Proper classes of short exact sequences and structure theory of modules

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Introduction

In this thesis we investigate proper classes of short exact sequences, especially those proper classes induced by supplement-like and complement-like submodules in the category of left R-modules over an associative ring R and in the categories of type $\sigma[M]$ of left R-modules subgenerated by a module M.

Proper classes were introduced by Buchsbaum in [9] for an exact category. We use the axioms given by Mac Lane in [43] for abelian categories (see 3.1).
A proper class $\mathcal{P}$, in an abelian category $\mathbb{A}$ defines a closed subbifunctor of the $\mathrm{Ext}^1_{\mathbb{A}}$ functor as has been shown by Butler and Horrocks in [10] or by Buan in [8], i.e.

$$\mathrm{Ext}^1_{\mathcal{P}}(-,-) \subseteq \mathrm{Ext}^1_{\mathbb{A}}(-,-) : \mathbb{A}^{op} \times \mathbb{A} \to \mathcal{A}b,$$

where $\mathcal{A}b$ denotes the category of abelian groups, and for $A, C \in \mathbb{A}$, $\mathrm{Ext}^1_{\mathbb{A}}(C, A)$ denotes the class of isomorphism classes of short exact sequences

$$0 \to A \to B \to C \to 0.$$

$\mathrm{Ext}^1_{\mathbb{A}}(C, A)$ together with the Baer sum is an abelian group and for any proper class $\mathcal{P}$, $\mathrm{Ext}^1_{\mathcal{P}}(C, A)$ is a subgroup (see [43]). These ideas were the starting point of relative homological algebra. Proper classes are investigated by Mishina and Skornyakov in [47], Walker in [60], Manovcev in [45] and Sklyarenko in [56] for abelian groups, and by Generalov in [27, 28], Stenström in [57, 58] and Sklyarenko in [55] for module categories.

Supplement and complement submodules induce proper classes. This was noted for abelian groups by Harrison in [32] and for R-Mod by Stenström and Generalov in [57, 27, 28]. Recently this investigations were further developed by Al-Takhmann et al. and Mermut in [46, 1].

The first purpose of this thesis is to continue this investigation studying the proper classes induced by supplements and complements relative to a class of modules $\mathcal{C}$ closed under submodules and factor modules. We call such a class a $\{q, s\}$-closed class. We obtain some known results and new ones for special cases of $\mathcal{C}$. For instance, if $\mathcal{C}$ is the class of singular modules, then we obtain the δ-supplements introduced by Zhou in [64]. When $\mathcal{C}$ is R-Mod, we recover the classical supplements and complements. We also consider the case when $\mathcal{C}$ is the torsion class $\mathbb{T}_\tau$ associated to a hereditary preradical τ or the torsionfree class $\mathbb{F}_\rho$ associated to a cohereditary preradical ρ. Two interesting cases are when $\mathcal{C} = \sigma[M]$ and $\mathcal{C} = \sigma_f[M]$ considered as $\{q, s\}$-closed classes of R-Mod.

After collecting some preliminary results in Chapters 1, 2 and 3 about abelian categories, proper classes, supplements and complements in a module category, we define, in Chapter 4, $\mathcal{C}$-supplements.

A submodule $K \subseteq N$ is called $\mathcal{C}$-small if for any submodule $X \subseteq N$ the equality $X + K = N$ and $N/X \in \mathcal{C}$ implies that $X = N$ and we write $K \ll_{\mathcal{C}} N$. The properties of $\mathcal{C}$-small submodules are similar to those of small submodules (see 8.3). We characterize the radical defined by the reject of the class of simple modules in $\mathcal{C}$. This radical, applied to a module N, is equal to the sum of all $\mathcal{C}$-small submodules of N. We call a submodule $K \subseteq N$ a $\mathcal{C}$-supplement in N if there is a submodule $K' \subseteq N$ with $K + K' = N$ and $K \cap K' \ll_{\mathcal{C}} K$. Then we prove that

$\mathcal{C}$-supplements induce a proper class. This proper class contains the proper class induced by the supplements. In [1] Al-Takhman et al. introduced a generalization of supplements using a radical τ. They call a submodule $K \subseteq N$ a τ-supplement in N if there is a submodule $K' \subseteq N$ such that $K + K' = N$ and $K \cap K' \subseteq \tau(K)$. They prove that the proper class induced by the τ-supplements is the proper class injectively generated by the class of τ-torsionfree modules, where a proper class $\mathcal{P}$, injectively generated by a class of modules $\mathcal{I}$, consists of those short exact sequences such that the modules in $\mathcal{I}$ are injective with respect to them. Every supplement is a Rad-supplement for $\tau = \text{Rad}$. In the same way $\mathcal{C}$-supplements are $\text{rad}_{\mathcal{C}}$-supplements with

$$\tau(N) = \text{rad}_{\mathcal{C}}(N) = \sum \{\text{all } \mathcal{C}\text{-small submodules of } N\}$$

for every module N.

In Chapter 5, dual to the concept of $\mathcal{C}$-small submodules, we call a submodule $K \subseteq N$ $\mathcal{C}$-essential if for any submodule $X \subseteq N$ the equality $X \cap K = 0$ and $X \in \mathcal{C}$ implies that $X = 0$. We write $K \subseteq_{\mathcal{C}e} N$. We characterize the idempotent preradical defined by the trace of the class of simple modules of $\mathcal{C}$ in a module N. This preradical is the intersecction of all $\mathcal{C}$-essential submodules of N. Next we define $\mathcal{C}$-complements as those submodules $K \subseteq N$ such that there is a submodule $K' \subseteq N$ with $K \cap K' = 0$ and $K \oplus K'/K \subseteq_{\mathcal{C}e} N/K$. We prove that $\mathcal{C}$-complements induce a proper class. Dual to the τ-supplements, Al-Takhman et al. introduce in [1] the concept of τ-complements for an idempotent preradical τ. The proper class induced by the τ-complements is the proper class projectively generated by the class of τ-torsion modules, where a proper class $\mathcal{P}$, projectively generated by a class of modules $\mathcal{Q}$, consists of the short exact sequences such that the modules in $\mathcal{Q}$ are projective with respect to them. Complements are Soc-complements for $\tau = \text{Soc}$ and $\mathcal{C}$-complements are $\text{tr}_{\mathcal{C}}$-complements for

$$\tau(N) = \text{tr}_{\mathcal{C}}(N) = \bigcap \{\text{all } \mathcal{C}\text{-essential submodules of } N\}$$

for every module N.

The second purpose of this work is to compare, in Chapter 6, the lattice of all proper classes with the lattice of cotorsion pairs. In [53] Salce introduced a cotorsion pair in the following way: Take a class $\mathcal{A}$ of abelian groups and consider $\mathcal{P}$, the proper class projectively generated by $\mathcal{A}$. Then he observes that

$$\text{Div}(\mathcal{P}) = \{X \in \text{Mod-}\mathbb{Z} \mid \text{Ext}(A, X) = 0 \, \forall A \in \mathcal{A}\},$$

where $\text{Div}(\mathcal{P})$ is the class of abelian groups D such that every short exact sequence beginning with D belongs to $\mathcal{P}$. He defines the cotorsion pair cogenerated by $\mathcal{A}$ by $({}^{\perp}\text{Div}(\mathcal{P}), \text{Div}(\mathcal{P}))$. In the same way he defines the cotorsion pair generated by a class $\mathcal{B}$ of abelian groups by $(\text{Flat}(\mathcal{R}), \text{Flat}(\mathcal{R})^{\perp})$ with $\mathcal{R}$ the proper class injectively generated by $\mathcal{B}$ and $\text{Flat}(\mathcal{R})$ the class of abelian groups G such that every short exact sequence ending at G belongs to $\mathcal{R}$. The work of Salce on cotorsion pairs was generalized to module categories and abelian categories and it has been extensively studied. We prove some results which show how some properties of those proper classes yield information about their associated cotorsion pairs (e.g. 14.22, 14.25). We define a correspondence between the lattice of injectively (projectively) generated proper classes and the lattice of cotorsion pairs using the construction of Salce. Let $\mathcal{P}$ be an injectively generated proper class in an abelian category. Define

$$\Phi(\mathcal{P}) = (\text{Flat}(\mathcal{P}), \text{Flat}(\mathcal{P})^{\perp}).$$

Φ is an order-reversing correspondence between injectively generated proper classes and cotorsion pairs which preserves arbitrary meets. We prove that this correspondence is bijective if we restrict Φ to the class of Xu proper classes, these are injectively generated proper classes $\mathcal{P}$, such that $\mathrm{Inj}(\mathcal{P}) = \mathrm{Flat}(\mathcal{P})^{\perp}$. For example, in R-Mod the proper class of pure exact sequences is a Xu proper class when the pure injective and the cotorsion modules coincide.

Finally we consider cotorsion pairs relative to a proper class $\mathcal{P}$ as introduced by Hovey in [37]. We call them $\mathcal{P}$-cotorsion pairs. They are pairs of complete orthogonal classes with respect to the functor $\mathrm{Ext}^1_{\mathcal{P}}$ instead of $\mathrm{Ext}^1_{\mathbb{A}}$. We show that, like the cotorsion pairs, they come from injectively (projectively) generated proper classes. Here we define three classes of objects which correspond to the $\mathcal{P}$-flats, $\mathcal{P}$-divisibles and $\mathcal{P}$-regulars in the absolute case.

Let $\mathcal{P}$ and $\mathcal{R}$ be two proper classes. An object X is called $\mathcal{P}$-$\mathcal{R}$-flat if every short exact sequence in $\mathcal{P}$ ending at X belongs to $\mathcal{R}$. $\mathcal{P}$-$\mathcal{R}$-divisibles are defined dually. We show that every $\mathcal{P}$-cotorsion pair is of the form

$$({}^{\perp_{\mathcal{P}}}(\mathcal{P}\text{-Div-}\mathcal{R}), \mathcal{P}\text{-Div-}\mathcal{R})$$

for a projectively generated proper class $\mathcal{R}$ and also of the form

$$(\mathcal{P}\text{-Flat-}\mathcal{R}, (\mathcal{P}\text{-Flat-}\mathcal{R})^{\perp_{\mathcal{P}}})$$

for an injectively generated proper class $\mathcal{R}$, where for a class $\mathcal{X}$,

$$\mathcal{X}^{\perp_{\mathcal{P}}} = \mathrm{Ker}\,(\mathrm{Ext}^1_{\mathcal{P}}(\mathcal{X}, -)) \text{ and } {}^{\perp_{\mathcal{P}}}\mathcal{X} = \mathrm{Ker}\,(\mathrm{Ext}^1_{\mathcal{P}}(-, \mathcal{X})).$$

We obtain properties of $\mathcal{P}$-$\mathcal{R}$-flats, $\mathcal{P}$-$\mathcal{R}$-divisibles and $\mathcal{P}$-$\mathcal{R}$-regulars and we show that this classes coincide with known concepts in module theory.

In the Appendix we include the construction of the relative functors $\mathrm{Ext}^n_{\mathcal{P}}$ and the definitions of the homological dimensions relative to a proper class $\mathcal{P}$ due to Mac Lane and Alizade (see [2, 43]).

One can define proper classes in more general categories. The definition of exact categories, introduced by Quillen in [50], is the reformulation of the axioms of a proper class, where a short exact sequence in $\mathcal{P}$ corresponds to a coflation (f, g) in the exact category. In a preabelian category Generalov defines a proper class of cokernels (kernels) using some equivalent axioms to those of Mac Lane (see [29]). In a triangulated category Beligiannis defines a proper class of triangles using some axioms analogous to those of an exact category (see [4]).

Notation

$\mathrm{Ext}^1_{\mathbb{A}}$	the extension functor, iii
$\mathbb{A}$	an abelian category, 1
$\mathcal{A}b$	the category of abelian groups, 1
R-Mod	the category of left R-modules, 1
$\mathrm{Proj}(\mathbb{A})$	the class of projective objects of $\mathbb{A}$, 2
$\mathrm{Inj}(\mathbb{A})$	the class of injective objects of $\mathbb{A}$, 2
$\sigma[M]$	the full subcategory of R-Mod subgenerated by M, 3
$\mathcal{A}^\circ$	$\{X \in \mathbb{A} \mid \mathrm{Hom}_{\mathbb{A}}(X, \mathcal{A}) = 0\}$, 4
$\mathcal{A}^\bullet$	$\{Y \in \mathbb{A} \mid \mathrm{Hom}_{\mathbb{A}}(\mathcal{A}, Y) = 0\}$, 4
$\mathrm{rad}_{\mathcal{C}}(N)$	$\bigcap\{\mathrm{Ker}\, f \mid f : N \to C,\, C \in \mathcal{C}\}$, 6
$\mathrm{tr}_{\mathcal{C}}(N)$	$\sum\{\mathrm{Im}\, f \mid f : C \to N,\, C \in \mathcal{C}\}$, 6
$\mathcal{P}, \mathcal{R}$	proper classes, 9
$\mathcal{A}$bs	the proper class of all short exact sequences, 9
$\mathcal{S}$plit	the proper class of all splitting short exact sequences, 9
$\mathrm{Proj}(\mathcal{P})$	the class of $\mathcal{P}$-projective objects, 12
$\mathrm{Inj}(\mathcal{P})$	the class of $\mathcal{P}$-injective objects, 12
$\mathrm{Flat}(\mathcal{P})$	the class of $\mathcal{P}$-flat objects, 12
$\mathrm{Div}(\mathcal{P})$	the class of $\mathcal{P}$-divisible objects, 13
$\mathrm{Reg}(\mathcal{P})$	the class of $\mathcal{P}$-regular objects, 13
$\pi^{-1}(\mathcal{Q})$	the proper class projectively generated by a class $\mathcal{Q}$, 14
$\iota^{-1}(\mathcal{I})$	the proper class injectively generated by a class $\mathcal{I}$, 15
τ-Compl	the proper class of τ-complements = the proper class projectively generated by $\mathbb{T}_\tau$, 15
τ-Suppl	the proper class of τ-supplements = the proper class injectively generated by $\mathbb{F}_\tau$, 16
$\mathcal{P}$ure	the proper class of pure short exact sequences, 17
$K \ll N$	K is a small submodule of N, 19
$K \subseteq_e N$	K is an essential submodule of N, 19
$K \subseteq_{cc} N$	K is a coclosed submodule of N, 20
Cocls	the proper class of coclosed submodules, 20
$K \ll_{\mathcal{C}} N$	K is a $\mathcal{C}$-small submodule of N, 25
$\mathcal{C}$-Suppl	the proper class of $\mathcal{C}$-supplements, 28
Suppl	the proper class of supplements, 31
$\mathcal{S}(\mathcal{C})$	the class of simple modules in $\mathcal{C}$, 32
$\mathrm{rad}_{\mathcal{S}(\mathcal{C})}(N)$	$\bigcap\{\mathrm{Ker}\, f \mid f : N \to S,\, S \in \mathcal{S}(\mathcal{C})\}$, 32
$\mathrm{rad}_{\mathcal{S}(\mathcal{C})}$-Suppl	the proper class of $\mathrm{rad}_{\mathcal{S}(\mathcal{C})}$-supplements = the proper class injectively generated by $\{N \in \sigma[M] \mid \mathrm{rad}_{\mathcal{S}(\mathcal{C})}(N) = 0\}$, 33

Co-Neat	$=$ Rad-Suppl the proper class injectively generated by $\{N \in \sigma[M] \mid \mathrm{Rad}(N) = 0\}$, 33
$\mathscr{I}$	the class of M-injective modules in $\sigma[M]$, 36
$\mathscr{P}$	the class of projective modules in $\sigma[M]$, 36
$\mathscr{M}$	the class of M-small modules in $\sigma[M]$, 36
$\mathscr{S}$	the class of M-singular modules in $\sigma[M]$, 36
α	$= \mathrm{rad}_{\mathcal{S}(\mathscr{I})}$, 36
β	$= \mathrm{rad}_{\mathcal{S}(\mathscr{P})}$, 36
γ	$= \mathrm{rad}_{\mathcal{S}(\mathscr{M})}$, 36
δ	$= \mathrm{rad}_{\mathcal{S}(\mathscr{S})}$, 36
$K \subseteq_{\mathcal{C}e} N$	K is a $\mathcal{C}$-essential sumodule of N, 39
$\mathcal{C}$-Compl	the proper class of $\mathcal{C}$-complements, 41
Compl	the proper class of complements, 44
Neat	$=$ Soc-Compl the proper class projectively generated by the class of simple modules, 45
$\mathrm{tr}_{\mathcal{S}(\mathcal{C})}(N)$	$\sum\{\mathrm{Im}\, f \mid f : S \to N,\, S \in \mathcal{S}(\mathcal{C})\}$, 44
$\mathrm{tr}_{\mathcal{S}(\mathcal{C})}$-Compl	the proper class of $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}$-complements $=$ the proper class projectively generated by $\{N \in \sigma[M] \mid \mathrm{tr}_{\mathcal{S}(\mathcal{C})}(N) = N\}$, 45
${}^{\perp}\mathcal{D}$	$\{X \in \mathbb{A} \mid \mathrm{Ext}^1_{\mathbb{A}}(X, \mathcal{D}) = 0\}$, 53
$\mathcal{D}^{\perp}$	$\{X \in \mathbb{A} \mid \mathrm{Ext}^1_{\mathbb{A}}(\mathcal{D}, X) = 0\}$, 53
$\Phi(\mathcal{P})$	$({}^{\perp}(\mathrm{Flat}(\mathcal{P})^{\perp}), \mathrm{Flat}(\mathcal{P})^{\perp})$, 54
$\Psi(\mathcal{R})$	$({}^{\perp}\mathrm{Div}(\mathcal{R}), ({}^{\perp}\mathrm{Div}(\mathcal{R}))^{\perp})$, 54
$\widetilde{\Phi}(\mathcal{F}, \mathcal{C})$	$\iota^{-1}(\mathcal{C})$, 57
$\widetilde{\Psi}(\mathcal{F}, \mathcal{C})$	$\pi^{-1}(\mathcal{F})$, 57
${}^{\perp_{\mathcal{P}}}\mathcal{D}$	$\{X \in \mathbb{A} \mid \mathrm{Ext}^1_{\mathcal{P}}(X, \mathcal{D}) = 0\}$, 60
$\mathcal{D}^{\perp_{\mathcal{P}}}$	$\{X \in \mathbb{A} \mid \mathrm{Ext}^1_{\mathcal{P}}(\mathcal{D}, X) = 0\}$, 60
$\mathcal{P}$-Flat-$\mathcal{R}$	the class of $\mathcal{P}$-$\mathcal{R}$-flat objects, 61
$\mathrm{FT}(\mathcal{P})$	$= \mathcal{P}$ure-Flat-$\mathcal{P}$, 61
$\mathcal{P}$-Div-$\mathcal{R}$	the class of $\mathcal{P}$-$\mathcal{R}$-divisible objects, 65
$\mathrm{PT}(\mathcal{P})$	$= \mathcal{P}$ure-Div-$\mathcal{P}$, 65
$\mathrm{Ext}^1_{\mathcal{P}}$	the extension functor relative to the proper class $\mathcal{P}$, 78

Chapter 1

Preliminaries

We will be mainly interested in the category R-Mod of left R-modules over an associative ring with unit and in categories $\sigma[M]$ of left R-modules subgenerated by a module M. However we provide in this section the background material we make use of in the more general setting of abelian categories. This methods will be applied in the subsequent sections to our module categories.

1 Abelian categories

In this section we recall some basic definitions and elementary properties of abelian categories. For a more comprehensive treatment see [44, Chapter VIII] or [25].

1.1. Abelian categories. A category $\mathbb{A}$ is an **abelian category** if it satisfies the following axioms:

(Ab1) For every pair of objects $X, Y \in \mathbb{A}$, $\mathrm{Hom}_{\mathbb{A}}(X, Y)$ is an abelian group and the composition is bilinear.

(Ab2) $\mathbb{A}$ has finite direct sums.

(Ab3) Every morphism has a kernel and a cokernel.

(Ab4) Every monomorphism is a kernel and every epimorphism is a cokernel.

Some of the most important features of abelian categories are the method of "diagram chasing", the fact that a morphism which is both a monomorphism and an epimorphism is an isomorphism, short exact sequences are determined by one of their morphisms and pullbacks of epimorphisms are epimorphisms. Examples of abelian categories are $\mathcal{A}b$, R-Mod and $\sigma[M]$.

In the subsequent we fix an abelian category $\mathbb{A}$.

1.2. Short exact sequences. A **short exact sequence** in $\mathbb{A}$ is a sequence of objects and morphisms in $\mathbb{A}$ of the form

$$E : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

such that $f = \mathrm{Ker}\, g$ and $g = \mathrm{Coker}\, f$. A **morphism of short exact sequences** is a tripel of morphisms (f_1, f_2, f_3) in $\mathbb{A}$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 0 \end{array}$$

commutative. A short exact sequence E is called **split exact** (or it **splits**) if there exists a morphism $f' : B \to A$ such that $f'f = \mathrm{id}_A$. Equivalently, there exists a morphism $g' : C \to B$ such that $gg' = \mathrm{id}_C$.

We recall some basic lemmas. The references at the beginning of each proposition are of the works where the proofs can be found.

1.3. The five lemma. [41, 8.3.13] *Consider a commutative diagram in $\mathbb{A}$ with exact rows*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5. \end{array}$$

(i) *If f_1 is an epimorphism and f_2 and f_4 monomorphisms, then f_3 is a monomorphism.*

(ii) *If f_5 is a monomorphism and f_2 and f_4 epimorphisms, then f_3 is an epimorphism.*

(iii) *If f_1, f_2, f_4, f_5 are isomorphisms, so is f_3.*

1.4. Homotopy lemma. [24, Proposition 3.2] *For a commutative diagram in $\mathbb{A}$ with exact rows*

$$\begin{array}{ccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 \\ \downarrow & & \downarrow & & \downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3, \end{array}$$

the following are equivalent:

(a) *There exists a morphism $A_2 \to B_1$ making the upper triangle in the left square commutative.*

(b) *There exists a morphism $A_3 \to B_2$ making the lower triangle in the right square commutative.*

1.5. Projectives. An object $P \in \mathbb{A}$ is called **projective** if it satisfies the equivalent conditions:

(a) The functor $\mathrm{Hom}_{\mathbb{A}}(P, -)$ is exact.

(b) $\mathrm{Ext}^1_{\mathbb{A}}(P, X) = 0$ for all $X \in \mathbb{A}$.

The class of all projective objects of $\mathbb{A}$ is denoted by $\mathrm{Proj}(\mathbb{A})$. The category has **enough projectives** if for every object $C \in \mathbb{A}$ there is an epimorphism

$$P \to C \quad \text{with} \quad P \in \mathrm{Proj}(\mathbb{A}).$$

1.6. Injectives. An object $I \in \mathbb{A}$ is called **injective** if it satisfies the equivalent conditions:

(a) The functor $\mathrm{Hom}_{\mathbb{A}}(-, I)$ is exact.

(b) $\mathrm{Ext}^1_{\mathbb{A}}(Y, I) = 0$ for all $Y \in \mathbb{A}$.

The class of all injective objects of $\mathbb{A}$ is denoted by $\mathrm{Inj}(\mathbb{A})$. The category has **enough injectives** if for every object $A \in \mathbb{A}$ there is a monomorphism

$$A \to I \quad \text{with} \quad I \in \mathrm{Inj}(\mathbb{A}).$$

1.7. (Co-)complete abelian categories. An abelian category $\mathbb{A}$ is called **complete** if $\mathbb{A}$ has arbitrary products and **cocomplete** if $\mathbb{A}$ has arbitrary coproducts.

1.8. Grothendieck categories. An abelian category $\mathbb{A}$ is a **Grothendieck category** if $\mathbb{A}$ admits a generator and direct limits are exact. If $\mathbb{A}$ is a Grothendieck category, then every object in $\mathbb{A}$ has an injective hull, i.e. $\mathbb{A}$ has enough injectives.

1.9. Locally finitely presented Grothendieck categories. Recall that an object A in a Grothendieck category $\mathbb{A}$ is **finitely generated** if for any family of subobjects A_i, $i \in I$, satisfying $A = \sum_I A_i$, there is a finite subset $J \subseteq I$ such that $A = \sum_J A_i$. The subcategory of finitely generated objects is denoted by $\mathrm{fg}\,\mathbb{A}$. The category $\mathbb{A}$ is called **locally finitely generated** if every object $X \in \mathbb{A}$ is a direct sum $X = \sum_I X_i$ of finitely generated subobjects X_i or, equivalently, $\mathbb{A}$ posesses a family of finitely generated generators. A finitely generated object $C \in \mathbb{A}$ is **finitely presented** if in every short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathbb{A}$ with B finitely generated, A is also finitely generated. The subcategory of finitely presented objects is denoted by $\mathrm{fp}\,\mathbb{A}$. The category $\mathbb{A}$ is said to be **locally finitely presented** if every object $X \in \mathbb{A}$ is a direct limit $\varinjlim X_i$ of finitely presented objects X_i or, equivalently, $\mathbb{A}$ posesses a family of finitely presented generators.

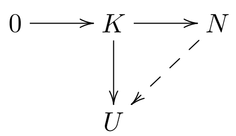
1.10. Locally noetherian categories. An object A in a Grothendieck category $\mathbb{A}$ is called **noetherian** if the lattice of subobjects of A satisfies the acending chain condition (**acc**) or, equivalently, every subobject of A is finitely generated. A Grothendieck category $\mathbb{A}$ is **locally noetherian** if $\mathbb{A}$ has a family of noetherian generators. In this case every object $X \in \mathbb{A}$ is the direct sum $X = \sum_I X_i$ of noetherian subobjects X_i. A locally finitely generated Grothendieck category is locally noetherian iff the direct sum of injective objects is injective. A classical result due to Matlis says that if $\mathbb{A}$ is locally noetherian, then every injective object is a direct sum of indecomposable injectives.

1.11. The category R-Mod. Let R be an associative ring with unit element. We denote by R-Mod (Mod-R) the category of unital left (right) R-modules. Morphisms between left R-modules are written on the left and for two morphisms $f : M \to N$, $g : N \to K$ we write their composition gf. $\mathrm{Hom}_R(M, N)$ denotes the abelian group of morphisms from M to N, and $\mathrm{End}_R(M)$ is the endomorphism ring of M. $E(M)$ stands for the R-injective hull of M. R-Mod is a complete and cocomplete locally finitely presented Grothendieck category with a projective generator R.

1.12. The category $\sigma[M]$. An R-module N is said to be **generated** by an R-module M if there exists an epimorphism $\varphi : M^{(\Lambda)} \to N$ for a set Λ. N is said to be **subgenerated** by M if N is isomorphic to a submodule of an M-generated module. The full subcategory of R-Mod whose objects are all R-modules subgenerated by M is denoted by $\sigma[M]$. The subcategory $\sigma[M]$ is closed under direct sums, kernels and cokernels. A module N is called a **subgenerator** in $\sigma[M]$ if $\sigma[M] = \sigma[N]$. If $M = R$, then $\sigma[M] = R$-Mod. The product of a family $\{N_\lambda\}_\Lambda$ in $\sigma[M]$ is denoted by $\prod_\Lambda^M N_\lambda$. It is obtained by

$$\prod_\Lambda^M N_\lambda = \mathrm{Tr}(\sigma[M], \prod_\Lambda N_\lambda).$$

$\sigma[M]$ is the smallest full Grothendieck subcategory of R-Mod containing M. An R-module U is called M-injective if every diagram



can be completed commutatively by a morphism $N \to U$. M-projective modules are defined dually. A module $N \in \sigma[M]$ is M-injective iff N is injective in the category $\sigma[M]$ but in general an M-projective module need not be projective in the category $\sigma[M]$. The **M-injective hull** of a module $N \in \sigma[M]$ is given by $\widehat{N} = \mathrm{Tr}(M, E(N))$, where $E(N)$ is the R-injective hull of N. $\sigma[M]$ is locally finitely presented iff for every finitely presented module $F \in R$-Mod and every morphism $f : F \to A$, with $A \in \sigma[M]$, there is a factorization of f through a finitely presented module in $\sigma[M]$ (see [49, 1.6]). In general $\sigma[M]$ need not have finitely presented modules or projective modules (see [61, 18.12] and [49, 1.7]). The category $\sigma[M]$ is locally noetherian iff M is locally noetherian, i.e. every finitely generated submodule of M is noetherian (see [61, 27.3] for a characterization of locally noetherian modules).

1.13. The category $\sigma_f[M]$. We denote by $\sigma_f[M]$ the full subcategory of $\sigma[M]$ whose objects are submodules of finitely M-generated modules. The category $\sigma_f[R]$ consist of the submodules of finitely generated R-modules. $\sigma_f[M]$ is closed under finite products, submodules and factor modules. $\sigma_f[M]$ is an abelian category and contains all finitely generated modules of $\sigma[M]$. The category $\sigma_f[M]$ is not in general a Grothendieck category.

2 Torsion pairs and preradicals

In this section we recall the definition and some properties of torsion pairs (also called torsion theories) and preradicals in abelian categories. Torsion pairs in abelian categories were introduced by Dickson in [16]. There is some additional axiom on the abelian category. $\mathbb{A}$ is called **subcomplete** if the class of subobjects of any object A in $\mathbb{A}$ is a set and for any set of subobjects $\{U_\alpha\}_\Lambda$ of A, $\oplus_\Lambda U_\alpha$ and $\prod_\Lambda A/U_\alpha$ exist (see also [40]). For our purposes we will always assume this axiom being closer to a module category, in which we will later concretize the results of this more general setting.

2.1. Hom-orthogonal classes. For a class of objects $\mathcal{A}$ in $\mathbb{A}$ we define the classes

$$\mathcal{A}^\circ = \{X \in \mathbb{A} \mid \mathrm{Hom}_\mathbb{A}(X, \mathcal{A}) = 0\},$$

$$\mathcal{A}^\bullet = \{Y \in \mathbb{A} \mid \mathrm{Hom}_\mathbb{A}(\mathcal{A}, Y) = 0\}.$$

2.2. Torsion pairs. A **torsion pair** in $\mathbb{A}$ is a pair $(\mathbb{T}, \mathbb{F})$ of classes of objects of $\mathbb{A}$ such that

(i) $\mathbb{T} = \mathbb{F}^\circ$,

(ii) $\mathbb{F} = \mathbb{T}^\bullet$.

If $(\mathbb{T}, \mathbb{F})$ is a torsion pair, then $\mathbb{T}$ is called the **torsion class** and $\mathbb{F}$ the **torsionfree class**. For a torsion pair $(\mathbb{T}, \mathbb{F})$ we have that $\mathbb{T}$ is closed under factors, extensions and direct sums and $\mathbb{F}$ is closed under extensions, subobjects and products.
For any class of objects $\mathcal{A} \subseteq \mathbb{A}$, the pairs

$$(\mathcal{A}^\circ, \mathcal{A}^{\circ\bullet}) \quad \text{and} \quad (\mathcal{A}^{\bullet\circ}, \mathcal{A}^\bullet)$$

are torsion pairs, called the torsion pair **generated** and **cogenerated** by the class $\mathcal{A}$, respectively.

2.3. Preradicals.[1] A **preradical** τ is an endofunctor $\tau : \mathbb{A} \to \mathbb{A}$ such that

(i) for every $C \in \mathbb{A}$, $\tau(C) \subseteq C$,

(ii) for every morphism $f : C \to C'$, $\tau(f) = f \mid_{\tau(C)} : \tau(C) \to \tau(C')$.

[1]Preradicals can be defined in more general categories (see [13]).

This can be expressed in the commutative diagram

$$\begin{array}{ccc} \tau(C) & \xrightarrow{\tau(f)=f|_{\tau(C)}} & \tau(C') \\ \big\downarrow & & \big\downarrow \\ C & \xrightarrow[f]{} & C'. \end{array}$$

2.4. Properties of preradicals. *Let τ be a preradical, $A \subseteq B$ and $\{A_i\}_I$ a family of objects in $\mathbb{A}$. Then*

(i) *If $\tau(A) = A$, then $A \subseteq \tau(B)$.*

(ii) *If $\tau(B/A) = 0$, then $\tau(B) \subseteq A$.*

(iii) $\tau(\oplus_I A_i) = \oplus_I \tau(A_i)$.

(iv) $\tau(\prod_I A_i) \subseteq \prod_I \tau(A_i)$.

2.5 Definition. A preradical τ is called

idempotent if $\tau(\tau(C)) = \tau(C)$ for all $C \in \mathbb{A}$,

radical if $\tau(C/\tau(C)) = 0$ for all $C \in \mathbb{A}$.

For each preradical τ in $\mathbb{A}$, there is an induced endofunctor

$$1/\tau : \mathbb{A} \to \mathbb{A}, \qquad C \mapsto C/\tau(C).$$

2.6. Hereditary preradicals. A preradical τ is called **hereditary** if it satisfies the equivalent conditions:

(a) τ is left exact.

(b) $\tau(A) = A \cap \tau(B)$ for all $A \subseteq B$.

(c) τ is idempotent and $\mathbb{T}_\tau$ is closed under subobjects.

2.7. Cohereditary preradicals. A preradical τ is called **cohereditary** if it satisfies the equivalent conditions:

(a) $1/\tau$ is right exact.

(b) $\tau(B/A) = (\tau(B) + A)/A$ for all $A \subseteq B$.

(c) τ is a radical and $\mathbb{F}_\tau$ is closed under factors.

(d) τ respects epimorphisms.

Associated to a preradical τ we have two clases of objects of $\mathbb{A}$

$$\mathbb{T}_\tau := \{X \in \mathbb{A} \mid \tau(X) = X\},$$

$$\mathbb{F}_\tau := \{Y \in \mathbb{A} \mid \tau(Y) = 0\}$$

the **pretorsion class** and the **pretorsionfree class**. $\mathbb{T}_\tau$ is closed under direct sums and factors and $\mathbb{F}_\tau$ is closed under products and subobjects. The pair $(\mathbb{T}_\tau, \mathbb{F}_\tau)$ is a torsion pair iff τ is an idempotent radical.

We will show how the closure properties of some classes of modules yield information about the preradicals associated to them. From now on let $\mathbb{A} = \sigma[M]$.

2.8. Classes of modules. Let $\mathcal{C}$ be a class of modules in $\sigma[M]$. $\mathcal{C}$ is called

s-closed	if $\mathcal{C}$ is closed under submodules,
q-closed	if $\mathcal{C}$ is closed under factor modules,
{q,s}-closed	if $\mathcal{C}$ is s-closed and q-closed,
pretorsion	if $\mathcal{C}$ is closed under direct sums and is q-closed,
pretorsionfree	if $\mathcal{C}$ is closed under products and is s-closed,
torsion	if $\mathcal{C}$ is pretorsion and closed under extensions,
torsionfree	if $\mathcal{C}$ is pretorsionfree and closed under extensions,
hereditary pretorsion	if $\mathcal{C}$ is pretorsion and s-closed,
cohereditary pretorsionfree	if $\mathcal{C}$ is pretorsionfree and q-closed.

2.9. Reject and trace. Let $\mathcal{C}$ be a class of modules in $\sigma[M]$. We define two preradicals by setting for each $N \in \sigma[M]$

$$\mathrm{rad}_{\mathcal{C}}(N) = \mathrm{Rej}(N, \mathcal{C}) = \bigcap \{\mathrm{Ker}\, f \mid f : N \to C,\, C \in \mathcal{C}\},$$

$$\mathrm{tr}_{\mathcal{C}}(N) = \mathrm{Tr}(\mathcal{C}, N) = \sum \{\mathrm{Im}\, f \mid f : C \to N,\, C \in \mathcal{C}\}.$$

2.10. Properties of $\mathrm{rad}_{\mathcal{C}}$. *Let $\mathcal{C}$ be a class of modules in $\sigma[M]$.*

(i) $\mathrm{rad}_{\mathcal{C}}$ *is a radical.*

(ii) *If $\mathcal{C}$ is a pretorsionfree class, then* $\mathrm{rad}_{\mathcal{C}}(N) = 0$ *iff* $N \in \mathcal{C}$.

(iii) *If $\mathcal{C}$ is a torsionfree class, then $rad_{\mathcal{C}}$ is an idempotent radical.*

(iv) *If $\mathcal{C}$ is a cohereditary pretorsionfree class, then for all $K \subseteq N$,*

$$\mathrm{rad}_{\mathcal{C}}(N/K) = (\mathrm{rad}_{\mathcal{C}}(N) + K)/K.$$

(v) *If $\mathcal{C}$ is a class of M-injective modules, then* $\mathrm{rad}_{\mathcal{C}}$ *is an idempotent preradical.*

Proof. (i) For every $C \in \mathcal{C}$ and every morphism $f : N/\mathrm{rad}_{\mathcal{C}}(N) \to C$, define $U_{f,C}$ by $\mathrm{Ker}\, f = U_{f,C}/\mathrm{rad}_{\mathcal{C}}(N)$. Note that

$$U_{f,C} = \mathrm{Ker}\,(N \to N/U_{f,C}) \text{ with}$$
$$N/U_{f,C} \simeq f(N/\mathrm{rad}_{\mathcal{C}}(N)) \subseteq C \in \mathcal{C},$$

i.e. $U_{f,C}$ is the kernel of a morphism from N to C. Therefore

$$\mathrm{rad}_{\mathcal{C}}(N/\mathrm{rad}_{\mathcal{C}}(N)) = \cap_{f,C}(U_{f,C}/\mathrm{rad}_{\mathcal{C}}(N)) =$$

$$= (\cap_{f,C} U_{f,C})/\mathrm{rad}_{\mathcal{C}}(N) = \mathrm{rad}_{\mathcal{C}}(N)/\mathrm{rad}_{\mathcal{C}}(N) = 0.$$

(ii) $\mathrm{rad}_{\mathcal{C}}(N) = 0$ iff N is cogenerated by $\mathcal{C}$. Thus if $\mathcal{C}$ is closed under products and submodules every module cogenerated by $\mathcal{C}$ belongs to $\mathcal{C}$. Clearly for every module $N \in \mathcal{C}$ we have $\mathrm{rad}_{\mathcal{C}}(N) = 0$.

(iii) Consider the short exact sequence

$$0 \to \mathrm{rad}_{\mathcal{C}}(N)/\mathrm{rad}_{\mathcal{C}}(\mathrm{rad}_{\mathcal{C}}(N)) \to N/\mathrm{rad}_{\mathcal{C}}(\mathrm{rad}_{\mathcal{C}}(N)) \to N/\mathrm{rad}_{\mathcal{C}}(N) \to 0.$$

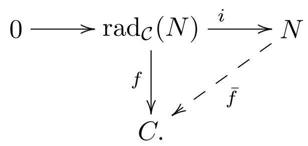
Since $\mathrm{rad}_{\mathcal{C}}(\mathrm{rad}_{\mathcal{C}}(N)/\mathrm{rad}_{\mathcal{C}}(\mathrm{rad}_{\mathcal{C}}(N)) = 0$ and $\mathrm{rad}_{\mathcal{C}}(N/\mathrm{rad}_{\mathcal{C}}(N) = 0$,
$\mathrm{rad}_{\mathcal{C}}(N/\mathrm{rad}_{\mathcal{C}}(\mathrm{rad}_{\mathcal{C}}(N))) = 0$. From 2.4 (ii) it follows that $\mathrm{rad}_{\mathcal{C}}(N) \subseteq \mathrm{rad}_{\mathcal{C}}(\mathrm{rad}_{\mathcal{C}}(N))$.

(iv) Since

$$\begin{aligned} N/\mathrm{rad}_{\mathcal{C}}(N)\Big/(\mathrm{rad}_{\mathcal{C}}(N)+K)/\mathrm{rad}_{\mathcal{C}}(N) &\simeq N/(\mathrm{rad}_{\mathcal{C}}(N)+K) \\ &\simeq N/K\Big/(\mathrm{rad}_{\mathcal{C}}(N)+K)/K \end{aligned}$$

and $\mathcal{C}$ is closed under factors, $\mathrm{rad}_{\mathcal{C}}(N/(\mathrm{rad}_{\mathcal{C}}(N+K))) = 0$. From 2.4 (ii) it follows that $\mathrm{rad}_{\mathcal{C}}(N/K) \subseteq (\mathrm{rad}_{\mathcal{C}}(N)+K)/K$.

