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STRUCTURE OF CLOSED LINEAR TRANSLATION INVARIANT SUBSPACES OF A($\mathbb{C}$) AND KERNELS OF ANALYTIC CONVOLUTION OPERATORS

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Let $A(\mathbb{C})$ denote the vector space of all entire functions on $\mathbb{C}$, endowed with the compact-open topology. Every continuous linear functional μ on $A(\mathbb{C})$ induces a continuous linear map T_μ on $A(\mathbb{C})$ by

$$T_\mu(f) : z \mapsto < \mu_w, f(z+w) > , \ f \in A(\mathbb{C}).$$

These operators are called convolution operators and can also be regarded as differential operators of infinite order with constant coefficients. From this point of view, the structure of ker T_μ has already been investigated by Ritt [18] in 1917. A first answer to the more general question about the structure of the closed linear translation invariant subspaces of $A(\mathbb{C})$ was given by Schwartz [19]. Concerning the representation of the elements of ker T_μ by exponential monomials, Gelfond [9], Dickson [4] and Ehrenpreis [8] showed that, for every convolution operator T_μ on $A(\mathbb{C})$, ker T_μ has a finite dimensional decomposition for which the finite dimensional blocks are spanned by exponential polynomials.

The aim of the present note is to report on some progress concerning the study of such questions which has been made by the work of Berenstein and Taylor [1],[2], Taylor [21], the author [15] and Meise and Schwerdtfeger [16]. Even though the results are rather general, we restrict our attention here to the special situation introduced above since this allows a clear exposition of the ideas without too many technicalities. For a more general survey on part of this work (up to 1980) we refer to the article of Berenstein and Taylor [3].

This report is divided in three sections. In the first one, we introduce the convolution operators on $A(\mathbb{C})$ and show that the question on the structure of the closed linear translation invariant subspaces of $A(\mathbb{C})$ is equivalent - up to duality - to the structure of the quotients of the space $\mathrm{Exp}(\mathbb{C})$ of entire functions of exponential type by its closed ideals. In section 2 we explain how a result of Schwartz [19] on closed ideals in $\mathrm{Exp}(\mathbb{C})$ and the minimum modulus theorem, together with the approach of Berenstein and Taylor [1], lead to a fairly explicit model for $\mathrm{Exp}(\mathbb{C})/I$, where I is a closed non-zero ideal with infinite codimension. Then we describe in section 3 how an observation of the author [15] can be used to derive from this model the following result: Every infinite dimensional closed linear proper subspace W of $A(\mathbb{C})$ which is translation invariant has a Schauder basis consisting of exponential polynomials. With respect to this basis W is isomorphic to a nuclear power series space of infinite type. W is a complemented subspace of $A(\mathbb{C})$. Since this applies, in particular, to the kernels of convolution operators, it shows that the finite dimensional decomposition of ker T_μ mentioned above actually comes from grouping a certain Schauder basis. The model of $\mathrm{Exp}(\mathbb{C})/I$ obtained so far is then applied to derive, for a certain class of convolution operators T_μ, a necessary condition that the exponential monomials form a basis of ker T_μ. Concluding we use this condition to get some examples.

1. INTRODUCTION AND FORMULATION OF THE PROBLEM

1.1 CONVOLUTION OPERATORS ON $A(\mathbb{C})$

By $A(\mathbb{C})$ we denote the space of all entire functions on $\mathbb{C}$, endowed with the usual compact-open topology. The strong dual of $A(\mathbb{C})$ will be denoted by $A(\mathbb{C})_b'$; its elements will be called analytic functionals.
If μ is an analytic functional, then it is easy to check that μ induces a continuous linear operator T_μ on $A(\mathbb{C})$ by the following definition

(1) $$T_\mu(f) : z \mapsto \langle \mu_w, f(z+w) \rangle , \; z \in \mathbb{C}, \; f \in A(\mathbb{C}).$$

These operators are called convolution operators. They can also be characterized as those continuous linear operators on $A(\mathbb{C})$ which commute with all the translation operators $\tau_a : f \mapsto f(\cdot + a)$, $a \in \mathbb{C}$. If $f \in A(\mathbb{C})$ has the Taylor expansion $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and if we put $\mu_n := \langle \mu, z^n \rangle$, $n \in \mathbb{N}_0$, then one can show that, for all $z \in \mathbb{C}$,

(2) $$T_\mu(f)[z] = \sum_{n=0}^{\infty} \frac{1}{n!} \mu_n f^{(n)}(z) = \sum_{k=0}^{\infty} \Big(\sum_{n=0}^{\infty} \frac{(n+k)!}{n!k!} f_{n+k} \mu_n \Big) z^k .$$

Hence every convolution operator can be regarded as a differential operator of infinite order with constant coefficients.

