H^1(\eta, E^{sm}_\eta)$ can be described in terms of Galois theory as

$$H^1(K^S/K, E^{sm}(K^s)),$$

where K^S is a separable closure of K. In particular, it is a torsion group, for it is a direct limit of torsion groups.

By Lemma 1.6.3, every class in $[X_\eta] \in H^1(\eta, E^{sm}_\eta)$ is representable by a smooth genus-1 X_η curve over η. Using the minimal model theory for relative curves, we see $X_\eta \to \eta$ can be extended to a unqiue proper curve $X \to C$ with regular total space and without (-1)-curves in the fibers.

An element $[X] \in H^1(C, E^{sm})$ corresponds under this construction to a locally trivial torsor. Geometrically, this means that its minimal regular model X is locally for the étale topolgy of C isomorphic to E. This can be characterized by two other equivalent conidtions:

(i) For every closed point $s \in C$, the fibre X_s has at least one reduced component.
(ii) Étale locally over C, the morphism $X \to C$ has sections.

The equivalence follows immediately from Hensel's lemma.

In this section, we are going to treat the following problem: Given a lifting $\mathcal{E}/\mathcal{C}$ of a Jacobian elliptic fibration E/C, can we lift the surfaces that are associated to the elements of $H^1(C, E^{sm})$?

Let $\mathcal{E}/\mathcal{C}$ be an adic Jacobian elliptic fibration over $\mathrm{Spf}(R)$. We write $\mathcal{A}$ for the adic $\mathrm{Spf}(R)$-formal group scheme $\mathcal{E}^{sm}$ As a first step, we want to understand the lifting behaviour of $A = \mathcal{A} \otimes_R k$-torsors. Those are torsors corresponding to elements in $H^1_{et}(C, A)$.

We let $\mathcal{A}_n = \mathcal{A} \otimes_R (R/\mathfrak{m}^{n+1})$ and $\mathcal{C}_n = \mathcal{C} \otimes_R (R/\mathfrak{m}^{n+1})$. Denote by $i\colon \mathcal{C}_n \to \mathcal{C}_{n+1}$ the natural inclusion, and by $s\colon \mathcal{A}_{n+1}(\mathcal{C}_{n+1}) \to i_*A_n(\mathcal{C}_{n+1})$ the specialization map on sections. Furthermore, set $\mathscr{L} = \mathrm{Lie}(\mathcal{A}/\mathcal{C})$. We have the following exact sequence:

$$\begin{aligned} 0 \to i_*\mathcal{A}_n(\mathcal{C}_{n+1})/s(\mathcal{A}_{n+1}(\mathcal{C}_{n+1})) \to H^1(\mathcal{C}_{n+1}, i_*\mathscr{L}) \to \\ \to H^1_{et}(\mathcal{C}_{n+1}, \mathcal{A}_{n+1}) \xrightarrow{s} H^1_{et}(\mathcal{C}_{n+1}, i_*\mathcal{A}_n) \to H^2(\mathcal{C}_{n+1}, i_*\mathscr{L}) \quad (2.7.0.1) \end{aligned}$$

which was derived in Chapter 1 Section 6. Since $\mathscr{L}$ is a coherent sheaf, we have $H^2(\mathcal{C}_{n+1}, i_*\mathscr{L}) = 0$ for dimension reasons. It follows that we can lift cohomology classes. A class in $H^1_{et}(\mathcal{C}_{n+1}, \mathcal{A}_{n+1})$ is represented by an algebraic space T. However, its reduction is representable by a scheme, so we conclude by [Knu71, Corollary 3.5], that T itself is in fact a scheme.

2.7.1. Proposition. *Let $\mathcal{E}/\mathcal{C}$ be an adic Jacobian elliptic fibration over* $\mathrm{Spf}(R)$*, such that the reduction E/C is regular and minimal. Let X/C be the minimal regular model of a class $[X] \in H^1(C, E^{sm})$. Then X has a formal lifting.*

Proof. As before, we use the following notations:

$$\mathcal{A} = \mathcal{E}^{sm}, \mathcal{A}_n = \mathcal{A} \otimes_R R/\mathfrak{m}^{n+1} \text{ and } \mathcal{C}_n = \mathcal{C} \otimes_R R/\mathfrak{m}^{n+1}.$$

By what we said above, we know that we can always lift a $\mathcal{A}_n$-torsor $\mathcal{T}_n$ to a $\mathcal{A}_{n+1}$-torsor. Hence we find a formal scheme $\mathcal{T}$ which is adic over $\mathcal{C}$, and represents a lifting of the class $[X]$.

It remains to compatify $\mathcal{T}$ in such a way that we obtain a lifting of X. In order to do so, we use of that X and E are locally in the étale topology over C isomorphic.

Let $s \in C$ be a closed point, such that X_s is non-smooth. We can choose an étale neighborhood $\mathcal{U}$ of x, having the property that $\mathcal{T}_\mathcal{U} = \mathcal{T} \times_\mathcal{C} \mathcal{U}$ has a section. Such an $\mathcal{U}$ exists, because we can find an étale and affine neighborhood $U \subset C$ of x, such that $T \times_C U$ is split. Now, given a lifting $\mathcal{U}$ of U, we can also can lift the section of $T \times_C U$, because $\mathcal{T}$ is smooth.

So we find that $\mathcal{T}_\mathcal{U}$ is isomorphic to an open subscheme of $\mathcal{E}_\mathcal{U} = \mathcal{E} \times_\mathcal{C} \mathcal{U}$: namely to the smooth part of its identity component.

Denote by $\mathcal{T}^\circ$ the open subscheme of $\mathcal{T}$ obtained by restricting $\mathcal{T}$ to the lifting of the open subscheme $C - \{x\}$. As an algebraic space, $\mathcal{T}$ is given by the equivalence relation $\mathcal{V} \times_{\mathcal{T}} \mathcal{V}$ on

$$\mathcal{V} = \mathcal{T}_{\mathcal{U}} \amalg \mathcal{T}^{\circ}.$$

We are going to substitute $\mathcal{T}_{\mathcal{U}}$ by $\mathcal{E}_{\mathcal{U}}$. These schemes only differ at non-smooth fibres of the latter. Hence we can define an equivalence relation on

$$\mathcal{V}' = \mathcal{E}_{\mathcal{U}} \amalg \mathcal{T}^{\circ}$$

by setting $\mathcal{R} = \mathcal{V}'_{\text{diag}} \amalg \{\text{non-diagonal components of } \mathcal{V}\}$. Dividing out $\mathcal{V}'$ by $\mathcal{R}'$ we obtain the compatification $\mathcal{X}$ of $\mathcal{T}$ we were looking for in the category of formal algebraic spaces. However, since the reduction of $\mathcal{X}$ is the scheme X, it follows that it is a formal scheme itself ([Knu71, Corollary 3.5]). □

Having established a formal lifting result, we want to understand under what circumstances we can construct projective liftings. We restrict ourselves to fibrations X/C with semistable Jacobian E/C.

The treatment is based on the following lemma, which is a slight generalization of [Ray70, Lemme XIII]:

2.7.2. Lemma. *Let A be a commutative group scheme over some base scheme S, such that the morphism*

$$[m]_A \colon A \to A$$

is a surjection of sheaves in the $fppf$-topology. Let $[T] \in H^1_{fppf}(S, A)[m]$ be a m-torsion cohomology class and let T be its representing algebraic space.. Then we have a canonical morphism $\varphi\colon T \to A$, which is locally in the $fppf$-toplogy isomorphic to $[m]_A$.