(v) Let $f : \mathrm{rad}_{\mathcal{C}}(N) \to C$ with $C \in \mathcal{C}$. Consider the diagram



Since C is M-injective, there exist $\bar{f} : N \to C$ such that $f = \bar{f}i$. From the definition of $\mathrm{rad}_{\mathcal{C}}(N)$ follows that $f = \bar{f}i = 0$, i.e $\mathrm{rad}_{\mathcal{C}}(N) \subseteq \mathrm{rad}_{\mathcal{C}}(\mathrm{rad}_{\mathcal{C}}(N))$.

□

2.11. Properties of $\mathrm{tr}_{\mathcal{C}}$. *Let $\mathcal{C}$ be a class of modules in $\sigma[M]$.*

(i) $\mathrm{tr}_{\mathcal{C}}$ *is an idempotent preradical.*

(ii) *If $\mathcal{C}$ is a pretorsion class, then $\mathrm{tr}_{\mathcal{C}}(N) = N$ iff $N \in \mathcal{C}$.*

(iii) *If $\mathcal{C}$ is a torsion class, then $\mathrm{tr}_{\mathcal{C}}$ is an idempotent radical.*

(iv) *If $\mathcal{C}$ is a hereditary pretorsion class, then for all $K \subseteq N$,*

$$\mathrm{tr}_{\mathcal{C}}(K) = K \cap \mathrm{tr}_{\mathcal{C}}(N).$$

(v) *If $\mathcal{C}$ is a class of projective modules in $\sigma[M]$, then $tr_{\mathcal{C}}$ is a radical.*

Proof. (i) Let $f : C \to N$ with $C \in \mathcal{C}$. Then $\mathrm{Im}\, f \subseteq \mathrm{tr}_{\mathcal{C}}(N)$. Therefore $f : C \to \mathrm{tr}_{\mathcal{C}}(N)$, i.e. $\mathrm{Im}\, f \subseteq \mathrm{tr}_{\mathcal{C}}(\mathrm{tr}_{\mathcal{C}}(N))$. Thus $\mathrm{tr}_{\mathcal{C}}(N) \subseteq \mathrm{tr}_{\mathcal{C}}(\mathrm{tr}_{\mathcal{C}}(N))$.

(ii) $\mathrm{tr}_{\mathcal{C}}(N) = N$ iff N is generated by C. If $\mathcal{C}$ is closed under direct sums and factors, then every module generated by $\mathcal{C}$ belongs to $\mathcal{C}$. Clearly for every $C \in \mathcal{C}$, $\mathrm{tr}_{\mathcal{C}}(C) = C$.

(iii) Put $\mathrm{tr}_{\mathcal{C}}(N/\mathrm{tr}_{\mathcal{C}}(N)) = U/\mathrm{tr}_{\mathcal{C}}(N)$ and consider the short exact sequence

$$0 \to \mathrm{tr}_{\mathcal{C}}(N) \to U \to U/\mathrm{tr}_{\mathcal{C}}(N) \to 0.$$

Since $\mathrm{tr}_{\mathcal{C}}(\mathrm{tr}_{\mathcal{C}}(N)) = \mathrm{tr}_{\mathcal{C}}(N)$ and

$$\mathrm{tr}_{\mathcal{C}}(U/\mathrm{tr}_{\mathcal{C}}(N)) = \mathrm{tr}_{\mathcal{C}}(\mathrm{tr}_{\mathcal{C}}(N/\mathrm{tr}_{\mathcal{C}}(N))) = \mathrm{tr}_{\mathcal{C}}(N/\mathrm{tr}_{\mathcal{C}}(N)),$$

$\mathrm{tr}_{\mathcal{C}}(U) = U$. From 2.4 (i) follows that $U \subseteq \mathrm{tr}_{\mathcal{C}}(N)$, i.e $\mathrm{tr}_{\mathcal{C}}(N/\mathrm{tr}_{\mathcal{C}}(N) = 0$.

(iv) Since $\mathrm{tr}_{\mathcal{C}}(N)$ belongs to $\mathcal{C}$, also $K \cap \mathrm{tr}_{\mathcal{C}}(N)$. Therefore $\mathrm{tr}_{\mathcal{C}}(K \cap \mathrm{tr}_{\mathcal{C}}(N)) = K \cap \mathrm{tr}_{\mathcal{C}}(N) \subseteq K$. From 2.4 (i) follows that $K \cap \mathrm{tr}_{\mathcal{C}}(N) \subseteq \mathrm{tr}_{\mathcal{C}}(K)$.

(v) Let $f : C \to N/\mathrm{tr}_{\mathcal{C}}(N)$ be any morphism with $N \in \sigma[M]$ and $C \in \mathcal{C}$. Consider the diagram

$$\begin{array}{ccccc} & & C & & \\ & \swarrow^{\bar{f}} & \downarrow f & & \\ N & \xrightarrow[p]{} & N/\mathrm{tr}_{\mathcal{C}}(N) & \longrightarrow & 0. \end{array}$$

Since C is projective in $\sigma[M]$, there exist $\bar{f} : C \to N$ such that $p\bar{f} = f$. Since $\mathrm{Im}\,\bar{f} \subseteq \mathrm{tr}_{\mathcal{C}}(N)$, $\mathrm{Im}\,f = p(\mathrm{Im}\,\bar{f}) = 0$. Therefore $\mathrm{tr}_{\mathcal{C}}(N/\mathrm{tr}_{\mathcal{C}}(N)) = 0$. □

Chapter 2

Proper classes

3 Proper classes

In this section we introduce the main subject of this work, namely proper classes. Proper classes were introduced by Buchsbaum in [9]. They have been extensively studied in different contexts. We refer to [55, 29, 46, 47] for complete surveys and further reading.

Let $\mathcal{P}$ be a class of short exact sequences in an abelian category $\mathbb{A}$. If

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

belongs to $\mathcal{P}$, then f is called a **$\mathcal{P}$-monomorphism** and g a **$\mathcal{P}$-epimorphism**.

3.1. Proper classes. A class $\mathcal{P}$ of short exact sequences in $\mathbb{A}$ is called a **proper class** if it satisfies the following axioms:

(P1) $\mathcal{P}$ is closed under isomorphisms.

(P2) $\mathcal{P}$ contains all splitting short exact sequences of $\mathbb{A}$.

(P3) If f and f' are $\mathcal{P}$-monomorphisms, then $f'f$ is a $\mathcal{P}$-monomorphism.

(P4) If $f'f$ is a $\mathcal{P}$-monomorphism, then f is a $\mathcal{P}$-monomorphism.

(P5) If g and g' are $\mathcal{P}$-epimorphisms, then $g'g$ is a $\mathcal{P}$-epimorphism.

(P6) If gg' is a $\mathcal{P}$-epimorphism, then g is a $\mathcal{P}$-epimorphism.

The class of all short exact sequences of $\mathbb{A}$ (denoted by $\mathcal{A}$bs) and the class of all splitting short exact sequences (denoted by $\mathcal{S}$plit) are examples of proper classes. Note that the intersection of any family of proper classes is again a proper class.

3.2 Remark. Let $\mathcal{P}$ be a proper class in $\mathbb{A}$. Note that the axioms (P3) and (P4) imply:

(PB) In every pullback diagram

$$\begin{array}{ccccccccc} E' : 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ E : 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

if E belongs to $\mathcal{P}$, then E' belongs to $\mathcal{P}$. Moreover the axioms (P5) and (P6) imply:

(PO) In every pushout diagram

$$\begin{array}{ccccccccc}
E: 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
E': 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

if E belongs to $\mathcal{P}$, then E' belongs to $\mathcal{P}$.

3.3. Proposition. *Let $\mathcal{P}$ be a class of short exact sequences in an abelian category $\mathbb{A}$. The following are equivalent:*

(a) *$\mathcal{P}$ is a proper class.*

(b) *$\mathcal{P}$ satisfies the axioms* (P1), (P2), (P5), (P6) *and* (PB).

(c) *$\mathcal{P}$ satisfies the axioms* (P1), (P2), (P3), (P4) *and* (PO).

Proof. By the Remark 3.2 it follows (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c). We prove only (b) $\Rightarrow$ (a). The proof of (c) $\Rightarrow$ (a) is analogous (see [42, A.1]). We show that if $\mathcal{P}$ satisfies (b), then it satisfies also the axioms (P3) and (P4).

(P4) Let $f : A \to B$ and $g : B \to D$ be morphisms such that gf is a $\mathcal{P}$-monomorphism. We can form the pullback diagram

$$\begin{array}{ccccccccc}
E': 0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & \operatorname{Coker} f & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle g} & & \downarrow & & \\
E: 0 & \longrightarrow & A & \xrightarrow[gf]{} & D & \longrightarrow & \operatorname{Coker} gf & \longrightarrow & 0.
\end{array}$$

By hypothesis $E \in \mathcal{P}$, thus from (PB) follows that $E' \in \mathcal{P}$, i.e. f is a $\mathcal{P}$-monomorphism.

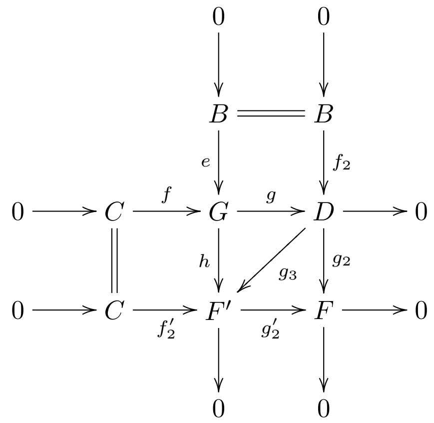
(P3) We use in this proof the following notation. For a pair of morphisms $f : X \to Y$ and $g : X \to Z$ we denote by $(f, g) : X \to Y \oplus Z$, the morphism defined by $(f, g)(x) = (f(x), g(x)) \in Y \oplus Z$. And for a pair of morphisms $h : Y \to X$ and $k : Z \to X$ we denote by $[h, k] : Y \oplus Z \to X$, the morphism defined by $[h, k](y, z) = h(y) + k(z) \in X$.
Let $f_1 : A \to B$ and $f_2 : B \to D$ be $\mathcal{P}$-monomorphisms. Set $C = \operatorname{Coker} f_1$ and $F = \operatorname{Coker} f_2$. Thus the sequences

$$E_1 : 0 \to A \xrightarrow{f_1} B \xrightarrow{g_1} C \to 0 \text{ and } E_2 : 0 \to B \xrightarrow{f_2} D \xrightarrow{g_2} F \to 0$$

belong to $\mathcal{P}$. From the pushout diagram

$$\begin{array}{ccccccccc}
E_2: 0 & \longrightarrow & B & \xrightarrow{f_2} & D & \xrightarrow{g_2} & F & \longrightarrow & 0 \\
 & & {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_3} & & \| & & \\
E_2': 0 & \longrightarrow & C & \xrightarrow[f_2']{} & F' & \xrightarrow[g_2']{} & F & \longrightarrow & 0
\end{array}$$

we obtain that $E_2' \in \mathcal{P}$. Consider the commutative diagram



with the lower right square a pullback. The lower row is in $\mathcal{P}$, therefore the sequences

$$0 \to B \xrightarrow{e} G \xrightarrow{h} F' \to 0 \text{ and } 0 \to C \xrightarrow{f} G \xrightarrow{g} D \to 0$$

belong to $\mathcal{P}$. Since $g_2 = g_2' g_3$, the middle row splits, i.e. there exists $\overline{g} : D \to G$ and $\overline{f} : G \to C$ such that $\mathrm{id}_C = \overline{f} f$, $\mathrm{id}_D = g\overline{g}$ and $f_2' \overline{f} = g_3 g - h$. Thus we have an isomorphism $(\overline{f}, g) : G \to C \oplus D$. This yields an isomorphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \xrightarrow{e} & G & \xrightarrow{h} & F' & \longrightarrow & 0 \\
 & & \| & & \big\downarrow{\scriptstyle (\overline{f},g)} & & \| & & \\
0 & \longrightarrow & B & \xrightarrow[(g_1,-f_2)]{} & C \oplus D & \xrightarrow[{[f_2', g_3]}]{} & F' & \longrightarrow & 0.
\end{array}$$

Therefore the lower row belongs to $\mathcal{P}$. Form the pullback diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B \oplus D & \xrightarrow{g_1 \oplus \mathrm{id}_D} & C \oplus D & \longrightarrow & 0 \\
 & & \| & & \big\downarrow{\scriptstyle [\mathrm{id}_B, 0]} & & \big\downarrow{\scriptstyle [\mathrm{id}_C, 0]} & & \\
0 & \longrightarrow & A & \xrightarrow[f_1]{} & B & \xrightarrow[g_1]{} & C & \longrightarrow & 0.
\end{array}$$

Since the lower row belongs to $\mathcal{P}$, the upper row belongs also to $\mathcal{P}$. Thus the composition

$$B \oplus D \xrightarrow{g_1 \oplus \mathrm{id}_D} C \oplus D \xrightarrow{[f_2', g_3]} F'$$

is a $\mathcal{P}$-epimorphism. Note that this last composition equals the composition

$$B \oplus D \xrightarrow{[f_2, \mathrm{id}_D]} D \xrightarrow{g_3} F'.$$

Therefore it is also a $\mathcal{P}$-epimorphism. It follows from (P6) that g_3 is a $\mathcal{P}$-epimorphism whose kernel is $f_2 f_1$, i.e. $f_2 f_1$ is a $\mathcal{P}$-monomorphism. This proves (P3). □

3.4. $\oplus$-closed proper classes. A proper class $\mathcal{P}$ in a cocomplete abelian category $\mathbb{A}$ is called **$\oplus$-closed**, if for any family of short exact sequences $\{E_i\}_I$ of $\mathcal{P}$, their direct sum $\oplus E_i$ belongs to $\mathcal{P}$.

3.5. $\prod$-closed proper classes. A proper class $\mathcal{P}$ in a complete abelian category $\mathbb{A}$ is called **$\prod$-closed**, if for any family of short exact sequences $\{E_i\}_I$ of $\mathcal{P}$, their direct product $\prod E_i$ belongs to $\mathcal{P}$.

3.6. Inductively closed proper classes. A proper class $\mathcal{P}$ in a Grothendieck category $\mathbb{A}$ is called **inductively closed** if for any direct system of short exact sequences $\{E_i\}_I$ of $\mathcal{P}$, their direct limit $\varinjlim E_i$ belongs to $\mathcal{P}$. An example of an inductively closed proper class is the class of all short exact sequences $\mathcal{A}$bs. The intersection of all inductively closed proper classes containing an arbitrary proper class $\mathcal{P}$ is called the **inductive closure** of $\mathcal{P}$ and denoted by $\overline{\mathcal{P}}$ (see [23]).

3.7. $\mathcal{P}$-injectives. An object $I \in \mathbb{A}$ is called **$\mathcal{P}$-injective** if it satisfies the equivalent conditions:

(a) The functor $\mathrm{Hom}_{\mathbb{A}}(-, I)$ is exact on all sequences of $\mathcal{P}$.

(b) $\mathrm{Ext}^1_{\mathcal{P}}(Y, I) = 0$ for all $Y \in \mathbb{A}$.

(a) means that I is injective with respect to every sequence in $\mathcal{P}$ and (b) that every sequence in $\mathcal{P}$ of the form

$$0 \to I \to B \to Y \to 0$$

splits. The class of all $\mathcal{P}$-injective objects is denoted by $\mathrm{Inj}(\mathcal{P})$. $\mathrm{Inj}(\mathcal{P})$ is closed under direct products and direct summands. Every injective object of $\mathbb{A}$ is $\mathcal{P}$-injective.

3.8. $\mathcal{P}$-essential extensions. A $\mathcal{P}$-monomorphism $A \to B$ is called a **$\mathcal{P}$-essential extension** of A if for any subobject $A' \subseteq B$ from $A \cap A' = 0$ and $A \hookrightarrow B/A'$ a $\mathcal{P}$-monomorphism follows that $A' = 0$.

3.9. $\mathcal{P}$-injective hull. A $\mathcal{P}$-essential extension $A \to J$ is called a **$\mathcal{P}$-injective hull** of A if J is $\mathcal{P}$-injective.

3.10. Proposition. [58, 4.5] *If $\mathcal{P}$ is an inductively closed proper class with enough $\mathcal{P}$-injectives, then every object in $\mathbb{A}$ has a $\mathcal{P}$-injective hull.*

3.11. $\mathcal{P}$-projectives. An object $P \in \mathbb{A}$ is called **$\mathcal{P}$-projective** if it satisfies the equivalent conditions:

(a) The functor $\mathrm{Hom}_{\mathbb{A}}(P, -)$ is exact on all sequences of $\mathcal{P}$.

(b) $\mathrm{Ext}^1_{\mathcal{P}}(P, X) = 0$ for all $X \in \mathbb{A}$.

(a) means that P is projective with respect to every sequence in $\mathcal{P}$ and (b) that every sequence in $\mathcal{P}$ of the form

$$0 \to X \to B \to P \to 0$$

splits. The class of all $\mathcal{P}$-projective objects is denoted by $\mathrm{Proj}(\mathcal{P})$. $\mathrm{Proj}(\mathcal{P})$ is closed under direct sums and direct summands. Every projective object of $\mathbb{A}$ is $\mathcal{P}$-projective.

3.12. $\mathcal{P}$-flats. An object $Q \in \mathbb{A}$ is called **$\mathcal{P}$-flat** if

$$\mathrm{Ext}^1_{\mathcal{P}}(Q, X) = \mathrm{Ext}^1_{\mathbb{A}}(Q, X) \text{ for all } X \in \mathbb{A},$$

i.e. every short exact sequence ending at Q belongs to $\mathcal{P}$. We denote the class of all $\mathcal{P}$-flat objects by $\mathrm{Flat}(\mathcal{P})$. Every projective object is $\mathcal{P}$-flat. If the category $\mathbb{A}$ has enough projectives, then an object Q is $\mathcal{P}$-flat iff there is a sequence

$$0 \to A \to P \to Q \to 0$$

in $\mathcal{P}$ with P projective.

3.13. $\mathcal{P}$-divisibles. An object $J \in \mathbb{A}$ is called **$\mathcal{P}$-divisible** if

$$\mathrm{Ext}^1_{\mathcal{P}}(Y, J) = \mathrm{Ext}^1_{\mathbb{A}}(Y, J) \text{ for all } J \in \mathbb{A},$$

i.e. every short exact sequence beginning with J belongs to $\mathcal{P}$. We denote the class of all $\mathcal{P}$-divisible objects by $\mathrm{Div}(\mathcal{P})$. Every injective object is $\mathcal{P}$-divisible. If the category $\mathbb{A}$ has enough injectives, then an object J is $\mathcal{P}$-divisible iff there is a sequence

$$0 \to J \to I \to C \to 0$$

in $\mathcal{P}$ with I injective.

$\mathcal{P}$-flats and $\mathcal{P}$-divisible objects are called $\mathcal{P}$-coprojective and $\mathcal{P}$-coinjective in [55, 46], ω-flat and ω-divisible in [47, 39] and $\mathcal{Q}$-flat and $\mathcal{I}$-coflat in [61] for the proper class projectively (injectively) generated by a class of modules $\mathcal{Q}$ ($\mathcal{I}$).

3.14. $\mathcal{P}$-regulars. An object $T \in \mathbb{A}$ is called **$\mathcal{P}$-regular** if every short exact sequence

$$0 \to A \to T \to C \to 0$$

belongs to $\mathcal{P}$, i.e. every short exact sequence with middle term T belongs to $\mathcal{P}$. We denote the class of all $\mathcal{P}$-regular objects by $\mathrm{Reg}(\mathcal{P})$.

In [61] $\mathcal{P}$-regular objects are called $\mathcal{Q}$-regular for the proper class projectively generated by a class of modules $\mathcal{Q}$ and $\mathcal{I}$-coregular for the proper class injectively generated by a class of modules $\mathcal{I}$.

3.15. The lattice of proper classes. Denote the class of all proper classes in $\mathbb{A}$ by $\wp_{\mathbb{A}}$. There is a partial order in $\wp_{\mathbb{A}}$ given by the inclusion

$$\mathcal{P}_1 \leq \mathcal{P}_2 \text{ iff } \mathcal{P}_1 \subseteq \mathcal{P}_2.$$

The minimal element is the class $\mathcal{S}$plit and the maximal is $\mathcal{A}$bs. For a family of proper classes $\{\mathcal{P}_\lambda\}_{\lambda \in \Lambda}$ the infimum of the family is given by the intersection

$$\wedge \mathcal{P}_\lambda = \cap \mathcal{P}_\lambda.$$

This is again a proper class (see [43]). The supremum of the family is given by the formula

$$\vee \mathcal{P}_\lambda = \cap \{\mathcal{P} \mid \cup \mathcal{P}_\lambda \subseteq \mathcal{P}, \mathcal{P} \text{ proper class}\}.$$

$(\wp_{\mathbb{A}}, \leq, \vee, \wedge, \mathcal{P}_0, \mathcal{P}_{\mathbb{A}})$ is a complete (big) lattice[1]. Following [48] we define some operations with proper classes. Let $\mathcal{P}_1$, $\mathcal{P}_2 \in \wp_{\mathbb{A}}$. We define

(i) $\mathcal{P}_1 \wedge \mathcal{P}_2 = \mathcal{P}_1 \cap \mathcal{P}_2$,

(ii) $\mathcal{P}_1 \vee \mathcal{P}_2 = \cap \{\mathcal{P} \mid \mathcal{P}_1 \cup \mathcal{P}_2 \subseteq \mathcal{P}, \mathcal{P} \text{ proper class}\}$,

(iii) $\mathcal{P}_1 \circ \mathcal{P}_2 = \{0 \to A \xrightarrow{f} B \to C \to 0 \mid f = f_1 f_2, \text{ with } f_i \text{ a } \mathcal{P}_i\text{-mono}\}$,

(iv) $\mathcal{P}_1 * \mathcal{P}_2$ given by $\mathrm{Ext}_{\mathcal{P}_1 * \mathcal{P}_2}(C, A) = \mathrm{Ext}_{\mathcal{P}_1}(C, A) + \mathrm{Ext}_{\mathcal{P}_2}(C, A)$.

In general $\mathcal{P}_1 \circ \mathcal{P}_2$ and $\mathcal{P}_1 * \mathcal{P}_2$ need no be proper classes. Finally, for a class of short exact sequences $\mathcal{R}$ (not necessarily proper) we define the proper class generated by $\mathcal{R}$ as follows:

$$< \mathcal{R} > := \cap \{\mathcal{P} \mid \mathcal{R} \subseteq \mathcal{P}, \mathcal{P} \text{ proper class}\}.$$

[1] $\wp_{\mathbb{A}}$ may not be a set.

4 Various proper classes

We recall how proper classes can be obtained from left or right exact additive functors between abelian categories. Let $F : \mathbb{A} \to \mathbb{B}$ be a right or left exact covariant (contravariant) additive functor between abelian categories. The class of short exact sequences

$$0 \to A \to B \to C \to 0$$

in $\mathbb{A}$ such that

$$0 \to F(A) \to F(B) \to F(C) \to 0$$
$$(0 \to F(C) \to F(B) \to F(A) \to 0)$$

is exact in $\mathbb{B}$ is a proper class (see [58]). We consider the case when F is one of the functors $\mathrm{Hom}_{\mathbb{A}}(X, -)$, $\mathrm{Hom}_{\mathbb{A}}(-, X)$ or $X \otimes -$ and also when F is a left exact preradical τ or the right exact endofunctor $1/\rho$, with ρ a cohereditary preradical of $\sigma[M]$.

4.1. Projectively generated proper classes. Let $\mathcal{Q}$ be a class of objects of $\mathbb{A}$. For each object $P \in \mathcal{Q}$ we have a covariant left exact functor

$$\mathrm{Hom}_{\mathbb{A}}(P, -) : \mathbb{A} \to \mathcal{A}b.$$

Thus the class of short exact sequences

$$E : 0 \to A \to B \to C \to 0$$

in $\mathbb{A}$ such that

$$0 \to \mathrm{Hom}_{\mathbb{A}}(P, A) \to \mathrm{Hom}_{\mathbb{A}}(P, B) \to \mathrm{Hom}_{\mathbb{A}}(P, C) \to 0$$

is exact, is a proper class. The intersection of the proper classes so obtained, running over the objects $P \in \mathcal{Q}$, is called the proper class **projectively generated** by $\mathcal{Q}$ and denoted by $\pi^{-1}(\mathcal{Q})$. Equivalently, a short exact sequence E belongs to $\pi^{-1}(\mathcal{Q})$ iff every object of $\mathcal{Q}$ is projective with respect to E.

Short exact sequences in the category $\sigma[M]$ belonging to $\pi^{-1}(\mathcal{Q})$, the proper class projectively generated by a class $\mathcal{Q}$ in $\sigma[M]$, are called $\mathcal{Q}$-pure in [61] in analogy with pure exact sequences. In fact, proper classes are also called purities by some authors (see [58, 27, 47]).

4.2. Proposition. If $\mathcal{P}$ is a proper class projectively generated by a class $\mathcal{Q}$, then an object $J \in \mathbb{A}$ is $\mathcal{P}$-divisible iff $\mathrm{Ext}^1_{\mathbb{A}}(P, J) = 0$ for all $P \in \mathcal{Q}$.

Proof. Let $\mathcal{P} = \pi^{-1}(\mathcal{Q})$ be a projectively generated proper class. Suppose that J is $\mathcal{P}$-divisible. Thus for all $P \in \mathcal{Q}$

$$\mathrm{Ext}^1_{\mathbb{A}}(P, J) = \mathrm{Ext}^1_{\mathcal{P}}(P, J) = 0.$$

Conversely, suppose that J is such that $\mathrm{Ext}^1_{\mathbb{A}}(P, J) = 0$ for all $P \in \mathcal{Q}$. Let

$$E : 0 \to J \to B \to C \to 0$$

be a short exact sequence in $\mathbb{A}$ and $f : Q \to C$ any morphism with $Q \in \mathcal{Q}$. Form the pullback diagram

$$\begin{array}{ccccccccc} E' : 0 & \longrightarrow & J & \longrightarrow & B' & \longrightarrow & Q & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\ E : 0 & \longrightarrow & J & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

By assumption E' splits, since $Q \in \mathcal{Q}$. Then we find a morphism $Q \to B$ lifting f, i.e. $E \in \mathcal{P}$. This proves that J is $\mathcal{P}$-divisible. □

4.3. The proper class τ-Compl. Let τ be an idempotent preradical in $\sigma[M]$ and $\mathbb{T}_\tau$ its associated torsion class. We define the proper class

$$\tau\text{-Compl} = \pi^{-1}(\mathbb{T}_\tau).$$

This is the proper class projectively generated by $\mathbb{T}_\tau$.

4.4. τ-complements [12, 10.6]. Let τ be an idempotent preradical in $\sigma[M]$. For a submodule $L \subseteq N \in \sigma[M]$ the following are equivalent:

(a) Every $X \in \mathbb{T}_\tau$ is projective with respect to the projection $N \to N/L$.

(b) There exists a submodule $L' \subseteq N$ such that

  (i) $L \cap L' = 0$ and

  (ii) $(L + L')/L = \tau(N/L)$.

(c) There exists a submodule $L' \subseteq N$ such that

  (i) $L \cap L' = 0$ and

  (ii) $(L + L')/L \supseteq \tau(N/L)$.

If L satisfies this conditions it is called a **τ-complement** in N.

4.5. Enough $\mathcal{P}$-projectives. A proper class $\mathcal{P}$ is said to have **enough $\mathcal{P}$-projectives** if for every object $C \in \mathbb{A}$ there exist a $\mathcal{P}$-epimorphism

$$P \to C \quad \text{with} \quad P \in \text{Proj}(\mathcal{P}).$$

If $\mathcal{P}$ has enough $\mathcal{P}$-projectives, then $\mathcal{P}$ is projectively generated (see [55, Proposition 1.1]).

Proper classes with enough projectives are also called projective proper classes (e.g. [55]).

4.6. Injectively generated proper classes. Let $\mathcal{I}$ be any class of objects in $\mathbb{A}$. For each object $I \in \mathcal{I}$ we have a contravariant left exact functor

$$\text{Hom}_{\mathbb{A}}(-, I) : \mathbb{A} \to \mathcal{A}b.$$

Thus the class of short exact sequences

$$0 \to A \to B \to C \to 0$$

in $\mathbb{A}$ such that

$$0 \to \text{Hom}_{\mathbb{A}}(C, I) \to \text{Hom}_{\mathbb{A}}(B, I) \to \text{Hom}_{\mathbb{A}}(A, I) \to 0$$

is exact, is a proper class. Dual to 4.1, the intersection over the $I \in \mathcal{I}$ of all proper classes so obtained is called the proper class **injectively generated** by $\mathcal{I}$ and denoted by $\iota^{-1}(\mathcal{I})$. A short exact sequence E belongs to $\iota^{-1}(\mathcal{I})$ iff every object of $\mathcal{I}$ is injective with respect to E.

Short exact sequences in the category $\sigma[M]$ belonging to $\iota^{-1}(\mathcal{I})$, the proper class injectively generated by a class $\mathcal{I}$ in $\sigma[M]$, are called $\mathcal{I}$-copure in [61].

4.7. Proposition. If $\mathcal{P}$ is a proper class injectively generated by a class $\mathcal{I}$, then an object $Q \in \mathbb{A}$ is $\mathcal{P}$-flat iff $\text{Ext}^1_{\mathbb{A}}(Q, I) = 0$ for all $I \in \mathcal{I}$).

Proof. The proof is dual to 4.2. □

4.8. The proper class τ-Suppl. Let τ be a radical in $\sigma[M]$ and $\mathbb{F}_\tau$ its associated torsionfree class. We define the proper class

$$\tau\text{-Suppl} = \iota^{-1}(\mathbb{F}_\tau).$$

This is the proper class injectively generated by $\mathbb{F}_\tau$.

4.9. τ-supplements [12, 10.11]. Let τ be a radical in $\sigma[M]$. For a submodule $L \subseteq N \in \sigma[M]$ the following are equivalent:

(a) Every $X \in \mathbb{F}_\tau$ is injective with respect to the inclusion $L \to N$.

(b) There exists a submodule $L' \subseteq N$ such that

 (i) $L + L' = N$ and

 (ii) $L \cap L' = \tau(L)$.

(c) There exists a submodule $L' \subseteq N$ such that

 (i) $L + L' = N$ and

 (ii) $L \cap L' \subseteq \tau(L)$.

If L satisfies this conditions it is called a **τ-supplement** in N.

4.10. Enough $\mathcal{P}$-injectives. A proper class $\mathcal{P}$ is said to have **enough $\mathcal{P}$-injectives** if for every object $A \in \mathbb{A}$ there exist a $\mathcal{P}$-monomorphism:

$$A \to I \quad \text{with} \quad I \in \text{Inj}(\mathcal{P}).$$

If $\mathcal{P}$ has enough $\mathcal{P}$-injectives, then $\mathcal{P}$ is injectively generated.

Proper classes with enough injectives are also called injective proper classes (e.g. [55]).

4.11 Proposition. *The intersection of any family of injectively (projectively) generated proper classes is injectively (projectively) generated.*

Proof. Consider $\{\mathcal{P}_\lambda\}_\Lambda$ a family of injectively generated proper classes. i.e. $\mathcal{P}_\lambda = \iota^{-1}(\mathcal{I}_\lambda)$ with $\mathcal{I}_\lambda \subseteq \mathbb{A}$. We claim that $\mathcal{P} = \cap\mathcal{P}_\lambda = \iota^{-1}(\cup\mathcal{I}_\lambda)$. Since $\mathcal{I}_\lambda \subseteq \cup\mathcal{I}_\lambda$, then $\iota^{-1}(\cup\mathcal{I}_\lambda) \subseteq \iota^{-1}(\mathcal{I}_\lambda)$ for every $\lambda \in \Lambda$. Thus $\iota^{-1}(\cup\mathcal{I}_\lambda) \subseteq \mathcal{P}$. Conversely each object in $\cup\mathcal{I}_\lambda$ is contained in one of the $\mathcal{I}_\lambda$, thus is injective with respect to each short exact sequence in $\mathcal{P} \subseteq \mathcal{P}_\lambda$. Therefore $\mathcal{P} \subseteq \iota^{-1}(\cup\mathcal{I}_\lambda)$. The proof for the projectively generated case is analogous. □

We turn now our attention to the functor $X \otimes -$. For each R-module X, the tensor product defines a right exact functor $X \otimes - : \text{Mod-}R \to \mathcal{A}b$.

4.12 Remark. In the more general setting of abelian categories with a tensor product $\otimes$, i.e. a monidal category (see [44, §VII]), each object $X \in \mathbb{A}$ defines a functor $X \otimes - : \mathbb{A} \to \mathbb{A}$. If the functor $X \otimes -$ has a right adjoint, $\text{rhom}(X, -)$, then it is right exact. If for every $X \in \mathbb{A}$, $X \otimes -$ has a right adjoint, then $\mathcal{C}$ is called a **right-closed** category. A **left-closed** category is defined dually using the functor $- \otimes X$.

Let $\mathbb{A}$ be an abelian monoidal category [35].

4.13. Flatly generated proper classes. Let $\mathcal{F}$ be a class of objects in $\mathbb{A}$ such that the functor $\text{rhom}(F, -)$ is defined for each object $F \in \mathcal{F}$ [59]. Then we have a covariant right exact functor $F \otimes -$. The class of short exact sequences

$$0 \to A \to B \to C \to 0$$

in $\mathbb{A}$ such that the sequence

$$0 \to F \otimes A \to F \otimes B \to F \otimes C \to 0$$

is exact for all $F \in \mathcal{F}$ is proper. This class is called the proper class **flatly generated** by $\mathcal{F}$ and is denoted by $\tau^{-1}(\mathcal{F})$.

A fundamental concept (in both module theory and representation theory) is the proper class of pure exact sequences. It is based on the existence of a family of finitely presented generators of the category $\mathbb{A}$ and the fact that an object $X \in \mathbb{A}$ is finitely presented iff $\mathrm{Hom}_{\mathbb{A}}(X, -)$ commutes with direct limits. This is the case when $\mathbb{A}$ is a locally finitely presented Grothendieck category (see [33, 26]). We assume throughout this section that $\mathbb{A}$ is such a category.

4.14. The Cohn purity. Let $\mathrm{fp}\,\mathbb{A}$ be the class of all finitely presented objects of $\mathbb{A}$. The proper class $\pi^{-1}(\mathrm{fp}\,\mathbb{A})$ is called the **Cohn purity** and is denoted by $\mathcal{P}\mathrm{ure}$. An element of $\mathcal{P}\mathrm{ure}$ is called a **pure exact sequence**.

4.15. Proposition. *Every pure exact sequence is the direct limit of splitting sequences.*

Proof. Let $E : 0 \to A \to B \to C \to 0$ be pure and write $C = \varinjlim C_j$ with C_j finitely presented. For each canonical morphism $C_i \to \varinjlim C_j$ form the pullback diagram

$$\begin{array}{ccccccccc}
E_i : 0 & \longrightarrow & A & \longrightarrow & B_i & \longrightarrow & C_i & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
E : 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0.
\end{array}$$

The sequences E_i are pure and since C_i is finitely presented, they split. $\{E_j\}_I$ form a direct system whose direct limit is E. □

Thus the Cohn purity $\mathcal{P}\mathrm{ure}$ is minimal among all inductively closed proper classes.

4.16. Purity in R-Mod [61, 34.5]. *Let R be a ring and*

$$E : 0 \to A \to B \to C \to 0$$

a short exact sequence in R-Mod. *The following are equivalent:*

(a) *The sequence E is pure.*

(b) *The sequence*

$$0 \to F \otimes_R A \to F \otimes_R B \to F \otimes_R C \to 0$$

is exact for all finitely presented right (or for all) right R-modules F.

(c) *The sequence*

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(C, \overline{\mathbb{Q}}) \to \mathrm{Hom}_{\mathbb{Z}}(B, \overline{\mathbb{Q}}) \to \mathrm{Hom}_{\mathbb{Z}}(A, \overline{\mathbb{Q}}) \to 0$$

with $\overline{\mathbb{Q}} = \mathbb{Q}/\mathbb{Z}$

  (i) *remains exact after applying $- \otimes_R P$ with P finitely presented or*

  (ii) *it splits in* Mod-R.

(d) *Every finite system of equations over A which is solvable in B is solvable in A.*

(e) *For every commutative diagram*

$$\begin{array}{ccccc} & & R^n & \xrightarrow{g} & R^k \\ & & {\scriptstyle f}\downarrow & & \downarrow \\ 0 & \longrightarrow & A & \longrightarrow & B \end{array}$$

with $n, k \in \mathbb{N}$, there exist a morphism $h : R^k \to A$ such that $f = hg$.

(f) *The sequence E is the direct limit of splitting sequences.*

4.17. The proper class $\mathcal{P}_\tau$. Let τ be a hereditary preradical in $\sigma[M]$. The class of all short exact sequences

$$E : 0 \to A \to B \to C \to 0$$

such that

$$\tau(E) : 0 \to \tau(A) \to \tau(B) \to \tau(C) \to 0$$

is exact, is a proper class and is denoted by $\mathcal{P}_\tau$. $\mathcal{P}_\tau$ is a proper class by the construction at the beginning of §4.

4.18. The proper class $\mathcal{P}_{1/\rho}$. Let ρ be a cohereditary preradical in $\sigma[M]$. The class of all short exact sequences

$$E : 0 \to A \to B \to C \to 0$$

such that

$$1/\rho(E) : 0 \to A/\rho(A) \to B/\rho(B) \to C/\rho(C) \to 0$$

is exact, is a proper class and is denoted by $\mathcal{P}_{1/\rho}$. Since ρ is cohereditary, $1/\rho$ is a right exact endofunctor (see 2.7). $\mathcal{P}_{1/\rho}$ is a proper class by the construction at the beginning of §4.

Chapter 3

Proper classes related to supplements and complements

Supplement and complement submodules induce proper classes. This was noted for abelian groups in [32] and for R-Mod in [57, 27, 28]. Recently this investigations were continued in [46, 1].

5 Supplements and complements

In this section we recall the definition of small, essential, closed, coclosed, complement and supplement submodules. The basic properties of small and essential submodules are listed for example in [12, 2.2] and [61, 17.3]. They will follow from a more general concept of small and essential submodules investigated in the next section.

5.1. Small submodules. A submodule $K \subseteq N$ is called **small** in N if for every submodule $X \subseteq N$, the equality $K + X = N$ implies $X = N$. We denote this by $K \ll N$.

5.2. Small epimorphisms. An epimorphism $f : N \to N'$ is called a **small epimorphism** if $\operatorname{Ker} f \ll N$.

5.3. Supplements. A submodule $L \subseteq N$ is called a **supplement** in N if there exists a submodule $L' \subseteq N$ such that

(i) $L + L' = N$ and

(ii) $L \cap L' \ll L$.

Equivalently, L is minimal in the set of submodules $\{K \subseteq N \mid L' + K = N\}$.

5.4. Essential submodules. A submodule $K \subseteq M$ is called **essential** in N if for every submodule $X \subseteq N$, the equality $K \cap X = 0$ implies $X = 0$. We denote this by $K \subseteq_e N$.

5.5. Complements. A submodule $L \subseteq N$ is called a **complement** in N if there exists a submodule $L' \subseteq N$ such that

(i) $L \cap L' = 0$ and

(ii) $(L \oplus L')/L \subseteq_e N/L$.

Equivalently, L is maximal in the set of submodules $\{K \subseteq N \mid L' \cap K = 0\}$. Note that 5.5 is not the common definition of complement but it is of course equivalent to it. We prefer this definition because it shows the duality with the supplements (see 5.3). By Zorn's lemma every module has a complement.