1.2 THE CONVOLUTION ALGEBRA $(A(\mathbb{C})_b', *)$ AND THE FOURIER-BOREL ISOMORPHISM

If μ and ν are analytic functionals, we define their convolution product $\mu * \nu \in A(\mathbb{C})'$ by

(1) $$\langle \mu * \nu, f \rangle := \sum_{n=0}^{\infty} f_n \Big(\sum_{k+j=n} \frac{n!}{k!j!} \mu_k \nu_j \Big),$$

where $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $\mu_n := \langle \mu, z^n \rangle$, $\nu_n := \langle \nu, z^n \rangle$. It is easy to check that $(A(\mathbb{C})_b', *)$ is a commutative locally convex algebra with unit and that

(2) $$<\mu*\nu,f>_{A(\mathbb{C})} = <\mu\otimes\nu,f^{\Delta}>_{A(\mathbb{C}^2)},$$

where $f^{\Delta} : (z,w) \mapsto f(z+w)$.

In order to remark that the algebra $A(\mathbb{C})_b'$ is isomorphic to an algebra of functions, we put

$$\mathrm{Exp}(\mathbb{C}) = \{f \in A(\mathbb{C}) \mid \text{there exists } A>0 \text{ with } \sup_{z\in\mathbb{C}} |f(z)|e^{-A|z|} < \infty\}$$

and endow $\mathrm{Exp}(\mathbb{C})$ with its natural inductive limit topology. It is easy to see that $\mathrm{Exp}(\mathbb{C})$ is a commutative locally convex algebra with unit and that the "Fourier-Borel" map $F : A(\mathbb{C})_b' \to \mathrm{Exp}(\mathbb{C})$, defined by

(3) $$F(\mu)[z] := <\mu_w, e^{zw}>, \quad z\in\mathbb{C},\ \mu\in A(\mathbb{C})_b',$$

is a topological algebra isomorphism. Obviously we have for all $z\in\mathbb{C}$

(4) $$F(\mu)[z] = \sum_{n=0}^{\infty} \frac{1}{n!}\mu_n z^n.$$

1.3 THE PROBLEM

Our aim is to get a satisfactory description of the kernel of a given convolution operator T_μ. In the special case that T_μ is a differential operator, everybody knows how to do this. In the more general situation, the same method also provides certain elements in $\ker T_\mu$. We introduce the following notation:

If $\mu\in A(\mathbb{C})'$, $\mu\neq 0$, is given, then we put $V(\mu) := \{a\in\mathbb{C} \mid F(\mu)[a] = 0\}$.

For $a\in V(\mu)$, we denote by m_a the multiplicity of the zero a of $F(\mu)$. Hence we have $F(\mu)^{(j)}(a) = 0$ for $0\le j<m_a$ and $F(\mu)^{(m_a)}(a)\neq 0$. For $a\in V(\mu)$ and $0\le j<m_a$ we denote by $E_{j,a}$ the so-called exponential monomials

$$E_{j,a} : z \mapsto z^j e^{az}.$$

From 1.2(3), we get for all $z\in\mathbb{C}$ and all $k\in\mathbb{N}_0$

(1) $$F(\mu)^{(k)}[z] = <\mu_w, w^k e^{zw}>.$$

This implies for $a\in V(\mu)$ and $0\le j<m_a$ that

$$\begin{aligned} T_\mu(E_{j,a})[z] &= <\mu_w, (z+w)^j e^{a(z+w)}> \\ &= \sum_{k=0}^{j} \binom{j}{k} z^{k-j} e^{az} <\mu_w, w^k e^{aw}> \\ &= \sum_{k=0}^{j} \binom{j}{k} z^{k-j} e^{az} F(\mu)^{(k)}[a] = 0 \end{aligned}$$

and hence $E_{j,a}\in\ker T_\mu$. Note that

$$E := \{E_{j,a} \mid a\in V(\mu),\ 0\le j<m_a\}$$

is a free set. The functions in span(E) are called exponential solutions (or exponential polynomials) of the convolution operator T_μ.

We remark that, for every $f \in \operatorname{span}(E)$ (resp. $\ker T_\mu$) and every $a \in \mathbb{C}$, the function $\tau_a(f)$ is again in span (E) (resp. $\ker T_\mu$), i.e. span(E) and $\ker T_\mu$ are translation invariant linear subspaces of $A(\mathbb{C})$. Hence we have the following two natural questions:

(a) Is it possible to obtain all elements of $\ker T_\mu$ from the exponential monomials by a certain procedure?

Or more generally:

(b) How can one describe the structure of the closed linear translation invariant subspaces of $A(\mathbb{C})$?

It is classical to attack these questions by applying duality theory to get a different interpretation. As the first observation, we note that every convolution operator is the adjoint of a multiplication operator.

