Vector Bundles as Generators on Schemes and Stacks

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Summary

The present work is dedicated to the investigation of the resolution property of quasicompact and quasiseparated schemes, or more generally of algebraic stacks with pointwise affine stabilizer groups. Such a space X has the resolution property if every quasicoherent sheaf of finite type admits a surjection from a locally free sheaf of finite rank.

Locally this is satisfied by definition, but globally this is a non-trivial problem. There exist counter examples in the category of schemes, but they are non-separated and even fail to have affine diagonal. This is a mild separateness condition and Totaro showed that it is in fact necessary [Tot04]. Therefore it is natural to stick to schemes and algebraic stacks with affine diagonal. In this class the resolution property holds for all regular, noetherian schemes, all quasiprojective schemes, or more generally all Deligne-Mumford stacks with quasiprojective coarse moduli space.

As our first main result we verify the resolution property for a large class of surfaces in the first part of the present work. Namely, we show that all two-dimensional schemes that are proper over a noetherian ring satisfy the resolution property. This class includes many singular, non-normal, non-reduced and non-quasiprojective surfaces. The case of normal separated algebraic surfaces was settled by Schröer and Vezzosi [SV04] and we generalize their methods of gluing local resolutions to the non-normal and non-reduced case, using the pinching techniques of Ferrand [Fer03] in combination with deformation theory of vector bundles.

In the second part of the present work our main result states that for a large class of algebraic stacks the resolution property is equivalent to a stronger form: There exists a *single* locally free sheaf $\mathcal{E}$ such that the collection of sheaves, obtained by taking appropriate locally free subsheaves of direct sums, tensor products and duals of $\mathcal{E}$, is sufficiently large in order to resolve arbitrary quasicoherent sheaves of finite type. Next, we interpret this geometrically: A sheaf $\mathcal{E}$ has this property if and only if its associated frame bundle has *quasiaffine* total space.

This yields a natural generalization of the concept of ample line bundles on separated schemes to vector bundles of higher rank on arbitrary quasicompact algebraic stacks with affine diagonal.

As an immediate consequence of this result we infer a generalization of Totaro's Theorem to non-normal stacks which says that X has the resolution property if and only if $X \simeq [U/GL_n]$ for some quasiaffine scheme U acted on by the general linear group [Tot04, Thm 1.1].

Zusammenfassung

Die vorliegende Arbeit ist dem Studium der Auflösungseigenschaft quasikompakter und quasiseparierter Schemata, oder allgemeiner algebraischer Stacks mit punktweise affinen Stabilisatorgruppen, gewidmet. Ein solcher Raum X hat die Auflösungseigenschaft, falls jede quasikohärente Garbe von endlichem Typ eine Surjektion von einer lokal freien Garbe von endlichem Rang besitzt.

Dies ist nach Definition stets lokal erfüllt, im globalen Fall allerdings ein nicht-triviales Problem. Es existieren hierfür Gegenbeispiele in der Kategorie der Schemata, allerdings sind dies nicht-separierte Schemata, die nicht einmal affine Diagonale besitzen. Letzteres ist eine schwache Form von Separiertheit und nach Totaro sogar eine notwendige Bedingung für die Auflösungseigenschaft [Tot04]. Daher ist es eine natürliche Einschränkung, nur algebraische Stacks mit affiner Diagonale zu betrachten. In dieser Klasse gilt die Auflösungseigenschaft für alle $\mathbb{Q}$-faktoriellen und noetherschen Schemata, alle quasiprojektiven Schemata, oder allgemeiner für alle Deligne-Mumford-Stacks mit quasiprojektivem grobem Modulraum.

Als unser erstes Hauptresultat verifizieren wir im ersten Teil der vorliegenden Arbeit die Auflösungseigenschaft für eine große Klasse von Flächen. Wir zeigen nämlich, dass jedes zweidimensionale Schema, das eigentlich über einem noetherschen Grundring ist, die Auflösungseigenschaft erfüllt. Diese Klasse beinhaltet viele singuläre, nicht-normale, nicht-reduzierte und nicht-quasiprojektive Flächen. Der Fall normaler algebraischer Flächen wurde von Schröer und Vezzosi [SV04] bewiesen und wir verallgemeinern deren Methode, lokale Auflösungen zusammenzufügen, im nicht-normalen und nicht-reduzierten Fall mittels Ferrands Verklebetechniken von Schemata [Fer03] und der Deformationstheorie von Vektorbündeln.

Unser Hauptresultat im zweiten Teil der Arbeit besagt, dass für eine große Klasse von algebraischen Stacks, welche alle Schemata und alle noetherschen algebraischen Stacks mit affinen Stabilisatoren einschließt, die Auflösungseigenschaft äquivalent zu einer viel stärken Form ist: Es existiert eine *einzige* lokal frei Garbe $\mathcal{E}$ mit der Eigenschaft, dass die assoziierte Familie der Garben, welche als gewisse lokal freie Untergarben nach iterierter Bildung von direkten Summen, Tensorprodukten und Dualen von $\mathcal{E}$ entstehen, schon hinreichend groß ist, um beliebige quasikohärente Garben von endlichem Typ aufzulösen. Als nächstes interpretieren wir dies geometrisch: Diese zu einer Garbe $\mathcal{E}$ assozierte Familie von lokal freien Garben hat genau dann jene Eigenschaft, wenn das zugehörige Rahmenbündel einen quasiaffinen Totalraum besitzt.

Dies führt zu einer natürlichen Verallgemeinerung des Konzepts ampler Gradenbündel auf Schemata hinzu Vektorbündeln höheren Rangs auf beliebigen quasikompakten algebraischen Stacks mit affiner Diagonale.

Als unmittelbare Konsequenz dieses Resultats folgern wir eine Verallgemeinerung von Totaro's Theorem für nicht-normale Stacks, welches besagt, dass X genau dann die Auflösungseigenschaft besitzt, wenn X als Quotient $X \simeq [U/GL_n]$ dargestellt werden kann, wobei U ein quasiaffines Schema ist, auf dem die allgemeine lineare Gruppe operiert.

Introduction

A central object in algebraic geometry is the projective space $\mathbb{P}^n$. Classically, it is the moduli space that parametrizes all lines in the affine space $\mathbb{A}^{n+1}$ meeting the origin. So, one might define it as the quotient $(\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{G}_m$, where the multiplicative group $\mathbb{G}_m$ acts freely by scalar multiplication on $\mathbb{A}^{n+1} \setminus \{0\}$.

In modern language, developed by Grothendieck and his school, the *scheme* $\mathbb{P}^n$ is characterized by the set of morphisms of schemes $T \to \mathbb{P}^n$, where T runs over all schemes, by Yoneda's Lemma. This set parametrizes all quotient maps $\mathcal{O}_T^{\oplus n+1} \twoheadrightarrow \mathcal{L}$, where $\mathcal{L}$ varies over all invertible sheaves on T. In particular, on $\mathbb{P}^n$ itself exists a universal globally generated invertible sheaf $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)$. Its global sections correspond to the hyperplanes in $\mathbb{P}^n$.

The tensor powers $\mathcal{L}^{\otimes m}$, $m \in \mathbb{Z}$, define a family of invertible sheaves with two distinguished properties; a *geometric* and an *algebraic* one:

(i) They induce a quotient presentation of $\mathbb{P}^n$. The corresponding vector bundles $\ell_m \colon L_m \to \mathbb{P}^n$ have rank 1, so that the associated principal homogeneous spaces $p_m \colon E_m \to \mathbb{P}^n$ are obtained by restriction to the complement of the zero section. Indeed, one checks that these bundle projections coincide with the original quotient map $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ and the structure group $\mathbb{G}_m$ operates on the fibers via $x \mapsto x^{-m}$. In particular, E_m is *quasiaffine* and one recovers $\mathbb{P}^n$ as the quotient $E_m/\mathbb{G}_m$.

(ii) Another property of the family $\mathcal{L}^{\otimes m}$, $m \in \mathbb{Z}$, is directly related to the category of (quasi-) coherent sheaves on $\mathbb{P}^n$. For every coherent sheaf $\mathcal{F}$, a sufficiently large twist $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$, $m \gg 0$, is globally generated. Equivalently, for every quasicoherent sheaf $\mathcal{F}$ there exists a surjective homomorphism $\bigoplus_{i\in I} (\mathcal{L}^\vee)^{\otimes n_i} \twoheadrightarrow \mathcal{F}$ with a set of positive integers $n_i \in \mathbb{N}$. This means that the collection $(\mathcal{L}^\vee)^{\otimes m}$, $m \in \mathbb{N}$, defines a *generating family* for the category of quasicoherent sheaves on $\mathrm{QCoh}(\mathbb{P}^n)$.

Properties (i) and (ii) descend along every immersion of schemes $X \hookrightarrow \mathbb{P}^n$ and therefore make sense for every very ample line bundle $\mathcal{L}$ on X.

However, both properties have natural generalizations of independent interest, even for non-quasi-projective schemes or algebraic spaces and algebraic stacks. For that, let us first briefly discuss suitable generalizations of the property (i).

Let X be a quasicompact and quasiseparated scheme (or an algebraic space, or more generally an algebraic stack). To every locally free sheaf $\mathcal{E}$ on X of rank n (abusively, we shall call this a *vector bundle*) corresponds a principal homogeneous space $p \colon E \to X$ with structure group GL_n, the *frame bundle*. One recovers X from E and the GL_n-action as the quotient $X \simeq E/GL_n$. In particular, the geometry of X is the GL_n-equivariant geometry of E. Therefore, it is a natural problem to determine those vector bundles $\mathcal{E}$ whose associated frame bundle E has "simple" geometry. This strategy becomes important when studying algebraic stacks, rather than algebraic spaces or schemes. Loosely speaking, algebraic stacks locally look like quotients of algebraic spaces by group scheme actions, so it is natural to ask, when there exists a quotient presentation globally, or equivalently, when does there exists a vector bundle $\mathcal{E}$ whose associated frame bundle E is representable by an algebraic space. In that case, one calls X a *quotient stack*. However, algebraic spaces are only étale locally affine schemes and therefore have complicated geometry. Slightly unconventional, we shall call X a *global quotient stack* if there exists a vector bundle $\mathcal{E}$ whose frame bundle E is quasiaffine, meaning that the global geometry of X is largely encoded in the group action of GL_n on E.

Let us now discuss the common generalization of the *algebraic* property (ii) above. An algebraic stack X has the *resolution property* or *enough locally free sheaves of finite type* if every quasicoherent sheaf is a quotient of a filtered direct limit of locally free sheaves of finite type. Equivalently, if X is noetherian, every coherent sheaf is a quotient of a coherent locally free sheaf, and it follows that every coherent sheaf can be resolved by a complex of vector bundles, which is infinite unless X is smooth.

The upshot is that many homological properties of vector bundles carry over to a large class of coherent sheaves, leading to essential simplifications in the theory of perfect complexes [TT90] in algebraic K-theory. In particular, it ensures that Grothendieck's K-group $K_0^{\mathrm{naive}}(X)$ and Quillen's extension thereof $K_*^{\mathrm{naive}}(X)$ coincide with the right K-groups $K_*(X)$, invented by Thomason. This has direct applications for the interplay between homological and geometrical problems. For example, it appears in the study of triangulated categories of singularities [Orl06] and of derived equivalences of schemes and stacks [Kaw04].

When considering the resolution property for the classifying stack BG of an affine group scheme G, this gives a necessary condition for the equivariant embeddability of schemes into projective spaces generalizing the work of Sumihiro [Sum75] and to Hilbert's 14th problem — the finite generation of invariant rings [Tho87, §3] .

For an introduction to the resolution property of schemes and stacks, we refer the reader to Totaro's article [Tot04] and to [Tho87] for the case of quotient stacks.

It turns out that both generalizations of (i) and (ii) are *equivalent* in a very natural way. By Thomason's equivariant resolution theorem [Tho87, 2.18], it is known that every global quotient stack X has the resolution property. Strikingly, Totaro showed that the converse also holds if X is normal, noetherian and has affine stabilizer groups at closed points [Tot04, Thm. 1.1]. The latter restriction is reasonable since every global quotient stack has *affine diagonal* [Tot04, 1.3] and hence affine stabilizer groups over all points. Besides, the resolution property is not meaningful for the geometry of an algebraic stack having non-affine stabilizers; e.g. the category of quasicoherent sheaves on the classifying stack BE of an elliptic curve is trivial.

We shall see in this work that actually the normal hypothesis can be removed and even that the noetherian assumption is unnecessary (at least if X is an quotient stack like an algebraic space or a scheme, or if X is of finite presentation over the base). However, our original motivation was to understand the structure of the family of locally free sheaves which appear in the resolution property. To our knowledge the size and the tensor structure thereof has been ignored so far.

In analogy to the family of invertible sheaves $\mathcal{L}^{\otimes n}$, $n \in \mathbb{Z}$, above, we shall associate to a vector bundle $\mathcal{E}$ on X a family of vector bundles that are obtained by taking subsheaves of finite direct sums, tensor powers and duals of $\mathcal{E}$; we call these *tensorial constructions* adopting the notion of Broshi [Bro10].

Our main result states that on an algebraic stack X a vector bundle $\mathcal{E}$ has quasiaffine frame bundle E if and only if the latter family is a generating family for the category of quasicoherent sheaves on X. If the base is of characteristic 0 or if $\mathcal{E}$ splits as a direct sum of invertible sheaves, then we shall see that E is quasiaffine if the subfamily $\mathcal{E}^{\otimes i} \otimes (\mathcal{E}^\vee)^{\otimes j}$, where $i, j \geq 0$, is a generating family. In case that $\mathcal{E}$ is invertible, one recovers the original properties (i) and (ii).

These results are even new if X is a scheme, whereas our proof depends heavily on the existence of the classifying stack BGL_n and we see no way for providing a purely scheme theoretic proof. In fact, we show that it suffices to prove the case $X = BGL_n$ and use the well-known result of the representation theory of GL_n, that every rational GL_n-representation can be reconstructed by taking subrepresentations of suitable tensorial constructions of the standard representation.

The resolution property is satisfied for a vast class of schemes and stacks. It is known that it holds for schemes that are projective or quasiprojective over a noetherian base ring due to the existence of an ample line bundle, or more generally if X has an ample family of line bundles (that is a family of invertible sheaves where the whole collection behaves like an ample line bundle, cf. [SGA 6, II.2.2] and [BS03]). Such schemes are called *divisorial* and the existence of an ample family characterizes the property that resolutions are made up from direct sums of line bundles which correspond to anti-effective Cartier divisors. This class includes all noetherian, regular and separated schemes by Kleiman's Theorem (see [Bor67], or independently by Illusie [SGA 6, II.2.2]) or more generally all noetherian $\mathbb{Q}$-factorial schemes with affine diagonal (as observed by Brenner and Schröer [BS03]).

However, there are many non-singular and non-quasiprojective schemes starting in dimension 2. It can happen that X has no effective Cartier divisors or, even worse, that there exist no non-trivial invertible sheaves at all, leaving no hope for constructing resolutions by line bundles (see [Sch99] for normal surfaces and [Eik92], [Ful93], [Pay09] for toric threefolds). Nevertheless, Schröer and Vezzosi showed in [SV04] that every normal separated algebraic surface has the resolution property by gluing local resolutions.

In the larger category of algebraic spaces the problem of the resolution property is much more difficult to solve. The étale topology is too fine to be reasonably connected to the topology generated by Weil- or Cartier divisors. It is not known whether the resolution property holds for proper algebraic spaces over a field that are smooth and have dimension ≥ 3, or for those that are normal and have dimension 2.

In the category of algebraic stacks one gains a further level of complexity due to the presence of stabilizer groups. Here appears the first example of a non-regular but normal noetherian algebraic stack that does not satisfy the resolution property, yet has affine diagonal: Edidin, Hassett, Kresch and Vistoli showed that a $\mathbb{G}_m$-gerbe that corresponds to a non-torsion element in the cohomological Brauer group of a scheme does not satisfy the resolution property (see Example 4.3.8). However, all known counterexamples of algebraic stacks with quasifinite diagonal do not have affine diagonal (see [SV04, §4] or [Tot04, §8]).

If the algebraic stack has finite diagonal then the coarse moduli space X_{cms} exists and its geometry is closely related to the geometry of X. In fact, if X_{cms} is a scheme, then the set of Cartier divisors on X is much better behaved. For example, Totaro verified a suitable generalization of Kleiman's Theorem for smooth orbifolds [Tot04, 1.2]. Many moduli stacks that appear in nature even have a quasiprojective coarse moduli space, so that X_{cms} has the resolution property, and for these stacks, being a quotient stack or having the resolution property is equivalent. For a recent discussion of that matter we refer the reader to [Kre09].

The situation is completely different in the analytic category. Schuster verified the resolution property for all compact complex surfaces [Sch82]. However, it fails for generic complex tori of dimension ≥ 3 as all rational higher Chern classes of holomorphic vector bundles vanish, so that there exists only few of them [Voi02, A.5].

Whereas in the algebraic setting, the question is widely open, and more difficult since the category of (quasi-) coherent sheaves is much more flexible. For example, the 2-functor $X \mapsto \mathrm{QCoh}(X)$ that assigns to each algebraic stack with affine diagonal its category of quasicoherent sheaves is a *faithful embedding* into the 2-category of tame, complete, abelian tensor categories by Lurie [Lur05].

Contrarily, the category of coherent sheaves on generic complex $K3$ surfaces or two-dimensional tori is independent of the complex structure by Verbitsky [Ver08]. For that reason, we work in the category of algebraic stacks throughout this work.

Totaro asked, whether the resolution property holds for noetherian schemes, algebraic spaces or more generally, algebraic stacks with quasifinite stabilizer groups and affine diagonal [Tot04, p. 3, Question 1]. This is not even known for normal toric varieties. Recently, Payne started the investigation of toric threefolds and constructed examples of proper schemes where non-trivial vector bundles of rank ≤ 3 do not exist [Pay09, 1.1]. As the resolution property descends along immersions to every subspace, a necessary condition for the resolution property of any space is its validity for *every* subspace. In case of proper threefolds a positive solution to Totaro's question would imply that many non-normal and non-reduced separated algebraic surfaces have the resolution property.

In the first part of the work we shall verify the last implication. Precisely, we show that the resolution property holds for *all* separated algebraic surfaces, generalizing the methods of Schröer and Vezzosi. However, its proof is not a simple reduction to the normal case by taking the normalization of the reduction.

One can even say, that the central issues in the study of the resolution property of schemes is the descent problem of vector bundles along proper birational maps and the behavior of the resolution property under deformations. On the one hand, the resolution property eventually holds after sufficiently many blow ups (e.g. by Chow's Lemma in the separated case). On the other hand, the counterexamples imply that it is not a birational invariant. It is not clear, whether the resolution property is stable under deformations. Voisin's analytic counterexample shows that this is in general wrong if one takes complex deformations into account, as a projective 3-torus can be deformed into a general one. However, an affirmative result for *algebraic* deformations would have useful applications. For example, in case that $X = B_A G$ is the classifying stack of an affine smooth group scheme G over an artinian ring A, it is known that the resolution property holds for the reduction X_{red}, and by Thomason the resolution property of the deformation X would imply the embeddability of G into GL_n. However, this seems to be unknown[1].

Unfortunately, the case of dimension ≥ 3 remains completely open. We believe that the resolution property always holds in codimension ≤ 2 for arbitrary separated algebraic schemes; this means, every point has an open neighborhood that satisfies the resolution property and whose complement has codimension ≥ 3.

A verification of this conjecture would simplify the case of general threefolds, yet our methods of the surface case are insufficient in the absence of properness.

[1] According to a recent discussion on `http://mathoverflow.net/questions/22078/` initiated by B. Conrad, this is not even clear for the ring of dual numbers $A = k[\varepsilon]$.

Description of the chapters

This work is divided in two parts and split in seven chapters. In the first part we shall investigate the resolution property of schemes and prove it for algebraic surfaces (this is essentially contained in [Gro10]).

We start in chapter 1 with a discussion of divisorial schemes, explain the classical approaches to prove the resolution property and show the limitations thereof. As a first result we prove that for every scheme which is separated and of finite type over a noetherian ring every point has a divisorial neighborhood U whose complement has codimension ≥ 2. We show that U has an ample family of invertible sheaves $\mathcal{L}_i$, $i \in I$, that is obtained by gluing an *ample* line bundle: There exists a proper, birational map $f\colon X' \to X$ and an ample line bundle $\mathcal{L}'$ on X' which is isomorphic to each $\mathcal{L}_i$ over $f^{-1}(U)$. This is accomplished by using Ferrand's pinching techniques (see appendix A) in combination with deformation theory of vector bundles.

The upshot is that by extending the ample family to the whole scheme, one obtaines a family of coherent sheaves $\mathcal{F}_i$ (which is called an *almost ample family*) that behaves like an ample family of line bundles away from closed subsets of codimension ≥ 2. It carries a weak form of positivity that can be measured cohomologically at least if X is proper. This will play an important role in order to handle cohomological obstructions in the following chapter.

Theorem (1.4.2)**.** *Let X be a scheme that is separated and of finite type over a noetherian ring. Then there exists an almost ample family $\mathcal{F}_1, \dots, \mathcal{F}_n$ of coherent sheaves.*

In chapter 2 we show as our first main result: Namely, that a large class of two-dimensional schemes has the resolution property:

Theorem (2.0.1)**.** *Let X be a 2-dimensional scheme that is proper over a noetherian ring. Then X has the resolution property.*

If the base ring is a field, the hypothesis "proper" "might be replaced by separated and of finite type" by Nagata compactification.

In order to prove the theorem, we generalize Schröer and Vezzosi's method of gluing local resolutions to the case of non-normal and non-reduced surfaces and describe the obstructions to gluing in terms of certain cohomology groups of coherent sheaves. The existence of the candidates for the right local resolutions is derived from an appropriate generalization of the Bourbaki Lemma for arbitrary noetherian local rings, which is proven using the basic element theory of Evans and Griffith [EG85].

Finally, we construct a series of vector bundles that descend from a suitably chosen Chow cover and will serve as the candidates for the first syzygies appearing in the gluing process. We close the chapter with the proof of the main theorem using the almost ample family constructed in chapter 1.

In the second part of this work we shall prove as our second main result the equivalence of the generalizations (i) and (ii) mentioned above in the category algebraic stacks.

For that, we lay the ground in chapter 3 and study generating families of finitely presented sheaves for the category of quasicoherent sheaves on algebraic stacks; we call them *generating sheaves*. Their existence is equivalent to the *completeness property* which was studied by Rydh [Ryd10b] and is always true for (pseudo-) noetherian stacks.

We also introduce a relative version of generating sheaves for quasicompact morphisms of algebraic stacks and prove that they share the analogous properties of relatively ample line bundles. In case that the morphism is the natural map $X \to X_{\mathrm{cms}}$ of an Deligne-Mumford stack to its coarse moduli space, one recovers the definition of a generating sheaf in the sense of Olsson and Starr [OS03]; in fact, our approach was greatly inspired by reading their paper.

At the end of the chapter, we study the influence of affine fppf coverings. First, we prove that generating families are always preserved under *finite* fppf coverings. Secondly, we show that every quasicompact and quasiseparated algebraic stack with affine diagonal has the *flat resolution property*:

Theorem (3.5.5). *Let X be a quasicompact algebraic stack with* affine diagonal. *Then every quasicoherent $\mathcal{O}_X$-module is a quotient of a* flat *quasicoherent $\mathcal{O}_X$-module.*

In chapter 4 we set up the framework for the study of the resolution property of algebraic stacks. Restricting the results of the previous chapter to the case of *locally free* generating sheaves, we introduce the notion of the *relative resolution property.* This view point is essential for comparing the (absolute) resolution property between different algebraic stacks and was already adopted by Thomason [Tho87, Remark 2.7] (in non-stacky language).

Generalizing the arguments of Thomason, we prove that a large class of algebraic stacks, having an affine covering by a regular noetherian scheme of low dimension, satisfy the resolution property. In case of classifying stacks, one recovers the classical results for group schemes. However, the arguments also apply to gerbes and even to more general stacks.

In the main part of this chapter we study the equivariant resolution property; that is, the resolution property of quotients $X = Y/G$, or equivalently the resolution property of morphisms $X \to BG$ where G is in arbitrary affine, flat and finitely presented group scheme. We transfer Thomason's work in the language of stacks, discuss separateness properties and lay the ground for actions by the general linear group which is needed in the following chapter.

In chapter 5 we shall characterize algebraic stacks being strongly representable by quasiaffine schemes if the structure sheaf is generating:

Theorem (5.3.2). *Let X be an algebraic stack over an algebraic space S with* affine *stabilizer groups at closed points. Then $X \to S$ is representable by a quasiaffine morphism if and only if $\mathcal{O}_X$ is generating over S.*

This is a central result and explains why quasiaffine schemes appear naturally when studying the resolution property. Following Totaro's arguments, our proof essentially reduces to the case that the algebraic stack is representable by an algebraic space and has a finite and finitely presented (but not necessarily flat) covering by a quasiaffine scheme. Using Ferrand's pinching results we show that such an algebraic spaces is always representable by an AF-scheme.

Actually, our method allows to prove a much more general result by providing a variant of Chevalley's theorem for AF-schemes (i.e. schemes where every finite set of points is contained in an affine open neighborhood):

Theorem (5.1.5). *Let $f\colon Z \to X$ be an integral and surjective morphism of algebraic spaces with finite topological fibres over a base algebraic space S. Then $Z \to S$ is an AF-morphism if and only if $X \to S$ is an AF-morphism.*

Chapter 6 is dedicated to the investigation of the tensor structures and the size of locally free generating families on quasicompact and quasiseparated algebraic stacks with affine stabilizer groups. After recalling the correspondence between vector bundles $\mathcal{E}$ and frame bundles $p\colon E \to X$, we associate to every vector bundle

$\mathcal{E}$ a set of vector bundles, obtained by applying all tensorial constructions t on $\mathcal{E}$ and call this the *tensor hull* $|\mathcal{E}|$. Moreover, we define the larger family $\langle\mathcal{E}\rangle$ that additionally contains all subsheaves of the various $t(\mathcal{E})$ which are locally split with respect to the smooth covering p; this we denote by the *local tensor hull* $\langle\mathcal{E}\rangle$.

As a first result here, we show that E is quasiaffine if and only if the local tensor hull $\langle\mathcal{E}\rangle$ is a generating family. We shall say that $\mathcal{E}$ is a *tensor generator* for X.

Theorem. *(6.2.12) Let $X \to S$ be a quasicompact and quasiseparated morphism of algebraic stacks and let $\mathcal{E}$ be a vector bundle on X. Then the following conditions are equivalent:*

(i) $\mathcal{E}$ is a tensor generator for X over S.

(ii) The frame bundle of $\mathcal{E}$ has quasiaffine total space over S.

Moreover, if these conditions are satisfied, then the diagonal $\Delta_{X/S}$ is affine.

The latter holds already for the tensor hull $|\mathcal{E}|$ if S is of characteristic zero or if $\mathcal{E}$ is a split direct sum of line bundles.

Coming back to the example from the beginning of this introduction, we see that if X is a scheme and $\mathcal{E}$ invertible, then $\mathcal{E}$ is a tensor generator if and only if $\mathcal{E}^{\otimes n}$ is ample for some $n \in \mathbb{Z}$.

Next, we generalize this result to finite families $(\mathcal{E}_1, \ldots, \mathcal{E}_n) =: \mathcal{E}_I$ of vector bundles. By a similar argument follows that the tensor product family $\langle\mathcal{E}_I\rangle := |\mathcal{E}_1| \otimes \cdots \otimes |\mathcal{E}_n|$ (calculated objectwise) is a generating family if and only if the fiber product $E_1 \times_X \cdots \times_X E_n$ of the associated frame bundles $E_i \to X$ is quasiaffine. In this case we call the family a *tensor generating family*. We will see that a sheaf is a tensor generator if and only if it is a direct sum of a tensor generating family.

In order to tackle the *a priori* possibly large locally free generating families which appear in the resolution property we dare to take *infinite* families of vector bundles $\mathcal{E}_I := (\mathcal{E}_i)_{i\in I}$ into account. Taking unions of the former tensor multiplied families we associate to $\mathcal{E}_I$ a big family of vector bundles $\langle\mathcal{E}_I\rangle$. With this definition it is easy to verify that the resolution property is equivalent to the existence of a possibly infinite tensor generating family. We shall prove that there exists always a *finite* tensor generating subfamily (at least if X is noetherian or finitely presented over the base). This is achieved by approximating infinite fiber products of the associated frame bundles $E_i \to X$ if X is of finite presentation over the base. If the latter is not satisfied we avoid this infinite limit argument and show by hand that for sufficiently large but finite $J \subset I$ the fiber product $\prod_{i\in J}(E_i/X)$ is eventually quasiaffine, i.e. $\mathcal{E}_J$ is a *finite* tensor generating family. However, the latter argument works only with additional assumptions on the representability of these finite fiber products.

In the final part of this chapter we show that the preceding results fit together in a natural way by giving a short proof of Totaro's Theorem, generalized to non-normal and non-noetherian algebraic stacks:

Theorem. *(6.3.1) Let $X \to S$ be a quasicompact and quasiseparated morphism of algebraic stacks with S quasicompact which satisfies one of the following hypothesis:*

(a) X is noetherian with affine stabilizer groups at closed S-points and S is affine.

(b) $X \to S$ is a quotient stack; for instance, if $X \to S$ is representable.

(c) $X \to S$ is of finite presentation and has relative affine stabilizer groups at geometric S-points.

Then the following assertions are equivalent:

(i) $X \to S$ has the resolution property.

(ii) $X \to S$ is a global quotient stack.

In the last chapter 7 we give a brief discussion of applications of our results and motivate further developments, including the study of finite flat scheme covers and the behavior of the resolution property with respect to deformations.

Conventional notations

By an *algebraic stack* we mean an Artin stack with separated and quasicompact diagonal in the sense of [LMB00]. In particular, it is quasiseparated. However, we often explicitly stress that a morphism or an algebraic stack should be quasiseparated when this is a necessary condition. Presumably, the separated condition on the diagonal can be weakened in many cases and be replaced by quasiseparated.

By a *global quotient stack*, we mean a quotient stack $[X/GL_n]$ for some quasiaffine scheme acted on by GL_n for some $n \in \mathbb{N}_0$.

A *family of sheaves* $(\mathcal{E})_{i\in I}$ will be often denoted by $\mathcal{E}_I$. If the latter appears in symbolic manipulations we always mean the objectwise definition. For example, the pullback $f^*\mathcal{E}_I$ by a morphism f is defined as the family $(f^*\mathcal{E}_i)_{i\in I}$.

Contents

Part 1

Construction of locally free resolutions on schemes

CHAPTER 1

Divisoriality in codimension one

1.1. Resolution by ideal sheaves

The problem of constructing a locally free resolution $\varphi\colon \mathcal{E} \twoheadrightarrow \mathcal{M}$ for a given quasicoherent sheaf $\mathcal{M}$ comprises the construction of the vector bundle $\mathcal{E}$ and the quotient map φ. Trivially, both can be solved locally, but in general there is no reason why the local resolutions should glue to a global one.

Let us describe the approach of constructing locally free resolutions by *invertible* sheaves following [BS03, 1.7] building on the idea of Kleiman [Bor67], or independently Illusie [SGA 6]. Apart from the case of normal, separated algebraic surfaces, which was settled in [SV04, 2.1], this provides a proof of the resolution property for schemes for all known cases. Furthermore, we will see the limitations of this construction, but also infer an important reduction step in Proposition 1.1.2.

Let X be a noetherian scheme, $\mathcal{M}$ a coherent sheaf and fix an arbitrary germ of $\mathcal{M}$ that is represented by a section $s\colon \mathcal{O}_U \to \mathcal{M}|_U$ for some dense affine open set $U \subset X$. Denote by $D = X - U$ the reduced complement given by a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$; it has codimension 1 if X has affine diagonal [EGA IV.4, 21.12.7]. By removing the singularities of the local sections of $\mathcal{I}$ near D, we can extend s to a homomorphism $\varphi_s\colon \mathcal{I}^n \to \mathcal{M}$ of coherent sheaves for a sufficiently large power $n \in \mathbb{N}$ whose image contains the previously chosen germ [EGA $\mathrm{I}_{2\mathrm{nd}}$, I.6.9.17]. Using that X is quasicompact and that $\mathcal{M}$ is locally generated by finitely many sections s_i, we can repeat this procedure finitely many times on the affine complements of suitable Weil divisors D_i and get a surjection $\bigoplus_i \mathcal{I}_i^{n_i} \twoheadrightarrow \mathcal{M}$ by summing up the extended sections φ_{s_i}. From this point of view, we may say that every noetherian scheme with affine diagonal has *enough* Weil divisors. However, the domain of the latter homomorphism is only locally free if each ideal $\mathcal{I}_i^{n_i}$ is invertible; that is $X - U$ carries the subscheme structure of an effective Cartier divisor.

This shows that schemes where a base of the Zariski topology is given by complements of effective Cartier divisors satisfy the resolution property and the resolutions can be made up by invertible sheaves; these schemes are called *divisorial* (c.f. [Bor63], [Bor67]). From the discussion above, one deduces that a vast class of schemes satisfies this property:

(1.1.1) Theorem ([EGA II], [Bor67] and [BS03, 1.7])**.** *Let X be a quasicompact scheme that meets one of the following requirements:*

(i) X is separated and has an ample line bundle; for example, if X is quasi-projective over some noetherian ring.

(ii) X is noetherian, locally $\mathbb{Q}$-factorial and has affine diagonal.

Then X is divisorial.

In general, it is not possible to construct locally free resolutions by invertible sheaves. Even worse, it can happen that there are no non-trivial line bundles at all due to the presence of singularities.

Nevertheless, we may hope to find resolutions by vector bundles of sufficiently large rank. A useful consequence of the previous discussion is the following reduction principle which enables us to reduce the construction of various locally free resolutions to a *single* one, as observed by Schröer and Vezzosi [SV04, Prop. 2.2].

(1.1.2) Proposition. *Let X be a quasicompact and quasiseparated scheme. Then there exists a quasicoherent $\mathcal{O}_X$-module of finite presentation $\mathcal{F}$ such that every quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$ of finite presentation admits a surjection $(\mathcal{F}^{\otimes n})^{\oplus m} \twoheadrightarrow \mathcal{M}$ for sufficiently large $n, m \in \mathbb{N}$.*

Proof. Let us first assume that X is noetherian. Then choose a finite covering by open affines $U_i \subset X$ whose reduced complements are given by coherent ideal sheaves $\mathcal{I}_i \subset \mathcal{O}_X$ and set $\mathcal{F} = \bigoplus_i \mathcal{I}_i$. Since every power $\mathcal{I}_i^n$ is a quotient of tensor product $\mathcal{I}^{\otimes n}$, we see that every finite direct sum $\bigoplus_j \mathcal{I}_{i_j}^{n_j}$ is a quotient of $(\mathcal{F}^{\otimes n})^{\oplus m}$ for sufficiently large $n, m \in \mathbb{N}$. However, by the previous discussion we can always resolve an arbitrary coherent sheaf by the former. This settles the noetherian case.

By noetherian approximation there exists an affine morphism $f\colon X \to X_0$ with X_0 noetherian [TT90, Thm. C.9]. Denote by $\mathcal{F}_0$ the coherent $\mathcal{O}_{X_0}$-module that satisfies the claimed property for X_0. Since $f_*\mathcal{M}$ is the direct limit of coherent submodules, we find a map $\varphi\colon (\mathcal{F}_0^{\otimes n})^{\oplus m} \to f_*\mathcal{M}$ such that the composition of $f^*\varphi$ with the surjective evaluation map $f^*f_*\mathcal{M} \to \mathcal{M}$ is surjective. So $\mathcal{F} = f^*\mathcal{F}_0$ satisfies the claimed property. □

1.2. Ample families

The collection of Cartier divisors on a divisorial schemes behaves like the hyperplane sections on quasiprojective schemes. Extending the notion of an ample line bundle one introduces the concept of ample families of line bundles:

(1.2.1) Definition. A family of invertible $\mathcal{O}_X$-modules $\mathcal{L}_i$, $i \in I$, on a quasicompact and quasiseparated scheme X is called *ample* if the following equivalent conditions are satisfied:

(i) The family $\{(\mathcal{L}_i^\vee)^{\otimes n} | \, i \in I, m \in \mathbb{N}\}$ is *generating* for X: For every quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$ there exist $i_j \in I$, $n_j \in \mathbb{N}$, where j runs over some set of indices J, and a surjection

$$\bigoplus_{j \in J} (\mathcal{L}_{i_j}^\vee)^{\otimes n_j} \twoheadrightarrow \mathcal{M}.$$

(ii) Same as in (i) but with $\mathcal{M}$ of finite type and I finite.

(iii) Same as in (i) but with $\mathcal{M}$ a sheaf of ideals of finite type and I finite.

(iv) The open sets X_s, where s runs over all global sections of $\mathcal{L}_i^{\otimes n}$, $i \in I$, $n \in \mathbb{N}$, form a base of the topology of X.

(v) Same as in (iv) but the X_s form a covering of X by affine open sets.

(vi) Same as in (v) but the X_s form a covering of X by quasiaffine open sets.

If X has an ample family of line bundles, then it is called *divisorial*.

The equivalence of (i)-(v) is shown in [SGA 6, II.2.2.3]. However, the proof works also for quasiaffine subsets X_s since one only needs that every quasicoherent sheaf is globally generated [EGA II, 5.1.2.c'] which shows that (vi) is also an equivalent condition.

There is also a relative version of this definition for a morphism of schemes as for ample line bundles, and this satisfies the permanence properties analogous to relatively ample line bundles.

1.3. Divisoriality by blowing up

For a general scheme, there is no reason for the existence of an ample line bundle. However, the next proposition establishes that every quasicompact and quasiseparated scheme is birational to a divisorial one, by blowing up a suitable set of Weil divisors. We call a map of schemes $f\colon Y \to X$ *birational* if there exists a dense open subset $U \subset X$ such that $f^{-1}(U)$ is dense in Y and $f|_{f^{-1}(U)}$ is an isomorphism.

(1.3.1) Proposition. *Let X be a quasicompact and quasiseparated scheme. Then there exists a blow-up $f\colon Y \to X$ of a finitely presented closed subscheme such that Y has an ample family of $\mathcal{O}_Y$-modules.*

Proof. By noetherian approximation there exists an affine morphism $p\colon X \to X_0$ with X_0 noetherian. If $f_0\colon Y_0 \to X_0$ is the blow-up of a finitely presented closed subscheme $Z_0 \subset X_0$ then the blow-up $f\colon Y \to X$ of the finitely presented subscheme $Z := p^{-1}(Z_0)$ is the schematic closure of $X - Z$ in $Y_0 \times_{X_0} X$. Therefore, it suffices to prove that Y_0 has an ample family. So we may assume that X is noetherian.