5.6. Closed submodules. A submodule $K \subseteq N$ is called **closed** in N if K has no proper essential extension in N, i.e. if $K \subseteq_e L \subseteq N$, then $K = L$.

It is well-known that closed submodules and complement submodules coincide (see [12, 1.10]).

5.7. Coclosed submodules. A submodule $K \subseteq N$ is called **coclosed** in N if for every submodule $X \subseteq K$, $K/X \ll N/X$ implies $X = K$. We denote this property by $K \subseteq_{cc} N$.

Recall the following lemma from [67].

5.8. Properties of coclosed submodules [67, Lemma A.4]. *Let $U \subseteq L \subseteq N$ be submodules of N.*

(i) *If U is coclosed in N, then U is coclosed in L.*

(ii) *If L is coclosed in N, then L/U is coclosed in N/U.*

(iii) *If U is coclosed in L and L is coclosed in N, then U is coclosed in N.*

(iv) *If U is coclosed in N and L/U is coclosed in N/U, then L is coclosed in N.*

5.9. The proper class Cocls. The class of all short exact sequences in $\sigma[M]$

$$0 \to A \xrightarrow{f} B \to C \to 0$$

such that $\operatorname{Im} f$ is coclosed in B is a proper class and it is denoted by Cocls.

Proof. The axioms (P1) and (P2) are easily verified. (P3)-(P6) follow from 5.8. □

Using the characterization of Bowe in [7, Theorem 1.2], Zöschinger dualizes E-neat monomorphisms in [65, Definition pp.307] and he calls them coneat epimorphisms. (We write Z-coneat for this). An epimorphism $g : B \to C$ is **Z-coneat** if for every factorization $g = \beta\alpha$ with β an small epimorphism ($\operatorname{Ker} \beta$ is a small submodule), then β is an isomorphism. Then he proves that.

5.10. Cocls-epimorphisms. [65, Hilfssatz 2.2(a)] *An epimorphism $g : B \to C$ is* Z-coneat *iff* $\operatorname{Ker} f$ *is coclosed in B, i.e. g is a* Cocls-*epimorphism.*

The following is the dual statement of 13.15.

5.11. Cocls-flats. *Let C be a module in $\sigma[M]$ and suppose that C has a projective cover $f : P \to C$. The following are equivalent:*

(a) *C is* Cocls-*flat.*

(b) *Every epimorphism $Q \to C$ with Q projective is a* Cocls-*epimorphism.*

(c) *The projective cover f of C is a* Cocls-*epimorphism.*

(d) *C is projective.*

Proof. (a) ⇒ (b), (b) ⇒ (c) and (d) ⇒ (a) are clear.

(c) ⇒ (d) Since f is a projective cover, then $\operatorname{Ker} f \ll P$. By assumption $\operatorname{Ker} f \subseteq_{cc} P$, thus $\operatorname{Ker} f = 0$, i.e. $P \simeq C$. Therefore C is projective. □

Recall that a module N is called fully non-M-small in $\sigma[M]$ if $N = \operatorname{rad}_{\mathscr{M}}(N)$ with $\mathscr{M}$ the class of M-small modules. From the characterization in [12, 8.11] we obtain.

5.12. Cocls-divisibles. *If the module N in $\sigma[M]$ is fully non-M-small, then N is* Cocls-*divisible.*

Proof. Let
$$0 \to N \xrightarrow{f} B \to C \to 0$$
be any short exact sequence. By [12, 8.11], $f(N)$ is coclosed in B, i.e. N is Cocls-divisible. □

6 τ-supplements

We have seen that the torsionfree class $\mathbb{F}_\tau$ associated to a radical τ in $\sigma[M]$ induces a proper class τ-Suppl (see 4.8). In this section we investigate this proper class. The following proposition is a generalization of [46, 3.4.4] where $\tau = \text{Rad}$ in R-Mod.

6.1. τ-Suppl-injectives. *A module $I \in \sigma[M]$ is* τ-Suppl-*injective iff I is a direct summand of a module of the form $E \oplus F$ with E M-injective and $\tau(F) = 0$.*

Proof. $\Rightarrow$) Consider the morphism
$$f : I \to \widehat{I} \oplus I/\tau(I), \qquad x \mapsto (x, x + \tau(I)).$$
With $\widehat{I}$ the injective hull of I in $\sigma[M]$. f is a monomorphism. For if $0 = f(x) = (x, x + \tau(I))$, then $x = 0$. Moreover f is a τ-Suppl-monomorphism. To see this let Y be a module such that $\tau(Y) = 0$ and $g : I \to Y$ any morphism. We show that g can be extended to $\widehat{I} \oplus I/\tau(I)$ making the diagram
$$\begin{array}{ccccc} 0 & \longrightarrow & I & \xrightarrow{f} & \widehat{I} \oplus I/\tau(I) \\ & & {\scriptstyle g}\downarrow & \swarrow {\scriptstyle \bar{g}} & \\ & & Y & & \end{array}$$
commutative. Define the morphism $\bar{g} : \widehat{I} \oplus I/\tau(I) \to Y$ by $\bar{g}(x, y + \tau(I)) = g(y)$. $\bar{g}$ is well-defined since if $(x, y + \tau(I)) = (x', y' + \tau(I))$, then $x = x'$ and $y - y' \in \tau(I)$. Since τ is a preradical, then $g(\tau(I)) \subseteq \tau(Y) = 0$. Therefore $g(y) = g(y')$. Clearly $\bar{g}f = g$. By assumption I is τ-Suppl-injective, hence $f : I \to \widehat{I} \oplus I/\tau(I)$ splits. Note that $\widehat{I}$ is M-injective and $\tau(I/\tau(I)) = 0$.
$\Leftarrow$) Since $E \oplus F$ is τ-Suppl-injective, also every direct summand is so. □

In [22] Rad-Suppl-divisibles in the category R-Mod are called **absolutely supplemented**. The following is a generalization of [22, 3.1.4] where $\tau = \text{Rad}$ in R-Mod.

6.2. τ-Suppl-divisibles. *Let N be a module in $\sigma[M]$. The following conditions are equivalent:*

(a) *N is* τ-Suppl-*divisible.*

(b) *N is a τ-supplement in every M-injective module I containing N.*

(c) *N is a τ-supplement in its M-injective hull $\widehat{N}$.*

Proof. (a) $\Rightarrow$ (b) N τ-Suppl-divisible means that every short exact sequence beginning with N belongs to τ-Suppl, thus (b) is clear.

(b) $\Rightarrow$ (c) Is clear.

(c) $\Rightarrow$ (a) Let $i : N \hookrightarrow H$ and $j : N \hookrightarrow \widehat{N}$ be the canonical inclusions. Since $\widehat{N}$ is M-injective, the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{\;i\;} & H \\ & & {\scriptstyle j}\big\downarrow & \swarrow {\scriptstyle \overline{j}} & \\ & & \widehat{N} & & \end{array}$$

can be completed commutatively by $\overline{j} : H \to \widehat{N}$. By assumption, there exists a submodule $N' \subseteq \widehat{N}$ such that $N + N' = \widehat{N}$ and $N \cap N' = \tau(N)$. We claim that N is a τ-supplement of $\overline{j}^{-1}(N')$. Let $h \in H$, then $\overline{j}(h) = n + n' = \overline{j}(n) + n'$ with $n \in N$ and $n' \in N'$. Therefore $h - n \in \overline{j}^{-1}(N')$. Since $h = (h - n) + n \in \overline{j}^{-1}(N') + N$, we obtain $\overline{j}^{-1}(N') + N = H$. Note that $N \cap \overline{j}^{-1}(N') \subseteq N \cap N' = \tau(N)$. This shows (a). □

6.3. Proposition. *Every τ-torsion module in $\sigma[M]$ is τ-*Suppl*-divisible.*

Proof. Let N be a τ-torsion module. Consider any short exact sequence

$$E : 0 \to N \to B \to C \to 0$$

and any morphism $f : N \to F$ with $F \in \mathbb{F}_\tau$. Then $f = 0$. Thus we can extend f trivially to B. Thus every τ-torsionfree module is injective with respect to E, i.e. $E \in \tau$-Suppl and therefore N is τ-Suppl-divisible. □

6.4. Lemma. *Let τ be a preradical in R-Mod. Then for every projective module P we have $\tau(P) = \tau(R)P$.*

Proof. Consider the epimorphism $f : R^{(P)} \to P$ given by $f((r_p)p) = \sum_p rp$. This is well-defined since the sums are finite. Note that $f(\tau(R^{(P)})) = f(\tau(R)^{(P)}) = \tau(R)P$. Since τ is a preradical, then $\tau(R)P \subseteq \tau(P)$. From the projectivity of P follows that there exists $g : P \to R^{(P)}$ such that $\mathrm{id}_P = fg$. Therefore $\tau(P) = fg(\tau(P)) \subseteq f(\tau(R^{(P)})) \subseteq \tau(R)P$. □

The following proposition is a generalization of [46, 3.7.2] where $\tau = \mathrm{Rad}$ in R-Mod.

6.5. τ-Suppl-flats. *Let τ be a radical in R-Mod. If $\tau(R) = 0$, then τ-*Suppl*-flat modules and projective modules coincide.*

Proof. Let N be a τ-Suppl-flat module. There is an epimorphism $f : F \to N$ from a free module F. Thus F is projective. Consider the following short exact sequence

$$E : 0 \to \mathrm{Ker}\, f \to F \xrightarrow{f} N \to 0.$$

By assumption $E \in \tau$-Suppl. Note that $\tau(\mathrm{Ker}\, f) \subseteq \tau(F)$. By 6.4 we obtain $\tau(\mathrm{Ker}\, f) \subseteq \tau(F) = \tau(R)F = 0F = 0$. Thus $\mathrm{Ker}\, f$ is τ-Suppl-injective. Therefore the sequence E splits which implies that $F \simeq \mathrm{Ker}\, f \oplus N$, i.e. N is projective. □

Recall the following definition from [1].

6.6. τ-supplemented modules [1, 2.1]. A module L in $\sigma[M]$ is called **τ-supplemented** if every submodule of L has a τ-supplement.

6.7. Properties of τ-supplemented modules [1, 2.2]. *Let L be a τ-supplemented module in $\sigma[M]$.*

(i) *Every submodule $K \subseteq L$ with $K \cap \tau(L) = 0$ is a direct summand. In particular, if $\tau(L) = 0$, then L is semisimple.*

(ii) *Every factor module and every direct summand of L is τ-supplemented.*

(iii) *$L/\tau(L)$ is a semisimple module.*

(iv) *$L = U \oplus N$ where N is semisimple and $\tau(U) \subseteq_e U$.*

6.8. τ-covers [1, 2.11]. An epimorphism $f : P \to L$ is called a **τ-cover** provided $\operatorname{Ker} f \subseteq \tau(P)$. If P is projective in $\sigma[M]$, then f is called a **projective τ-cover**.

6.9. Properties of τ-covers [1, 2.13].

(i) *If $f : P \to L$ is a projective τ-cover and $g : L \to N$ is a τ-cover, then $gf : P \to N$ is a projective τ-cover.*

(ii) *If each $f_i : P_i \to L_i$, $i \in I$ is a (projective) τ-cover, then the map $\oplus_I f_i : \oplus_I P_i \to \oplus_I L_i$ is a (projective) τ-cover.*

6.10. τ-supplemented modules and projective τ-covers [1, 2.14]. *Let $U \subseteq L \in \sigma[M]$. The following are equivalent:*

(a) *L/U has a projective τ-cover.*

(b) *U has a τ-supplement V which has a projective τ-cover.*

(c) *If $V \subseteq L$ and $L = U + V$, then U has a τ-supplement $V' \subseteq V$ such that V' has a projective τ-cover.*

6.11. τ-semiperfect modules. [1, 2.15]. A module $L \in \sigma[M]$ is called **τ-semiperfect** (**τ-perfect**) if every factor module of L (any direct sum of copies of L) has a projective τ-cover.

6.12. Characterization of τ-semiperfect modules [1, 2.16]. *Let L be a module in $\sigma[M]$. The following are equivalent:*

(a) *L is τ-semiperfect.*

(b) *L is τ-supplemented by supplements which have projective τ-covers.*

(c) *L is amply supplemented by supplements which have projective τ-covers.*

6.13. Proposition. *Let τ be a hereditary preradical (= left exact) in $\sigma[M]$ and $L \subseteq N$. If $L \subseteq \tau(N)$, then L is a τ-supplement in N.*

Proof. Note that $L + N = N$ and $L \cap N = L \subseteq \tau(N) \cap L = \tau(L)$. Thus L is a τ-supplement in N. □

7 τ-complements

The torsion class $\mathbb{T}_\tau$ associated to an idempotent preradical τ in $\sigma[M]$ induces a proper class τ-Compl (see 4.3). In this section we investigate this proper class.

7.1. τ-Compl-projectives. *Suppose that $\sigma[M]$ has enough projectives. A module $N \in \sigma[M]$ is* τ-Compl-*projective iff N is a direct summand of a module of the form $P \oplus Q$ with P projective and $\tau(Q) = Q$.*

Proof. $\Rightarrow$) Let N be τ-Compl-projective. By hypothesis there is an epimorphism $\pi : P \to N$. Consider the morphism

$$f : P \oplus \tau(N) \to N, \qquad f(p, n) = \pi(p) + n.$$

Observe that f is an epimorphism. Let Q be a module in $\mathbb{T}_\tau$, i.e. $\tau(Q) = Q$ and $g : Q \to N$ any morphism. Since τ is a preradical, $g(\tau(Q)) \subseteq \tau(N)$. Define $\bar{g} : Q \to P \oplus \tau(N)$ by $\bar{g}(q) = (0, q)$. We have $g(q) = f\bar{g}(q)$. This shows that the sequence

$$0 \to \operatorname{Ker} f \to P \oplus \tau(N) \xrightarrow{f} N \to 0$$

belongs to τ-Compl. Since N is τ-Compl-projective, the sequence splits, i.e. N is a direct summand of $P \oplus \tau(N)$.

$\Leftarrow$) Any module of the form $P \oplus Q$ with P projective and $\tau(Q) = Q$ is τ-Compl-projective, thus also every direct summand is so. $\square$

7.2. τ-Compl-divisibles. *If every M-injective module in $\sigma[M]$ is τ-torsion, then every* τ-Compl-*divisible module is M-injective.*

Proof. If N is τ-Compl-divisible, then the short exact sequence

$$0 \to N \hookrightarrow \widehat{N} \to \widehat{N}/N \to 0$$

belongs to τ-Compl. Since $\widehat{N}$ is τ-torsion and the τ-torsion modules are τ-Compl-projective, the sequence splits, i.e. N is a direct summand of an M-injective module, thus N is also M-injective. $\square$

7.3. Proposition. *Every τ-torsionfree module in $\sigma[M]$ is* τ-Compl-*flat.*

Proof. Let N be τ-torsionfree. Consider any short exact sequence

$$E : 0 \to A \to B \to N \to 0$$

and any morphism $f : T \to N$ with $T \in \mathbb{T}_\tau$. Then $f = 0$. Thus f can be trivially lifted to B, i.e. every τ-torsion module is projective with respect to E. Therefore $E \in \tau$-Compl. This means that N is τ-Compl-flat. $\square$

For $\tau = \operatorname{Soc}$, this implies that every module N with $\operatorname{Soc}(N) = 0$ is Neat-flat.

Chapter 4

Proper classes related to $\mathcal{C}$-supplements

In [64] Zhou introduces the concept of δ-small submodules generalizing small submodules. He developes a theory of generalized perfect, semiperfect and semiregular rings. A submodule $K \subseteq N$ is called **δ-small** in N if for every submodule $X \subseteq N$, the equality $K + X = N$ and N/X a singular module implies $X = N$. He denote this by $K \ll_\delta N$. Extending this idea we introduce the concept of $\mathcal{C}$-small submodules changing the class of singular modules for a class of modules closed under submodules and factor modules. Such classes of modules are called open classes in [15] and $\{q, s\}$-closed classes in [31].

Throughout this chapter let $\mathcal{C}$ be a $\{q, s\}$-closed class of modules.

For example, in $\sigma[M]$, the class $\mathbb{T}_\tau$ for a hereditary preradical τ, or the class $\mathbb{F}_\rho$ for a cohereditary preradical ρ are $\{q, s\}$-closed. In R-Mod, for any R-module the classes $\sigma[M]$ and $\sigma_f[M]$ are $\{q, s\}$-closed. Moreover for any class $\mathcal{X}$ in $\sigma[M]$ closed under isomorphisms and containing the zero module, the class

$$\bar{\mathcal{X}} = \{X \in \mathcal{X} \mid K \subseteq H \subseteq X \text{ and } H/K \in \mathcal{X} \text{ implies } K = H\}$$

is $\{q, s\}$-closed (see [31]).

8 $\mathcal{C}$-small submodules

8.1. $\mathcal{C}$-small submodules. A submodule $K \subseteq N$ is called **$\mathcal{C}$-small** in N if for every submodule $X \subseteq N$, the equality $K + X = N$ and $N/X \in \mathcal{C}$ implies $X = N$. We denote this by $K \ll_{\mathcal{C}} N$.

For any $\{q, s\}$-closed class $\mathcal{C} \subseteq \sigma[M]$ every small submodule in $\sigma[M]$ is $\mathcal{C}$-small. If $\mathcal{C} = \sigma[M]$, then the $\sigma[M]$-small submodules are the small submodules in $\sigma[M]$. For any $\{q, s\}$-closed class $\mathcal{C} \subseteq \sigma[M]$ if $N \in \mathcal{C}$, then a submodule $K \subseteq N$ is $\mathcal{C}$-small in N iff K is small in N.

8.2. Examples of $\mathcal{C}$-small submodules.

(i) If $\mathcal{C} = \mathscr{S}$, the class of M-singular modules in $\sigma[M]$, then the $\mathscr{S}$-small submodules are the δ-small submodules in $\sigma[M]$ as defined by Zhou in [64]. He proves that for a submodule $K \subseteq N \in \sigma[M]$ the following conditions are equivalent (see [64, 1.2]):

(a) $K \ll_\delta N$.

(b) If $X + K = N$, then $X \oplus Y = N$ for a projective semisimple submodule $Y \subseteq N$ with $Y \subseteq K$.

(c) If $X + K = N$ and N/X is Goldie torsion, then $X = K$.

(ii) If $\mathcal{C} = \mathcal{S}(\sigma[M])$, the class of simple modules in $\sigma[M]$, then a submodule $K \subseteq N \in \sigma[M]$ is $\mathcal{S}(\sigma[M])$-small iff $K \subseteq \mathrm{Rad}(N)$.

Proof. Let K be a $\mathcal{S}(\sigma[M])$-small submodule of N and $X \subseteq N$ any maximal submodule of N. Suppose $K \nsubseteq X$, then $X + K = N$ with N/X a simple module. Thus, by hypothesis $X = N$, which is a contradiction. Therefore $K \subseteq X$, i.e. $K \subseteq \mathrm{Rad}(N)$.
Conversely, suppose that $K \subseteq \mathrm{Rad}(N)$. For any submodule $X \subseteq N$ such that N/X is simple, X is a maximal submodule of N. Therefore $K \subseteq \mathrm{Rad}(N) \subseteq X$. This implies $X + K = X \neq N$, i.e. K is a $\mathcal{S}(\sigma[M])$-small submodule of N. □

(iii) Let $\mathcal{C}$ be either the class of noetherian or artinian modules in $\sigma[M]$. Since every simple module is noetherian and artinian, a noetherian-small or an artinian-small submodule is $\mathcal{S}(\sigma[M])$-small (see (ii)).

(iv) If $\mathcal{C} = \mathbb{F}_\tau$, with τ a cohereditary preradical in $\sigma[M]$, then a submodule $K \subseteq N \in \sigma[M]$ is $\mathbb{F}_\tau$-small iff for every submodule $X \subseteq N$ such that $X + K = N$ we have $X + \tau(N) = N$.

Proof. Let K be an $\mathbb{F}_\tau$-small submodule of N and $X \subseteq N$ a submodule of N such that $X + K = N$. Since τ is a radical and $\mathbb{F}_\tau$ is closed under factor modules (see 2.7), the factor module

$$N/(X + \tau(N)) \simeq N/\tau(N) \Big/ (X + \tau(N))/\tau(N) \in \mathbb{F}_\tau.$$

On the other hand $N = X + \tau(N) + K$. Since K is $\mathbb{F}_\tau$-small, we must have $N = X + \tau(N)$.
Conversely, let $X \subseteq N$ such that $X + K = N$ and $N/X \in \mathbb{F}_\tau$. Thus by 2.4 (ii), we have $\tau(N) \subseteq X$. By hypothesis $N = X + \tau(N) = X$, i.e. K is an $\mathbb{F}_\tau$-small submodule of N. □

8.3. Properties of $\mathcal{C}$-small submodules. *Let K, L and N be modules in $\sigma[M]$.*

(i) *If $K \subseteq L \subseteq N$, then $L \ll_\mathcal{C} N$ iff $K \ll_\mathcal{C} N$ and $L/K \ll_\mathcal{C} N/K$.*

(ii) *If $K_1, K_2 \subseteq N$, then $K_1 + K_2 \ll_\mathcal{C} N$ iff $K_1 \ll_\mathcal{C}$ and $K_2 \ll_\mathcal{C} N$.*

(iii) *If $K_1, ..., K_n$ are $\mathcal{C}$-small submodules of N, then $K_1 + \cdots + K_n$ is also $\mathcal{C}$-small in N.*

(iv) *If $K \ll_\mathcal{C} N$ and $f : N \to N'$, then $f(K) \ll_\mathcal{C} N'$.*

(v) *If $K \subseteq L \subseteq N$ and L is a direct summand in N, then $K \ll_\mathcal{C} N$ iff $K \ll_\mathcal{C} L$.*

Proof. (i) $\Rightarrow$) Suppose that $K + X = N$ and $N/X \in \mathcal{C}$. Then $L + X = N$. Thus, by assumption $X = N$, i.e. $K \ll_\mathcal{C} N$. Suppose now that $L/K + Y/K = N/K$ and $N/Y \in \mathcal{C}$. Then $L + Y = N$. By assumption $Y = N$. Thus $Y/K = N/K$, i.e $L/K \ll_\mathcal{C} N/K$.
$\Leftarrow$) Suppose that $L + X = N$ and $N/X \in \mathcal{C}$. Then $L/K + (X + K)/K = N/K$. Since $\mathcal{C}$ is closed under factor modules, we have that

$$(N/X)/((X + K)/X) \simeq N/(X + K) \in \mathcal{C}.$$

Then by hypothesis $(X + K)/K = N/K$. Therefore $X + K = N$. Now $N/X \in \mathcal{C}$ implies $X = N$. Thus $L \ll_\mathcal{C} N$.

(ii) $\Rightarrow$) Suppose that $K_1 + X = N$ and $N/X \in \mathcal{C}$. Then $K_1 + X + K_2 = N$. By hypothesis we must have $X = N$, i.e. $K \ll_{\mathcal{C}} N$. Analogous $K_2 \ll_{\mathcal{C}} N$.
$\Leftarrow$) Let now $K_1 + K_2 + Y = N$ and $N/Y \in \mathcal{C}$. Then $N/(Y + K_2) \in \mathcal{C}$ since is a quotient of N/Y. Then $K_1 + Y = N$. Since K_1 is also a $\mathcal{C}$-small submodule of N, we have $Y = N$, i.e. $K_1 + K_2 \ll_{\mathcal{C}} N$.

(iii) Suppose that $(K_1 + \cdots + K_n) + X = N$ and $N/X \in \mathcal{C}$. The module $N/(K_2 + \cdots + K_n) + X$ is a factor module of N/X, thus $N/(K_2 + \cdots + K_n) + X \in \mathcal{C}$. since K_1 is $\mathcal{C}$-small, we have $(K_2 + \cdots + K_n) + X = N$. Continuing this way we obtain $X = N$, i.e. $K_1 + \cdots + K_n \ll_{\mathcal{C}} N$.

(iv) Suppose that $f(K) + X = N'$ and $N'/X \in \mathcal{C}$. Then $N = K + f^{-1}(X)$. Consider the morphism

$$\bar{f} : N/f^{-1}(X) \to N'/X, \qquad n + f^{-1}(X) \mapsto f(n) + X.$$

Note that $\bar{f}$ is well defined and is a monomorphism. Therefore $N/f^{-1}(X)$ is isomorphic to a submodule of N'/X which is in $\mathcal{C}$, thus $N/f^{-1}(X)$ is also in $\mathcal{C}$. Since K is a $\mathcal{C}$-small submodule of N, we must have $f^{-1}(X) = N$ and therefore $K \subseteq f^{-1}(X)$. Then $f(K) \subseteq X$ and $X = N'$ i.e $f(K) \ll_{\mathcal{C}} N'$.

(v) $\Rightarrow$) Let $H \oplus L = N$. Suppose $K + X = L$ and $L/X \in \mathcal{C}$. Then $N = H + L = H + K + X$. Note that $N/(H+X) \simeq K/K \cap (H+K)$ and $L/X \simeq K/X \cap K$. Thus $N/(H+X)$ is isomorphic to a factor module of L/X. This implies that $N/(H + X) \in \mathcal{C}$. By assumption $K \ll_{\mathcal{C}} N$, therefore $N = H + X$. Moreover $H \cap X \subseteq H \cap L = 0$, i.e. $N = H \oplus X$. Finally

$$L = N \cap L = (H \oplus X) \cap L = (H \cap L) \oplus X = X.$$

Which means $K \ll_{\mathcal{C}} L$.
$\Leftarrow$) This is always true without assuming L being a direct summand and follows from (iv) with $f = i : L \hookrightarrow N$ the inclusion morphism. □

8.4. $\mathcal{C}$-small epimorphisms. An epimorphism $f : P \to N$ is called a **$\mathcal{C}$-small epimorphism** if $\operatorname{Ker} f \ll_{\mathcal{C}} P$.

8.5. Proposition. *An epimorphism $f : P \to N$ is $\mathcal{C}$-small iff every (mono) morphism $g : L \to P$ with fg epimorphism and* $\operatorname{Coker} g \in \mathcal{C}$ *is an epimorphism.*

Proof. $\Rightarrow$) Let $X \subseteq P$ be such that $X + \ker f = P$ and $P/X \in \mathcal{C}$. Consider $i : X \hookrightarrow P$ the canonical inclusion. Then fi is an epimorphism with $\operatorname{Coker} i \in \mathcal{C}$ and by assumption, i is an epimorphism, i.e. $X = P$.
$\Leftarrow$) Let $p \in P$. There exists $l \in L$ with $f(g(l)) = f(p)$. Therefore $p = g(l) + (p - g(l)) \in \operatorname{Im} g + \ker f$. Thus $P = \operatorname{Im} g + \ker f$ and $\operatorname{Coker} g \in \mathcal{C}$. Since $\ker f \ll_{\mathcal{C}} P$, $\operatorname{Im} g = P$, i.e. g is an epimorphism. □

8.6. Properties of $\mathcal{C}$-small epimorphisms. *Let N,L and P be modules in $\sigma[M]$.*

(i) *If $f : P \to N$ and $g : N \to L$ are $\mathcal{C}$-small epimorphisms, then gf is a $\mathcal{C}$-small epimorphism.*

(ii) *If each $f_i : P_i \to N_i$, $i = 1, 2, ..., n$ is a $\mathcal{C}$-small epimorphism, then*

$$f_1 \oplus \cdots \oplus f_n : P_1 \oplus \cdots \oplus P_n \to N_1 \oplus \cdots \oplus N_n$$

is a $\mathcal{C}$-small epimorphism.

Proof. (i) We show that $\operatorname{Ker} gf \ll_{\mathcal{C}} P$. Observe that $\operatorname{Ker} f \subseteq \operatorname{Ker} gf \subseteq P$. Since $\operatorname{Ker} f \ll_{\mathcal{C}} P$, then in view of 8.3 (i), it is enough to show that

$$\operatorname{Ker} gf/\operatorname{Ker} f \ll_{\mathcal{C}} P/\operatorname{Ker} f.$$

Let $X \subseteq P$ such that $\operatorname{Ker} gf/\operatorname{Ker} f + X/\operatorname{Ker} f = P/\operatorname{Ker} f$ and $P/X \in \mathcal{C}$. Then $\operatorname{Ker} gf + X = P$. Since f is an epimorphism,

$$N = f(P) = f(\operatorname{Ker} gf) + f(X) = \operatorname{Ker} g + f(X).$$

On the other hand, $N \simeq P/\operatorname{Ker} f$ and $f(X) \simeq X/\operatorname{Ker} f \cap X = X/\operatorname{Ker} f$. Thus $N/f(X) \simeq P/\operatorname{Ker} f/X/\operatorname{Ker} f \simeq P/X$ which is in $\mathcal{C}$. Since $\operatorname{Ker} g \ll_{\mathcal{C}} N$, $f(X) = N$. It follows that $X = P$.

(ii) It follows from 8.3 (iii). □

8.7. Projective $\mathcal{C}$-covers. An epimorphism $f : P \to N$ is called a **projective $\mathcal{C}$-cover** of N if P is projective in $\sigma[M]$ and f is a $\mathcal{C}$-small epimorphism.

8.8. Properties of projective $\mathcal{C}$-covers. *Let M be a module. If $f_i : P_i \to N_i$, $i = 1, ..., n$ are projective $\mathcal{C}$-covers, then*

$$f_1 \oplus \cdots \oplus f_n : P_1 \oplus \cdots \oplus P_n \to N_1 \oplus \cdots \oplus N_n$$

is a projective $\mathcal{C}$-cover.

Proof. It follows from 8.6 (ii). □

9 $\mathcal{C}$-supplements

9.1. $\mathcal{C}$-supplements. A submodule $L \subseteq N$ is called a **$\mathcal{C}$-supplement** in N if there exists a submodule $L' \subseteq N$ such that

(i) $L + L' = N$ and

(ii) $L \cap L' \ll_{\mathcal{C}} L$.

Observe that if $\mathcal{C} = \sigma[M]$, then the $\mathcal{C}$-supplement submodules are precisely the supplement submodules and if $\mathcal{C}$ is the class of singular modules in R-Mod, then the $\mathcal{C}$-supplement submodules coincide with the δ-supplement submodules (see [64]).

9.2. The proper class $\mathcal{C}$-Suppl. The class of all short exact sequences in $\sigma[M]$

$$0 \to A \xrightarrow{f} B \to C \to 0,$$

such that $\operatorname{Im} f$ is a $\mathcal{C}$-supplement in B, is a proper class and it is denoted by $\mathcal{C}$-Suppl.

Proof. We prove that the axioms (P1)-(P6) of 3.1 are satisfied. To simplify the proof we assume that the short exact sequences are of the form

$$0 \to A \xrightarrow{i} B \xrightarrow{p} B/A \to 0$$

with i and p the canonical inclusion and projection respectively.

(P1) Let $E \in \mathcal{C}$-Suppl and $E' \simeq E$. There is a commutative diagram

$$\begin{array}{ccccccccc} E : 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & B/A & \longrightarrow & 0 \\ & & {\scriptstyle f_1}\big\downarrow & & {\scriptstyle f_2}\big\downarrow & & {\scriptstyle f_3}\big\downarrow & & \\ E' : 0 & \longrightarrow & C & \longrightarrow & D & \longrightarrow & D/C & \longrightarrow & 0 \end{array}$$

with f_i isomorphisms. Since $E \in \mathcal{C}$-Suppl, there exists $K' \subseteq N$ such that

$$A + A' = B \text{ and } A \cap A' \ll_{\mathcal{C}} A.$$

Then

$$D = f_2(B) = f_2(A + A') = f_2(A) + f_2(A') = f_1(A) + f_2(A') = C + f_2(A').$$

Set $f_2(A') = C'$. We have $f_1(A \cap A') = C \cap C' \subseteq C$. From 8.3 (iv) follows that $C \cap C' \ll_{\mathcal{C}} C$. Hence C is a $\mathcal{C}$-supplement in D, i.e. $E' \in \mathcal{C}$-Suppl.

(P2) Let E be a splitting short exact sequence. E is of the form

$$E : 0 \to A \to A \oplus A' \to A' \to 0.$$

Clearly A is a $\mathcal{C}$-supplement in $A \oplus A'$, i.e. $E \in \mathcal{C}$-Suppl.

(P3) Let f and g be $\mathcal{C}$-Suppl-monomorphisms. As agreed f and g are the inclusions in the diagram

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \big\downarrow & & & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & B/A & \longrightarrow & 0 \\ & & & & {\scriptstyle g}\big\downarrow & & & & \\ & & & & D & & & & \\ & & & & \big\downarrow & & & & \\ & & & & D/B & & & & \\ & & & & \big\downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

There exists submodules $A' \subseteq B$ and $B' \subseteq D$ such that

$$A + A' = B,\ A \cap A' \ll_{\mathcal{C}} A \text{ and } B + B' = D,\ B \cap B' \ll_{\mathcal{C}} B.$$

We prove that A is a $\mathcal{C}$-supplement of $A' + B'$ in D. Note that $D = A + (A' + B')$. Suppose that $X \subseteq A$ such that $A \cap (A' + B') + X = A$ and $A/X \in \mathcal{C}$. We must show that $A = X$. Note first that $(B \cap B') + X + A' \subseteq B$. Let $b \in B = A + A'$, then $b = a + a'$ and $a = a'' + b' + x$ with $a'' + b' \in A \cap (A' + B')$ and $x \in X$. Thus $b = a'' + b' + x + a'$ where $a' \in B$, i.e. $b \in (B \cap B') + X + A'$. Therefore $(B \cap B') + X + A' = B$. We claim that $B/(X + A')$ is in $\mathcal{C}$. Observe that

$$B/(X + A') = (A + A')/(X + A') \subseteq (A + A' + X)/(X + A') \simeq A/(A \cap (A' + X)).$$

The last module is a quotient of A/X, thus in $\mathcal{C}$. Since $B \cap B' \ll_{\mathcal{C}} B$ we must have $X + A' = B$. Note that $X + (A \cap A') \subseteq A$. Let $a \in A \subseteq B = X + A'$, then $a = x + a'$ with $a' \in A \cap A'$.

Thus $X + (A \cap A') = A$ and $A/X \in \mathcal{C}$. Since $A \cap A' \ll_{\mathcal{C}} A$ we must have $A = X$ which proves that A is a $\mathcal{C}$-supplement in D, i.e. gf is a $\mathcal{C}$-Suppl-monomorphism.

(P4) Suppose that gf is a $\mathcal{C}$-Suppl-monomorphism. We keep the notation of the diagram above. By assumption A is a $\mathcal{C}$-supplement in D. There exists $A' \subseteq D$ such that $A + A' = D$ and $A \cap A' \ll_{\mathcal{C}} A$. We prove that A is a $\mathcal{C}$-supplement of $A' \cap B$ in B. Note first that

$$B = D \cap B = (A + A') \cap B = A + (A' \cap B).$$

Since $A \cap A' \cap B \subseteq A \cap A' \ll_{\mathcal{C}} A$, this implies that $A \cap A' \cap B \ll_{\mathcal{C}} A$. Hence A is a $\mathcal{C}$-supplement in B, i.e. f is a $\mathcal{C}$-Suppl-monomorphism.

(P5) Suppose now that g and f are $\mathcal{C}$-Suppl-epimorphisms. Consider the diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & D/A & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{f} & B/A & \longrightarrow & 0 \\
 & & & & & & \downarrow{\scriptstyle g} & & \\
 & & & & & & B/D & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

We want to show that D is a $\mathcal{C}$-supplement in B. Since f and g are $\mathcal{C}$-Suppl-epimorphisms there exist submodules $A' \subseteq B$ and $D' \subseteq B$ such that

$$D/A + D'/A = B/A,\ D/A \cap D'/A \ll_{\mathcal{C}} D/A \text{ and } A + A' = B,\ A \cap A' \ll_{\mathcal{C}} A.$$

We claim that D is a $\mathcal{C}$-supplement of $A' \cap D'$ in B. Note that $A + (D' \cap A') = D'$. To see this let $d' \in D'$. Write $d' = a + a'$, then $a' = d' - a \in D'$. It follows that

$$B = D' + D = A + (D' \cap A') + D = D + (D' \cap A').$$

Now consider the morphism

$$\overline{\sigma} : D/A \to D/(D \cap D') \simeq (D + D')/D' = B/D' \text{ given by}$$

$$d + A \mapsto d + (D \cap D').$$

$\overline{\sigma}$ is a $\mathcal{C}$-small epimorphism, since $\operatorname{Ker} \overline{\sigma} = D \cap D'/A \ll_{\mathcal{C}} D/A$. Define the epimorphism

$$\sigma : D/(A \cap A') \to B/(A' \cap D'),\ d + (A \cap A') \mapsto d + (A' \cap D'),$$

and the isomorphisms

$$\alpha : A/(A \cap A') \simeq (A + A')/A' = B/A',$$

$$\beta : B/(A' \cap D') \simeq D'/(A' \cap D') \oplus A'/(A' \cap D') \simeq B/A' \oplus B/D' \text{ and}$$

$$\gamma : D/(A \cap A') = A + (D \cap A')/(A \cap A') \simeq A/(A \cap A') \oplus (D \cap A')/(A \cap A') \simeq$$

$$\simeq A/(A \cap A') \oplus D/A.$$

Observe that

$$\alpha \oplus \overline{\sigma} : A/(A \cap A') \oplus D/A \to B/A' \oplus B/D'$$

is a $\mathcal{C}$-small epimorphism, since

$$\mathrm{Ker}\,(\alpha \oplus \overline{\sigma}) = 0 \oplus (D \cap D')/A \ll_{\mathcal{C}} A/(A \cap A') \oplus D/A.$$

Then $\sigma = \beta^{-1}(\alpha \oplus \overline{\sigma})\gamma$ is a $\mathcal{C}$-small epimorphism since it is a composition of $\mathcal{C}$-small epimorphisms. Thus $\mathrm{Ker}\,\sigma = (D \cap D' \cap A')/(A \cap A') \ll_{\mathcal{C}} D/(A \cap A')$. On the other hand, since $A \cap A' \ll_{\mathcal{C}} A$, $A \cap A' \ll_{\mathcal{C}} D$. It follows from8.3 (i) that $D \cap D' \cap A' \ll_{\mathcal{C}} D$. This proves that D is a $\mathcal{C}$-supplement in B, i.e. gf is a $\mathcal{C}$-Suppl-epimorphism.

(P6) Let gf be a $\mathcal{C}$-Suppl-epimorphism. With the notation of the diagram above we prove that D/A is a $\mathcal{C}$-supplement in B/A. Since gf is a $\mathcal{C}$-Suppl-epimorphism there exists $D' \subseteq B$ such that

$$D + D' = B \text{ and } D \cap D' \ll_{\mathcal{C}} D.$$

We prove that D/A is a $\mathcal{C}$-supplement of $(D' + A)/A$ in B/A. Note that $D/A + (D' + A)/A = B/A$. Let $X \subseteq D$ such that $(D/A \cap (D' + A)/A) + X/A = D/A$ and $D/X \in \mathcal{C}$. Then $D = D \cap (D' + A) + X = (D \cap D') + A + X$. Since $D/X \in C$, the quotient $D/(X + A) \in \mathcal{C}$. Since $D \cap D' \ll_{\mathcal{C}} D$ we must have $D = X + A = X$. Hence $D/A \cap (D' + A)/A \ll_{\mathcal{C}} D/A$. This proves that D/A is a $\mathcal{C}$-supplement in B/A, i.e. g is a $\mathcal{C}$-Suppl-epimorphism. □

As a corollary of 9.2 we obtain for $\mathcal{C} = \sigma[M]$ that the supplement submodules induce a proper class as has been shown by Generalov in [28].

9.3. The proper class Suppl. The class of all short exact sequences in $\sigma[M]$

$$0 \to A \xrightarrow{f} B \to C \to 0$$

such that $\mathrm{Im}\, f$ is a supplement in B is a proper class and it is denoted by Suppl.