Proof. Recall that for a homomorphism of commutative groups scheme $A \to B$, we get a map on cohomology $H^1_{fppf}(S, A) \to H^1_{fppf}(S, B)$, On torsors, this map is given by

$$T \mapsto (T \times B)/A$$

where the quotient is taken by the diagonal action of A which on T is the given action and on B comes form the homomorphism $A \to B$.

We apply this to the multiplication map m_A. The class $[T] \in H^1_{fppf}(S, A)$ is by assumption mapped to the trivial class, which means that the torsor $(T \times A)/A$ is trivial. We have a canonical A-equivariant map $\varphi\colon T \to T'$ given by $\mathrm{Id}_T \times \epsilon_A \to T \times A$ composed with the quotient map.

Now assume T were a trivial torsor. We can choose a section such that φ becomes a group homomorphism. Under the identification $A \simeq T$ we get a homomorphism $A \to (A \times A)/A$. Using the assumed m-divisiblity of A, one sees that this homomorphism is surjective, and it is immideate that its kernel is $A[m]$. This proves the claim. □

Note that Lemma 2.7.2 applies in particular to the group scheme given by the smooth locus of generalized elliptic fibrations. If m is prime to p, we can moreover work in the étale toplogy instead of the $fppf$-topology. We make us of this to prove:

2.7.3. Proposition. *Let E/C be a generalized elliptic curve, over a smooth and proper curve C/k. Let X/C be an elliptic surface which is locally in the étale toplogy over C isomorphic to E. Assume that the cohomology class $[X] \in H^1_{et}(C, E^{sm})$ given by X is m-torsion, for some integer m prime to p.*

Let $\mathcal{E}/\mathcal{C}$ be a generalized elliptic curve, with projective total space, lifting E/C. Then there exists a projective lifting $\mathcal{X}/\mathcal{C}$ of X/C.

PROOF. With the usual notations, we set $T = \mathrm{Pic}^1_{X/C}$ and $A = \mathrm{Pic}^0_{X/C}$. There exists a formal lifting $\mathcal{T}$ of T, such that for every n, we have that $\mathcal{T} \otimes_R R/\mathfrak{m}^{n+1}$ gives an m-torsion class in $H^1_{et}(\mathcal{C}_n, \mathcal{E} \otimes_R R/\mathfrak{m}^{n+1})$. This follows by taking m-torsion in (2.7.0.1).

From Lemma 2.7.2 we get a quasi-finite map $\varphi\colon \mathcal{T} \to \mathcal{A}$ of formal schemes. Let $\mathcal{X}$ be the compactification of $\mathcal{T}$, constructed as in Proposition 2.7.1. Our aim is to extend φ to a finite map $\mathcal{X} \to \mathcal{E}$.

Locally in the étale toplogy over $\mathcal{C}$, this extension does in fact exist, because we can assume $\mathcal{T}$ to be split and choose a section, such that φ is just multiplication by m. However, multiplication by m gives a morphism of generalized elliptic curves.

It remains to show that the extension of φ descents to a morphism $\mathcal{X} \to \mathcal{E}$. This follows, because we can pick an étale covering, splitting $\mathcal{T}$, with the property that every connected component contains only one basepoint of a singular fibre. On the intersection of those local charts, the morphism is already defined.

Note that the extension of φ being a quasi-finite morphism between separated and proper schemes, is in fact finite. Whence, $\mathcal{X}$, having a finite morphism onto a projective formal scheme, is projective. Using Grothendieck's Algebraization Theorem (1.1.3) the claim follows. □

The main theorem follows directly form Theorem 2.6.7 and Proposition 2.7.3:

2.7.4. Theorem. *Let X/C be a semistable elliptic fibration with regular total space. Let E/C be the Jacobain of X/C. Assume that the cohomology class $[X] \in H^1_{et}(C, E^{sm})$ given by X is m-torsion, for some integer m prime to p, and that the modular invariant of E/C is separable. Then X/C has a projective lifting over R.*

CHAPTER 3

Liftability under tameness assumptions

The purpose of this chapter is to generalize the lifting results from Chapter 2 towards more general elliptic fibrations. So far we have only considered generalized elliptic curves E/C, in other words, we restricted ourselves to the case where the singular fibers of E are n-gons. This semistability assumption can be dropped under certain tameness assumptions on E.

In this chapter, R will be a regular complete local ring, with algebraically closed residue field k.

1. Tame coverings

We repeat the definition of tame coverings of schemes, and state an important theorem by Grothendieck and Murre about liftings of tame coverings. Let A be a discrete valuation ring, with fraction field K. For a finite Galois extension $K \subset L$, we denote by $S \subset K$ the normalization of A in K. Then the extension $K \subset L$ is called tame if the inertia group of every prime ideal of S has order prime to p, where p is the characterisitc of the residue field of A. In general, an extension $L \supset K \supset A$ is called tame if it is finite and separable and its Galois closure is tame.

3.1.1. Definition. Let $f\colon X' \to X$ be a finite morphisms of integral and normal schemes. Let D be a reduced closed subscheme of codimension 1 in X. Denote by U the complement of D. The morphism f is called *tamely ramified with respect to D* if f is étale over U and for every generic point η of D we have the following condition: Let $\xi \in X'$ be a point of codimension 1, mapping to η. This induces a map $\mathscr{O}_{X,\eta} \to \mathscr{O}_{X',\xi}$ which in turn gives an extension of the fraction fields. This extension has to be tame.

The concept of a tame morphism of curves is thus a straightforward generalization of the corresponding notion for global fields. Let $f\colon C' \to C$ be a finite separable morphism of smooth and proper curves. Let $D \subset C$ denote the branching locus. Then f is tame with respect to D if and only if the field extension corresponding to f is tame. When dealing with coverings of curves, we will often omit the specification of the branching locus, and just speak of a tame covering.

To state a lifting result for tame coverings, we define the category of $\mathrm{Rev}_D(S)$ whose objects are tame covering of S with respect to a divisor D, and whose morphisms are triangles

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

where $X_1, X_2 \to S$ are tame coverings with respect to D. Note that the composition $X_1 \to X_2 \to S$ is again tame ([GM71, Lemme 2.2.5]).

In the following, we restrict ourselves to the case where S is a smooth and proper curve over a complete regular local ring R. The subscheme given by the divisor $\mathcal{D}$ is assumed to be smooth over R.

3.1.2. Theorem ([GM71] Theorem 4.3.2)**.** *Let $\mathcal{C}$ be a smooth and proper curve over R. Fix a divisor $\mathcal{D}$ as above, consisting of regular connected component $\mathcal{D}_1, \dots, \mathcal{D}_n$. Denote by $D \subset C = \mathcal{C} \otimes_R k$ the reduction of $\mathcal{D}$. Then we have an equivalence of categories*

$$\mathrm{Rev}^{\mathcal{D}}(\mathcal{C}) \to \mathrm{Rev}^{D}(\mathcal{C} \otimes_R k).$$

In [GM71] Theorem 3.1.2 is stated in the formal category over a more general base. Working over proper bases, we can apply formal GAGA (Theorem 2.5.9) to obtain the same result in the algebraic category.

2. Tame Jacobian fibrations

Let $E \to C$ be a Jacobian elliptic fibration over a smooth and proper k-curve. The general theory of semistable reduction provides the existence of a separable covering $C' \to C$, such that the base change $E \times_C C'$ is birational to a semistable fibration.