Choose a finite covering $U_1, \ldots, U_n$ of dense affine open sets of X and provide each $Z_i = X - U_i$ with the reduced subscheme structure. Put $Y_0 = X, g_0 = \mathrm{id}_Y$ and let $g_k\colon Y_k \to Y_{k-1}$ be the blow-up of the inverse image $(g_{k-1} \circ \cdots \circ g_0)^{-1}(Z_k)$ for $k = 1, \ldots, n$. Consequently the inverse images of $Z_1, \ldots, Z_n$ under the composition $f\colon Y = Y_k \to Y_{k-1} \to \cdots \to Y_0 = X$ are effective Cartier divisors $D_1, \ldots, D_n$, and f can be written as a single blow-up of a finitely presented closed subscheme $Z \subset X$ by [RG71, 5.1.5].

Let $\mathcal{L}$ an f-ample $\mathcal{O}_Y$-module. We claim that $\{\mathcal{O}_Y(aD_i) \otimes \mathcal{L}^{\otimes m}\}_{i,a,m}$ is an ample family. Let $\mathcal{F}$ be a coherent $\mathcal{O}_Y$-module. Then for $m \gg 0$ the adjunction map $f^*f_*(\mathcal{F} \otimes \mathcal{L}^m) \twoheadrightarrow \mathcal{F} \otimes \mathcal{L}^m$ is surjective. Since $f_*(\mathcal{F} \otimes \mathcal{L}^m)$ is coherent and U_i affine, we can choose a resolution $\mathcal{O}_{U_i}^{\oplus a_i} \twoheadrightarrow f_*(\mathcal{F} \otimes \mathcal{L}^m)|_{U_i}$ for each i which gives rise to a surjection $\mathcal{O}_{f^{-1}(U_i)}^{\oplus a_i} \to f^*f_*(\mathcal{F} \otimes \mathcal{L}^m)|_{f^{-1}(U_i)}$. Each complement $Y - f^{-1}(U_j) = D_j$ is a Cartier divisor, so that the former map extends to Y as map $\mathcal{O}_Y(-b_iD_i)^{\oplus a_i} \to f^*f_*(\mathcal{F} \otimes \mathcal{L}^m)$ [EGA $\mathrm{I}_{2\mathrm{nd}}$, I.6.8.1]. It follows that the composition $\mathcal{O}_Y(-b_iD_i)^{\oplus a_i} \to \mathcal{F} \otimes \mathcal{L}^m$ is surjective on $f^{-1}(U_i)$. As $Y = \bigcup_i f^{-1}(U_i)$ this completes the proof. ☐

Since there exist regular quasicompact and quasiseparated 2-dimensional schemes which do not have the resolution property [SV04, §4] we infer the following result. It is not a consequence of Chow's Lemma because we do not restrict to separated finite type schemes.

(1.3.2) Corollary. *The resolution property of quasicompact and quasiseparated schemes is not a birational invariant.*

1.4. Divisoriality in codimension one

We show that on any separated scheme X that is of finite type over a noetherian ring, each point has a divisorial neighborhood U that contains all points of codimension ≤ 1. More precisely, we seek to construct an ample family that possesses enough positivity for later purpose. This will be a crucial part in the proof that all separated algebraic surfaces satisfy the resolution property.

(1.4.1) Definition. Let X be a noetherian scheme. We call a subset $U \subset X$ *thick* if it contains all points of codimension ≤ 1. A coherent sheaf $\mathcal{L}$ on a noetherian scheme X is called *almost invertible* if it is invertible on some thick open subset. A family of almost invertible sheaves is called an *almost invertible family* if the corresponding thick subsets cover X. An almost invertible family $\mathcal{F}_i$, $i \in I$, with thick open sets V_i is called an *almost ample family* if the following conditions are satisfied:

(i) For every coherent sheaf $\mathcal{M}$ there exists a surjection $\bigoplus_{j\in J} \mathcal{F}_{i_j}^{\otimes n_j} \twoheadrightarrow \mathcal{M}$ for some finite set of indices $i_j \in I$ and powers $n_j \in \mathbb{N}$.

(ii) For every $i \in I$ there exists a blow-up $f\colon X' \to X$ such that X' carries an ample line bundle $\mathcal{L}'$ which is isomorphic to $(f^*\mathcal{F}_i)^\vee$ over $f^{-1}(V_i)$.

Note, that this notion is slightly abusive, since a family of invertibles is called ample, if the tensor powers of their *duals* form a generating family (cf. 1.2.1.(i)).

The whole section is devoted to the proof of the following theorem and will be a direct consequence of Proposition 1.4.13.

(1.4.2) Theorem. *Let X be a scheme that is separated and of finite type over a noetherian ring. Then there exists an almost ample family $\mathcal{F}_1, \ldots, \mathcal{F}_n$ of coherent sheaves.*

(1.4.3) *Remark.* In order to show the resolution property of X, we infer that it suffices to resolve the various sheaves $\mathcal{F}_i^{\otimes n}$ for all sufficiently large $n \in \mathbb{N}$.

The proof of 1.4.13 involves an application of Chow's Lemma, Ferrand's characterization of vector bundles on glued schemes (see appendix A.1) and a part of deformation theory for vector bundles [Ill05, §5]. The latter two techniques are necessary as we are taking non-normal and non-reduced schemes into account. Using this methods we will establish the following existence result as an intermediate step.

(1.4.4) Proposition. *Let X be a separated scheme of finite type over a noetherian ring. Then every point $x \in X$ has a* divisorial thick *neighborhood $V \subset X$ and an ample family of $\mathcal{O}_V$-modules $\mathcal{L}_i$, $i \in I$, can be chosen with the following property:*

There exists a projective morphism $f\colon X' \to X$, that is finite over V and an isomorphism near x such that on its domain exists an ample $\mathcal{O}_{X'}$-module $\mathcal{L}'$ which restricts to ${f_V}^*\mathcal{L}_i$ over $f^{-1}(V)$ for all $i \in I$.

The proof of 1.4.4 is given at the end of section 1.4 and structured as follows: First, take a suitable Chow cover $X_2 \to X$, then apply Stein factorization to get a decomposition $X_2 \to X_1 \to X$ and finally decompose the finite map $X_1 \to X$ over its schematic image $X_1 \twoheadrightarrow X_0 \hookrightarrow X$. Then X can be viewed as a deformation of X_0, and X_1 as the pinching of X_0 along closed subschemes. By removing sufficiently many closed subsets of codimension ≥ 2 from X, one can arrange that the obstruction for lifting and gluing of an invertible sheaf vanish, so that a chosen ample line bundle on X_2 descends to X as family of invertible sheaves on a thick open neighborhood of x.

Gluing ample families along closed subschemes. Consider first a finite birational morphism $f\colon X' \to X$ of noetherian schemes with schematically dense image. Let $Y \subset X$ be the closed subscheme defined by the conductor ideal $\operatorname{Ann}\operatorname{coker}(\mathcal{O}_X \to f_*\mathcal{O}_{X'}) \subset \mathcal{O}_X$ and put $Y' = f^{-1}(Y)$. We have a cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{v} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{u} & X \end{array}$$

which is also cocartesian. For details we refer to Appendix A.

Let us denote by $\mathrm{LFF}(Z) \subset \mathrm{QCoh}(Z)$ the subcategory of locally free sheaves of finite type on a scheme Z. It is shown that there is an adjoint equivalence between $\mathrm{LFF}(X)$ and $\mathrm{LFF}(Y) \times_{\mathrm{LFF}(Y')} \mathrm{LFF}(X')$.

The objects in the latter category are given by triples $(\mathcal{E}, \sigma, \mathcal{F}')$ with a locally free $\mathcal{O}_Y$-module $\mathcal{E}$ of finite type, a locally free $\mathcal{O}_{X'}$-module $\mathcal{F}'$ of finite type and an isomorphism $\sigma\colon g^*\mathcal{E} \to v^*\mathcal{F}'$. The morphisms consists of pairs of morphisms of quasicoherent sheaves satisfying a compatibility condition after pullback on Y'. The natural functor $T\colon \mathrm{LFF}(X) \to \mathrm{LFF}(Y) \times_{\mathrm{LFF}(Y')} \mathrm{LFF}(X')$ that maps a vector bundle $\mathcal{F}$ to the triple

$$(u^*\mathcal{F},\ g^*u^*\mathcal{F} \xrightarrow{\text{can}} v^*f^*\mathcal{F},\ f^*\mathcal{F})$$

has a right adjoint $S\colon \mathrm{LFF}(Y) \times_{\mathrm{LFF}(Y')} \mathrm{LFF}(X') \to \mathrm{LFF}(X)$. Now by Theorem A.1 the adjoint functors S and T define an equivalence.

We use this description of quasicoherent sheaves on a pinched scheme to glue invertible sheaves having a section outside a closed subset of the conductor subscheme. We call a subset $Z \subset X$ of a topological space X *nowhere dense* if Z does not contain any generic point of X.

(1.4.5) Proposition. *With the previous notation, suppose that there exists an invertible $\mathcal{O}_Y$-module $\mathcal{M}$, an invertible $\mathcal{O}_{X'}$-module $\mathcal{L}'$ and sections $t \in \mathrm{H}^0(Y, \mathcal{M})$ and $s' \in \mathrm{H}^0(X', \mathcal{L}')$ satisfying $g^{-1}(Y_t) = X'_{s'} \cap Y'$. Then outside a nowhere dense closed subset $Z \subset Y$ the sheaves $\mathcal{M}$ and $\mathcal{L}'$ glue to an invertible $\mathcal{O}_X$-module $\mathcal{L}$ having a global section s that restricts to t and s' respectively. More precisely, there exists an invertible $\mathcal{O}_{X-Z}$-module $\mathcal{L}$ such that*

(i) $\mathcal{L}|_{Y-Z} \simeq \mathcal{M}|_{Y-Z}$ *and* ${f_{X-Z}}^*\mathcal{L} \simeq \mathcal{L}'|_{X'-f^{-1}(Z)}$.

(ii) *There is a section* $s \in \mathrm{H}^0(X - Z, \mathcal{L})$ *with* $s|_{Y-Z} = t|_{Y-Z}$ *and* ${f_{X-Z}}^*s = s'|_{X'-f^{-1}(Z)}$.

Proof. Let $\sigma\colon g^*\mathcal{O}_Y \xrightarrow{\simeq} v^*\mathcal{O}_{X'}$ be the natural isomorphism induced by $fv = ug$. Then the natural isomorphisms $u^*\mathcal{O}_X \xrightarrow{\simeq} \mathcal{O}_Y$, $f^*\mathcal{O}_X \xrightarrow{\simeq} \mathcal{O}_{X'}$ give rise to a canonical isomorphism $\psi\colon T(\mathcal{O}_X) \xrightarrow{\simeq} (\mathcal{O}_Y, \sigma, \mathcal{O}_{X'})$.

Consider now the given sections as morphism $t\colon \mathcal{O}_Y \to \mathcal{M}$ and $s'\colon \mathcal{O}_{X'} \to \mathcal{L}'$. We try to find an isomorphism of $\mathcal{O}_{Y'}$-modules $\tau\colon g^*\mathcal{M} \xrightarrow{\simeq} v^*\mathcal{L}'$ that fits in a commutative diagram

$$\begin{array}{ccc} v^*\mathcal{O}_{X'} & \xrightarrow{v^*s'} & v^*\mathcal{L}' \\ \sigma\uparrow & & \uparrow\tau \\ g^*\mathcal{O}_Y & \xrightarrow{g^*t} & g^*\mathcal{M} \end{array} \tag{1.4.5.1}$$

so that the pair $\varphi := (t, s')$ defines a morphism $(\mathcal{O}_Y, \sigma, \mathcal{O}_{X'}) \to (\mathcal{M}, \tau, \mathcal{L}')$ in $\mathrm{LFF}(Y) \times_{\mathrm{LFF}(Y')} \mathrm{LFF}(X')$. This would give a morphism of $\mathcal{O}_X$-modules

$$s\colon \mathcal{O}_X \to ST(\mathcal{O}_X) \xrightarrow{S(\psi)} S(\mathcal{O}_Y, \sigma, \mathcal{O}_{X'}) \xrightarrow{S(\varphi)} \mathcal{L}.$$

with $T(s) = \varphi \circ \psi$, i.e. $s|_Y = t$ and $f^*s = s'$ as desired.

The goal is to construct τ on an open subset $Y' - g^{-1}(Z') \subset Y'$ where $Z \subset Y$ is a suitable closed subset that is nowhere dense in Y (i.e. contains no generic point of Y).

First, observe that there exists a dense open subset $U' \subset Y'$ where both $g^*\mathcal{M}$ and $v^*\mathcal{L}'$ are isomorphic since they are invertible. Then the closed set $Z_1 = g(Y' - U')$ contains no generic point of Y as g is quasifinite and we obtain an isomorphism by restriction:

$$\tau_1\colon g^*\mathcal{M}|_{Y'-f^{-1}(Z_1)} \xrightarrow{\simeq} v^*\mathcal{L}'|_{Y'-f^{-1}(Z_1)}.$$

So outside Z_1 we can define an isomorphism τ but it does not necessarily fit in a commutative diagram (1.4.5.1). If both horizontal maps vanish, the commutativity is trivial, so it suffices to check the commutativity over $Y'_{g^*t} \cup Y'_{v^*s'}$ which is equal to $g^{-1}(Y_t) = X'_{s'} \cap Y'$ by hypothesis.

For that define $Z_2 := \overline{Y_t} - Y_t$ where the closure is taken in Y. It is a closed subset of Y that does not contain any generic points of Y. Then $Z := Z_1 \cup Z_2$ is a nowhere dense closed subset of Y, so that $Y - Z$ decomposes as a disjoint union of open subsets $U \subset Y_t$ and $V \subset Y - \overline{Y_t}$.

Consequently $Y' - g^{-1}(Z)$ is the disjoint union of two open sets $g^{-1}(U) \subset g^{-1}(Y_t)$ and $g^{-1}(V) \subset Y'$, such that both g^*t and v^*s' are isomorphism on the former and both vanish on the latter. Therefore we may define τ with $\tau = v^*s' \circ \sigma \circ (u^*t)^{-1}$ on $g^{-1}(U)$ and with $\tau = \tau_1$ on $g^{-1}(V)$, so that τ is an isomorphism and fits in the commutative diagram (1.4.5.1) over $X - Z$. □

We use Proposition 1.4.5 to show that the pair of an sufficiently ample line bundle and a section descend along a finite birational morphism:

(1.4.6) Lemma. *Let $f\colon X' \to X$ be a finite birational morphism of quasicompact schemes with schematically dense image, such that X' is separated and carries an ample $\mathcal{O}_{X'}$-module $\mathcal{L}'$, and let $Y \subset X$ be the closed subscheme defined by the conductor ideal.*

Then for every closed point $x \in X$ with sufficiently small open neighborhood $W \subset X$ there exists a nowhere dense closed subset $Z \subset Y - W$ and on $V := X - Z$ an invertible $\mathcal{O}_V$-module $\mathcal{L}$ with a section $s \in \mathrm{H}^0(V, \mathcal{L})$, such that ${f_V}^*\mathcal{L} \simeq \mathcal{L}'^{\otimes n}|_{f^{-1}(V)}$ for some $n \in \mathbb{N}$ and $x \in V_s \subset W$.

Proof. *Case $x \in Y$:* By shrinking W we may assume that W is quasiaffine, and hence $Y \cap W$ is quasiaffine. Then by removing an appropriate nowhere dense closed subset $Z_1 \subset Y$ disjoint to $Y \cap W$ we may assume that Y is quasiaffine, too.

So $\mathcal{O}_Y$ is ample and there is a section $t \in \mathrm{H}^0(Y, \mathcal{O}_Y)$ such that $x \in Y_t \subset Y \cap W$. Replacing W with $W \cap (X - (Y - Y_t))$ we may assume that $Y_t = Y \cap W$. Then $f^{-1}(x) = g^{-1}(x) \subset g^{-1}(Y_t) = Y' \cap W'$, where $W' := f^{-1}(W)$. Now $\mathcal{L}'$ is ample, too, so we find a section $s \in \mathrm{H}^0(X', \mathcal{L}'^{\otimes n})$ for some positive integer n such that $X'_{s'} \subset W'$ and $X'_{s'}$ contains $f^{-1}(x)$ and all generic points of Y' that are contained in $g^{-1}(Y_t)$ [EGA II, 4.5.4].

It follows that $f^{-1}(x) \subset X'_{s'} \cap Y' \subset W' \cap Y' = g^{-1}(Y_t)$ and that the difference $g^{-1}(Y_t) - (X'_{s'} \cap Y')$ contains no generic point of Y'. Then its image under g does not contain any generic points of Y either since g is quasifinite, so that its closure $Z_2 \subset Y$ is nowhere dense in Y and satisfies $x \notin Z_2$. Thus $g^{-1}(Y_t) = X'_{s'} \cap Y'$ over $X - f^{-1}(Z_2)$ and by replacing X with $X - Z_2$ we may assume that $Z_2 = \emptyset$.

Consequently, Proposition 1.4.5 applies so that $\mathcal{O}_Y$ and $\mathcal{L}'^{\otimes n}$ glue together along g outside a nowhere dense closed subset $Z_3 \subset Y$. By replacing X with $X - Z_3$ we may assume that $Z_3 = \emptyset$. So the proposition says there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ satisfying $\mathcal{L}|_Y \simeq \mathcal{O}_Y$ and $f^*\mathcal{L} \simeq \mathcal{L}'^{\otimes n}$ which has a global section s with $s|_Y = t$ and $f^*s = s'$. It follows that $x \in X_s$ since $X_s \cap Y = Y_t$ and $X_s \subset W$ since $f^{-1}(X_s) = X'_{f^*s} = X'_{s'} \subset W' = f^{-1}(W)$.

Case $x \in X - Y$: We may assume $W \subset X - Y$, a fortiori $W' := f^{-1}(W) \subset X' - Y'$. We apply Proposition 1.4.5 again for $\mathcal{M} = \mathcal{O}_Y$ with $0 = t \in \mathrm{H}^0(Y, \mathcal{O}_Y)$ and $s' \in \mathrm{H}^0(X', \mathcal{L}'^{\otimes n})$ with $x = f^{-1}(x) \in X'_{s'} \subset W'$ and proceed as above. □

Consequently, every ample sheaf on X' descends to an ample family on X outside a nowhere dense closed subset of Y:

(1.4.7) Proposition. *Let $f\colon X' \to X$ be a finite birational morphism of quasicompact schemes with schematically dense image, such that X' is separated and carries an ample $\mathcal{O}_{X'}$-module $\mathcal{L}'$, and let $Y \subset X$ be the closed subscheme defined by the conductor ideal* $\operatorname{Ann}\operatorname{coker}(\mathcal{O}_X \to f_*\mathcal{O}_{X'})$.

Then there exists an open set $V \subset X$ such that the complement $X - V$ is contained in Y, $V \cap Y$ is affine and dense in Y, and a finite ample family of $\mathcal{O}_V$-modules $\mathcal{L}_i$, $i \in I$, such that ${f_V}^*\mathcal{L}_i \simeq \mathcal{L}'^{\otimes n}$ for all $i \in I$ and some $n \in \mathbb{N}$.

PROOF. Applying Lemma 1.4.6 for open affine neighborhoods we find for each $x_i \in X$ a closed set $Z_i \subset Y$ with $\operatorname{codim}(Z_i, Y) \geq 1$ and on $V_i := X - Z_i$ an invertible $\mathcal{O}_{V_i}$-module $\mathcal{L}_i$ with ${f_{V_i}}^*\mathcal{L}_i \simeq (\mathcal{L}')^{\otimes n_i}|_{f^{-1}(V_i)}$ for some positive integer n_i, and a section $s_i \in \mathrm{H}^0(V_i, \mathcal{L}_i)$ such that $(V_i)_{s_i}$ is a quasiaffine neighborhood of x.

Since X is quasicompact, finitely many $(V_i)_{s_i}$, $i \in I$, suffice to form an open covering of X. Then the finite union $Z := \bigcup_{i \in I} Z_i$ is a nowhere dense closed subset of Y. Since $Y - Z$ is dense in Y it contains a dense affine open subset. By enlarging Z we may therefore assume that $Y - Z$ is affine open and dense in Y. We define $V := X - Z$ and replace the sheaves $\mathcal{L}_i$ and s_i with their restrictions to $V = \bigcap_{i \in Z} (V_i)_{s_i}$. Moreover, replacing each $\mathcal{L}_i$ with an appropriate tensor power, we may assume that all n_i are equal. ☐

The following example of a non-divisorial surface with projective normalization illustrates that it is in general not true that an ample sheaf descends to an ample sheaf or an ample family along a finite birational morphism without removing appropriate closed subsets. Nevertheless, the resolution property holds by the forthcoming Theorem 2.0.1.

(1.4.8) Example (A non-divisorial proper algebraic surface whose normalization is projective)**.** We work over an algebraically closed field k, say $k := \mathbb{C}$ for simplicity. Let E be an elliptic curve and consider the surface $X := E \times \mathbb{P}^1$. Choose distinct fibers E_0 and E_∞ over $\mathbb{P}^1$. Let $t_x\colon E \to E$ be the translation with respect to a rational point $x \in E$ of infinite order. Then define the finite map $g\colon E_0 \coprod E_\infty \simeq E \coprod E \to E$ as the identity on E_0 and as t_x on E_∞.

By Ferrand [Fer03, Théorème 5.4], the pushout of the closed immersion $i\colon E_0 \coprod E_\infty \hookrightarrow X$ along g exists in the category of schemes.

$$\begin{array}{ccc} E_0 \coprod E_\infty & \xrightarrow{\ i\ } & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ E & \xrightarrow{\ j\ } & S \end{array}$$

This square is cartesian and cocartesian, j is a closed immersion and f is finite with schematically dense image. It follows that S is an integral, proper surface with normalization f.

Assume that S is divisorial. Then for a chosen point $y \in j(E) \subset S$ there exists an effective Cartier divisor $C \subset S$ with $y \notin C$ and $S - C$ affine. In particular, $C \cap j(E)$ is non-empty and zero-dimensional. It follows that the line bundle $\mathcal{L} := j^*\mathcal{O}_S(C)$ has positive degree and hence is ample. Now, the isomorphism $g^*\mathcal{L} \simeq i^*f^*\mathcal{O}_S(C)$ induces an isomorphism $\mathcal{L} \simeq {t_x}^*\mathcal{L}$. But then x must have finite order by the theory of abelian varieties [Mum70, p. 60, Application 1], contradicting the choice of x.

Deformations of ample families. Next we deal with birational nilpotent immersions $f\colon X' \hookrightarrow X$ of noetherian schemes. Then $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$ is surjective and its kernel $\mathcal{I}$ is nilpotent since X is noetherian. In order to lift a locally free $\mathcal{O}_{X'}$-module $\mathcal{E}'$ to locally free $\mathcal{O}_X$-module $\mathcal{E}$ it suffices to consider the case $\mathcal{I}^2 = 0$, so that $\mathcal{I}$ is an $\mathcal{O}_{X'}$-module. Then the obstruction to the existence of a lift is an element of $\mathrm{H}^2(X', \mathcal{I} \otimes_{\mathcal{O}_{X'}} \mathcal{E}nd(\mathcal{E}'))$ by [Ill05, Theorem 5.3]. Moreover, having a second locally free $\mathcal{O}_{X'}$-module $\mathcal{F}'$ the obstruction of lifting a morphism $s\colon \mathcal{E}' \to \mathcal{F}'$ is an element of $\mathrm{H}^1(X', \mathcal{I} \otimes_{\mathcal{O}_{x'}} \mathcal{H}om(\mathcal{E}', \mathcal{F}'))$.

(1.4.9) Lemma. *Let $f\colon X' \hookrightarrow X$ be a birational nilimmersion of noetherian schemes and let $Y \subset X$ be the closed subscheme defined by the kernel of $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$. Then there exists an open subset $V \subset X$ with $X - V \subset Y$ and $V \cap Y$ dense and affine in Y, such that for every locally free $\mathcal{O}_{f^{-1}(V)}$-module $\mathcal{E}'$ there exists a locally free $\mathcal{O}_V$-module $\mathcal{E}$ with $f_V^*\mathcal{E} \simeq \mathcal{E}'$, and every section $s \in \mathrm{H}^0(f^{-1}(V), \mathcal{E}')$ extends to a section $\mathrm{H}^0(V, \mathcal{E})$.*

Proof. Let $Z \subset Y$ be the complement of any affine dense open neighborhood in $U \subset Y$ and set $V = X - Z$. Then Z is nowhere dense in X, and $V \cap Y = U$. We conclude that the restriction $V' := f^{-1}(V) \hookrightarrow V$ of f is a birational nilimmersion whose ideal of definition $\mathcal{J} = \mathcal{I}|_V$ has affine support. In particular, it has cohomological dimension 0. By the previous discussion, we see that every vector bundle $\mathcal{E}'$ on V' has a lift $\mathcal{E}$ to V, and every map $\mathcal{O}_{V'} \to \mathcal{E}'$ extends to a map $\mathcal{O}_V \to \mathcal{E}$. □

This settles the descent of ample line bundles along infinitesimal thickenings.

(1.4.10) Lemma. *Let $f\colon X' \hookrightarrow X$ be a birational nilimmersion of noetherian schemes and let $Y \subset X$ be the closed subscheme defined by the kernel of $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$. Then there exists an open subset $V \subset X$ such that $X - V \subset Y$, $V \cap Y$ is dense and affine in Y, such that for every ample family $\mathcal{L}'_i$, $i \in I$, of $\mathcal{O}_{f^{-1}(V)}$-modules there exists an ample family $\mathcal{L}_i$, $i \in I$, of $\mathcal{O}_V$-modules with $f_V^*\mathcal{L}_i \simeq \mathcal{L}'_i$.*

Proof. Due to Lemma 1.4.9 an ample family of $\mathcal{O}_{f^{-1}(V)}$-modules lifts to a family of invertible $\mathcal{O}_V$-modules $\mathcal{L}_i$ and every global section $s' \in \mathrm{H}^0(f^{-1}(V), \mathcal{L}_i'^{\otimes n})$ extends to a section $s \in \mathrm{H}^0(V, \mathcal{L}_i^{\otimes n})$. If the non-vanishing set $f^{-1}(V)_{s'} = f^{-1}(V_s)$ is affine, then this also holds for V_s. Using property 1.2.1.(v)) we infer that $\mathcal{L}_i$, $i \in I$, is an ample family of $\mathcal{O}_V$-modules. □

The case of a general proper morphism. Finally, we consider an arbitrary birational proper morphism f of schemes.

(1.4.11) Proposition. *Let $f\colon X' \to X$ be a proper birational morphism of noetherian schemes such that X' is separated and carries an ample $\mathcal{O}_{X'}$-module $\mathcal{L}'$. Let $U \subset X$ be the maximal dense open set where f is an isomorphism. Then there exists an open neighborhood $U \subset V \subset X$ and an ample family $\mathcal{L}_i$, $i \in I$, of $\mathcal{O}_V$-modules such that*

(i) $V - U$ is affine,

(ii) V is thick,

(iii) f is finite over V,

(iv) there exists an $n \in \mathbb{N}$ such that ${f_V}^*\mathcal{L}_i \simeq \mathcal{L}'^{\otimes n}|_{f^{-1}(V)}$ for all $i \in I$.

Proof. Applying Stein factorization to $f = g \circ f'$ and factoring g over its scheme theoretic image $g = i \circ g'$, reduces to the case that f is birational morphism which is in addition either a nilimmersion, a finite morphism with schematically dense image or a proper Stein morphism. The latter case is trivial since a birational Stein map of finite type is an isomorphism over an open subset containing all points of codimension ≤ 1 by Lemma 1.4.12 below. The other cases were treated in Lemma 1.4.10 and Proposition 1.4.7. □

(1.4.12) Lemma. *Let $f\colon Y \to X$ be morphism of locally noetherian schemes that is birational and of finite type. Then f is quasifinite over all points of codimension ≤ 1.*

Proof. Let $x \in X$ be a point of codimension 1. By applying base change with $\operatorname{Spec} \mathcal{O}_{X,x} \to X$ we may assume that X is local of Krull dimension 1 with closed point x. Moreover, we may assume that X and Y are irreducible by a second base change. Let $y \in f^{-1}(x)$ an arbitrary point. Then $\dim \mathcal{O}_{Y,y} \geq 1$ because y cannot be a generic point of Y, whereas $0 \leq \dim \mathcal{O}_{Y,y} + \operatorname{deg.tr}_{k(x)} k(y) \leq \dim \mathcal{O}_{X,x} = 1$ by [EGA IV.2, 5.6.5.2] . Consequently, holds $\operatorname{deg.tr}_{k(x)} k(y) = 0$ which shows that $f^{-1}(x)$ is finite. □

As an application of Proposition 1.4.11 and Chow's Lemma we are now able to prove Proposition 1.4.4.

Proof of Proposition 1.4.4. Let A be the noetherian base ring. By Chow's Lemma there is a quasiprojective A-scheme X' carrying an ample line bundle $\mathcal{L}'$ and a proper morphism $f\colon X' \to X$, which is an isomorphism over a dense affine open neighborhood $x \in U \subset X$, such that $f^{-1}(U)$ is schematically dense in X' (this version of Chow's Lemma is due to Nagata, and stated in this scheme theoretic version in [Voj07, 2.5], or in [Con07, 2.6]). Hence Proposition 1.4.11 applies. □

Existence of almost ample families and proof of Theorem 1.4.2. As a consequence of Proposition 1.4.4, we deduce that every coherent sheaf can be resolved by coherent sheaves of rank one which are invertible on thick open subsets and satisfy a weak positivity property like ample line bundles. The following is a detailed version of Theorem 1.4.2:

(1.4.13) Proposition. *Let X be a scheme that is separated and of finite type over a noetherian ring. Then there exists a family of coherent $\mathcal{O}_X$-modules $\mathcal{F}_i$, $i \in I$, having the following properties:*

- *(i) For every $i \in I$ there exists an thick open subset $V_i \subset X$ such that $\mathcal{F}_i|_{V_i}$ is invertible and $X = \bigcup_{i \in I} V_i$.*
- *(ii) For every $i \in I$ there exists a proper birational map $f\colon X' \to X$ such that X' carries an ample line bundle $\mathcal{L}'$ which is isomorphic to $(f^*\mathcal{F}_i)^\vee$ over $f^{-1}(V)$.*
- *(iii) For every coherent $\mathcal{O}_X$-module $\mathcal{M}$ there exists a surjection*
$$\bigoplus_{j \in J} \mathcal{F}_{i_j}^{\otimes n_j} \twoheadrightarrow \mathcal{M}$$
for some finite set of indices $i_j \in I$ and powers $n_j \in \mathbb{N}$.

Proof of Proposition 1.4.13 and Theorem 1.4.2. For every $x \in X$ take an open neighborhood $V \subset X$ as in Proposition 1.4.4 having an ample family of $\mathcal{O}_V$-modules $\mathcal{L}_{V,j}$, $j \in J$. Pick an ideal $\mathcal{I} \subset \mathcal{O}_X$ defining some subscheme structure on $X - V$ and choose for each $j \in J$ a coherent $\mathcal{O}_X$-module $\mathcal{L}_j$ extending $\mathcal{L}_{V,j}$ to X. In that way we get a family $\mathcal{F}_i := \mathcal{I}^m \cdot \mathcal{L}_j^\vee$, where i runs over all elements (x, m, j) of $I := X \times \mathbb{N} \times J$. By construction holds (i) and (ii), and property (iii) is satisfied since the surjection of $\mathcal{O}_V$-modules of definition 1.2.1.(i) extend to maps of $\mathcal{O}_X$-modules [EGA $\mathrm{I}_{2\mathrm{nd}}$, I.6.9.17]. □

(1.4.14) *Remark.* Each $\mathcal{F}_i$ in the family of Proposition 1.4.13 can be chosen to be an ideal sheaf if X is S_1, or to be an anti-effective almost Cartier divisor (in the sense of [Har07, Def. 2]) if X satisfies S_2.

After studying the cohomological behavior of ample line bundles, we will show later in Proposition 1.5.9 that almost ample families satisfy a cohomological vanishing property for the top cohomology on proper schemes. This will be crucial for constructing locally free resolutions on arbitrary proper surfaces as the cohomological obstructions for gluing local resolutions lie in H^2 (for details see the proof of Theorem 2.0.1).

1.5. Cohomologically ample families of coherent sheaves

We introduce a specific cohomological vanishing condition for families of coherent sheaves, in order to study the cohomogical behavior of ample line bundles with respect to *alterations* (i.e. a proper, surjective and generically finite map of schemes). Our definition of cohomological ampleness is similar to the one given in [Ste98].

(1.5.1) Definition. Let X be a scheme that is proper over a noetherian ring and d a natural number. A family of coherent $\mathcal{O}_X$-modules $\mathcal{E}_n$, $n \in \mathbb{N}$, is called a *cohomologically d-ample family* if for all coherent $\mathcal{O}_X$-modules $\mathcal{F}$ there exists a positive integer n_0 such that for all $n \geq n_0$ and $i \geq d+1$ holds $\mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{E}_n) = 0$.

A coherent $\mathcal{O}_X$-module $\mathcal{F}$ is called *cohomologically d-ample* if $\mathcal{F}^{\otimes n}$, $n \in \mathbb{N}$, is a cohomologically d-ample family of coherent sheaves.

The main advantage of this concept is that the vanishing condition for the top cohomology is preserved and reflected by alterations:

(1.5.2) Theorem. *Let $f\colon Y \to X$ be an alteration of d-dimensional schemes that are proper over a noetherian ring and let $\mathcal{E}_n$, $n \in \mathbb{N}$, be a family of coherent $\mathcal{O}_X$-modules. Then $(\mathcal{E}_n)$ is $(d-1)$-ample if and only if $(f^*\mathcal{E}_n)$ is $(d-1)$-ample.*

The proof is given at the end of this section and divided in several steps. First, we show that cohomologically ample families are preserved by finite morphisms.

(1.5.3) Lemma. *Let $f\colon Y \to X$ be a finite morphism of schemes that are proper over a noetherian ring, $\mathcal{E}_n$, $n \in \mathbb{N}$, a family of coherent $\mathcal{O}_X$-modules and $d \in \mathbb{N}$ some positive integer. If $(\mathcal{E}_n)$ is cohomologically d-ample, so too is $(f^*\mathcal{E}_n)$.*

Proof. The projection formula $f_*\mathcal{F} \otimes \mathcal{E}_n = f_*(\mathcal{F} \otimes f^*\mathcal{E}_n)$ holds since f_* is exact. Therefore it induces an isomorphism $\mathrm{H}^i(Y, \mathcal{F} \otimes f^*\mathcal{E}_n) \simeq \mathrm{H}^i(X, f_*\mathcal{F} \otimes \mathcal{E}_n)$ for all $i, n \geq 0$, which proves the assertion. □

Using Grothendieck duality we infer that the vanishing property, involving only the top dimensional cohomology, is also reflected by finite surjections.

(1.5.4) Lemma. *Let $f\colon Y \to X$ be a finite surjective map of d-dimensional schemes that are proper over a noetherian ring and let $\mathcal{E}_n$, $n \in \mathbb{N}$, be a family of coherent $\mathcal{O}_X$-modules. If $(f^*\mathcal{E}_n)$ is cohomologically $(d-1)$-ample, so too is $(\mathcal{E}_n)$.*

Proof. Due to the following commutative square of finite surjective maps and using Lemma 1.5.3 it suffices to check the case where either f is a surjective immersion or X is reduced.

$$\begin{array}{ccc} Y_{\mathrm{red}} & \longrightarrow & Y \\ \downarrow{\scriptstyle f_{\mathrm{red}}} & & \downarrow{\scriptstyle f} \\ X_{\mathrm{red}} & \longrightarrow & X \end{array}$$

Case 1: f is a surjective immersion. The closed immersion $Y \hookrightarrow X$ is given by a nilpotent ideal $\mathcal{I} \subset \mathcal{O}_X$ since X is noetherian. We may assume that $\mathcal{I}^2 = 0$. Then applying $\cdot \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}_n$ to the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to 0$$

induces an exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}_n \xrightarrow{u} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}_n \xrightarrow{v} \mathcal{F}/\mathcal{I}\mathcal{F} \otimes_{\mathcal{O}_X/\mathcal{I}} \mathcal{E}_n/\mathcal{I}\mathcal{E}_n \to 0,$$

which splits in two exact sequences of $\mathcal{O}_X$-modules

$$\begin{aligned} 0 \to \ker u \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}_n \to \ker v \to 0, \\ 0 \to \ker v \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}_n \to \mathcal{F}|_Y \otimes_{\mathcal{O}_Y} \mathcal{E}_n|_Y \to 0. \end{aligned} \tag{1.5.4.1}$$

Since $\mathcal{I}^2 = 0$ holds $\mathcal{I} \simeq \mathcal{I}/\mathcal{I}^2 \simeq \mathcal{O}_X/\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{I}$ as sheaves of abelian groups, so that

$$\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}_n \simeq (\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F})|_Y \otimes_{\mathcal{O}_Y} \mathcal{E}_n|_Y$$

is an $\mathcal{O}_Y$-module. Finally the result follows by considering the long exact cohomology sequences associated to (1.5.4.1) and using that $\mathrm{H}^d(Y, (\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F})|_Y \otimes_{\mathcal{O}_Y} \mathcal{E}_n|_Y)$ and $\mathrm{H}^d(Y, \mathcal{F}|_Y \otimes_{\mathcal{O}_Y} \mathcal{E}_n|_Y)$ vanish for sufficiently large n.

Case 2: X is reduced. There is a natural short exact sequence

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{X'} \to \mathcal{C} \to 0,$$

where $\mathcal{C}$ is generically locally free since f is generically flat and finitely presented. Applying $\mathcal{H}om_{\mathcal{O}_X}(\cdot, \mathcal{F})$ gives the exact sequence

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{C}, \mathcal{F}) \to f_*f^!\mathcal{F} \xrightarrow{\alpha} \mathcal{F} \to \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{C}, \mathcal{F}).$$

Now the support of $\operatorname{coker} \alpha$ has dimension $\leq d-1$, so tensoring with $\mathcal{E}$, using the projection formula and taking cohomology gives a surjection

$$\mathrm{H}^d(Y, f^!(\mathcal{F}) \otimes f^*\mathcal{E}) \twoheadrightarrow \mathrm{H}^d(X, \mathcal{F} \otimes \mathcal{E}),$$

which implies the assertion. □

Even arbitrary alterations reflect the vanishing condition for the top dimensional cohomology. This follows again from Grothendieck duality.