9.4. $\mathcal{C}$-Suppl-divisibles. *For a module N in $\sigma[M]$, the following are equivalent:*

(a) *N is $\mathcal{C}$-*Suppl-*divisible.*

(b) *N is a $\mathcal{C}$-supplement in every M-injective module I containing N.*

(c) *N is a $\mathcal{C}$-supplement in its M-injective hull $\widehat{N}$.*

Proof. (a) ⇒ (b) and (b) ⇒ (c) are clear.

(c) ⇒ (a) Let $i : N \hookrightarrow H$ and $j : N \hookrightarrow \hat{N}$ be the canonical inclusions. Since $\hat{N}$ is M-injective, the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{i} & H \\ & & {\scriptstyle j}\downarrow & \swarrow {\scriptstyle \bar{f}} & \\ & & \hat{N} & & \end{array}$$

can be completed commutatively by $\bar{f} : H \to \hat{N}$. By assumption, there exists a submodule $N' \subseteq \hat{N}$ such that $N + N' = \hat{N}$ and $N \cap N' \ll_{\mathcal{C}} N$. We claim that N is a $\mathcal{C}$-supplement of $f^{-1}(N')$. Let $h \in H$, then $f(h) = n + n' = f(n) + n'$ with $n \in N$ and $n' \in N'$. Therefore $h - n \in f^{-1}(H')$. Since $h = (h - n) + n \in f^{-1}(N') + N$, we obtain $f^{-1}(N') + N = H$. Note that $N \cap f^{-1}(N') \subseteq N \cap N' \ll_{\mathcal{C}} N$. From 8.3 (i) follows that $N \cap f^{-1}(N') \ll_{\mathcal{C}} N$, i.e. N is a $\mathcal{C}$-supplement in H. □

If $\mathcal{C} = \sigma[M]$, i.e. $\mathcal{C}$-Suppl = Suppl we obtain.

9.5. Suppl-divisibles [22, 3.1.4]. *Let N be a module in $\sigma[M]$. The following are equivalent:*

(a) *N is* Suppl-*divisible.*

(b) *N is a supplement in every M-injective module I containing N.*

(c) *N is a supplement in its M-injective hull $\hat{N}$.*

10 $\mathbf{rad}_{\mathcal{S}(\mathcal{C})}$-supplements

10.1. The radical $\mathbf{rad}_{\mathcal{S}(\mathcal{C})}$. Let $\mathcal{C}$ be a $\{q, s\}$-closed class in $\sigma[M]$. Denote by $\mathcal{S}(\mathcal{C})$ the class of simple modules of $\mathcal{C}$. Put for any $N \in \sigma[M]$,

$$\operatorname{rad}_{\mathcal{S}(\mathcal{C})}(N) = \operatorname{Rej}(N, \mathcal{S}(\mathcal{C})) = \bigcap\{\operatorname{Ker} f \mid f : N \to S,\ S \in \mathcal{S}(\mathcal{C})\}.$$

10.2. Properties of $\mathbf{rad}_{\mathcal{S}(\mathcal{C})}$.

(i) $\operatorname{rad}_{\mathcal{S}(\mathcal{C})}$ *is a radical.*

(ii) $\operatorname{Rad} \leq \operatorname{rad}_{\mathcal{S}(\mathcal{C})}$.

(iii) $\operatorname{rad}_{\mathcal{S}(\mathcal{C})}(N) = N$ *iff N has no nonzero simple factor modules in $\mathcal{C}$.*

Proof. (i) It follows from 2.10 (i).

(ii) Is clear.

(iii) $\Rightarrow$) Suppose that $N/K \in \mathcal{S}(\mathcal{C})$. Consider $p : N \to N/K$ the canonical projection. By assumption $N \subseteq \operatorname{Ker} p = K$, i.e. $N/K = 0$.
$\Leftarrow$) Let $f : N \to S$ be a morphism with $0 \neq S \in \mathcal{S}(\mathcal{C})$. Clearly S is a simple factor module of N in $\mathcal{C}$, thus $f = 0$, i.e $N = \operatorname{Ker} f$. Therefore $N = \operatorname{rad}_{\mathcal{S}(\mathcal{C})}(N)$.

□

10.3. $\mathbf{rad}_{\mathcal{S}(\mathcal{C})}$ and $\mathcal{C}$-small submodules. *Let N be a module in $\sigma[M]$.*

(i) $\operatorname{rad}_{\mathcal{S}(\mathcal{C})}(N) = \sum\{L \subseteq N \mid L \ll_{\mathcal{C}} N\}$.

(ii) $\operatorname{rad}_{\mathcal{S}(\mathcal{C})}(N) = N$ *iff every finitely generated submodule of N is $\mathcal{C}$-small in N.*

(iii) *If every proper submodule of N is contained in a maximal submodule of N, then* $\operatorname{rad}_{\mathcal{S}(\mathcal{C})}(N) \ll_{\mathcal{C}} N$.

Proof. (i) Let L be a $\mathcal{C}$-small submodule of N and $K \subseteq N$ such that $N/K \in \mathcal{S}(\mathcal{C})$. Suppose $L \nsubseteq K$. Since K is a maximal submodule, then $K + L = N$. Now N/K is in $\mathcal{C}$, thus L $\mathcal{C}$-small implies that $N = K$ contradicting the maximality of K. Therefore $L \subseteq K$.
Now let $n \in \operatorname{rad}_{\mathcal{S}(\mathcal{C})}(N)$ and $U \subseteq N$ such that $Rn + U = N$ with $N/U \in \mathcal{S}(\mathcal{C})$. Suppose that $U \neq N$. Take a maximal submodule V of N with $U \subseteq V$ and $n \notin V$. Since $\mathcal{C}$ is closed under factor modules, N/V is simple and belongs to $\mathcal{C}$. Thus $n \in \operatorname{rad}_{\mathcal{S}(\mathcal{C})}(N) \subseteq V$ which is a contradiction. Then we must have $U = N$ and therefore Rn a $\mathcal{C}$-small submodule of N.

(ii) $\Rightarrow$) If $\operatorname{rad}_{\mathcal{S}(\mathcal{C})}(N) = N$ then $\operatorname{Rad}(N) = N$. Thus every finitely generated submodule of N is in $\operatorname{Rad}(N)$ and hence it must be small in N and also $\mathcal{C}$-small.
$\Leftarrow$) Conversely for every $x \in N$, $Rx \subseteq N$ is $\mathcal{C}$-small. Therefore $Rx \subseteq \operatorname{rad}_{\mathcal{S}(\mathcal{C})}(N)$. Thus

$N = \text{rad}_{\mathcal{S}(\mathcal{C})}(N)$.

(iii) Let $rad_{\mathcal{S}(\mathcal{C})}(N) + X = N$ with $N/X \in \mathcal{C}$. Suppose $X \neq N$. There is a maximal submodule U of N containing X. Thus $N/U \simeq (N/X)/(U/X) \in \mathcal{C}$ and is simple. Therefore $\text{rad}_{\mathcal{S}(\mathcal{C})}(N) \subseteq U$. Then we must have $N = U$ which is a contradiction and so $X = N$. It follows that $\text{rad}_{\mathcal{S}(\mathcal{C})}(N) \ll_{\mathcal{C}} N$. □

We consider next the class τ-Suppl for the radical $\tau = \text{rad}_{\mathcal{S}(\mathcal{C})}$ (see 4.9).

10.4. The proper class $\text{rad}_{\mathcal{S}(\mathcal{C})}$-Suppl. The class of all short exact sequences

$$0 \to A \xrightarrow{f} B \to C \to 0$$

such that $\text{Im} f$ is a $\text{rad}_{\mathcal{S}(\mathcal{C})}$-supplement in B is a proper class and it is denoted by $\text{rad}_{\mathcal{S}(\mathcal{C})}$-Suppl. By definition

$$\text{rad}_{\mathcal{S}(\mathcal{C})}\text{-Suppl} = \iota^{-1}(\mathbb{F}_{\text{rad}_{\mathcal{S}(\mathcal{C})}}) = \iota^{-1}\{N \in \sigma[M] \mid \text{rad}_{\mathcal{S}(\mathcal{C})}(N) = 0\}.$$

If $\mathcal{C} = \sigma[M]$, i.e. $\text{rad}_{\mathcal{S}(\mathcal{C})} = \text{Rad}$ we obtain.

10.5. The proper class Co-Neat.

$$\text{Co-Neat} = \iota^{-1}\{N \in \sigma[M] \mid \text{Rad}(N) = 0\}.$$

As a consequence of 4.9 we have for $\tau = \text{Rad}$:

10.6. Characterization of coneat submodules. [1, 1.14] *For a submodule $L \subseteq N$, the following are equivalent:*

(a) *The inclusion $L \to N$ is a* Co-Neat-*monomorphism.*

(b) *There exists a submodule $L' \subseteq N$ such that*

  (i) *$L + L' = N$ and*

  (ii) *$L \cap L' = \text{Rad}(L)$.*

(c) *There exists a submodule $L' \subseteq N$ such that*

  (i) *$L + L' = N$ and*

  (ii) *$L \cap L' \subseteq \text{Rad}(L)$.*

If the conditions are satisfied, then L is called a Rad**-supplement** in N.
The relationship between the proper classes $\text{rad}_{\mathcal{S}(\mathcal{C})}$-Suppl and $\mathcal{C}$-Suppl is analogous to the one between Suppl and Co-Neat = Rad-Suppl.

10.7. Proposition. *Let $\mathcal{C}$ be a $\{q, s\}$-closed class in $\sigma[M]$. For any module M,*

$$\mathcal{C}\text{-Suppl} \subseteq \text{rad}_{\mathcal{S}(\mathcal{C})}\text{-Suppl}.$$

Proof. If L is a $\mathcal{C}$-supplement of N, then there is a submodule $L' \subseteq N$ such that $L + L' = N$ and $L \cap L' \ll_{\mathcal{C}} L$. By 10.3 (i), the submodule $\text{rad}_{\mathcal{S}(\mathcal{C})}(L)$ is the sum of all $\mathcal{C}$-small submodules of L, therefore $L \cap L' \subseteq \text{rad}_{\mathcal{S}(\mathcal{C})}(L)$, i.e. L is a $\text{rad}_{\mathcal{S}(\mathcal{C})}$-supplement in N. □

Thus we get in case $\mathcal{C} = \sigma[M]$:

10.8. Proposition [46, 3.4.1]. *For any module M,*

$$\text{Suppl} \subseteq \text{Co-Neat} \subseteq \iota^{-1}\{\text{all (semi-)simple modules in } \sigma[M]\}.$$

10.9. $\mathcal{C}$-Suppl-flats. *Let $\mathcal{C}$ be a $\{q,s\}$-closed class of modules in R-Mod. If $\text{rad}_{\mathcal{S}(\mathcal{C})}(R) = 0$, then every $\mathcal{C}$-Suppl-flat module is projective.*

Proof. Follows from 6.5. □

In case $\mathcal{C} = R$-Mod we obtain.

10.10. Suppl-flats [46, 3.7.2]. *If $\text{Rad}(R) = 0$, then every Suppl-flat module is projective.*

10.11. $\text{rad}_{\mathcal{S}(\mathcal{C})}$ and ($\mathcal{C}$-)supplements. *Let K be a submodule of N.*

(i) *If $\text{rad}_{\mathcal{S}(\mathcal{C})}(K) \ll K$, then K is a $\mathcal{C}$-supplement in N iff K is a supplement.*

(ii) *If $\text{rad}_{\mathcal{S}(\mathcal{C})}(K) \ll_{\mathcal{C}} K$, then K is a $\mathcal{C}$-supplement in N iff K is a $\text{rad}_{\mathcal{S}(\mathcal{C})}$-supplement.*

(iii) *If $\text{rad}_{\mathcal{S}(\mathcal{C})}(N) = N$, then every finitely generated submodule of N has a $\mathcal{C}$-supplement in N.*

Proof. (i) It is clear that supplements are always $\mathcal{C}$-supplements. Conversely, if K is a $\mathcal{C}$-supplement, there exists $K' \subseteq N$ with $K + K' = N$ and $K \cap K' \ll_{\mathcal{C}} K$. Then $K \cap K' \subseteq \text{rad}_{\mathcal{S}(\mathcal{C})}(K) \ll K$. Therefore $K \cap K' \ll K$, i.e. K is a supplement in N.

(ii) ⇒) Follows from 10.7.
⇐) Let now K be a $\text{rad}_{\mathcal{S}(\mathcal{C})}$-supplement in N. Then there exist $K' \subseteq N$ with $K + K' = N$ and $K \cap K' = \text{rad}_{\mathcal{S}(\mathcal{C})}(K)$ which, by assumption, is $\mathcal{C}$-small in K, thus $K \cap K' \ll_{\mathcal{C}} K$, i.e. K is a $\mathcal{C}$-supplement in N.

(iii) Let $K \subseteq N$ be a finitely generated submodule. Then, by 10.3 (ii), K is $\mathcal{C}$-small. Thus $K + N = N$ and $K \cap N = K \ll_{\mathcal{C}} N$, i.e N is a $\mathcal{C}$-supplement of K. □

10.12. $\mathcal{C}$-coclosed submodules. A submodule $L \subseteq N$ in $\sigma[M]$ is called a **$\mathcal{C}$-coclosed submodule** of N if for every submodule $X \subseteq N$, $L/X \ll_{\mathcal{C}} N/X$ implies $X = L$.

10.13. Lemma. *Let $\mathcal{C}$ be a $\{q,s\}$-closed class of modules in $\sigma[M]$ closed under products. A $\mathcal{C}$-small submodule L of a module N is a $\text{rad}_{\mathcal{S}(\mathcal{C})}$-supplement in N iff $\text{rad}_{\mathcal{S}(\mathcal{C})}(L) = L$.*

Proof. Let $L \ll_{\mathcal{C}} N$. If L is a $\text{rad}_{\mathcal{S}(\mathcal{C})}$-supplement in N, then there exists a submodule $L' \subseteq N$ such that $L + L' = N$ and $L \cap L' = \text{rad}_{\mathcal{S}(\mathcal{C})}(L)$. Since $L \ll_{\mathcal{C}} N$, $L' = N$ provided N/L' is in $\mathcal{C}$. Suppose we have $N/L' \in \mathcal{C}$. Then $L' = N$ and hence $L = L \cap N = \text{rad}_{\mathcal{S}(\mathcal{C})}(L)$. To prove that $N/L' \in \mathcal{C}$ note that

$$N/L' = (L + L')/L' \simeq L/L \cap L' = L/\text{rad}_{\mathcal{S}(\mathcal{C})}(L).$$

Thus $\text{rad}_{\mathcal{S}(\mathcal{C})}(N/L') = 0$. It follows from 2.10 (ii) that $N/L' \in \mathcal{C}$.
Conversely, suppose that $L \ll_{\mathcal{C}} N$ and $\text{rad}_{\mathcal{S}(\mathcal{C})}(L) = L$. Then $L + N = N$ and $L \cap N = L = \text{rad}_{\mathcal{S}(\mathcal{C})}(L)$, i.e. L is a $\text{rad}_{\mathcal{S}(\mathcal{C})}$-supplement in N. □

10.14. $\mathcal{C}$-coclosed and $\text{rad}_{\mathcal{S}(\mathcal{C})}$-supplement submodules. *Let $\mathcal{C}$ be a $\{q,s\}$-closed class of modules in $\sigma[M]$ closed under products. The following are equivalent:*

(a) *Every nonzero $\text{rad}_{\mathcal{S}(\mathcal{C})}$-supplement in a module in $\sigma[M]$ is a $\mathcal{C}$-coclosed submodule.*

(b) *Every nonzero $\mathcal{C}$-small submodule of any module in $\sigma[M]$ has a maximal submodule.*

Proof. (a) $\Rightarrow$ (b) If $L \ll_{\mathcal{C}} N$ and $L = \mathrm{Rad}(L) \subseteq \mathrm{rad}_{\mathcal{S}(\mathcal{C})}(L)$, then $\mathrm{rad}_{\mathcal{S}(\mathcal{C})}(L) = L$. Thus, by 10.13, L is a $\mathrm{rad}_{\mathcal{S}(\mathcal{C})}$-supplement in N. But, by hypothesis, L is $\mathcal{C}$-coclosed in N and hence not $\mathcal{C}$-small. This is a contradiction, thus $\mathrm{Rad}(L) \neq L$, i.e. L has at least one maximal submodule.

(b) $\Rightarrow$ (a) Let L be a $\mathrm{rad}_{\mathcal{S}(\mathcal{C})}$-supplement in $N \in \sigma[M]$. For any submodule $U \subseteq L \subseteq N$, L/U is a $\mathrm{rad}_{\mathcal{S}(\mathcal{C})}$-supplement in N/U, since in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & N & \longrightarrow & N/L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & L/U & \longrightarrow & N/U & \longrightarrow & N/L & \longrightarrow & 0, \end{array}$$

the first row is a short exact sequence in $\mathrm{rad}_{\mathcal{S}(\mathcal{C})}$-Suppl. Thus, by 3.1 (P5) and (P6), the bottom row belongs also to $\mathrm{rad}_{\mathcal{S}(\mathcal{C})}$-Suppl, i.e. L/U is a $\mathrm{rad}_{\mathcal{S}(\mathcal{C})}$-supplement in N/U. Suppose now that $L/U \ll_{\mathcal{C}} N/U$, then $\mathrm{rad}_{\mathcal{S}(\mathcal{C})}(L/U) = L/U$ by 10.13. By hypothesis L/U has a maximal submodule. Hence $L/U \not\ll_{\mathcal{C}} N/U$ for all $U \subseteq L$ implies that L is $\mathcal{C}$-coclosed in N. □

Recall that a module $N \in \sigma[M]$ is called M**-small** if $N \ll L$ for some $L \in \sigma[M]$. Equivalently, $N \ll \widehat{N}$, where $\widehat{N}$ is the M-injective hull of N (see 1.12). For $\mathcal{C} = \sigma[M]$ we obtain.

10.15. Coneat and coclosed submodules. [1, 1.16] *Let M be a module. The following are equivalent:*

(a) *Every nonzero coneat submodule of a module in $\sigma[M]$ is a coclosed submodule.*

(b) *Every nonzero M-small module in $\sigma[M]$ is a Max module (resp. has a maximal submodule).*

10.16. $\mathcal{C}$-hollow modules. A module $N \in \sigma[M]$ is called **$\mathcal{C}$-hollow** if every proper submodule of N is $\mathcal{C}$-small in N.

10.17 Proposition. *Let $\mathcal{C}$ be a $\{q, s\}$-closed class of modules in $\sigma[M]$. If $N \in \sigma[M]$ is $\mathcal{C}$-hollow, then N/K is $\mathcal{C}$-hollow for every $K \subseteq N$.*

Proof. Let $H/K \subseteq N/K$ be a proper submodule. Then H is a proper submodule of N. Then, by assumption, H is a $\mathcal{C}$-small submodule of N. By Proposition 8.3 (i), H/K is $\mathcal{C}$-small in N/K. Therefore N/K is $\mathcal{C}$-hollow. □

10.18 Proposition. *Let $\mathcal{C}$ be a $\{q, s\}$-closed class of modules in $\sigma[M]$. If $N = \sum_{i \in I} N_i$ with N_i $\mathcal{C}$-hollow. Then $N/rad_{\mathcal{S}(\mathcal{C})}(N)$ is semisimple.*

Proof. We have $N/rad_{\mathcal{S}(\mathcal{C})}(N) = \sum_{i \in I}(N_i + rad_{\mathcal{S}(\mathcal{C})}(N))/rad_{\mathcal{S}(\mathcal{C})}(N)$. On the other hand, $(N_i + rad_{\mathcal{S}(\mathcal{C})}(N))/rad_{\mathcal{S}(\mathcal{C})}(N) \simeq N_i/(rad_{\mathcal{S}(\mathcal{C})}(N) \cap N_i)$ which is a simple module or zero since for a proper submodule $K_i/(rad_{\mathcal{S}(\mathcal{C})}(N) \cap N_i) \subseteq N_i/(rad_{\mathcal{S}(\mathcal{C})}(N) \cap N_i)$ we have that K_i is $\mathcal{C}$-small thus $K_i \subseteq rad_{\mathcal{S}(\mathcal{C})}(N)$. Therefore $N/rad_{\mathcal{S}(\mathcal{C})}(N)$ is semisimple. □

10.19 Proposition. *Let $\mathcal{C}$ be a $\{q, s\}$-closed class of modules in $\sigma[M]$. If $N/rad_{\mathcal{S}(\mathcal{C})}(N)$ is semisimple and $rad_{\mathcal{S}(\mathcal{C})}(N) \ll_{\mathcal{C}} N$, then every proper submodule K of N with $N/K \in \mathcal{C}$ is contained in a maximal submodule.*

Proof. Let $K \subseteq N$ be a proper submodule such that $N/K \in \mathcal{C}$ and $p : N \longrightarrow N/rad_{\mathcal{S}(\mathcal{C})}(N)$ the canonical projection. Since $rad_{\mathcal{S}(\mathcal{C})}(N) \ll_{\mathcal{C}} N$ we have $p(K) \neq N/rad_{\mathcal{S}(\mathcal{C})}(N)$. Therefore $p(U)$ is contained in a maximal submodule $X \subseteq N/rad_{\mathcal{S}(\mathcal{C})}(N)$. Then U is contained in the maximal submodule $p^{-1}(X) \subseteq N$. □

The following special cases of $\mathrm{rad}_{\mathcal{S}(\mathcal{C})}$ are of interest.

10.20. The radicals $\alpha, \beta, \gamma, \delta$. We denote by

$\alpha = \mathrm{rad}_{\mathcal{S}(\mathscr{I})}$ for $\mathscr{I}$ the class of M-injective modules,

$\beta = \mathrm{rad}_{\mathcal{S}(\mathscr{P})}$ for $\mathscr{P}$ the class of M-projective modules,

$\gamma = \mathrm{rad}_{\mathcal{S}(\mathscr{M})}$ for $\mathscr{M}$ the class of M-small modules,

$\delta = \mathrm{rad}_{\mathcal{S}(\mathscr{S})}$ for $\mathscr{S}$ the class of M-singular modules.

In the category $\sigma[M]$ the class of simple modules splits into four disjoint classes, namely

$$\mathcal{S}(\mathscr{I}) \cap \mathcal{S}(\mathscr{P}),\ \mathcal{S}(\mathscr{I}) \cap \mathcal{S}(\mathscr{S}),\ \mathcal{S}(\mathscr{M}) \cap \mathcal{S}(\mathscr{P}),\ \text{and}\ \mathcal{S}(\mathscr{M}) \cap \mathcal{S}(\mathscr{S}).$$

More precisely.

10.21. Lemma [17, 12, 4.2, 8.2].

(i) *A simple module in $\sigma[M]$ is either M-injective or M-small.*

(ii) *A simple module in $\sigma[M]$ is either M-projective or M-singular.*

Proof. Let S be a simple module in $\sigma[M]$.
(i) Suppose that S is not M-injective. Then $S \neq \widehat{S}$. Let $X \subseteq \widehat{S}$ such that $S + X = \widehat{S}$. Since $S \subseteq_e \widehat{S}$, then $S \cap X \neq 0$. Thus $S \subseteq X$. Therefore $X = \widehat{S}$, i.e. S is small in its injective hull and so it is M-small.

(ii) Suppose that S is not M-singular. Consider a short exact sequence

$$0 \to A \to B \to S \to 0.$$

Note that the maximal submodule A is not essential in B, thus it is a direct summand. Therefore the sequence splits, i.e. S is M-projective. □

We fix some notation:

$K \ll_\gamma N$ if K is an $\mathscr{M}$-small submodule of N,

$K \ll_\delta N$ if K is an $\mathscr{S}$-small submodule of N.

10.22. Properties of $\alpha, \beta, \gamma, \delta$.

(i) $\mathrm{Rad} = \alpha \cap \gamma = \beta \cap \delta$,

(ii) *α is an idempotent radical.*

(iii) $\beta(N) \supseteq \cap\{L \subseteq N \mid L$ *maximal and direct summand*$\}$.

(iv) $\gamma(N) = \sum\{L \subseteq N \mid L \ll_\gamma N\}$.

(v) $\delta(N) = \sum\{L \subseteq N \mid L \ll_\delta N\}$.

Proof. (i) It follows from 10.21.

(ii) It follows from 2.10 (v).

(iii) Let $f : N \to S$ with $S \in \mathcal{S}(\mathscr{P})$. Let $L = \mathrm{Ker}\, f$, then L is a maximal submodule of N and a direct summand. This proves the assertion.
(iv) and (v) follow from 10.3 (i). □

For σ, τ radicals in $\sigma[M]$ we have always that

$$(\sigma \cap \tau)\text{-Suppl} \subseteq \sigma\text{-Suppl} \cap \tau\text{-Suppl}.$$

For the other inclusion we need the following concepts.

10.23. Diuniform modules (see [51, Definition 42]). A module $N \in \sigma[M]$ is called **diuniform** if $K \cap L = 0$ implies $K = 0$ or $L = 0$ with K and L fully invariant submodules of N.

10.24. Proposition. *Let σ, τ be radicals in $\sigma[M]$. Consider the following conditions:*

(a) *Every module in $\mathbb{F}_{\sigma\cap\tau}$ is diuniform.*

(b) $\mathbb{F}_{\sigma\cap\tau} = \mathbb{F}_\sigma \cup \mathbb{F}_\tau$.

(c) $(\sigma \cap \tau)$-Suppl=σ-Suppl $\cap$ τ-Suppl.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Proof. (a) $\Rightarrow$ (b) It is clear that $\mathbb{F}_\sigma \cup \mathbb{F}_\tau \subseteq \mathbb{F}_{\sigma\cap\tau}$. Let $X \in \mathbb{F}_{\sigma\cap\tau}$. Then $(\sigma \cap \tau)(X) = 0$. Thus by assumption $\sigma(X) = 0$ or $\tau(X) = 0$, i.e. $X \in \mathbb{F}_\sigma \cup \mathbb{F}_\tau$.

(b) $\Rightarrow$ (c) By definition $(\sigma \cap \tau)$-Suppl=$\iota^{-1}(\mathbb{F}_{\sigma\cap\tau})$. From the proof of 4.11 it follows that σ-Suppl $\cap$ τ-Suppl=$\iota^{-1}(\mathbb{F}_\sigma) \cap \iota^{-1}(\mathbb{F}_\tau) = \iota^{-1}(\mathbb{F}_\sigma \cup \mathbb{F}_\tau)$. Thus by assumption $\iota^{-1}(\mathbb{F}_{\sigma\cap\tau}) = \iota^{-1}(\mathbb{F}_\sigma \cup \mathbb{F}_\tau)$. □

10.25. GCO modules (see [17]). A module M is called a **GCO-module** if every singular simple module is M-injective or M-projective. Equivalently, M is a GCO-module iff every M-singular simple module is M-injective (see [17, 16.4]).

We obtain a new characterization of GCO-modules:

10.26. Proposition. *Let M be a module. The following conditions are equivalent:*

(a) *M is a GCO-module,*

(b) $\alpha \leq \delta$,

(c) α-Suppl $\subseteq$ δ-Suppl.

Proof. (a) $\Rightarrow$ (b) By assumption $\mathcal{S}(\mathscr{S}) \subseteq \mathcal{S}(\mathscr{I})$, thus for every module N, $\alpha(N) = \mathrm{rad}_{\mathcal{S}(\mathscr{I})}(N) \subseteq \mathrm{rad}_{\mathcal{S}(\mathscr{S})}(N) = \delta(N)$.

(b) $\Rightarrow$ (c) Is clear.

(c) $\Rightarrow$ (a) Let S be a simple M-singular module. Since S is simple, then $\alpha(S) = 0$ or $\alpha(S) = S$. Suppose first $\alpha(S) = 0$, then S is cogenerated by the simple M-injective modules, hence S is M-injective.
Now suppose that $\alpha(S) = S$. Then the short exact sequence

$$0 \to S \to \widehat{S} \to \widehat{S}/S \to 0$$

belongs to α-Suppl. Therefore it belongs also to δ-Suppl, i.e. there exists a submodule $S' \subseteq \hat{S}$ such that $S + S' = \widehat{S}$ and $S \cap S' = \delta(S)$. Since S is M-singular, $\delta(S) = 0$. But $S \subseteq_e \widehat{S}$ implies $S' = 0$ and thus $S = \widehat{S}$, i.e. S is M-injective. □

10.27. Proposition. *Let M be a module. Consider the following conditions:*

(a) *every M-injective simple module is M-singular,*

(b) $\delta \leq \alpha$,

(c) δ-Suppl $\subseteq \alpha$-Suppl.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Proof. (a) $\Rightarrow$ (b) By assumption $\mathcal{S}(\mathscr{I}) \subseteq \mathcal{S}(\mathscr{S})$, thus for every module N, $\delta(N) = \text{rad}_{\mathcal{S}(\mathscr{S})}(N) \subseteq \text{rad}_{\mathcal{S}(\mathscr{I})}(N) = \alpha(N)$.

(b) $\Rightarrow$ (c) Is clear. $\square$

Chapter 5

Proper classes related to $\mathcal{C}$-complements

In this chapter we consider the dual notion to $\mathcal{C}$-small submodules and $\mathcal{C}$-supplements. As above, $\mathcal{C}$ is a $\{q, s\}$-closed class of modules.

11 $\mathcal{C}$-essential submodules

11.1. $\mathcal{C}$-essential submodules. A submodule $K \subseteq N$ is called a **$\mathcal{C}$-essential** submodule of N if for every submodule $X \subseteq N$, the equality $K \cap X = 0$ and $X \in \mathcal{C}$ implies $X = 0$. We denote this by $K \subseteq_{\mathcal{C}e} N$.

For any $\{q, s\}$-closed class $\mathcal{C}$ in $\sigma[M]$ every essential submodule is $\mathcal{C}$-essential. If $\mathcal{C} = \sigma[M]$, then the $\sigma[M]$-essential submodules are the essential submodules in $\sigma[M]$. For any $\{q, s\}$-closed class $\mathcal{C}$ in $\sigma[M]$, if $N \in \mathcal{C}$, then a submodule $K \subseteq N$ is $\mathcal{C}$-essential in N iff K is essential in N.

11.2. Examples of $\mathcal{C}$-essential submodules.

(i) If $\mathcal{C} = \mathcal{S}(\sigma[M])$, the class of simple modules in $\sigma[M]$, then a submodule $K \subseteq N$ is $\mathcal{S}(\sigma[M])$-essential iff $\mathrm{Soc}(N) \subseteq K$.

Proof. Let K be a $\mathcal{S}(\sigma[M])$-essential submodule of N and S any simple submodule of N. Suppose that $S \nsubseteq K$, then $S \cap K = 0$. Thus, by hypothesis, $S = 0$, which is a contradiction. Therefore $S \subseteq K$, i.e. $\mathrm{Soc}(N) \subseteq K$. Conversely, suppose that $\mathrm{Soc}(N) \subseteq K$. For any submodule $X \subseteq N$ such that $X \cap K = 0$ with X simple, we have $X \subseteq \mathrm{Soc}(N) \subseteq K$. Therefore $0 = X \cap K = X$, i.e. K is a $\mathcal{S}(\sigma[M])$-essential submodule of N. □

(ii) Let $\mathcal{C}$ be either the class of noetherian or artinian modules in $\sigma[M]$. Since every simple module is noetherian and artinian, a noetherian-essential or an artinian-essential submodule is $\mathcal{S}(\sigma[M])$-essential (see (i)).

(iii) If $\mathcal{C} = \mathbb{T}_\tau$, with τ a hereditary preradical in $\sigma[M]$, then a submodule $K \subseteq N \in \sigma[M]$ is $\mathbb{T}_\tau$-essential iff for every submodule $X \subseteq N$ such that $X \cap K = 0$ we have $X \cap \tau(N) = 0$.

Proof. Let K be a $\mathbb{T}_\tau$-essential submodule of N and $X \subseteq N$ a submodule of N such that $X \cap K = 0$. Since τ is an idempotent preradical and $\mathbb{T}_\tau$ is closed under submodules (see 2.6), the submodule

$$X \cap \tau(N) \in \mathbb{T}_\tau.$$

On the other hand $X \cap \tau(N) \cap K = 0$. Since K is $\mathbb{T}_\tau$-essential, we must have $X \cap \tau(N) = 0$. Conversely, let $X \subseteq N$ be a submodule such that $X \cap K = 0$ and $X \in \mathbb{T}_\tau$. Thus by 2.4 (i), we have $X \subseteq \tau(N)$. By hypotheis, $0 = X \cap \tau(N) = X$, i.e. K is a $\mathbb{T}_\tau$-essential submodule of N. □

11.3. Properties of $\mathcal{C}$-essential submodules. *Let K, L and N be modules in $\sigma[M]$.*

(i) *If $K \subseteq L \subseteq N$, then $K \subseteq_{\mathcal{C}e} N$ iff $K \subseteq_{\mathcal{C}e} L$ and $L \subseteq_{\mathcal{C}e} N$.*

(ii) *If $L \subseteq_{\mathcal{C}e} N$ and $f : H \to N$, then $f^{-1}(L) \subseteq_{\mathcal{C}e} H$.*

(iii) *If $K_1 \subseteq_{\mathcal{C}e} L_1 \subseteq N$ and $K_2 \subseteq_{\mathcal{C}e} L_2 \subseteq N$, then $K_1 \cap K_2 \subseteq_{\mathcal{C}e} L_1 \cap L_2$.*

(iv) *If $K_i \subseteq_{\mathcal{C}e} N_i$ for each $i \in I$, then $\oplus_I K_i \subseteq_{\mathcal{C}e} \oplus_I N_i$.*

Proof. (i) $\Rightarrow$) Let $X \subseteq L$ such that $X \cap K = 0$ and $X \in \mathcal{C}$. By assumption $K \subseteq_{\mathcal{C}e} N$ which implies $X = 0$. Hence $K \subseteq_{\mathcal{C}e} L$. Let $Y \subseteq N$ such that $Y \cap L = 0$ and $Y \in \mathcal{C}$. Then $K \cap Y = K \cap Y \cap L = 0$. Since $K \subseteq_{\mathcal{C}e} N$, $Y = 0$, i.e. $L \subseteq_{\mathcal{C}e} N$.
$\Leftarrow$) Let $X \subseteq N$ such that $X \cap K = 0$ and $X \in \mathcal{C}$. Then $L \cap X \cap K = 0$ and $L \cap X \in \mathcal{C}$. Since $K \subseteq_{\mathcal{C}e} L$, then $L \cap X = 0$. From $L \subseteq_{\mathcal{C}e} N$ follows that $X = 0$, i.e. $K \subseteq_{\mathcal{C}e} N$.

(ii) Let $X \subseteq H$ such that $X \cap f^{-1}(L) = 0$ and $X \in \mathcal{C}$. Note that $f(X) \cap N = 0$. Since $L \subseteq_{\mathcal{C}e} N$ and $f(X) \in \mathcal{C}$, $f(X) = 0$. Therefore $X \subseteq \operatorname{Ker} f \subseteq f^{-1}(L)$. It follows that $X = X \cap f^{-1}(L) = 0$, i.e. $f^{-1}(L) \subseteq_{\mathcal{C}e} H$.

(iii) Let $X \subseteq L_1 \cap L_2$ such that $X \cap K_1 \cap K_2 = 0$ and $X \in \mathcal{C}$. Since $X \cap K_1 \subseteq L_2$ and $\mathcal{C}$ is closed under submodules, then $X \cap K_1 \in \mathcal{C}$. From $K_2 \subseteq_{\mathcal{C}e} L_2$ follows that $X \cap K_1 = 0$. Since $K_1 \subseteq_{\mathcal{C}e} L_1$, then $X = 0$. Therefore $K_1 \cap K_2 \subseteq_{\mathcal{C}e} L_1 \cap L_2$.

(iv) We proof first that if $K_1 \subseteq_{\mathcal{C}e} L_1$ and $K_2 \subseteq_{\mathcal{C}e} L_2$, then $K_1 \oplus K_2 \subseteq_{\mathcal{C}e} L_1 \oplus L_2$. By (ii) for the projection $L_1 \oplus L_2 \to L_1$, follows that $K_1 \oplus L_2 \subseteq_{\mathcal{C}e} L_1 \oplus L_2$. Analogous $L_1 \oplus K_2 \subseteq_{\mathcal{C}e} L_1 \oplus L_2$. Then

$$K_1 \oplus K_2 = (K_1 \oplus L_2) \cap (L_1 \oplus K_2) \subseteq_{\mathcal{C}e} L_1 \oplus L_2$$

This also shows that the result is true for any finite family of sumodules $K_i \subseteq_{\mathcal{C}e} L_i \subseteq N$.
Let $X \subseteq \oplus_I N_i$ be in $\mathcal{C}$. If $X \neq 0$, there exists $0 \neq x \in \oplus_{j=1}^n N_{i_j}$ since this is a finite sum, $\oplus_{j=1}^n K_{i_j} \subseteq_{\mathcal{C}e} \oplus_{j=1}^n N_{i_j}$, then we must have $Rx \cap \oplus_{j=1}^n K_{i_j} \neq 0$. Therefore $0 \neq Rx \cap \oplus_{j=1}^n K_{i_j} \subseteq X \cap \oplus_I N_i$. Thus $\oplus_I K_i \subseteq_{\mathcal{C}e} \oplus_I N_i$. □

11.4. $\mathcal{C}$-essential monomorphisms. A monomorphism $f : N \to J$ is called a **$\mathcal{C}$-essential monomorphism** if $\operatorname{Im} f \subseteq_{\mathcal{C}e} J$.

11.5. Proposition. *A monomorphism $f : N \to J$ is $\mathcal{C}$-essential iff each morphism $h : J \to H$ with hf a monomorphism and* $\operatorname{Ker} h \in \mathcal{C}$ *is a monomorphism.*

Proof. $\Rightarrow$) Let $x \in \operatorname{Ker} h \cap \operatorname{Im} f$. Then $0 = h(x) = h(f(j))$ for a $j \in J$. Since hf is mono, $j = 0$, i.e. $x = f(j) = 0$. This implies $\operatorname{Ker} h \cap \operatorname{Im} f = 0$. By assumption, $\operatorname{Im} f$ is $\mathcal{C}$-essential, thus $\operatorname{Ker} h = 0$.
$\Leftarrow$) Let $X \subseteq J$ such that $X \cap \operatorname{Im} f = 0$ and $X \in \mathcal{C}$. Consider the canonical projection $p : J \to J/X$. Note that $\operatorname{Ker} p = X$. The composition pf is a monomorphism. Then, by assumption, p is a monomorphism, i.e. $X = 0$. This means that $\operatorname{Im} f \subseteq_{\mathcal{C}e} J$. □

12 $\mathcal{C}$-complements

12.1. $\mathcal{C}$-complements. A submodule $L \subseteq N$ is called a **$\mathcal{C}$-complement** in N if there exists a submodule $L' \subseteq N$ such that

(i) $L \cap L' = 0$ and

(ii) $(L + L')/L \subseteq_{\mathcal{C}e} N/L$.

12.2. The proper class $\mathcal{C}$-Compl. The class of all short exact sequences in $\sigma[M]$

$$0 \to A \xrightarrow{f} B \to C \to 0$$

such that $\operatorname{Im} f$ is a $\mathcal{C}$-complement in B is a proper class and it is denoted by $\mathcal{C}\text{-Compl}_{\sigma[M]}$.

Proof. We prove that the axioms (P1)-(P6) of 3.1 are satisfied. We keep the convention that short exact sequences are of the form

$$0 \to A \xrightarrow{i} B \xrightarrow{p} B/A \to 0$$

with i and p the canonical inclusion and projection, respectively.