In the case of Jacobian elliptic fibrations, we can understand the covering $C' \to C$ explicitly in terms of torsion subschemes of E. Denote by K the fraction field of C. The general principle is, that the existence of enough rational torsion points $E(K)[n]$ enforces good or semistable reduction. This goes under the name of reduction criteria. We can derive the results we need just from knowing the possible component groups of Néron models:

3.2.1. Proposition. *Let E/C be an Jacobian elliptic fibration with regular total space. Let $x \in C$ be a closed point, and let R be the local henselian ring of C at x. Set $E_x = E \times_C \mathrm{Spec}(R)$.*

Then E_x has good reduction, if and only if $E_x[n] \simeq (\mathbb{Z}/n\mathbb{Z})^2$ for every n prime to p. Furthermore, E_x has semistable reduction, if $E_x[n]$ contains the group $\mathbb{Z}/n\mathbb{Z}$ for some $n > 4$ and prime to p.

Proof. If E_x is a smooth elliptic curve $E_x[n] \simeq (\mathbb{Z}/n\mathbb{Z})^2$ follows, because we are over a strictly henselian ring. To show the opposite direction, we make use of the fact that the group scheme given by the smooth locus of the special fibre $E_{\overline{x}}$ of E_x is one of the following groups (see [Sil94, Corollary 9.2 (d)]):

(i) a smooth elliptic curve (good reduction),
(ii) an extension $1 \to \mathbb{G}_m \to E^s_{\overline{x}} \to \mathbb{Z}/m\mathbb{Z} \to 0$ (semistable reduction),
(iii) an extension $0 \to \mathbb{G}_a \to E^s_{\overline{x}} \to A \to 0$ where A is an abelian group of order less or equal to 4. (additive reduction).

The latter two cases are easily excluded because the n-torsion subscheme of the identity components are to small (Recall $\mathbb{G}_m[n] \simeq \mu_n$ and $\mathbb{G}_a[n] = 0$) and the component groups are finite.

To proof the second statement, one excludes the additive case for the very same reason. □

3.2.2. Corollary. *Let E/C be a Jacobian elliptic fibration. Choose some integer $n > 4$ and prime to p. Let K be the fraction field of C, and denote by $L = K(E[n])$ the splitting field of $E[n]$. Then the minimal regular model E'/C' of $E \otimes_K L$ is semistable.*

Proof. We have $E'[n] \simeq (\mathbb{Z}/n\mathbb{Z})^2$. The global existence of n-torsion sections implies the existence of n-torsion sections over the henselian local rings, so Proposition 3.2.1 applies. By the assumptions on n, semistability follows. □

In view of Corollary 3.2.2, we are offered a natural strategy to construct a lifting of a Jacobian elliptic fibration E/C: First, we choose a covering $C' \to C$, such that the minimal regular model E' of the basechange $E \times_C C'$ is semistable.

Assume we can find a lifting $\mathcal{C}' \to \mathcal{C}$ of the covering $C' \to C$, along with a lifting $\mathcal{E}'/\mathcal{C}'$ of E'/C'. Then we can try to obtain a lifting $\mathcal{E}/\mathcal{C}$ of E/C by descent along $\mathcal{C}' \to \mathcal{C}$.

For an arbitrary Jacoiban elliptic fibration E/C, we can defined its modular invariant as follows: Over the fraction field K of C it holds that $E \times_C \operatorname{Spec}(K)$ is a smooth elliptic curve. Hence we get a morphism $\operatorname{Spec}(K) \to \operatorname{Spec}(k[t])$, given by its modular invariant. The unique extension $C \to \mathbb{P}^1_k$ is definied to be the modular invariant of E/C.

It is now easy to give a necessary condition on the lifting $\mathcal{E}'$ in order to make the descent possible: If $\mathcal{E}'/\mathcal{C}'$ is obtain by base change from $\mathcal{E}/\mathcal{C}$, then the modular invariant of $\mathcal{E}'$ has to factor over the modular invariant of $\mathcal{E}$. [1]

In case the lifting $\mathcal{E}'$ is parametrized by some moduli space $\mathcal{M}_\Gamma$, the above observation is just that we have to require the following diagram to be commutative:

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{M}_\Gamma \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathbb{P}^1_R \end{array} \qquad (3.2.2.1)$$

To what extend is the commutativity of (3.2.2.1) also sufficient? To answer this question, we shall first clarify our use of the term “descend”. This is because we are not going to lift the pullback $E \times_C C'$, but rather a desingularization thereof. Therefore we cannot hope to treat the problem purely in terms of descend theory.

Instead, we require the covering $\mathcal{C}' \to \mathcal{C}$ to be Galois. This makes it possible to reinterpret descend data as an action of the Galois group in the following way: For a finite and étale Galois morphism $S' \to S$ of group G (i.e. a G-torsor) we know [BLR90, 6.2. Example B] that descent data on a S'-scheme X' is equivalent to a Galois action of G on X'. By a Galois action we mean a group action of G, such that for ever $\sigma \in G$ the following diagram commutes:

$$\begin{array}{ccc} X' & \xrightarrow{\sigma} & X' \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\sigma} & S' \end{array}$$

The object we obtain by descend is just the quotient X'/G which is an S-scheme.

Our strategy is now, to lift a suitable desingularized model of the pullback $E \times_C C'$ along with its Galois action, and to form the quotient afterwards. First, we establish the existence of the Galois action on certain models having split n-torsion subschemes.

3.2.3. Lemma. *Let $E \to C$ be a Jacobian elliptic fibration. Let $n > 4$ be some integer prime to p. Let K be the fraction field of C, and let $L = K(E[n])$ be the splitting field of the n-torsion subscheme. Let $C' \to C$ be the Galois covering of group G, obtain as normal proper model of $L \supset K$.*

Then there exists a generalized elliptic curve E'/C' with $E'_L \simeq E \otimes_K L$, and we can choose a $\Gamma(n)$-structure on E'. Furthermore, the Galois action of G on E'_L extends to an action on E'.

The quotient of E' by this action, considered as elliptic fibration over C, is birational to E and has only rational double point singularities.

[1] To be more precise, one should say that this factorization is given over the open subset of $\mathcal{C}$ where the modular invariant of $\mathcal{E}$ is defined.

Proof. Let $\widetilde{E}$ be the minimal regular model of $E \times_C C'$. Since $\widetilde{E}_L[n]$ is split, the singular fibers of $\widetilde{E}$ are m-gons with $n|m$.

Thus $\widetilde{E}[n]$ is a finite and étale group scheme, lying in the smooth locus of $\widetilde{E}$. The desired model E' is obtained by contracting every component of every fibre which is not intersected by $\widetilde{E}[n]$.

Now, we interpret the action of G on E'_L as rational action on E'. For σ in G we get a diagram:

$$\begin{array}{ccccc} E' & \overset{f_\sigma}{\dashrightarrow} & \sigma^* E' & \longrightarrow & E' \\ & \searrow & \downarrow & \square & \downarrow \\ & & C' & \xrightarrow{\sigma} & C' \end{array}$$

The rational map $f_\sigma \colon E' \dashrightarrow \sigma^* E'$ is given by the factorization of the C-linear rational map $E' \dashrightarrow E'$ given by σ over the pullback $\sigma^* E'$.

The pair E' and $\sigma^*(E')$ satisfies the assumptions of Lemma 3.2.4 below, which allowes us to extend the birational map $f_\sigma \colon E' \dashrightarrow \sigma^* E'$ to an isomorphism. The compositions

$$E' \xrightarrow{f_\sigma} \sigma^* E' \to E'$$

define a Galois action of Γ on E', for it is enough that the axioms required by a group action hold generically, due to separability.