(1.5.5) Proposition. *Let $f\colon Y \to X$ be a proper birational map of d-dimensional schemes that are proper over a noetherian ring, and let $(\mathcal{E}_n)$ be a family of coherent $\mathcal{O}_X$-modules. If $(f^*\mathcal{E}_n)$ is cohomologically $(d-1)$-ample, so too is $(\mathcal{E}_n)$.*

Proof. We may assume that X and Y are integral, and then suppose that the base ring A is complete, local and integral. By the Theorem on Formal Functions and Grothendieck's Vanishing Theorem it suffices to consider the case that the dimension of the closed fiber of $g\colon X \to \operatorname{Spec} A$ is of dimension d and not less. Since X is irreducible it follows that g maps X to the closed point of A. Therefore we may assume that A is a field. It follows that the normalizations of X and Y are finite maps. In particular, we may suppose that X and Y are normal and that f is birational. Thus, the assertion follows from the succeeding lemma. □

(1.5.6) Lemma. *Let $f\colon Y \to X$ be a proper birational map of d-dimensional, normal schemes that are proper over a field. Then there exists a surjection for every coherent $\mathcal{O}_X$-module $\mathcal{F}$*

$$\mathrm{H}^d(Y, \mathcal{H}om(f^*\omega_X, \omega_Y) \otimes f^*\mathcal{F}) \twoheadrightarrow \mathrm{H}^d(X, \mathcal{F}).$$

Proof. By Serre duality holds $\mathrm{H}^d(X, \mathcal{F})^\vee \simeq \operatorname{Hom}(\mathcal{F}, \omega_X)$. Since X is normal the dualizing module ω_X is torsionfree and generically invertible and the birationality of f implies the injectivity of the natural map

$$\operatorname{Hom}(\mathcal{F}, \omega_X) \to \operatorname{Hom}(f^*\mathcal{F}, f^*\omega_X) \to \operatorname{Hom}(f^*\mathcal{F}, \mathcal{H}om(\mathcal{H}om(f^*\omega_X, \omega_Y), \omega_Y)).$$

By adjunction the group on the right is isomorphic to

$$\operatorname{Hom}(f^*\mathcal{F} \otimes \mathcal{H}om(f^*\omega_X, \omega_Y), \omega_Y),$$

which is isomorphic to $\mathrm{H}^d(Y, f^*\mathcal{F} \otimes \mathcal{H}om(f^*\omega_X, \omega_Y))^\vee$ using Serre duality. □

We will see that it is convenient to pass to other families of coherent sheaves that differ on closed subsets of large codimension. The following Lemma 1.5.7 enables us to control the loss of positivity during that process. Recall that a scheme X has *cohomological dimension* $\leq d$ if $\mathrm{H}^i(X,\mathcal{F}) = 0$ for all $i \geq d+1$ and arbitrary quasicoherent $\mathcal{O}_X$-modules $\mathcal{F}$.

(1.5.7) Lemma. *Let X be a d-dimensional noetherian scheme and $\mathcal{F}$, $\mathcal{F}'$ be coherent $\mathcal{O}_X$-modules which are isomorphic outside a closed subscheme $Z \subset X$ of cohomological dimension k. Then*

$$\mathrm{H}^i(X,\mathcal{F}) \simeq \mathrm{H}^i(X,\mathcal{F}') \quad \text{for all } i \geq 2+k.$$

Proof. Since X is noetherian any isomorphism $\varphi_0\colon \mathcal{F}|_{X-Z} \xrightarrow{\simeq} \mathcal{F}'|_{X-Z}$ extends to a morphism $\varphi\colon \mathcal{I}^n\mathcal{F} \to \mathcal{F}'$ [EGA $\mathrm{I}_{\mathrm{2nd}}$, I.6.9.17], where the ideal $\mathcal{I} \subset \mathcal{O}_X$ provides the reduced subscheme structure on Z. Then φ as well as $\mathcal{I}^n\mathcal{F} \hookrightarrow \mathcal{F}$ are isomorphisms outside Z. So their kernels and cokernels have support of cohomological dimension $\leq k$ and we conclude $\mathrm{H}^i(X,\mathcal{F}) \xleftarrow{\simeq} \mathrm{H}^i(X,\mathcal{I}^n\mathcal{F}) \xrightarrow{\simeq} \mathrm{H}^i(X,\mathcal{F}')$ for all $i \geq 2+k$ by taking the long cohomology sequences. □

(1.5.8) *Remark.* If Z is affine then the cohomological dimension is 0 by Serre's criterion for affines. If Z is of Krull dimension d then its cohomological dimension is $\leq d$ by Grothendieck's Vanishing Theorem.

Proof of Theorem 1.5.2. By Lemma 1.5.3 we may assume that f is a Stein map, hence it is an isomorphism in codimension ≤ 1. Given an arbitrary coherent $\mathcal{O}_Y$-module $\mathcal{F}$, it follows from Lemma 1.5.7 that the canonical map $f_*\mathcal{F} \otimes \mathcal{E}_n \to f_*(\mathcal{F} \otimes f^*\mathcal{E}_n)$ induces an isomorphism $\mathrm{H}^d(X, f_*\mathcal{F} \otimes \mathcal{E}_n) \simeq \mathrm{H}^d(X, f_*(\mathcal{F} \otimes f^*\mathcal{E}_n))$.
Therefore it suffices to show the existence of an isomorphism of abelian groups $\mathrm{H}^d(Y,\mathcal{M}) \simeq \mathrm{H}^d(X, f_*\mathcal{M})$ for an arbitrary coherent $\mathcal{O}_Y$-module $\mathcal{M}$. To see this, let $x \in X$ with $\dim \mathcal{O}_{X,x} \geq 1$. Then $\dim f^{-1}(x) \leq \dim \mathcal{O}_{X,x} - 1$ since f is birational. Hence $(R^q f_*\mathcal{M})_x = 0$ for all $q \geq \dim \mathcal{O}_{X,x}$. It follows that $\operatorname{codim}(\operatorname{Supp} R^q f_*\mathcal{M}, X) \geq q+1$ and this implies $\mathrm{H}^p(X, R^q f_*\mathcal{M}) = 0$ for all $p \geq 0$, $q \geq 1$ with $p+q \geq d$. Applying the Grothendieck spectral sequence settles the result. □

Application to almost ample families. As announced after Remark 1.4.14, we add a vanishing property for the top cohomology to the properties of almost ample families (see Proposition 1.4.13). It is a direct consequence of the following Lemma.

(1.5.9) Proposition. *Let X be a d-dimensional scheme that is proper over a noetherian ring and has an almost ample family of coherent sheaves $\mathcal{F}_i$, $i \in I$. Then each $\mathcal{F}_i^\vee$ is cohomologically $(d-1)$-ample.*

(1.5.10) Lemma. *Let $f\colon Y \to X$ be a proper birational map of d-dimensional schemes that are proper over a noetherian ring such that Y has an ample invertible sheaf $\mathcal{L}$. Let $V \subset X$ a thick open subset and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module such that $f^*\mathcal{F}$ is isomorphic to $\mathcal{L}$ over $f^{-1}(V)$. Then $\mathcal{F}$ is cohomologically $(d-1)$-ample.*

Proof. By applying Stein factorization to f we may assume that f is a Stein map using Lemma 1.5.4. Then the maximal open subset $U \subset X$ where f is an isomorphism is thick by Lemma 1.4.12 and Zariski's Main Theorem [EGA III.1, 4.4.1].

It follows that $(f_*\mathcal{L})^{\otimes n}$ and $\mathcal{F}^{\otimes n}$ are isomorphic away from a closed subset of dimension $\leq d-2$. Hence by Lemma 1.5.7 it is equivalent to show that $f_*\mathcal{L}$ is cohomologically $(d-1)$-ample. By Serre's Vanishing Theorem it is even 0-ample. $\square$

CHAPTER 2

Algebraic surfaces have enough locally free sheaves

We shall prove the resolution property for every scheme X that has low dimension and is proper over a noetherian ring A. If $\dim X \leq 1$, then X is projective over A. Therefore, we consider the case that the irreducible components of X have dimension ≤ 2. Note that X is not necessarily reduced and has many irreducible components of dimension less or equal than 2. Furthermore we impose no regularity assumption on X or on the base ring.

(2.0.1) Theorem. *Let X be a 2-dimensional scheme that is proper over a noetherian ring. Then X has the resolution property.*

The proof of 2.0.1 is given at the very end of the chapter on page 26.

Since the resolution property descends along immersions, it holds for all 2-dimensional schemes X that are embeddable into 2-dimensional schemes which are proper over a base ring.

(2.0.2) Corollary. *Let X be a 2-dimensional scheme that is separated and of finite type over a field. Then X has the resolution property.*

PROOF. Since X is separated and of finite type over a noetherian ring, there exists a compactification $X \subset \overline{X}$ by Nagata's embedding Theorem. In particular, if the base ring is a field, then $\overline{X}$ is also of dimension 2 and Theorem 2.0.1 applies. $\square$

The strategy of the proof is as follows. By Theorem 1.4.2 it suffices to resolve the members of an almost ample family. Since X is of dimension ≤ 2, these are invertible up to finitely many points of codimension 2.

Generalizing the methods of [SV04] to non-normal and non-reduced noetherian schemes, we describe in section 2.1 a technique for constructing locally free resolutions by gluing local extension classes and formulate cohomological obstructions thereof. The existence of the right choice of the local classes is accomplished by providing a variant of the Bourbaki Lemma for non-reduced local rings, using the theory of basic elements of Evans and Griffith [EG85].

In order to constrol the cohomological obstructions, we construct vector bundles that have enough positivity to be suitable candidates for a first syzygy sheaf in the gluing process. Their existence is proven in section 2.2 by constructing a 1-ample family of vector bundles on a suitable Chow cover that are trivial on all exceptional and ramification components and hence descend on the original surface. As we do neither restrict to normal or reduced schemes our choice of the Chow cover involves pinching techniques and deformation theory of vector bundles.

Finally, we collect all preceding result to prove Theorem 2.0.1 in section 2.3.

All constructions we are going to perform do in fact not need that X has dimension 2. They may be adapted to state conditions for the existence of resolutions by sheaves whose singular locus has codimension ≥ 3 if X has any dimension. However, we were not able to control the obstructions in this general case. The presence of Serre's Vanishing Theorem for a projective Chow cover is the underlying reason why we stick to proper surfaces. However, we guess that a careful use of local cohomology and appropriate duality theorems might solve this problem.

(2.0.3) Conjecture. *Let X be a separated scheme of finite type over a noetherian ring.*

(i) Every point has an open neighborhood $U \subset X$ that has the resolution property and contains all points of codimension ≤ 2.

(ii) Every coherent sheaf $\mathcal{M}$ is a quotient of a coherent sheaf $\mathcal{E}$ which is locally free in codimension ≤ 2.

The verification of this conjecture would be quite useful to tackle the resolution property for separated threefolds as it would suffice to resolve coherent sheaves whose singular locus consists of finitely many points of codimension 3.

2.1. Gluing resolutions

In this section we formulate conditions that are sufficient for the existence of locally free resolutions of coherent sheaves which are locally free away from finitely many closed points of codimension 2. The crucial ingredient is the gluing of short exact sequences, which was already successfully applied in [Sch82] for the verification of the resolution property for complex compact surfaces and suitably modified in [SV04] to tackle the case of normal algebraic surfaces.

Gluing extension classes. To motivate the construction, observe that a surjection $\varphi\colon \mathcal{E} \twoheadrightarrow \mathcal{M}$ is uniquely determined by its *first syzygy* $\mathcal{S} := \ker \varphi$ and an extension class $\gamma \in \mathrm{Ext}^1(\mathcal{M}, \mathcal{S})$. So, if a reasonable candidate $\mathcal{S}$ for *some* (not necessarily locally free) resolution of $\mathcal{M}$ exists, then one can try to glue local extensions of $\mathcal{M}$ by $\mathcal{S}$ to a global one (which may differ from φ). The cohomological obstructions appearing here are well known, but described next for the convenience of the reader.

(2.1.1) Proposition. *Let X be a scheme and $\mathcal{M}$, $\mathcal{S}$ be quasicoherent $\mathcal{O}_X$-modules.*

(i) For each local extension $\gamma \in \mathrm{H}^0(X, \mathcal{E}xt(\mathcal{M}, \mathcal{S}))$ there exists an obstruction $o(\gamma) \in \mathrm{H}^2(X, \mathcal{H}om(\mathcal{M}, \mathcal{S}))$ whose vanishing is necessary and sufficient for the existence of a lift $\gamma' \in \mathrm{Ext}^1(\mathcal{M}, \mathcal{S})$.

(ii) If the obstruction $o(\gamma)$ vanishes, then the set of all liftings is a torsor under $\mathrm{H}^1(X, \mathcal{H}om(\mathcal{M}, \mathcal{S}))$, which identifies the locally trivial extensions.

Proof. This is a consequence of the 5-term sequence associated to the spectral sequence of $E_2^{pq} = \mathrm{H}^p(X, \mathcal{E}xt^q(\mathcal{M}, \mathcal{S})) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{M}, \mathcal{S})$. ☐

(2.1.2) *Remark.* If $\mathcal{M}$ is locally free outside a 0-dimensional subset $Z \subset X$, the local extensions of $\mathcal{M}$ by $\mathcal{S}$ appear in a simple form. In fact, there is a canonical isomorphism $\bigoplus_{z \in Z} \mathrm{Ext}^1_{\mathcal{O}_{X,z}}(\mathcal{M}_z, \mathcal{S}_z) \simeq \mathrm{H}^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{M}, \mathcal{S}))$, so that we may choose elements of $\mathrm{Ext}^1_{\mathcal{O}_{X,z}}(\mathcal{M}_z, \mathcal{S}_z)$, $z \in Z$, independently to a get a global section of $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{M}, \mathcal{S})$.

(2.1.3) *Remark.* We will use a more convenient formulation of the obstruction space $\mathrm{H}^2(X, \mathcal{H}om(\mathcal{M}, \mathcal{S}))$ for gluing local resolutions of $\mathcal{M}$ by $\mathcal{S}$. Note that the natural pairing $\mathcal{H}om(\mathcal{M}, \mathcal{O}_X) \otimes \mathcal{S} \xrightarrow{\varepsilon} \mathcal{H}om(\mathcal{M}, \mathcal{S})$ is an isomorphism, where $\mathcal{M}$ is locally free. But it is in general not an isomorphism, since the tensor product may have torsion sections. Nevertheless, if $\mathcal{M}$ is locally free up to finitely many points of codimension 2, we deduce from Lemma 1.5.7 the isomorphism

$$\mathrm{H}^2(X, \mathcal{H}om(\mathcal{M}, \mathcal{S})) \simeq \mathrm{H}^2(X, \mathcal{M}^\vee \otimes \mathcal{S}). \qquad (2.1.3.1)$$

One difficulty for gluing local resolutions is to guess the choice of the right syzygy sheaf $\mathcal{S}$ and to solve the problem that the obstruction $o \in \mathrm{H}^2(\mathcal{H}om(\mathcal{M},\mathcal{S}))$ depends on both sheaves $\mathcal{M}$ and $\mathcal{S}$. Therefore we divide the construction of a locally free resolution of $\mathcal{M}$ in two steps, where we can specify the first syzygy sheaf and dissolve this dependency.

Local resolutions. Let $(A,\mathfrak{m})$ be a noetherian local ring of Krull dimension 2. We denote by $X = \operatorname{Spec} A$ the associated prime spectrum and by $U := X - \{\mathfrak{m}\}$ the punctured spectrum. By abuse of notation we identify quasicoherent $\mathcal{O}_X$-modules with their A-modules of global sections.

Let M be an A-module of finite type that is locally free of constant rank on U. Then for every free resolution $\varphi\colon A^{\oplus n} \twoheadrightarrow M$ the first syzygy module $S := \ker\varphi$ is locally free on U and satisfies $\det S|_U \simeq \det M^\vee|_U$. In particular, if S has generically rank 1, then $S|_U \simeq \det S|_U$ is uniquely determined by M.

In general, S has higher rank, but we can choose a free submodule such that its quotient has rank 1 and is locally free on U. In [SV04, Theorem 2.1] this was accomplished by invoking the Bourbaki Lemma [Bou65, p. 76]. It says that for a normal noetherian ring, every torsionfree module of rank r has a free submodule of rank $r-1$ such that its quotient is isomorphic to an ideal, hence has rank 1.

Since we work over arbitrary noetherian rings, we need an appropriate generalization for non-reduced rings (see [BV75] for torsionfree modules).

(2.1.4) Lemma (Modified Bourbaki Lemma). *Let $k \in \mathbb{N}$, A be a noetherian ring and M be a finitely generated A-module such that M is free of rank $r \geq k$ at all primes of height $\leq k$. Then there is a free submodule F of M of rank $r-k$, such that M/F is free of rank k at all primes of height $\leq k$.*

Proof. This is an application of basic element theory (c.f. [EG85]). Denote by $\mu(M)$ the minimal number of generators. A submodule $N \subset M$ is called *w-fold basic* at a prime ideal $\mathfrak{p} \subset A$ if $\mu(M/N)_\mathfrak{p} \leq \mu(M_\mathfrak{p}) - w$. A set of generators $x_1,\dots,x_s$ of N is called *basic up to height k* if N is $\min(s, k - \operatorname{ht}\mathfrak{p} + 1)$-fold basic in M at each prime ideal $\mathfrak{p} \subset A$ of height less or equal to k.

Assume $r \geq k+1$. Let $x_1,\dots,x_s$ be a choice of generators of M, $s \geq r$. If $\mathfrak{p} \subset A$ is a prime ideal of height less or equal to k, then $w := \min(s, k - \operatorname{ht}\mathfrak{p} + 1) \leq k+1$, thus $\mu(M_\mathfrak{p}) - w \geq r - (k+1) \geq 0$, and we conclude that $x_1,\dots,x_s$ are basic up to height k. Hence, by [EG85, Theorem 2.3], there is a one element set $\{y\}$ which is basic up to height k and induces a short exact sequence

$$0 \longrightarrow Ay \longrightarrow M \longrightarrow M/Ay \longrightarrow 0. \qquad (2.1.4.1)$$

Then for prime ideals $\mathfrak{p} \subset A$ with $\operatorname{ht}\mathfrak{p} \leq k$ the localizations

$$0 \longrightarrow A_\mathfrak{p} y \longrightarrow M_\mathfrak{p} \longrightarrow (M/Ay)_\mathfrak{p} \longrightarrow 0$$

give rise to exact sequences

$$A_\mathfrak{p} y \otimes_A k(\mathfrak{p}) \to M_\mathfrak{p} \otimes_A k(\mathfrak{p}) \to (M/Ay)_\mathfrak{p} \otimes_A k(\mathfrak{p}).$$

By choice of y holds $\mu(M/Ay)_\mathfrak{p} \leq \mu(M)_\mathfrak{p} - \min\{1, k - \operatorname{ht}\mathfrak{p} + 1\} = r-1$ and we infer that $A_\mathfrak{p} y \otimes_A k(\mathfrak{p})$ is nonzero. Thus, y may serve as part of a basis for the free module $M_\mathfrak{p}$ and we conclude that $(M/Ay)_\mathfrak{p}$ is free of rank $r-1 \geq k$.

By induction there is a free submodule $F' \subset M/Ay$ of rank $r-1-k$ such that $(M/Ay)/F'$ is free of rank k at all primes of height $\leq k$. Pulling back the short exact sequence (2.1.4.1) along the inclusion $F' \hookrightarrow M/Ay$ gives a submodule $F \subset M$ that is an extension of F' by Ay, thus free of rank $r-k$, and satisfies $M/F' \simeq (M/Ay)/F'$. $\square$

Using the modified Bourbaki Lemma, we conclude that for a 2-dimensional noetherian local ring, every free resolution of an A-module of finite type breaks up as follows.

(2.1.5) Proposition. *Let* $(A, \mathfrak{m})$ *be a noetherian local ring of dimension* 2 *and* $U = \operatorname{Spec} A - \{\mathfrak{m}\}$. *Then for every finitely generated* A*-module* M *that is locally free of rank* $r \geq 1$ *on* U *there exists an exact diagram of finitely generated* A*-modules that are locally free of constant rank on* U

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & A^{\oplus n-r-1} & = & A^{\oplus n-r-1} & & & & \\
 & & \downarrow & & \downarrow{\scriptstyle \iota_2} & & & & \\
0 & \longrightarrow & S & \xrightarrow{\iota} & A^n & \xrightarrow{\psi} & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \psi_2} & & \| & & \\
0 & \longrightarrow & L & \xrightarrow{\iota_1} & N & \xrightarrow{\psi_1} & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

such that $L|_U \simeq \det M^\vee|_U$ *and* $n \in \mathbb{N}_0$.

PROOF. Every choice of a generating set for M gives rise to a short exact sequence $0 \longrightarrow S \longrightarrow A^{\oplus n} \longrightarrow M \longrightarrow 0$, such that S is locally free in codimension ≤ 1 of rank $n - r$. Therefore Lemma 2.1.4 implies that there exists a short exact sequence $0 \longrightarrow A^{\oplus n-r-1} \longrightarrow S \xrightarrow{p} L \longrightarrow 0$, such that L is locally free in codimension ≤ 1 of rank 1. Finally, the pushout of the first exact sequence by p gives the desired commutative diagram. □

The upshot of the previous proposition is that the surjection ψ decomposes as two surjections $\psi = \psi_1 \circ \psi_2$, such that $\ker \psi_2$ is free and $\ker \psi_1$ is a coherent extension of $\det M^\vee|_U$. Here, the module N is not free in general, but has projective dimension ≤ 1.

(2.1.6) Definition. Let X be a noetherian scheme. We say that a coherent $\mathcal{O}_X$-module $\mathcal{N}$ has property F_k, or is *free in codimension* $\leq k$, if $\mathcal{N}_x$ is free for all $x \in X$ with $\dim \mathcal{O}_{X,x} \leq k$ and if $\operatorname{pd}(\mathcal{N}_x) \leq 1$ otherwise.

(2.1.7) *Remark.* If X is a Cohen-Macaulay scheme of dimension 2, then the Auslander-Buchsbaum formula implies, that a coherent sheaf satisfies F_k if and only if it has locally finite projective dimension and property S_k is fulfilled.

Global resolutions. Building on the ideas of Proposition 2.1.5 we seek to construct a locally free resolution $\mathcal{E} \twoheadrightarrow \mathcal{M}$ by constructing two surjections $\psi_1 \colon \mathcal{N} \twoheadrightarrow \mathcal{M}$, $\psi_2 \colon \mathcal{E} \twoheadrightarrow \mathcal{N}$, where $\mathcal{N}$ satisfies F_1. These surjections arise as an extension of $\mathcal{M}$ by a modification of $\det \mathcal{M}^\vee$ respectively by an extension of $\mathcal{N}$ by a locally free sheaf.

(2.1.8) Proposition. *Let* X *be a* 2*-dimensional, noetherian scheme, and* $\mathcal{M}$ *a coherent* $\mathcal{O}_X$*-module that is locally free (of constant rank) outside a closed set* $Z \subset X$ *with* $\operatorname{codim}(Z, X) = 2$, *and denote by* $\mathcal{F}$ *some chosen coherent extension of* $\det \mathcal{M}^\vee|_{X-Z}$.

Then there exists an obstruction $o \in \mathrm{H}^2(X, \mathcal{H}om(\mathcal{M}, \mathcal{F}))$, *whose vanishing is necessary and sufficient for the existence of a short exact sequence of* $\mathcal{O}_X$*-modules*

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{N} \longrightarrow \mathcal{M} \longrightarrow 0,$$

where

(i) $\mathcal{L}$ *is some coherent extension of* $\det \mathcal{M}^\vee|_{X-Z}$*, possibly different from* $\mathcal{F}$*,*

(ii) $\mathcal{N}$ *satisfies* F_1*.*

Proof. Denote by r the rank of $\mathcal{M}$. By Proposition 2.1.5 there exists for each $z \in Z$ an extension

$$\gamma_z\colon \quad 0 \longrightarrow L_z \longrightarrow N_z \longrightarrow M_z \longrightarrow 0,$$

such that $L_z|_{\operatorname{Spec}\mathcal{O}_{X,z}-\{z\}} \simeq \det \mathcal{M}^\vee|_{\operatorname{Spec}\mathcal{O}_{X,z}-\{z\}}$ and $\operatorname{pd}(N_z) \leq 1$. Then the family L_z, $z \in Z$, and $\det \mathcal{M}^\vee|_{X-Z}$ glue to a coherent $\mathcal{O}_X$-module $\mathcal{L}$, i.e. $\mathcal{L}_z \simeq L_z$ and $\mathcal{L}|_{X-Z} \simeq \det \mathcal{M}^\vee|_{X-Z}$, and the extension classes glue to a global section γ of $\mathcal{E}xt^1(\mathcal{M}, \mathcal{L})$.

Since $\mathcal{H}om(\mathcal{M}, \mathcal{L})|_{X-Z} \simeq \mathcal{H}om(\mathcal{M}, \det \mathcal{M}^\vee)|_{X-Z}$ we deduce from the hypothesis and Lemma 1.5.7 that $\mathrm{H}^2(X, \mathcal{H}om(\mathcal{M}, \mathcal{L})) \simeq \mathrm{H}^2(X, \mathcal{H}om(\mathcal{M}, \mathcal{F}))$. Thus we can identify the obstruction $o(\gamma) \in \mathrm{H}^2(X, \mathcal{H}om(\mathcal{M}, \mathcal{L}))$ for gluing the extensions γ with an element $o \in \mathrm{H}^2(X, \mathcal{H}om(\mathcal{M}, \mathcal{F}))$. If the obstruction vanishes, we obtain a coherent $\mathcal{O}_X$-module $\mathcal{N}$, which satisfies all asserted properties. □

If $\mathcal{M}$ has rank 1 then equation (2.1.3.1) implies that the obstruction for gluing is an element of

$$\mathrm{H}^2(X, \mathcal{H}om(\mathcal{M}, \mathcal{F})) \simeq \mathrm{H}^2(X, (\mathcal{M}^\vee)^{\otimes 2}). \tag{2.1.8.1}$$

Hence, it just depends on $\mathcal{M}$ and not on the choice of the coherent extension of $\det \mathcal{M}^\vee_{X-Z}$. The following corollary describes the situation where $\mathcal{M}^\vee$ has enough positivity to ensure the vanishing of the whole obstruction space.

(2.1.9) Corollary. *Let* X *be a* 2*-dimensional scheme that is proper over a noetherian ring and let* $\mathcal{M}$ *be a coherent* $\mathcal{O}_X$*-module. If* $\mathcal{M}$ *is invertible in codimension* ≤ 1 *and* $\mathcal{M}^\vee$ *is* 1*-ample, then* $\mathcal{M}^{\otimes n}$ *is a quotient of a coherent* $\mathcal{O}_X$*-module satisfying* F_1 *for sufficiently large* $n \in \mathbb{N}$*.*

We focus next on locally free resolutions of coherent sheaves that satisfy F_1. If such a resolution exists, then its first syzygy sheaf $\mathcal{S}$ is locally free. Conversely we deduce by a similar argument as in the proof of 2.1.8 the following proposition.

(2.1.10) Proposition. *Let* X *be a* 2*-dimensional scheme and* $\mathcal{M}$ *a coherent* $\mathcal{O}_X$*-module satisfying* F_1*. Then for every locally free* $\mathcal{O}_X$*-module* $\mathcal{S}$ *of constant rank there exists an obstruction* $o \in \mathrm{H}^2(X, \mathcal{H}om(\mathcal{M}, \mathcal{S}^{\oplus m}))$*, for some* $m \in \mathbb{N}$*, whose vanishing is necessary and sufficient for the existence of a locally free resolution*

$$0 \longrightarrow \mathcal{S}^{\oplus m} \longrightarrow \mathcal{E} \longrightarrow \mathcal{M} \longrightarrow 0.$$

The following corollary describes the situation, where $\mathcal{S}$ has enough positivity, so that the whole obstruction space vanishes. It is an immediate consequence of the previous proposition and equation (2.1.3.1).

(2.1.11) Corollary. *Let* X *be a* 2*-dimensional scheme that is proper over a noetherian ring. If* X *has a* 1*-ample family of locally free* $\mathcal{O}_X$*-modules of constant rank then every coherent* $\mathcal{O}_X$*-module satisfying* F_1 *is a quotient of a locally free* $\mathcal{O}_X$*-module.*

(2.1.12) *Problem.* Proposition 2.1.10 also holds for noetherian Deligne-Mumford stacks. However, it is not clear to the author if this is true for Lemma 2.1.8 since its proof depends on the fact that local sections are defined on open subsets and not just on étale neighborhoods.

(2.1.13) *Remark.* We are already able to prove the resolution property for an arbitrary 2-dimensional scheme X that is proper over a field as long as there exists a dense open subset of regular points (which is always true if X is generically reduced). To see this, observe that it suffices to resolve a coherent sheaf $\mathcal{M}$ which is invertible up to a closed subset $Z \subset X$ of finitely many points. By removing a regular point $x \notin Z$ from X we get a scheme $U = X - \{x\}$ that is separated and of finite type but has cohomological dimension ≤ 1 by Lichtenbaum's vanishing theorem [Kle67]. In particular, all obstruction spaces for gluing local extension classes vanish, so that we get a locally free resolution $\mathcal{E}_U \twoheadrightarrow \mathcal{M}|_U$ by Proposition 2.1.8 and 2.1.10. Since X is para-factorial near x, this map extends as $\mathcal{E} \to \mathcal{M}$ with $\mathcal{E}$ locally free near x. Even if this is not surjective near x we can repeat this procedure a second time for another point $x' \in X$ and adding the analog map $\mathcal{E}' \to \mathcal{M}$ gives the desired locally free resolution $\mathcal{E}' \oplus \mathcal{E} \twoheadrightarrow \mathcal{M}$.

2.2. Positive vector bundles on non-projective surfaces

Given a 2-dimensional scheme that is proper over a noetherian ring, we seek to construct a family of locally free $\mathcal{O}_X$-modules, that inherits enough positivity from ample line bundles on a Chow cover to ensure a cohomological vanishing property. These sheaves are the candidates for the first syzygy that appeared in Proposition 2.1.10.

(2.2.1) Theorem. *Let X be a 2-dimensional scheme that is proper over a noetherian ring. Then there exists a cohomologically 1-ample family of rank 2 vector bundles $\mathcal{E}_n$, $n \in \mathbb{N}$.*

The proof is given at the end of this section on page 25.

(2.2.2) *Remark.* The sheaves $\mathcal{E}_n$ are non-trivial in general. Otherwise the structure sheaf is 1-ample or equivalently X is of cohomological dimension $\mathrm{cd}(X) \leq 1$ in contradiction to $\mathrm{cd}(X) = 2$ by Lichtenbaum's vanishing theorem if the base ring is a field [Kle67].

Observe, that 1-ampleness can be checked on proper birational covers by Proposition 1.5.5.

First, we describe birational proper morphisms $f\colon X' \to X$, where we have sufficiently enough control about the descend of locally free sheaves. Then we prove that we can choose among those a Chow cover f; i.e. we can arrange that on X' exists an ample line bundle. Therefore it suffices to prove the theorem under the assumption that there exists an ample line bundle, but with the additional condition that the family of locally free sheaves satisfies a descent condition.

Descent of vector bundles along proper birational maps. We present first some conditions on a birational proper morphism $f\colon X \to Y$ of 2-dimensional schemes which ensure that locally free $\mathcal{O}_X$-modules descend to locally free $\mathcal{O}_Y$-modules.

We call the 1-dimensional subscheme consisting of the union of all contracted subschemes, the *exceptional curve* $E \subset X$, provided with the reduced subscheme structure. The ideal sheaf $\mathrm{Ann}_{\mathcal{O}_Y} \mathrm{coker}(\mathcal{O}_Y \to f_*\mathcal{O}_X) \subset \mathcal{O}_Y$ is called the *conductor ideal of* f and defines the *branching subscheme* $B \subset Y$. Its preimage by f is called the *ramification subscheme* $R \subset X$.

(2.2.3) Lemma. *Let $f\colon X \to Y$ be a proper birational morphism of 2-dimensional noetherian schemes with exceptional curve $E \subset X$ and ramification subscheme $R \subset X$. Suppose that there exists an effective Cartier divisor $D \subset X$ with $\mathrm{Supp}\, D = E$ and $\mathcal{O}_E(-D)$ ample.*

Then for each $r \geq 0$ there is an $n \geq 1$ such that every locally free $\mathcal{O}_X$-module of rank r, whose restriction to nD and R is trivial, belongs to the essential image of $f^*\colon \mathrm{LFF}(Y) \to \mathrm{LFF}(X)$.

PROOF. In case that f is Stein, there is no ramification, hence the statement reduces to Proposition 1.2 of [SV04]. If f is finite with schematically dense image, the exceptional curve is empty and the assertion follows from Theorem A.1. If f is a nilimmersion (a closed surjective immersion), the support of its defining ideal sheaf has dimension at most 1 as f is birational. Hence every locally free $\mathcal{O}_X$-module is isomorphic to the pullback of a locally free $\mathcal{O}_Y$-module by [Ill05, 5.3].

For the general case we factor f over its schematic image and apply Stein factorization to get a composition $f = uvw\colon X \xrightarrow{w} X' \xrightarrow{v} Y_0 \overset{u}{\hookrightarrow} Y$, where w is a proper Stein map, v is finite with schematically dense image and u is a nilimmersion. □

A projective auxiliary surface. Next we construct for a given 2-dimensional scheme an auxiliary projective model that carries an effective Cartier divisor which is anti-ample on the exceptional curve.

We are going to use the concept of generalized divisors described in [Har94, §2]. Therefore we impose condition S_1 on X so that the sheaf $\mathcal{K}_X$ of total quotient rings of $\mathcal{O}_X$ is well behaved (c.f. [Har94, Prop. 2.1]). Note that this property is stable under blowups as effective Cartier divisors carry no associated points of the surrounding subscheme and that the inverse image of Cartier divisors is well defined [EGA IV.4, 21.4.5(ii)].

(2.2.4) Lemma. *Let X be an 2-dimensional scheme, that satisfies S_1 and is separated and of finite type over a noetherian ring. Then there is a proper birational morphism $f\colon X' \to X$ with schematically dense image, such that its domain X' carries an ample line bundle, satisfies S_1 and the exceptional subscheme $E \subset X'$ supports an effective Cartier divisor $D \subset X'$ such that the restriction $\mathcal{O}_E(-D)$ is ample.*

PROOF. Denote the base ring by A. Choose a dense affine open subscheme $U \subset X$. Then U is schematically dense and $\operatorname{codim}(X - U, X) = 1$ because X has affine diagonal over $\mathbb{Z}$. By Chow's Lemma (in this form due to Nagata, c.f. [Con07, 2.6]) there exists an ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ defining a closed subscheme $Z \subset X$, whose support is contained in $X - U$, such that the blow-up $f'\colon X' \to X$ of X with center Z gives a scheme X' that is projective over A.

By the blow-up construction the inverse image ideal $\mathcal{I}' := \mathcal{I}_S \cdot \mathcal{O}_{X'} \subset \mathcal{O}_{X'}$ is invertible and hence defines an effective Cartier divisor $C' \subset X'$ which contains the exceptional curve $E' \subset X'$ and $\mathcal{I}' = \mathcal{O}_{X'}(-C')$ is f'-ample. If C' is supported on E', then the proof is already complete since $\mathcal{O}_{X'}(-C')$ is ample on the fibers and hence on E'.

In general, C' is supported on a larger subset. Let $C'_v \subset C'$ (respectively $C'_h \subset C'$) be the closed subsets of curves that are contracted (respectively mapped to curves). Then $C' = C'_v \cup C'_h$. A priori C'_v and C'_h are not Cartier divisors. To remedy this situation we insert a further blow-up. Denote by $\mathcal{I}'_v \subset \mathcal{O}_{X'}$ respectively $\mathcal{I}'_h \subset \mathcal{O}_{X'}$ their defining ideals, and let $f''\colon X'' \to X'$ be the blow-up of X' with center $\mathcal{I}'_v + \mathcal{I}'_h$.

It follows that the strict transforms C''_v, C''_h of C'_v respectively C'_h are separated by the effective Cartier divisor T'' given by the ideal $\mathcal{O}_{X''}(-T'') := (\mathcal{I}'_v + \mathcal{I}'_h) \cdot \mathcal{O}_{X''}$ (c.f. [Har77, Exercise II.7.12]). Note that T'' is f''-exceptional since the center $\mathcal{I}'_v + \mathcal{I}'_h$ defines a 0-dimensional subscheme of X'.

The pullback $f''^*\mathcal{O}_{X'}(-C') = \mathcal{I}' \cdot \mathcal{O}_{X''} =: \mathcal{O}_{X''}(-C'')$ defines now an effective Cartier divisor C'' such that the horizontal components of $C'' - T''$ are *disjoint* to the vertical ones relative to $f := f' \circ f''$. It follows that $C'' - T'' = F''_h + F''_v$ is the disjoint sum of two Cartier divisors, a horizontal one F''_h and a vertical one F''_v. In particular, holds $\mathcal{O}_{X''}(C''-T'') = \mathcal{O}_{X''}(F''_v)\otimes\mathcal{O}_{X''}(F''_h)$.

The upshot of this argumentation is that the f-exceptional components E'' carry the structure of an effective Cartier divisor $D'' := F''_v + 2T''$ and it remains to show that $\mathcal{O}_{E''}(-D'')$ is ample.

First, note that $\mathcal{O}_{X''}(-T'')$ is f''-ample and $\mathcal{O}_{X'}(-C')$ is f'-ample by the construction of blow-ups, thus

$$\mathcal{O}_{X''}(-T'')\otimes f''^{*}\mathcal{O}_{X'}(-C') \simeq \mathcal{O}_{X''}(-T''-C'') \simeq \mathcal{O}_{X''}(-D''-F''_h)$$

is f-ample. Let be $E'' = \bigcup_i E''_i$ the disjoint union of f-exceptional fibers. Then $\mathcal{O}_{E''_i}(-D''-F''_h)$ is ample for each i and hence $\mathcal{O}_{E''}(-D''-F''_h)$ is ample. Now F''_h has no common components with E'', so $\mathcal{O}_{E''}(F''_h)$ has non-negative degree. Therefore $\mathcal{O}_{E''}(-D'') = \mathcal{O}_{E''}(-D''-F''_h)\otimes\mathcal{O}_{E''}(F''_h)$ is an ample $\mathcal{O}_{E''}$-module. □

The projective case. We seek to prove Theorem 2.2.1 under the condition that X possesses an ample line bundle $\mathcal{L}$ and satisfies S_1.