(P1) Let $E \in \mathcal{C}$-Compl and $E' \simeq E$. There is a commutative diagram

$$\begin{array}{ccccccccc}
E: 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & B/A & \longrightarrow & 0 \\
 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\
E': 0 & \longrightarrow & C & \longrightarrow & D & \longrightarrow & D/C & \longrightarrow & 0
\end{array}$$

with f_i isomorphisms. Since $E \in \mathcal{C}$-Compl, there exists $A' \subseteq B$ such that

$$A \cap A' = 0 \text{ and } (A + A')/A \subseteq_{\mathcal{C}e} B/A.$$

Then

$$C \cap f_2(A') = f_1(A) \cap f_2(A') = f_2(A) \cap f_2(A') = f_2(A \cap A') = 0.$$

Set $C' = f_2(A')$. We need to show that $(C + C')/C \subseteq_{\mathcal{C}e} D/C$. Let $X/C \subseteq D/C$ such that $(C + C')/C \cap X/C = 0$ and $X/C \in \mathcal{C}$. Then

$$(C + C') \cap X = C.$$

Since f_2 is an isomorphism,

$$f_2(A) = f_2(A + A') \cap f_2(f_2^{-1}(X)) = f_2((A + A') \cap f_2^{-1}(X)).$$

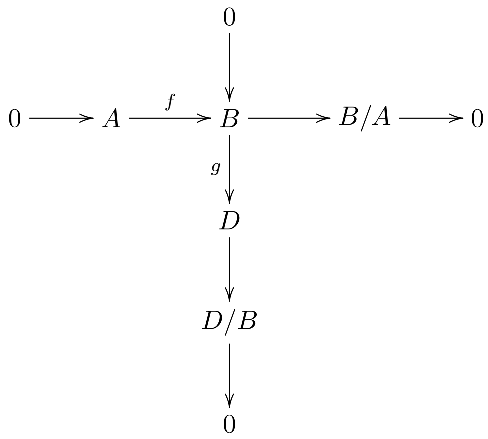
It follows that $(A + A') \cap f_2^{-1}(X) = A$. On the oder hand $f_2^{-1}(X)/A \in \mathcal{C}$ since $f_2^{-1}(X)/A \simeq X/C$ which is in $\mathcal{C}$. By assumption $(A + A')/A \subseteq_{\mathcal{C}e} B/A$, thus $f_2^{-1}(X)/A = 0$. Therefore $X/C = 0$, i.e. $(C + C')/C \subseteq_{\mathcal{C}e} D/C$. This shows that $E' \in \mathcal{C}$.

(P2) Let E be a split short exact sequence. E is of the form

$$0 \to A \to A \oplus A' \to A' \to 0.$$

Then $A \cap A' = 0$ and $(A \oplus A')/A \simeq A' \subseteq_{\mathcal{C}e} A'$. Thus A is a $\mathcal{C}$-complement in $A \oplus A'$, i.e. $E \in \mathcal{C}$-Compl.

(P3) Let f and g be $\mathcal{C}$-Compl-monomorphisms. Consider the diagram



There exist submodules $A' \subseteq B$ and $B' \subseteq D$ such that

$$A \cap A' = 0,\ (A \oplus A')/A \subseteq_{\mathcal{C}e} B/A \text{ and } B \cap B' = 0,\ (B \oplus B')/B \subseteq_{\mathcal{C}e} D/B.$$

We prove that A is a $\mathcal{C}$-complement of $A' + B'$ in D. Note that $A \cap (A' + B') = 0$. On the other hand

$$(B \oplus B')/A/B/A \simeq (B \oplus B')/B \subseteq_{\mathcal{C}e} D/B \simeq D/A/B/A.$$

From 11.3 (ii) follows that $(B \oplus B')/A \subseteq_{\mathcal{C}e} D/A$. Observe that

$$(B \oplus B')/B = B/A \oplus (A + B')/A \text{ and}$$

$$(A + A' + B')/A = (A + A')/A \oplus (A + B')/A.$$

Since $(A + A')/A \subseteq_{\mathcal{C}e} B/A$ and $(A + B')/A \subseteq_{\mathcal{C}e} (A + B')/A$, it follows from 11.3 (iv) that

$$(A + A')/A \oplus (A + B')/A \subseteq_{\mathcal{C}e} B/A \oplus (A + B')/A = (B \oplus B')/A \subseteq_{\mathcal{C}e} D/A.$$

Therefore $(A + A' + B')/A \subseteq_{\mathcal{C}e} D/A$, i.e. gf is a $\mathcal{C}$-Compl-epimorphism.

(P4) Let gf be a $\mathcal{C}$-Compl-monomorphism. Keep the notation of the diagram above. There exists a submodule $A' \subseteq D$ such that

$$A \cap A' = 0 \text{ and } (A \oplus A')/A \subseteq_{\mathcal{C}e} D/A.$$

We claim that A is a $\mathcal{C}$-complement of $A' \cap B$ in B. Note that $A \cap (A' \cap B) = 0$. Let $X/A \subseteq B/A$ such that $[A + (A' \cap A)]/A \cap X/A = 0$ and $X/A \in \mathcal{C}$. Then

$$A = [A + (A' \cap B)] \cap X = (A + A') \cap (X \cap B).$$

Since $(X \cap B)/A = X/A \in \mathcal{C}$, $X = X \cap B = A$, i.e.

$$[A + (A' \cap B)]/A \subseteq_{\mathcal{C}e} B/A.$$

This proves that f is a $\mathcal{C}$-Compl-monomorphism.

(P5) Suppose that g and f are $\mathcal{C}$-Compl-epimorphisms. Consider the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & D/A & & \\
 & & & & \downarrow & & \\
0 \longrightarrow & A & \longrightarrow & B & \xrightarrow{f} & B/A & \longrightarrow 0 \\
 & & & & & \downarrow g & \\
 & & & & & B/D & \\
 & & & & & \downarrow & \\
 & & & & & 0. &
\end{array}$$

There exist submodules $D'/A \subseteq B/A$ and $A' \subseteq B$ such that

$$D/A \cap D'/A = 0,\ (D + D')/A/D/A \subseteq_{\mathcal{C}e} B/A/D/A \text{ and}$$

$$A \cap A' = 0,\ (A + A')/A \subseteq_{\mathcal{C}e} B/A.$$

We show that D is a $\mathcal{C}$-complement of $D' \cap A'$ in B. From 11.3 (ii) follows that $(D+D')/D \subseteq_{\mathcal{C}e} B/D$ (we will use this later). First observe that $D \cap D' = A$, then $D \cap D' \cap A' = A \cap A' = 0$. Let $X \subseteq B$ such that $[D + (D' \cap A')]/D \cap X/D = 0$ and $X/D \in \mathcal{C}$. Thus $[D + (D' \cap A')] \cap X = D$. Therefore

$$D + (D' \cap A' \cap X) = [D + (D' \cap A')] \cap X = D.$$

This implies that $D' \cap A' \cap X \subseteq D$, then $D' \cap A' \cap X \subseteq D \cap D' = A$. On the other hand $(A + A') \cap (D' \cap X) = A + (D' \cap A' \cap X) = A$. Thus $(A + A')/A \cap (D' \cap X)/A = 0$. Since $(A + A')/A \subseteq_{\mathcal{C}e} B/A$, $D' \cap X = A$ provided $(D' \cap X)/A$ is in $\mathcal{C}$. Suppose we have this, then $(D + D') \cap X = D + (D' \cap X) = D + A = D$. Therefore $(D + D')/D \cap X/D = 0$. Since $(D + D')/D \subseteq_{\mathcal{C}e} B/D$ and $X/D \in \mathcal{C}$, then $X = D$, i.e. gf is a $\mathcal{C}$-Compl-epimorphism. It remains only to prove that $(D' \cap X)/A$ is in $\mathcal{C}$. Observe that

$$(X \cap D')/A = (X \cap D')/(D \cap D') \simeq [(X \cap D') + D]/D \subseteq X/D.$$

By assumption, X/D is in $\mathcal{C}$, thus also every submodule. This completes the proof.

(P6) Let gf be a $\mathcal{C}$-Compl-epimorphism. With the notation above we prove that D/A is a $\mathcal{C}$-complement in B/A. By assumption, there is a submodule $D' \subseteq B$ such that

$$D \cap D' = 0 \text{ and } (D + D')/D \subseteq_{\mathcal{C}e} B/D.$$

We claim that D/A is a $\mathcal{C}$-complemet of $(D'+A)/A$ in B/A. Observe that $D/A \cap (D'+A)/A = 0$ and

$$[D/A + (D' + A)/A]/D/A = [(D + D')/A]/D/A \simeq$$

$$\simeq (D + D')/D \subseteq_{\mathcal{C}e} B/D \simeq B/A/D/A$$

which proves that g is a $\mathcal{C}$-Compl-epimorphism. □

As a corollary we obtain for $\mathcal{C} = \sigma[M]$ that complement submodules induce a proper class (see [27]).

12.3. The proper class Compl. The class of all short exact sequences in $\sigma[M]$

$$0 \to A \xrightarrow{f} B \to C \to 0$$

such that $\operatorname{Im} f$ is a complement in B is a proper class and it is denoted by Compl.

12.4. $\mathcal{C}$-closed submodules. A submodule $K \subseteq N$ is called **$\mathcal{C}$-closed** in N if whenever $K \subseteq_{\mathcal{C}e} H \subseteq N$, then $K = H$, i.e. K has no proper $\mathcal{C}$-essential extension in N.

If $\mathcal{C} = \sigma[M]$, then the $\mathcal{C}$-closed submodules are the closed submodules. It is well known that a submodule $K \subseteq N$ is a complement iff it is closed. Since a complement is a $\mathcal{C}$-complement we obtain that $\mathcal{C}$-complements always exist but they are not always $\mathcal{C}$-closed. However we can prove that if $K \subseteq N$ is $\mathcal{C}$-closed then K is a $\mathcal{C}$-complement in N. In order to do that we will need the following definition.

12.5. Definition. Let $K \subseteq N$. We say that the submodule $K \subseteq N$ has the property $(*)$ in N if for every $H \subseteq_{\mathcal{C}e} N$ with $K \subseteq H$ we have $H/K \subseteq_{\mathcal{C}e} N/K$.

12.6. Proposition. *Consider the following conditions for a submodule K of N:*

(a) *K is $\mathcal{C}$-closed in N.*

(b) *K has the property $(*)$ in N.*

(c) *K is a $\mathcal{C}$-complement in N.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Proof. (a) $\Rightarrow$ (b) Suppose K is $\mathcal{C}$-closed. Let $K \subseteq H \subseteq_{\mathcal{C}e} N$. If $X \subseteq N$ is such that $X/K \cap H/K = 0$ with $X/K \in \mathcal{C}$. Then $X \cap H = K$. Since H is a $\mathcal{C}$-essential submodule of N it follows from 11.3 (iii) that $K = X \cap H \subseteq_{\mathcal{C}e} X$. This implies by assuption that $K = X$. Thus $H/K \subseteq_{\mathcal{C}e} N/K$.

(b) $\Rightarrow$ (c) Suppose now that K has the property $(*)$ in N. Let K' be a complement of K in N. Then $K \oplus K' \subseteq_e N$ implying $K \oplus K' \subseteq_{\mathcal{C}e} N$. Then, by asumption, $(K \oplus K')/K \subseteq_{\mathcal{C}e} N/K$, i.e. K is a $\mathcal{C}$-complement in N. □

In the case $\mathcal{C} = \sigma[M]$ the conditions (a)-(c) are equivalent. For the general implication (c) $\Rightarrow$ (a) we need some maximality condition on K, which we don't know.

13 $\mathbf{tr}_{\mathcal{S}(\mathcal{C})}$-complements

13.1. The idempotent preradical $\mathbf{tr}_{\mathcal{S}(\mathcal{C})}$. Let $\mathcal{C}$ be a $\{q, s\}$-closed class of modules in $\sigma[M]$ and $\mathcal{S}(\mathcal{C})$ the class of simple modules in $\mathcal{C}$. For any $N \in \sigma[M]$ put

$$\operatorname{tr}_{\mathcal{S}(\mathcal{C})}(N) = \operatorname{Tr}(\mathcal{S}(\mathcal{C}), N) = \sum \{\operatorname{Im} f \mid f : S \to N,\ S \in \mathcal{S}(\mathcal{C})\}.$$

13.2. Properties of $\mathbf{tr}_{\mathcal{S}(\mathcal{C})}$.

(i) $\operatorname{tr}_{\mathcal{S}(\mathcal{C})}$ *is an idempotent preradical.*

(ii) $\operatorname{tr}_{\mathcal{S}(\mathcal{C})} \leq \operatorname{Soc}$.

(iii) $\operatorname{tr}_{\mathcal{S}(\mathcal{C})}(N) = 0$ *iff N has no nontrivial simple submodules in $\mathcal{C}$.*

Proof. (i) Follows from 2.11 (i).

(ii) Is clear.

(iii) For any $f : S \to N$ with $S \in \mathcal{S}(\mathcal{C})$, $\operatorname{Im} f \simeq S$. Thus if N has no nontrivial simple submodules in $\mathcal{C}$ we must have $f = 0$. Conversely if $S \subseteq N$ belongs to $\mathcal{C}$, then $S \subseteq \operatorname{tr}_{\mathcal{S}(\mathcal{C})}(N) = 0$. $\square$

13.3 Lemma. *Let $\mathcal{C}$ be a $\{q,s\}$-closed class of modules in $\sigma[M]$ and S a simple module. If S is subgenerated by $\mathcal{C}$, then $S \in \mathcal{C}$.*

Proof. Let S be a simple module subgenerated by $\mathcal{C}$ and $\widehat{S}$ ist M-injective hull. Consider the $\mathcal{C}$-generated module

$$L := \operatorname{Tr}(\mathcal{C}, \widehat{S}) = \sum\{f(C) \mid f : C \to \widehat{S},\, C \in \mathcal{C}\}.$$

Since S is $\mathcal{C}$-subgenerated, $S \subseteq L \subseteq \widehat{S}$. S is an essential submodule of $\widehat{S}$, therefore $S \subseteq_e L$. Since $S \neq 0$ and L is $\mathcal{C}$-generated, there is a nonzero morphism $f : C \to L$ for some $C \in \mathcal{C}$, hence $f(C) \cap S \neq 0$. It follows that $S \subseteq f(C)$, i.e. S is a subfactor module of C, therefore $S \in \mathcal{C}$. $\square$

13.4. $\operatorname{tr}_{\mathcal{S}(\mathcal{C})}$ and $\mathcal{C}$-essential submodules. *Let $\mathcal{C}$ be a $\{q,s\}$-closed class and N be a module in $\sigma[M]$. Then*

$$\operatorname{tr}_{\mathcal{S}(\mathcal{C})}(N) = \bigcap\{K \subseteq N \mid K \subseteq_{\mathcal{C}e} N\}.$$

Proof. Let S be submodule of N such that $S \in \mathcal{S}(\mathcal{C})$ and $K \subseteq_{\mathcal{C}e} N$. Then $S \cap K \neq 0$. S simple implies that $S \subseteq K$, hence we have the inclusion " $\subseteq$ ".
For the other inclusion note that

$$\bigcap\{K \subseteq N \mid K \subseteq_{\mathcal{C}e} N\} \subseteq \bigcap\{K \subseteq N \mid K \subseteq_e N\} = \operatorname{Soc}(N).$$

Therefore $\cap\{K \subseteq N \mid K \subseteq_{\mathcal{C}e} N\}$ is a semisimple module. Consider the submodule $U = \sum\{K \subseteq N \mid K \in \mathcal{C}\}$. Then we have clearly that $U \subseteq_{\mathcal{C}e} N$. Then $\cap\{K \subseteq N \mid K \subseteq_{\mathcal{C}e} N\} \subseteq U$. If S is a simple submodule of $\cap\{K \subseteq N \mid K \subseteq_{\mathcal{C}e} N\}$, then S is a submodule of a finite sum of submodules of U which are in $\mathcal{C}$. Therefore, by 13.3, S is in $\mathcal{C}$. This completes the proof. $\square$

We consider next the class τ-Compl for the idempotent preradical $\tau = \operatorname{tr}_{\mathcal{S}(\mathcal{C})}$ (see 4.4).

13.5. The proper class $\operatorname{tr}_{\mathcal{S}(\mathcal{C})}$-Compl. The class of all short exact sequences

$$0 \to A \xrightarrow{f} B \to C \to 0$$

such that $f(A)$ is a $\operatorname{tr}_{\mathcal{S}(\mathcal{C})}$-complement in B is a proper class and it is denoted by $\operatorname{tr}_{\mathcal{S}(\mathcal{C})}$-Compl. By definition

$$\operatorname{tr}_{\mathcal{S}(\mathcal{C})}\text{-Compl} = \pi^{-1}(\mathbb{T}_{\operatorname{tr}_{\mathcal{S}(\mathcal{C})}}) = \pi^{-1}\{N \in \sigma[M] \mid \operatorname{tr}_{\mathcal{S}(\mathcal{C})}(N) = N\}.$$

If $\mathcal{C} = \sigma[M]$, i.e. $\operatorname{tr}_{\mathcal{S}(\mathcal{C})} = \operatorname{Soc}$, we obtain:

13.6. The proper class Neat.

$$\text{Neat} = \pi^{-1}\{\text{all (semi-)simple modules in } \sigma[M]\} =$$

$$= \pi^{-1}\{N \in \sigma[M] \mid \operatorname{Soc}(N) = N\}.$$

For $\tau = \mathrm{Soc}$ we obtain from 4.4:

13.7. Characterization of neat submodules. *For a submodule $L \subseteq N$, the following are equivalent:*

(a) *The inclusion $L \to N$ is a* Neat-*monomorphism.*

(b) *There exists a submodule $L' \subseteq N$ such that*

  (i) $L \cap L' = 0$ *and*

  (ii) $L \oplus L'/L = \mathrm{Soc}(N/L)$.

(c) *There exists a submodule $L' \subseteq N$ such that*

  (i) $L \cap L' = 0$ *and*

  (ii) $L \oplus L'/L \supseteq \mathrm{Soc}(N/L)$.

If the conditions are satisfied, then L is called a Soc-**complement** in N.

13.8 Remark. In [21, pp. 39] Enochs defines a different concept of a neat morphism in the category of R-modules over an integral domain R. (We write E-neat for this concept to avoid confusion with the neat concept of 13.6). He calls a morphism $f : A \to B$ **E-neat** if for every proper submodule $D \subseteq F$ of any module F and any morphism $g : D \to A$ there exist a submodule $D' \subseteq F$ and a morphism $\overline{g} : D' \to B$ such that $D \subsetneq D'$ and $fg = \overline{g}i$, where $i : D \hookrightarrow D'$ is the canonical inclusion. Then he calls a submodule $A \subseteq B$ E-neat if the canonical inclusion $A \hookrightarrow B$ is a E-neat morphism. In [7, Example (4) pp. 4] Bowe points out that over any ring R a submodule $A \subseteq B$ of an R-module B is E-neat iff A has no proper essential extension in B, i.e. A is closed (= complement) in B. In [7, Theorem 1.2] Bowe gives some equivalent conditions for a morphism to be E-neat. In particular a monomorphism f is E-neat iff for every factorization $f = \beta\alpha$ with α an essential monomorphism, α is an isomorphism. The two concepts coincide iff R is a C-ring (see [27, Theorem 5]), i.e. the E-neat monomorphisms (= Compl-monomorphisms) and the Neat-monomorphisms coincide iff R is a C-ring.

The relationship between the proper classes $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}$-Compl and $\mathcal{C}$-Compl is analogous to the one between Compl and Neat = Soc-Compl.

13.9. Proposition. *Let $\mathcal{C}$ be a $\{q, s\}$-closed class of modules in $\sigma[M]$. For any module M*

$$\mathcal{C}\text{-Compl} \subseteq \mathrm{tr}_{\mathcal{S}(\mathcal{C})}\text{-Compl}.$$

Proof. If L is a $\mathcal{C}$-complement of N, then there is a submodule $L' \subseteq N$ such that $L \cap L' = 0$ and $L \oplus L'/L \subseteq_{\mathcal{C}e} N/L$. By 13.4, the submodule $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}(N/L)$ is the intersection of all $\mathcal{C}$-essential submodules of N/L, therefore $L \oplus L'/L \supseteq \mathrm{tr}_{\mathcal{S}(\mathcal{C})}(N/L)$, i.e. L is a $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}$-complement in N. □

In case $\mathcal{C} = \sigma[M]$ we obtain the result [57, corollary of Proposition 5]:

13.10. Proposition. *For any module M,*

$$\mathrm{Compl} \subseteq \mathrm{Neat}.$$

13.11. $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}$ and ($\mathcal{C}$-)complements. *Let $\mathcal{C}$ be a $\{q, s\}$-closed class of modules in $\sigma[M]$. Let K be a submodule of N.*

(i) *If $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}(N/K) \subseteq_e N/K$, then K is a $\mathcal{C}$-complement in N iff K is a complement.*

(ii) *If* $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}(N/K) \subseteq_{\mathcal{C}e} N/K$, *then* K *is a* $\mathcal{C}$*-complement in* N *iff* K *is a* $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}$*-complement.*

Proof. (i) It is clear that complements are always $\mathcal{C}$-complements. Conversely, if K is a $\mathcal{C}$-complement in N, then there exists a submodule $K' \subseteq N$ such that $K \cap K' = 0$ and $K \oplus K'/K \subseteq_e N/K$. Then $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}(N/K) \subseteq K \oplus K'/K$. This implies that $K \oplus K'/K \subseteq_e N/K$, i.e. K is a complement in N.

(ii) $\Rightarrow$) Follows from 13.9.
$\Leftarrow$) If K is a $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}$-complement then there exists a submodule $K' \subseteq N$ such that $K \cap K' = 0$ and $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}(N/K) \subseteq K \oplus K'/K$. Since $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}(N/K) \subseteq_{\mathcal{C}e} N/K$, then by 11.3 (i) $K \oplus K'/K \subseteq_{\mathcal{C}e} N/K$, i.e K is a $\mathcal{C}$-complement. □

13.12. $\mathcal{C}$-singular modules. A module N in $\sigma[M]$ is called **$\mathcal{C}$-singular** if $N \simeq B/A$ with $A \subseteq_{\mathcal{C}e} B \in \sigma[M]$.

The class of $\mathcal{C}$-singular modules is closed under submodules, direct sums and factor modules. A module $N \in \sigma[M]$ is called **non-$\mathcal{C}$-singular** if N has no nonzero $\mathcal{C}$-singular submodules.

13.13. $\mathcal{C}$-Compl-flats. *If the module* N *is non-*$\mathcal{C}$*-singular, then* N *is* $\mathcal{C}$-Compl-*flat.*

Proof. Let
$$0 \to A \to B \to N \to 0$$
be any short exact sequence. Suppose that A is not a $\mathcal{C}$-complement in B. Thus, by 12.6, A is not $\mathcal{C}$-closed in B. Then there exists $B' \subseteq B$ such that $A \subseteq_{\mathcal{C}e} B'$ and $A \neq B'$. By definition B'/A is $\mathcal{C}$-singular and $0 \neq B'/A \subseteq B/A \simeq N$. But this contradicts that N is non-$\mathcal{C}$-singular, thus A must be a $\mathcal{C}$-complement in B, i.e. N is $\mathcal{C}$-Compl-flat. □

As a corollary we obtain for $\mathcal{C} = \sigma[M]$:

13.14. Compl-flats. *If the module* $N \in \sigma[M]$ *is non-*M*-singular, then* N *is* Compl-*flat.*

Since closed submodules and complement submodules coincide, we have the following characterization of the Compl-divisible modules.

13.15. Compl-divisibles [22, 4.1.4]. *Let* N *be a module in* $\sigma[M]$. *The following are equivalent:*

(a) N *is* Compl-*divisible.*

(b) N *is a complement in every* M*-injective module* I *containing* N.

(c) N *is a complement in its* M*-injective hull* $\hat{N}$.

(d) N is M-injective.

The following lemma generalizes [12, 4.3].

13.16. Lemma. *Let* M *be a module.*

(i) *Every* $\mathcal{C}$*-singular module is a submodule of an* M*-generated* $\mathcal{C}$*-singular module.*

(ii) *Every finitely generated* $\mathcal{C}$*-singular module belongs to* $\sigma[M/L]$ *for some* $L \subseteq_{\mathcal{C}e} M$.

(iii) $\{M/K \mid K \subseteq_{\mathcal{C}e} M\}$ *is a generating set for the* M*-generated* $\mathcal{C}$*-singular modules.*

Proof. (i) Let $K \subseteq_{\mathcal{C}e} L \in \sigma[M]$. The M-injective hull $\widehat{L}$ of L is M-generated and $L \subseteq_e \widehat{L}$. Since essential submodules are $\mathcal{C}$-essential, by 11.3 (i), $K \subseteq_{\mathcal{C}e} \widehat{L}$. Hence $L/K \subseteq \widehat{L}/K$.

(ii) A finitely generated $\mathcal{C}$-singular module is of the form N/K for a finitely generated $N \in \sigma[M]$ and $K \subseteq_{\mathcal{C}e} N$. N is a $\mathcal{C}$-essential submodule of a finitely M-generated module $\widetilde{N}$, i.e. there exists an epimorphism $g : M^k \to \widetilde{N}$, $k \in \mathbb{N}$ and $U := g^{-1}(N)$ and $V := g^{-1}(K)$ are $\mathcal{C}$-essential submodules of M^k. Let $\varepsilon_i : M \to M^k$ be the canonical inclusions. Then $L := \cap_{i \leq k} \varepsilon_i^{-1}(V)$ is a $\mathcal{C}$-essential submodule of M and L^k is contained in the kernel of the composition

$$U \xrightarrow{g} N \to N/K.$$

This implies that $N/K \in \sigma[M/L]$.

(iii) follows from (ii). □

We call a module $N \in \sigma[M]$, **$\mathcal{C}$-semiartinian** if for every nonzero factor N/K of N, $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}(N/K) \neq 0$ (see [17, 3.12]).

13.17. $\mathcal{C}$-closed and $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}$-complement submodules. *For a module M, the following are equivalent:*

(a) *Every $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}$-complement of M is $\mathcal{C}$-closed.*

(b) *A submodule of M is $\mathcal{C}$-closed iff it is a $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}$-complement.*

(c) *For every $L \in \sigma[M]$, $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}$-complements of L are $\mathcal{C}$-closed.*

(d) *For every $\mathcal{C}$-essential submodule $U \subseteq M$, $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}(M/U) \neq 0$.*

(e) *Every $\mathcal{C}$-singular module is $\mathcal{C}$-semiartinian.*

Proof. (a) ⇔ (b) Is clear since $\mathcal{C}$-closed submodules are $\mathcal{C}$-complements and these are $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}$-complements.

(c) ⇒ (a) Is obvious.

(a) ⇒ (d) Let $U \subseteq_{\mathcal{C}e} M$ be a proper submodule. Then U is not $\mathcal{C}$-closed and hence not a $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}$-complement. Thus there is a morphism $g : S \to M/U$ where S is simple and in $\mathcal{C}$, that can not be extended to a morphism $S \to M$. In particular, this implies that $\mathrm{Im}\, g \neq 0$ and $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}(M/U) \neq 0$.

(d) ⇒ (e). Let $U \subseteq V \subseteq M$. If $U \subseteq_{\mathcal{C}e} M$, then $V \subseteq_{\mathcal{C}e} M$ and by (d), for every factor M/V of M/U, $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}(M/V) \neq 0$, i.e. M/U is $\mathcal{C}$-semiartinian. By 13.16, the set $\{M/U \mid U \subseteq_{\mathcal{C}e} M\}$ is a generating set for all the $\mathcal{C}$-singular M-generated modules and every $\mathcal{C}$-singular module is a submodule of an M-generated $\mathcal{C}$-singular module. Thus, since for every proper factor M/V of M/U, $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}(M/V) \neq 0$, this is also true for all $\mathcal{C}$-singular modules.

(e) ⇒ (c). Let $K \subseteq L$ be a $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}$-complement in L and suppose that it has a proper $\mathcal{C}$-essential extension $\overline{K} \subseteq L$. Then $\overline{K}/K$ is a $\mathcal{C}$-singular module and hence, by assumption, contains a simple submodule in $\mathcal{C}$, $S = N/K$ where $K \subseteq_{\mathcal{C}e} N \subseteq \overline{K}$. Since K is a $\mathrm{tr}_{\mathcal{S}(\mathcal{C})}$-complement, it follows that the morphism $N \to N/K = S$ splits. This contradicts $K \subseteq_{\mathcal{C}e} N$, showing that K is $\mathcal{C}$-closed in L. □

For $\mathcal{C} = \sigma[M]$ we obtain as a collorary.

13.18. Neat and closed submodules [1, 1.9]. *For a module M the following are equivalent:*

(a) *Every neat submodule of M is closed.*

(b) *A submodule of M is closed iff it is neat.*

(c) *For every $L \in \sigma[M]$, closed submodules of L are neat.*

(d) *For every essential submodule $U \subseteq M$,* $\mathrm{Soc}(M/U) \neq 0$.

(e) *Every M-singular module is semiartinian.*

In [62] Wu called a ring R, an **SAP-ring** if every simple R-module is absolutely pure, i.e. $\mathcal{P}$ure-divisible. We extend his characterization of SAP-rings (see [62, 3.1]) to those modules M such that every simple module in $\sigma[M]$ is $\mathcal{P}$-divisible for a projectively generated proper class $\mathcal{P}$ and we obtain a characterization of cosemisimple modules.

13.19. Proposition. *Let $\mathcal{P}$ be a proper class projectively generated by a class $\mathcal{Q}$ of modules in $\sigma[M]$. The following are equivalent:*

(a) *Every simple module is $\mathcal{P}$-divisible.*

(b) $\mathrm{Rad}(K) = \mathrm{Rad}(N) \cap K$ *for every submodule K of any module $N \in \sigma[M]$ such that $N/K \in \mathcal{Q}$.*

Proof. (a) $\Rightarrow$ (b) Let K be a submodule of N such that $N/K \in \mathcal{Q}$ and K_0 any maximal submodule of K. Consider the following pushout diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & N/K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & K/K_0 & \longrightarrow & N' & \longrightarrow & N/K & \longrightarrow & 0. \end{array}$$

Since K/K_0 is simple, by hypothesis, it is $\mathcal{P}$-divisible, hence the bottom sequence belongs to $\mathcal{P}$. Since $N/K \in \mathcal{Q}$ and $\mathcal{P}$ is projectively generated by $\mathcal{Q}$, we can find a morphism $f_0 : N \to K/K_0$ such that $f_0(K) = K/K_0$. Therefore $\mathrm{Ker}\, f_0 \cap K = K_0$. Thus for every maximal submodule K_0 of K we find a maximal submodule $N_0 = \mathrm{Ker}\, f_0$ of N such that $N_0 \cap K = K_0$. On the other hand if N_1 is a maximal submodule of N, then $K \subseteq N_1$ or $K + N_1 = N$. In the last case, $(K + N_1)/N_1 \simeq K/K \cap N_1$ is a simple module, therefore $K \cap N_1$ is a maximal submodule of K. It follows that $\mathrm{Rad}(K) = (\cap N_\alpha) \cap K = \mathrm{Rad}(N) \cap K$, with α running over the maximal submodules N_α of N.

(b) $\Rightarrow$ (a) Let S be a simple module,

$$0 \to S \to B \to C \to 0$$

a short exact sequence in $\sigma[M]$ and $f : Q \to C$ any morphism with $Q \in \mathcal{Q}$. Consider the following pullback diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & B' & \longrightarrow & Q & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow f & & \\ 0 & \longrightarrow & S & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

By hypothesis $\mathrm{Rad}(B') \cap S = \mathrm{Rad}(S) = 0$. Suppose that $\mathrm{Rad}(B') \neq 0$. Then there is a maximal submodule K of B' such that $B' = S \oplus K$. If $\mathrm{Rad}(B') = 0$, then S is not a small submodule of B', thus there is a proper submodule $H \subset B'$ such that $B' = S \oplus H$. In both cases the top sequence of the above diagram splits. Thus there is a morphism $Q \to B$ which lifts f, i.e. the bottom sequence belongs to $\mathcal{P}$. This proves that S is $\mathcal{P}$-divisible. $\square$

As a corollary for $\sigma[M] = R$-Mod and $\mathcal{Q}$ the class of finitely presented modules, i.e. $\mathcal{P}$ is the class of pure exact sequences, we obtain the characterization of SAP-rings by Wu:

13.20. SAP-rings [62, 3.1]. *Let R be a ring. The following are equivalent:*

(a) *R is a* SAP-*ring.*

(b) $\mathrm{Rad}(K) = \mathrm{Rad}(N) \cap K$ *for every submodule K of any module $N \in \sigma[M]$ such that N/K is finitely presented.*

Recall that a module M is **cosemisimple** iff every simple module in $\sigma[M]$ is M-injective, i.e. $\mathcal{S}$plit-divisible. Thus for $\mathcal{P} = \mathcal{S}$plit the proper class of split exact sequences which is projectively generated by $\sigma[M]$, we obtain:

13.21. Cosemisimple modules. *Let M be a module. The following are equivalent:*

(a) *M is cosemisimple.*

(b) $\mathrm{Rad}(K) = \mathrm{Rad}(N) \cap K$ *for every submodule K of any module $N \in \sigma[M]$, i.e.* Rad *is a hereditary preradical.*

Dually we prove when the simple modules are $\mathcal{P}$-flat for an injectively generated proper class.

13.22. Proposition. *Let $\mathcal{P}$ be a proper class injectively generated by a class $\mathcal{I}$ of modules in $\sigma[M]$. The following are equivalent:*

(a) *Every simple module is $\mathcal{P}$-flat.*

(b) $\mathrm{Soc}(N/K) = (\mathrm{Soc}(N) + K)/K$ *for every K submodule of any module N such that $K \in \mathcal{I}$.*

Proof. (a) $\Rightarrow$ (b) Let K be a submodule of N such that $K \in \mathcal{I}$ and H_0/K any simple submodule of N/K. Consider the following pullback diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & N' & \longrightarrow & H_0/K & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & N/K & \longrightarrow & 0. \end{array}$$

Since H_0/K is a simple module, by hypothesis, it is $\mathcal{P}$-flat. Thus the top sequence belongs to $\mathcal{P}$. Since $K \in \mathcal{I}$ and $\mathcal{P}$ is injectively generated by $\mathcal{I}$, we can find a morphism $f_0 : H_0/K \to N$ such that $(f_0(H_0/K) + K)/K = H_0/K$. Therefore for every simple submodule H_0/K of N/K we find a simple submodule $N_0 = f_0(H_0/K)$ of N such that $(N_0 + K)/K = H_0/K$. On the other hand, for every simple submodule H of N, the module $(H + K)/K$ is zero or isomorphic to it, hence simple. It follows that

$$\mathrm{Soc}(N/K) = ((\sum H_\alpha) + K)/K = (\mathrm{Soc}(N) + K)/K,$$

with H_α running over the simple submodules of N.

(b) $\Rightarrow$ (a) Let S be a simple module,

$$0 \to A \to B \to S \to 0$$

a short exact sequence and $f : A \to I$ any morphism with $I \in \mathcal{I}$. Consider the following pushout diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & S & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & I & \longrightarrow & B' & \longrightarrow & S & \longrightarrow & 0. \end{array}$$

By hypothesis, $S \simeq B'/I = \mathrm{Soc}(B'/I) = (\mathrm{Soc}(B') + I)/I$, thus $B' = \mathrm{Soc}(B') + I$. Therefore there is a simple submodule $T \subseteq B'$ such that $T \nsubseteq I$, i.e. $B' = T \oplus I$, otherwise every simple submodule of B' is contained in I, which would imply $I = B'$ contradicting the maximality of I. It follows that the bottom sequence splits, thus we can find a morphism $B \to I$ extending f, i.e. the top sequence belongs to $\mathcal{P}$. This proves that S is $\mathcal{P}$-flat. □

In [52] Ramamurthi called a ring R, an **SF-ring** if every simple R-module is flat. As a corollary of 13.22 for $\sigma[M] = R$-Mod and $\mathcal{P} = \mathcal{P}$ure, the class of pure exact sequences, which is injectively generated by the pure injective modules, we obtain:

13.23. SF-rings. *Let R be a ring. The following are equivalent:*

(a) *R is a* SF-*ring.*

(b) $\mathrm{Soc}(N/K) = (\mathrm{Soc}(N) + K)/K$ *for every K submodule of any module N such that K is pure injective.*

Recall that a module M is **semisimple** iff every simple module in $\sigma[M]$ is projective. Thus for $\mathcal{P} = \mathcal{S}$plit the proper class of split exact sequences, which is injectively generated by $\sigma[M]$, we obtain:

13.24. Semisimple modules. *Let M be a module. The following are equivalent:*

(a) *M is semisimple.*

(b) $\mathrm{Soc}(N/K) = (\mathrm{Soc}(N) + K)/K$ *for every K submodule of any module $N \in \sigma[M]$, i.e.* Soc *is a cohereditary preradical.*

Chapter 6

Cotorsion pairs related to proper classes

14 Cotorsion pairs, covers and envelopes

In this section we recall the definition and some basic properties of cotorsion pairs (also called cotorsion theories), covers and envelopes. Cotorsion pairs for abelian groups were introduced by Salce in [53]. They can be easily extended to abelian categories and also to more general categories (see [5]). In this section let $\mathbb{A}$ be an abelian category. We associate to each proper class in an abelian category $\mathbb{A}$ a cotorsion pair in $\mathbb{A}$, which defines a correspondence between the class of all proper classes of $\mathbb{A}$ and the class of all cotorsion pairs of $\mathbb{A}$. This correspondence is bijective if it is restricted to the class of Xu proper classes, which we introduce in this section.

14.1. Ext-orthogonal classes. Let $\mathcal{D}$ be a class of objects of an abelian category $\mathbb{A}$. We define the classes

$$^{\perp}\mathcal{D} = \{X \in \mathbb{A} \mid \mathrm{Ext}^1_{\mathbb{A}}(X, \mathcal{D}) = 0\},$$
$$\mathcal{D}^{\perp} = \{X \in \mathbb{A} \mid \mathrm{Ext}^1_{\mathbb{A}}(\mathcal{D}, X) = 0\}.$$

14.2. Cotorsion pairs. A pair $(\mathcal{F}, \mathcal{C})$ of classes of objects of $\mathbb{A}$ is called a **cotorsion pair** if

(i) $\mathcal{F} = {}^{\perp}\mathcal{C}$,

(ii) $\mathcal{C} = \mathcal{F}^{\perp}$.

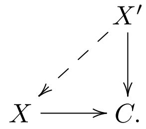
In a cotorsion pair $(\mathcal{F}, \mathcal{C})$, the class $\mathcal{C}$ is called the **cotorsion class** and $\mathcal{F}$ the **cotorsion free class**. The class $\mathcal{F}$ is closed under extensions, direct summands and contains all projective objects. The class $\mathcal{C}$ is closed under extensions, direct summands and contains all injective objects.
For any class of objects $\mathcal{D} \subseteq \mathbb{A}$ the pairs

$$({}^{\perp}\mathcal{D}, ({}^{\perp}\mathcal{D})^{\perp}) \quad \text{and} \quad ({}^{\perp}(\mathcal{D}^{\perp}), \mathcal{D}^{\perp})$$

are cotorsion pairs called, the cotorsion pair **generated** and **cogenerated** by the class $\mathcal{D}$ respectively. Examples of cotorsion pairs are $(\mathrm{Proj}(\mathbb{A}), \mathbb{A})$, $(\mathbb{A}, \mathrm{Inj}(\mathbb{A}))$. These are called the **trivial cotorsion pairs**. A non-trivial example is the pair $(\mathrm{Flat}, \mathrm{Cot})$, the flat cotorsion pair in R-Mod.

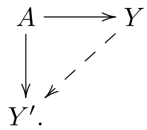
14.3. Covers and envelopes. Let $\mathcal{X}$ be a class of objects of $\mathbb{A}$. A morphism $X \to C$ with $X \in \mathcal{X}$ is an **$\mathcal{X}$-precover** of C if the map $\mathrm{Hom}_{\mathbb{A}}(X', X) \to \mathrm{Hom}_{\mathbb{A}}(X', C)$ is surjective for all

$X' \in \mathcal{X}$. This can be expressed by the diagram



An $\mathcal{X}$-precover $f : X \to C$ is an **$\mathcal{X}$-cover** of C if every $h \in \mathrm{End}(X)$ such that $fh = f$ is an automorphism. A class $\mathcal{X} \subseteq \mathbb{A}$ is a **precover class** (**cover class**) if every object $C \in \mathbb{A}$ has an $\mathcal{X}$-precover ($\mathcal{X}$-cover). An $\mathcal{X}$-precover of C, $f : X \to C$, is called **special** provided that f is an epimorphism and $\mathrm{Ker}\, f \in \mathcal{X}^{\perp}$.