To finish the proof, we have to quotient out by the action of G. The quotient E'/G will have some quotient singularities. We claim, that those are rational singularities. This is done in Lemma 3.2.5 below. □

3.2.4. Lemma. *Let E/S and E'/S be two generalized elliptic curves over an integral base scheme S, such that $E[n] \simeq E'[n] \simeq (\mathbb{Z}/n\mathbb{Z})^2$ holds for some integer $n \geq 3$ and prime to p. Let η by the generic point of S. Then every isomorphism $\varphi_\eta \colon \mathcal{E} \to \mathcal{E}'$ extends to an isomorphism $\mathcal{E} \to \mathcal{E}'$.*

Proof. We can choose $\Gamma(n)$-structures γ and γ' on E and E' such that their restrictions are compatible with the morphism φ_η. In other words, φ_η is now an isomorphism of pairs $(\mathcal{E}_\eta, \gamma_\eta) \xrightarrow{\sim} (E'_\eta, \gamma'_\eta)$. The pairs $(\mathcal{E}, \gamma)$ and $(\mathcal{E}', \gamma')$ hence define morphisms into $\mathcal{M}_{\Gamma(n)}$ which become identical when restricted to η. Since $\mathcal{M}_{\Gamma(n)}$ is a separable scheme, we conclude that $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic, namely by an extension of ϕ_η. □

Until now, we did not need the assumption on the tameness of $C' \to C$. However, this is a necessary condition to control the quotient singularities:

3.2.5. Lemma. *Let $C' \to C$ be a tame Galois covering of group G, let $E' \to C'$ be a fibration, such that E' has only rational double point singularities. Assume we have a Galois action of G on E'. Then the quotient E'/G has only rational double point singularities.*

Proof. Let ξ be a singular point of E'/G. Let $y \in C$ be the image of x. Set $C_y = \operatorname{Spec}(\mathscr{O}^\wedge_{C,y})$. We consider the pullback $C' \times_C C_y$. It is a disjoint union of components C'_{x_i} where $\{x_1, \ldots, x_n\} \subset C'$ are the the points mapping to y. The morphisms $C'_{x_i} \to C_x$ are finite and their Galois groups are the stabilizer groups G_{x_i}. Note that the groups G_{x_i} depending on x_i are conjugate, and that the C_{x_i} are isomorphic. If we pull back the quotient map $E' \to E'/G$ via $C_y \to C$, we get

$$E' \times_{C'} (\coprod C'_{x_i}) \to \coprod (E' \times_{C'} C'_{x_i})/G_{x_i} \to E'/G \times_C C_y.$$

The second map is given by identifying the isomorphic copies $(E' \times_{C'} C'_{x_i})/G_{x_i}$. In particular, we see that $(E'/G) \times_C C_y$ is isomorphic to $(E' \times_{C'} C'_{x_1})/G_{x_1}$.

Now the claim follows because the tameness assumption implies that the order of $G_{x'_1}$ is prime to p. It is a well know statement from singularity theory, that the quotient of a rational singularity is again rational if the group acting has order prime to p. For example, the proof in [DPT80, exp° 10 §2] works in our context, because it only requires that the quotient map is equipped with a trace morphism.

Having seen that E'/G has only rational singularities we can use the special structure of rational curves, lying in the fibers of elliptic fibrations, to conclude that those rational singularities are in fact rational double points. This argument is given in [Sei87, Lemma 1.2]. □

So far, we have seen that in order to obtain a lifting of E/C from the some model of a base change E'/C' it is necessary to lift the diagram:

$$\begin{array}{ccc} C' & \longrightarrow & \mathcal{M}_\Gamma \otimes k \\ \downarrow & & \downarrow \\ C & \longrightarrow & \mathbb{P}^1_k \end{array} \tag{3.2.5.1}$$

In general this problem can be very difficult because the lifting behaviour of wildly ramified maps is hard to control. The situation is better if the maps in (3.2.5.1) are tame. The next lemma provides an important example of a tame covering:

3.2.6. Lemma. *Assume $p \geq 5$ and $n \geq 3$, prime to p. Then the map*

$$\mathcal{M}_{\Gamma(n)} \otimes R \to \mathbb{P}^1_R$$

is tame with respect to the divisor $\mathcal{D}$ which is the sum of the divisors given by the equations $j = 0, 1728, \infty$.

Proof. First, we check that $\mathcal{M}_{\Gamma(n)} \otimes k \to \mathbb{P}^1_k$ is tame. We have a group action of $\mathrm{GL}(2, \mathbb{Z}/n\mathbb{Z})$ of $\mathcal{M}_{\Gamma(n)} \otimes R$ given on points: For $g \in \mathrm{GL}(2, \mathbb{Z}/n\mathbb{Z})$ this action is defined by

$$(E, \gamma) \mapsto (E, g \circ \gamma).$$

Clearly, dividing out by this action is just forgetting the $\Gamma(n)$-structure. This is how $\mathcal{M}_{\Gamma(n)} \otimes k \to \mathbb{P}^1_k$ is defined outside the cusp. However since we work with normal curves, we can interpret $\mathcal{M}_{\Gamma(n)} \otimes k \to \mathbb{P}^1_k$ globally as a quotient map.

The subgroup $\{\pm 1\} \subset \mathrm{GL}(2, \mathbb{Z}/n\mathbb{Z})$ acts trivially on $\mathcal{M}_{\Gamma(n)}$ because we have an isomorphism

$$(E, \gamma) \to (E, -\gamma)$$

given by the involution. It is immediate that this is the entire kernel of the $\mathrm{GL}(2, \mathbb{Z}/n\mathbb{Z})$ action. Thus we have

$$G = \mathrm{Gal}((\mathcal{M}_{\Gamma(n)} \otimes k)/\mathbb{P}^1_k) = \mathrm{GL}(2, \mathbb{Z}/n\mathbb{Z})/\{\pm 1\}.$$

Using the modular description of the action, it is easy to find the inertia group I_x of some point $x \in \mathbb{P}^1_k$: It is $\mathrm{Aut}(E_x)/\{\pm 1\}$ where E_x is the unique elliptic curve whose modular invariant is of value x.

In particular, we get $I_x = \{1\}$ if x is ordinary, $I_x = \mathbb{Z}/3\mathbb{Z}$ if $x = 1728$, $I_x = \mathbb{Z}/4\mathbb{Z}$ if $x = 0$. Note that $p \geq 5$ is assumed. If $x = \infty$ we find $I_x = \mu_n \simeq \mathbb{Z}/n\mathbb{Z}$. We see that in each case, the order of I_x is prime to p, which implies tameness.

To see that $\mathcal{M}_{\Gamma(n)} \otimes R \to \mathbb{P}^1_R$ is tame, observe that this map is étale outside of $\mathcal{D}$, because the reduction is étale. Now, the statement follows from Theorem 3.1.2 applied with the divisor $\mathcal{D}$. □

The tameness of $\mathcal{M}_{\Gamma(N)} \otimes R \to \mathbb{P}^1_R$ is fundamental for the rest of this Chapter. Hence we will have to assume $p \geq 5$. We are now ready to give the first main application:

3.2.7. Proposition. *Let $E \to C$ be a Jacobian elliptic fibration and let $p \geq 5$. Assume that the modular invariant of E/C is tame. Let $n \geq 3$ be some integer prime to p, such that the splitting field of $E[n]$ gives rise to a tame Galois covering $C' \to C$ of group G.*

Then there exists a Jacobian elliptic fibration $\widetilde{\mathcal{E}}/\mathcal{C}$ over R, such that the reduction $\widetilde{E}/C$ is a birational model of E/C, having only rational double point singularities.