(2.2.5) Lemma. *Let Y be a scheme of dimension 1 and $\mathcal{L}$ an ample $\mathcal{O}_Y$-module. Then for every $r\in\mathbb{N}$ and $n\gg 0$ there exists a short exact sequence*

$$0 \longrightarrow \mathcal{O}_Y^{\oplus r} \longrightarrow (\mathcal{L}^n)^{\oplus r} \longrightarrow \mathcal{O}_D \longrightarrow 0, \qquad (2.2.5.1)$$

where $D\subset Y$ is an effective Cartier divisor with $\mathcal{O}_Y(D)\simeq\mathcal{L}^{nr}$.

Proof. This follows by using sufficiently many global sections of $\mathcal{L}$ that are nonzero at the embedded points of Y. □

(2.2.6) Proposition. *Let X be an r-dimensional scheme that is proper over a noetherian ring, satisfies S_1 and has an ample $\mathcal{O}_X$-module $\mathcal{L}$. Then for every 1-dimensional closed subscheme $Y\subset X$ and $n\gg 0$ there exists a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r, whose restriction $\mathcal{E}|_Y$ is trivial and that fits in a short exact sequence*

$$0\to\mathcal{E}\to(\mathcal{L}^n)^{\oplus r}\to\mathcal{L}^m|_H\to 0, \qquad (2.2.6.1)$$

where $m\geq n$ and $H\subset X$ is an effective Cartier divisor satisfying $\mathcal{O}_X(H)=\mathcal{L}^{rn}$.

Proof. By enlarging Y we may assume that Y meets every irreducible component of X. Since X is proper over some noetherian base ring, satisfies S_1 and $\mathcal{L}$ is ample, it follows that every regular section of $\mathcal{L}^{nr}|_Y$ lifts to a regular section of $\mathcal{L}^{nr}$ for sufficiently large $n\in\mathbb{N}$.

By replacing $\mathcal{L}$ with an appropriate multiple we may assume that $n=1$. Let

$$0\to\mathcal{O}_Y^{\oplus r}\to\mathcal{L}^{\oplus r}|_Y\xrightarrow{\varphi}\mathcal{O}_D\to 0$$

be as in Lemma 2.2.5, where $D\subset Y$ is an effective Cartier divisor defined by a regular section $s\in \mathrm{H}^0(Y,\mathcal{L}^r|_Y)$. We seek to extend φ to a surjective map of $\mathcal{O}_X$-modules.

First, observe that s lifts to a regular section of $\mathcal{L}^r$, so that D extends to an effective Cartier divisor $H\subset X$ with $\mathcal{O}_X(H)\simeq\mathcal{L}^r$ and $\mathcal{O}_D\simeq\mathcal{O}_H\otimes\mathcal{O}_Y$. Moreover, D is supported on points so that for each $a\in\mathbb{N}$ we can identify $\mathcal{O}_D\simeq\mathcal{L}^a|_D\simeq\mathcal{L}^a\otimes\mathcal{O}_H\otimes\mathcal{O}_Y$ using that $\mathcal{L}$ is invertible. This shows that for each $a\in\mathbb{N}$ the sheaf $\mathcal{L}^a|_H$ is a coherent extension of the codomain of φ.

Next, we seek to construct a surjection $\Phi\colon\mathcal{L}^{\oplus r}\twoheadrightarrow\mathcal{L}^a|_H$ that extends φ. For that consider the decomposition $\varphi=\sum_{i=1}^r\varphi_i$ with maps $\varphi_i\colon\mathcal{L}|_Y\to\mathcal{L}^a|_{H\cap Y}$. We are going to lift the φ_i to maps $\Phi_i\colon\mathcal{L}\to\mathcal{L}^a$ successively, so that the cokernel of $\sum_{i=1}^s\Phi_i$ cuts down to zero for $s=r$. The subtle point is that we have to find a common a that works for all i. Therefore we consider for each i a family of extensions $\Phi_{i,m}\colon\mathcal{L}\to\mathcal{L}^{a_i\cdot m}|_H$ for an appropriate increasing sequence $a_i\in\mathbb{N}$, $m\in\mathbb{N}$.

Precisely, since $\mathcal{L}$ is ample and X is proper over A, every map $\psi\colon \mathcal{L}|_Y \to \mathcal{L}^a|_{H\cap Y}$ extends to a map $\Psi\colon \mathcal{L} \to \mathcal{L}^a|_H$ for $a \gg 0$. Moreover, we can achieve that Ψ is non-zero over any given finite subset of points $Z \subset H$ and that there is a section $s\colon \mathcal{O}_H \to \mathcal{L}^a|_H$ that is equal to 1 over $D \cup Z$, by taking larger a. The upshot is that for all $m \geq 2$ the composition $s^{\otimes m-1} \circ \Psi$ defines a family of maps $\Psi_m\colon \mathcal{L} \to \mathcal{L}^{am}$ that extend ψ and have stable cokernel support, i.e. we have equality of closed subsets

$$\operatorname{Supp}\operatorname{coker}\Psi_m = \operatorname{Supp}\operatorname{coker}\Psi_{m+1} = \cdots$$

and these are disjoint to $D \cup Z$.

Using this we lift each φ_i, $i = 1,\ldots,s$, by induction to a family of maps $\Phi_{i,m}\colon \mathcal{L} \to \mathcal{L}^{a_i m}$, where $a_i = a_{i-1}b_i$ for some $b_i \geq 1$, $a_0 := 1$, so that $S(\Phi_{i,m})$ is disjoint to the generic points of $S(\Phi_{1,m}) \cap \cdots \cap S(\Phi_{i-1,m})$ for all $m \geq 2$. It follows that $S(\Phi_{1,m}) \cap \cdots \cap S(\Phi_{r,m}) = \emptyset$ because $\dim(H) = r-1$.

Now let $m_r := 2$ and $m_i := b_{i+1}m_{i+1}$, $i = r-1,\ldots,1$. Then $n := a_i m_i$ ist constant for all $i = 1,\ldots,r$. It follows that $\Phi := \sum_{i=1}^r \Phi_{i,m_i}\colon \mathcal{L}^{\oplus r} \twoheadrightarrow \mathcal{L}^n|_H$ is the desired surjective extension of φ.

In particular $\mathcal{E} := \ker\Phi$ fits in a short exact sequence

$$0 \to \mathcal{E} \to \mathcal{L}^{\oplus r} \xrightarrow{\Phi} \mathcal{L}^n|_H \to 0 \qquad (2.2.6.2)$$

and we conclude that $\mathcal{E}$ is locally free of rank 2 because $\operatorname{pd}(\mathcal{L}^n|_H) = 1$. If we apply $\cdot \otimes \mathcal{O}_Y$, we obtain the previous short exact sequence back since $\mathcal{E}|_Y$ has no sections supported on $H \cap Y = D$. □

(2.2.7) Corollary. *Let X be an r-dimensional scheme that is projective over a noetherian ring and satisfies S_1. Then for every subscheme $Y \subset X$ with $\dim(Y) \leq 1$ there exists a family of locally free $\mathcal{O}_X$-modules $\mathcal{E}_n$, $n \in \mathbb{N}$, of rank r with the following properties:*

- *(i) For each $n \in \mathbb{N}$ holds $\mathcal{E}_n|_Y \simeq \mathcal{O}_Y^{\oplus r}$.*
- *(ii) For every coherent $\mathcal{O}_X$-module $\mathcal{F}$ and $n \gg 0$ holds $\mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{E}_n) = 0$ for $i \geq 2$; i.e. $(\mathcal{E}_n)$ is 1-ample.*

Proof. Take for $\mathcal{E}_n$, $n \geq n_0 \gg 0$, the locally free sheaf constructed in Proposition 2.2.6. By renumbering we may assume that $n_0 = 1$. Then property (i) is clear and (ii) follows from tensoring $\mathcal{F}$ with the exact sequence (2.2.6.1), applying $\Gamma(X,\cdot)$ and invoking Serre's vanishing theorem. □

If X is 2-dimensional and projective, then the corollary above specializes to Theorem 2.2.1. By the previous reduction arguments we can remove the projectivity assumption to prove the general case.

Proof of Theorem 2.2.1. If $X' \to X$ denotes the S_1-ization of X, then every locally free $\mathcal{O}_{X'}$-module descends to X by Lemma 2.2.3; thus we may assume that X has no embedded points using Theorem 1.5.2.

Then by Lemma 2.2.4 there exists a proper birational map $f\colon X' \to X$ with X' projective and satisfying S_1. Let $Y' \subset X'$ be the union of the ramification subscheme and the exceptional fiber. By Corollary 2.2.7 for each $m \in \mathbb{N}$ there exists a cohomologically 1-ample family of vector bundles $(\mathcal{E}'_n)$ on X' that is trivial on mY'. Taking m sufficiently large we can achieve that the family $(\mathcal{E}'_n)$ descends to a family of vector bundles $(\mathcal{E}_n)$ on X by Lemma 2.2.3. Then by Theorem 1.5.2 we conclude that $(\mathcal{E}_n)$ is 1-ample. □

2.3. Proof of the resolution property

Finally, we collect the results of the preceding sections and deduce our main theorem.

PROOF OF THEOREM 2.0.1. By Proposition 1.4.13 there exists an almost ample family of coherent sheaves $\mathcal{F}_i$, $i \in I$. It suffices therefore to find for each $i \in \mathbb{N}$ and $n \gg 0$ a locally free resolution $\mathcal{E} \twoheadrightarrow \mathcal{F}_i^{\otimes n}$.

Since $\mathcal{F}_i^{\vee}$ is 1-ample by Proposition 1.5.9, we deduce from Proposition 2.1.9 the existence of such a resolution with $\mathcal{E}$ not locally free but satisfying F_1. However, we can resolve $\mathcal{E}$ by an algebraic vector bundle $\mathcal{E}'$ using Corollary 2.1.11 because there exists a 1-ample family of locally free $\mathcal{O}_X$-modules of constant rank due to Theorem 2.2.1. □

We finish now the verification of the resolution property for surfaces and turn to a more general setting in order to study the geometric significance of the resolution property.

Part 2

Generating vector bundles on algebraic stacks

CHAPTER 3

Generating subcategories of quasicoherent sheaves

We shall investigate generators in the category of quasicoherent sheaves of modules on an algebraic stack and their variation with respect to morphisms of algebraic stacks. In the following chapter we specialize to locally free generators to study the resolution property of algebraic stacks. However, there is no further complexity if we take arbitrary finitely presented sheaves into account. In fact, the existence of a set of finitely presented generators is true in broad generality.

Also, we introduce the notion of a relatively generating family of finitely presented quasicoherent sheaves. This meets the requirements of all intended applications with geometric significance. Another natural restriction we make, is the exclusive consideration of quasicompact and quasiseparated morphisms, as their associated pushforward functor of sheaves of modules preserves quasicoherence.

This approach was already successively used by M. Olsson and J. Starr [OS03] and Kresch [Kre09] to study locally free generating sheaves of Deligne-Mumford stacks relative to their coarse moduli spaces, or in more down-to-earth terms by Thomason [Tho87] to study the equivariant resolution property. We understand our treatment of relatively generating families as a natural generalization to arbitrary algebraic stacks.

It is no surprise that generating families satisfy the analogous permanence properties of relatively ample line bundles. However, our proofs are formal in nature and based on adjoint functors. In fact, we will derive all properties from the flat base change theorem and from the left-cancellation property of injective set maps. Most of the results of this section are in this generality even new for schemes.

After a brief repetition of quasicoherent sheaves in section 3.1, and the definitions of generating families and subcategories in section 3.2, we investigate finitely presented generators of quasicoherent sheaves in section 3.3. The existence thereof is equivalent to the *completeness property* whose investigation was initiated by Rydh [Ryd10b] while studying noetherian approximation for algebraic stacks. We continue in section 3.4 the discussion by introducing a relative version of finitely presented generators and take the time to verify the expected permanence properties with respect to composition of morphisms, base change and flat descent.

In the final section 3.5 we show that generating families are preserved by finite, flat coverings (3.5.1) and also discuss the usage of arbitrary affine flat coverings. For example, we verify the *flat resolution property* for an algebraic stack with affine diagonal; that is, every quasicoherent sheaf is a quotient of a flat quasicoherent sheaf on an algebraic stack with affine diagonal (3.5.5). However, there is no reason to hope that these flat sheaves are finitely presented on algebraic stacks with non-quasifinite diagonal in general, so that we cannot derive a proof of the resolution property without any additional assumption (see 4.3.8 for a counterexample).

3.1. The category of quasicoherent sheaves

We briefly recall some definitions and properties of quasicoherent sheaves of modules on an algebraic stack X.

There are several ringed topoi associated to X: three big topoi X_{Zar}, X_{fppf}, $X_{\mathrm{\acute{E}T}}$, the small lisse-étale topos $X_{\text{lis-ét}}$ (developed in [LMB00] and corrected in [Ols07]) and the small étale topos $X_{\text{ét}}$. However, unless X is a Deligne-Mumford stack (for example an algebraic space or a scheme), the latter might be empty. In spite of this diversity, there are natural equivalences of the category of quasicoherent $\mathcal{O}_X$-modules with respect to X_{Zar}, X_{fppf}, $X_{\mathrm{\acute{E}T}}$, $X_{\text{lis-ét}}$, respectively to $X_{\text{ét}}$ if X is Deligne-Mumford [Lie08, A.1]. If X is a scheme, then this equivalence extends to the small Zariski-Topos X_{zar}. We will always consider an algebraic stack with its lisse-étale topos, but by abuse of notation we frequently switch to the small étale topos (resp. Zariski-topos) if X is a Deligne-Mumford stack (resp. a scheme).

Recall that the category $\mathrm{Mod}(\mathcal{O}_X)$ is an abelian tensor category; i.e. it is endowed with a tensor product $\otimes_{\mathcal{O}_X}$ that is symmetric, with an identity element $\mathcal{O}_X$, and with an internal bi-functor $\mathcal{H}om_{\mathcal{O}_X}(\cdot,\cdot)$, which is pointwise right adjoint to the tensor product. These data are supposed to satisfy some compatibility conditions.

The left tensor functor $\mathcal{E}\otimes_{\mathcal{O}_X}\colon \mathrm{Mod}(X) \to \mathrm{Mod}(X)$ preserves quasicoherence for every quasicoherent $\mathcal{O}_X$-module $\mathcal{E}$ so that the abelian full subcategory $\mathrm{QCoh}(X) \subset \mathrm{Mod}(\mathcal{O}_X)$ inherits the tensor structure from $\mathrm{Mod}(\mathcal{O}_X)$.

If $\mathcal{E}$ is of finite presentation, then also $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\cdot)\colon \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_X)$ preserves quasicoherence and hence induces by restriction a right-adjoint of $\mathcal{E}\otimes_{\mathcal{O}_X}\cdot\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$. If $\mathcal{E}$ is not finitely presented, then there is still a way to define such a right adjoint by composing $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\cdot)$ with the quasi-coherator Q_X. The latter is the right adjoint of the natural embedding $\mathrm{QCoh}(X) \to \mathrm{Mod}(X)$, which exists due to the special adjoint functor theorem as long as as $\mathrm{QCoh}(X)$ has a *set* of generators (see [TT90, B.12] or [SGA 6, II.3]). This condition is true for noetherian stacks or more generally for those satisfying the completeness property (the proof of [SGA 6, II.3.2] translates literally using the definitions of §3.3.2), but the general case is unclear (for the author). A more serious drawback is that Q_X does *not* commute with localizations. Therefore we stick to finitely presented sheaves in this context.

Given a morphisms $f\colon X \to Y$ of algebraic stacks, there are two induced functors $f_*\colon \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_Y)$ and $f^*\colon \mathrm{Mod}(\mathcal{O}_Y) \to \mathrm{Mod}(\mathcal{O}_X)$ which define an adjoint pair (f^*, f_*). Recall, that the pullback functor $f^*\colon \mathrm{Mod}(\mathcal{O}_Y) \to \mathrm{Mod}(\mathcal{O}_X)$ preserves the tensor structures, i.e. there exists isomorphisms $f^*\mathcal{O}_X \simeq \mathcal{O}_Y$ and $f^*(\cdot \otimes_{\mathcal{O}_Y} \cdot) \simeq f^*(\cdot) \otimes_{\mathcal{O}_Y} f^*(\cdot)$ that satisfy suitable coherence conditions. If f is quasicompact and quasiseparated, then f^* and f_* restrict to an adjoint pair (f^*, f_*) of functors between quasicoherent sheaves $f_*\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ and $f^*\colon \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ (we refer the reader to [Ols07, 6.5] for more details).

3.2. Generating subcategories

For the readers convenience we briefly recall the classical notion of a generator in an arbitrary category before we specialize to our situation in section 3.3.

Let $\mathcal{C}$ be a category and $\mathcal{G}$ a full subcategory. Recall that $\mathcal{G}$ is called a *generating subcategory for* $\mathcal{C}$ if for every pair of morphisms $f, g\colon Y \to X$ in $\mathcal{C}$, such that for all objects Z of $\mathcal{G}$ and maps $p\colon Z \to Y$ in $\mathcal{G}$ holds $fp = gp$, follows $f = g$ already[1]. We call a set of objects G of $\mathcal{C}$ a *generating set of objects*, or slightly abusive a *generating*

[1]This definition of a generating subcategory slightly differs from the one given in [SGA 4.1, I.7.1], where it is called a *by epimorphisms generating subcategory*.

family of objects, if the full subcategory it spans is a generating subcategory. Let us give a list of equivalent definitions:

(3.2.1) Lemma. *Let $\mathcal{C}$ be a category and G a set of objects. Then the following statements are equivalent:*

(i) G is a generating set for $\mathcal{C}$.

(ii) The functor $\gamma_*\colon \mathcal{C} \to \mathbf{Set}^G$, $X \mapsto (\mathrm{Hom}(Y,X))_{Y\in G}$ is faithful.

(iii) Let $\mathcal{G}$ be the full subcategory of $\mathcal{C}$, spanned by G. Then the restricted Yoneda functor $\mathcal{C} \to \mathrm{PrSh}(\mathcal{G})$, $X \mapsto \mathrm{Hom}(\cdot, X)|_{\mathcal{G}^{opp}}$ is faithful[2]*.*

If $\mathcal{C}$ is cocomplete, we can enlarge the list of equivalent conditions as follows:

(iv) For every object X of $\mathcal{C}$ the evaluation map

$$\varepsilon_X\colon \coprod_{Y\in G}\ \coprod_{f\in\mathrm{Hom}(Y,X)} Y \to X \tag{3.2.1.1}$$

is an epimorphism.

(v) For every object X of $\mathcal{C}$, there exists a family of objects Y_i, $i \in I$, in G and an epimorphism $\varphi\colon \coprod_{i\in I} Y_i \to X$.

The proof is formal and left to the reader.

In general, the generating subcategories $\mathcal{G}$, we are interested in, are not closed under infinite coproducts, as one obviously looses finiteness properties. Therefore it makes sense to keep the coproduct in (v). However, all coproducts will exist in $\mathcal{C}$ in our applications, so that we are allowed to work with a *canonical* morphism as in (iv) instead of the one in (v), whose choice depends on X.

It turns out that the functorial approach in (ii) is the right one to compare generating sets between different categories. In fact, the cocompleteness of $\mathcal{C}$ ensures, that γ_* has a left-adjoint given by

$$\gamma^*\colon \mathbf{Set}^G \to \mathcal{C}, \quad (A_X)_{X\in G} \mapsto \coprod_{X\in G}\ \coprod_{a\in A_X} X.$$

The evaluation map ε becomes then the counit of the adjunction (γ^*, γ_*). The delta functions $(\delta_X)_{X\in G}$ form a generating set for $\mathbf{Set}^G$ and are mapped by γ to G up to isomorphism.

The upshot is that we can identify a family G of objects in $\mathcal{C}$ by an adjoint pair (γ^*, γ_*), and G is generating if and only if γ_* is faithful. The collection of faithful functors is closed under composition and satisfies the left-cancellation property. Those that have a left adjoint are characterized as follows:

(3.2.2) Proposition. *([Par70, 2.12.3]) Let $f_*\colon \mathcal{C} \to \mathcal{C}'$, $f^*\colon \mathcal{C}' \to \mathcal{C}$ be functors such that f^* is a left adjoint of f_*, and denote by $\varepsilon\colon f^*f_* \Rightarrow \mathrm{id}_{\mathcal{C}}$ the counit. Then the following statements are equivalent:*

(i) f_* is faithful.

(ii) f_* reflects epimorphisms.

(iii) If $\psi\colon Y \to f_*(X)$ is an epimorphism, then $\psi^\sharp\colon f^*(Y) \to X$ is also an epimorphism, where $\psi^\sharp = \varepsilon_X \circ f^*(\psi)$.

(iv) f^* preserves generating sets.

(v) For each object X of $\mathcal{C}$, the counit $\varepsilon_X\colon f^*f_*(X) \to X$ is an epimorphism.

---

[2] We denote by $\mathrm{PrSh}(\cdot)$ the category of presheaves on a given category.

The codomain of the functor in Lemma 3.2.1.(ii) still depends on the generating set. However, if we want to compare generating sets of two *given* categories (e.g. QCoh(X) and QCoh(Y)), we need a more general definition:

(3.2.3) Definition. A family of functors $(f_i\colon \mathcal{C} \to \mathcal{D})_{i\in I}$ is called *faithful* if for any objects X, Y of $\mathcal{C}$ the map $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \prod_{i\in I} \mathrm{Hom}_{\mathcal{D}}(f_i(X), f_i(Y))$ is injective, or equivalently, if the induced functor $\mathcal{C} \to \mathcal{D}^I$ is faithful. Explicitly, for every pair $s_1 \neq s_2$ of different arrows $: X \to Y$ in $\mathcal{C}$ there exists an element $i \in I$ such that $f_i(s_1) \neq f_i(s_2)$ are different arrows $f_i(X) \to f_i(Y)$ in $\mathcal{D}$.

Faithful families will play a crucial role in the forthcoming argumentations. We adopted this point of view because is is a natural generalization of the geometrical fact that families of hyperplane sections in the projective space $\mathbb{P}^n$ separate arbitrary pairs of points or equivalently that $\mathcal{O}_{\mathbb{P}^n}(-d)$, $d \in \mathbb{N}$, is a generating family for QCoh($\mathbb{P}^n$).

(3.2.4) *Remark.* The collection of families of functors with fixed source and target is endowed with a natural composition. Namely, for two families of functors $(f_i\colon \mathcal{C} \to \mathcal{D})_{i\in I}$ and $(g_j\colon \mathcal{D} \to \mathcal{E})_{i\in I}$, we define the composed family as $(g_j \circ f_j\colon \mathcal{C} \to \mathcal{E})_{(i,j)\in I\times J}$. A straightforward formal calculation shows that this composition inherits the left-cancellation property for faithful families of functors from the one of faithful functors.

Proposition 3.2.2 generalizes to faithful sets of functors as follows:

(3.2.5) Proposition. *Let $(f_i)_{i\in I}$ be a family of adjoint functors $f_{i*}\colon \mathcal{C} \to \mathcal{C}'$, $f_i{}^*\colon \mathcal{C}' \to \mathcal{C}$ with $\mathcal{C}$ cocomplete. Then the following conditions are equivalent:*

(i) $(f_i)_{i\in I}$ is a faithful family of functors.

(ii) $(f_i)_{i\in I}$ reflects epimorphisms. That is, if $u\colon X \to Y$ is a morphisms in $\mathcal{C}$, such that each $f_{i*}(u)$ is an epimorphism, then u is an epimorphism.

(iii) If $(\psi_{f_i}\colon Y'_i \twoheadrightarrow f_{i*}(X))_{i\in I}$ is a family of epimorphisms for an abject X in $\mathcal{C}'$, then $\coprod_{i\in I} f_i{}^*(Y'_i) \to X$ is an epimorphism in $\mathcal{C}$.

(iv) If G' is a generating set for $\mathcal{C}'$, then the family $(f_i{}^*(Y'))_{i\in I,\, Y'\in G'}$ defines a generating set for $\mathcal{C}$.

(v) For each object X in $\mathcal{C}$, the coproduct of counits $\coprod_{i\in I} f_i{}^* f_{i*}(X) \to X$ is an epimorphism.

3.3. Globally generating families

3.3.1. Finitely presented generators.

We apply the previous categorical definitions of generating families to introduce the definition of a generating family of finitely presented (and hence quasicoherent) $\mathcal{O}_X$-modules for an algebraic stack.

(3.3.1) *Convention.* A family of quasicoherent $\mathcal{O}_X$-modules $(\mathcal{E}_i)_{i\in I}$ on an algebraic stack X is denoted by $\mathcal{E}_I$ to keep the notation streamlined. Also, every algebraic operation that applies to an $\mathcal{O}_X$-module, translates to families of $\mathcal{O}_X$-modules by pointwise application. For example, if $f\colon Y \to X$ is a morphism of algebraic stacks, then $f^*\mathcal{E}_I$ denotes the family $(f^*\mathcal{E}_i)_{i\in I}$.

Before we give a list of equivalent definitions, let us introduce some notation.

(3.3.2) Definition. For a family of $\mathcal{O}_X$-modules $\mathcal{E}_I$, we call the induced family $\mathcal{E}_I^+$ of all finite direct sums $\bigoplus_{j\in J} \mathcal{E}_i^{\oplus n_j}$, $J \subseteq I$, $n_j \in \mathbb{N}$, the *additive closure* of $\mathcal{E}_I$. Clearly holds $\mathcal{E}_i \subseteq \mathcal{E}_I^+$ as sets and if $\mathcal{E}_i \supseteq \mathcal{E}_I^+$, then we say that $\mathcal{E}_I$ is *additively closed*.

(3.3.3) Lemma. *Let X be a quasicompact and quasiseparated algebraic stack and $\mathcal{E}_I$ a family of finitely presented $\mathcal{O}_X$-modules. Then the following assertions are equivalent:*

(i) *The family of functors*

$$\mathrm{Hom}(\mathcal{E}_i, \cdot)\colon \mathrm{QCoh}(X) \to \mathrm{Mod}_{\mathbb{Z}}, \tag{3.3.3.1}$$

where i runs over all elements in I, is a faithful family.

(ii) *For every quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$, the canonical evaluation map is an epimorphism*

$$\bigoplus_{i\in I} \bigoplus_{\varphi\in \mathrm{Hom}(\mathcal{E}_i,\mathcal{M})} \mathcal{E}_i \twoheadrightarrow \mathcal{M}. \tag{3.3.3.2}$$

(iii) *For every quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$, there exists a subset $J \subseteq I$, a family of positive integers $n\colon J \to \mathbb{N}_0$ and an epimorphism*

$$\bigoplus_{i\in J} \mathcal{E}_i^{\oplus n_i} \twoheadrightarrow \mathcal{M}. \tag{3.3.3.3}$$

(iv) *For every quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$, there exists a filtered direct system of sheaves $\mathcal{E}_\alpha$ in the additive closure $\mathcal{E}_I^+$, and an epimorphism*

$$\varinjlim_{\alpha\in A} \mathcal{E}_\alpha \twoheadrightarrow \mathcal{M}. \tag{3.3.3.4}$$

(v) *Every quasicoherent $\mathcal{O}_X$-module is the direct limit of quasicoherent $\mathcal{O}_X$-submodules of finite type, and for every quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$ of finite type there exists an epimorphism*

$$\mathcal{E} \twoheadrightarrow \mathcal{M}, \tag{3.3.3.5}$$

where $\mathcal{E}$ is the additive closure $\mathcal{E}_I^+$.

PROOF. The equivalence of (i), (ii) and (iii) is formal. (iii) ⇒ (iv) is trivial and (iv) ⇒ (iii) holds because an inductive limit is a quotient of the direct sum of its members. Finally, (v) ⇔ (iii) is straightforward. □

(3.3.4) Definition. Let X be a quasicompact and quasiseparated algebraic stack. A family $\mathcal{E}_I$ of quasicoherent $\mathcal{O}_X$-modules of finite presentation is called a *generating family (or subcategory) for X*, if one of the equivalent assertions (i)-(v) of Lemma 3.3.3 is satisfied. Equivalently, if the full subcategory of QCoh(X), spanned by $\mathcal{E}_I$, is a generating subcategory for QCoh(X) in the sense of section 3.2.

(3.3.5) Definition. By replacing QCoh(X) in Definition 3.3.4 with a subcategory $\mathcal{C}$ (e.g. all flat, or finitely presented, or finite-type quasicoherent $\mathcal{O}_X$-modules), we say that $\mathcal{E}_I$ is a *generating family for* $\mathcal{C}$

Every algebraic stack or scheme is patched together by affine schemes in a suitable topology. However, the existence of a generating family is a *global property* of the whole stack. In case that one faces an affine scheme, the structure of a generating family is simple, as expected.

(3.3.6) Example. Let X be an affine scheme. Then $\mathcal{O}_X$ is a generator for X. Furthermore, it is an projective object in QCoh(X). So any family of quasicoherent $\mathcal{O}_X$-modules of finite presentation is generating if and only if $\mathcal{O}_X$ is a direct summand of some member $\mathcal{E}_i$ by 3.3.3.(v).

The local structure of an algebraic stack is also connected to the linear representations of the stabilizer groups. However, we postpone the discussion of the generators of those until we have set up the framework of relatively generating families in 3.4.

As this work specializes later on to the study of *finite-type locally free* generators we provide the following definition.

(3.3.7) Definition. A quasicompact and quasiseparated algebraic stack X has the *resolution property* (or *enough locally free sheaves*), if the subcategory of locally free $\mathcal{O}_X$-modules of finite type is a generating subcategory for $\mathrm{QCoh}(X)$.

However, before we deal with this question, we remind the reader that even the existence of an arbitrary generating family (of finitely presented sheaves) lacks a complete treatment. It is not known for an arbitrary quasicompact and quasiseparated algebraic stack X, unless X is noetherian, or more generally, if X satisfies the completeness property, as seen below 3.3.9.

3.3.2. Completeness property. — D. Rydh introduced in [Ryd10b] the *completeness property* for quasicompact and quasiseparated algebraic stacks, in order to investigate the usual completeness-, presentation- and extension properties of quasicoherent $\mathcal{O}_X$-modules and quasicoherent (integral) $\mathcal{O}_X$-algebras, which are classically know to hold for all quasicompact and quasiseparated schemes [EGA $\mathrm{I}_{\mathrm{2nd}}$] or for noetherian stacks [LMB00].

For that, consider the following properties of $\mathrm{QCoh}(X)$ for an algebraic stack X:

Completeness

(C1) Every quasicoherent $\mathcal{O}_X$-module is the direct limit of quasicoherent $\mathcal{O}_X$-submodules of finite type.

(C2) Every quasicoherent $\mathcal{O}_X$-module is a filtered direct limit of quasicoherent $\mathcal{O}_X$-modules of finite presentation.

(C3) Every quasicoherent $\mathcal{O}_X$-module is a quotient of a filtered direct limit of quasicoherent $\mathcal{O}_X$-modules of finite presentation.

Presentation — Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module of finite type.

(P1) There exists a finitely presented quasicoherent $\mathcal{O}_X$-module $\mathcal{P}$ and an epimorphism $\mathcal{P} \twoheadrightarrow \mathcal{F}$.

(P2) $\mathcal{F} = \varinjlim_\lambda \mathcal{F}_\lambda$ for a filtered direct system of finitely presented quasicoherent $\mathcal{O}_X$-modules with surjective bonding maps.

Extension — Let $U \subseteq X$ be a quasicompact open subset.

(E1) If $\mathcal{G}$ is a quasicoherent $\mathcal{O}_U$-module of finite type (resp. finite presentation), then there exists a quasicoherent $\mathcal{O}_X$-module $\mathcal{H}$ of finite type (resp. finite presentation), such that $\mathcal{H}|_U = \mathcal{G}$.

(E2) If $\mathcal{F}$ is a quasicoherent $\mathcal{O}_X$-module and $u\colon \mathcal{G} \to \mathcal{F}|_U$ a homomorphism of quasicoherent $\mathcal{O}_U$-modules with $\mathcal{G}$ of finite type (resp. of finite presentation), then there exists a quasicoherent $\mathcal{O}_X$-module $\mathcal{H}$ of finite type (resp. of finite presentation) and a map $v\colon \mathcal{H} \to \mathcal{F}$ extending $\mathcal{G}$ and u.

(3.3.8) Definition (Completeness property)**.** A quasicompact and quasiseparated algebraic stack X has the *completeness property* if all properties above are satisfied. By [Ryd10b, 4.3] condition (C3) already determines the other ones and the analogous properties for the categories of quasicoherent $\mathcal{O}_X$-algebras respectively quasicoherent integral $\mathcal{O}_X$-algebras are also fulfilled.

Using the equivalence of 3.3.3.(iii) and (C3) we may regard the completeness property as a natural hypothesis which prevents us to study the empty set.

(3.3.9) Proposition. *A quasicompact and quasiseparated algebraic stack X has the completeness property if and only if there exists a generating family of finitely presented $\mathcal{O}_X$-modules.*

Proof. The subcategory of all finitely presented $\mathcal{O}_X$-modules is additively closed. So (C3) and 3.3.3.(iv) are equivalent. □

(3.3.10) *Remark.* The completeness property holds for a vast class of algebraic stacks: Quasicompact and quasiseparated schemes [EGA I, §6.9], noetherian algebraic stacks [LMB00, 15.4], quasicompact and quasiseparated Deligne-Mumford stacks [Ryd10b, 4.11] or more generally all stacks of approximation type ([Ryd10b, 4.12]).

3.3.3. Embeddings of affine maps.

We finish the discussion of global generators with an application to the embeddability of affine maps.

Recall that every affine scheme $Y = \operatorname{Spec} A$ that is of finite type, say over a field k, can be embedded in an affine space $\mathbb{A}^n_k$ for sufficiently large $n \in \mathbb{N}$, by choosing a finite set of generators for the k-algebra A. Clearly, if one choses for k an arbitrary ring, the same proof applies.

We show that this situation even translates to the general setting that the base is an *arbitrary* algebraic stack X that has a generating family. Even more, this factorization principle *characterizes* generating families.

(3.3.11) Theorem. *Let X be a quasicompact and quasiseparated algebraic stack, and let $\mathcal{E}_I$ be a family of finitely presented $\mathcal{O}_X$-modules that is closed under finite tensor products and finite direct sums. Then the following are equivalent:*

(i) *$\mathcal{E}_I$ is a generating family for X.*

(ii) *For every affine morphism $f\colon Y \to X$ there exists a sheaf $\mathcal{E} = \bigoplus_{i\in J} \mathcal{E}_i$, for some subset $J \subset I$, and a factorization*

$$\begin{array}{ccc} Y & \overset{i}{\hookrightarrow} & \mathbb{V}(\mathcal{E}) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p} & \\ & X & \end{array}$$

such that i is a closed immersion. If f is of finite type, then I can be chosen to be finite.

Proof. Recall that, Sym is a left adjoint to the forgetful functor F from the category of quasicoherent $\mathcal{O}_X$-algebras in the category of quasicoherent $\mathcal{O}_X$-modules. In particular, Sym preserves epimorphisms and inductive limits.

First, assume that $\mathcal{E}_I$ is generating. Since f is affine, f_*O_Y is a quasicoherent $\mathcal{O}_X$-module, so there exists a direct sum $\mathcal{E} = \bigoplus_{j\in J} \mathcal{E}_j$ and surjection $\varphi\colon \mathcal{E} \twoheadrightarrow f_*\mathcal{O}_Y$ of quasicoherent $\mathcal{O}_X$-modules. Then $\operatorname{Sym}(\varphi)\colon \operatorname{Sym}\mathcal{E} \twoheadrightarrow f_*\mathcal{O}_Y$ is a surjection of quasicoherent $\mathcal{O}_X$-algebras that corresponds to a closed immersion $Y \hookrightarrow \operatorname{Spec}_X \operatorname{Sym}\mathcal{E}$. If f is of finite type, then $f_*\mathcal{O}_Y$ is a locally finitely generated $\mathcal{O}_X$-algebra. Since X is quasicompact, we can chose a finite subset $K \subset J$ such that the restriction of φ to the direct summands indexed by K still induces a surjection $\operatorname{Sym}(\bigoplus_{i\in K} \mathcal{E}_i) \twoheadrightarrow f_*\mathcal{O}_Y$ of quasicoherent $\mathcal{O}_X$-algebras.

Conversely, let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. Applying the factorization on $\operatorname{Spec}_X(\operatorname{Sym}\mathcal{F}) \to X$ we obtain a surjective epimorphism $\operatorname{Sym}(\bigoplus_{i\in I} \mathcal{E}_i) \twoheadrightarrow \operatorname{Sym}(\mathcal{F})$ of quasicoherent $\mathcal{O}_X$-algebras which is also an epimorphism of quasicoherent $\mathcal{O}_X$-modules. The canonical homomorphism of quasicoherent $\mathcal{O}_X$-modules $\mathcal{F} \to \operatorname{Sym}(\mathcal{F})$ is split injective, thus has a section. Then the composition gives a surjection $\operatorname{Sym}(\bigoplus_{i\in I} \mathcal{E}_i) \twoheadrightarrow \mathcal{F}$.

To see that the domain is a direct sum of tensor products, let A be the set of finite subsets of I. Then $\bigoplus_{i\in I}\mathcal{E}_i$ is the direct limit $\varinjlim_{\alpha\in A}\mathcal{E}_\alpha$ for $\mathcal{E}_\alpha = \bigoplus_{i\in\alpha}\mathcal{E}_i$ and we get a chain of surjections.

$$\bigoplus_{\alpha}\bigoplus_{n\geq 0}\mathcal{E}_\alpha^{\otimes n} \twoheadrightarrow \varinjlim_{\alpha}\bigoplus_{n\geq 0}\mathcal{E}_\alpha^{\otimes n} \twoheadrightarrow \varinjlim_{\alpha}\operatorname{Sym}\mathcal{E}_\alpha \xrightarrow{\sim} \operatorname{Sym}(\varinjlim_{\alpha}\mathcal{E}_\alpha)$$

This shows that $\mathcal{E}_I$ is a generating family, as required. □

In particular, this yields an embedding criterion characterizing the resolution property of the base:

(3.3.12) Corollary. *A noetherian algebraic stack S has the resolution property if and only if every affine morphism $X \to S$ of finite type factors over S by a closed embedding into some vector bundle $\mathbb{V}(\mathcal{E}) \to S$.*

3.4. Relatively generating families

We introduce the notion of *relative* generating families of finitely presented quasicoherent sheaves of modules for a quasicompact and quasiseparated morphism of algebraic stacks. Their permanence properties are discussed subsequently in §3.4.1.