Let $\mathcal{Y}$ be a class of objects of $\mathbb{A}$. A morphism $g : A \to Y$ with $Y \in \mathcal{Y}$ is a **$\mathcal{Y}$-preenvelope** of A if the map $\mathrm{Hom}_{\mathbb{A}}(Y, Y') \to \mathrm{Hom}_{\mathbb{A}}(A, Y')$ is surjective for all $Y' \in \mathcal{Y}$. This can be expressed by the diagram



A $\mathcal{Y}$-preenvelope $g : A \to Y$ is a **$\mathcal{Y}$-envelope** of A if every $h \in \mathrm{End}(Y)$ such that $hg = g$ is an automorphism. A class $\mathcal{Y} \subseteq \mathbb{A}$ is a **preenvelope class** (**envelope class**) if every object $A \in \mathbb{A}$ has a $\mathcal{Y}$-preenvelope ($\mathcal{Y}$-envelope). A $\mathcal{Y}$-preenvelope of A, $g : Y \to A$, is called **special** provided that g is a monomorphism and $\mathrm{Coker}\, g \in {}^{\perp}\mathcal{Y}$.

14.4. Lemma [5, V.3.3]. *Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair in an abelian category $\mathbb{A}$ with enough injectives and projectives. The following are equivalent:*

(a) *Every object in $\mathbb{A}$ has a special $\mathcal{F}$-precover.*

(b) *Every object in $\mathbb{A}$ has a special $\mathcal{C}$-preenvelope.*

If $(\mathcal{F}, \mathcal{C})$ satisfies the conditions in 14.4, then the cotorsion pair is called **complete**.

14.5. Wakamatsu's lemma. *Let $\mathcal{X} \subseteq \mathbb{A}$ be a class of objects closed under extensions.*

(i) *If $g : A \to X$ is an $\mathcal{X}$-envelope of A, then g is special.*

(ii) *If $f : X \to C$ is an $\mathcal{X}$-cover of C, then f is special.*

14.6. The lattice of cotorsion pairs. The class of all cotorsion pairs is partially ordered by inclusion of the second component, i.e.

$$(\mathcal{F}_1, \mathcal{C}_1) \leq (\mathcal{F}_2, \mathcal{C}_2) \quad \text{iff} \quad \mathcal{C}_1 \subseteq \mathcal{C}_2$$

or equivalently if and only if $\mathcal{F}_2 \subseteq \mathcal{F}_1$ (see [30]). The minimal element is $(\mathbb{A}, \mathrm{Inj}(\mathbb{A}))$ and the maximal $(\mathrm{Proj}(\mathbb{A}), \mathbb{A})$. The infimum of a family $\{(\mathcal{F}_i, \mathcal{C}_i)\}_I$ is given by

$$\wedge_I (\mathcal{F}_i, \mathcal{C}_i) = ({}^{\perp}(\cap_I \mathcal{C}_i), \cap_I \mathcal{C}_i)$$

and the supremum by

$$\vee_I (\mathcal{F}_i, \mathcal{C}_i) = (\cap_I \mathcal{F}_i), (\cap_I \mathcal{F}_i)^{\perp}).$$

14.7. The maps Φ and Ψ. Let $\mathcal{P}$ be a proper class of short exact sequences in an abelian category $\mathbb{A}$. We define

$$\Phi(\mathcal{P}) = ({}^{\perp}(\mathrm{Flat}(\mathcal{P})^{\perp}), \mathrm{Flat}(\mathcal{P})^{\perp})$$

which assings to $\mathcal{P}$ the cotorsion pair cogenerated by $\mathrm{Flat}(\mathcal{P})$ (see 3.12).
Let $\mathcal{R}$ be a proper class of short exact sequences in $\mathbb{A}$. We define

$$\Psi(\mathcal{R}) = ({}^{\perp}\mathrm{Div}(\mathcal{R}), ({}^{\perp}\mathrm{Div}(\mathcal{R}))^{\perp})$$

which assings to $\mathcal{R}$ the cotorsion pair generated by $\mathrm{Div}(\mathcal{R})$.

14.8. Lemma. *Let $\mathcal{P}$ be an injectively generated proper class in $\mathbb{A}$. Then*

$$\mathrm{Flat}(\mathcal{P}) = {}^{\perp}(\mathrm{Flat}(\mathcal{P})^{\perp}).$$

Proof. To prove this, note that every $\mathcal{P}$-injective object I is in $\mathrm{Flat}(\mathcal{P})^{\perp}$, since for a $\mathcal{P}$-flat object Q we have

$$\mathrm{Ext}^1_{\mathbb{A}}(Q, I) = \mathrm{Ext}^1_{\mathcal{P}}(Q, I) = 0.$$

Now if $Q \in {}^{\perp}(\mathrm{Flat}(\mathcal{P})^{\perp})$, then $\mathrm{Ext}^1_{\mathbb{A}}(Q, I) = 0$ for all $I \in \mathrm{Flat}(\mathcal{P})^{\perp}$ and in particular this is true for all $\mathcal{P}$-injective objects I. Thus in view of 4.7, Q is $\mathcal{P}$-flat, i.e. $\mathrm{Flat}(\mathcal{P}) = {}^{\perp}(\mathrm{Flat}(\mathcal{P})^{\perp})$. □

With a dual argument we obtain.

14.9. Lemma. *Let $\mathcal{R}$ be a projectively generated proper class in $\mathbb{A}$. Then*

$$\mathrm{Div}(\mathcal{R}) = ({}^{\perp}\mathrm{Div}(\mathcal{R}))^{\perp}.$$

This implies that for every injectively generated proper classs $\mathcal{P}$, $\mathrm{Flat}(\mathcal{P})$ is the first term of a cotorsion pair, namely $\Phi(\mathcal{P}) = (\mathrm{Flat}(\mathcal{P}), \mathrm{Flat}(\mathcal{P})^{\perp})$ and for a projectively generated proper class $\mathcal{R}$, $\mathrm{Div}(\mathcal{R})$ is the second term of a cotorsion pair $\Psi(\mathcal{R}) = ({}^{\perp}\mathrm{Div}(\mathcal{R}), \mathrm{Div}(\mathcal{R}))$.

14.10. Φ and Ψ properties. *Let $\mathcal{P}_1, \mathcal{P}_2$ and $\{\mathcal{P}_i\}_I$ be injectively generated proper classes and $\mathcal{R}_1, \mathcal{R}_2$ and $\{\mathcal{R}_i\}_I$ projectively generated proper classes in $\mathbb{A}$.*

(i) *If $\mathcal{P}_1 \subseteq \mathcal{P}_2$, then $\Phi(\mathcal{P}_1) \supseteq \Phi(\mathcal{P}_2)$.*

(ii) *If $\mathcal{R}_1 \subseteq \mathcal{R}_2$, then $\Psi(\mathcal{R}_1) \subseteq \Psi(\mathcal{R}_2)$.*

(iii) $\Phi(\wedge_I \mathcal{P}_i) = \vee_I \Phi(\mathcal{P}_i)$.

(iv) $\Psi(\wedge_I \mathcal{R}_i) = \wedge_I \Phi(\mathcal{R}_i)$.

(v) $\Phi(\mathcal{S}\mathrm{plit}) = (\mathrm{Proj}(\mathbb{A}), \mathbb{A}) = \Psi(\mathcal{A}\mathrm{bs})$.

(vi) $\Phi(\mathcal{A}\mathrm{bs}) = (\mathbb{A}, \mathrm{Inj}(\mathbb{A})) = \Psi(\mathcal{S}\mathrm{plit})$.

(vii) $\Phi(\mathcal{P}\mathrm{ure}) = (\mathrm{Flat}, \mathrm{Cot})$.

Proof. (i) $\mathcal{P}_1 \subseteq \mathcal{P}_2$ implies $\mathrm{Flat}(\mathcal{P}_1) \subseteq \mathrm{Flat}(\mathcal{P}_2)$, hence

$$\Phi(\mathcal{P}_1) = (\mathrm{Flat}(\mathcal{P}_1), \mathrm{Flat}(\mathcal{P}_1)^{\perp}) \supseteq (\mathrm{Flat}(\mathcal{P}_2), \mathrm{Flat}(\mathcal{P}_2)^{\perp}) = \Phi(\mathcal{P}_2).$$

(ii) $\mathcal{R}_1 \subseteq \mathcal{R}_2$ implies $\mathrm{Div}(\mathcal{R}_1) \subseteq \mathrm{Div}(\mathcal{R}_2)$, hence

$$\Psi(\mathcal{R}_1) = ({}^{\perp}\mathrm{Div}(\mathcal{R}_1), \mathrm{Div}(\mathcal{R}_1)) \subseteq ({}^{\perp}\mathrm{Div}(\mathcal{R}_2), \mathrm{Div}(\mathcal{R}_2)) = \Psi(\mathcal{R}_2).$$

(iii) By definition,

$$\Phi(\wedge_I \mathcal{P}_i) = \Phi(\cap_I \mathcal{P}_i) = (\mathrm{Flat}(\cap_I \mathcal{P}_i), \mathrm{Flat}(\cap_I \mathcal{P}_i)^{\perp}) \text{ and}$$

$$\vee_I \Phi(\mathcal{P}_i) = \vee_I (\mathrm{Flat}(\mathcal{P}_i), \mathrm{Flat}(\mathcal{P}_i)^{\perp}) = (\cap_I \mathrm{Flat}(\mathcal{P}_i), (\cap_I \mathrm{Flat}(\mathcal{P}_i))^{\perp}).$$

We show that $\mathrm{Flat}(\cap_I \mathcal{P}_i) = \cap_I \mathrm{Flat}(\mathcal{P}_i)$. Let $X \in \mathrm{Flat}(\cap_I \mathcal{P}_i)$ and $J_i \in \mathrm{Inj}(\mathcal{P}_i)$ for $i \in I$. Then

$$\mathrm{Ext}^1_{\mathbb{A}}(X, J_i) = \mathrm{Ext}^1_{\cap \mathcal{P}_i}(X, J_i) \subseteq \mathrm{Ext}^1_{\mathcal{P}_i}(X, J_i) = 0.$$

From 4.7 it follows that $X \in \mathrm{Flat}(\mathcal{P}_i)$. Since i was arbitrary, $X \in \cap_I \mathrm{Flat}(\mathcal{P}_i)$. Conversely, let $X \in \cap_I \mathrm{Flat}(\mathcal{P}_i)$ and $J \in \mathrm{Inj}(\cap_I \mathcal{P}_i)$. Then, for all $i \in I$,

$$\mathrm{Ext}^1_{\mathbb{A}}(X, J) = \mathrm{Ext}^1_{\mathcal{P}_i}(X, J), \text{ therefore}$$

$$\mathrm{Ext}^1_{\mathbb{A}}(X,J) = \cap_I \mathrm{Ext}^1_{\mathcal{P}_i}(X,J) = \mathrm{Ext}^1_{\cap \mathcal{P}_i}(X,J) = 0.$$

Again from 4.7 it follows that $X \in \mathrm{Flat}(\cap_I \mathcal{P}_i)$. This proves (iii).

(iv) By definition,

$$\Psi(\wedge_I \mathcal{R}_i) = \Psi(\cap_I \mathcal{R}_i) = ({}^{\perp}\mathrm{Div}(\cap_I \mathcal{R}_i), \mathrm{Div}(\cap_I \mathcal{R}_i)) \text{ and}$$

$$\wedge_I \Psi(\mathcal{R}_i) = \wedge_I ({}^{\perp}\mathrm{Div}(\mathcal{R}_i), \mathrm{Div}(\mathcal{R}_i)) = ({}^{\perp}(\cap_I \mathrm{Div}(\mathcal{R}_i)), \cap_I \mathrm{Div}(\mathcal{R}_i)).$$

We show that $\mathrm{Div}(\cap_I \mathcal{R}_i) = \cap_I \mathrm{Div}(\mathcal{R}_i)$. Let $X \in \mathrm{Div}(\cap_I \mathcal{R}_i)$ and $P_i \in \mathrm{Proj}(\mathcal{R}_i)$ for $i \in I$. Then

$$\mathrm{Ext}^1_{\mathbb{A}}(P_i, X) = \mathrm{Ext}^1_{\cap \mathcal{R}_i}(P_i, X) \subseteq \mathrm{Ext}^1_{\mathcal{R}_i}(P_i, X) = 0.$$

It follows from 4.2 that $X \in \mathrm{Div}(\mathcal{R}_i)$. Since i was arbitrary, $X \in \cap_I \mathrm{Div}(\mathcal{R}_i)$. Conversely, let $X \in \cap_I \mathrm{Div}(\mathcal{R}_i)$ and $P \in \mathrm{Proj}(\cap_I \mathcal{R}_i)$. Then, for all $i \in I$,

$$\mathrm{Ext}^1_{\mathbb{A}}(P, X) = \mathrm{Ext}^1_{\mathcal{R}_i}(P, X), \text{ therefore}$$

$$\mathrm{Ext}^1_{\mathbb{A}}(P, X) = \cap_I \mathrm{Ext}^1_{\mathcal{R}_i}(P, X) = \mathrm{Ext}^1_{\cap \mathcal{R}_i}(P, X) = 0.$$

Again from 4.2 it follows that $X \in \mathrm{Div}(\cap_I \mathcal{R}_i)$. This proves (iv).

(v)-(vii) are clear. □

14.11. Proposition. *Every cotorsion pair in* $\mathbb{A}$ *is of the form* $(\mathrm{Flat}(\mathcal{P}), \mathrm{Flat}(\mathcal{P})^{\perp})$ *for an injectively generated proper class* $\mathcal{P}$.

Proof. Let $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair. Set $\mathcal{P} = \iota^{-1}(\mathcal{Y})$. Then we need to show that $\mathrm{Flat}(\mathcal{P}) = \mathcal{X}$. Take $X \in \mathcal{X}$ and any short exact sequence

$$E : 0 \to A \to B \to X \to 0.$$

To prove that $E \in \mathcal{P}$ consider any morphism $A \to Y$ with $Y \in \mathcal{Y}$. By building a pushout we obtain the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & B' & \longrightarrow & X & \longrightarrow & 0.
\end{array}$$

Thus the lower row splits and we can find a morphism $B \to Y$ which extends $A \to Y$. This implies that $E \in \mathcal{P}$ and therefore $X \in \mathrm{Flat}(\mathcal{P})$.
Conversely, let $P \in \mathrm{Flat}(\mathcal{P})$. Then any short exact sequence

$$0 \to Y \to B \to P \to 0$$

with $Y \in \mathcal{Y}$ belongs to $\mathcal{P}$ and the elements of $\mathcal{Y}$ are $\mathcal{P}$-injectives, thus the sequence splits, i.e. $P \in \mathcal{X}$. □

14.12. Proposition. *Every cotorsion pair in* $\mathbb{A}$ *is of the form* $({}^{\perp}\mathrm{Div}(\mathcal{R}), \mathrm{Div}(\mathcal{R}))$ *for a projectively generated proper class* $\mathcal{R}$.

Proof. Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair. Set $\mathcal{R} = \pi^{-1}(\mathcal{F})$. Dual to 14.11 one proves that $(\mathcal{F}, \mathcal{C}) = ({}^{\perp}\mathrm{Div}(\mathcal{R}), \mathrm{Div}(\mathcal{R}))$. □

14.13. The maps $\widetilde{\Phi}$ and $\widetilde{\Psi}$. We define for every cotorsion pair $(\mathcal{F}, \mathcal{C})$ in $\mathbb{A}$ the maps

$$\widetilde{\Phi}(\mathcal{F}, \mathcal{C}) = \iota^{-1}(\mathcal{C}) \text{ and}$$

$$\widetilde{\Psi}(\mathcal{F}, \mathcal{C}) = \pi^{-1}(\mathcal{F}).$$

$\widetilde{\Phi}$ assings to each cotorsion pair an injectively generated proper class and $\widetilde{\Psi}$ a projectively generated proper class.

14.14. Properties of $\widetilde{\Phi}$ and $\widetilde{\Psi}$. *Let $(\mathcal{F}_1, \mathcal{C}_1)$ and $(\mathcal{F}_2, \mathcal{C}_2)$ be cotorsion pairs in $\mathbb{A}$.*

(i) *If $(\mathcal{F}_1, \mathcal{C}_1) \subseteq (\mathcal{F}_2, \mathcal{C}_2)$, then $\widetilde{\Phi}(\mathcal{F}_1, \mathcal{C}_1) \supseteq \widetilde{\Phi}(\mathcal{F}_2, \mathcal{C}_2)$.*

(ii) *If $(\mathcal{F}_1, \mathcal{C}_1) \subseteq (\mathcal{F}_2, \mathcal{C}_2)$, then $\widetilde{\Psi}(\mathcal{F}_1, \mathcal{C}_1) \subseteq \widetilde{\Psi}(\mathcal{F}_2, \mathcal{C}_2)$.*

(iii) $\widetilde{\Phi}(\vee_I(\mathcal{F}_i, \mathcal{C}_i)) \subseteq \wedge_I \widetilde{\Phi}(\mathcal{F}_i, \mathcal{C}_i)$.

(iv) $\widetilde{\Psi}(\wedge_I(\mathcal{F}_i, \mathcal{C}_i)) \subseteq \wedge_I \widetilde{\Psi}(\mathcal{F}_i, \mathcal{C}_i)$.

(v) $\widetilde{\Phi}(\mathrm{Proj}(\mathbb{A}), \mathbb{A}) = \mathcal{S}\mathrm{plit} = \widetilde{\Psi}(\mathbb{A}, \mathrm{Inj}(\mathbb{A}))$.

(vi) $\widetilde{\Phi}(\mathbb{A}, \mathrm{Inj}(\mathbb{A})) = \mathcal{A}\mathrm{bs} = \widetilde{\Psi}(\mathrm{Proj}(\mathbb{A}), \mathbb{A})$.

Proof. (i) $(\mathcal{F}_1, \mathcal{C}_1) \subseteq (\mathcal{F}_2, \mathcal{C}_2)$ iff $\mathcal{C}_1 \subseteq \mathcal{C}_2$. Therefore

$$\widetilde{\Phi}(\mathcal{F}_2, \mathcal{C}_2) = \iota^{-1}(\mathcal{C}_2) \subseteq \iota^{-1}(\mathcal{C}_1) = \widetilde{\Phi}(\mathcal{F}_1, \mathcal{C}_1).$$

(ii) $(\mathcal{F}_1, \mathcal{C}_1) \subseteq (\mathcal{F}_2, \mathcal{C}_2)$ iff $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Therefore

$$\widetilde{\Psi}(\mathcal{F}_1, \mathcal{C}_1) = \pi^{-1}(\mathcal{F}_1) \subseteq \pi^{-1}(\mathcal{F}_2) = \widetilde{\Psi}(\mathcal{F}_2, \mathcal{C}_2).$$

(iii) By definition, $\widetilde{\Phi}(\vee_I(\mathcal{F}_i, \mathcal{C}_i)) = \widetilde{\Phi}(\cap_I \mathcal{F}_i, (\cap_I \mathcal{F}_i)^{\perp}) = \iota^{-1}((\cap_I \mathcal{F}_i)^{\perp})$ and

$$\wedge_I \widetilde{\Phi}(\mathcal{F}_i, \mathcal{C}_i) = \wedge_I \iota^{-1}(\mathcal{C}_i) = \cap_I \iota^{-1}(\mathcal{C}_i).$$

We must show that $\iota^{-1}((\cap_I \mathcal{F}_i)^{\perp}) \subseteq \cap_I \iota^{-1}(\mathcal{C}_i)$. Note that for each $i \in I$,

$$\mathcal{C}_i = \mathcal{F}_i^{\perp} \subseteq (\cap_I \mathcal{F}_i)^{\perp}.$$

Thus $\iota^{-1}((\cap_I \mathcal{F}_i)^{\perp}) \subseteq \iota^{-1}(\mathcal{F}_i^{\perp}) = \iota^{-1}(\mathcal{C}_i)$. Therefore $\iota^{-1}((\cap_I \mathcal{F}_i)^{\perp}) \subseteq \cap_I \iota^{-1}(\mathcal{C}_i)$.

(iv) By definition,

$$\widetilde{\Psi}(\wedge_I(\mathcal{F}_i, \mathcal{C}_i)) = \widetilde{\Psi}({}^{\perp}(\cap_I \mathcal{C}_i), \cap_I \mathcal{C}_i) = \pi^{-1}({}^{\perp}(\cap_I \mathcal{C}_i)) \text{ and}$$

$$\wedge_I \widetilde{\Psi}(\mathcal{F}_i, \mathcal{C}_i) = \wedge_I \pi^{-1}(\mathcal{F}_i) = \cap_I \pi^{-1}(\mathcal{F}_i).$$

We must show $\pi^{-1}({}^{\perp}(\cap_I \mathcal{C}_i)) \subseteq \cap_I \pi^{-1}(\mathcal{F}_i)$. For each $i \in I$

$$\mathcal{F}_i = {}^{\perp}\mathcal{C}_i \subseteq {}^{\perp}(\cap_I \mathcal{C}_i).$$

Thus $\pi^{-1}({}^{\perp}(\cap_I \mathcal{C}_i)) \subseteq \pi^{-1}({}^{\perp}\mathcal{C}_i) = \pi^{-1}(\mathcal{F}_i)$. Hence $\pi^{-1}({}^{\perp}(\cap_I \mathcal{C}_i)) \subseteq \cap_I \pi^{-1}(\mathcal{F}_i)$.

(v) and (vi) are easy to verify. □

14.15. Proposition. *Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair, $\mathcal{P} = \iota^{-1}(\mathcal{I})$ a proper class injectively generated by a class of objects $\mathcal{I}$ and $\mathcal{R} = \pi^{-1}(\mathcal{Q})$ a proper class projectively generated by a class of objects $\mathcal{Q}$.*

(i) $\Phi(\widetilde{\Phi}(\mathcal{F},\mathcal{C})) = (\mathcal{F},\mathcal{C})$.

(ii) $\Psi(\widetilde{\Psi}(\mathcal{F},\mathcal{C})) = (\mathcal{F},\mathcal{C})$.

(iii) $\widetilde{\Phi}(\Phi(\mathcal{P})) \subseteq \mathcal{P}$.

(iv) $\widetilde{\Psi}(\Psi(\mathcal{R})) \subseteq \mathcal{R}$.

Proof. (i) $\Phi(\widetilde{\Phi}(\mathcal{F},\mathcal{C})) = \Phi(\iota^{-1}(\mathcal{C})) = (\mathrm{Flat}(\iota^{-1}(\mathcal{C})), (\mathrm{Flat}(\iota^{-1}(\mathcal{C}))^{\perp})$. By the proof of 14.11, $\mathcal{F} = \mathrm{Flat}(\iota^{-1}(\mathcal{C}))$.

(ii) $\Psi(\widetilde{\Psi}(\mathcal{F},\mathcal{C})) = \Psi(\pi^{-1}(\mathcal{F})) = ({}^{\perp}\mathrm{Div}(\pi^{-1}(\mathcal{F})), \mathrm{Div}(\pi^{-1}(\mathcal{F}))$. By the proof of 14.12, $\mathcal{C} = \mathrm{Div}(\pi^{-1}(\mathcal{F}))$.

(iii) $\widetilde{\Phi}(\Phi(\mathcal{P})) = \widetilde{\Phi}(\mathrm{Flat}(\mathcal{P}), \mathrm{Flat}(\mathcal{P})^{\perp}) = \iota^{-1}(\mathrm{Flat}(\mathcal{P})^{\perp})$. Since $\mathcal{I} \subseteq \mathrm{Flat}(\mathcal{P})^{\perp}$, $\iota^{-1}(\mathrm{Flat}(\mathcal{P})^{\perp}) \subseteq \iota^{-1}(\mathcal{I}) = \mathcal{P}$.

(iv) $\widetilde{\Psi}(\Psi(\mathcal{R})) = \widetilde{\Psi}({}^{\perp}\mathrm{Div}(\mathcal{R}), \mathrm{Div}(\mathcal{R})) = \pi^{-1}({}^{\perp}\mathrm{Div}(\mathcal{R}))$. Since $\mathcal{Q} \subseteq {}^{\perp}\mathrm{Div}(\mathcal{R})$, $\pi^{-1}({}^{\perp}\mathrm{Div}(\mathcal{R})) \subseteq \pi^{-1}(\mathcal{Q}) = \mathcal{R}$. □

In [34] Herzog and Rothmaler call a ring R a **Xu ring** if every cotorsion module is pure injective, i.e. $\mathrm{Inj}(\mathcal{P}\mathrm{ure}) = \mathrm{Flat}^{\perp}$. This suggests the following definition.

14.16. Xu proper classes. An injectively generated proper class $\mathcal{P}$ in $\mathbb{A}$ is called a **Xu proper class** if

$$\mathrm{Inj}(\mathcal{P}) = \mathrm{Flat}(\mathcal{P})^{\perp}.$$

For example, the proper class $\mathcal{A}$bs, the class of all short exact seqences in an abelian category $\mathbb{A}$, is a Xu proper class. The following proposition is a generalization of [63, 3.5.1] where $\mathcal{P} = \mathcal{P}\mathrm{ure}$ in R-Mod. There it was proved when $\mathcal{P}$ure is a Xu proper class.

14.17. Characterization of Xu proper classes. *Let $\mathcal{P}$ be an injectively generated proper class and suppose that every object N in $\mathbb{A}$ has a $\mathcal{P}$-injective hull $\mathcal{P}(N)$. The following are equivalent:*

(a) *$\mathcal{P}$ is a Xu proper class.*

(b) $\mathrm{Inj}(\mathcal{P})$ *is closed under extensions.*

(c) *For every object N, $\mathcal{P}(N)/N \in \mathrm{Flat}(\mathcal{P})$.*

Proof. The proof in [63, 3.5.1] also holds here. □

14.18. Lemma. *If $\mathcal{P}$ is a Xu proper class in $\mathbb{A}$, then $\widetilde{\Phi}(\Phi(\mathcal{P})) = \mathcal{P}$.*

Proof. Note that $\mathcal{P} = \iota^{-1}(\mathcal{I}) = \iota^{-1}(\mathrm{Inj}(\mathcal{P})) = \iota^{-1}(\mathrm{Flat}(\mathcal{P})^{\perp}) = \widetilde{\Phi}(\Phi(\mathcal{P}))$. □

14.19. Lemma. *For every cotorsion pair $(\mathcal{F},\mathcal{C})$, $\widetilde{\Phi}(\mathcal{F},\mathcal{C})$ is a Xu proper class.*

Proof. By definition $\widetilde{\Phi}(\mathcal{F},\mathcal{C}) = \iota^{-1}(\mathcal{C})$. Note that $\mathrm{Flat}(\iota^{-1}(\mathcal{C}))^{\perp} = \mathcal{F}^{\perp} = \mathcal{C} \subseteq \mathrm{Inj}(\iota^{-1}(\mathcal{C}))$ where the first equality follows from 14.11. By 4.7, $\mathrm{Inj}(\iota^{-1}(\mathcal{C})) \subseteq \mathrm{Flat}(\iota^{-1}(\mathcal{C}))^{\perp}$. Therefore $\mathrm{Flat}(\iota^{-1}(\mathcal{C}))^{\perp} = \mathrm{Inj}(\iota^{-1}(\mathcal{C}))$, i.e. $\iota^{-1}(\mathcal{C})$ is a Xu proper class. □

14.20. Proposition. *Φ is a bijective correspondence between the class of all Xu proper classes in $\mathbb{A}$ and the class of all cotorsion pairs in $\mathbb{A}$.*

Proof. From 14.15 (i), 14.18 and 14.19 follows that Φ and $\widetilde{\Phi}$ are inverse correspondences. □

We recall a corollary from [18] to see how the properties of $\mathcal{P}$ imply the existence of Flat($\mathcal{P}$)-covers.

14.21. Proposition [18, Corollary 10]. *Let R be a ring and $\mathcal{C}$ a class of pure injective modules. Then every module has a ${}^{\perp}\mathcal{C}$-cover.*

14.22. Proposition. *Let $\mathcal{P}$ be an injectively generated proper class in R-*Mod*. If $\mathcal{P}$ is inductively closed, then every module has a* Flat($\mathcal{P}$)*-cover.*

Proof. Let $\mathcal{P}$ be an inductively closed proper class in R-Mod, then $\mathcal{P}$ure $\subseteq \mathcal{P}$. This implies that $\mathrm{Inj}(\mathcal{P}) \subseteq \mathrm{Inj}(S) =$ pure injectives. Since ${}^{\perp}\mathrm{Inj}(\mathcal{P}) = \mathrm{Flat}(\mathcal{P})$, then the result follows from 14.21. □

In this context let us recall the following theorem due to Bican et al. in [6].

14.23. Proposition [6, Theorem 6]. *Let $\mathcal{P}$ be a proper class projectively generated by a set of finitely presented modules over any ring. Then every module has a* Flat($\mathcal{P}$)*-cover.*

Note that both 14.21 and 14.23 imply the existence of flat covers for all modules over any ring.

14.24 Example. Let $R = \mathbb{Z}$. Then every $\mathbb{Z}$-module has a Flat(Compl)-cover. The class of Flat(Compl) abelian groups coincide with the class of flat abelian groups (= torsionfree). Thus we obtain the known result that every abelian group has a torsionfree-cover (see [19]).

Proof. We show that the proper class Compl in $\mathbb{Z}$-Mod satisfies the hypothesis of Proposition 14.22. By [46, 4.1.1] the proper class Compl is injectively generated by the simple groups $\mathbb{Z}/\mathbb{Z}_p$, p prime. By [46, 4.4.4] the proper class Compl is the inductive closure of the proper class Suppl, thus Compl is inductively closed (see Definition 3.6). Therefore the result follows from 14.22. An abelian group G belongs to Flat(Compl) iff $G \simeq F/K$, with F a free abelian group and K a complement in F (see 3.12). On the other hand, an abelian group H is flat iff $H \simeq D/L$ with D a free abelian group and L a pure subgroup of D. The complement subgroups are precisely the neat subgroups (see [46, 4.1.1]). Since pure subgroups are neat and a neat subgroup of a torsionfree abelian group (in particular a free abelian group) is pure (see [38, Theorem 14]), the classes Flat(Compl) and Flat coincide. □

Let τ be a left exact radical of $\sigma[M]$ and $\mathbb{F}_\tau$ the torsionfree class. We consider the cotorsion pair $(\mathrm{Flat}(\mathcal{P}), \mathrm{Flat}(\mathcal{P})^{\perp})$ associated to the proper class $\mathcal{P} = \iota^{-1}(\mathbb{F}_\tau) = \tau$-Suppl. Recall that a module $X \in \sigma[M]$ is called τ-semiperfect if every factor module of X has a projective τ-cover see [1, 2.15].

14.25. Proposition. *Let τ be a left exact radical and $\mathcal{P} = \iota^{-1}(\mathbb{F}_\tau)$. If every $\mathcal{P}$-injective module is τ-semiperfect, then $(\mathrm{Flat}(\mathcal{P}), \mathrm{Flat}(\mathcal{P})^{\perp})$ is a complete cotorsion pair.*

Proof. We prove that every module $X \in \sigma[M]$ has a special $\mathrm{Flat}(\mathcal{P})^{\perp}$-preenvelope. We have the following sequence in $\mathcal{P}$,

$$0 \to X \xrightarrow{i} \widehat{X} \oplus X/\tau(X) \to \mathrm{Coker}\ i \to 0$$

where i is defined as in the proof of 6.1 with $\widehat{X} \oplus X/\tau(X)$ a $\mathcal{P}$-injective module. Thus $\widehat{X} \oplus X/\tau(X) \in \mathrm{Flat}(\mathcal{P})^{\perp}$. In order for this sequence to be a special $\mathrm{Flat}(\mathcal{P})^{\perp}$-preenvelope we must show $\mathrm{Coker}\ i \in \mathrm{Flat}(\mathcal{P})$. Let

$$0 \to K \to P \to \mathrm{Coker}\ i \to 0$$

be the τ-projective cover of Coker i. This sequence belongs to $\mathcal{P}$ (see 6.13). Consider any short exact sequence E ending at Coker i and the following pullback diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & K & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & P & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
E:0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & \operatorname{Coker} i & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Since P is projective the upper row splits. It follows from 3.1 (P6) that the morphism $B \to$ Coker i is a $\mathcal{P}$-epimorphism, thus $E \in \mathcal{P}$. This proves that Coker $i \in \operatorname{Flat}(\mathcal{P})$. □

15 $\mathcal{P}$-cotorsion pairs

In [37] Hovey introduced the notion of cotorsion pairs relative to a proper class. They are defined as complete orthogonal classes with respect to the functor $\operatorname{Ext}^1_{\mathcal{P}}$ instead of $\operatorname{Ext}^1_{\mathbb{A}}$.

15.1. $\operatorname{Ext}^1_{\mathcal{P}}$-orthogonal classes. Let $\mathcal{P}$ be a proper class of short exact sequences and $\mathcal{D}$ a class of objects in $\mathbb{A}$. We define the classes

$$^{\perp_{\mathcal{P}}}\mathcal{D} = \{X \in \mathbb{A} \mid \operatorname{Ext}^1_{\mathcal{P}}(X, \mathcal{D}) = 0\},$$

$$\mathcal{D}^{\perp_{\mathcal{P}}} = \{X \in \mathbb{A} \mid \operatorname{Ext}^1_{\mathcal{P}}(\mathcal{D}, X) = 0\}.$$

15.2. $\mathcal{P}$-cotorsion pairs. A pair $(\mathcal{F}, \mathcal{C})$ of classes of objects of $\mathbb{A}$ is called a **$\mathcal{P}$-cotorsion pair** if

(i) $\mathcal{F} = {}^{\perp_{\mathcal{P}}}\mathcal{C}$,

(ii) $\mathcal{C} = \mathcal{F}^{\perp_{\mathcal{P}}}$.

Clearly the pairs $(\operatorname{Proj}(\mathcal{P}), \mathbb{A})$ and $(\mathbb{A}, \operatorname{Inj}(\mathcal{P}))$ are $\mathcal{P}$-cotorsion pairs and for any class $\mathcal{D} \subseteq \mathbb{A}$ the pairs

$$({}^{\perp_{\mathcal{P}}}\mathcal{D}, ({}^{\perp_{\mathcal{P}}}\mathcal{D})^{\perp_{\mathcal{P}}}) \quad \text{and} \quad ({}^{\perp_{\mathcal{P}}}(\mathcal{D}^{\perp_{\mathcal{P}}}), \mathcal{D}^{\perp_{\mathcal{P}}})$$

are $\mathcal{P}$-cotorsion pairs, called the $\mathcal{P}$-cotorsion pair **generated** and **cogenerated** by the class $\mathcal{D}$ respectively.
Let $\mathcal{P}$ be a proper class in an abelian category $\mathbb{A}$ and $\mathcal{X}$ a class of objects of $\mathbb{A}$. We say that the class $\mathcal{X}$ is closed under **$\mathcal{P}$-extensions** if in every short exact sequence of $\mathcal{P}$

$$0 \to A \to B \to C \to 0$$

such that A and C belong to $\mathcal{X}$, B belongs also to $\mathcal{X}$.

15.3. Proposition. *If $(\mathcal{F}, \mathcal{C})$ is a $\mathcal{P}$-cotorsion pair, then:*

(i) *$\mathcal{F}$ and $\mathcal{C}$ are closed under $\mathcal{P}$-extensions.*

(ii) *$\mathcal{F}$ and $\mathcal{C}$ are closed under summands.*

(iii) *$\mathcal{F}$ contains all $\mathcal{P}$-projectives and $\mathcal{C}$ all the $\mathcal{P}$-injectives.*

Proof. (i) By [43, XII.5.1], for an object $Y \in \mathcal{C}$ and $X \in \mathcal{F}$ there are exact sequences of abelian groups

$$\cdots \to \mathrm{Hom}_{\mathbb{A}}(A, Y) \to \mathrm{Ext}^1_{\mathcal{P}}(C, Y) \to \mathrm{Ext}^1_{\mathcal{P}}(B, Y) \to \mathrm{Ext}^1_{\mathcal{P}}(A, Y) \to \cdots$$

$$\cdots \to \mathrm{Hom}_{\mathbb{A}}(X, C) \to \mathrm{Ext}^1_{\mathcal{P}}(X, A) \to \mathrm{Ext}^1_{\mathcal{P}}(X, B) \to \mathrm{Ext}^1_{\mathcal{P}}(X, C) \to \cdots$$

Thus (i) follows easily.

(ii) $\mathrm{Ext}^1_{\mathcal{P}}(-,-)$ is an additive functor.

(iii) Is clear. □

One may ask when a cotorsion pair $(\mathcal{X}, \mathcal{Y})$ ($\mathcal{A}$bs-cotorsion pair) in an abelian category $\mathbb{A}$ is a $\mathcal{P}$-cotorsion pair for a given proper class $\mathcal{P}$. This is the case when $\mathrm{Ext}^1_{\mathcal{P}}(X, Y) = \mathrm{Ext}^1_{\mathbb{A}}(X, Y)$ for $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Thus if $\mathcal{X} \subseteq \mathrm{Flat}(\mathcal{P})$ and $\mathcal{Y} \subseteq \mathrm{Div}(\mathcal{P})$, $(\mathcal{X}, \mathcal{Y})$ is a $\mathcal{P}$-cotorsion pair iff it is an $\mathcal{A}$bs-cotorsion pair.

In [55] Sklyarenko introduced the concepts of modules of flat type and of pure type relative to a proper class $\mathcal{P}$. A module C is called of **flat type relative to** $\mathcal{P}$ if for all $A \in R$-Mod

$$\mathrm{Ext}_{\mathcal{P}}(C, A) \subseteq \mathrm{Ext}_{\mathcal{P}\mathrm{ure}}(C, A),$$

i.e. every short exact sequence in $\mathcal{P}$ endig at C belongs to $\mathcal{P}$ure (= pure exact sequences). Analogously a module A is called of **pure type relative to** $\mathcal{P}$ if every short exact sequence in $\mathcal{P}$ begining with A belongs to $\mathcal{P}$ure. Based on this idea we introduce $\mathcal{P}$-$\mathcal{R}$-flat and $\mathcal{P}$-$\mathcal{R}$-divisible objects.

15.4. $\mathcal{P}$-$\mathcal{R}$-flats. Let $\mathcal{P}$ and $\mathcal{R}$ be proper classes in an abelian category $\mathbb{A}$. An object $Q \in \mathbb{A}$ is called **$\mathcal{P}$-$\mathcal{R}$-flat** if for all $A \in \mathbb{A}$

$$\mathrm{Ext}^1_{\mathcal{P}}(Q, A) \subseteq \mathrm{Ext}^1_{\mathcal{R}}(Q, A).$$

This means that every short exact sequence

$$0 \to A \to B \to Q \to 0$$

in $\mathcal{P}$ ending at Q belongs to $\mathcal{R}$. We denote the class of all $\mathcal{P}$-$\mathcal{R}$-flats by $\mathcal{P}$-Flat-$\mathcal{R}$. If $\mathcal{R} = \mathcal{P}$ure, then we call a $\mathcal{P}$-$\mathcal{P}$ure-flat object an object of flat type relative to $\mathcal{P}$ and we denote the class of all objects of flat type relative to $\mathcal{P}$ by $\mathrm{FT}(\mathcal{P})$.

15.5. Examples of $\mathcal{P}$-$\mathcal{R}$-flats.

(i) $\mathbb{A}$=$\mathcal{P}$-Flat-$\mathcal{A}$bs=$\mathcal{S}$plit-Flat-$\mathcal{R}$=$\mathcal{P}$-Flat-$\mathcal{P}$.

(ii) Flat($\mathcal{R}$)=$\mathcal{A}$bs-Flat-$\mathcal{R}$.

(iii) Flat=$\mathcal{A}$bs-Flat-$\mathcal{P}$ure.