Proof. Let E' denote the model of $E \times_C C'$ as constructed in Lemma 3.2.3, with a chosen $\Gamma(n)$-structure α_0. We have the usual diagram:

$$\begin{array}{ccc} C' & \longrightarrow & \mathcal{M}_{\Gamma(n)} \otimes k \\ \big\downarrow & & \big\downarrow \\ C & \xrightarrow[j_0]{} & \mathbb{P}^1_k \end{array}$$

We are going to construct a lifting:

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{M}_{\Gamma(n)} \otimes R \\ \big\downarrow & & \big\downarrow \\ \mathcal{C} & \longrightarrow & \mathbb{P}^1_R \end{array} \tag{3.2.7.1}$$

To do this, let $\mathcal{D} \subset \mathbb{P}^1_R$ be the divisor defined by the equations

$$j = 0, j = 1728 \text{ and } j = \infty.$$

We saw that $\mathcal{M}_{\Gamma(n)} \otimes R \to \mathbb{P}^1_R$ is tame with respect to $\mathcal{D}$. (Note $p \geq 5$). Denote by $D \subset \mathbb{P}^1_k$ the reduction of $\mathcal{D}$. Let $B \subset \mathbb{P}^1_k$ denote the reduced branching locus of $C' \to C \to \mathbb{P}^1_k$. We choose a lifting $\mathcal{B} \subset \mathbb{P}^1_R$ such that the following condition is satisfied: If there is an open subset $U \subset \mathbb{P}^1_k$ such that $B \cap U = D \cap U$, and if $\mathcal{U} \subset \widehat{\mathbb{P}^1_R}$ is the lifting of U, then $\mathcal{B} \cap \mathcal{U} = \mathcal{D} \cap \mathcal{U}$. This condition roughly says that components of $\mathcal{B}$ and $\mathcal{D}$ coinciding on the reduction should coincide after lifting as well.

Applying Theorem 3.1.2 with base $\mathbb{P}^1_R$ and $\mathcal{B} \cup \mathcal{D}$ we find a lifting $\mathcal{C}' \to \mathbb{P}^1_R$ which factors over a Galois covering $\mathcal{C}' \to \mathcal{C}$ of group G. Furthermore, the equivalence of categories gives us a morphism of coverings $\mathcal{C}' \to \mathcal{M}_{\Gamma(n)} \otimes R$ over $\mathbb{P}^1_R$.

Denote by $\mathcal{E}'/\mathcal{C}'$ the generalized elliptic curve given by $\mathcal{C}' \to \mathcal{M}_{\Gamma(n)} \otimes R$. The next step is to lift the G action on E' to $\mathcal{E}'$. For every $\sigma \in G$ we choose a pullback $\sigma^*\mathcal{E}'$ and write σ^*E' for the reduction. The action of G on E' can be described by a set of isomorphism index by the elements of G

$$f_\sigma \colon E' \to \sigma^*E'.$$

The action itself is obtained by composing f_σ with the projection $\sigma^*E' \to E'$.

We are going to lift the f_σ: First of all, the generalized elliptic curves $\mathcal{E}'$ and $\sigma^*\mathcal{E}'$ have the same modular invariants. From (3.2.7.1) we get that the modular invariants factor over $\mathcal{C}$, thus $j(\mathcal{E}')$ is invariant under the action of Galois. Now by Corollary 2.5.4, it follows that there exist liftings F_σ of the f_σ.

It remains to show that the collection of σ-linear automorphisms

$$\mathcal{E}' \xrightarrow{F_\sigma} \sigma^*\mathcal{E}' \to \mathcal{E}'$$

index by the elements of G gives in fact an *homomorphism* of groups $G \to \mathrm{Aut}_{\mathcal{C}}(\mathcal{E}')$. By separability, it is enough to check the required axioms generically. Let $\mathcal{V} \subset \mathcal{C}$ be an open subset such that the induced covering $\mathcal{V}' = \mathcal{C}' \times_{\mathcal{C}} \mathcal{V} \to \mathcal{U}$ is étale and therefore a G-torsor.

Let $\mathcal{V}'' = \mathcal{V}' \times_{\mathcal{V}} \mathcal{V}'$ and denote by $p_i \colon \mathcal{V}'' \to \mathcal{V}$ the projections on the i-th factor ($i = 1, 2$). We have an identification $\mathcal{V}'' \simeq \mathcal{V}' \times G$ on points given by:

$$(\sigma, x) \mapsto (\sigma(x), x)$$

The composition $\mathcal{V}' \times G \to \mathcal{V}'' \xrightarrow{p_i} \mathcal{V}'$ gives the first projection $\mathcal{V}' \times G \to \mathcal{V}'$ if $i = 1$ and the group action $\mathcal{V}' \times G \to \mathcal{V}'$ if $i = 2$. Under this identification, we see that the family of F_σ's gives an $\mathcal{V}''$-isomorphism i.e an covering datum:

$$p_1^* \mathcal{E}' \xrightarrow{\varphi} p_2^* \mathcal{E}'.$$

We have to check that this is in fact a descend datum. So we define

$$\mathcal{V}''' = \mathcal{V}' \times_{\mathcal{V}} \mathcal{V}' \times_{\mathcal{V}} \mathcal{V}'$$

and denote by $p_{ij} \colon \mathcal{V}''' \to \mathcal{V}''$ the projection onto the factor with indices i and j for i, j and $i, j = 1, 2, 3$. The cocycle condition can now be stated as:

$$p_{13}^* \varphi = p_{23}^* \varphi \circ p_{12}^* \varphi.$$

The above is an equation of $\mathcal{V}'''$-isomorphisms between the semistable elliptic curves $p_{i,j}^* p_k^* E'|_{\mathcal{V}'}$. On the reduction, this equation is satisfied because the f_σ correspond to descent data. Now the assertion follows by the uniqueness statement in Corollary 2.5.4.

The quotient of $\mathcal{E}'$ modulo the action of G is the desired model $\tilde{E}$. $\square$

3.2.8. *Remark* Let E/C be a Jacobian elliptic fibration. Given $p \geq 5$, it is always possible to find a prime number ℓ such that $L = K(E[\ell])$ is tame over the fraction field K of C: Namely, we have a faithfull action of $G = \mathrm{Gal}(L/E)$ on $E[\ell](L) \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$, hence an embedding $G \to \mathrm{GL}(2, \mathbb{Z}/\ell\mathbb{Z})$. The latter group is of order $\ell(\ell-1)^2(\ell+1)$. It is sufficient to choose ℓ such that p does not divide $\ell(\ell-1)^2(\ell+1)$. However, we have

$$\gcd_{\ell \text{ prime}} \{\ell(\ell-1)^2(\ell+1)\} = 48.$$

Compare the proof of [Sil94, Theorem 10.2].

As a consequence we get a lemma which will be useful in the next Section:

3.2.9. Lemma. *Let E/C be a Jacobian elliptic fibration and let $p \geq 5$. Denote by K the fraction field of C. Then for every integer m prime to p, we have that the field extension $K(E[m]) \supset K$ is tame.*

Proof. By the Remark we find a prime $\ell > 4$ different form p, such that $K(E[\ell]) \supset K$ is tame. Denote the associated covering of proper and smppth curves by $B \to C$. We claim that the extension $K(E[m\ell]) \supset K(E[\ell])$ is also tame: From Corollary 3.2.2 we get that a regular minimal model $\widetilde{E}$ of $E \otimes_K K(E[\ell])$ is semistable.

Let $\{x_1, \ldots, x_n\}$ be the point of C, over which the fibers $\widetilde{E}_x$ are singular. It exists a Kummer covering $B' \to B$ (see [GM71, 1.2.1]) of group $\mathbb{Z}/m\mathbb{Z}$ which is totaly ramified over the x_i. Denote by $\widetilde{E_{B'}}$ the regular minimal model of $\widetilde{E} \times_B B'$. The m-torsion subscheme of $\widetilde{E_{B'}}$ is now finite and étale so we find an étale covering $B'' \to B$ such that $E_{B'} \times_{B'} B''$ is split. By construction, $B'' \to C$ is tame because it is the compostion of two tame coverings.