(3.4.1) Lemma. *Let $f\colon X \to Y$ be a quasicompact and quasiseparated morphism of algebraic stacks and $\mathcal{E}_I = (\mathcal{E}_i)_{i\in I}$ a family of quasicoherent $\mathcal{O}_X$-modules of finite presentation. Then the following are equivalent:*

(i) For each quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$, the sum of evaluation maps below is an epimorphism in $\operatorname{QCoh}(X)$*:*

$$\operatorname{eval}_{\mathcal{E}_I,f}(\mathcal{M})\colon \bigoplus_i \mathcal{E}_i \otimes_{\mathcal{O}_X} f^*f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_i,\mathcal{M}) \quad\twoheadrightarrow\quad \mathcal{M}. \tag{3.4.1.1}$$

(ii) Let i run over all elements in I. Then the following family of functors is faithful:

$$f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_i,\cdot)\colon \operatorname{QCoh}(X) \to \operatorname{QCoh}(Y). \tag{3.4.1.2}$$

(3.4.2) Definition. Let $f\colon X \to Y$ be a quasicompact and quasiseparated morphism of algebraic stacks. A family $\mathcal{E}_I = (\mathcal{E}_i)_{i\in I}$ of finitely presented $\mathcal{O}_X$-modules is called an *weakly f-generating family for X* if one of the equivalent conditions in 3.4.1 is satisfied.

We call $\mathcal{E}_I$ *f-generating* if for every morphism of algebraic stacks $Y' \to Y$ the restricted family $\mathcal{E}_I|_{Y'\times_Y X}$ is weakly generating relative to $f_{Y'}\colon Y'\times_Y X \to Y'$ (we will see in 3.4.17, that we can equivalently restrict to affine schemes Y'). Moreover, these notions coincide if the target Y has quasiaffine diagonal by Proposition 3.4.16.

(3.4.3) *Remark.* —

(i) Definition 3.4.2 depends only on the 2-isomorphism class of f because 2-isomorphisms of functors preserve and reflect faithfulness.

(ii) To verify condition 3.4.2.(ii), we are allowed to compose the family of functors in question with an arbitrary faithful functor $\operatorname{QCoh}(X) \to \mathcal{C}$ or even an arbitrary family of faithful functors by 3.2.4.

(iii) For instance, if $Y = \operatorname{Spec} A$ is representable by an affine scheme (e.g. f is the canonical morphism with target $\operatorname{Spec}\mathbb{Z}$), then Definition 3.4.2 mutates in the definition of a (global) generating family for X (3.3.4) using that the forgetful functor $\operatorname{QCoh}(\operatorname{Spec} A) \simeq \operatorname{Mod}_A \to \operatorname{Mod}_{\mathbb{Z}}$ is faithful.

(iv) If $\mathcal{E}_I$ consists entirely of locally free sheaves then $\mathcal{E}_I$ then the evaluation map (3.4.1.2) can be written as $\bigoplus_{i\in I} \mathcal{E}_i \otimes_{\mathcal{O}_X} f^* f_*(\mathcal{E}_i^\vee \otimes_{\mathcal{O}_X} \mathcal{M}) \twoheadrightarrow \mathcal{M}$.

Ample line bundles form the prominent example, we seek to generalize:

(3.4.4) Example (Ample line bundles). Let $f\colon X \to Y$ be a quasicompact and quasiseparated morphism of schemes.

If an invertible sheaf $\mathcal{L}$ is f-ample then $(\mathcal{L}^{-n})_{n\in\mathbb{N}}$ is an f-generating family and f is separated, and the converse also holds. More generally, a family of invertible sheaves $(\mathcal{L}_i)_{i\in I}$ is f-ample if and only if $\{\mathcal{L}_i^{-n} \mid n \in \mathbb{N},\, i \in I\}$ is an f-generating family. It follows that f has affine diagonal but is not necessarily separated.

An important special case, is that $\mathcal{O}_X$ is f-ample if and only if it is f-generating, i.e. $f_*\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is faithful. We will study this case more intensively in section 5.

3.4.1. Permanence properties. This section is devoted to the verification of the expected permanence properties of (weakly) relatively generating sheaves, that were defined in 3.4.2.

3.4.1.1. *Local on the base.* Our first task is to show that for a family of quasi-coherent sheaves, the property of being (weakly) generating is local on the base for fpqc coverings.

(3.4.5) Proposition (fpqc-local on the base). *Let S be an algebraic stack, let $f\colon X \to Y$ be an S-morphism of algebraic S-stacks and $\mathcal{E}_I$ a family of quasicoherent $\mathcal{O}_X$-modules. Given a faithfully flat family $(s_\alpha\colon S_\alpha \to S)$, we obtain for each α a 2-cartesian base change square of algebraic stacks*

$$\begin{array}{ccc} X_\alpha & \xrightarrow{v_\alpha} & X \\ {\scriptstyle f_\alpha}\downarrow & \square & \downarrow{\scriptstyle f} \\ Y_\alpha & \xrightarrow{u_\alpha} & Y \end{array} \tag{3.4.5.1}$$

Suppose that each ${v_\alpha}^*\mathcal{E}_I$ is (weakly) f_α-generating, that f is quasicompact and quasiseparated and that each $\mathcal{E}_i$ is finitely presented; the latter two conditions are automatic if (u_α) is a fpqc-covering. Then $\mathcal{E}_I$ is (weakly) f-generating.

Proof. It suffices to prove the case of weakly generating sheaves. Consider for each α and i the following diagram:

$$\begin{array}{ccc} \mathrm{QCoh}(X) & \xrightarrow{{v_\alpha}^*} & \mathrm{QCoh}(X_\alpha) \\ {\scriptstyle \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_i,\cdot)}\downarrow & & \downarrow{\scriptstyle \mathcal{H}om_{\mathcal{O}_{X_\alpha}}({v_\alpha}^*\mathcal{E}_i,\cdot)} \\ \mathrm{QCoh}(X) & \xrightarrow{{v_\alpha}^*} & \mathrm{QCoh}(X_\alpha) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle {f_\alpha}_*} \\ \mathrm{QCoh}(Y) & \xrightarrow{{u_\alpha}^*} & \mathrm{QCoh}(Y_\alpha) \end{array}$$

The upper square is 2-commutative since $\mathcal{E}_i$ and ${v_\alpha}^*\mathcal{E}_i$ are of finite presentation and ${v_\alpha}^*$ commutes with the internal hom's by flatness. The lower square is 2-commutative by flat base change [LMB00, 13.1.9]. Thus, the whole diagram is 2-commutative.

Since (v_α) is a faithfully flat family, we infer that $(v_\alpha^*\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(X_\alpha))$ is a faithful family of functors. Therefore the statement is a consequence of the stability under composition and the left cancellation property for faithful families of functors (see Remark 3.2.4). $\square$

As an intermediate consequence we conclude that all *quasiaffine* morphism have relatively generating structure sheaf. This includes the collection of all morphisms that are affine, finite, quasi-finite separated morphisms, finite-type monomorphisms, quasicompact open immersions and closed immersions.

(3.4.6) Corollary. *Let $f\colon X \to Y$ be a quasiaffine morphism of algebraic stacks. Then $\mathcal{O}_X$ is f-generating.*

PROOF. The statement is local on Y, so we may assume that Y is affine. Then X is a quasiaffine scheme and the result follows from [EGA II, 5.1.2]. □

(3.4.7) *Remark.* If the relative stabilizer groups are affine, then the reverse implication is also true as we will see in Theorem 5.3.8 .

(3.4.8) Corollary. *Let $f\colon X \to Y$ be a morphism of algebraic stacks with quasiaffine diagonal; for example, if f has quasifinite and separated diagonal. Then $\mathcal{O}_X$ is Δ_f-generating.*

3.4.1.2. *Composition.* Next, we define a suitable "composition" of families that are generating relative to two composable morphisms.

(3.4.9) Proposition (Composition and left-cancellation property)**.** *Suppose we have a 2-commutative triangle of algebraic stacks*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{h} & \downarrow{\scriptstyle g} \\ & & Z \end{array} \tag{3.4.9.1}$$

and let $\mathcal{E}_I$ and $\mathcal{F}_J$ be families of quasicoherent $\mathcal{O}_X$-modules respectively $\mathcal{O}_Y$-modules.

(i) If $\mathcal{E}_I$ is (weakly) f-generating and $\mathcal{F}_J$ is (weakly) g-generating, then

$$\mathcal{E}_I \otimes f^*\mathcal{F}_J := (\mathcal{E}_i \otimes_{\mathcal{O}_X} f^*\mathcal{F}_j)_{(i,j)\in I\times J}$$

is a (weakly) h-generating family of $\mathcal{O}_X$-modules.

(ii) If $\mathcal{E}_I \otimes f^*\mathcal{F}_J$ is (weakly) h-generating, $\mathcal{F}_J$ is Δ_g-generating (resp. g is quasiseparated) and $\mathcal{F}_J$ is locally free of finite type, then $\mathcal{E}_I$ is (weakly) f-generating.

PROOF. *Part* (i): Note that the statement follows from the weak case by restricting (3.4.9.1) along an arbitrary morphism of algebraic stacks $Z' \to Z$ and using the isomorphism $(\mathcal{E}_I \otimes_{\mathcal{O}_X} f^*\mathcal{F}_J)|_{X'} \simeq \mathcal{E}_I|_{X'} \otimes_{X'} f'^*\mathcal{F}_J|_{Y'}$.

By assumption f and g are quasicompact and quasiseparated, so the same holds for h. In particular, the lower right triangle in the diagram below is well-defined and 2-commutative.

Let $i \in I$ and $j \in J$ be arbitrary. If $\mathcal{E}_i$ and $\mathcal{F}_j$ are of finite presentation, then also $\mathcal{E}_i \otimes f^*\mathcal{F}_j$ is of finite presentation. Consider then the following diagram

$$\begin{array}{ccccc} \mathrm{QCoh}(X) & \xrightarrow{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_i,\cdot)} & \mathrm{QCoh}(X) & \xrightarrow{f_*} & \mathrm{QCoh}(Y) \\ & \searrow^{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_i\otimes f^*\mathcal{F}_j,\cdot)} & \downarrow{\scriptstyle \mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{F}_j,\cdot)} & & \downarrow{\scriptstyle \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}_j,\cdot)} \\ & & \mathrm{QCoh}(X) & \xrightarrow{f_*} & \mathrm{QCoh}(Y) \\ & & & \searrow^{h_*} & \downarrow{\scriptstyle g_*} \\ & & & & \mathrm{QCoh}(Z) \end{array}$$

The upper left triangle is 2-commutative by adjunction of $\mathcal{E}_i \otimes$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_i, \cdot)$ in $\mathrm{QCoh}(X)$. The square is 2-commutative since it corresponds by adjunction to a 2-commutative square, which encodes the compatibility of f^* with tensor products. Thus, the whole diagram is 2-commutative, and the assertion follows from Lemma 3.4.1.(ii) using that faithful families of functors are stable under composition by Remark 3.2.4.

Part (ii) : First note that $\mathcal{E}_I$ consists of finitely presented $\mathcal{O}_X$-modules. Since Δ_g is quasiseparated, the hypotheses on h imply that f is quasicompact and quasiseparated. So the lower triangle in the diagram above is well-defined if we extend the lower right corner by the inclusion $\mathrm{QCoh}(Z) \subset \mathrm{Mod}_{\mathcal{O}_X}$, which is a faithful (and full) functor. So by Remark 3.4.3.(ii) we conclude the weak case of the assertion.

The proof of the general statement is a consequence of (i) by a standard argument, yet stated here for the readers convenience. Use that f factors up to 2-isomorphism as the composition of the upper horizontal morphisms of the following two 2-cartesian squares.

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_f} & X \times_Z Y \\ \downarrow{\scriptstyle f} & \square & \downarrow{\scriptstyle f\times 1} \\ Y & \xrightarrow{\Delta_g} & Y \times_Z Y \end{array} \qquad \begin{array}{ccc} X \times_Z Y & \xrightarrow{q} & Y \\ \downarrow{\scriptstyle p} & \square & \downarrow{\scriptstyle g} \\ X & \xrightarrow{h} & Z \end{array} \tag{3.4.9.2}$$

Since $\mathcal{F}_J$ is Δ_g-generating, $f^*\mathcal{F}_J$ is Γ_f-generating. So if $\mathcal{E}_I$ is h-generating, then $p^*\mathcal{E}_I$ is q-generating and hence ${\Gamma_f}^* p^*\mathcal{E}_I \otimes f^*\mathcal{F}_J \simeq \mathcal{E}_I \otimes f^*\mathcal{F}_J$ is f-generating by part (i). □

With a view towards quasiaffine maps we consider the special case that one of the families in Proposition 3.4.9 just consists of the structure sheaf.

(3.4.10) Corollary. *Suppose we have a 2-commutative square of algebraic stacks*

$$\begin{array}{ccc} U & \xrightarrow{u} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{v} & Y \end{array} \tag{3.4.10.1}$$

and a family $\mathcal{E}_I$ of quasicoherent $\mathcal{O}_X$-modules.

(i) If $\mathcal{E}_I$ is weakly f-generating and $\mathcal{O}_U$ is weakly u-generating, then $u^*\mathcal{E}_I$ is weakly g-generating.

(ii) If $\mathcal{E}_I$ is f-generating, $\mathcal{O}_U$ is u-generating and $\mathcal{O}_V$ is Δ_v-generating, then $u^*\mathcal{E}_I$ is g-generating.

Note that the assumptions on u and v always hold if u and v are quasiaffine.

Proof. Split the square in two 2-commutative triangles and use Proposition 3.4.9 twice. □

(3.4.11) Corollary (Reduction)**.** *Let $f\colon X \to Y$ be a morphism of algebraic stacks. If $\mathcal{E}_I$ is an (weakly) f-generating family of quasicoherent $\mathcal{O}_X$-modules, then $\mathcal{E}_I|_{X_{red}}$ is a (weakly) generating family of $\mathcal{O}_{X_{red}}$-modules relative to $f_{red}\colon X_{red} \to Y_{red}$.*

(3.4.12) *Remark.* Conversely, if $\mathcal{E}_I|_{X_{\mathrm{red}}}$ is f_{red}-generating, then it is not in general true, that $\mathcal{E}_I$ is f-generating. For example, let X be the spectrum of the ring of dual numbers $A = k[X]/(X^2)$ for some ring k. Then the A-module $A/(X)$ is not a generator for the category of A-modules, but the restriction to the reduction $A/(X) \otimes_A A/(X) \simeq A/(X) \simeq k$ is clearly a generator for the category of k-modules.

However, if $\mathcal{E}_I$ is the family, associated with a locally free tensor generator $\mathcal{E}$, then the converse holds by (v).

3.4.1.3. *Base change.* We show next that even weakly relatively generating families are stable under base change if the base has *quasiaffine diagonal* using characterization B.6 in terms of *locally quasiaffine* maps. By this, we mean a morphism $U \to V$ whose domain has a fpqc-covering $(U_i \to U)$ such that each composition $U_i \to V$ is quasiaffine (cf. B.5).

(3.4.13) Proposition (Base change)**.** *Let S be an algebraic stack, $f\colon X \to Y$ an S-morphism of algebraic S-stacks and $S' \to S$ be a morphism of algebraic stacks. Suppose that $Y_{(S')} \to Y$ is locally quasiaffine (if S or Y have quasiaffine diagonal, this always holds by B.6).*

If $\mathcal{E}_I$ is a weakly f-generating family of quasicoherent $\mathcal{O}_X$-modules, then $\mathcal{E}_{I(S')}$ is a weakly generating family of quasicoherent $\mathcal{O}_{X_{(S')}}$-modules relative to the base change $f_{(S')}\colon X_{(S')} \to Y_{(S')}$.

Proof. We may assume $S = Y$. By Corollary 3.4.10 the class of weakly relatively generating sheaves is stable under quasiaffine base change. On the other hand it is fpqc-local on the target by Proposition 3.4.5. Therefore it is stable under arbitrary locally quasiaffine morphisms. □

(3.4.14) *Remark.* The assumption that $Y_{(S')} \to Y$ is locally quasiaffine is necessary. For example, the left-cancellation property 3.4.9.(ii) implies, that every diagonal $\Delta_{X/Y}\colon X \to X \times_Y X$ of a morphism of algebraic stacks $X \to Y$ has *always* weakly relatively generating structure sheaf. However, as the following example illustrates, the attribute "weakly" cannot be omitted.

(3.4.15) Example. Let $A \xrightarrow{\pi} S$ be an abelian scheme. Then the natural map $p\colon S \to BA_S$ induces a 2-cartesian square

$$\begin{array}{ccc} A & \longrightarrow & BA_S \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \Delta} \\ S & \xrightarrow{(p,p)} & BA_S \times_S BA_S \end{array} \qquad (3.4.15.1)$$

Although $\mathcal{O}_{BA_S}$ is weakly Δ-generating, $\mathcal{O}_X$ is not weakly π-generating (unless A is finite).

However, if the target stack has quasiaffine diagonal, then the definitions of generating and weakly generating families are equivalent.

(3.4.16) Corollary. *Let $f\colon X \to Y$ be morphism of algebraic stacks. If $\mathcal{E}_I$ is a weakly generating family of $\mathcal{O}_X$-modules for f and if Y has quasiaffine diagonal (over $\mathbb{Z}$ or equivalently over some algebraic space S), then $\mathcal{E}_I$ is also f-generating.*

In particular, this holds if the target stack is an affine scheme. This allows us to set up a local criterion for generating sheaves.

(3.4.17) Proposition. *Let $f\colon X \to Y$ be a morphism of algebraic stacks and $\mathcal{E}_I$ a family of quasicoherent $\mathcal{O}_X$-modules. Then the following are equivalent:*

(i) *$\mathcal{E}_I$ is f-generating.*

(ii) *For every morphism* $\operatorname{Spec} A \to Y$, *the restriction $\mathcal{E}_I|_{X_A}$ is a generating family on $X_A =$* $\operatorname{Spec} A \times_Y X$.

(iii) *There exists a fpqc-covering family* $\operatorname{Spec} A_\alpha \to Y$ *such that each $\mathcal{E}_I|_{X_A}$ is a generating family on X_{A_α}.*

Suppose that f is quasicompact and quasiseparated, and that $\mathcal{E}_I$ is of finite presentation. Then the list of equivalent conditions enlarges by the following ones:

- *(iv) As in (ii) but with A being a complete local ring.*
- *(v) As in (iii) but $\operatorname{Spec} A_\alpha \to Y$ being an arbitrary faithfully flat family.*

Proof. This is straightforward using 3.4.5 and 3.4.16. $\square$

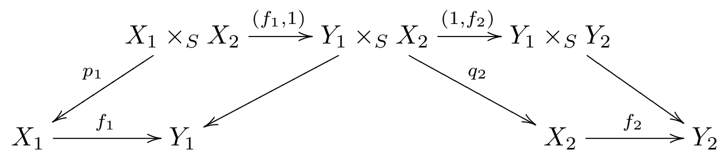
3.4.1.4. *Products.* We finish this section with the discussion of products of morphisms and products of generating sheaves thereof.

(3.4.18) Proposition (Products)**.** *Let S be an algebraic stack (resp. with quasi-affine diagonal), $f_\alpha\colon X_\alpha \to Y_\alpha$, $\alpha = 1, 2$, be two S-morphisms of algebraic S-stacks. If $\mathcal{E}_{I_\alpha}$ are (weakly) f_α-generating families on X_α, then the product family*

$$\mathcal{E}_{I_1} \boxtimes_S \mathcal{E}_{I_2} := ({p_1}^*\mathcal{E}_{i_1} \otimes_{\mathcal{O}_{X\times_S X'}} {p_2}^*\mathcal{E}_{i_2})_{(i_1,i_2)\in I_1\times I_2},$$

is a (weakly) $f_1 \times_S f_2$-generating family on $X_1 \times_S X_2$, where $p_\alpha\colon X_1 \times_S X_2 \to X_\alpha$ are the projections.

Proof. Decompose $f_1 \times_S f_2$ as the composition



By Proposition 3.4.13 we conclude that ${p_1}^*\mathcal{E}_{I_1}$ is a (weakly) $(f_1, 1)$-generating family on $X_1 \times_S X_2$ and ${q_2}^*\mathcal{E}_{I_2}$ is a (weakly) $(1, f_2)$-generating family on $Y_1 \times X_2$. Hence ${p_1}^*\mathcal{E}_{I_1} \otimes (f_1, 1)^*{q_2}^*\mathcal{E}_{I_2}$ is a (weakly) $f_1 \times_S f_2$-generating family on $X_1 \times_S X_2$. Since $q_2 \circ (f_1, 1) \simeq p_2$ this shows the result. $\square$

This establishes the full permanence properties of relative (weakly) generating families of finitely presented quasicoherent sheaves.

3.5. Flat affine descent

3.5.1. Finite locally free descent. From Proposition 3.4.9 and Corollary 3.4.6 follows that relative (weakly) generating families are preserved under pullbacks by quasiaffine morphisms. Our next concern is to show, that this also holds for pushforwards f_* in case that the morphism f in question is finite, faithfully flat and of finite presentation (or equivalently, finite, surjective and locally free). We infer this result by a careful study of the right adjoint $f^!$.

(3.5.1) Proposition. *Let $f\colon X \to Y$ a finite, faithfully flat and finitely presented morphism, and $g\colon Y \to Z$ be an arbitrary morphism of algebraic stacks. If $\mathcal{E}_I$ is a (weakly) $g\circ f$-generating family of $\mathcal{O}_X$-modules, then $f_*\mathcal{E}_I$ is a (weakly) g-generating family of $\mathcal{O}_Y$-modules.*

Proof. The assumptions on f are stable under arbitrary base change, so it suffices to prove the weak case. For that, let us assume that $\mathcal{E}_I$ is a weakly $g \circ f$-generating family of $\mathcal{O}_X$-modules. Since $g \circ f$ is quasicompact and f is surjective, g is quasicompact, and since $g \circ f$ is quasiseparated and f is quasicompact, g is quasiseparated. Moreover, f_* preserves finitely presented sheaves because f is finite and locally free.

We invoke Grothendieck duality for finite morphisms. Recall that f_* has a right adjoint $f^!$ defined by $f_*f^!(\cdot) = \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \cdot)$. Then the adjunction formula

$f_*\mathcal{H}om_{\mathcal{O}_X}(\cdot, f^!(\cdot)) \simeq \mathcal{H}om_{\mathcal{O}_Y}(f_*(\cdot), \cdot)$ implies that for each $i \in I$ we have an isomorphism of functors $\mathrm{QCoh}(X) \to \mathrm{QCoh}(Z)$

$$g_* \circ \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{E}_i, \cdot) \simeq g_* \circ (f_* \circ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_i, \cdot) \circ f^!).$$

It follows that f_* preserves relative weakly generating families if (and only if) $f^!$ is faithful. The latter holds if and only if the counit of the adjunction $f_*f^!(\mathcal{M}) \to \mathcal{M}$ is surjective for any quasicoherent $\mathcal{O}_Y$-module $\mathcal{M}$. By applying $\mathcal{H}om_{\mathcal{O}_Y}(\cdot, \mathcal{M})$ to the canonical map $\varphi_f : \mathcal{O}_Y \to f_*\mathcal{O}_X$, we see that this happens precisely, if φ_f is a locally split monomorphism of quasicoherent $\mathcal{O}_X$-modules (not necessarily algebras!). The latter is true by faithfully flatness of f. □

(3.5.2) *Remark.* Since f is not assumed to be étale, the counit $f_*f^! \Rightarrow 1$ is not a split surjection. However, the proof shows, that we just need that it is locally split with respect to the fppf covering f.

3.5.2. Affine faithfully flat descent and flat resolutions. Our next task is to prove a more general descent statement for faithfully flat and affine morphisms. As these morphisms do not preserve finitely presented sheaves, unless they are finite and finitely presented, we cannot expect a result as in Proposition 3.5.1. However, it allows us to perform fruitful reductions later on, as seen below in Theorem 3.5.5.

(3.5.3) Definition. Let $f : X \to Y$ be a flat morphism of algebraic stacks. Then we say that an injective map of quasicoherent $\mathcal{O}_Y$-modules $\varphi : \mathcal{E} \to \mathcal{F}$ is *f-locally split* if $f^*\varphi$ is a split monomorphism of quasicoherent $\mathcal{O}_X$-modules.

(3.5.4) Proposition. *Let $f : X \to Y$ be an affine and faithfully flat morphism of algebraic stacks and let $\mathcal{M}$ be a quasicoherent $\mathcal{O}_Y$-module that is endowed with an epimorphism $\psi : \mathcal{E} \twoheadrightarrow f^*\mathcal{M}$ of quasicoherent $\mathcal{O}_X$-modules.*

(i) The unit $\delta : \mathcal{M} \hookrightarrow f_*f^*\mathcal{M}$ of the adjunction (f^*, f_*) is an f-split monomorphism.

(ii) The inverse image of $\mathcal{M}$ under $f_*(\psi) : f_*\mathcal{E} \twoheadrightarrow f_*f^*\mathcal{M}$ surjects on $\mathcal{M}$ and embeds in $f_*\mathcal{E}$ locally split with respect to f.

Proof. The unit is f-locally split injective by fpqc-descent, for f^* is a left adjoint of f_*, the restriction $f^*\delta : f^*\mathcal{M} \to f^*f_*f^*\mathcal{M}$ has a left-inverse, given by the evaluation map $\varepsilon : f^*f_*f^*\mathcal{M} \to f^*\mathcal{M}$.

Since u_* is exact, $\varphi := f_*(\psi) : f_*\mathcal{E} \twoheadrightarrow f_*f^*\mathcal{M}$ is an epimorphism. Consider now the fiber square in $\mathrm{QCoh}(X)$

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{q} & \mathcal{M} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \delta} \\ f_*\mathcal{E} & \xrightarrow{\varphi} & f_*f^*\mathcal{M} \end{array} \tag{3.5.4.1}$$

Since φ is surjective, it follows that q is surjective. The pullback of (3.5.4.1) along f gives a commutative square:

$$\begin{array}{ccc} u^*\mathcal{F} & \xrightarrow{f^*q} & u^*\mathcal{M} \\ \downarrow{\scriptstyle f^*p} & & \downarrow{\scriptstyle f^*\delta} \\ u^*f_*\mathcal{E} & \xrightarrow{f^*\varphi} & u^*f_*f^*\mathcal{M} \end{array} \tag{3.5.4.2}$$

It is also cartesian, using the flatness of f and $\mathcal{F} = \ker(f_*\mathcal{E} \oplus \mathcal{M} \xrightarrow{\varphi-\delta} f_*f^*\mathcal{M})$. Hence, $\varepsilon \circ f^*\varphi$ induces a left-inverse for f^*p. So p is locally split injective by fpqc-descent, as asserted. □

As an application of Proposition 3.5.4 we prove the following resolution statement in Theorem 3.5.5 which enables us to restrict to flat sheaves rather than finding resolutions of arbitrary quasicoherent sheaves. It is possible to deduce it from [ATJLL97, Lemma 1.2.1] for schemes using the derived category of quasicoherent sheaves. For the readers (and authors) convenience we give a quick proof for arbitrary algebraic stacks.

(3.5.5) Theorem (Flat resolutions)**.** *Let X be a quasicompact algebraic stack with* affine diagonal*. Then every quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$ is a quotient of a* flat *quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$.*

PROOF. Let $\mathcal{M}$ be a quasicoherent $\mathcal{O}_X$-module. Since X is quasicompact, there is a smooth presentation $u\colon U \to X$ with U affine. Therefore we can choose a free resolution $\mathcal{O}_U^{\oplus I} \twoheadrightarrow u^*\mathcal{M}$ for some index set I. Since X has affine diagonal, u is an affine morphism, and it follows that $u_*\mathcal{O}_U^{\oplus I}$ is a flat quasicoherent $\mathcal{O}_X$-module. So by Proposition 3.5.4 it contains an u-locally split quasicoherent $\mathcal{O}_X$-submodule $\mathcal{F}$ that surjects on $\mathcal{M}$. Since $u^*\mathcal{F}$ is a direct summand of the flat sheaf $u^*u_*\mathcal{O}_U^{\oplus I}$, it is a pure subsheaf and hence flat. So by descent we conclude that $\mathcal{F}$ is flat. □

(3.5.6) *Remark.* The proof can be seen as a generalization of the fact that for an affine algebraic group scheme $G \to S$, every representation is a quotient of a subrepresentation that is contained in a direct sum of the regular representation by applying the previous argument to the covering $S \to B_S G$.

(3.5.7) *Remark.* From here we would immediately deduce the resolution property for an algebraic stack with affine diagonal if every flat quasicoherent sheaf $\mathcal{F}$ is the direct limit of locally free sheaves of finite rank. This is true for affine schemes by Lazard's Theorem [Laz69, 1.2], but also for locally Dedekind schemes since $\mathcal{F}$ is the union of torsion-free submodules of finite-type. Deligne showed in [SGA 4.1, Exp. V, Appendix 8, 8.2.12] that Lazard's Theorem remains valid for general ringed sites if one enlarges the notion of inductive limits by "local inductive limits". However, the latter are to general to serve our purpose.

As a trivial application we infer that on reduced stacks with affine diagonal it suffices to resolve torsion-free sheaves.

(3.5.8) Corollary (Torsionfree resolutions)**.** *Let X be a* reduced *quasicompact algebraic stack with* affine diagonal.

(i) *Every quasicoherent sheaf is a quotient of a torsion-free quasicoherent sheaf.*

(ii) *Suppose that X has the completeness property. Then every quasicoherent sheaf of finite type is a quotient of a quasicoherent and torsion-free sheaf of finite type.*

PROOF. A flat quasicoherent $\mathcal{O}_X$-module is torsion-free, so the first result follows from 3.5.5. If X has the completeness property, and $\mathcal{F} \to \mathcal{M}$ is an epimorphism of quasicoherent $\mathcal{O}_X$-modules, where $\mathcal{M}$ is of finite type and $\mathcal{F}$ is torsion-free, then $\mathcal{F}$ is the direct limit of quasicoherent $\mathcal{O}_X$-submodules $\mathcal{F}_\alpha \subset \mathcal{F}$ of finite type, which are also torsion-free. It follows that there exists an α such that the composition $\mathcal{F}_\alpha \to \mathcal{F} \to \mathcal{M}$ is an epimorphism. □

CHAPTER 4

The resolution property of stacks

We define in section 4.1 the resolution property of morphisms and give a brief overview of the properties thereof. In section 4.2 we verify the resolution property for regular stacks that have a low dimensional regular cover. In the last section 4.3 we discuss the presence of group actions and show that the notion of the relative resolution property appears here in a natural way.

4.1. The relative resolution property

By restricting the previous results about relatively generating families of finitely presented sheaves to families of locally free sheaves, one obtains the concept of generating vector bundles for an arbitrary quasicompact and quasiseparated morphism $f\colon X \to Y$. These families share the same permanence properties as relatively ample line bundles or ample families of line bundles.

(4.1.1) Definition. We say that a morphism $f\colon X \to Y$ of algebraic stacks has the *resolution property*, or that X *has the resolution property over* Y *(relative to* f*)*, if there exists an f-generating family of locally free $\mathcal{O}_X$-modules of finite type (see Def. 3.4.2).

(4.1.2) *Remark.* Since the rank of locally free sheaves is locally constant, it follows that the rank of the generating sheaves is constant if X is connected. Even if X is disconnected, but has finitely many disconnected components, every generating family of locally free finite-type $\mathcal{O}_X$-modules is equivalent to a generating family of locally free $\mathcal{O}_X$-modules of finite and constant rank by adding componentwise appropriate direct summands of finite free $\mathcal{O}_X$-modules. In particular, this holds if Y or just X is quasicompact.

A technical problem is the *size* of a generating family, which makes it difficult to associate geometric properties to the resolution property. We will solve this problem in section 6.2 by introducing a stronger variant of the resolution property, where the generating families in question arise in a natural way by a single vector bundle.

An illuminative example, where one does not encounters such problems, is the class of morphisms, where the structure sheaf alone defines a relatively generating singleton. This also serves as a preparation for the solution of the problem mentioned above, and will be described in chapter 5.

For the sake of completeness, we give a list of permanence properties for morphisms satisfying the relative resolution property. Properties (i)-(v) below follow immediately from the discussion in section 3.4.1 and (vi) is a consequence of Proposition 3.5.1.

(4.1.3) Proposition. — *Let S be an algebraic stack.*

(i) Every morphism $f\colon X \to Y$, having a relatively ample family of line bundles, satisfies the resolution property. This includes (quasi-) projective, (quasi-) affine, finite, quasi-finite finite-type separated morphisms, finite-type monomorphisms, closed and quasicompact open immersions.

(ii) If $f\colon X \to Y$ and $g\colon Y \to Z$ are two morphisms having the resolution property, then $g \circ f$ has the resolution property.

(iii) If an S-morphism $f\colon X \to Y$ has the resolution property, then for every base change morphism of algebraic stacks $S' \to S$, $f_{(S')}$ has the resolution property.

(iv) Let $f\colon X \to Y$ and $f'\colon X' \to Y'$ be two S-morphisms having the resolution property. Then $f \times_S g\colon X \times_S X' \to Y \times_S Y'$ has the resolution property.

(v) If the composition $g \circ f$ of two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ has the resolution property, and if Δ_g has the resolution property (for example, if Δ_g is quasiaffine), then f has the resolution property.

(vi) If the composition $g \circ f$ of two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ has the resolution property, and if f is finite, faithfully flat and finitely presented, then g has the resolution property.

(4.1.4) *Remark.* The verification of 4.1.3.(vi) answers a question raised by D. Rydh [Ryd09] affirmatively whether or not a finite, faithfully flat and finitely presented morphism preserves the resolution property. It appears to be previously known only for finite étale coverings and for finite, and faithfully flat morphisms of classifying stacks $BH \to BG$, given by a closed embedding $H \to G$ of group schemes that are flat, separated and of finite type over a noetherian and separated base scheme [Tho87, 2.13 and 2.14]. We postpone a discussion of its consequences to section 7.1.

Note that the resolution property of a morphism $f\colon X \to Y$ is *not local* on the target Y in general (cf. example 4.3.8). However, if one has a fixed family of quasicoherent sheaves, the property of being a relatively generating family of finite-type locally free sheaves is local on Y.

(4.1.5) Proposition (*fpqc*-descent)**.** *Let S be an algebraic stack. Let $f\colon X \to Y$ be an S-morphism, let $\mathcal{E}_I$ be a family of quasicoherent $\mathcal{O}_X$-modules and let $s_\alpha\colon S_\alpha \to S$ be a faithfully flat family. If each $\mathcal{E}_I|_{X_{(S_\alpha)}}$ is a generating locally free family for $f_{S_\alpha}\colon X_{(S_\alpha)} \to Y_{(S_\alpha)}$, and either the family (s_α) is fpqc, or f is quasicompact and quasiseparated and $\mathcal{E}_I$ locally free of finite type, then $\mathcal{E}_I$ is a f-generating locally free family.*

Proof. Locally free sheaves satisfy fpqc-descent, so the result is a special case of Proposition 3.4.5. □

4.2. Algebraic stacks with regular covers of low dimension

We discuss existence result for algebraic stacks X of low dimension; by this we mean the Krull dimension of a given covering $U \to X$ whose domain U is a scheme. If X itself is a scheme, then we know already that X has an ample family if X is $\mathbb{Q}$-factorial, noetherian and has affine diagonal (1.1.1), or if X has dimension ≤ 2 and is separated, of finite type over a field, or proper over a noetherian ring (see Corollary 2.0.2 resp. Theorem 2.0.1).

If an algebraic stack X has an affine fpqc-covering $U \to X$ with U an affine scheme, then we can try to transfer the resolution property of U to X by the method described in section 3.5.2. A priori, this provides us resolutions by big flat quasicoherent sheaves. However, if U is regular of dimension ≤ 2, then the latter are unions of finite-type subsheaves which happen to be locally free if they can be arranged to be torsion-free, respectively satisfy condition S_2.

(4.2.1) Lemma. *Let X be an algebraic stack, such that there exists an affine and faithfully flat morphism $U \to X$ whose domain U is a regular, noetherian scheme of dimension ≤ 1. Then every torsion-free quasicoherent $\mathcal{O}_X$-module is the filtered direct limit of locally free $\mathcal{O}_X$-submodules of finite type. In particular, X has the resolution property and affine diagonal.*

Proof. By fpqc-descent follows that X is noetherian. So every quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$ is the direct limit of coherent quasicoherent $\mathcal{O}_X$-submodules $\mathcal{F}_\alpha$. If $\mathcal{F}$ is locally torsion-free, each $\mathcal{F}_\alpha$ is torsion-free and hence locally free by the hypothesis on U and fpqc-descent.

For the last statements we argue as follows. First, note that U has affine diagonal since U is of dimension ≤ 1. Then $U \times_X U$ is not only affine over U, but also affine over $U \times_{\mathbb{Z}} U$. Thus X has affine diagonal, so every quasicoherent $\mathcal{O}_X$-module of finite-type is quotient of a locally free $\mathcal{O}_X$-module of finite type 3.5.8. □

This gives a first list of algebraic stacks that satisfy the resolution property. The first example is due to Thomason.

(4.2.2) Example. Suppose that S is a regular, noetherian scheme S of dimension ≤ 1 (e.g. $\mathbb{Z}$ or a field). Let $G \to S$ be an affine, flat and finitely presented group scheme. Then $B_S G$ has the resolution property.

(4.2.3) Example. Let $f\colon X \to S$ be a gerbe with affine diagonal and suppose again that S is regular, noetherian of dimension ≤ 1. So, locally over S, f is of the previous form. In particular, X is a reduced noetherian algebraic stack, and the family of coherent torsion-free sheaves is a generating family by 3.5.8. However, the latter are all locally free, as this is true locally over S. So X is has the resolution property. We warn the reader, that the resolution property is in general not local over S (see 4.3.8).

Applying this strategy to the case that the covering U has dimension 2, one obtains a similar result. The following is proved in [Tho87, Lemma 2.5] in case that $X = B_S G$ for some affine, flat and finitely presented group scheme $G \to S$. But the proof translates literally to this more general setting.

(4.2.4) Lemma. *Let X be an algebraic stack, and let $u\colon U \to X$ be an affine fppf presentation by a noetherian algebraic space U satisfying S_2, such that $u_*\mathcal{O}_U$ is a projective $\mathcal{O}_X$-module (for example if u is smooth with geometrically integral fibers). If U has the resolution property, then every coherent $\mathcal{O}_X$-module is a quotient of a coherent $\mathcal{O}_X$-module satisfying S_2. In particular, if U is regular of dimension ≤ 2, then X has the resolution property.*

(4.2.5) Example. Let S be an algebraic space.