(iv) Proj($\mathcal{P}$)=$\mathcal{P}$-Flat-$\mathcal{S}$plit.

(v) Proj($\mathbb{A}$)=$\mathcal{A}$bs-Flat-$\mathcal{S}$plit.

(vi) FT($\mathcal{P}$)=$\mathcal{P}$-Flat-$\mathcal{P}$ure.

15.6. $\mathcal{C}$-Compl-$\mathcal{R}$-flat modules. Let $\mathcal{C}$ be a $\{q,s\}$-closed class, Q be a module in $\sigma[M]$ and $\mathcal{R}$ a proper class in $\sigma[M]$. If every $\mathcal{C}$-singular factor module of Q is $\mathcal{R}$-flat, then Q is $\mathcal{C}$-Compl-$\mathcal{R}$-flat.

Proof. Let

$$0 \to A \to B \to Q \to 0$$

be a short exact sequence in $\mathcal{C}$-Compl, i.e. there exists a submodule $A' \subseteq B$ such that $A \cap A' = 0$ and $A \oplus A'/A \subseteq_{\mathcal{C}e} B/A \simeq Q$. Then, by 11.3 (ii), $A \oplus A' \subseteq_{\mathcal{C}e} B$. Note that $B/A \oplus A' \simeq B/A \big/ A \oplus A'/A$ is isomorphic to a $\mathcal{C}$-singular factor module of Q which, by hypothesis, is $\mathcal{R}$-flat. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i} & & \| & & \downarrow & & \\
0 & \longrightarrow & A \oplus A' & \xrightarrow[j]{} & B & \longrightarrow & B/A \oplus A' & \longrightarrow & 0
\end{array}$$

with i and j the canonical inclusions. Since $B/A \oplus A'$ is $\mathcal{R}$-flat, the lower row belongs to $\mathcal{R}$. On the other hand, i is an split monomorphism, thus it is also an $\mathcal{R}$-monomorphism. Therefore the morphism $A \to B$ must be an $\mathcal{R}$-monomorphism, i.e. Q is $\mathcal{C}$-Compl-$\mathcal{R}$-flat. $\square$

The following proposition is a generalization of [55, Proposition 10.2, Lemma 10.1, Corollary 10.2, Proposition 10.3] where $\mathcal{P}$ is any proper class and $\mathcal{R} = \mathcal{P}$ure in R-Mod.

15.7. Properties of $\mathcal{P}$-$\mathcal{R}$-flat objects. *Let $\mathcal{P}$ and $\mathcal{R}$ be proper classes in an abelian category $\mathbb{A}$.*

(i) $\mathcal{P}$-Flat-$\mathcal{R} \cap$ Flat$(\mathcal{P}) \subseteq$ Flat$(\mathcal{R})$.

(ii) *If $\mathcal{R} \subseteq \mathcal{P}$, then* $\mathcal{P}$-Flat-$\mathcal{R} \cap$ Flat$(\mathcal{P}) =$ Flat$(\mathcal{R})$.

(iii) *If the sequence*

$$0 \to A \to B \to C \to 0$$

belongs to $\mathcal{P}$, then $A, C \in$ $\mathcal{P}$-Flat-$\mathcal{R}$ *implies $B \in$* $\mathcal{P}$-Flat-$\mathcal{R}$.

(iv) *If the sequence*

$$0 \to A \to B \to C \to 0$$

belongs to $\mathcal{R}$, then $B \in$ $\mathcal{P}$-Flat-$\mathcal{R}$ *implies $C \in$* $\mathcal{P}$-Flat-$\mathcal{R}$.

(v) $Q_1 \oplus Q_2 \in \mathcal{P}$-Flat-$\mathcal{R}$ *iff* Q_1, $Q_2 \in \mathcal{P}$-Flat-$\mathcal{R}$.

(vi) *Suppose $\mathbb{A}$ is cocomplete. If $\mathcal{R}$ is $\bigoplus$-closed and $Q = \bigoplus_I Q_i$, then $Q \in$* $\mathcal{P}$-Flat-$\mathcal{R}$ *iff each* $Q_i \in \mathcal{P}$-Flat-$\mathcal{R}$.

(vii) *Suposse $\mathbb{A}$ is a Grothendieck category. If $\mathcal{R}$ is inductively closed, then the direct limit of $\mathcal{P}$-$\mathcal{R}$-flats is $\mathcal{P}$-$\mathcal{R}$-flat.*

(viii) *Every object of $\mathbb{A}$ is $\mathcal{P}$-$\mathcal{R}$-flat iff $\mathcal{P} \subseteq \mathcal{R}$.*

Proof. (i) Let $Q \in \mathcal{P}$-Flat-$\mathcal{R} \cap$ Flat$(\mathcal{P})$. Then for every $X \in C$

$$\mathrm{Ext}^1_{\mathbb{A}}(Q, X) = \mathrm{Ext}^1_{\mathcal{P}}(Q, X) \subseteq \mathrm{Ext}^1_{\mathcal{R}}(Q, X).$$

Thus $Q \in$ Flat$(\mathcal{R})$.

(ii) Let $Q \in \mathrm{Flat}(\mathcal{R})$. Then

$$\mathrm{Ext}^1_{\mathbb{A}}(Q,X) = \mathrm{Ext}^1_{\mathcal{R}}(Q,X) \subseteq \mathrm{Ext}^1_{\mathcal{P}}(Q,X).$$

Thus $Q \in \mathrm{Flat}(\mathcal{P})$. Clearly

$$\mathrm{Ext}^1_{\mathcal{P}}(Q,X) \subseteq \mathrm{Ext}^1_{\mathcal{R}}(Q,X).$$

Thus $Q \in \mathcal{P}$-Flat-$\mathcal{R}$.

(iii) Let

$$0 \to X \to Y \xrightarrow{g} B \to 0$$

be a short exact sequence in $\mathcal{P}$. Consider the composition

$$X \hookrightarrow g^{-1}(A) \hookrightarrow Y.$$

Since this composition is a $\mathcal{P}$-monomorphism, from 3.1 (P4), it follows that $X \hookrightarrow g^{-1}(A)$ is a $\mathcal{P}$-monomorphism. Since $A \in \mathcal{P}$-Flat-$\mathcal{R}$, the short exact sequence

$$0 \to X \hookrightarrow g^{-1}(A) \xrightarrow{g|_{g^{-1}(A)}} A \to 0$$

belongs to $\mathcal{R}$. On the other hand, the composition $Y \to B \to C$ is a $\mathcal{P}$-epimorphism, by 3.1 (P5). Then, since $C \in \mathcal{P}$-Flat-$\mathcal{R}$, the sequence

$$0 \to g^{-1}(A) \hookrightarrow Y \to C \to 0$$

belongs to $\mathcal{R}$. Finally, by 3.1 (P3), the composition $X \hookrightarrow g^{-1}(A) \hookrightarrow Y$ is an $\mathcal{R}$-monomorphism, i.e. $B \in \mathcal{P}$-Flat-$\mathcal{R}$.

(iv) Let $B \in \mathcal{P}$-Flat-$\mathcal{R}$ and

$$0 \to X \to Y \to C \to 0$$

any short exact sequence in $\mathcal{P}$. Consider the pullback diagram

$$\begin{array}{cccccccccc}
E' : 0 & \longrightarrow & X & \longrightarrow & Y' & \longrightarrow & B & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
E : 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & C & \longrightarrow & 0.
\end{array}$$

Note that $E \in \mathcal{P}$ implies $E' \in \mathcal{P}$. Since $B \in \mathcal{P}$-Flat-$\mathcal{R}$, the sequence E' belongs to $\mathcal{R}$. By assumption $B \to C$ is an $\mathcal{R}$-epimorphism. It follows from 3.1 (P5) and (P6) that $Y \to C$ is an $\mathcal{R}$-epimorphism, i.e. $C \in \mathcal{P}$-Flat-$\mathcal{R}$.

(v) $\Rightarrow$) Since the splitting sequence

$$0 \to Q_1 \to Q_1 \oplus Q_2 \to Q_2 \to 0$$

belongs to $\mathcal{R}$, by (iv), it follows that $Q_2 \in \mathcal{P}$-Flat-$\mathcal{R}$. Analogously $Q_1 \in \mathcal{P}$-Flat-$\mathcal{R}$.
$\Leftarrow$) It follows from (iii).

(vi) $\Rightarrow$) Since the splitting sequence

$$0 \to \bigoplus_{i \neq j} Q_i \to Q \to Q_j \to 0$$

belongs to $\mathcal{R}$, it follows by (iv) that $Q_j \in \mathcal{P}$-Flat-$\mathcal{R}$.
$\Leftarrow$) Let

$$E : 0 \to A \to B \xrightarrow{g} Q \to 0$$

be a short exact sequence in $\mathcal{P}$. For each $i \in I$ the composition

$$A \hookrightarrow g^{-1}(Q_i) \hookrightarrow B$$

is a $\mathcal{P}$-monomorphism, thus from 3.1 (P4) follows that $A \hookrightarrow g^{-1}(Q_i)$ is a $\mathcal{P}$-monomorphism, i.e. the short exact sequence

$$E_i : 0 \to A \to g^{-1}(Q_i) \xrightarrow{g|g^{-1}(Q_i)} Q_i \to 0$$

belongs to $\mathcal{P}$. Since $Q_i \in \mathcal{P}$-Flat-$\mathcal{R}$, $E_i \in \mathcal{R}$. By assumption $\mathcal{R}$ is $\bigoplus$-closed, thus

$$\bigoplus_i E_i : 0 \to \bigoplus_I A \to \bigoplus_I g^{-1}(Q_i) \to Q \to 0$$

belongs to $\mathcal{R}$. The $\mathcal{R}$-epimorphism of $\bigoplus_i E_i$ is the composition

$$\bigoplus_I g^{-1}(Q_i) \to B \to Q.$$

Thus, by 3.1 (P6), the morphism $B \to Q$ is an $\mathcal{R}$-epimorphism, i.e. $E \in \mathcal{R}$.

(vii) Let

$$E : 0 \to A \to B \to Q \to 0$$

be a short exact sequence in $\mathcal{P}$ and $Q = \varinjlim Q_i$ with $Q_i \in \mathcal{P}$-Flat-$\mathcal{R}$. For each canonical morphism $Q_i \to \varinjlim Q_i$ form the pullback diagram

$$\begin{array}{ccccccccc}
E_i : 0 & \longrightarrow & A & \longrightarrow & B_i & \longrightarrow & Q_i & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
E : 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & Q & \longrightarrow & 0.
\end{array}$$

The sequences $E_i \in \mathcal{P}$ and since $Q_i \in \mathcal{P}$-Flat-$\mathcal{R}$, $E_i \in \mathcal{R}$. $\{E_i\}_I$ forms a direct system whose direct limit is E. Since $\mathcal{R}$ is inductively closed, $E \in \mathcal{R}$.

(viii) $\Rightarrow$) Is clear since every short exact sequence in $\mathcal{P}$ belongs to $\mathcal{R}$.
$\Leftarrow$) Is also clear. $\square$

15.8. Proposition. *Let $\mathcal{R}$ be an injectively generated proper class and $\mathcal{P}$ any proper class. Then $Q \in \mathcal{P}$-Flat-$\mathcal{R}$ iff $\mathrm{Ext}^1_{\mathcal{P}}(Q, X) = 0$ for all $X \in \mathrm{Inj}(\mathcal{R})$, i.e. ${}^{\perp_{\mathcal{P}}}\mathrm{Inj}(\mathcal{R})$=$\mathcal{P}$-Flat-$\mathcal{R}$.*

Proof. $\Rightarrow$) Clearly if $Q \in \mathcal{P}$-Flat-$\mathcal{R}$ and $X \in \mathrm{Inj}(\mathcal{R})$, then $\mathrm{Ext}^1_{\mathcal{P}}(Q, X) \subseteq \mathrm{Ext}^1_{\mathcal{R}}(Q, X) = 0$.
$\Leftarrow$) Suppose that Q is such that $\mathrm{Ext}^1_{\mathcal{P}}(Q, X) = 0$ for all $X \in \mathrm{Inj}(\mathcal{R})$. Consider any short exact sequence in $\mathcal{P}$ ending at Q,

$$0 \to A \to B \to Q \to 0,$$

and a morphism $f : A \to X$ for $X \in \mathrm{Inj}(\mathcal{R})$. Form the pushout diagram

$$\begin{array}{ccccccccc}
E : 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & Q & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow & & \downarrow & & \| & & \\
E' : 0 & \longrightarrow & X & \longrightarrow & B' & \longrightarrow & Q & \longrightarrow & 0.
\end{array}$$

The sequence E' belongs also to $\mathcal{P}$. By assumption E' splits, thus there exists a morphism $B \to X$ extending f. This means that X is injective with respect to E. Thus $E \in \mathcal{R}$. $\square$

15.9. Corollary. *Let $\mathcal{R}$ be an injectively generated proper class and $\mathcal{P}$ any proper class. Then*

$$\mathcal{P}\text{-Flat-}\mathcal{R} = {}^{\perp_{\mathcal{P}}}((\mathcal{P}\text{-Flat-}\mathcal{R})^{\perp_{\mathcal{P}}}).$$

Proof. Clearly $\mathcal{P}\text{-Flat-}\mathcal{R} \subseteq {}^{\perp_{\mathcal{P}}}((\mathcal{P}\text{-Flat-}\mathcal{R})^{\perp_{\mathcal{P}}})$. For the other inclusion note that $\text{Inj}(\mathcal{R}) \subseteq (\mathcal{P}\text{-Flat-}\mathcal{R})^{\perp_{\mathcal{P}}}$. For if $X \in \text{Inj}(\mathcal{R})$ and $Q \in \mathcal{P}\text{-Flat-}\mathcal{R}$, then

$$\text{Ext}^1_{\mathcal{P}}(Q, X) \subseteq \text{Ext}^1_{\mathcal{R}}(Q, X) = 0.$$

Therefore ${}^{\perp_{\mathcal{P}}}((\mathcal{P}\text{-Flat-}\mathcal{R})^{\perp_{\mathcal{P}}}) \subseteq {}^{\perp_{\mathcal{P}}}\text{Inj}(\mathcal{R}) = \mathcal{P}\text{-Flat-}\mathcal{R}$. □

15.10. Proposition. *Let $\mathcal{P}$ be a proper class of short exact sequences. Every $\mathcal{P}$-cotorsion pair is of the form $(\mathcal{P}\text{-Flat-}\mathcal{R}, (\mathcal{P}\text{-Flat-}\mathcal{R})^{\perp_{\mathcal{P}}})$ for some injectively generated proper class $\mathcal{R}$.*

Proof. Let $(\mathcal{X}, \mathcal{Y})$ be a $\mathcal{P}$-cotorsion pair. Set $\mathcal{R} = \iota^{-1}(\mathcal{Y})$. We show that $\mathcal{X} = \mathcal{P}\text{-Flat-}\mathcal{R}$. Take $X \in \mathcal{X}$ and a short exact sequence

$$0 \to A \to B \to X \to 0$$

in $\mathcal{P}$. If $f : A \to Y$ is a morphism with $Y \in \mathcal{Y}$, form the pushout diagram

$$\begin{array}{rccccccccc}
E : 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\
E' : 0 & \longrightarrow & Y & \longrightarrow & B' & \longrightarrow & X & \longrightarrow & 0.
\end{array}$$

By assumption the sequence E' splits, thus we can find a morphism $B \to Y$ which extends f. Thus $E \in \mathcal{R}$, i.e. $X \in \mathcal{P}\text{-Flat-}\mathcal{R}$.
Conversely, if $X \in \mathcal{P}\text{-Flat-}\mathcal{R}$. Since the elements of $\mathcal{Y}$ are $\mathcal{R}$-injectives, for every $Y \in \mathcal{Y}$

$$\text{Ext}^1_{\mathcal{P}}(X, Y) \subseteq \text{Ext}^1_{\mathcal{R}}(X, Y) = 0,$$

thus $X \in \mathcal{X}$. □

For each proper class $\mathcal{P}$ in a locally finitely presented Grothendieck category $\mathbb{A}$, we obtain a "classical" $\mathcal{P}$-cotorsion pair

$$(\text{FT}(\mathcal{P}), \text{FT}(\mathcal{P})^{\perp_{\mathcal{P}}}) = (\mathcal{P}\text{-Flat-}\mathcal{P}\text{ure}, (\mathcal{P}\text{-Flat-}\mathcal{P}\text{ure})^{\perp_{\mathcal{P}}})$$

with $\mathcal{P}$ure the class of pure short exact sequences. If $\mathcal{P} = \mathcal{A}$bs, then we recover the Flat-Cotorsion pair $(\text{Flat}, \text{Cot})$.

15.11. $\mathcal{P}$-$\mathcal{R}$-divisibles. Let $\mathcal{P}$ and $\mathcal{R}$ be proper classes in an abelian category $\mathbb{A}$. An object $J \in \mathbb{A}$ is called **$\mathcal{P}$-$\mathcal{R}$-divisible** if for all $C \in \mathbb{A}$

$$\text{Ext}^1_{\mathcal{P}}(C, J) \subseteq \text{Ext}^1_{\mathcal{R}}(C, J).$$

This means that every short exact sequence

$$0 \to J \to B \to C \to 0$$

in $\mathcal{P}$ beginning with J belongs to $\mathcal{R}$. We denote the class of all $\mathcal{P}$-$\mathcal{R}$-divisibles by $\mathcal{P}$-Div-$\mathcal{R}$. If $\mathcal{R} = \mathcal{P}$ure we use the name of object of pure type relative to $\mathcal{P}$ for a $\mathcal{P}$-$\mathcal{P}$ure-divisible object and we denote the class of all objects of pure type relative to $\mathcal{P}$ by $\text{PT}(\mathcal{P})$.

15.12. Examples of $\mathcal{P}$-$\mathcal{R}$-divisibles.

(i) $\mathbb{A}$=$\mathcal{P}$-Div-$\mathcal{A}$bs=$\mathcal{S}$plit-Div-$\mathcal{R}$=$\mathcal{P}$-Div-$\mathcal{P}$.

(ii) Div($\mathcal{R}$)=$\mathcal{A}$bs-Div-$\mathcal{R}$.

(iii) AP=$\mathcal{A}$bs-Div-$\mathcal{P}$ure.

(iv) Inj($\mathcal{P}$)=$\mathcal{P}$-Div-$\mathcal{S}$plit.

(v) Inj($\mathbb{A}$)=$\mathcal{A}$bs-Div-$\mathcal{S}$plit.

(vi) PT($\mathcal{P}$)=$\mathcal{P}$-Div-$\mathcal{P}$ure.

15.13. $\mathcal{C}$-Suppl-$\mathcal{R}$-divisible modules. Let $\mathcal{C}$ be a $\{q, s\}$-closed class, J be a module in $\sigma[M]$ and $\mathcal{R}$ a proper class in $\sigma[M]$. If every $\mathcal{C}$-small submodule of J is $\mathcal{R}$-divisible, then J is $\mathcal{C}$-Suppl-$\mathcal{R}$-divisible.

Proof. Let

$$0 \to J \to B \to C \to 0$$

be a short exact sequence in $\mathcal{C}$-Suppl, i.e. there exists a submodule $J' \subseteq B$ such that $J+J' = B$ and $J \cap J' \ll_{\mathcal{C}} J$. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J \cap J' & \longrightarrow & B & \xrightarrow{p} & B/J \cap J' & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow{\scriptstyle q} & & \\ 0 & \longrightarrow & J & \longrightarrow & B & \longrightarrow & C \simeq B/J & \longrightarrow & 0 \end{array}$$

with p and q the canonical projections. By hypothesis $J \cap J'$ is $\mathcal{R}$-divisible, i.e. the top row belongs to $\mathcal{R}$. Since q is an split epimorphism, it is also an $\mathcal{R}$-epimorphism. Therefore the morphism $B \to C$ must be an $\mathcal{R}$-epimorphism, i.e. J is $\mathcal{C}$-Suppl-$\mathcal{R}$-divisible. □

The following proposition is a generalization of [55, Lemma 10.4, Corollary 10.5, Proposition 10.7, Proposition 10.8] where $\mathcal{P}$ is any proper class and $\mathcal{R} = \mathcal{P}$ure in R-Mod.

15.14. Properties of $\mathcal{P}$-$\mathcal{R}$-divisible objects. *Let $\mathcal{P}$ and $\mathcal{R}$ be proper classes in an abelian category $\mathbb{A}$.*

(i) $\mathcal{P}$-Div-$\mathcal{R} \cap \text{Div}(\mathcal{P}) \subseteq \text{Div}(\mathcal{R})$.

(ii) *If $\mathcal{R} \subseteq \mathcal{P}$, then* $\mathcal{P}$-Div-$\mathcal{R} \cap \text{Div}(\mathcal{P}) = \text{Div}(\mathcal{R})$.

(iii) *If the sequence*

$$0 \to A \to B \to C \to 0$$

belongs to $\mathcal{P}$, then $A, C \in$ $\mathcal{P}$-Div-$\mathcal{R}$ implies $B \in$ $\mathcal{P}$-Div-$\mathcal{R}$.

(iv) *If the sequence*

$$0 \to A \to B \to C \to 0$$

belongs to $\mathcal{R}$ and $B \in$ $\mathcal{P}$-Div-$\mathcal{R}$, then $A \in$ $\mathcal{P}$-Div-$\mathcal{R}$.

(v) $J_1 \oplus J_2 \in$ $\mathcal{P}$-Div-$\mathcal{R}$ *iff* J_1, $J_2 \in$ $\mathcal{P}$-Div-$\mathcal{R}$.

(vi) *Suppose $\mathbb{A}$ is a Grothendieck category. If $\mathcal{R}$ is inductively closed and*

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_\lambda \subseteq \cdots$$

is a chain of $\mathcal{P}$-submodules of $A = \cup_\Lambda A_\lambda$. Then if each A_λ is $\mathcal{P}$-$\mathcal{R}$-divisible, so is A.

(vii) *Suppose $\mathbb{A}$ is a Grothendieck category. If $\mathcal{R}$ is inductively closed and $J = \bigoplus_I J_i$, then J is $\mathcal{P}$-$\mathcal{R}$-divisible iff every J_i is $\mathcal{P}$-$\mathcal{R}$-divisible.*

(viii) *Suppose* $\mathbb{A}$ *is complete. If* $\mathcal{R}$ *is* $\prod$*-closed and* $J = \prod_I J_i$*, then* J *is* $\mathcal{P}$*-*$\mathcal{R}$*-divisible iff every* J_i *is* $\mathcal{P}$*-*$\mathcal{R}$*-divisible.*

(ix) *Every object of* $\mathbb{A}$ *is* $\mathcal{P}$*-*$\mathcal{R}$*-divisible iff* $\mathcal{P} \subseteq \mathcal{R}$*.*

Proof. (i) Let $A \in \mathcal{P}\text{-Div-}\mathcal{R} \cap \mathrm{Div}(\mathcal{P})$. For every $C \in \mathbb{A}$,

$$\mathrm{Ext}^1_{\mathbb{A}}(C, A) = \mathrm{Ext}^1_{\mathcal{P}}(C, A) \subseteq \mathrm{Ext}^1_{\mathcal{R}}(C, A).$$

Therefore $C \in \mathrm{Div}(\mathcal{R})$.

(ii) Let $A \in \mathrm{Div}(\mathcal{R})$, then for every $C \in \mathbb{A}$

$$\mathrm{Ext}^1_{\mathbb{A}}(C, A) = \mathrm{Ext}^1_{\mathcal{R}}(C, A) \subseteq \mathrm{Ext}^1_{\mathcal{P}}(C, A).$$

Thus $A \in \mathrm{Div}(\mathcal{P})$. Clearly $\mathrm{Ext}^1_{\mathcal{P}}(C, A) \subseteq \mathrm{Ext}^1_{\mathcal{R}}(C, A)$. Thus $A \in \mathcal{P}\text{-Div-}\mathcal{R}$.

(iii) Let $0 \to B \to X \to Y \to 0$ be a short exact sequence in $\mathcal{P}$. Consider the composition

$$X \twoheadrightarrow X/A \twoheadrightarrow X/B \simeq Y.$$

Since this composition is a $\mathcal{P}$-epimorphism, it follows from 3.1 (P6) that $X/A \twoheadrightarrow Y$ is a $\mathcal{P}$-epimorphism. Since $B/A \simeq C \in \mathcal{P}\text{-Div-}\mathcal{R}$, the sequence

$$0 \to C \simeq B/A \hookrightarrow X/A \twoheadrightarrow X/B \simeq Y \to 0$$

belongs to $\mathcal{R}$. On the other hand, by 3.1 (P3), the composition $A \to B \to X$ is a $\mathcal{P}$-monomorphism. Then, since $A \in \mathcal{P}\text{-Div-}\mathcal{R}$, the sequence

$$0 \to A \to X \to X/A \to 0$$

belongs to $\mathcal{R}$. Finally, by 3.1 (P5), the composition $X \twoheadrightarrow X/A \twoheadrightarrow Y$ is an $\mathcal{R}$-epimorphism, i.e. $B \in \mathcal{P}\text{-Div-}\mathcal{R}$.

(iv) Let

$$E : 0 \to A \to X \to Y \to 0$$

be a short exact sequence in $\mathcal{P}$. Consider the pushout diagram

$$\begin{array}{ccccccccc}
E : 0 & \longrightarrow & A & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
E' : 0 & \longrightarrow & B & \longrightarrow & X' & \longrightarrow & Y & \longrightarrow & 0.
\end{array}$$

Then $E' \in \mathcal{P}$. Since $B \in \mathcal{P}\text{-Div-}\mathcal{R}$, then $E' \in \mathcal{R}$. By assumption the morphism $A \to B$ is an $\mathcal{R}$-monomorphism, thus, by 3.1 (P3) and (P4), it follows that $A \to X$ is an $\mathcal{R}$-monomorphism, i.e. $A \in \mathcal{P}\text{-Div-}\mathcal{R}$.

(v) $\Rightarrow$) The split exact sequence

$$0 \to J_1 \to J_1 \oplus J_2 \to J_2 \to 0$$

belongs to $\mathcal{R}$. By (iv), it follows that $J_1 \in \mathcal{P}\text{-Div-}\mathcal{R}$. Analogous $J_2 \in \mathcal{P}\text{-Div-}\mathcal{R}$.
$\Leftarrow$) It follows from (iii).

(vi) Let

$$E : 0 \to A \to X \to Y \to 0$$

be a short exact sequence in $\mathcal{P}$. For each $\lambda \in \Lambda$ the composition $A_\lambda \hookrightarrow A \hookrightarrow X$ is, by 3.1 (P3), a $\mathcal{P}$-monomorphism, i.e. the sequence

$$E_\lambda : 0 \to A_\lambda \to X \to X/A_\lambda \to 0$$

belongs to $\mathcal{P}$. Since $A_\lambda \in \mathcal{P}$-Div-$\mathcal{R}$, $E_\lambda \in \mathcal{R}$. Since $\mathcal{R}$ is inductively closed, $\varinjlim E_\lambda = E \in \mathcal{R}$, i.e. $A \in \mathcal{P}$-Div-$\mathcal{R}$.

(vii) $\Rightarrow$) Follows from (v).
$\Leftarrow$) Let

$$E : 0 \to J \to B \to C \to 0$$

be a short exact sequence in $\mathcal{P}$. For each finite subset $K \subseteq I$, it follows from (v) that $\bigoplus_K J_k$ is $\mathcal{P}$-$\mathcal{R}$-divisible. Since

$$J = \bigoplus_I J_i = \bigcup_{K \subseteq I} (\bigoplus_K J_k),$$

the assertion follows from (vi).

(viii) $\Rightarrow$) For each $k \in I$ the split sequence

$$0 \to J_k \to \prod_I J_i \to \prod_{i \neq k} J_i \to 0$$

belongs to $\mathcal{P}$. Thus, from (iv), it follows that $J_k \in \mathcal{P}$-Div-$\mathcal{R}$.
$\Leftarrow$) Let

$$E : 0 \to J \to B \to C \to 0$$

be a short exact sequence in $\mathcal{P}$. For each $k \in I$ the composition

$$B \twoheadrightarrow B/\prod_{i \neq k} J_i \twoheadrightarrow B/J \simeq C$$

is a $\mathcal{P}$-epimorphism. Thus, by 3.1 (P6), the morphism $B/\prod_{i \neq k} J_i \twoheadrightarrow C$ is a $\mathcal{P}$-epimorphism, i.e. the short exact sequence

$$E_k : 0 \to J_k \simeq J/\prod_{i \neq k} J_i \to B/\prod_{i \neq k} J_i \to C \to 0$$

belongs to $\mathcal{P}$. Since $J_k \in \mathcal{P}$-Div-$\mathcal{R}$, $E_k \in \mathcal{R}$. By assumption $\mathcal{R}$ is $\prod$-closed, therefore the sequence

$$\prod_I E_k : 0 \to J \to \prod_I (B/\prod_{i \neq k} J_i) \to \prod_I C \to 0$$

belongs to $\mathcal{R}$. Finally the $\mathcal{R}$-monomorphism of $\prod_I E_k$ is the composition

$$J \to B \to \prod_I (B/\prod_{i \neq k} J_i).$$

Thus, by 3.1 (P4), the morphism $J \to B$ is an $\mathcal{R}$-monomorphism, i.e. $E \in \mathcal{R}$.

(ix) $\Rightarrow$) Is clear since every short exact sequence in $\mathcal{P}$ belongs to $\mathcal{R}$.
$\Leftarrow$) Is also clear. □

15.15. Proposition. *Let $\mathcal{R}$ be a projectively generated proper class and $\mathcal{P}$ any proper class. Then $J \in \mathcal{P}$-Div-$\mathcal{R}$ iff $\mathrm{Ext}^1_{\mathcal{P}}(X, J) = 0$ for all $X \in \mathrm{Proj}(\mathcal{R})$, i.e. $\mathrm{Proj}(\mathcal{R})^{\perp_{\mathcal{P}}} = \mathcal{P}$-Div-$\mathcal{R}$.*

Proof. $\Rightarrow$) Clearly, if $J \in \mathcal{P}$-Div-$\mathcal{R}$ and $X \in \mathrm{Proj}(\mathcal{R})$, then $\mathrm{Ext}^1_{\mathcal{P}}(X, J) \subseteq \mathrm{Ext}^1_{\mathcal{R}}(X, J) = 0$.
$\Leftarrow$) Suppose that J is such that $\mathrm{Ext}^1_{\mathcal{P}}(X, J) = 0$ for all $X \in \mathrm{Proj}(\mathcal{R})$. Consider any short exact sequence in $\mathcal{P}$ beginning with J,

$$0 \to J \to B \to C \to 0,$$

and a morphism $f : X \to C$ for $X \in \mathrm{Proj}(\mathcal{R})$. Form the pullback diagram

$$\begin{array}{ccccccccc} E' : 0 & \longrightarrow & J & \longrightarrow & B' & \longrightarrow & X & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow f & & \\ E : 0 & \longrightarrow & J & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

The sequence E' belongs also to $\mathcal{P}$. By assumption E' splits, thus there exist a morphism $X \to B$ lifting f. This means that X is projective with respect to E. Thus $E \in \mathcal{R}$. $\square$

15.16. Corollary. *Let $\mathcal{R}$ be a projectively generated proper class. Then*

$$\mathcal{P}\text{-Div-}\mathcal{R} = ({}^{\perp_{\mathcal{P}}}(\mathcal{P}\text{-Div-}\mathcal{R}))^{\perp_{\mathcal{P}}}.$$

Proof. Clearly $\mathcal{P}$-Div-$\mathcal{R} \subseteq ({}^{\perp_{\mathcal{P}}}(\mathcal{P}\text{-Div-}\mathcal{R}))^{\perp_{\mathcal{P}}}$. For the other inclusion note that $\mathrm{Proj}(\mathcal{R}) \subseteq {}^{\perp_{\mathcal{P}}}(\mathcal{P}\text{-Div-}\mathcal{R})$. For, if $X \in \mathrm{Proj}(\mathcal{R})$ and $J \in \mathcal{P}$-Div-$\mathcal{R}$, then

$$\mathrm{Ext}^1_{\mathcal{P}}(X, J) \subseteq \mathrm{Ext}^1_{\mathcal{R}}(X, J) = 0.$$

Therefore $({}^{\perp_{\mathcal{P}}}(\mathcal{P}\text{-Div-}\mathcal{R}))^{\perp_{\mathcal{P}}} \subseteq \mathrm{Proj}(\mathcal{R})^{\perp_{\mathcal{P}}} = \mathcal{P}$-Div-$\mathcal{R}$. $\square$

15.17. Proposition. *Let $\mathcal{P}$ be a proper class. Every $\mathcal{P}$-cotorsion pair is of the form $({}^{\perp_{\mathcal{P}}}(\mathcal{P}$-Div-$\mathcal{R}), \mathcal{P}$-Div-$\mathcal{R})$ for some projectively generated proper class $\mathcal{R}$.*

Proof. Let $(\mathcal{X}, \mathcal{Y})$ be a $\mathcal{P}$-cotorsion pair. Set $\mathcal{R} = \pi^{-1}(\mathcal{X})$. We show that $\mathcal{Y} = \mathcal{P}$-Div-$\mathcal{R}$. Take $Y \in \mathcal{Y}$ and a short exact sequence

$$0 \to Y \to B \to C \to 0$$

in $\mathcal{P}$. If $f : X \to C$ is a morphism with $X \in \mathcal{X}$, form the pullback diagram

$$\begin{array}{ccccccccc} E' : 0 & \longrightarrow & Y & \longrightarrow & B' & \longrightarrow & X & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow f & & \\ E : 0 & \longrightarrow & Y & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

By assumption the sequence E' splits, thus we can find a morphism $X \to B$ which lifts f. Thus $E \in \mathcal{R}$, i.e. $Y \in \mathcal{P}$-Div-$\mathcal{R}$.
Conversely, if $Y \in \mathcal{P}$-Div-$\mathcal{R}$. Since the elements of $\mathcal{X}$ are $\mathcal{R}$-projectives, for every $X \in \mathcal{X}$,

$$\mathrm{Ext}^1_{\mathcal{P}}(X, Y) \subseteq \mathrm{Ext}^1_{\mathcal{R}}(X, Y) = 0,$$

thus $Y \in \mathcal{Y}$. $\square$

In [61] Wisbauer studied $\mathcal{Q}$-regular and $\mathcal{I}$-coregular modules in the category $\sigma[M]$ for a projectively generated proper class $\mathcal{P} = \pi^{-1}(\mathcal{Q})$ and an injectively generated proper class $\mathcal{P} = \iota^{-1}(\mathcal{I})$ (see 3.14). The definition of a $\mathcal{P}$-regular module can be rephrased by saying that every short exact sequence in $\mathcal{A}$bs with the middle module T belongs to $\mathcal{P}$. Extending this approach we introduce $\mathcal{P}$-$\mathcal{R}$-regular objects.

15.18. $\mathcal{P}$-$\mathcal{R}$-regular objects. Let $\mathcal{P}$ and $\mathcal{R}$ be proper classes. An object $T \in \mathbb{A}$ is called **$\mathcal{P}$-$\mathcal{R}$-regular** if every short exact sequence

$$0 \to A \to T \to C \to 0$$

in $\mathcal{P}$ belongs to $\mathcal{R}$.

15.19. Examples of $\mathcal{P}$-$\mathcal{R}$-regular objects. Let $\mathcal{P}$ and $\mathcal{R}$ be proper classes.

(i) In an abelian category $\mathbb{A}$, every object of $\mathbb{A}$ is $\mathcal{P}$-$\mathcal{P}$-regular, $\mathcal{P}$-$\mathcal{A}$bs-regular and $\mathcal{S}$plit-$\mathcal{R}$-regular.

Proof. By definition, an object $T \in \mathbb{A}$ is $\mathcal{P}$-$\mathcal{R}$-regular if every short exact sequence in $\mathcal{P}$ with middle term T belongs to $\mathcal{R}$. Thus it is clear that T is $\mathcal{P}$-$\mathcal{P}$-regular. Since every short exact sequence $\mathcal{P}$ is contained in $\mathcal{A}$bs, T is $\mathcal{P}$-$\mathcal{A}$bs-regular. Since the proper class $\mathcal{S}$plit is contained in any proper class $\mathcal{R}$, T is $\mathcal{S}$plit-$\mathcal{R}$-regular. □

(ii) In $\sigma[M]$ a module is $\mathcal{A}$bs-$\mathcal{S}$plit-regular iff it is semisimple.

Proof. A module N in $\sigma[M]$ is semisimple iff every submodule $K \subseteq N$ is a direct summand iff every short exact sequence

$$0 \to K \to N \to N/K \to 0$$

belongs to $\mathcal{S}$plit iff T is $\mathcal{A}$bs-$\mathcal{S}$plit-regular. □

(iii) In R-Mod a module is Compl-$\mathcal{S}$plit-regular iff it is extending (see [12, §7]).

Proof. By definition a module N in R-Mod is extending iff every complement (= closed) submodule $K \subseteq N$ is a direct summand iff every short exact sequence in Compl

$$0 \to K \to N \to N/K \to 0$$

belongs to $\mathcal{S}$plit iff N is Compl-$\mathcal{S}$plit-regular. □

(iv) In [14] Crivei investigates a similar question introducing $\mathcal{P}$-extending and $\mathcal{P}$-lifting modules for an arbitrary proper class $\mathcal{P}$. A module N is called **$\mathcal{P}$-extending** if every submodule K of N has an essential extension L such that L is a $\mathcal{P}$-submodule of N. Equivalently N is $\mathcal{P}$-extending iff every closed (=complement) submodule of N is a $\mathcal{P}$-submodule of N. That is, every short exact sequence in Compl with middle module N belongs to $\mathcal{P}$ (see [14, 2.3]).
In R-Mod a module is Compl-$\mathcal{P}$-regular iff it is $\mathcal{P}$-extending (see [14, 2.3]).

Proof. Let $\mathcal{P}$ be a proper class in R-Mod. By definition a module N in R-Mod is $\mathcal{P}$-extending iff every complement (= closed) submodule $K \subseteq N$ is a $\mathcal{P}$-submodule of N iff every short exact sequence in $\text{Compl}_{R\text{-Mod}}$

$$0 \to K \to N \to N/K \to 0$$

belongs to $\mathcal{P}$ iff N is Compl-$\mathcal{P}$-regular. □

(v) In R-Mod the module N is amply supplemented and Suppl-$\mathcal{S}$plit-regular iff N is amply supplemented and Cocls-$\mathcal{S}$plit-regular iff N is lifting (see [12, 22.3]).

Proof. By [12, 22.3], a module N in R-Mod is lifting iff N is amply supplemented and every coclosed submodule $K \subseteq N$ is a direct summand iff N is amply supplemented and every supplement $K \subseteq N$ is a direct summand iff N is amply supplemented and every short exact sequence in Cocls or in Suppl

$$0 \to K \to N \to N/K \to 0$$

splits iff N is amply supplemented and Cocls-$\mathcal{S}$plit-regular or Suppl-$\mathcal{S}$plit-regular. □

(vi) In R-Mod a module is Compl-$\mathcal{P}$ure-regular iff it is purely extending (see [11, Lemma 1.1]).

Proof. By [11, Lemma 1.1] a module N in R-Mod is purely extending iff every complement (= closed) submodule $K \subseteq N$ is pure in N iff every short exact sequence in Compl

$$0 \to K \to N \to N/K \to 0$$

belongs to $\mathcal{P}$ure iff N is Compl-$\mathcal{P}$ure-regular. □

(vii) In $\sigma[M]$ a module is $\mathcal{A}$bs-$\mathcal{P}$ure-regular iff it is regular (see [61, §37]).

Proof. This is just the definition of a regular module in $\sigma[M]$. □

(viii) In R-Mod if the ring R is noetherian and the module N is flat, then N is Cocls-$\mathcal{P}$ure-regular iff N is $\mathcal{P}$ure-Cocls-regular (see [66, Satz 3.4(b)]).