Clearly, $K(E[m])$ in contained in the field of fractions of B'', hence tame as well. $\square$

3. Non-Jacobian fibrations of period prime to p

Given an arbitrary elliptic fibration X/C, let K be the fraction field of C. Then the generic fibre X_K is a smooth genus-1 curve over $\mathrm{Spec}(K)$. The elliptic curve $E_K = \mathrm{Pic}^0_{X_K}$ is called the Jacobian of X_K. The curve X_K has a natural E_K-torsor structure by identifying X_K with $\mathrm{Pic}^1_{X_K}$. The isomorphism type of X_K (and hence of X) is uniquely determind by this torsor structure.

The set of isomorphism classes of E_K-torsors is described by the cohomology groups

$$H^1_{et}(\mathrm{Spec}(K), E_K) = H^1(K^s/K, E(K^s))$$

where the right hand side is a Galois cohomology group with respect to a fixed separable closure K^s of K. The study of this group is called Ogg-Safarevich-theory (see [Šaf61] or [LT58] for details).

Let us assume that the class of X is m-torsion in $H^1(K^s/K, E(K^s))$ for some m prime to p. By considering the Kummer sequence

$$0 \to E(K)/mE(K) \to H^1(K^s, E(K^s)[m]) \to H^1(K^s, E(K^s))[m] \to 0$$

it follows that the class of X in $H^1(K^s, E(K^s))[m]$ can be represented by a cocycle z taking values in $E(K^s)[m]$. Following the conventions in [Ser97] we write the Galois action of G on the group $E(K^s)$ as:

$$x \mapsto {}^\sigma x \ \text{ for } \sigma \in G \text{ and } x \in E(K^s).$$

A cocycle z is a map $G \to E(K^s)$ written $\sigma \mapsto z_\sigma \in E(K^s)$ and fulfilling the cocycle condition:

$$z_{\sigma\tau} = z_\sigma + {}^\sigma z_\tau.$$

Let σ be an element of G. The cocycle z defines a σ-linear action ρ_z on E' by

$$\rho_z(\sigma)x = z_\sigma + {}^\sigma x.$$

This formula gives in fact a group action on $E(K^s)$ as follows form the cocycle condition for z.

Let $C' \to C$ be a normal and proper model of the field extension $K(E_K[m]) \supset K$, where $K(E[m])$ denotes the splitting field of the m-torsion points of E_K. If $m \geq 4$, it follows form Lemma 3.2.3 that it exists a model E'/C' of $E_K \otimes K'$ which is a generalized elliptic curve with $E'[m] \simeq (\mathbb{Z}/m\mathbb{Z})^2$. If $m < 4$, we can pick a multiple $m' \geq 4$ and prime to p, and use $K(E[m'])$ instead.

In Lemma 3.2.3 we saw that the Galois action of G which on $E'(K^s)$ given by $z \mapsto {}^\sigma z$ extends to an action $\varrho\colon G \to \mathrm{Aut}_{C'}(E')$. A section s of E' over C' gives rise to a translation $t_s\colon E' \to E'$. Hence we can define $\rho_z\colon E' \to E'$ by setting

$$\rho_z(\sigma) = t_{z_\sigma} \circ \varrho(\sigma).$$

The quotient E'/Γ of E' by this action is birationally equivalent to X. Our strategy is now to lift E' along with this action, and form the quotient afterwards. To get started, we need the following:

3.3.1. Proposition. *Let X/C be an elliptic fibration, and let $p \geq 5$. Let E_K be the Jacobian of the generic fibre of X/C. Assume its modular invariant to be tame. Assume further, that the class $[X] \in H^1(K^s/K, E(K^s))$ is m-torsion, with m prime to p.*

Then there exists a birational model $\widetilde{X}/C$ of X, having only rational singularities, that lifts to a fibration $\widetilde{\mathcal{X}}/\mathcal{C}$ over R.

Proof. From Lemma 3.2.9 we get a tame Galois covering $C' \to C$ of group G, such that $E \times_C C'$ has a model E' which is a generalized elliptic curve with $E'[m] \simeq (\mathbb{Z}/n\mathbb{Z})^2$.

Exactly as in the proof of Proposition 3.2.7, we find a lifting $\mathcal{C}' \to \mathcal{C}$, which is again Galois with group G, and a lifting $\mathcal{E}'/\mathcal{C}'$ such that the G-action ϱ on E' extends to $\mathcal{E}'$.

The subgroup scheme $\mathcal{E}'[m]$ is étale over $\mathcal{C}$. The singular geometric fibers of E' are d-gons, where d is a multiple of m. By Proposition 2.6.5 it follows that the same is true for $\mathcal{E}'$. Hence we conclude by Proposition 2.3.1, that $\mathcal{E}'[m]$ is also finite.

It follows that $\mathcal{E}'[m] \simeq (\mathbb{Z}/m\mathbb{Z})^2$ because the splitting given on the reduction lifts. In particular, there exist a unique extension of the map $z\colon G \to E'[m]$, which we also denote by Z. The lifting is indeed a cocycle because the element

$$Z_{\sigma\tau} - (Z_\sigma + {}^\sigma Z_\tau)$$

specializes to zero, and therefore is already equal to zero due to the unramifiedness of $\mathcal{E}'[m]$. Hence we set

$$\rho_Z(\sigma)x = t_{Z_\sigma} \circ \varrho(\sigma)$$

which is the lifting of ρ_z we wanted to construct. Forming the quotient we obtain the desired model $\widetilde{\mathcal{X}}/\mathcal{C}$. □

3.3.2. *Remark* In Proposition 3.3.1 we made an assumption on the order of the class $[X]$ in $H^1(K^s/K, E_K)$. This number is called the *period* of X_K. There is an other invariant, which is closely related to the period and has a more geometric interpretation, namely the *index*. It is defined to be the greatest common divisor of the degrees of irreducible effective divisor on X_K. Geometrically speaking, those divisor correspond to multisections of a minimal regular model X/C of X_K.

The period always divides the index, and both numbers have the same prime factors (see [LT58, Proposition 5]). The assumption in Proposition 3.3.1 can thus equivalently be state by using the index instead of the period.

4. Simultaneous desingularization

Having constructed liftings of singular models, the question of simultaneous desingularization comes up. For families of surfaces, this theory was developed by M. Artin in [Art74]. It is formulated in the category of algebraic spaces. We give the basic definitions:

Let $f\colon \mathcal{X} \to S$ be a morphism of algebraic spaces which is flat and of finite type such that the fibres of f are normal algebraic spaces of pure dimension 2. We define a *resolution* of f to be a proper S-morphism of algebraic spaces $f'\colon X' \to \mathcal{X}$ such that the fibres of f' are minimal resolutions of the fibres of f.