(i) Let $X = [U/G/S]$ be a quotient stack, where $G \to S$ is a smooth affine group scheme with connected fibers acting on a noetherian algebraic S-space U that has the resolution property and satisfies S_2. Then every coherent $\mathcal{O}_X$-module is quotient of a coherent $\mathcal{O}_X$-module satisfying S_2. If U is regular of dimension ≤ 2, then X has the resolution property.

(ii) In particular, if $S = U$ is a noetherian algebraic space that has the resolution property and satisfies S_2, then one recovers Thomason's case $X = B_S G$ [Tho87, Lemma 2.5].

(4.2.6) *Remark.* So far we considered algebraic stacks that admit a *regular* covering of low dimension. The non-regular case is substantially more difficult, even if the covering is the spectrum of a Gorenstein artinian ring, e.g. $k[\varepsilon]$ with k being a field and $X = BG$ for a smooth affine group scheme G over $k[\varepsilon]$. We will postpone the discussion to section 7.2.

4.3. The equivariant resolution property

The framework of algebraic stacks allows us to integrate arbitrary actions of group schemes in the discussion of the resolution property. We will see that the classifying stack of GL_n plays here a prominent role because the frame bundles of vector bundles are GL_n-torsors in a natural way (see section 6.1.1).

Let $X \to S$ be a quasicompact and quasiseparated morphism of algebraic stacks, and let $\pi\colon G \to S$ be a separated finitely presented and faithfully flat group space with a right-action on X over S (if S is not an algebraic space, we refer the reader to Romagny's exposition of group actions on stacks [Rom05]). Then the stack quotient $[X/G/S]$ exists as an algebraic S-stack (see [Rom05, 4.1] for the case that S is a scheme and [LMB00, 10.13.1] for the case that X and S are algebraic spaces).

Most of the succeeding results are due to Thomason [Tho87] expressed without the language of algebraic stacks. However, by considering the resolution property always as a property of morphisms, we hope to give a streamlined exposition of that matter. Furthermore, we explain why one should consider the resolution property as a property of morphisms rather than as an absolute one.

4.3.1. The resolution property for quotients.

Let us discuss the relationship between the resolution property of $X \to S$ and the quotient $[X/G/S] \to S$. The canonical fppf S-morphism $p\colon X \to [X/G/S]$ and the classifying S-morphism $[X/G/S] \to B_S G$ fit in a 2-cartesian diagram of S-morphisms

$$\begin{array}{ccc} X & \xrightarrow{\ p\ } & [X/G/S] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ S & \xrightarrow{\ q\ } & B_S G \end{array} \qquad (4.3.0.1)$$

where $q\colon S \to B_S G$ is the fppf presentation that corresponds to the trivial torsor. In particular, every property of f' that is fppf local and stable on the target can be detected on f. Even if one is interested in the resolution property of schemes, this square is only defined in the 2-category of stacks since $B_S G$ is never representable (unless G is trivial).

Having the permanence properties of the relative resolution property in mind (cf. 4.1.3), we proceed as follows. If f' has the resolution property, so too has f. Conversely, if f has the resolution property given by a family of locally free $\mathcal{O}_X$-modules $\mathcal{E}_I$ of finite type, then f' has the resolution property if each $\mathcal{E}_i$ descends to the quotient $[X/G/S]$. The descend datum of quasicoherent $\mathcal{O}_X$-module $\mathcal{E}$ consists of an isomorphism $\sigma\colon m^*\mathcal{E} \xrightarrow{\sim} {\mathrm{pr}_2}^*\mathcal{E}$, where $m\colon G \times_S X \to X$ denotes the action and $\mathrm{pr}_2\colon G \times_S X \to X$ the projection, that satisfies the usual cocycle condition on $G \times_S G \times_S X$. It is also called a *G-linearization of $\mathcal{E}$* (cf. [MF82, Def. 1.6] or [HL97, 4.2.3]).

As a second observation, we remark that the structure morphism $[X/G/S] \to S$ factors over the classifying map f':

$$\begin{array}{ccc} [X/G/S] & \xrightarrow{f'} & B_S G \\ & \searrow \quad \swarrow & \\ & S & \end{array} \tag{4.3.0.2}$$

Therefore $[X/G/S] \to S$ has the resolution property if both $B_S G \to S$ and f' have the resolution property since the resolution property of morphisms is stable under composition. Conversely, according to the left cancellation property, f' has the resolution property if both $[X/G/S] \to S$ and the diagonal $\Delta_{B_S G/S}$ have the resolution property. The latter is true in all cases, we are interested in, since $\Delta_{B_S G/S}$ or equivalently $G \to S$ is (quasi-) affine.

Summarizing, we verified the following assertions.

(4.3.1) Proposition. *With the preceding notations, the following are equivalent:*

(i) The classifying map $f' \colon [X/G/S] \to B_S G$ has the resolution property.

(ii) $f \colon X \to S$ has the resolution property, given by a family of f-generating and G-linearized locally free $\mathcal{O}_X$-modules $(\mathcal{E}_i)_{i \in I}$ of finite type.

Moreover, if $B_S G \to S$ has the resolution property, then these conditions imply:

(iii) The structure map $[X/G/S] \to S$ has the resolution property.

The reverse implication also holds if the diagonal $\Delta_{B_S G/S}$ of $B_S G \to S$ has the resolution property (e.g. if $G \to S$ is quasiaffine).

If $f \colon X \to S$ has the (ordinary) resolution property, we see that the problem of the resolution property of $[X/G/S] \to S$ divides in two parts:

(A) Firstly, the construction of a f-generating family of vector bundles on X that admit a G-linearization; that is, the resolution property of the classifying morphism $[X/G/S] \to B_S G$.

(B) Secondly, the verification of the resolution property of the classifying stack $B_S G \to S$.

We will address ourselves to this problems in section 4.3.3, respectively 4.3.4. Let us state our main example of quotient stacks that satisfy the resolution property:

(4.3.2) Example. Let $f \colon X \to S$ be a quasiaffine morphism of algebraic stacks such that X has a right action of $GL_{n,S} = GL_{n,\mathbb{Z}} \times_{\mathbb{Z}} S$ over S. Then $[X/GL_{n,S}] \to S$ has the resolution property because $\mathcal{O}_X$ has a canonical $GL_{n,S}$-linearization and $BGL_{n,S}$ has the resolution property by Example 4.2.2. Sighlty unconventional, we say that X is a *global quotient stack (over S)* or f is a *relative global quotient stack*, adopting the recent notation of D. Rydh [Ryd10b].

(4.3.3) *Remark.* There exist various definitions of weaker types of quotient stacks X over an algebraic space S. They have in common that a separated, flat and finitely presented group space $G \to S$ acts on an algebraic space $U \to S$ such that $X \simeq [U/G/S]$. If $G \to S$ admits a group homomorphism $G \to GL_{n,S}$ that is a closed immersion, then $[U/G/S] \simeq [U'/GL_{n,S}]$ by Morita equivalence, where $U' = U \times^G GL_{n,S}$ is the balanced product $(U \times_S GL_{n,S})/G$ which is again an algebraic space over S. Such an immersion exists if $B_S G$ has the resolution property over $\mathbb{Z}$ by Thomason (see Theorem 4.3.14).

The discussion of the equivariant resolution property has also useful consequences for the non-equivariant resolution property.

(4.3.4) Corollary. *If $[X/G/S] \to S$ has the resolution property and $G \to S$ is quasiaffine, then $X \to S$ has the resolution property. In particular, if $B_S G$ has the resolution property over $\mathbb{Z}$, then this holds for S too.*

For example, this might be helpful to reduce the resolution property of toric varieties to the equivariant one.

(4.3.5) Example. Let X be a toric scheme (over $\mathbb{Z}$) and denote by T the torus acting on X. If $[X/T]$ has the resolution property, then X has the resolution property.

However, it is not even clear if $[X/T]$ has sufficiently many vector bundles at all. We refer the reader to the extensive discussion of the case of non-quasiprojective toric threefolds by S. Payne [Pay09].

4.3.2. Separateness properties of quotients. Having the previous notations at hand, we describe shortly the relation between separateness properties of $X \to S$ and $[X/G/S] \to S$, or more general conditions on the diagonal.

For that, let $\mathcal{P}$ be a property of morphisms of algebraic stacks which is closed under 2-isomorphism, stable under base change and fppf local on the base. By standard arguments follows that $\mathcal{P}$ contains all 1-isomorphisms and is closed under composition. For example, the collection of closed immersions, immersions and quasicompact, separated, affine, quasiaffine, finite and quasifinite morphisms, etc.

Then by imposing $\mathcal{P}$ on the diagonal of a morphism of algebraic stacks, we get a derived collection that inherits the permanence properties of $\mathcal{P}$; for instance, with the preceding list, one obtains the collection of separated, locally separated, quasiseparated morphisms and those with separated, affine, quasiaffine, finite or quasifinite diagonal.

(4.3.6) Proposition. *With the preceding notations, the following are equivalent:*

(i) The diagonal of the classifying map $[X/G/S] \to B_S G$ has $\mathcal{P}$.

(ii) The diagonal of $X \to S$ has $\mathcal{P}$.

Moreover, if $G \to S$ has property $\mathcal{P}$, these conditions imply the following one:

(iii) The diagonal of $[X/G/S] \to S$ has $\mathcal{P}$.

The reverse implication also holds, if $\Delta_{\Delta_{B_S G/S}}$ has $\mathcal{P}$.

PROOF. The equivalence (ii) $\Leftrightarrow$ (i) follows from (4.3.0.1) by fppf descent. If $G \to S$ has $\mathcal{P}$, then the diagonal of $B_S G \to S$ has $\mathcal{P}$ by fppf descent. So (i) $\Rightarrow$ (iii) follows from (4.3.0.2), for $\mathcal{P}$ is closed under composition. The remaining implication (iii) $\Rightarrow$ (i) is again a consequence of (4.3.0.2) according to the left cancellation property of $\mathcal{P}$. □

(4.3.7) Example. —

(i) Every (relative) quotient stack $[U/G/S] \to S$ has quasiaffine diagonal, where $U \to S$ is representable and $G \to S$ a quasiaffine group scheme. It follows from B.3 that the relative stabilizer groups of $[U/G/S] \to S$ are affine.

(ii) If $U \to S$ has affine diagonal (e.g. if $U \to S$ is separated), then the quotient $[U/G/S] \to S$ has affine diagonal if $G \to S$ is an affine group scheme. In particular, a *global quotient stack* $[U/GL_{n,S}] \to S$ has affine diagonal (this was proven by Totaro [Tot04, Prop. 1.3] for the absolute case with X noetherian).

By Totaro's Theorem the resolution property of a morphism $X \to S$ is equivalent to $X \to S$ being a global quotient stack under very natural hypothesis (Theorem 6.3.1). Hence, the affiness of the diagonal $\Delta_{X/S}$ is a necessary condition for the resolution property to hold. In fact all schematic counter examples to the resolution property known so far do not have affine diagonal (see [SV04] or the proof of [Tot04, 8.1]).

However, this condition is not sufficient if one takes non-quasifinite stabilizers into account.

(4.3.8) Example. Given an algebraic space S, Edidin, Hassett, Kresch and Vistoli [EHKV01, Example 3.12] observed that every non-torsion element in $\mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_m)$ gives rise to a $\mathbb{G}_m$-gerbe $X \to S$ which is not a quotient stack, in particular not a global quotient stack and hence does not satisfy the resolution property by Totaro's Theorem. However, by Grothendieck there exists a normal, affine algebraic surface S that admits such an element. Therefore $X \to S$ and thus X do not satisfy the resolution property. Moreover, we see that the resolution property for morphisms is in general not local on the target because $X \to S$ is, locally over S, the classifying stack of $\mathbb{G}_m$ which satisfies the resolution property.

4.3.3. Existence of linearizations. Let us give a brief overview of the existence of G-linearizations on vector bundles. Given a family of vector bundles that is generating for a morphism $f \colon X \to S$, we seek to construct a modified family of locally free $\mathcal{O}_X$-modules of finite type, that is still generating, carries in addition a G-linearization and hence descends along the quotient map $X \to [X/G/S]$.

If $G \to S$ is *finite, surjective and locally free* (equivalently finite and faithfully flat of finite presentation, or just finite and faithfully flat if S is locally noetherian) this is well-known. For instance, this is accomplished by *averaging* the generating family:

(4.3.9) Lemma. *Suppose that $G \to S$ is finite and locally free and let $q \colon X \to [X/G/S]$ be the quotient map. Then for every locally free $\mathcal{O}_X$-module $\mathcal{E}$ the quasicoherent $\mathcal{O}_X$-module $q^*q_*\mathcal{E}$ is locally free of finite type, has a canonical G-linearization and the evaluation map $q^*q_*\mathcal{E} \twoheadrightarrow \mathcal{E}$ is surjective.*

Even though this construction is canonical and works for more general group schemes $G \to S$, it is only useful to us as long as $G \to S$ is proper, because we restrict to finitely presented generating sheaves. However, the resolution property loses geometric significance if we do not restrict to algebraic stacks with affine stabilizer groups (see Theorem 5.3.2), so that it seems reasonable to consider only (quasi-) affine group schemes as far as we make no further assumption of the regularity of the group action.

The construction of G-linearizations for affine but non-finite group schemes is significantly more difficult. To the author's knowledge there is no known general method, to derive G-linearized locally free $\mathcal{O}_X$-module of finite rank from a given one. However, in case that $G \to S$ is smooth and affine with connected fibers and X a normal noetherian scheme, there is a strong result of Raynaud and Sumihiro [Sum75, 1.6] for the existence of a G-linearization on tensor powers of line bundles:

(4.3.10) Theorem. *([Sum75, 1.6]) Let S be a quasicompact and quasiseparated scheme and let $G \to S$ be a smooth affine group scheme with connected fibers that acts on a normal noetherian S-scheme X. Then for every invertible $\mathcal{O}_X$-module $\mathcal{L}$ and $n \in \mathbb{N}$ sufficiently large, $\mathcal{L}^{\otimes n}$ admits a G-linearization.*

In the proof the isomorphism $m^*\mathcal{L}^{\otimes n} \xrightarrow{\sim} {\mathrm{pr}_2}^*\mathcal{L}^{\otimes n}$ is constructed by first taking an arbitrary choice, which is possible since $\mathrm{Pic}(G \times_S X)/\mathrm{Pic}(X)$ is torsion, and adding a suitable correction factor later on to meet the cocycle condition. The normality assumption is essential (see [Tot04, 9.1] for a counterexample). However, it remains unclear if there exists a suitable generalization to locally free sheaves of higher rank.

(4.3.11) *Problem.* With the hypothesis in Theorem 4.3.10, does there exist for every locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank ≥ 2 a tensorial construction t such that $t(\mathcal{E})$ admits a G-linearization?

Returning to our original motivation this leads to the following result:

(4.3.12) Corollary ([Tho87, 2.10])**.** *Let S be a quasicompact and quasiseparated scheme and X a normal noetherian scheme over S with an action of a smooth affine group scheme $G \to S$ with connected fibers. Suppose that $X \to S$ has an ample family of invertible sheaves $\mathcal{L}_1, \dots, \mathcal{L}_r$ (for instance, if X is locally $\mathbb{Q}$-factorial and has affine diagonal over $\mathbb{Z}$). Then for a sufficiently large integer m the tensor powers $\mathcal{L}_i^m$ descend on $[X/G/S]$ to an ample family for $[X/G/S] \to B_S G$.*

(4.3.13) Example. Hironaka constructed a smooth three-dimensional algebraic space X which is not a scheme, by taking a suitable glued smooth scheme U that carries a free action of $\mathbb{Z}/2\mathbb{Z}$ and defining $X = U/(\mathbb{Z}/2\mathbb{Z})$ [Knu71, p. 15-17]. It follows from the corollary that $X \to B(\mathbb{Z}/2\mathbb{Z})$ and hence X has the resolution property.

4.3.4. The resolution property of classifying stacks. For later application, let us give an overview about the techniques for proving the resolution property of $B_S G \to S$ for an affine, flat and finitely presented group scheme $G \to S$, where S is an arbitrary algebraic space (or algebraic stack with quasiaffine diagonal, if you like) following Thomason [Tho87, §2]. We also call this the *equivariant resolution property of S*.

As the base S has quasiaffine diagonal (over $\mathbb{Z}$), we can equivalently study the resolution property of the $B_S G \to \operatorname{Spec}\mathbb{Z}$ assuming that $S \to \operatorname{Spec}\mathbb{Z}$ has the resolution property (4.1.3.(ii),(v)).

The case that $G \to S$ is flat, finite and finitely presented is well-known [Tho87, 2.14]. Note that this follows also from 4.1.3.(vi) because the trivial torsor $S \to B_S G$ is a *finite* fppf covering.

If S is regular noetherian of dimension ≤ 1, then $B_S G$ and hence $B_S G \to S$ has the resolution property by Example 4.2.2. As the resolution property is stable under base change, we conclude that $B_S G \to S$ has the resolution property for every *split* group scheme $G \to S$.

In particular, if $G = GL_n$ is the general linear algebraic group, then $BGL_{n,S} \to S$ has the resolution property. So we can ask, when this holds for $B_S G \to S$ assuming that $G \subset GL_{n,S}$ is a closed subgroup space. In fact, this is a *necessary* condition by work of Thomason (if S is affine).

(4.3.14) Theorem ([Tho87, 3.1])**.** *Let S be an algebraic space and $G \to S$ an affine, flat and finitely presented group space. If $B_S G$ has the resolution property (over $\mathbb{Z}$), then there exists a vector bundle $\mathcal{V}$ on S and a group homomorphism which is a closed immersion $G \hookrightarrow GL_S(\mathcal{V})$. If S is affine, then one may take $\mathcal{V}$ to be a free module.*

In order to study the relation between the equivariant resolution property of two group schemes $H \to S$ and $G \to S$ we shall look at all morphisms

$B(\varphi)\colon B_SH \to B_SG$ that are induced by group homomorphisms $\varphi\colon H \to G$ and ask when these have the resolution property. If φ is a closed immersion, then this is related to the quotient $G/H \to S$ using fppf descent and the following Lemma.

(4.3.15) Lemma. *Let $\varphi\colon H \to G$ be a homomorphism of S-group spaces that is a closed immersion. Then there exists a 2-cartesian diagram over S*

$$\begin{array}{ccc} G/H & \longrightarrow & B_SH \\ \downarrow & & \downarrow{\scriptstyle B(\varphi)} \\ S & \longrightarrow & B_SG \end{array} \qquad (4.3.15.1)$$

where the lower fppf covering map corresponds to the trivial G-torsor.

PROOF. This is a special case of the square (4.3.0.1). □

So, in order to study the resolution property of B_SH we can apply the same strategy as described in Proposition 4.3.1. Again, we meet the problem to determine which generating families of vector bundles for $G/H \to S$ descend on B_SH, i.e. admit a G-linearization. Since the structure sheaf admits a canonical one, we obtain as a special case the following result.

(4.3.16) Corollary. *With the notations of 4.3.15 follows that $G/H \to S$ is (quasi-) affine if and only if $B(\varphi)$ is (quasi-) affine.*

(4.3.17) Example. Let $GL_n \times GL_m \hookrightarrow GL_{n+m}$ be the diagonal embedding. Then $B(GL_n \times GL_m) \to BGL_{n+m}$ is affine since $GL_{n+m}/GL_n \times GL_m$ is an affine *Stiefel* scheme.

In contrast to the general situation of Proposition 4.3.1, the study of G-linearizations of non-quasiaffine quotients $G/H \to S$ is simplified by the existence of the fppf quotient map $G \to G/H$. We know that $G/H \to S$ is representable, has affine diagonal (4.3.6), and is of finite presentation by fppf descent since $G \to S$ is. As we originally have the case $G = GL_n$ in mind, it is worth to study the smooth connected case.

(4.3.18) Lemma ([Tho87, 2.11])**.** *Let $G \to S$ be a smooth, affine group space with connected fibers over a regular, noetherian scheme S with affine diagonal and $H \subset G$ a closed subgroup space that is flat and finitely presented over S. Suppose that $G/H \to S$ is schematic. Then $B_SH \to B_SG$ is schematic and has an ample family.*

PROOF. By fppf descent G/H is regular, noetherian, has affine diagonal and hence an ample family which is also ample for $G/H \to S$. □

(4.3.19) *Problem.* When is G/H a scheme? Or, does the regular algebraic space G/H have an ample family?

(4.3.20) *Remark.* Among others, Thomason used these techniques in [Tho87, 2.18, 2.1] to prove that the equivariant resolution property even holds for a larger class of group schemes $G \to S$.

He also relates the equivariant resolution property to equivariant embeddings of schemes and to Hilbert's 14th problem. For that we refer the reader to [Tho87, §3].

CHAPTER 5

Morphisms with generating structure sheaf

This section is devoted to the study of those algebraic stacks that have generating structure sheaf, or equivalently, where every quasicoherent sheaf is globally generated. In fact, we show, under reasonable conditions on the stabilizer groups, that this *characterizes* an algebraic stack as being strongly representable by a quasiaffine scheme (see Theorem 5.3.2 below and Theorem 5.3.8 for the relative case).

This is well-known for schemes [EGA II], but even for algebraic spaces not obvious because the Grothendieck topology thereof is not determined by open subspaces, so that the non-vanishing sets of sections are in general to large. For normal noetherian algebraic spaces, and even algebraic stacks, our result can be read off the proof of Totaro's Theorem [Tot04, 1.1]. It uses the fact that every normal noetherian algebraic space can be written as a quotient of a normal noetherian scheme by a finite group action.

However, for removing the hypothesis "normal", one encounters pinching problems. Using a pinching result for AF-schemes (where every finite set of points is contained in an affine open neighborhood, c.f. 5.1.1) of D. Ferrand [Fer03] and the methods of noetherian approximation [Ryd10b], we managed to remove the normal and noetherian hypothesis. In fact, we provide by Theorem 5.1.5 a suitable generalization of Chevalley's Theorem to AF-schemes and AF-morphisms (c.f. Def. 5.1.1).

In Theorem 5.3.2 we show then that a quasicompact and quasiseparated algebraic stack X with affine stabilizer groups at closed points is representable by a quasiaffine scheme if and only if $\mathcal{O}_X$ is a generator, using the method of [EHKV01, 2.12] to show the triviality of the stabilizer groups. This generalizes the well-known fact that the structure sheaf of quasicompact and separated scheme is ample if and only if the scheme is quasiaffine [EGA II, 5.1.2].

Even if we are just interested in noetherian algebraic stacks, we will see in section 6.2.5 that there exists natural construction involving infinite inverse limits of stacks which are not noetherian in general, yet give a clarifying picture when studying large generating families of vector bundles.

We also provide a relative version of this result for morphisms with relatively affine stabilizers at *geometric points* (cf. Theorem 5.3.8). This includes all morphisms with quasiaffine diagonal; for instance, those that are representable, a relative Deligne-Mumford stack, or more generally have quasifinite diagonal (see Example 5.3.7).

5.1. A variant of Chevalley's theorem for AF-schemes

Chevalley's Theorem for algebraic spaces asserts that the target X of an integral and surjective morphism $f\colon Z \to X$ of algebraic spaces is affine if (and only if) the source Z is affine (see [Knu71, III.4.1] for f finite and X noetherian and separated, and see [Ryd10b, 8.1] for the non-noetherian case). In particular, it follows that X is representable by a scheme.

This theorem is false in general if one replaces the property affine by quasiaffine. For example, there exists non-normal algebraic surfaces (over $\mathbb{C}$ if desired) with quasi-affine normalization (as announced in [EGA II, 6.6.13], but postponed to the unpublished EGA V; take $\mathbb{A}^2 - \{0\}$ and glue the coordinate axes along the inverse map $x \mapsto x^{-1}$, see [Aut, Lemma 0272] for details).

To the authors knowledge there exist no result which establishes the representability of the target X if the source Z is quasiaffine. If f is flat and finite, this was recently settled in [Ryd10a, Lemma B.1], and it follows that the target is also quasiaffine. Later we will be confronted with finite normalizations which are *not* flat, so we need a more general criterion.

(5.1.1) Definition. A scheme X is an *AF-scheme* (*affine finie*) if every finite set of points is contained in an affine open neighborhood. A morphism of algebraic spaces $X \to Y$ is an *AF-morphism* if for every $\operatorname{Spec} A \to Y$ the restriction X_A is an AF-scheme. In particular, an AF-morphism is *schematic*.

Examples of AF-morphisms are quasiaffine or quasi-projective morphisms (see [EGA II]). AF-morphisms are closed under composition and stable under base change.

(5.1.2) *Problem.* Does the resolution property hold for every quasicompact and quasiseparated AF-scheme? Does there exists a descent result for AF-morphisms?

The following Proposition serves as a preparation for Theorem 5.1.5.

(5.1.3) Proposition. *Let S be an algebraic space and $f\colon Z \to X$ be an integral and surjective morphism of algebraic spaces. If $Z \to S$ is quasiaffine, then $X \to S$ is an AF-morphism and hence* schematic *and separated. If X is locally noetherian and normal, or if f is flat, then $X \to S$ is quasiaffine.*

Proof. Applying a base change for a given $\operatorname{Spec} A \to S$, we may assume that S is affine. It follows that Z is quasiaffine and we can replace S by $\operatorname{Spec} \mathbb{Z}$. Then X is quasicompact since $f\colon Z \to X$ is surjective and Z is quasicompact. Also, X is separated because f is surjective and universally closed and Z is separated. By [Ryd10b, Thm. D] f factors over a finite and finitely presented morphism $Z_0 \to X$, where Z_0 is a quasiaffine scheme.

Hence we may assume that f is finitely presented, replacing Z by Z_0. We say that X has a *finite and finitely presented quasiaffine covering* (given by f). This property is stable under quasiaffine base change, so it ascends along finite coverings and closed immersions. If there exists a an AF-scheme X_0 and an affine morphism $X \to X_0$, then X is representable by an AF-scheme.

So we may suppose that X is of finite type over $\mathbb{Z}$ and that f is finite by usual noetherian approximation arguments (the proof of [Ryd10b, 8.1] applies literally).

If the reduction X_{red} is an AF-scheme (resp. a quasiaffine scheme), then X is an AF-scheme (resp. a quasiaffine scheme) by Chevalley's Theorem and using that $|X_{\mathrm{red}}| \to |X|$ is a homeomorphism.

Hence, we assume that X is reduced and Nagata. By noetherian induction we may assume that every proper closed subspace is representable by an AF-scheme. The normalization $g\colon X' \to X$ is finite since X is Nagata. The source X' is normal and has a finite cover by a quasiaffine scheme. Suppose for a moment that X' is representable by a quasiaffine scheme. In particular, it is an AF-scheme [EGA II, 4.5.4]. Since $X' \to X$ is finite and has schematically dense image, we may consider X as the pinching of X' along the conductor subspace $Y = \operatorname{Supp} \operatorname{Ann} \operatorname{coker}(\mathcal{O}_X \to g_*\mathcal{O}_{X'}) \subset X$ (see Lemma 5.1.7 below). However, by a theorem of D. Ferrand [Fer03, 5.4], the pushout $X_0 := X' \coprod_{g^{-1}(Y)} Y$ exists already in the category of ringed spaces and is an AF-scheme since X' and Y are AF-schemes. Since X is the pushout $X' \coprod_{g^{-1}(Y)} Y$ in the category of algebraic spaces, the universal property implies the existence of a morphism of algebraic spaces $p\colon X \to X_0$ that satisfies $p \circ g = q$, where $q\colon X' \to X_0$ is the quotient map of schemes. Invoking Chevalley's Theorem, we infer that this is in fact an affine morphism. Thus, X is representable by an AF-scheme and *a posteriori* holds $X_0 \simeq X$.

This reduces to the case that X is a normal noetherian algebraic space. By [LMB00, 16.6.2] we know that X is a quotient of a normal noetherian scheme X' by finite group G. Since $X' \to X$ is finite, we infer that X' has finite quasiaffine covering. If X' is a quasiaffine scheme, then it is well-known that the quotient $X = X'/G$ is also quasiaffine (see [Ryd07, 4.8] for the non-noetherian case).

Therefore we reduced the proof of the first statement in the proposition to the proof of the second one. So let us finally assume that X is a noetherian normal and separated scheme. Then the result follows from the norm trick: $\mathcal{O}_X$ is ample by [EGA II, 6.6.2]. So X is quasiaffine [EGA II, 5.1.2], as required. □

As an application of Proposition 5.1.3, we show a new characterization of schematic points on algebraic spaces with respect to integral surjective coverings:

(5.1.4) Corollary. *Let $f\colon Y \to X$ be an integral surjective morphism of algebraic spaces. A point $x \in X$ is schematic if and only if the fiber $f^{-1}(x)$ is contained in an open subspace $U \subseteq Y$, which is representable by a quasiaffine (equivalently affine) scheme.*

Proof. Integral morphisms are affine, so the condition is clearly necessary. Conversely, assume that a quasiaffine neighborhood $f^{-1}(x) \subseteq U \subseteq X$ exists. Replacing f by the restriction to the open subspace $X - f(Y - U) \subseteq X$, and hence Y by $U - f^{-1}(f(Y - U))$, we may assume that Y is quasiaffine. Thus, Proposition 5.1.3 implies that X is a scheme. □

(5.1.5) Theorem. *Let $f\colon Z \to X$ be an integral and surjective morphism of algebraic spaces with finite topological fibres over a base algebraic space S. Then $Z \to S$ is an AF-morphism if and only if $X \to S$ is an AF-morphism.*

Proof. We may assume that the base S is affine. Since finite morphisms are representable and affine, the condition is clearly sufficient. So let us assume that Z is an AF-scheme. From Corollary 5.1.4 follows that X is representable by a scheme. Let $x_i \in X$ be a finite set of points. By applying the argument in the proof of 5.1.4 for the finite set $\{x_i\}$ we may assume that Z is quasiaffine. Then the result is a consequence of Proposition 5.1.3. □

(5.1.6) *Remark.* In particular, if $Z \to S$ is quasiaffine, then $X \to S$ is *schematic* and separated.

The following lemma is folklore but stated for lack of reference.

(5.1.7) Lemma. *Every cartesian square of algebraic spaces below with g birational, finite and schematically dominant and u the closed immersion defined by the conductor ideal, is also cocartesian.*

$$\begin{array}{ccc} Y' & \xrightarrow{v} & X' \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{u} & X \end{array} \tag{5.1.7.1}$$

Proof. Note first that if X is an affine scheme, then also X', Y and Y' are affine schemes, as all maps above are affine. In this case, the assumptions on u and f guarantee that the square is cocartesian, as one checks by a ring-theoretic calculation. As we see next, the general case follows from this, since all hypothesis above are stable under flat base change.

So, let us suppose we have two morphism of algebraic spaces $\alpha\colon X' \to Z$ and $\beta\colon Y \to Z$ satisfying $\alpha v = \beta g$. We have to show that both factor over a unique map $\gamma\colon X \to Z$.

The uniqueness of γ is local over Z and over X. Therefore, by taking suitable étale coverings of Z, respectively of X, this reduces to the affine case, which was settled above.

To show the existence, we use the uniqueness, and may therefore assume that Z is an affine scheme by using again an étale covering of Z. Then, by taking an affine étale covering $U \to X$, we deduces the existence of a map $U \to Z$. By taking the affine étale covering $U \times_X U \to X$, we infer that $U \to Z$ is compatible over $U \times_X U$ and therefore get our desired morphism $X \to Z$. □

5.2. Properties of morphisms with generating structure sheaf

Let us first collect the permanence properties of morphisms $f\colon X \to Y$, where $\mathcal{O}_X$ is weakly f-generating, respectively f-generating (see Def. 3.4.2), before we show in the next section that they coincide with quasiaffine morphisms if the relative stabilizer groups are affine. As a special case of Lemma 3.4.1 we have a list of equivalent definitions.

(5.2.1) Proposition. *Let $f\colon X \to Y$ be a quasicompact and quasiseparated morphism of algebraic stacks. Then the following are equivalent:*

(i) $\mathcal{O}_X$ is weakly f-generating.

(ii) The pushforward $f_*\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is faithful.

(iii) For every quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$ the evaluation map $f^*f_*\mathcal{M} \to \mathcal{M}$ is surjective.

One may drop the attribute "weakly" in (i) if the analogous conditions in (ii) and (iii) are satisfied for arbitrary base changes f.

The collection of morphisms of algebraic stacks whose domain has relatively weakly generating structure sheaf satisfies the following permanence properties:

It is local on the base for fpqc-coverings (3.4.5) and hence contains all quasiaffine morphisms (3.4.6). It is closed under 2-isomorphism and composition, and satisfies the left-cancellation property with respect to quasiseparated morphisms (3.4.9). Moreover, it is stable under arbitrary base change and finite products if the base has quasiaffine diagonal (3.4.13 and 3.4.18); recall, that the latter holds for all quasiseparated schemes, algebraic spaces, Deligne-Mumford stacks, or more generally all algebraic stacks with quasifinite diagonal (we refer to appendix B).

In particular, for every morphism whose target has quasiaffine diagonal the structure sheaf of the domain is weakly relatively generating if and only if it is relatively generating (3.4.16).

The permanence properties of the collection of morphisms with relatively generating structure sheaf is much better behaved: It is also fpqc-local on the base (3.4.5), contains all quasiaffine morphisms (3.4.6) and is closed under composition (3.4.9), but by definition stable under *arbitrary* base change. It follows that it is also closed under arbitrary finite products (3.4.18) and satisfies the left-cancellation property with respect to morphisms whose diagonal has generating structure sheaf (3.4.9). Moreover, this property can be checked over affines (3.4.17).

5.3. Relation to quasiaffine schemes

We shall prove the characterization of quasiaffine algebraic stacks (Theorem 5.3.2) and provide a proof for the relative case of quasiaffine morphisms later on with a suitable definition of relative pointwise affine stabilizer groups (Theorem 5.3.8).

Let us first tackle the case of a classifying stack. This will be important to prove the triviality of the stabilizer groups in the general case.

(5.3.1) Proposition. *Let S be an algebraic space with affine diagonal over $\mathbb{Z}$, and let $G \to S$ be a separated, flat and finitely presented group S-space with classifying stack $q\colon B_S G \to S$. Then $\mathcal{O}_{B_S G}$ is weakly q-generating if and only if q^* and q_* induce an equivalence of categories:*

$$\mathrm{QCoh}(B_S G) \underset{q^*}{\overset{q_*}{\rightleftarrows}} \mathrm{QCoh}(S). \tag{5.3.1.1}$$

In particular, if $G \to S$ is affine, then this can only happen if q is an isomorphism, or representable, or equivalently if G is the trivial S-group space.

Proof. We freely use the identification of quasicoherent $\mathcal{O}_{B_S G}$-modules with quasicoherent G-equivariant $\mathcal{O}_S$-modules (as explained in [AOV08, 2.1]). Denote by $p\colon S \to B_S G$ the fppf presentation associated with the trivial G-torsor. All $\mathcal{O}$-modules and sheaves are assumed to be quasicoherent $\mathcal{O}$-modules by abuse of notation. First of all, we recall some properties of p and q. We have $q \circ p \simeq 1_S$, hence $q_* \circ p_* \simeq 1_{\mathrm{QCoh}(S)}$. Now p^* is the functor that forgets the G-structure and p_* endows an $\mathcal{O}_S$-module with the trivial G-action. q_* is the functor that takes G-invariants and one checks that $q^* \simeq p_*$. It follows that $q_* q^* \simeq q_* p_* \simeq 1_{\mathrm{QCoh}(S)}$, hence q^* is a quasi-right-inverse of q_*. For a G-sheaf $\mathcal{F}$ follows that subsheaf of G-invariants $q^* q_* \mathcal{F}$ embeds in $\mathcal{F}$ via the canonical evaluation map $q^* q_* \mathcal{F} \to \mathcal{F}$.

Now $\mathcal{O}_{B_S G}$ is q-generating if and only if for every $\mathcal{F}$, the evaluation map $q^* q_* \mathcal{F} \twoheadrightarrow \mathcal{F}$ is an epimorphism. So by the previous discussion this holds if and only if this is an isomorphism of quasicoherent sheaves. Equivalently, if q^* is a quasi-left-inverse of q_*. This verifies the first assertion.

The last assertion can be deduced from Tannaka theory for algebraic stacks as presented in [Lur05]. Since $G \to S$ is affine, $B_S G \to S$ has affine diagonal, so $B_S G$ has affine diagonal over $\mathbb{Z}$. Moreover, the algebraic space S is local for the étale topology in the sense of [Lur05, Def. 4.2] by Remark 4.4 loc. cit. So, if the tensor functor $q^*\colon \mathrm{QCoh}(S) \to \mathrm{QCoh}(B_S G)$ is a 2-isomorphism, then $q\colon B_S G \to S$ must be already a 2-isomorphism by Theorem 5.11 loc. cit.. Then Δ_q is an isomorphism, but the restriction of Δ_q along $p \times_S p\colon S \to B_S G \times_S B_S G$ is 2-isomorphic to $G \to S$ and we conclude the assertion. □

In order to transfer the previous model case to an arbitrary quasicompact algebraic stack X over an algebraic space S, we recall how the the stabilizer groups determine the local structure of algebraic stacks.

For an S-morphism $f\colon T \to X$, the *stabilizer* of f, denoted by G_f or $\mathrm{Aut}_{X(T)}(x)$, is the algebraic group S-space defined by the fiber products

$$\begin{array}{ccccc} G_f & \longrightarrow & IX & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta_{X/S}} \\ T & \xrightarrow{f} & X & \xrightarrow{\Delta_{X/S}} & X \times_S X \end{array} \tag{5.3.1.2}$$

Here $IX = I(X/S)$ is called the *inertia stack*; it is a group stack over X. The structure morphisms $G_f \to T$ and $IX \to X$ are representable, separated and of finite type since $\Delta_{X/S}$ is [LMB00, 4.2].

(5.3.2) Theorem. *Let X be an algebraic stack over an algebraic space S with* affine *stabilizer groups at closed points. Then $X \to S$ is representable by a quasiaffine morphism if and only if $\mathcal{O}_X$ is generating over S.*

PROOF. Clearly, the condition is necessary, for a quasiaffine map of schemes has relatively ample structure sheaf [EGA II, 5.1.2(e')], so it requires to verify the sufficiency.

We first show that X is representable by an algebraic space using Lemma 5.3.3 below. The arguments generalize parts of [EHKV01, 2.12] in the non-noetherian case. By assumption X is quasicompact and has separated diagonal of finite type.