<u>1.4 LEMMA.</u> *For* $\mu \in A(\mathbb{C})'$ *define* $M_\mu : A(\mathbb{C})_b' \to A(\mathbb{C})_b'$ *by* $M_\mu(\nu) := \mu * \nu$. *Then* ${}^tM_\mu = T_\mu$ *if we identify* $(A(\mathbb{C})_b')'$ *with* $A(\mathbb{C})$.

<u>PROOF.</u> For $j \in \mathbb{N}_0$, let $\varepsilon(j)$ denote the analytic functional satisfying $< \varepsilon(j), z^n > = \delta_{j,n}$ for all $n \in \mathbb{N}_0$. Then it follows from 1.2(1) that for $f : z \to \sum_{n=0}^{\infty} f_n z^n$ we have

$$< {}^tM_\mu(f), \varepsilon(j) > = < f, M_\mu(\varepsilon(j)) > = < f, \mu * \varepsilon(j) > =$$

$$= \sum_{n=j}^{\infty} f_n \frac{n!}{j!(n-j)!} \mu_{n-j} = \sum_{n=0}^{\infty} f_{n+j} \frac{(n+j)!}{j!n!} \mu_n .$$

By 1.1(2), this implies

$$ {}^tM_\mu(f)[z] = \sum_{k=0}^{\infty} < {}^tM_\mu(f), \varepsilon(k) > z^k$$

$$= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} \frac{(n+k)!}{n!k!} f_{n+k} \mu_n \right) z^k = T_\mu(f)[z],$$

and hence ${}^tM_\mu = T_\mu$.

<u>1.5 PROPOSITION.</u> *Let* W *be a closed linear subspace of* $A(\mathbb{C})$. *Then* W *is translation invariant if and only if* $W^\perp$ *is an ideal in the convolution algebra* $(A(\mathbb{C})_b', *)$.

<u>PROOF.</u> Assume that W is translation invariant. Since, for every $f \in A(\mathbb{C})$, we have $\lim_{h \to 0} \frac{f(\cdot+h) - f(\cdot)}{h} = f'$ in the topology of $A(\mathbb{C})$, we see that $f \in W$ implies that

$f^{(n)}$ is in W for all $n \in \mathbb{N}$. Hence we get for each $\mu \in W^{\perp}$ and each $f \in W$ with $f(z) = \sum_{n=0}^{\infty} f_n z^n$, that for all $n \in \mathbb{N}_0$,

$$(1) \qquad 0 = \langle \mu, f^{(n)} \rangle = \langle \mu, \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} f_{n+k} z^k \rangle = \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} f_{n+k} \mu_k .$$

By 1.2(1), it follows that, for each $\nu \in A(\mathbb{C})'$,

$$\langle \mu * \nu, f \rangle = \sum_{n=0}^{\infty} f_n \left(\sum_{k+j=n} \frac{n!}{k!j!} \mu_k \nu_j \right)$$

$$(2) \qquad = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (k+j)! f_{k+j} \frac{\mu_k}{k!} \frac{\nu_j}{j!}$$

$$= \sum_{j=0}^{\infty} \frac{\nu_j}{j!} \left(\sum_{k=0}^{\infty} \frac{(k+j)!}{k!} f_{k+j} \mu_k \right) = 0.$$

Hence $W^{\perp}$ is a closed ideal in $(A(\mathbb{C})_b', *)$.

To prove the converse, let us assume that $W^{\perp}$ is an ideal in $A(\mathbb{C})$. Then we get from 1.4 that, for all $\mu \in W^{\perp}$, all $f \in W$ and all $a \in \mathbb{C}$,

$$0 = \langle \mu * \delta_a, f \rangle = \langle M_{\delta_a}(\mu), f \rangle = \langle \mu, T_{\delta_a}(f) \rangle = \langle \mu, \tau_a(f) \rangle .$$

Hence $\tau_a(f)$ is in $W^{\perp\perp} = W$ for all $a \in \mathbb{C}$, i.e. W is translation invariant.

1.6 REFORMULATION OF THE PROBLEM

Since $A(\mathbb{C})$ is a nuclear Fréchet space, well-known duality results show that, for each closed linear subspace W of $A(\mathbb{C})$,

$$(1) \qquad W = W^{\perp\perp} \simeq (A(\mathbb{C})_b' / W^{\perp})_b' .$$

Applying the Fourier-Borel isomorphism, we get

$$(2) \qquad W \simeq (\mathrm{Exp}(\mathbb{C}) / F(W^{\perp}))_b' .$$

Hence it follows from 1.5 that - up to the computation of a dual space - question 1.3(b) is equivalent to determining the quotient of the algebra $\mathrm{Exp}(\mathbb{C})$ by a closed ideal. In case that $W = \ker T_\mu$, it follows from 1.4 that $F(W^{\perp}) = F(\overline{\mathrm{Im}(M_\mu)})$, which is the closure of the principal ideal $F(\mu) \cdot \mathrm{Exp}(\mathbb{C})$.