Proof. By [66, Satz 3.4(b)] if the ring R is noetherian and the module N is flat, then the coclosed submodules and the pure submodules of N coincide. Thus every short exact sequence

$$0 \to K \to N \to N/K \to 0$$

belongs to $\mathcal{P}$ure iff it belongs to Cocls, i.e. N is $\mathcal{P}$ure-Cocls-regular iff N is Cocls-$\mathcal{P}$ure-regular. □

(ix) Every module in $\sigma[M]$ is Co-Neat-Cocls-regular iff every non-zero M-small module in $\sigma[M]$ is a Max module (has a maximal submodule) (see [1, 1.16]).

Proof. By [1, 1.16], every non-zero M-small module N is a Max module iff every non-zero coneat submodule K is coclosed in N iff every short exact sequence in Co-Neat

$$0 \to K \to N \to N/K \to 0$$

belongs to Cocls iff N is Co-Neat-Cocls-regular. □

15.20. Properties of $\mathcal{P}$-$\mathcal{R}$-regular objects.

(i) *If the sequence*

$$0 \to T' \to T \to T'' \to 0$$

belongs to $\mathcal{P}$ *and* T *is* $\mathcal{P}$-$\mathcal{R}$*-regular, then* T' *and* T'' *are* $\mathcal{P}$-$\mathcal{R}$*-regular.*

(ii) *If* $\mathcal{R} = \pi^{-1}(\mathcal{Q})$*, then* T *is* $\mathcal{P}$-$\mathcal{R}$*-regular iff every* $Q \in \mathcal{Q}$ *is projective with respect to all* $\mathcal{P}$*-epimorphisms* $T \to C$*.*

(iii) *If* $\mathcal{R} = \iota^{-1}(\mathcal{I})$*, then* T *is* $\mathcal{P}$-$\mathcal{R}$*-regular iff every* $I \in \mathcal{I}$ *is injective with respect to all* $\mathcal{P}$*-monomorphisms* $A \to T$*.*

(iv) *If T is $\mathcal{P}$-$\mathcal{R}$-flat, then T is $\mathcal{P}$-$\mathcal{R}$-regular iff every $\mathcal{P}$-factor of T is $\mathcal{P}$-$\mathcal{R}$-flat.*

(v) *If T is $\mathcal{P}$-$\mathcal{R}$-divisible, then T is $\mathcal{P}$-$\mathcal{R}$-regular iff every $\mathcal{P}$-subobject of T is $\mathcal{P}$-$\mathcal{R}$-divisible.*

(vi) *Every object of $\mathbb{A}$ is $\mathcal{P}$-$\mathcal{R}$-regular iff $\mathcal{P} \subseteq \mathcal{R}$.*

Proof. (i) Let

$$0 \to A \to T' \to C \to 0$$

be a short exact sequence in $\mathcal{P}$. The composition

$$A \to T' \to T$$

is a $\mathcal{P}$-monomorphism. Since T is $\mathcal{P}$-$\mathcal{R}$-regular, this composition is an $\mathcal{R}$-monomorphism. Thus from 3.1 (P4) follows that $A \to T'$ is an $\mathcal{R}$-monomorphism, i.e. T' is $\mathcal{P}$-$\mathcal{R}$-regular.
A similar argument using that the composition $T \to T'' \to C$ is a $\mathcal{P}$-epimorphism shows hat T'' is $\mathcal{P}$-$\mathcal{R}$-regular.

(ii) $\Rightarrow$) Let

$$0 \to A \to T \to C \to 0$$

be a short exact sequence in $\mathcal{P}$ and $Q \to C$ any morphism with $Q \in \mathcal{Q}$. Form the pullback diagram

$$\begin{array}{ccccccccc}
E' : 0 & \longrightarrow & A & \longrightarrow & T' & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
E : 0 & \longrightarrow & A & \longrightarrow & T & \longrightarrow & C & \longrightarrow & 0.
\end{array}$$

By assumption the sequence E belongs to $\mathcal{R}$, thus E' belongs also to $\mathcal{R}$. Since Q is $\mathcal{R}$-projective, then the sequence E' splits. Therefore there is a morphism $Q \to T$ lifting $Q \to C$, i.e. Q is projective with respect to the $\mathcal{P}$-epimorphism $T \to C$.
$\Leftarrow$) In the pullback diagram above, the sequence E belongs, by definition, to $\mathcal{R} = \pi^{-1}(\mathcal{Q})$ iff every Q is projective with respect to it.

(iii) The proof is dual to (ii).

(iv) $\Rightarrow$) Let T'' be a $\mathcal{P}$-factor object of T, i.e. there is a $\mathcal{P}$-epimorphism $T \to T''$. Let

$$E : 0 \to A \to B \to T'' \to 0$$

be a short exact sequence in $\mathcal{P}$. Form the pullback diagram

$$\begin{array}{ccccccccc}
E' : 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & T & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
E : 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & T'' & \longrightarrow & 0.
\end{array}$$

Since T is $\mathcal{P}$-$\mathcal{R}$-regular, then the epimorphism $T \to T''$ is an $\mathcal{R}$-epimorphism. Also T is $\mathcal{P}$-$\mathcal{R}$-flat, thus the sequence E' belongs to $\mathcal{R}$. Thus, by 3.1 (P5) and (P6), the epimorphism $B \to T''$ is an $\mathcal{R}$-epimorphism, i.e. T'' is $\mathcal{P}$-$\mathcal{R}$-flat.
$\Leftarrow$) Let

$$E : 0 \to A \to T \to C \to 0$$

be a short exact sequence in $\mathcal{P}$. By assumption C is $\mathcal{P}$-$\mathcal{R}$-flat, then $E \in \mathcal{R}$, i.e. T is $\mathcal{P}$-$\mathcal{R}$-regular.

(v) The proof is dual to (iv).

(vi) $\Rightarrow$ Is clear since every short exact sequence in $\mathcal{P}$ belongs to $\mathcal{R}$.
$\Leftarrow$ Is also clear. $\square$

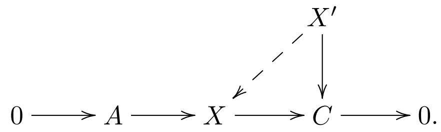
16 Covers and envelopes relative to a proper class.

We now turn our attention to relative aproximations by introducing (pre)covers and (pre)envelopes relative to a proper class $\mathcal{P}$. These notions were considered in [3] for the category of finitely generated Λ-modules over an artin algebra Λ.

16.1. Covers and envelopes relative to $\mathcal{P}$. Let $\mathcal{X}$ be a class of objects of $\mathbb{A}$ and $\mathcal{P}$ a proper class. An **$\mathcal{X}$-precover relative to $\mathcal{P}$** of an object $C \in \mathbb{A}$ is a short exact sequence

$$E : 0 \to A \to X \xrightarrow{f} C \to 0$$

in $\mathcal{P}$ with $X \in \mathcal{X}$ such that the map $\mathrm{Hom}_{\mathbb{A}}(X', X) \to \mathrm{Hom}_{\mathbb{A}}(X, C)$ is surjective for all $X' \in \mathcal{X}$. This can be expressed by the diagram

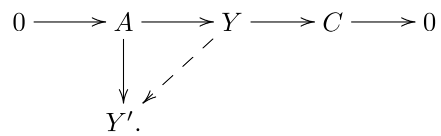


An $\mathcal{X}$-precover relative to $\mathcal{P}$ is an **$\mathcal{X}$-cover relative to $\mathcal{P}$** of C if every $h \in \mathrm{End}(X)$ such that $fh = f$ is an automorphism. A class $\mathcal{X}$ is a **$\mathcal{P}$-precover class** (**$\mathcal{P}$-cover class**) if every object $C \in \mathbb{A}$ has an $\mathcal{X}$-precover ($\mathcal{X}$-cover) relative to $\mathcal{P}$. An $\mathcal{X}$-precover E of C relative to $\mathcal{P}$ is called **special** if $\mathrm{Ker}\, f \in \mathcal{X}^{\perp_{\mathcal{P}}}$.

Let $\mathcal{Y}$ be a class of objects of $\mathbb{A}$ and $\mathcal{P}$ a proper class. A **$\mathcal{Y}$-preenvelope relative to $\mathcal{P}$** of an object $A \in \mathbb{A}$ is a short exact sequence

$$E : 0 \to A \xrightarrow{g} Y \to C \to 0$$

in $\mathcal{P}$ with $Y \in \mathcal{Y}$ such that the map $\mathrm{Hom}_{\mathbb{A}}(Y, Y') \to \mathrm{Hom}_{\mathbb{A}}(A, Y')$ is surjective for all $Y' \in \mathcal{Y}$. This can be expressed by the diagram



A $\mathcal{Y}$-preenvelope relative to $\mathcal{P}$ is a **$\mathcal{Y}$-envelope relative to $\mathcal{P}$** of A if every $h \in \mathrm{End}(Y)$ such that $hg = g$ is an automorphism. A class $\mathcal{Y}$ is a **$\mathcal{P}$-preenvelope class** (**$\mathcal{P}$-envelope class**) if every object $A \in \mathbb{A}$ has a $\mathcal{Y}$-preenvelope ($\mathcal{Y}$-envelope) relative to $\mathcal{P}$. A $\mathcal{Y}$-preenvelope E of A relative to $\mathcal{P}$ is called **special** if $\mathrm{Coker}\, g \in {}^{\perp_{\mathcal{P}}}\mathcal{Y}$.

16.2 Remark. Let $\mathbb{A}$ be locally finitely presented Grothendieck category. It was proved in [54, 4.5] that every object X in $\mathbb{A}$ has a pure injective envelope. Thus the class $\mathrm{Inj}(\mathcal{P}\mathrm{ure})$ is a $\mathcal{P}$ure-envelope class.

16.3. Proposition. *Let $\mathcal{P} = \iota^{-1}(\mathcal{I})$ be a proper class injectively generated by a class $\mathcal{I}$ in an abelian category $\mathbb{A}$ and $\mathcal{Y}$ a class in $\mathbb{A}$ closed under isomorphisms. The following are equivalent:*

(a) *$\mathcal{Y}$ is a $\mathcal{P}$-envelope class.*

(b) *$\mathcal{Y}$ is an envelope class and* $\mathrm{Inj}(\mathcal{P}) \subseteq \mathcal{Y}$.

Proof. (a) $\Rightarrow$ (b) It is clear that $\mathcal{Y}$ is an envelope class. Let I be a $\mathcal{P}$-injective object in $\mathbb{A}$. By hypothesis, I has a $\mathcal{Y}$-envelope relative to $\mathcal{P}$

$$E : 0 \to I \xrightarrow{i} Y \to C \to 0.$$

Since E belongs to $\mathcal{P}$ and I is $\mathcal{P}$-injective, the sequence E splits. Thus I is a direct summand of Y. Consider the composition $Y \xrightarrow{p} I \xrightarrow{i} Y$, with p the canonical projection. Note that $ipi = i$. Since E is a $\mathcal{Y}$-envelope relative to $\mathcal{P}$, ip must be an isomorphism, thus i is also an isomorphism, i.e. $I \simeq Y$. Therefore $\mathrm{Inj}(\mathcal{P}) \subseteq \mathcal{Y}$.

(b) $\Rightarrow$ (a) Let A be an object of $\mathbb{A}$. By hypothesis, A has a $\mathcal{Y}$-envelope $i : A \to Y$. Complete i to a short exact sequence

$$E : 0 \to A \xrightarrow{i} Y \to \mathrm{Coker}\, i \to 0.$$

Let I be an object of $\mathcal{I}$ and $f : A \to I$ any morphism. Since $\mathrm{Inj}(\mathcal{P}) \subseteq \mathcal{Y}$ and E is an $\mathcal{Y}$-envelope of A, there is a morphism $\overline{f} : Y \to I$ such that $\overline{f}i = f$. This implies that E belongs to $\mathcal{P}$. Thus E is a $\mathcal{Y}$-envelope relative to $\mathcal{P}$ of A. □

Dually we obtain.

16.4. Proposition. *Let $\mathcal{P} = \pi^{-1}(\mathcal{Q})$ be a proper class projectively generated by a class $\mathcal{Q}$ in an abelian category $\mathbb{A}$ and $\mathcal{X}$ a class in $\mathbb{A}$ closed under isomorphisms. The following are equivalent:*

(a) *$\mathcal{X}$ is a $\mathcal{P}$-cover class.*

(b) *$\mathcal{X}$ is a cover class and* $\mathrm{Proj}(\mathcal{P}) \subseteq \mathcal{X}$.

16.5. Lemma. *Let $\mathcal{P}$ be a proper class in an abelian category $\mathbb{A}$ with enough $\mathcal{P}$-projectives and $\mathcal{P}$-injectives and $(\mathcal{F}, \mathcal{C})$ a $\mathcal{P}$-cotorsion pair. The following are equivalent:*

(a) *Every object of $\mathbb{A}$ has a special $\mathcal{F}$-precover relative to $\mathcal{P}$.*

(b) *Every object of $\mathbb{A}$ has a special $\mathcal{C}$-preenvelope relative to $\mathcal{P}$.*

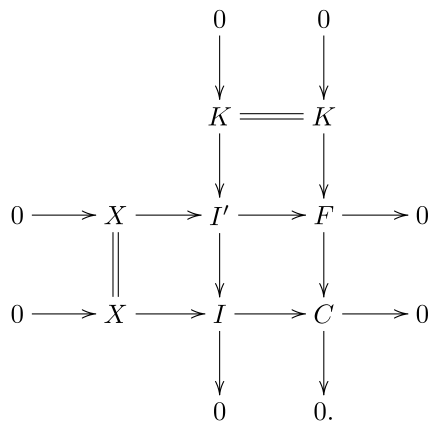
Proof. (a) $\Rightarrow$ (b) Let X be any object of $\mathbb{A}$. Since $\mathcal{P}$ has enough $\mathcal{P}$-injectives, there is a short exact sequence in $\mathcal{P}$

$$0 \to X \to I \to C \to 0$$

with I $\mathcal{P}$-injective. Since $\mathrm{Inj}(\mathcal{P}) \subseteq \mathcal{C}$, $I \in \mathcal{C}$. By hypothesis, C has a special $\mathcal{F}$-precover relative to $\mathcal{P}$

$$0 \to K \to F \to C \to 0.$$

Consider the following commutative diagram with all rows and columns in $\mathcal{P}$



Since K and I are in $\mathcal{C}$ and $\mathcal{C}$ is closed under $\mathcal{P}$-extensions (15.3 (i)), $I' \in \mathcal{C}$. Thus

$$0 \to X \to I' \to F \to 0$$

is a special $\mathcal{C}$-preenvelope relative to $\mathcal{P}$ of X.

(b) $\Rightarrow$ (a) is dual. $\square$

The following is a relative version of Wakamatsu's Lemma.

16.6. Relative Wakamatsu's lemma. *Let $\mathbb{A}$ be an abelian category and $\mathcal{X}$ a class of objects of $\mathbb{A}$ closed under $\mathcal{P}$-extensions.*

(i) *If $E_A : 0 \to A \to X_A \to Y_A \to 0$ is an $\mathcal{X}$-envelope of A relative to $\mathcal{P}$, then E_A is special.*

(ii) *If $E_C : 0 \to Y_C \to X_C \to C \to 0$ is an $\mathcal{X}$-cover of C relative to $\mathcal{P}$, then E_C is special.*

Proof. (i) Let

$$E : 0 \to X \to B \to Y_A \to 0$$

be a short exact sequence in $\mathcal{P}$ with $X \in \mathcal{X}$. Consider the following commutative pullback diagram with all rows and columns in $\mathcal{P}$

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X & = & X & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & X'_A & \to & B & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & X_A & \to & Y_A & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Since X and X_A are in $\mathcal{X}$ and $\mathcal{X}$ is $\mathcal{P}$-extension closed, X'_A must be in $\mathcal{X}$. By hypothesis, E_A is a $\mathcal{X}$-envelope relative to $\mathcal{P}$ of A, thus we can find a morphism $X_A \to X'_A$ which yields the following diagram commutative

$$\begin{array}{ccccccccc}
0 & \to & A & \to & X_A & \to & Y_A & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & X'_A & \to & B & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & X_A & \to & Y_A & \to & 0.
\end{array}$$

Since E_A is an $\mathcal{X}$-envelope relative to $\mathcal{P}$, the composition

$$X_A \to X'_A \to X_A$$

is an isomorphism. Thus the composition

$$Y_A \to B \to Y_A$$

is the identity of Y_A. This implies that the sequence E splits, therefore $Y_A \in {}^{\perp_{\mathcal{P}}}\mathcal{X}$.

(ii) is dual. $\square$

Recall that a $\mathcal{P}$-cotorsion pair $(\mathcal{F}, \mathcal{C})$ in an abelian category $\mathbb{A}$ is called **complete** if every object of $\mathbb{A}$ has a special $\mathcal{F}$-precover relative to $\mathcal{P}$ and a special $\mathcal{C}$-preenvelope relative to $\mathcal{P}$. In [37] Hovey proved, using the small object argument of Quillen [36, Section 2.1], that every *small* $\mathcal{P}$-cotorsion pair in a Grothendieck category is complete provided transfinite compositions of $\mathcal{P}$-monomorphisms are $\mathcal{P}$-monomorphisms.

16.7. Transfinite compositions. Let $\{(A_\alpha)_{\alpha<\lambda}, (j_{\alpha,\beta})_{\alpha<\beta<\lambda}\}$ be a direct system in a cocomplete abelian category $\mathbb{A}$ indexed by an ordinal λ and such that $\varinjlim_{\alpha<\gamma} A_\alpha \to \mathcal{A}_\gamma$ is an isomorphism for each $\gamma < \lambda$. The morphism

$$f : A_0 \to \varinjlim_{\alpha<\lambda} A_\alpha,$$

is called the **transfinite composition** of the morphisms $j_{\alpha,\alpha+1} : A_\alpha \to A_{\alpha+1}$.

16.8. Small $\mathcal{P}$-cotorsion pairs. Let $\mathbb{A}$ be a complete and cocomplete abelian category and $\mathcal{P}$ a proper class such that transfinite composition of $\mathcal{P}$-monomorphisms are $\mathcal{P}$-monomorphisms. A $\mathcal{P}$-cotorsion pair $(\mathcal{F}, \mathcal{C})$ is called **small** if the following conditions hold:

(i) There is a set $\{U_i\}$ of objects in $\mathcal{F}$ such that $\mathcal{P}$ is projectively generated by $\{U_i\}$.

(ii) $(\mathcal{F}, \mathcal{C})$ is cogenerated by a set, i.e. there is a set $\mathcal{G}$ of objects in $\mathcal{F}$ such that $\mathcal{C} = \mathcal{G}^{\perp_\mathcal{P}}$.

(iii) For each $G \in \mathcal{G}$, there is a $\mathcal{P}$-monomorphism i_G with cokernel G such that, if an object X is injective with respect to all i_G, $G \in \mathcal{G}$, then $X \in \mathcal{C}$.

The set of morphisms i_G together with the morphisms $0 \to U_i$ is referred to as a set of **generating monomorphisms** of $(\mathcal{F}, \mathcal{C})$.

16.9. Proposition [37, 6.5]. *Let $\mathbb{A}$ be a Grothendieck category and $\mathcal{P}$ a proper class such that transfinite composition of $\mathcal{P}$-monomorphisms are $\mathcal{P}$-monomorphisms. Let $\mathcal{I}$ be a set of $\mathcal{P}$-monomorphisms in $\mathbb{A}$. Then $\mathcal{I}$ is a set of generating monomorphisms for a small $\mathcal{P}$-cotorsion pair iff the following conditions hold:*

(i) *$\mathcal{I}$ contains the morphisms $0 \to U_i$ for some set $\{U_i\}$ which projectively generates $\mathcal{P}$.*

(ii) *For every object $X \in \mathbb{A}$ such that X is injective with respect to all $i \in \mathcal{I}$, we have* $\mathrm{Ext}^1_\mathcal{P}(\mathrm{Coker}\, j, X) = 0$ *for all $j \in \mathcal{I}$.*

Furthermore, in this case, $\mathcal{C}$ is the class of all X such that X is injective with respect to all $i \in \mathcal{I}$, $\mathcal{F}$ is the smallest class containing the cokernels of the morphisms of $\mathcal{I}$ that is closed under summands and transfinite extensions, and $(\mathcal{F}, \mathcal{C})$ is complete.

16.10. Corollary [37, 6.6]. *Let $\mathbb{A}$ be a Grothendieck category and $\mathcal{P}$ a proper class such that transfinite composition of $\mathcal{P}$-monomorphisms are $\mathcal{P}$-monomorphisms. Then every small $\mathcal{P}$-cotorsion pair is complete.*

Appendix

17 The functor $\mathrm{Ext}^n_{\mathcal{P}}$.

Throughout this section let $\mathbb{A}$ be an abelian category and $\mathcal{P}$ a proper class.

17.1. Morphisms of short exact sequences. A **morphism of short exact sequences** in $\mathcal{P}$, $\Gamma : E \to E'$ is a triple $\Gamma = (\alpha, \beta, \gamma)$ of morphisms in $\mathbb{A}$ such that the diagram

$$\begin{array}{ccccccccc} E : 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ E' : 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

is commutative.

17.2. Congruence of short exact sequences. Two short exact sequences in $\mathcal{P}$ beginning with A and ending at C

$$E : 0 \to A \to B \to C \to 0 \quad \text{and} \quad E' : 0 \to A \to B' \to C \to 0$$

are **congruent** if there is a morphism $(1_A, \beta, 1_C) : E \to E'$. We write $E \equiv E'$.

Note that the morphism β must be an isomorphism. Thus " $\equiv$ " is an equivalence relation on the set of short exact sequences in $\mathcal{P}$ beginning with A and ending at C.
Let $\alpha : A \to A'$ be a morphism and $E : 0 \to A \to B \to C \to 0 \in \mathcal{P}$. We define αE to be the short exact sequence obtained from the pushout diagram

$$\begin{array}{ccccccccc} E : 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \| & & \\ \alpha E : 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

By 3.1 (P5) and (P6), αE belongs to $\mathcal{P}$. Dually, for $\gamma : C' \to C$ we define $E\gamma$ to be the short exact sequence obtained from the pullback diagram

$$\begin{array}{ccccccccc} E\gamma : 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \gamma} & & \\ E : 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

By 3.1 (P3) and (P4), $E\gamma$ belongs to $\mathcal{P}$. The sequences αE and $E\gamma$ are unique up to congruence.

17.3. Diagonal and codiagonal morphism. Denote by $i_1, i_2 : Y \to Y \oplus Y$ the canonical inclusions. We denote by $\nabla_Y : Y \oplus Y \to Y$ the morphism which renders

$$\begin{array}{ccccc} Y & \xrightarrow{i_1} & Y \oplus Y & \xleftarrow{i_2} & Y \\ \| & & \downarrow{\scriptstyle \nabla} & & \| \\ Y & = & Y & = & Y \end{array}$$

commutative. We call ∇_Y the **codiagonal** morphism. Dually denote $\pi_1, \pi_2 : X \oplus X \to X$ the canonical projections. We denote by $\Delta_X : X \to X \oplus X$ the morphism which renders

$$\begin{array}{ccccc} X & = & X & = & X \\ \| & & \downarrow{\scriptstyle \Delta} & & \| \\ X & \xleftarrow{\pi_1} & X \oplus X & \xrightarrow{\pi_2} & X \end{array}$$

commutative. We call Δ_X the **diagonal** morphism.

17.4. The Baer sum. Given two short exact sequences in $\mathcal{P}$

$$E_i : 0 \to A_i \to B_i \to C_i \to 0$$

$i = 1, 2$ we define their **direct sum** to be the short exact sequence

$$E_1 \oplus E_2 : 0 \to A_1 \oplus A_2 \to B_1 \oplus B_2 \to C_1 \oplus C_2 \to 0.$$

By [43, XII.4.1], $E_1 \oplus E_2$ belongs to $\mathcal{P}$. Now let E_1 and E_2 be short exact sequences in $\mathcal{P}$ beginning with A and ending at C. The **Baer sum** of E_1 and E_2 is given by the formula

$$E_1 + E_2 = \nabla_A (E_1 \oplus E_2) \Delta_C.$$

Clearly $E_1 + E_2$ belongs to $\mathcal{P}$.

17.5. $\mathrm{Ext}^1_{\mathcal{P}}(C, A)$. Denote by $\mathrm{Ext}^1_{\mathcal{P}}(C, A)$ the congruence class of short exact sequences in $\mathcal{P}$ beginning with A and ending at C. We have the following properties:

(i) $\mathrm{Ext}^1_{\mathcal{P}}(C, A)$ is an abelian group,

  (1) the sum is given by the Baer sum,

  (2) for $E \in \mathrm{Ext}^1_{\mathcal{P}}(C, A)$ the inverse is $(-1_A)E$,

  (3) the zero element is the class of the split sequence

$$0 \to A \to A \oplus C \to C \to 0,$$

(ii) for $\alpha : A' \to A$ and $\gamma : C' \to C$ we have the identities

$$\alpha(E_1 + E_2) \equiv \alpha E_1 + \alpha E_2 \quad , \quad (E_1 + E_2)\gamma \equiv E_1\gamma + E_2\gamma,$$

(iii) for $\alpha_1, \alpha_2 : A \to A'$ and $\gamma_1, \gamma_2 : C' \to C$ we have the identities

$$(\alpha_1 + \alpha_2)E \equiv \alpha_1 E + \alpha_2 E \quad , \quad E(\gamma_1 + \gamma_2) \equiv E\gamma_1 + E\gamma_2.$$

This means that $\mathrm{Ext}^1_{\mathcal{P}}(-,-) : \mathbb{A}^{op} \times \mathbb{A} \to \mathcal{A}b$ is a bifunctor from $\mathbb{A}^{op} \times \mathbb{A}$ to the category $\mathcal{A}b$ of abelian groups and the morphisms

$$\alpha_* : \mathrm{Ext}^1_{\mathcal{P}}(C, A) \to \mathrm{Ext}^1_{\mathcal{P}}(C, A'), \qquad E \mapsto \alpha E$$

$$\gamma^* : \mathrm{Ext}^1_{\mathcal{P}}(C, A) \to \mathrm{Ext}^1_{\mathcal{P}}(C', A), \qquad E \mapsto E\gamma$$

are abelian group morphisms.

17.6. $\mathcal{P}$-proper n-fold exact sequences. An n-fold exact sequence

$$S : 0 \to A = B_n \to B_{n-1} \to B_{n-2} \to \cdots \to B_0 \to C = B_{-1} \to 0$$

is called **$\mathcal{P}$-proper** if for all $i = 0, ..., n-1$ the sequence

$$0 \to \mathrm{Ker}\,(B_{i+1} \to B_i) \to B_i \to \mathrm{Im}\,(B_i \to B_{i-1}) \to 0$$

belongs to $\mathcal{P}$.

17.7. The Yoneda composite. Consider a $\mathcal{P}$-proper n-fold exact sequence beginning with A and ending at K

$$S : 0 \to A \to B_{n-1} \to B_{n-2} \to \cdots \to B_0 \to K \to 0$$

and a $\mathcal{P}$-proper m-fold exact sequence beginning with K and ending at C

$$T : 0 \to K \to B'_{m-1} \to B'_{m-2} \to \cdots \to B'_0 \to C \to 0.$$

This two sequences may be glued together by the composite morphism

$$\omega : B_0 \to K \to B'_{m-1}$$

to give a $\mathcal{P}$-proper (n+m)-fold exact sequence $S \circ T$ called the **Yoneda composite** of S and T

$$S \circ T : 0 \to A \to B_{n-1} \to \cdots \to B_0 \xrightarrow{\omega} B'_{m-1} \to \cdots \to B'_0 \to C \to 0.$$

This composition is associative. We can write any $\mathcal{P}$-proper n-fold exact sequence S as the composition of n short exact sequences in $\mathcal{P}$

$$E_i : 0 \to \mathrm{Ker}(B_{i-1} \to B_{i-2}) \to B_{i-1} \to \mathrm{Im}(B_{i-1} \to B_{i-2}) \to 0$$

in the form $S = E_n \circ E_{n-1} \circ \cdots \circ E_1$. The E_i are unique up to isomorphism.

17.8. Morphisms of $\mathcal{P}$-proper n-fold exact sequences. A morphism $\Gamma : S \to S'$ of $\mathcal{P}$-proper n-fold exact sequences is an $(n+2)$-tuple $\Gamma = (\alpha, \alpha_{n-1}, \ldots, \alpha_0, \gamma)$ of morphisms in $\mathcal{C}$ such that the diagram

$$\begin{array}{ccccccccccccc} S : 0 & \to & A & \to & B_{n-1} & \to & B_{n-2} & \to \cdots \to & B_0 & \to & C & \to & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha_{n-1}} & & \downarrow{\scriptstyle\alpha_{n-2}} & & \downarrow{\scriptstyle\alpha_0} & & \downarrow{\scriptstyle\gamma} & & \\ S' : 0 & \to & A' & \to & B'_{n-1} & \to & B'_{n-2} & \to \cdots \to & B'_0 & \to & C' & \to & 0 \end{array}$$

is commutative.

17.9. Congruence of $\mathcal{P}$-proper n-fold exact sequences. Two $\mathcal{P}$-proper n-fold exact sequences S, S' beginning with A and ending at C are **cogruent** if given a factorization of $S = E_n \circ E_{n-1} \circ \cdots \circ E_1$, S' can be obtained from S by a finite sequence of replacements of the following types:

(i) Replace any E_i by a congruent short exact sequence in $\mathcal{P}$,

(ii) replace any pair of successive factors of the form $E''\beta \circ E'$ by $E'' \circ \beta E'$,

(iii) replace any pair of successive factors of the form $E'' \circ \beta E'$ by $E''\beta \circ E'$.

We write $S \equiv S'$.

The relation " $\equiv$ " is an equivalence relation on the set of $\mathcal{P}$-proper n-fold exact sequences beginning with A and ending at C.

17.10. Composite with morphisms. Let $\alpha : A \to A'$, $\gamma : C' \to C$ and

$$S : 0 \to A \to B_{n-1} \to B_{n-2} \to \cdots \to B_0 \to C \to 0$$

a $\mathcal{P}$-proper n-fold exact sequence. We define the $\mathcal{P}$-proper n-fold exact sequences αS and $S\gamma$ by

$$\alpha S = \alpha(E_n \circ E_{n-1} \circ \cdots \circ E_1) = (\alpha E_n) \circ E_{n-1} \circ \cdots \circ E_1,$$
$$S\gamma = (E_n \circ E_{n-1} \circ \cdots \circ E_1)\gamma = E_n \circ E_{n-1} \circ \cdots \circ (E_1\gamma).$$

This formulas define morphisms of $\mathcal{P}$-proper n-fold exact sequences

$$(\alpha, \alpha_{n-1}, \cdots, \alpha_0, 1_C) : S \to \alpha S \quad \text{and} \quad (1_A, \gamma_{n-1}, \cdots, \gamma_0, \gamma) : S\gamma \to S.$$

Any morphism $\Gamma = (\alpha, \alpha_{n-1}, \ldots, \alpha_0, \gamma) : S \to S'$ yields a congruence

$$\alpha S \equiv S'\gamma.$$

17.11. The Baer sum of $\mathcal{P}$-proper n-fold exact sequences. Two $\mathcal{P}$-proper n-fold exact sequences S and T have a direct sum

$$S \oplus T : 0 \to A \oplus A' \to B_{n-1} \oplus B'_{n-1} \to \cdots \to B_0 \oplus B'_0 \to C \oplus C' \to 0.$$

If $S \equiv S'$ and $T \equiv T'$, then $S \oplus T \equiv S' \oplus T'$. We define the **Baer sum** of S and T, both beginning with A and ending at C, by the formula

$$S + T = \nabla_A(S \oplus T)\Delta_C.$$

17.12. $\mathrm{Ext}^n_{\mathcal{P}}(C, A)$. Let $n > 1$. Denote by $\mathrm{Ext}^n_{\mathcal{P}}(C, A)$ the congruence class of $\mathcal{P}$-proper n-fold exact sequences beginning with A and ending at C. We have the following properties:

(i) $\mathrm{Ext}^n_{\mathcal{P}}(C, A)$ is an abelian group,

(1) the sum is given by the Baer sum,

(2) for $S \in \mathrm{Ext}^n_{\mathcal{P}}(C, A)$ the inverse is $(-1_A)S$,

(3) the zero element is the class of the $\mathcal{P}$-proper n-fold sequence

$$0 \to A \xrightarrow{1_A} A \to 0 \to \cdots \to 0 \to C \xrightarrow{1_C} C \to 0.$$

This makes $\mathrm{Ext}^n_{\mathcal{P}}(-,-) : \mathbb{A}^{op} \times \mathbb{A} \to \mathcal{A}b$ a bifunctor from $\mathbb{A}^{op} \times \mathbb{A}$ to the category $\mathcal{A}b$ of abelian groups.
We have defined for each $n > 1$ the functors $\mathrm{Ext}^n_{\mathcal{P}}(-,-)$. For $n = 1$, $\mathrm{Ext}^1_{\mathcal{P}}(-,-)$ was already defined in 17.5 and for $n = 0$ we set $\mathrm{Ext}^0_{\mathcal{P}}(-,-) = \mathrm{Hom}_{\mathbb{A}}(-,-)$.

17.13. Long exact sequences induced by $\mathrm{Ext}^n_{\mathcal{P}}$. Let

$$E : 0 \to A \to B \to C \to 0$$

be a short exact sequence in $\mathcal{P}$ and G any object of $\mathbb{A}$. There exist two exact sequences of abelian groups

$$\cdots \to \mathrm{Ext}^{n-1}_{\mathcal{P}}(A, G) \to \mathrm{Ext}^n_{\mathcal{P}}(C, G) \to \mathrm{Ext}^n_{\mathcal{P}}(B, G) \to \mathrm{Ext}^n_{\mathcal{P}}(A, G) \to \cdots$$

$$\cdots \to \mathrm{Ext}^{n-1}_{\mathcal{P}}(G, C) \to \mathrm{Ext}^n_{\mathcal{P}}(G, A) \to \mathrm{Ext}^n_{\mathcal{P}}(G, B) \to \mathrm{Ext}^n_{\mathcal{P}}(G, C) \to \cdots$$

Proof. See [43, XII.5.1]. □

18 $\mathcal{P}$-dimensions

18.1. $\mathcal{P}$-projective and $\mathcal{P}$-injective dimensions (see [2]). The **$\mathcal{P}$-projective dimension** of an object $C \in \mathbb{A}$ is defined by

$$\mathcal{P}\text{-proj.dim.}C = \min \{n \mid \mathrm{Ext}^{n+1}_{\mathcal{P}}(C, A) = 0 \text{ for all } A \in \mathbb{A}\}.$$

The **$\mathcal{P}$-injective dimension** of an object $A \in \mathcal{C}$ is defined by

$$\mathcal{P}\text{-inj.dim.}A = \min \{n \mid \mathrm{Ext}^{n+1}_{\mathcal{P}}(C, A) = 0 \text{ for all } C \in \mathbb{A}\}.$$

If there is no such n, we put $\mathcal{P}$-proj.dim.$C = \infty$ respectively $\mathcal{P}$-inj.dim.$C = \infty$.
The **global dimension** of the proper class $\mathcal{P}$ is defined by

$$\text{gl.dim.}\mathcal{P} = \sup\{\mathcal{P}\text{-proj.dim.}C \mid C \in \mathbb{A}\} = \sup\{\mathcal{P}\text{-inj.dim.}A \mid A \in \mathbb{A}\}.$$

18.2. $\mathcal{P}$-proper complexes. A sequence

$$\cdots \to X_{-1} \xrightarrow{\delta_{-1}} X_0 \xrightarrow{\delta_0} X_1 \xrightarrow{\delta_1} X_2 \to \cdots$$

(not necessarily exact) is called a **$\mathcal{P}$-proper complex** if for all $n \in \mathbb{Z}$, $\delta_n \delta_{n-1} = 0$ and the sequence

$$0 \to \mathrm{Ker}\,\delta_n \to X_n \to \mathrm{Im}\,\delta_n \to 0$$

belongs to $\mathcal{P}$.

18.3. $\mathcal{P}$-projective resolutions. A **$\mathcal{P}$-projective resolution** of an object $C \in \mathbb{A}$ is a $\mathcal{P}$-proper complex

$$\cdots \to P_{-1} \to P_0 \to C \to 0$$

with each $P_i \in \mathrm{Proj}(\mathcal{P})$.

18.4. $\mathcal{P}$-injective resolutions. A **$\mathcal{P}$-injective resolution** of an object $A \in \mathbb{A}$ is a $\mathcal{P}$-proper complex

$$0 \to A \to I_0 \to I_1 \to \cdots$$

with each $I_i \in \mathrm{Inj}(\mathcal{P})$.

If $\mathcal{P}$ is a projective proper class, then each object $X \in \mathbb{A}$ has a $\mathcal{P}$-projective resolution. Dually, if $\mathcal{P}$ is an injective proper class, then each object $X \in \mathbb{A}$ has a $\mathcal{P}$-injective resolution.

18.5. Proposition. *Let $\mathcal{P}$ be a projective proper class in $\mathbb{A}$. The following are equivalent for an object $C \in \mathbb{A}$:*

(a) $\mathcal{P}$-proj.dim.$C \leq n$

(b) *any $\mathcal{P}$-proper sequence*

$$0 \to C_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to C \to 0$$

with all P_i $\mathcal{P}$-projective has the first term C_n $\mathcal{P}$-projective,

(c) *C has a $\mathcal{P}$-projective resolution of length n:*

$$0 \to P_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to C \to 0.$$

Proof. See [43, VII.1]. □

18.6. Proposition. *Let $\mathcal{P}$ be an injective proper class in $\mathbb{A}$. The following are equivalent for an object $A \in \mathbb{A}$:*

(a) $\mathcal{P}$-inj.dim.$A \leq n$

(b) *any $\mathcal{P}$-proper sequence*

$$0 \to A \to I_0 \to I_1 \to \cdots \to I_{n-1} \to A_n \to 0$$

with all I_i $\mathcal{P}$-injective has the last term A_n $\mathcal{P}$-injective,

(c) *A has a $\mathcal{P}$-injective resolution of length n*

$$0 \to A \to I_0 \to I_1 \to \cdots \to I_{n-1} \to I_n \to 0.$$

Proof. See [43, VII.1]. □

18.7. $\mathcal{P}$-thick subcategories. A subcategory $\mathcal{T}$ of $\mathbb{A}$ is called **$\mathcal{P}$-thick** if it is closed under retracts and whenever two out of three entries in a short exact sequence of $\mathcal{P}$ are in $\mathcal{T}$ so is the third.

18.8. Gorenstein proper classes. Let $\mathcal{P}$ be a projective and injective proper class. $\mathcal{P}$ is called a **Gorenstein proper class** if the objects of finite $\mathcal{P}$-projective dimension and the objects of finite $\mathcal{P}$-injective dimension coincide.

18.9 Proposition. *Let $\mathcal{P}$ be a Gorenstein proper class. Then the class of objects of finite $\mathcal{P}$-projective dimension (= finite $\mathcal{P}$-injective dimension) is a $\mathcal{P}$-thick subcategory of $\mathbb{A}$.*

Proof. Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence in $\mathcal{P}$. For every $X \in \mathbb{A}$ there is a long exact sequence

$$\cdots \to \mathrm{Ext}^i_{\mathcal{P}}(C, X) \to \mathrm{Ext}^i_{\mathcal{P}}(B, X) \to \mathrm{Ext}^i_{\mathcal{P}}(A, X) \to \mathrm{Ext}^{i+1}_{\mathcal{P}}(C, X) \to \cdots$$

It is clear that if two of A, B, C have finite $\mathcal{P}$-projective dimension, then so does the third. □

18.10 Example. Let R be an Iwanaga-Gorenstein ring i.e. R is left and right noetherian and R has finite self-injective dimension on both the left and the right. It is known that over such rings the modules of finite projective dimension coincide with the modules of finite injective dimension [20]. In this case we take $\mathcal{P}$ to be all short exact sequences, thus $\mathcal{P}$-projective (-injective) means projective (injective).

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