We use that every point of X is algebraic in order to attach the classifying stack of the stabilizer groups (this holds by [Ryd10a, B.1], correcting the definition of [LMB00, 11.2]). For that, let $\xi \in |X|$ be a closed S-point. Then there exists a representative $x\colon \operatorname{Spec} k \to X$ which factors over a quasiaffine monomorphism $\mathcal{G}_\xi \hookrightarrow X$, where $\mathcal{G}_\xi$ is the residual gerbe of ξ which is of finite presentation over the residue field $k(\xi)$. It follows that there exists a finite field extension $k(\xi) \subset L$ such that $\mathcal{G}_\xi \otimes L \simeq BG_{x'}$, where $x'\colon \operatorname{Spec} L \to \operatorname{Spec} k(\xi) \to X$ is the induced representative of ξ. Since $\mathcal{O}_X$ is generating over X and the composition $BG_{x'} \to \mathcal{G}_\xi \to X$ is quasiaffine, we conclude that $\mathcal{O}_{BG_{x'}}$ is generating. So by our model case 5.3.1, we know that $G_{x'}$ must be trivial.

Thus, the inertia group morphism $\theta\colon IX \to X$ has 0-dimensional closed fibers. But it is also of finite type since $\Delta_{X/S}$ is, so that the fiber dimension $x \mapsto \dim_{k(x)} \theta^{-1}(x)$ is upper semi-continuous (the proof is local over X, but the proof for group *schemes* [SGA 3.1, IV_B 4.1] literally translates to algebraic group spaces). Then $IX \to X$ must be quasifinite, and hence is quasiaffine [OS03, 3.1]. It follows that for the non-closed S-points $x\colon \operatorname{Spec} k \to X$, the stabilizer group spaces G_x are quasiaffine algebraic group schemes, and hence also affine because the base is a field [FSR05, 7.5.3]. By repeating the previous argument they are *a posteriori* trivial. From Lemma 5.3.3 follows that X is representable.

This shows that $f\colon X \to S$ is a morphism of algebraic spaces. In order to verify that f is quasiaffine, we may suppose that S is affine by étale descent, and even set $S = \operatorname{Spec} \mathbb{Z}$. By [Ryd10b, Thm. B] there exists a *scheme* Z and a finite, finitely presented and surjective morphism $f\colon Z \to X$. It follows that $f^*\mathcal{O}_X$ is generating for Z. So if Z is a quasiaffine scheme, then Theorem 5.1.5 implies that X is representable by a scheme.

This reduces to the final case that X is a scheme. Since $\mathcal{O}_X$ is generating, every quasicoherent ideal sheaf is quotient of a free $\mathcal{O}_X$-module. It follows that the open subsets X_f, where f runs over the global sections of $\mathcal{O}_X$, define a base of the Zariski-topology. Since X is covered by affine open subschemes, there exists a subbase consisting of affines X_f. For these f holds that $\Gamma(X, \mathcal{O}_X)_f \to \Gamma(X_f, \mathcal{O}_{X_f})$ is bijective [EGA $\mathrm{I}_{\mathrm{2nd}}$, 6.8.3]. Thus, the affine hull $p\colon X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$ induces an isomorphism $p^{-1}(D(f)) \to X_f$, and is hence a quasicompact open immersion. This settles that X is quasiaffine, as required. □

The following representability criterion is probably well-known [EHKV01, 2.12], yet stated here for lack of reference for the non-noetherian case.

(5.3.3) Lemma. *An algebraic S-stack X is representable by an algebraic S-space if and only if the stabilizer groups vanish at every geometric S-point.*

PROOF. Trivially, the condition is necessary. Now X is representable by an algebraic S-space if and only if $\Delta_{X/S}$ is a monomorphism, equivalently if and only if for each S-point $T \to X$ the stabilizer group space $G_f \to T$ is the trivial group space [LMB00, 8.1.1]. Clearly, this is also equivalent to the fact that the inertia stack $\theta\colon IX \to X$ is trivial, i.e. that f is an isomorphism, or that the unit section $\varepsilon\colon X \to IX$ is an isomorphism.

The hypothesis on the stabilizers implies that θ is an isomorphism over every geometric point and hence is quasifinite. Since θ is representable, separated and of finite type, we conclude that θ is quasiaffine, and hence schematic. So by base change, we may assume that θ is a quasiaffine morphism of schemes and an isomorphism over all geometric points. Even now the latter holds over every point, using descent and a standard approximation argument since every fiber is of finite type and thus of finite presentation. Since θ is (locally) of finite type, we conclude that θ is a monomorphism [EGA IV.4, 17.2.6]. It follows that ε is an isomorphism, as required. □

(5.3.4) *Remark.* We infer that the second assertion in Proposition 5.3.1 is also valid if $G \to S$ is a separated, flat and finitely presented group space with affine fibers at closed points. Note that affine or quasiaffine morphisms cannot be detected fiber-wise; thus, this hypothesis is far more general.

5.3.1. The relative case.

In order to provide Theorem 5.3.2 in the relative setting, we have to define a suitable hypothesis on the *relative* stabilizer groups of a morphism $X \to Y$ which is local on the target. In contrast to the absolute case, we formulate these restrictions on the relative stabilizer groups over all *geometric* points, rather than just the closed ones, in order to obtain a well behaved property of morphisms.

(5.3.5) Definition. A morphism $f\colon X \to Y$ of algebraic stacks has *(relatively) affine stabilizers at all geometric points* if the *relative inertia stack* $I(X/Y) = X \times_{X \times_Y X} X \to X$ has affine geometric fibers.

(5.3.6) *Remark.* One checks, that this property is fppf local on the target and stable under arbitrary base change. Moreover, it is closed under composition, using that for two morphisms of algebraic stacks $f\colon X \to Y$, $g\colon Y \to Z$ holds the formula $I(X/Y) = \ker(I(X/Z) \to f^*I(Y/Z))$ and that the extension of two affine group schemes is affine.

(5.3.7) Example. Every morphism f with quasiaffine diagonal, or just with quasiaffine relative inertia, has relatively affine stabilizers at all points (5.3.1.2). This includes all morphisms that have quasifinite diagonal, like a relative Deligne-Mumford stack, or a relative algebraic space (i.e. a representable morphism).

As a formal consequence we obtain a relative version of Theorem 5.3.2.

(5.3.8) Theorem. *Let $f\colon X \to Y$ be a morphism of algebraic stacks with relatively affine stabilizers at geometric points. Then f is quasiaffine if and only if $\mathcal{O}_X$ is f-generating.*

Proof. All properties under consideration are fppf local on the target and stable. So, we may assume that Y is an algebraic space and Theorem 5.3.2 applies. □

Using that the diagonal of a morphism of algebraic stacks is always representable, we infer the following corollaries.

(5.3.9) Corollary. *A morphism of algebraic stacks $f\colon X \to Y$ has quasiaffine diagonal Δ_f if and only if $\mathcal{O}_X$ is Δ_f-generating.*

A morphism with quasiaffine diagonal has quasiaffine relative inertia and hence affine relative stabilizer groups at geometric points. This allows us to give a characterization of quasiaffine morphisms in purely sheaf theoretic terms.

(5.3.10) Corollary. *A morphism of algebraic stacks $f\colon X \to Y$ is quasiaffine if and only if $\mathcal{O}_X$ is generating for f and Δ_f.*

CHAPTER 6

Tensor generators and Totaro's Theorem

For an arbitrary quasicompact and quasiseparated algebraic stack with affine stabilizer groups at geometric points, we show that a vector bundle has quasiaffine frame bundle if and only if an associated family of vector bundles, the local tensor hull $\langle\mathcal{E}\rangle$, is a generating family for $\mathrm{QCoh}(X)$ (cf. Theorem 6.2.12). This unifies the following well-known facts from projective geometry and representation theory of algebraic groups.

(i) An invertible sheaf $\mathcal{L}$ on a quasicompact and separated scheme X is ample if and only if for every finite-type sheaf $\mathcal{F}$ and $n \in \mathbb{N}$ sufficiently large, the twist $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated.

(ii) Let G a closed algebraic subgroup scheme of $GL_{n,k}$ over a field k. Then every finite-dimensional, rational representation of G can be constructed from its original representation on k^n by the process of forming tensor products, direct sums, subrepresentations, quotients and duals [Wat79, 3.5].

First, we recall in section 6.1 the correspondence between GL_n-torsors and vector bundles of rank n. In section 6.2 we begin with the definition of tensorial constructions and introduce the concept of the (local) tensor hull. Then we study vector bundles, whose local tensor hull is a generating family, and call them *tensor generators*. After proving Theorem 6.2.12, we generalize this to finite and infinite *tensor generaring families*.

6.1. Correspondence between frame bundles and vector bundles

As a preparation for the following section we recall the correspondence between vector bundles and GL_n-torsors.

6.1.1. Frame bundles. Let $X \to S$ be a morphism of algebraic stacks. Recall that there is a one-to-one correspondence between locally free $\mathcal{O}_X$-modules of rank n and $GL_{n,S}$-torsors over X (in the fppf or equivalently étale topology, since $GL_{n,S} = GL_{n,\mathbb{Z}} \times_{\mathbb{Z}} S \to S$ is smooth and connected, or even in the Zariski topology [Ser95] if X is a scheme).

For a vector bundle $\mathcal{E}$ of rank n, the *frame bundle* $p\colon E = \mathrm{Isom}(\mathcal{E}, \mathcal{O}_X^{\oplus n}) \to X$ is the open substack of $\mathrm{Hom}_S(\mathcal{E}, \mathcal{O}_X^{\oplus n}) = \mathrm{Spec}_X \mathrm{Sym}(\mathcal{H}om(\mathcal{E}, \mathcal{O}_X^{\oplus n})^\vee) \to X$, that parametrizes isomorphisms $\mathcal{E} \xrightarrow{\simeq} \mathcal{O}_X^{\oplus n}$. It has a natural structure over S and the projection p is an affine and finitely presented morphism. Furthermore, it carries a right-action of $GL_{n,S}$ which turns it into a torsor (see [HL97, 4.2.3] for more details). In particular, $E \to X$ is smooth, affine and has connected fibers by fppf descent.

Conversely, given a $GL_{n,S}$-torsor $E \to X$, then $GL_{n,S}$ acts naturally on $\mathbb{A}^n_S$ from the left, and $V = E \times^{GL_{n,S}} \mathbb{A}^n_S = E \times_S \mathbb{A}^n_S / GL_{n,S} \to X$ becomes a vector bundle, whose associated sheaf of sections $\mathcal{E}$ is locally free.

6.1.2. Universal vector bundle. $BGL_{n,S} = BGL_{n,\mathbb{Z}} \times_{\mathbb{Z}} S$ carries a canonical locally free sheaf $\mathcal{V}$ of rank n whose associated frame bundle is the trivial $GL_{n,S}$-torsor. Using the correspondence between quasicoherent $\mathcal{O}_{BGL_{n,S}}$-modules and quasicoherent $\mathcal{O}_S$-modules endowed with a $\mathcal{O}_{GL_{n,S}}$-coaction, one checks that $\mathcal{V}$ is the standard representation of $GL_{n,S}$. In turn, one recovers the originally locally free sheaf via the classifying morphism $c\colon X \to BGL_{n,S}$ by $\mathcal{E} \simeq c^*\mathcal{V}$.

This gives a natural connection between properties of $\mathcal{V}$ and those of $\mathcal{E}$ as long as they are preserved under c. For example, one may define the cohomological Chern classes of $\mathcal{E}$ as the pullback of those of $\mathcal{V}$ (see [Hei05] for a recent discussion).

6.1.3. Algebraic G-bundles. Let us consider vector bundles whose associated frame bundle is induced by torsor for some affine group scheme $G \to S$.

(6.1.1) Definition. Let S be an affine scheme and $\varphi\colon G \to GL$ a closed subgroup S-scheme that is faithfully flat and finitely presented over S.

A vector bundle $\mathcal{E}$ on an algebraic S-stack X is called an *algebraic G-bundle (with respect to φ)* if the classifying morphism $X \to BGL_{n,S}$ factors over $B(\varphi)\colon B_S G \to BGL_n$. We say that G is a *structure group* of $\mathcal{E}$ (w.r.t. φ). If $B(\varphi)$ (or equivalently $GL_n/G \to S$) is quasiaffine, then we shall say that G is *admissible*.

If not stated otherwise, we will consider every vector bundle of rank n as an algebraic GL_n-bundle.

We will see that locally free sheaves of linearly reductive representation type appear naturally in the context of tensor generating sheaves (see Proposition 6.2.11).

(6.1.2) Example. If S is of characteristic 0, then every vector bundle has linearly reductive structure group $GL_{n,S} \simeq GL_{n,\mathbb{Q}} \times_{\mathbb{Q}} S$.

(6.1.3) Example. A vector bundle $\mathcal{E}$ decomposes as the direct sum of n invertible sheaves if and only if $\mathcal{E}$ is an algebraic $\mathbb{G}_m^n$-bundle with respect to the diagonal embedding $\varphi\colon \mathbb{G}_m^n \hookrightarrow GL_n$. In particular, $\mathcal{E}$ has linearly reductive structure group in this case.

6.1.4. Properties of the classifying morphism. Properties of the classifying map are related to those of the total space $q\colon E \to S$ of the frame bundle, due to the canonical fiber square

$$\begin{array}{ccc} E & \xrightarrow{p} & X \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle c} \\ S & \longrightarrow & BGL_{n,S} \end{array}$$

where the lower horizontal arrow is the smooth presentation corresponding to the trivial $GL_{n,S}$-torsor. Let now $\mathcal{P}$ be a property of morphisms of algebraic S-stacks (e.g. representable or quasiaffine). If $\mathcal{P}$ is stable, then it would descend from c to q, and the converse holds if $\mathcal{P}$ is fppf local. This proves the following Lemma:

(6.1.4) Lemma. *Let $\mathcal{P}$ be a property of morphisms of algebraic stacks that is stable and fppf local on the target.*

Let $X \to S$ be a morphism of algebraic stacks and $\mathcal{E}$ a vector bundle on X of constant rank. Then the following are equivalent:

(i) For the frame bundle $E \to X$ of $\mathcal{E}$, the total space $E \to S$ has $\mathcal{P}$.

(ii) The classifying morphism $X \to BGL_{n,S}$ has $\mathcal{P}$.

For example, these conditions are satisfied for representable, quasiaffine morphisms or morphisms with relatively generating structure sheaf. However, strongly representable (i.e. schematic) morphisms, or morphisms with the resolution property, do not satisfy fppf descent in general (see 4.3.8 for a counterexample of the latter).

6.1.5. Relation between frame bundles and the resolution property. Using the frame bundle correspondence, the study of the resolution property of a morphism of algebraic stacks $f\colon X \to S$ is equivalent to the study of the resolution property of a collection of morphisms $f_i\colon X \to S_i$ whose targets S_i are more stacky, but whose fibers are simpler in important cases. Consider the following first demonstration to illustrate this issue:

(6.1.5) Proposition. *Let $f\colon X \to S$ be morphism of algebraic stacks and $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of rank n with associated frame bundle $p\colon E \to X$. Then the following are equivalent:*

(i) $f\colon X \to S$ has the resolution property.

(ii) The classifying S-morphism $c\colon X \to BGL_{n,S}$ has the resolution property.

(iii) $f \circ p\colon E \to S$ has the resolution property given by a generating family of $GL_{n,S}$-linearized locally free $\mathcal{O}_X$-modules.

Proof. We know that $BGL_{n,S} \to S$ satisfies the resolution property by Example 4.3.2 and that it has affine diagonal. So everything follows from Proposition 4.3.1. □

We will investigate in section 6.2.2 those locally free sheaves more closely whose classifying morphisms $X \to BGL_{n,S}$ have the resolution property in its most simplest form; namely, where $\mathcal{O}_X$ is relatively generating.

6.2. Tensor generators and quasiaffine frame bundles

So far, we have ignored the *tensor structure* and the *size* of the generating families of locally free sheaves. They belong to an additive tensor subcategory and satisfy fpqc-descent. Instead of carrying an explicit family, we may restrict to study those members, which generate the remaining ones by tensorial constructions. Unless stated otherwise explicitly, we restrict to *locally free* generating sheaves.

6.2.1. Tensorial constructions and local tensor hulls. For a vector bundle $\mathcal{E}$ on an algebraic stack X we associate a family of vector bundles $\langle \mathcal{E} \rangle$ by *tensorial constructions*, which has the property that every member becomes globally generated after restricting to the frame bundle $E \to X$. We will see in section 6.2.2 that such a family is generating if and only if E is quasiaffine.

(6.2.1) Definition. By a *tensorial construction* we mean a finite composition t of the canonical (bi-) functors $\oplus$, $\otimes$, $(\cdot)^\vee$ that has one input and one output datum.

(6.2.2) Example. Examples are the constant functor $\mathcal{F} \mapsto \mathcal{O}$, the n-th direct sum $\mathcal{F} \mapsto \mathcal{F}^{\oplus n}$, the n-th tensor power $\mathcal{F} \mapsto \mathcal{F}^{\otimes n}$ or $\mathcal{F} \mapsto \bigoplus_{i=0}^n \bigoplus_{j=0}^n \mathcal{F}^{\otimes i} \otimes (\mathcal{F}^\vee)^{\otimes j}$.

(6.2.3) *Remark.* Tensorial constructions preserve isomorphisms and respect *globally* free sheaves. Moreover, they commute with pullbacks f^* for arbitrary morphisms $f\colon X \to Y$ of algebraic stacks.

(6.2.4) Definition. The *tensor hull* $|\mathcal{E}|$ of a vector bundle $\mathcal{E}$ on an algebraic stack X is the full subcategory of $\mathrm{QCoh}(X)$ whose objects are direct summands of tensorial constructed $t(\mathcal{E})$, for some tensorial construction t.

The *local tensor hull* $\langle\mathcal{E}\rangle$ of $\mathcal{E}$ is the full subcategory of $\mathrm{QCoh}(X)$ of all quasicoherent $\mathcal{O}_X$-modules $\mathcal{F}$ endowed with an injection $\mathcal{F} \hookrightarrow t(\mathcal{E})$ that is locally split (cf. Def. 3.5.3) relative to the frame bundle $E \to X$ for some tensorial construction t.

Clearly, holds $|\mathcal{E}| \subset \langle\mathcal{E}\rangle$ and the inclusion is proper because injections of vector bundles do not split in general. Before we discuss examples we give a list of properties whose verification is straightforward and hence omitted.

(6.2.5) *Remark.* Let $\langle\cdot\rangle$ be either $\langle\cdot\rangle$ or $|\cdot|$.

- (i) $\langle\mathcal{E}\rangle$ consists entirely of locally free $\mathcal{O}_X$-modules of finite type by fppf descent along the frame bundle.
- (ii) For every morphism of algebraic stacks $f\colon Y \to X$ exists a natural inclusion $f^*\langle\mathcal{E}\rangle \subseteq \langle f^*\mathcal{E}\rangle$ according to the universality of frame bundles. If f is an isomorphism, then this is an equivalence.
- (iii) Every isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{E}'$ induces an equivalence $\langle\mathcal{E}\rangle \xrightarrow{\sim} \langle\mathcal{E}'\rangle$.
- (iv) For every $n \in \mathbb{N}$ holds that every sheaf in $\langle\mathcal{O}^{\oplus n}\rangle$ is quotient of a free sheaf of finite type.
- (v) For every $\mathcal{F} \in \langle\mathcal{E}\rangle$ the pullback along the frame bundle $E \to X$ is globally generated by (ii)-(iv)

As the following example illustrates, the tensor hull of a line bundle is a familiar object when studying resolutions that are made up by line bundles.

(6.2.6) Example. Let X be a connected algebraic stack.

- (i) Let $\mathcal{L}$ be an invertible sheaf on X. Then the elements in $|\mathcal{L}|$ are finite direct sums of $\mathcal{L}^n$ for some $n \in \mathbb{Z}$. For that let $\mathcal{F} \subset t(\mathcal{L})$ be a direct summand for some tensorial construction t. Clearly holds $t(\mathcal{L}) = \bigoplus_{i\in I} \mathcal{L}^{n_i}$ for a finite family of integers $n_i \in \mathbb{Z}$. By intersecting this decomposition with $\mathcal{F}$, we get a decomposition $\mathcal{F} = \bigoplus_{i\in I} \mathcal{F}_i$, where $\mathcal{F}_i$ is a direct summand of $\mathcal{L}^{n_i}$, and hence zero or isomorphic to the latter.
- (ii) The previous example generalizes to families $\mathcal{L}_1, \dots, \mathcal{L}_r$ of invertible sheaves. One checks that every element in $|\bigoplus_{i=1}^r \mathcal{L}_i|$ can be written as a direct sum of tensor products $\mathcal{L}^{\bar{n}}$ with $\bar{n} \in \mathbb{Z}^r$.

The use of the local tensor hull will be important in the following section.

(6.2.7) Lemma. *Let G be an affine, flat, finitely presented and* linearly reductive *group scheme over an affine base S. Then for every vector bundle $\mathcal{E}$ on BG holds $|\mathcal{E}| = \langle\mathcal{E}\rangle$.*

Proof. It suffices to show that $|\mathcal{E}| \supset \langle\mathcal{E}\rangle$. For that, let $\mathcal{F}$ be locally free sheaf of finite type appearing in $\langle\mathcal{E}\rangle$. Then there exists a tensorial construction t and an injective map $\varphi\colon \mathcal{F} \subset t(\mathcal{E})$ which is locally split with respect to the frame bundle $E \to X$ of $\mathcal{E}$. We have to show that φ is split. First, note that $\mathcal{G} := \operatorname{coker}\varphi$ is also locally free of finite type by smooth descent. The obstruction for the splitting of φ is an element of $\mathrm{Ext}^1(\mathcal{G}, \mathcal{F}) \simeq \mathrm{H}^1(B_S G, \mathcal{G}^\vee \otimes \mathcal{F})$. However, the latter group is zero since $B_S G \to S$ is cohomologically affine, by definition of the linear reductivity of $G \to S$ [Alp09, 11.1]. ☐

6.2.2. Tensor generators and quasiaffine frame bundles. We show next that the local tensor hull of a vector bundle is a generating family if and only if the frame bundle of the latter has a quasiaffine total space.

(6.2.8) Definition. Let $f\colon X \to Y$ be a quasicompact and quasiseparated morphism of algebraic stacks with relatively affine stabilizers at geometric points. We shall call a vector bundle $\mathcal{E}$ a *tensor generator for X over S (with respect to f)*, if $\langle\mathcal{E}\rangle$ is an f-generating family. If the subfamily $|\mathcal{E}|$ is f-generating, then we call $\mathcal{E}$ a *strict tensor generator for X over S*.

The need for local tensor hulls, rather than tensor hulls is due to the existence of non-semisimple representations if one takes non-linearly reductive group schemes into account.

(6.2.9) Lemma. *Let S be a quasicompact and quasiseparated algebraic stack. The universal vector bundle $\mathcal{E}$ on $BGL_{n,S}$ is a tensor generator over S.*

Proof. We may assume that $S = \operatorname{Spec}\mathbb{Z}$. Set $G = GL_{n,S}$ for simplicity. In order to show that $\langle\mathcal{E}\rangle$ is a tensor generating family, we have to resolve an arbitrary quasicoherent sheaf $\mathcal{M}$ on BG. Denoting by $f\colon S \to BGL_{n,S}$ the trivial $GL_{n,S}$-torsor, we may assume that $\mathcal{M}$ is a flat f-locally split subsheaf of $(f_*\mathcal{O}_S)^{\oplus I}$ for some set I by Proposition 3.5.4. By intersecting the decomposition of the latter down to $\mathcal{M}$ we may even assume that I is a singleton. We exploit now the explicit description of the regular representation $f_*\mathcal{O}_S$.

For that, let us identify the category of quasicoherent sheaves on BG as the category of $\mathcal{O}_G$-comodules. Here $\mathcal{O}_G = \mathbb{Z}[x_{ij}][\det^{-1}]$ is the corresponding Hopf algebra, where $\det \in \mathbb{Z}[x_{ij}]$ is the monic determinant polynomial of degree n. Its comultiplication $\mu\colon \mathcal{O}_G \to \mathcal{O}_G \otimes \mathcal{O}_G$ is given by $x_{ij} \mapsto \bigoplus_{k=1}^n x_{ik} \otimes x_{kj}$ so that $\mu\colon \mathcal{O}_G \to (\mathcal{O}_G)_0 \otimes \mathcal{O}_G$ becomes a homomorphism of comodules, where the subscript 0 denotes the trivial coaction. The coinverse $\iota\colon \mathcal{O}_G \to \mathcal{O}_G$ is given by $x_{ij} \mapsto \det^{-1} c_{ji}$, where $c_{ij} \in \mathbb{Z}[x_{ij}]$ are the cofactors.

Let us give a coordinate free presentation of $\mathcal{O}_G$. We identify the standard representation $\mathcal{E}$ as the free module $\mathcal{E}_0 := \mathbb{Z}^{\oplus n} = \langle e_i\rangle$ that is endowed with the natural comodule structure $\varrho\colon e_j \mapsto \bigoplus_{k=1}^n e_k \otimes x_{kj}$. This coaction extends naturally to a coaction on $\operatorname{Sym}\mathcal{E}$ and on $\bigwedge^n \mathcal{E}$. The determinant realizes the homomorphism of comodules $(\mathcal{O}_S)_0 \to \bigwedge^n \mathcal{E}$. Then the quotient map $\mathbb{Z}[x_{ij}][T] = (\mathbb{Z}[x_{1,j}] \otimes \cdots \otimes \mathbb{Z}[x_{n,j}])[T] \to \mathbb{Z}[x_{ij}][\det^{-1}]$, sending T to $\det^{-1}$, can be written as a surjective homomorphism of comodules $\bigoplus_{i\geq 0} \operatorname{Sym}(\mathcal{E})^{\otimes n} \otimes (\bigwedge^n \mathcal{E})^{\otimes -i} \twoheadrightarrow \mathcal{O}_G$.

It follows that the preimage of the non-equivariant split subcomodule $\mathcal{M} \subset \mathcal{O}_G$ is a non-equivariant split subsheaf $\mathcal{M}' \subset:= \bigoplus_{i,j\geq 0} \operatorname{Sym}^j(\mathcal{E}^{\oplus n}) \otimes (\bigwedge^n \mathcal{E})^{\otimes -i}$. Using that Sym^j and $\bigwedge^n$ are quotients of tensor products, we infer that $\mathcal{M}'$ is a quotient of a non-equivariant split subsheaf $\mathcal{M}'' \subset \bigoplus_{i,j\geq 0}(\mathcal{E}^{\oplus n})^{\otimes j} \otimes (\mathcal{E}^\vee)^{\otimes ni}$. In particular, $\mathcal{M}''$ is an element of $\langle\mathcal{E}\rangle$. If S is of characteristic 0 or if $n = 1$, then $GL_{n,S} = GL_{n,\mathbb{Q}} \times_{\mathbb{Q}} S$ is linearly reductive over S [Alp09, Ex. 11.4], so that the inclusion $\mathcal{M}'' \hookrightarrow \bigoplus_{i,j\geq 0}(\mathcal{E}^{\oplus n})^{\otimes j} \otimes (\mathcal{E}^\vee)^{\otimes ni}$ is split. Therefore, $\mathcal{M}''$ is already an element of $|\mathcal{E}|$. □

(6.2.10) Lemma. *Let $\varphi\colon G \subset GL_{n,S}$ be a closed subgroup scheme that is flat, finitely presented over an* affine *base S such that $GL_{n,S}/G$ is quasiaffine. If $G \to S$ is linearly reductive, then for the induced map $\Phi\colon B_SG \to BGL_{n,S}$ and the universal bundle $\mathcal{E}^{univ}$ on $BGL_{n,S}$ follows that $\Phi^*\mathcal{E}^{univ}$ is a strict tensor generator on B_SG.*

Proof. Since $|\mathcal{E}|$ is a generating family on $BGL_{n,S}$ by 6.2.9, it follows that $\Phi^*|\mathcal{E}|$ is a generating family on B_SG because Φ is quasiaffine by Lemma 4.3.16.

Therefore $\Phi^*\mathcal{E}$ is a tensor generator on $B_S G$ using 6.2.5.(ii). Since G is linearly reductive, we conclude that $\mathcal{E}$ is already a strict tensor generator by 6.2.7. □

Finally, we have set up the framework to give a characterization of generating locally free sheaves in terms of the associated frame bundle.

(6.2.11) Proposition. *Let $f\colon X \to Y$ be a morphism of algebraic S-stacks and let $\mathcal{E}$ be a vector bundle on X of rank n, with associated frame bundle $p\colon E \to X$ and classifying morphism $c\colon X \to BGL_{n,S}$. Then the following assertions are equivalent:*

(i) The local tensor hull $\langle \mathcal{E} \rangle$ is a generating family for X over Y.

(ii) $\mathcal{O}_E$ is generating for $fp\colon E \to X \to Y$.

(iii) $\mathcal{O}_X$ is generating for $(f,c)\colon X \to BGL_{n,Y} = Y \times_S BGL_{n,S}$.

Moreover, condition (i) is always satisfied if

(iv) the tensor hull $|\mathcal{E}|$ is a generating family for X over Y,

and the converse holds if $\mathcal{E}$ has a admissible linearly reductive structure group (cf. 6.1.1); for instance, if $\mathcal{E}$ is a direct sum of invertible sheaves, or if S is of characteristic 0.

Proof. First note, that (ii) $\Leftrightarrow$ (iii) follows from fppf descent, and (iv) $\Rightarrow$ (i) is trivial as $|\mathcal{E}| \subset \langle \mathcal{E} \rangle$.

The implication (i) $\Rightarrow$ (ii) is easy. Suppose that $\langle \mathcal{E} \rangle$ is a f-generating family. In particular, f is quasicompact and quasiseparated. Then $p^* \langle \mathcal{E} \rangle$ is pf-generating because f is affine. It follows that the larger family $\langle f^*\mathcal{E} \rangle \simeq \langle \mathcal{O}_E^{\oplus} \rangle$ is also a fp-generating family consisting of globally generated sheaves. Thus, $\mathcal{O}_E$ is fp-generating.

So let us prove now (iii) $\Rightarrow$ (i): Let $\varphi\colon G \hookrightarrow GL_n$ be the embedding of the structure group of $\mathcal{E}$. The Y-morphism $(c,f)\colon X \to BGL_{n,Y}$ factors as the top row of the following 2-commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{(f,c)} & Y \times_S B_S G & \xrightarrow{\mathrm{pr}_2} & B_S G & \xrightarrow{\Phi} & BGL_{n,S} \\ & \searrow^{f} & \downarrow{\scriptstyle \mathrm{pr}_1} & \square & \downarrow & \swarrow & \\ & & Y & \longrightarrow & S & & \end{array} \qquad (6.2.11.1)$$

By Lemma 6.2.9 we know that $\langle \mathcal{E}^{\mathrm{univ}} \rangle$ is a generating family for $BGL_{n,S} \to S$. Since G is admissible, Φ is quasiaffine by 4.3.16. So $\Phi^* \langle \mathcal{E}^{\mathrm{univ}} \rangle$ is a generating family for $B_S G \to S$. Thus, the base change ${\mathrm{pr}_2}^* \langle \mathcal{E}^{\mathrm{univ}} \rangle$ is a generating family for pr_1. Hence, the composition $\{\mathcal{O}_X\} \otimes_{\mathcal{O}_X} (f,c)^* {\mathrm{pr}_2}^* \Phi^* \langle \mathcal{E}^{\mathrm{univ}} \rangle$ is a tensor generating family for $\mathrm{pr}_1 \circ (f,c) \simeq f$. However, for the latter family holds by 6.2.5

$$(f,c)^* {\mathrm{pr}_2}^* \Phi^* \langle \mathcal{E}^{\mathrm{univ}} \rangle \simeq c^* \langle \mathcal{E}^{\mathrm{univ}} \rangle \subseteq \langle c^* \mathcal{E}^{\mathrm{univ}} \rangle \simeq \langle \mathcal{E} \rangle .$$

We conclude that $\langle \mathcal{E} \rangle$ is an f-generating family, as required.

For the final implication (iii) $\Rightarrow$ (iv) we may replace $\langle \cdot \rangle$ by $|\cdot|$ in the argumentation above by Lemma 6.2.10. □

(6.2.12) Theorem. *Let $X \to S$ be a quasicompact and quasiseparated morphism of algebraic stacks and let $\mathcal{E}$ be a vector bundle on X. Then the following conditions are equivalent:*

(i) $\mathcal{E}$ is a tensor generator for X over S.

(ii) The frame bundle of $\mathcal{E}$ has quasiaffine total space over S.

Moreover, if these conditions are satisfied, then the diagonal $\Delta_{X/S}$ is affine.

PROOF. The equivalence is a consequence of 6.2.11 and Theorem 5.3.8. If these conditions hold, then $X \simeq [E/GL_{n,S}]$ where E is the total space of the frame bundle associated to $\mathcal{E}$. Therefore X is a global quotient stack over S and hence has affine diagonal $\Delta_{X/S}$ by 4.3.7. □

(6.2.13) Corollary. *A morphism $X \to S$ of algebraic stacks is a global quotient stack if and only if there exists a tensor generator for $X \to S$.*

6.2.3. Properties of tensor generators. Due to the universality of the frame bundle construction, we infer from Theorem 6.2.12 immediately that tensor generators are stable under pullbacks by quasiaffine maps, stable under base change and fpqc-local on the target, as this holds for quasiaffine maps.

(6.2.14) Proposition. *Let $X \to S$ be a morphism of algebraic stacks and $\mathcal{E}, \mathcal{F}$ be vector bundles on X of constant rank. Then the following properties hold for tensor generators relative to $X \to S$:*

(i) If $t(\mathcal{E})$ is a tensor generator for some tensorial construction t, then $\mathcal{E}$ is a tensor generator.

(ii) $\mathcal{E}$ is a tensor generator if and only if $\mathcal{E}^\vee$ is a tensor generator.

(iii) If $\mathcal{E}$ is a tensor generator, then $\mathcal{E} \oplus \mathcal{F}$ is a tensor generator.

(iv) If $\mathcal{E} \otimes \mathcal{F}$ is a tensor generator, then $\mathcal{E} \oplus \mathcal{F}$ is a tensor generator.

(v) $\mathcal{E}$ is a tensor generator for $X \to S$ if and only if $\mathcal{E}|_{X_{red}}$ is a tensor generator for $X_{red} \to S$.

PROOF. For a vector bundle $\mathcal{V}$ let us denote by $F(\mathcal{V}) \to X$ the associated frame bundle.

(i): From the modular description of frame bundles one deduces a map $F(\mathcal{E}) \to F(t\mathcal{E})$ over X, using that tensorial constructions preserve isomorphisms and globally free sheaves (6.2.5). By the left cancellation property for affine maps this must be affine. Therefore, if $F(t\mathcal{E})$ is quasi-affine over S it follows that $F(\mathcal{E})$ is quasiaffine over S.

(ii): There is a canonical isomorphism $F(\mathcal{E}) \xrightarrow{\simeq} F(\mathcal{E}^\vee)$.

(iii): This follows from the following Lemma 6.2.15.

(iv): Using the modular description of frame bundles, we infer the existence of an affine map $F(\mathcal{E}) \times_X F(\mathcal{F}) \to F(\mathcal{E} \otimes \mathcal{F})$ as above (i).

(v). $F(\mathcal{E}) \to S$ is quasiaffine if and only if $F(\mathcal{E}_{\text{red}}) \simeq F(\mathcal{E})_{\text{red}} \to S_{\text{red}}$ is quasiaffine, and the closed immersion is always (quasi-) affine. □

The following Lemma is due to Rydh [Ryd09].

(6.2.15) Lemma. *Let $X \to S$ be a morphism of algebraic stacks and $\mathcal{E}_0 := \mathcal{E}_1 \oplus \mathcal{E}_2$ a decomposition of vector bundles on X of rank $n_i \in \mathbb{N}$ with associated frame bundles $E_i \to X$, $i = 0, 1, 2$. Then $E_1 \times_X E_2 \to S$ is quasiaffine if and only if $E_0 \to S$ is quasiaffine.*

In particular, this holds if E_1 or E_2 is quasiaffine over S.

PROOF. If E_1 is quasiaffine over S, the same holds for $E_1 \times_X E_2$ since the bundle projection $E_2 \to X$ is affine. So let us assume that the latter holds. Note that the fiber product is a $GL_{n_1} \times_S GL_{n_2}$-torsor. Using fppf descent, we conclude that the classifying map $X \to B(GL_{n_1} \times_S GL_{n_2})$ is quasiaffine (4.3.0.1). Composing it with the affine morphism $B(GL_{n_1} \times_S GL_{n_2}) \to BGL_{n_0}$ that corresponds to the diagonal embedding (4.3.17), it follows that the composition $X \to BGL_{n_0}$, which classifies $\mathcal{E}_0$, is quasiaffine.

Conversely, if $X \to BGL_{n_0}$ is quasiaffine, then $X \to B(GL_{n_1} \times_S GL_{n_2})$ is quasiaffine by the left-cancellation property for quasiaffine morphisms of algebraic stacks. □

6.2.4. Finite tensor generating families. Being a tensor generator is stable under adding finitely many vector bundles by 6.2.14.(iii). Clearly, this property does not descend to direct summands, in general. However, we shall see that if $\mathcal{E}$ decomposes as a direct sum of $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n$, then the whole collection $\mathcal{E}_1, \ldots, \mathcal{E}_n$ behaves like a tensor generator.

(6.2.16) Proposition. *Let $X \to S$ be a morphism of algebraic stacks with affine stabilizers at geometric points and $\mathcal{E}_1, \ldots, \mathcal{E}_n$ a* finite *family of vector bundles of constant rank on X. Then the following are equivalent:*

- *(i) $\bigoplus_{i=1}^n \mathcal{E}_i$ is a tensor generator for X over S.*
- *(ii) $\bigotimes_{i=1}^n \langle \mathcal{E}_i \rangle$ is a generating family for X over S.*
- *(iii) The fiber product $\prod_{i=1}^n (E_i/X)$ of the frame bundles $E_i \to X$ of the $\mathcal{E}_i$'s is quasiaffine over S.*

Moreover, condition (ii) is always satisfied if

- *(iv) $\bigotimes_{i=1}^n |\mathcal{E}_i|$ is a generating family for X over S,*

and the converse holds, if each $\mathcal{E}_i$ has an admissible linearly reductive structure group.

Proof. The proof is an extension of the proof of Theorem 6.2.12. The equivalence (i) ⇔ (iii) from Lemma 6.2.15 and Theorem 6.2.12 by induction. For (ii) ⇒ (iii) check that $\bigotimes_i \langle \mathcal{E}_i \rangle$ pulls back on $\prod_i (E_i/X)$ to a family of globally generated sheaves.