Since we have to allow base changes in order to find resolutions, it is useful to work with the functor $\mathrm{Res}_{\mathcal{X}/S}$ on the category of S-schemes defined by

$$\mathrm{Res}_{\mathcal{X}/S}(S') = \text{ set of resolution of } f \times_S S' \text{ of } f.$$

Artin showed that $\mathrm{Res}_{\mathcal{X}/S}$ is representable by a (not necessarily separated) algebraic space over S. Denote the object representing $\mathrm{Res}_{\mathcal{X}/S}$ by T. Over geometric points of the base, we know that there always exists a unique resolution. Thus $R \to S$ is bijective on geometric points. Let $s \in S$ be a closed point of S. We consider the map $T_s^h \to S_s^h$ between the henselization of S at s and the henselization of T_s at the unique point lying over s. For us, the case of rational double points is of great importance:

3.4.1. Theorem (Theorem 2 [Art74])**.** *Let $f\colon \mathcal{X} \to S$ a family of surfaces, having only rational double point singularities. Then the local map $T_s^h \to S_s^h$ is finite and surjective.*

In the case where S is the spectrum of a complete discrete valuation domain, we get the following by exchanging T_s^h with its normalization:

3.4.2. Lemma. *Let $f\colon \mathcal{X} \to S$ be as in Theorem 3.4.1. Assume that S is the spectrum of a complete discrete valuation domain. Then there exists a finite flat covering $S' \to S$ and an algebraic space $\mathcal{X}' \to S'$ such that the special fibre of $\mathcal{X}'$ is a resolution of the special fibre of $\mathcal{X}$.*

We can apply Lemma 3.4.2 to deformations of Weierstraß models: Let $W \to C$ be a minimal Weierstraß fibration, meaning that it has only rational double point singularities. Let $\mathcal{W} \to \mathcal{C}$ be a lifting over some local complete discrete valuation ring R. In this situation, we can apply Lemma 3.4.2 because from the fact that W has only rational double point singularities, it follows that the generic fibre of $\mathcal{W}$ over R has only rational double point singularities as well (see [Lie08, Proposition 6.1]). So there exists a finite extension $R' \supset R$ such that the base change of $\mathcal{W}/\mathcal{C}$ has a simultaneous resolution $\widetilde{\mathcal{W}} \to \mathcal{W}$ in the category of algebraic spaces.

Another application is to desingularize the models of elliptic fibrations we have lifted in Section 2 and 3. Because minimal regular models are unique, a desingularized lifting will be a lifting of a minimal regular model. Indeed, from Proposition 3.2.7 and Remark 3.2.8 we get:

3.4.3. Theorem. *Let $p \geq 5$. Let $E \to C$ be a Jacobian elliptic fibration which is minimal and has regular total space. Assume that its modular invariant is tame. Then there exists a Jacobian lifting $\widetilde{\mathcal{E}}/\mathcal{C}$ in the category of algebraic spaces.*

Similarly, from Proposition 3.3.1 and Remark 3.3.2 we get:

3.4.4. Theorem. *Let $p \geq 5$. Let X/C be an elliptic fibration which is minimal and has regular total space. Let E_K the Jacobian of the generic fibre, and let E/C be the minimal regular model of E_K thereof. Assume the modular invariant $j_0\colon C \to \mathbb{P}^1_k$ is tame and that X/C possesses a multisection of degree prime to p.*

Then there exists a lifting $\mathcal{X}/\mathcal{C}$ of X/C in the category of algebraic spaces after a finite flat extension $R' \supset R$.

It is not easy to determine in which cases the simultaneous desingularization does fail to be a scheme. Note however, that liftability in the category of algebraic spaces is stronger than liftability in the formal category: Given an algebraic spaces, fibred over a local ring R, such that the special fiber is a scheme, its completion with respect to the special fibre is a formal scheme [Knu71, 5. Theorem 2.5].

We can use this to conclude that in esay situations, desingularization in the category of schemes is possbile:

3.4.5. Proposition. *Let $\mathcal{X}$ be an algebriac space which is proper over some complete notherian local ring R. Assume that the special fibre $X = \mathcal{X} \otimes_R R/m$ is a projective scheme and that $H^2(X, \mathscr{O}_X) = 0$. Then $\mathcal{X}$ is a projective R-scheme.*

Proof. We are going to show that an R-ample line bundle exists on $\mathcal{X}$. To that purpose, denote by $\widehat{\mathcal{X}}$ the adic formal $\mathrm{Spf}(R)$-scheme obtained by completion along the special fibre.

Fix an ample line bundle $\mathscr{L}_0$ on X. By assumption ($H^2(X, \mathscr{O}_X) = 0$), lifting of line bundles is unobstructed, hence there exists a linebundle $\widehat{\mathscr{L}}$ on $\widehat{\mathcal{X}}$.

By Grothendieck's Algebraization Theorem, there exists an embedding $\widehat{\imath}\colon \widehat{\mathcal{X}} \to \widehat{\mathbb{P}}^n_R$ for some n. Grothendieck's existence theorem is valid in the category of algebraic spaces ([Knu71, 5. Theorem 6.3]). Applying this to the graph of $\widetilde{\imath}$ we get an embedding $i\colon \mathcal{X} \to \mathbb{P}^n_R$. □

The cohomological assumption in Proposition 3.4.5 is satisified for example by a rational elliptic surface. We mention one more consequence of Grothendieck's Algebrization Theorem:

3.4.6. Proposition. *Let E/C be a Jacobian elliptic fibration, with irreducible geometric fibers. If E/C has a formal lifting (in particular if it has a lifting in the catgory of algebraic spaces) along with its zero section, then it also has a projective lifting.*

Bibliography

[Art74] M. Artin. Algebraic construction of Brieskorn's resolutions. *J. Algebra*, 29:330–348, 1974. xiii, 57

[Băd01] Lucian Bădescu. *Algebraic surfaces*. Universitext. Springer-Verlag, New York, 2001. Translated from the 1981 Romanian original by Vladimir Maşek and revised by the author. 24

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990. 51

[BM77] E. Bombieri and D. Mumford. Enriques' classification of surfaces in char. *p*. II. In *Complex analysis and algebraic geometry*, pages 23–42. Iwanami Shoten, Tokyo, 1977. 15, 17

[BS03] Carmen Laura Basile and Alexei Skorobogatov. On the Hasse principle for bielliptic surfaces. In *Number theory and algebraic geometry*, volume 303 of *London Math. Soc. Lecture Note Ser.*, pages 31–40. Cambridge Univ. Press, Cambridge, 2003. 25

[CD89] François R. Cossec and Igor V. Dolgachev. *Enriques surfaces. I*, volume 76 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1989. 15

[Con07] Brian Conrad. Arithmetic moduli of generalized elliptic curves. *J. Inst. Math. Jussieu*, 6(2):209–278, 2007. 27, 35

[Del81] P. Deligne. Relèvement des surfaces $K3$ en caractéristique nulle. In *Algebraic surfaces (Orsay, 1976–78)*, volume 868 of *Lecture Notes in Math.*, pages 58–79. Springer, Berlin, 1981. Prepared for publication by Luc Illusie. xv

[DI87] Pierre Deligne and Luc Illusie. Relèvements modulo p^2 et décomposition du complexe de de Rham. *Invent. Math.*, 89(2):247–270, 1987. xiv

[DPT80] Michel Demazure, Henry Charles Pinkham, and Bernard Teissier, editors. *Séminaire sur les Singularités des Surfaces*, volume 777 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980. Held at the Centre de Mathématiques de l'École Polytechnique, Palaiseau, 1976–1977. 53

[DR73] P. Deligne and M. Rapoport. Les schémas de modules de courbes elliptiques. In *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 143–316. Lecture Notes in Math., Vol. 349. Springer, Berlin, 1973. xii, 27, 28, 29, 30, 31, 32, 41, 43, 44

[EGA I] A. Grothendieck. Éléments de géométrie algébrique. I. Le langage des schémas. *Inst. Hautes Études Sci. Publ. Math.*, (4):228, 1960. 3

[EGA II] A. Grothendieck. Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. *Inst. Hautes Études Sci. Publ. Math.*, (8):222, 1961. 7

[EGA III.1] A. Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. *Inst. Hautes Études Sci. Publ. Math.*, (11):167, 1961. 2

[EGA IV.3] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, (28):255, 1966. 31

[GM71] Alexander Grothendieck and Jacob P. Murre. *The tame fundamental group of a formal neighbourhood of a divisor with normal crossings on a scheme*. Lecture Notes in Mathematics, Vol. 208. Springer-Verlag, Berlin, 1971. 49, 50, 55