The final implication (iii) ⇒ (ii) goes as follows. Let $\varphi_i : G_i \hookrightarrow GL_{n_i,S}$ be the embedding of structure groups of the $\mathcal{E}_i$'s with induced map $\Phi_i : BG_i \to BGL_{n_i,S}$. By assumption the classifying morphism $X \to B(\prod_i GL_{n_i,S})$ is quasiaffine, where $n_i \in \mathbb{N}$ denotes the rank of $\mathcal{E}_i$. It factors over $c: X \to B(\prod_i G_i)$ which is also quasiaffine because the diagonal of the representable morphism $B(\prod \varphi_i) : B(\prod_i G_i) \to B(\prod_i GL_{n_i,S})$ is quasiaffine.

There is a natural isomorphism $b: B(\prod_i G_i) \simeq \prod_i BG_i$ which induces the classifying morphism $c_j : X \to BG_j$ of $\mathcal{E}_j$, by composing $b \circ c$ with the projection $\mathrm{pr}_j : \prod_i BG_i \to BG_j$. Since each $\Phi_i{}^* \langle \mathcal{E}_i^{\mathrm{univ}} \rangle$ is a tensor generating family for $BG_i \to S$, the tensor product $\bigotimes_i \mathrm{pr}_i{}^* \Phi_i{}^* \langle \mathcal{E}_i^{\mathrm{univ}} \rangle$ is a tensor generating family for $\prod_i BG_i \to S$ by Proposition 3.4.18. Thus, $\bigotimes_i c^* b^* \mathrm{pr}_i{}^* \Phi_i{}^* \langle \mathcal{E}_i^{\mathrm{univ}} \rangle$ is a tensor generating family for $X \to S$. Using the natural embedding $c^* b^* \mathrm{pr}_i{}^* \Phi_i{}^* \langle \mathcal{E}_i^{\mathrm{univ}} \rangle \subset \langle c_i{}^* \Phi_i{}^* \mathcal{E}_i^{\mathrm{univ}} \rangle = \langle \mathcal{E}_i \rangle$ one finishes this part.

Now (ii) ⇔ (iv) is trivial and (iv) ⇔ (iii) follows by replacing $\langle \cdot \rangle$ above with $|\cdot|$ and using that the structure groups $G_i \to S$ are linearly reductive. □

(6.2.17) Definition. We shall say that a finite family of vector bundles $\mathcal{E}_1, \ldots, \mathcal{E}_n$ on an algebraic stack X over S is a *tensor generating family for X over S*, if it satisfies one of the equivalent conditions of 6.2.16 above.

On the one hand 6.2.16 characterizes a tensor generating family as the direct summands of a tensor generator as claimed in the beginning.

(6.2.18) Corollary. *A finite family $\mathcal{E}_1, \ldots, \mathcal{E}_n$ is a tensor generating family if and only if $\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n$ is a tensor generator.*

On the other hand Proposition 6.2.16 gives a geometric interpretation of a finite generating family.

(6.2.19) Corollary. *Let $X \to S$ be a morphism of algebraic stacks. Then there exists a* finite *tensor generating family of vector bundles $\mathcal{E}_i$ of rank n_i for $X \to S$ if and only if $X \simeq [U/\prod_i GL_{n_i,S}]$ for some quasiaffine morphism $U \to S$.*

Even the case of invertible sheaves was not proven before in this generality, yet presumed by Totaro [Tot04, p. 5]. In case that $X \to S$ is a morphism of schemes, we obtain a *characterization of ample families.* The global case of a reduced and separated scheme that is of finite type over an algebraically closed field was settled by Hausen [Hau02, 1.1].

(6.2.20) Corollary. *A map of schemes $X \to S$ has a relatively ample family of n invertible sheaves if and only if $X \simeq U/\mathbb{G}^n_{m,S}$ (over S) for some quasiaffine S scheme U.*

6.2.5. Infinite tensor generating subfamilies.

Having discussed the properties of finite tensor generating families for $X \to S$, we shall show the existence thereof. Clearly, the resolution property for $X \to S$ is a necessary condition, so we will assume it here. However, it just implies the existence of some generating family of vector bundles and this is usually infinite. Therefore, we decided to incorporate infinite families in the definition of a tensor generating family.

(6.2.21) Definition. Let $X \to S$ be a quasicompact and quasiseparated morphism with affine stabilizers at geometric points. A family of vector bundles $\mathcal{E}_I$ on X is called a *tensor generating family for X over S*, if the union of the families $\bigotimes_{j \in J} \langle \mathcal{E}_j \rangle$ is a generating family, where J runs over all (or equivalently all sufficiently large) finite subsets of I. Let us denote the latter by $\langle \mathcal{E}_I \rangle$

(6.2.22) Lemma. *A morphism of algebraic stacks $X \to S$ with affine stabilizer groups at geometric points has the resolution property if and only if $X \to S$ has a tensor generating family.*

Proof. Clearly, the condition is sufficient. However, it is also necessary since every family of sheaves $\mathcal{E}_I$ is contained in $\langle \mathcal{E}_I \rangle$. □

Under reasonable conditions on $X \to S$, it requires to prove that for a tensor generating family $\mathcal{E}_I$ there always exists a finite subfamily $\mathcal{E}_J$, $J \subset I$, which is still tensor generating or equivalently, that every finite but sufficiently large subfamily $\mathcal{E}_J$ is tensor generating.

At first, we show that this always holds if X is of finite presentation by forming the infinite fiber product of the associated frame bundles over X. This stack has the property that the given family becomes *simultaneously* trivial.

We restrict to a quasicompact base stack S. Then the question is local over S, so that we may assume that S is affine. Unless stated otherwise, all global statements are relative over the base S.

Throughout this section denote by $E_i \to X$ the associated frame bundle of $\mathcal{E}_i$ and set $E_J := \prod_{j \in J}(E_j/X)$ for all $J \subset I$ with projection $p_J \colon E_J \to X$. Clearly, this is a correct definition if J is finite. For J infinite, E_J is can be defined as the filtered inverse limit of all E_K with $K \subset J$ finite. A filtered inverse system of affine X-morphisms is given by the projections $\pi_{KL} \colon E_K \to E_L$, where $L \subset K$ runs over all finite subsets of J. These maps are affine, so the limit exists and is an algebraic stack [Ryd10b, Thm. C]. Moreover, the projections $p_J \colon E_J \to X$ are affine because all p_K and p_{KL} are affine.

(6.2.23) Proposition. *Let $X \to S$ be a quasicompact and quasiseparated morphism with affine stabilizer groups at geometric points and $\mathcal{E}_I$ a family of vector bundles. Then for the following conditions the implications (i) $\Rightarrow \cdots \Rightarrow$ (iv) hold and (i) $\Leftarrow$ (iv) is also true if $X \to S$ is of finite presentation:*

(i) E_J is quasiaffine over S for sufficiently large but finite $J \subset I$.

(ii) $\mathcal{E}_J$ is a finite tensor generating family over S for sufficiently large but finite $J \subset I$.

(iii) $\mathcal{E}_I$ is a tensor generating family over S.

(iv) The infinite fiber product $E_I = \prod_{i \in I}(E_i/X)$ is quasiaffine over S.

Proof. Suppose that the family is infinite. We claim that the infinite fiber product $E_I = \prod_{i \in I}(E_i/X)$ is representable by a quasiaffine scheme.

We know that the projection $p_I : E_I \to X$ is affine, *a fortiori* quasiaffine. Therefore $p^*\mathcal{E}_I$ is a tensor generating family for E_I. This is clear if I is the singleton by Remark 6.2.5 but the general case reduces to this.

Now, each $p^*\mathcal{E}_i$ is trivial on E_I because p factors over $E_i \to X$. So $\mathcal{O}_{E_I}$ is a generator for E_I. The latter might be a huge non-noetherian and non-finitely presented stack. However, it has (relative) affine stabilizers groups at geometric points and is quasicompact and quasiseparated over S since $p_J : E_I \to X$ is affine. So by Theorem 5.3.8 we conclude that E_J is representable by a quasiaffine scheme.

Using that X is of finite presentation, it follows that E_J is already quasiaffine for sufficiently large $J \subset I$ because each E_J is of finite presentation using [Ryd10b, Thm. C]. In other words $\mathcal{E}_J$ is a finite tensor generating family for X by Proposition 6.2.16. □

Unfortunately, the author does not know if the finite presentation hypothesis on X is really necessary in the proof. We use this assumption to guarantee that each E_J is of finite presentation, so that it becomes eventually quasiaffine for sufficiently large J while descending from the limit E_I along the bonding maps.

We will show next, that one may replace "finite presentation" by the much weaker hypothesis "quasicompact and quasiseparated" as long as E_J is eventually representable. For example, the latter condition is satisfied if X is noetherian using [Tot04, Lemma 4.1] or trivially, if X is representable. Even the general noetherian case will suffice for most of the applications. However, we have to make an effort to ascend along the bonding maps by hand and are not allowed to apply the limit trick again.

(6.2.24) Proposition. *Let X be an algebraic S-stack with S quasicompact and $\mathcal{E}_I$ a tensor generating family. Suppose that E_J is representable for some finite subset $J \subset I$. Then for sufficiently large but finite $J' \subset I$ follows that $E_{J'}$ is quasiaffine; that is, $\mathcal{E}_{J'}$ is a finite tensor generating subfamily by 6.2.16.*

We give a sequence of preparatory lemmata and give a proof at the end of this section on page 73. The strategy, which was also successfully applied by Totaro in the normal case, is to find a distinguished finitely presented sheaf satisfying the property that whenever it is generated by a finite family, the latter is already a tensor generating family. The non-normal case essentially reduces to the normal case by a pinching process which is encoded in our variant of Chevalley's Theorem 5.1.5.

In order to keep the proof short we feel free to introduce new terminology for temporarily use. Again we may assume that S is affine.

(6.2.25) Lemma. *Suppose that $\langle \mathcal{E}_I \rangle$ generates a finitely presented sheaf $\mathcal{M}$. Then the pullback of $\mathcal{M}$ along $E_J \to X$ is globally generated for sufficiently large $J \subset I$.*

Proof. By hypothesis there exists a resolution $\varphi\colon \bigoplus_{\alpha=1}^{n} \mathcal{F}_\alpha \twoheadrightarrow \mathcal{M}$, where each $\mathcal{F}_\alpha$ is an element of $\langle \mathcal{E}_I \rangle$. This means that $\mathcal{F}_\alpha$ is a finite tensor product of sheaves $\mathcal{F}_{j,\alpha}$ lying in $\langle \mathcal{E}_j \rangle$ for some $j \in I$. In particular, the various indices j belong to a finite subset $J \subset I$. By 6.2.5 the pullback of each $\mathcal{F}_{j,\alpha}$ along the projection $E_j \to X$ is globally generated on E_j. Thus, ${p_J}^*\mathcal{F}$ is globally generated, so too is $\mathcal{M}$. □

In this case, we shall say that $\mathcal{E}_J$ *tensor generates* $\mathcal{M}$ by abuse of language.

(6.2.26) Lemma. *Let X be a quasicompact and quasiseparated algebraic space and $\mathcal{E}_I$ a tensor generating family. Then there exists a finitely presented sheaf $\mathcal{G}$ that satisfies the following property:*

(i) If $\mathcal{E}_J$ tensor generates $\mathcal{G}$ for some finite $J \subset I$, then E_J has a finite, finitely presented and surjective covering by a quasiaffine scheme.

(ii) If X is a scheme, then E_J is quasiaffine.

We call $\mathcal{G}$ a $\mathcal{E}_I$*-initial* if condition 6.2.26.(i) is satisfied. By Theorem 5.1.5 we know that E_J is always representable by an AF-scheme, and if X (and hence E_J) is normal and noetherian, then E_J is already quasiaffine. Before we give a proof, let us first describe the behavior under pushforwards along finite maps.

(6.2.27) Lemma. *Let $f\colon X' \to X$ be a finite, finitely presented and surjective morphism of algebraic spaces, and let $\mathcal{E}_I$ be a tensor generating family on X. If $\mathcal{G}$ is a $f^*\mathcal{E}_I$ initial sheaf on X', then $f_*\mathcal{G}$ is $\mathcal{E}_I$-initial.*

Proof. Let $\varphi\colon \mathcal{E} \twoheadrightarrow f_*\mathcal{G}$ be a surjection with $\mathcal{E} \in \langle \mathcal{E}_I \rangle$ and J as above. Form the cartesian square

$$\begin{array}{ccc} E_J \times_X Y & \longrightarrow & X' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ E_J & \xrightarrow{p_J} & X \end{array}$$

whose vertical arrows are finite, finitely presented and surjective morphisms. This induces a surjection $f^*\mathcal{E} \twoheadrightarrow f^*f_*\mathcal{G} \twoheadrightarrow \mathcal{G}$ since f is quasiaffine.

Now $E_J \times_X Y$ is canonically isomorphic to the fiber product $E'_J = \prod_{j\in J}(E'_j/X')$ of the frame bundles $E'_j \to X'$ associated with $\mathcal{E}'_j := f^*\mathcal{E}_j$. So by assumption on $\mathcal{G}$, E'_J has a finite, finitely presented and surjective covering $h\colon U \to E'_J$ by a quasiaffine scheme U. Clearly the composition $f' \circ h$ is also finite, finitely presented and surjective. □

Proof of Lemma 6.2.26. There exists a scheme Y and a finite, finitely presented and surjective morphism $f\colon Y \to X$ [Ryd10b, Thm. B]. So by Lemma 6.2.27 it suffices to prove (ii). Since X is quasicompact and quasiseparated *scheme*, by Proposition 1.1.2 there exists a finitely presented sheaf $\mathcal{G}$ such that every finitely presented sheaf $\mathcal{M}$ has a resolution $\bigoplus_i \mathcal{G}^{\otimes n_i} \twoheadrightarrow \mathcal{M}$ for finitely many $n_i \in \mathbb{N}$. So if $\mathcal{E}_J$ tensor generates $\mathcal{G}$ for some finite J, then every sheaf on E_J is globally generated. It follows that E_J is quasiaffine. □

Proof of Proposition 6.2.24. Let us suppose that E_J is an algebraic space. By Lemma 6.2.26 there exists a ${p_J}^*\mathcal{E}_I$-initial sheaf $\mathcal{G}$ on E_J. Since X has the resolution property, it satisfies the completeness property by Proposition 3.3.9. Thus, ${p_J}_*\mathcal{G}$ is the direct limit of finitely presented $\mathcal{O}_X$-modules. Therefore we can choose a morphism $\mathcal{E}' \to {p_J}_*\mathcal{G}$ for some $\mathcal{E}' \in \langle \mathcal{E}_K \rangle$, such that the adjoint map

${p_J}^*\mathcal{E}' \to {p_J}^*{p_J}_*\mathcal{G} \twoheadrightarrow \mathcal{G}$ is surjective. Then $E_J \times_X E_K$ is canonically isomorphic to the total space of the frame bundles associated with ${p_J}^*\mathcal{E}_K$ and hence representable by an AF-scheme by the assumptions on $\mathcal{G}$ and Theorem 5.1.5. In particular, it is a scheme. So there exists an $(p_{JK})^*\mathcal{E}_I$-initial sheaf $\mathcal{G}'$ on $E_J \times_X E_K$ that satisfies 6.2.26.(ii).

By repeating the previous argument for $\mathcal{G}'$, resp. p_J, replaced by $\mathcal{G}$, resp. $p_{JK}\colon E_J \times_X E_K \to X$, there exists a finite subset $L \subset I$ such $E_J \times_X E_K \times_X E_L$ is quasiaffine. We conclude that $E_{J\cup K\cup L}$ is a closed substack of the latter since each $E_i \times_X E_i \to E_i$ has a section (by the diagonal). Thus, $E_{J\cup K\cup L}$ is also quasiaffine. □

(6.2.28) *Remark.* The existence of the finitely presented sheaf $\mathcal{G}$ on given quasicompact and quasiseparated algebraic space X of Lemma 6.2.26 simplifies the investigation of the resolution property of X because it suffices to construct a *single* locally free resolution for the verification thereof.

6.3. Totaro's Theorem

With the preceding results, we can easily derive a proof of Totaro's theorem which characterizes global quotient stacks in terms of the resolution property. Totaro proved this for normal noetherian algebraic stacks [Tot04, 1.1]. However, our results imply that the normal hypothesis is unnecessary. Besides, the preceding methods even allow to prove a relative version for arbitrary finitely presented morphisms $X \to S$ of algebraic stacks without any noetherian hypothesis.

(6.3.1) Theorem. *Let $X \to S$ be a quasicompact and quasiseparated morphism of algebraic stacks with S quasicompact which satisfies one of the following hypothesis:*

- *(a) X is noetherian with affine stabilizer groups at closed S-points and S is affine.*
- *(b) $X \to S$ is a quotient stack; for instance, if $X \to S$ is representable.*
- *(c) $X \to S$ is of finite presentation and has relative affine stabilizer groups at geometric S-points.*

Then the following assertions are equivalent:

- *(i) $X \to S$ has the resolution property.*
- *(ii) $X \to S$ is a global quotient stack.*

Proof. The implication (ii) ⇒ (i) is essentially due to Thomason (see Example 4.3.2) and holds for arbitrary quasicompact and quasiseparated morphisms $X \to S$. So the new part is (i) ⇒ (ii). By Lemma 6.2.22 we know that the family of all vector bundles $\mathcal{E}_I$ is a tensor generating family. If we can show that $\mathcal{E}_I$ has a finite *tensor* generating subfamily $\mathcal{E}_J$, then $\bigoplus_{j\in J}\mathcal{E}_j$ is a tensor generator by Proposition 6.2.16 and the associated frame bundle gives the desired global quotient stack structure by Proposition 6.2.12.

Let us prove that each hypothesis (a)-(c) on $X \to S$ is sufficient to verify the existence of such a finite subfamily $\mathcal{E}_J$. In case (a) we may invoke Proposition 6.2.23. In the other cases $X \to S$ is not of finite presentation, but it suffices to show that the fiber product $\prod_n (E_{n_i}/X)$ of the frame bundles $E_i \to X$ of $\mathcal{E}_i$ is eventually representable for sufficiently large but finite $J \subset I$ by Proposition 6.2.24. This condition is true in case (b) because the vector bundle inducing the quotient stack structure belongs to $\mathcal{E}_I$. However, the final case (c) was already settled by Totaro. He showed in [Tot04, Lemma 4.1] that X is a quotient stack, even if X is not normal. □

CHAPTER 7

Future prospects and applications

Originally, the resolution property of a scheme or an algebraic stack with affine diagonal X has been formulated by means of the category of quasicoherent sheaves $\mathrm{QCoh}(X)$ relative to their full subcategory of vector bundles $\mathrm{VB}(X)$. The equivalence of the resolution property with the existence of locally free tensor generators suggests to consider it rather as a property of a *single* vector bundle. On the one hand, it manifests as an algebraic property — the associated tensor hull is a generating family for $\mathrm{QCoh}(X)$. On the other hand, it is expressed in geometrical terms – the associated frame bundle has quasiaffine total space.

The next step would be an extensive investigation of tensor generators and the search for further analogies with ample line bundles. An open question is the existence of cohomological criteria or of appropriate embedding theorems.

A positive solution might be useful to extend the known existence results to a larger class of schemes and stacks. So far the resolution property remains difficult to verify, even for concrete objects like toric varieties. The latter are not just a testing ground by Włodarczyk's embedding theorem: Every normal variety, satisfying the condition that every pair of points admits an affine open neighborhood, can be embedded into a toric varietey [Wło93].

One remarkable advantage of the concept of tensor generators is the reduction of the resolution property to a single locally free sheaf. This insight will be applied in Theorem 7.2.2 below to give a sufficient condition for the stability of the resolution property under deformation.

Before, we prove that every quasicompact algebraic stack with quasifinite diagonal satisfies the resolution property étale locally (7.1.2).

7.1. Finite flat scheme covers

In 4.1.3.(vi) we showed that every finite, faithfully flat and finitely presented morphism of algebraic stacks $f\colon Z \to X$ preserves the resolution property. In particular, X has the resolution property if there exists such a covering f with Z being a scheme that is quasiaffine, or has an ample line bundle or an ample family of line bundles.

The existence of finite and flat scheme covers for an algebraic stack X is difficult to verify, in general. However, if X a separated Deligne-Mumford stack of finite type over a field k whose coarse moduli space is a quasiprojective scheme, then this holds by [KV04, Thm. 1] using a Bertini-type argument; the cover Z is quasiprojective, so that X has the resolution property. Many well-known moduli stacks allow a locally closed embedding in such stacks, e.g. the moduli stacks $\mathcal{K}_{g,n}(X,d)$ of n-pointed genus g stable maps of degree d into a tame Deligne-Mumford stack X with projective moduli space [AGOT07], [AV02]. For a recent discussion of that matter we refer the reader to [Kre09].

Conversely, a quasicompact algebraic stack with quasifinite diagonal admits a finite, finitely presented scheme covering that is flat over a dense quasicompact open subset ([EHKV01, 2.7] and [Ryd10b, Thm. B]).

Generic flatness and Theorem 6.3.1 imply the following result:

(7.1.1) Proposition. *Let X be a quasicompact algebraic stack with quasifinite diagonal. Then there exists a dense open subset $U \subset X$ which is a global quotient stack.*

A consequence of 4.1.3.(vi) is that every algebraic stack with quasifinite diagonal has the resolution property étale locally.

(7.1.2) Theorem. *Every quasicompact algebraic stack X with quasifinite diagonal has étale locally the resolution property, and hence is étale locally a global quotient stack.*

Proof. In [Ryd10a, Thm. 7.4] is was shown that X has étale locally a finite, flat and finitely presented covering by a quasiaffine scheme using the Keel-Mori trick [KM97]. So by 4.1.3.(vi) X has étale locally the resolution property. From Theorem 6.3.1 follows then that X is étale locally a global quotient stack. □

7.2. The resolution property and deformations

Given an algebraic stack X, it is often significantly simpler to verify the resolution property of the reduction X_{red} (see Remark 2.1.13 for the case of algebraic surfaces, or section 4.2 for classifying stacks of group schemes).

We shall see that the obstruction for lifting the resolution property along the nilpotent immersion $X_{\mathrm{red}} \hookrightarrow X$ lies in a second cohomology group of a sheaf on X_{red}.

(7.2.1) Lemma. *Let $X_0 \hookrightarrow X$ be an first order deformation of quasicompact and quasiseparated algebraic stacks given by a quasicoherent ideal $\mathcal{I} \subset \mathcal{O}_X$.*

If $\mathcal{E}_0$ is a tensor generator on X_0, then there exists an obstruction lying in $\mathrm{H}^2(X_0, \mathcal{I} \otimes \mathcal{E}nd(\mathcal{E}_0))$ whose vanishing is necessary and sufficient for the existence of a tensor generator $\mathcal{E}$ satisfying $\mathcal{E}|_{X_0} = \mathcal{E}_0$.

Proof. The obstruction controls the lifting of the vector bundle [Ill05, §5]. Since $\mathcal{E}_0|_{X_{\mathrm{red}}} = \mathcal{E}|_{X_{\mathrm{red}}}$ is a tensor generator, also $\mathcal{E}$ is a tensor generator by 6.2.14.(v). □

As an immediate consequence of Theorem 6.3.1 this yields a criterion for lifting the resolution property along nilpotent immersions.

(7.2.2) Theorem. *Let X_0 be a quasicompact and quasiseparated algebraic stack satisfying the hypothesis of Theorem 6.3.1, and let $i: X_0 \hookrightarrow X$ be a first order deformation given by a quasicoherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$.*

If X_0 has the resolution property, then there exists a vector bundle $\mathcal{E}_0$ on X_0 and an obstruction $o \in \mathrm{H}^2(X_0, \mathcal{I} \otimes \mathcal{E}_0)$ whose vanishing is sufficient for the resolution property of X to hold.

Clearly, this is just a first step for the understanding of the resolution property with respect to deformations. As an application, we infer that the resolution property is stable under infinitesimal thickenings of points and curves.

(7.2.3) Corollary. *Let X be a quasicompact and quasiseparated algebraic space. If X_{red} has the resolution property and the support of the nilradical $\mathcal{N}il(X)$ is of dimension ≤ 1, then X has the resolution property.*

Obviously, the obstruction vanishes if all higher cohomology groups are already zero.

(7.2.4) Corollary. *Let X be a quasicompact and quasiseparated algebraic stack satisfying the hypothesis of Theorem 6.3.1.*

Suppose that X is cohomologically affine (i.e. there exists a cohomologically affine morphism $X \to S$ to some affine scheme S). Then X has the resolution property if and only if X_{red} has the resolution property.

APPENDIX A

Pinching flat quasicoherent sheaves

For the convenience of the reader we recall the results of Ferrand [Fer03] on pinched schemes.

Let $f\colon X' \to X$ be a finite morphism of schemes which is an isomorphism over a dense open set $U \subset X$ and has schematically dense image (i.e. $f^\sharp\colon \mathcal{O}_X \to f_*\mathcal{O}_{X'}$ is injective). Define the conductor ideal $\mathcal{C} := \mathrm{Ann}_{\mathcal{O}_X}(f_*\mathcal{O}_{X'}/\mathcal{O}_X)$, the inverse image ideal $\mathcal{C}' := \mathcal{C} \cdot \mathcal{O}_{X'}$ and denote by Y resp. X' the closed subschemes, they define. This gives a cartesian diagram

$$\begin{array}{ccc} X' & \xleftarrow{v} & Y' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xleftarrow{u} & Y \end{array} \tag{A.0.1}$$

which is also cocartesian by [Fer03], 4.3. and 1.2. Therefore, we may interpret X to be the gluing of X' along the finite morphism $g\colon Y' \to Y$.

Let $\mathcal{C}(Z)$ be the category of quasicoherent $\mathcal{O}_Z$-modules on a scheme Z, that are flat, or flat and of finite type, or locally free of finite type. There is a complete description of $\mathcal{C}(X)$ as the fiber product of $\mathcal{C}(Y)$ with $\mathcal{C}(X')$ over $\mathcal{C}(Y')$. We follow closely the notation of Ferrand, work out in detail [Fer03, Complément 7.4] and start with the definition of the functors, that induce the announced equivalence.

From the equality $ug = fv$ we infer an isomorphism of functors $\sigma\colon g^*u^* \xrightarrow{\simeq} v^*f^*$. Thus, by the universal property of the fibered product of categories, we obtain a covariant functor

$$T\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y) \times_{\mathrm{QCoh}(Y')} \mathrm{QCoh}(X')$$
$$\mathcal{F} \mapsto (u^*\mathcal{F},\ \sigma_{\mathcal{F}},\ f^*\mathcal{F}).$$

Next, define a right adjoint functor:

$$S\colon \mathrm{QCoh}(Y) \times_{\mathrm{QCoh}(Y')} \mathrm{QCoh}(X') \to \mathrm{QCoh}(X).$$

For that consider the adjunction maps

$$\psi_v(\mathcal{M}')\colon \mathcal{M}' \to v_*v^*\mathcal{M}'$$
$$\psi_g(\mathcal{N})\colon \mathcal{N} \to g_*g^*\mathcal{N}$$

and the identity $h := fv = ug$. Then we define S to map a triple $(\mathcal{N}, \tau, \mathcal{M}')$, where $\mathcal{N} \in \mathrm{QCoh}(Y)$, $\mathcal{M}' \in \mathrm{QCoh}(X')$ and $\tau\colon g^*\mathcal{N} \xrightarrow{\simeq} v^*\mathcal{M}'$, to the fiber product $u_*\mathcal{N} \times_{h_*v^*\mathcal{M}'} f_*\mathcal{M}'$, so that it fits in the cartesian square

$$\begin{array}{ccccc} S(\mathcal{N}, \tau, \mathcal{M}') & \longrightarrow & & & f_*\mathcal{M}' \\ \downarrow & & & & \downarrow{\scriptstyle f_*(\psi_v(\mathcal{M}'))} \\ u_*\mathcal{N} & \xrightarrow{u_*(\psi_g(\mathcal{N}))} & u_*g_*g^*\mathcal{N} & \xrightarrow{h_*(\tau)} & h_*v^*\mathcal{M}' \end{array} \tag{A.0.2}$$

That is, $S(\mathcal{N}, \tau, \mathcal{M}')$ is defined to be the kernel of

$$u_*\mathcal{N} \oplus f_*\mathcal{M}' \to u_*g_*g^*\mathcal{N} \oplus f_*v_*v^*\mathcal{M}' \xrightarrow{h_*(\tau)-\mathrm{id}} h_*v^*\mathcal{M}'.$$

Then a straightforward calculation shows that S is right adjoint to T. Having the adjoint functors at hand, we are able to state the following result:

(A.1) Theorem. *Let X, X', Y, Y' be schemes and $f\colon X' \to X$, $g\colon Y' \to Y$ affine morphisms and $v\colon Y' \to X'$ a closed immersion, defining a cocartesian square*

$$\begin{array}{ccc} X' & \xleftarrow{\;v\;} & Y' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xleftarrow{\;u\;} & Y \end{array}$$

and consider the adjoint functors defined above

$$T \quad : \quad \mathrm{QCoh}(X) \quad \rightleftarrows \quad \mathrm{QCoh}(Y) \times_{\mathrm{QCoh}(Y')} \mathrm{QCoh}(X') \quad : \quad S.$$

Then the square is also cartesian and the following holds:

(i) The counit of the adjunction $TS \to \mathrm{id}$ is an isomorphism.

(ii) A quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$ is zero if and only if $T(\mathcal{M}) = 0$

(iii) For every quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$, the unit of adjunction $\mathcal{M} \to ST(\mathcal{M})$ is surjective, its kernel is annihilated by $\mathcal{I}$ and contained in $\mathcal{I}\mathcal{M}$, where $\mathcal{I} \subset \mathcal{O}_X$ is the quasicoherent sheaf of ideals defining $Y \subset X$.

(iv) Given a scheme Z, denote by $\mathcal{C}(Z)$ the category of quasicoherent $\mathcal{O}_Z$-modules, that are of finite type (respectively flat, flat and of finite type, locally free of finite type). Then the functor S induces by restriction a functor

$$S_{\mathcal{C}}\colon \mathcal{C}(X') \times_{\mathcal{C}(Y')} \mathcal{C}(Y) \to \mathcal{C}(X),$$

which is an equivalence of categories in the respected cases.

Proof. All properties are local with respect to X, so if we cover X by open affines, we get a family of squares like (A.1) of affine schemes as all involved morphisms are affine. Thus, we may assume that all involved schemes are affine and may refer to [Fer03, Theorem 2.2]. □

APPENDIX B

Algebraic stacks with quasiaffine diagonal

Let us briefly recall the collection of morphisms $f\colon X \to Y$ of algebraic stacks that have quasiaffine diagonal $\Delta\colon X \to X \times_Y X$. An algebraic stack X has quasiaffine diagonal if the structure morphism $X \to \operatorname{Spec} \mathbb{Z}$ has quasiaffine diagonal. This condition can be considered as a weak separatedness condition for algebraic stacks. For example, every (locally) separated morphism has quasiaffine diagonal. More generally, if the diagonal is quasicompact and quasi-finite, then it is already quasiaffine by [LMB00, A.2] because the diagonal is always locally of finite type and assumed to be separated. In particular, every (quasiseparated) scheme, algebraic space or Deligne-Mumford stack has quasiaffine diagonal.

By standard arguments [LMB00] one deduces the usual permanence properties, which follow from those of quasiaffine morphisms:

(B.1) Proposition. *Let S be an algebraic stack.*

(i) If $f\colon X \to Y$ and $g\colon Y \to Z$ have quasiaffine diagonal, then $g \circ f$ has quasiaffine diagonal.

(ii) If $f\colon X \to Y$ is an S-morphism with quasiaffine diagonal, then for every base change morphism of algebraic stacks $S' \to S$, $f_{(S')}\colon X_{(S')} \to Y_{(S')}$ has quasiaffine diagonal.

(iii) Let $f\colon X \to Y$ and $f'\colon X' \to Y'$ be S-morphisms with quasiaffine diagonal. Then $f \times_S g\colon X \times_S X' \to Y \times_S Y'$ has quasiaffine diagonal.

(iv) If the composition $g \circ f$ of two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ has quasiaffine diagonal then f has quasiaffine diagonal.

(v) If $f\colon X \to Y$ is an S-morphism and $S_\alpha \to S$ a fpqc-covering family, so that each $f_{(S_\alpha)}\colon X_{(S_\alpha)} \to Y_{(S_\alpha)}$ has quasiaffine diagonal, then f has quasiaffine diagonal.

(B.2) Corollary. *Let $f\colon X \to S$ be a morphism of algebraic stacks. If X has quasiaffine diagonal over $\mathbb{Z}$, then f has quasiaffine digonal. Conversely, if $S \to \operatorname{Spec} \mathbb{Z}$ and f have quasiaffine diagonal, then $X \to \operatorname{Spec} \mathbb{Z}$ has quasiaffine diagonal.*

Algebraic stacks with quasiaffine diagonal have always affine stabilizer groups:

(B.3) Proposition. *Let X be an algebraic stack (over $\mathbb{Z}$). If X has quasiaffine diagonal, then for every point $x\colon$ Spec $\to X$ the stabilizer group scheme G_x is an affine algebraic group over k.*

Proof. The diagonal $\Delta\colon X \to X \times_{\mathbb{Z}} X$ is quasiaffine and of finite type and the same holds for the group scheme $G_x \to \operatorname{Spec} k$ since it is the pullback of Δ along $(x, x)\colon \operatorname{Spec} k \to X \times_{\mathbb{Z}} X$. This proves the assertion using the well-known fact that every quasiaffine algebraic group scheme is already affine (see [FSR05, 7.5.3]). □

(B.4) *Remark.* The converse statement does not hold in general. By [Ray70, X. 13] there exists a smooth group scheme $G \to S$, that is not quasiaffine (even not quasiprojective), but has affine closed fibres. So $X = BG_S$ gives a counterexample.

The next proposition gives a characterization of algebraic stacks with quasiaffine diagonal in terms of morphisms.

(B.5) Definition. A morphism $X \to Y$ of algebraic stacks is called *locally quasiaffine* if there exists a fpqc-covering family $(U_i \to X)$ such that each $U_i \to X \to Y$ is a quasiaffine morphism of algebraic stacks.

(B.6) Proposition. *An algebraic stack X has quasiaffine diagonal if and only if every morphism of algebraic stacks $Y \to X$ is locally quasiaffine.*

Before we prove this as a special case of B.9, we briefly discuss the permanence properties of locally quasiaffine morphisms. They are local on the domain and inherit all functorial properties of quasiaffine morphisms, except for localness on the codomain Y. In fact they are only local on Y for *quasiaffine* covering maps $V_j \to Y$.

(B.7) Proposition. *Let S be an algebraic stack.*

(i) *Every 1-isomorphism is locally quasiaffine.*

(ii) *A morphism of algebraic stacks is locally quasiaffine if and only if one (equivalently every) 2-isomorphic morphism is locally quasiaffine.*

(iii) *If $f: X \to Y$ is a morphism and $u_i: U_i \to X$ a fpqc-covering family, so that each $f \circ u_i$ is locally quasiaffine, then f is locally quasiaffine.*

(iv) *Every fpqc-covering family $U_i \to X$ of locally quasiaffine morphisms can be refined to become a fpqc-covering family of quasiaffine morphisms.*

(v) *If $f: X \to Y$ is a locally quasiaffine S-morphism, then for every base change morphism of algebraic stacks $S' \to S$, $f_{(S')}: X_{(S')} \to Y_{(S')}$ is locally quasiaffine.*

(vi) *Let $f: X \to Y$ and $g: Y \to Z$ be locally quasiaffine morphisms. Then $g \circ f$ is also locally quasiaffine.*

(vii) *Let $f: X \to Y$ and $f': X' \to Y'$ be locally quasiaffine S-morphisms. Then $f \times_S g: X \times_S X' \to Y \times_S Y'$ is locally quasiaffine.*

(viii) *If the composition $g \circ f$ of two morphisms $f: X \to Y$ and $g: Y \to Z$ is locally quasiaffine, and if g has locally quasiaffine diagonal, then f is locally quasiaffine.*

(ix) *If $f: X \to Y$ is an S-morphism and $S_\alpha \to S$ a fpqc-covering family of quasiaffine morphisms, so that each $f_{(S_\alpha)}: X_{(S_\alpha)} \to Y_{(S_\alpha)}$ is locally quasiaffine, then f is locally quasiaffine.*

Proof. The proof is formal and hence left to the reader. □

(B.8) Lemma. *For every algebraic stack X the structure morphism $X \to \mathbb{Z}$ is locally quasiaffine. Hence, for every Y with quasiaffine diagonal, every morphism $X \to Y$ is locally quasiaffine.*

Proof. Every algebraic stack X has a fpqc-covering family $U_i \to X$ of morphisms with U_i affine. Then the structure maps $U_i \to \operatorname{Spec} \mathbb{Z}$ are affine and hence quasiaffine, so $X \to \mathbb{Z}$ is always locally quasiaffine. The second claim follows from B.7.(vi). □

(B.9) Proposition. *Let X be an algebraic stack. Then the following conditions are equivalent:*

(i) X has quasiaffine diagonal over $\mathbb{Z}$.

(ii) X has locally quasiaffine diagonal over $\mathbb{Z}$.

(iii) Every morphism $Y \to X$ of algebraic stacks is locally quasiaffine.

(iv) There exists a fpqc-covering family $U_i \to X$ of affine schemes U_i, where each $u_i\colon U_i \to X$ is a quasiaffine morphism.

Proof. (i) ⇒ (ii) is trivial and (ii) ⇒ (iii) follows from B.8 and B.7.(vi). But (iii) ⇒ (iv) is also clear by refining an arbitrary fpqc-covering $V_j \to X$ of affine schemes V_j. It suffices therefore to show (iv) ⇒ (i). So let $u_i\colon U_i \to X$ be the given covering. Then each $U_i \times_X U_j$ is a (quasi-)affine scheme. Now the products $u_i \times u_j\colon U_i \times_{\mathbb{Z}} U_j \to X \times_{\mathbb{Z}} X$ form an fpqc-covering of $X \times_{\mathbb{Z}} X$. The pullback of $\Delta_{X/\mathbb{Z}}$ along $u_i \times u_j$ is $U_i \times_X U_j \to U_i \times_{\mathbb{Z}} U_j$. This is a quasiaffine morphisms since the domain is quasiaffine, so by fpqc descent follows that $\Delta_{X/\mathbb{Z}}$ is quasiaffine. □

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