[GM98] Barry Green and Michel Matignon. Liftings of Galois covers of smooth curves. *Compositio Math.*, 113(3):237–272, 1998. 14

[Gro74] Alexandre Grothendieck. *Groupes de Barsotti-Tate et cristaux de Dieudonné*. Les Presses de l'Université de Montréal, Montreal, Que., 1974. Séminaire de Mathématiques Supérieures, No. 45 (Été, 1970). 19

[Har03] David Harbater. Patching and Galois theory. In *Galois groups and fundamental groups*, volume 41 of *Math. Sci. Res. Inst. Publ.*, pages 313–424. Cambridge Univ. Press, Cambridge, 2003. 37

[Hus04] Dale Husemöller. *Elliptic curves*, volume 111 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2004. With appendices by Otto Forster, Ruth Lawrence and Stefan Theisen. 33

[Igu55] Jun-ichi Igusa. On some problems in abstract algebraic geometry. *Proc. Nat. Acad. Sci. U. S. A.*, 41:964–967, 1955. 4

[Ill05] Luc Illusie. Grothendieck's existence theorem in formal geometry. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 179–233. Amer. Math. Soc., Providence, RI, 2005. With a letter (in French) of Jean-Pierre Serre. 38

[JLR09] Tyler J. Jarvis, William E. Lang, and Jeremy R. Ricks. Integral models of extremal rational elliptic surfaces. *arXiv0908.1831J*, 2009. 20

[Kat81] N. Katz. Serre-Tate local moduli. In *Algebraic surfaces (Orsay, 1976–78)*, volume 868 of *Lecture Notes in Math.*, pages 138–202. Springer, Berlin, 1981. 19

[Kle05] Steven L. Kleiman. The Picard scheme. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 235–321. Amer. Math. Soc., Providence, RI, 2005. 11

[KM85] Nicholas M. Katz and Barry Mazur. *Arithmetic moduli of elliptic curves*, volume 108 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985. 4, 5, 7, 18, 19, 21, 29

[Knu71] Donald Knutson. *Algebraic spaces*. Lecture Notes in Mathematics, Vol. 203. Springer-Verlag, Berlin, 1971. 45, 46, 58

[KU85] Toshiyuki Katsura and Kenji Ueno. On elliptic surfaces in characteristic p. *Math. Ann.*, 272(3):291–330, 1985. xi

[Lan95] William E. Lang. Examples of liftings of surfaces and a problem in de Rham cohomology. *Compositio Math.*, 97(1-2):157–160, 1995. Special issue in honour of Frans Oort. xii

[Lie08] Christian Liedtke. Algebraic surfaces of general type with small c_1^2 in positive characteristic. *Nagoya Math. J.*, 191:111–134, 2008. 58

[Lie09] Christian Liedtke. Algebraic surfaces in positive characteristic. *arXiv0912.4291v1*, 2009. xv

[Lie10] Christian Liedtke. Moduli and lifting of enriques surfaces. *arXiv:1007.0787v1*, 2010. xv

[Liu03] Q Liu. Reduction and lifting of finite covers of curves. In *Proceedings of the 2003 Workshop on Cryptography and Related Mathematics, Chuo University*, pages 161–180. 2003. xiv

[LLR04] Qing Liu, Dino Lorenzini, and Michel Raynaud. Néron models, Lie algebras, and reduction of curves of genus one. *Invent. Math.*, 157(3):455–518, 2004. 7

[LMB00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000. 31

[LT58] Serge Lang and John Tate. Principal homogeneous spaces over abelian varieties. *Amer. J. Math.*, 80:659–684, 1958. 56, 57

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994. 22, 24

[Mil80] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980. 10

[Mir90] Rick Miranda. Persson's list of singular fibers for a rational elliptic surface. *Math. Z.*, 205(2):191–211, 1990. 28

[Mum70] David Mumford. *Abelian varieties*. Tata Institute of Fundamental Research Studies in Mathematics, No. 5. Published for the Tata Institute of Fundamental Research, Bombay, 1970. 18

[Oor79] Frans Oort. Abelian varieties: moduli and lifting properties. In *Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978)*, volume 732 of *Lecture Notes in Math.*, pages 477–495. Springer, Berlin, 1979. xv

[Pri00] Rachel J. Pries. Construction of covers with formal and rigid geometry. In *Courbes semi-stables et groupe fondamental en géométrie algébrique (Luminy, 1998)*, volume 187 of *Progr. Math.*, pages 157–167. Birkhäuser, Basel, 2000. 37, 38

[PS00] Amílcar Pacheco and Katherine F. Stevenson. Finite quotients of the algebraic fundamental group of projective curves in positive characteristic. *Pacific J. Math.*, 192(1):143–158, 2000. 8, 9

[Ray70] Michel Raynaud. *Faisceaux amples sur les schémas en groupes et les espaces homogènes*. Lecture Notes in Mathematics, Vol. 119. Springer-Verlag, Berlin, 1970. 10, 46

[Ray78] M. Raynaud. Contre-exemple au "vanishing theorem" en caractéristique $p > 0$. In *C. P. Ramanujam—a tribute*, volume 8 of *Tata Inst. Fund. Res. Studies in Math.*, pages 273–278. Springer, Berlin, 1978. xv

[RŠ76] A. N. Rudakov and I. R. Šafarevič. Inseparable morphisms of algebraic surfaces. *Izv. Akad. Nauk SSSR Ser. Mat.*, 40(6):1269–1307, 1439, 1976. xv

[Šaf61] I. R. Šafarevič. Principal homogeneous spaces defined over a function field. *Trudy Mat. Inst. Steklov.*, 64:316–346, 1961. 56

[Saï04] Mohamed Saïdi. Wild ramification and a vanishing cycles formula. *J. Algebra*, 273(1):108–128, 2004. 38

[Sch68] Michael Schlessinger. Functors of Artin rings. *Trans. Amer. Math. Soc.*, 130:208–222, 1968. 1

[Sei87] Wolfgang K. Seiler. Global moduli for polarized elliptic surfaces. *Compositio Math.*, 62(2):187–213, 1987. 53

[Sei88] Wolfgang K. Seiler. Deformations of Weierstrass elliptic surfaces. *Math. Ann.*, 281(2):263–278, 1988. xv

[Ser79] Jean-Pierre Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg. xiv

[Ser97] Jean-Pierre Serre. *Galois cohomology*. Springer-Verlag, Berlin, 1997. Translated from the French by Patrick Ion and revised by the author. 56

[SGA 1] Alexander Grothendieck. *Revêtements étales et groupe fondamental. Fasc. II: Exposés 6, 8 à 11*, volume 1960/61 of *Séminaire de Géométrie Algébrique*. Institut des Hautes Études Scientifiques, Paris, 1963. xiv, 2

[SGA III.2] M. Artin, J. E. Bertin, M. Demazure, P. Gabriel, A. Grothendieck, M. Raynaud, and J.-P. Serre. *Schémas en groupes. Fasc. 7: Exposés 23 à 26*, volume 1963/64 of *Séminaire de Géométrie Algébrique de l'Institut des Hautes Études Scientifiques*. Institut des Hautes Études Scientifiques, Paris, 1965/1966. 9

[Sil94] Joseph H. Silverman. *Advanced topics in the arithmetic of elliptic curves*, volume 151 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. 50, 55

[Sil09] Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, second edition, 2009. 8

[Tat75] J. Tate. Courbes elliptiques : Formulaire. In *Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 52–83. Lecture Notes in Math., Vol. 476. Springer, Berlin, 1975. 28, 30