In the next section, we shall see that this reformulation has the advantage that we can apply results from complex analysis to study the quotient spaces $\mathrm{Exp}(\mathbb{C})/I$, where I is a closed ideal in $\mathrm{Exp}(\mathbb{C})$.

2. THE MAIN TOOLS

In order to derive a fairly explicit description of $\mathrm{Exp}(\mathbb{C})/I$ for all closed ideals I in $\mathrm{Exp}(\mathbb{C})$, we use some (rather special) properties of the ideals and the *solutions of* T_μ *are dense in ker* T_μ.

Proof. a) We may assume $I = f\,\mathrm{Exp}(\mathbb{C})$, where $f \neq 0$. Since $I \subset I_{loc}$ and since I_{loc} is closed, it suffices to show $I_{loc} \subset I$; to prove this, let $g \in I_{loc}$ be given. The considerations in 2.1 show that $g = h \cdot f$ for some $h \in A(\mathbb{C})$. Obviously, it suffices to show $h \in \mathrm{Exp}(\mathbb{C})$. At this point, we remark that, from 2.3 and $g \in \mathrm{Exp}(\mathbb{C})$, we obtain positive numbers D and M such that

$$\sup_{t\in[0,2\pi]} |h(r_n e^{it})|\, e^{-Dr_n} \leq M \text{ for all large } n \in \mathbb{N}.$$

This and the estimate on r_n given in 2.3, together with an application of the maximum principle to the annulus $\{z \in \mathbb{C} \mid r_n \leq |z| \leq r_{n+1}\}$, show that $h \in \mathrm{Exp}(\mathbb{C})$.

b) Since the Fourier-Borel transform is a topological algebra isomorphism, we get from 1.4 and part a) that ${}^tT_\mu = M_\mu$ is injective and that $\mathrm{im}\,{}^tT_\mu = \mathrm{im}\, M_\mu$ is closed. Hence the surjectivity of T_μ is a consequence of a classical result of Dieudonné and Schwartz (see Horvâth [10],p. 308).

By the Hahn-Banach theorem, the linear subspace E of the exponential solutions of T_μ is dense in ker T_μ iff $E^\perp = (\mathrm{kern}\ T_\mu)^\perp$. Since $\mathrm{im}\,{}^tT_\mu = \mathrm{im}\, M_\mu$ is closed, it suffices to prove $E^\perp \subset \mathrm{im}\, M_\mu$; so let $\nu \in E^\perp$. Then, for all $a \in V(\mu) = V(F(\mu))$ and $0 \leq j < m_a$,

$$0 = \langle \nu, E_{j,a} \rangle = \langle \nu, z^j e^{az} \rangle = \sum_{n=0}^{\infty} \nu_{n+j} \frac{a^n}{n!} = \sum_{n=0}^{\infty} \frac{(n+j)!}{n!} \frac{\nu_{n+j}}{(n+j)!} a^n$$

$$= F(\nu)^{(j)}(a),$$

and this shows $F(\nu) \in I_{loc}(F(\mu)\mathrm{Exp}(\mathbb{C})) = F(\mu)\mathrm{Exp}(\mathbb{C})$. Hence there exists $\lambda \in A(\mathbb{C})'$ with $\nu = \mu * \lambda = M_\mu(\lambda)$, i.e. $\nu \in \mathrm{im}\, M_\mu = E^\perp$.

Now we are ready to sketch how to obtain a fairly explicit model for $\mathrm{Exp}(\mathbb{C})/I$, following the approach of Berenstein and Taylor [1]; it suffices to consider the closed non-zero ideals I of $\mathrm{Exp}(\mathbb{C})$ which are of infinite codimension.

2.5 THE STRUCTURE OF $\mathrm{Exp}(\mathbb{C})/I$

Let I be a closed ideal in $\mathrm{Exp}(\mathbb{C})$ which is different from {0} and $\mathrm{Exp}(\mathbb{C})$. We define $\rho : \mathrm{Exp}(\mathbb{C}) \to \prod_{a\in V(I)} \mathcal{O}_a/I_a$ by $\rho(f) := ([f]_a + I_a)_{a\in V(I)}$. It is easy to see that $\ker \rho = I_{loc}$ and since $I = I_{loc}$ by 2.2, this gives $\ker \rho = I$. At this point the structure of $\mathrm{Exp}(\mathbb{C})/I$ will be clear if im ρ, as a locally convex space, is described in such a way that $\rho : \mathrm{Exp}(\mathbb{C}) \to \mathrm{im}\,\rho$ is a topological homomorphism. We will do this in several steps.

(1) The slowly decreasing property

By 2.2b), we have $I = I_{loc}(f_1, f_2)$, where we can assume $f_1 \neq 0$. In view of 2.4b), we shall assume from now on that $V(I)$ is infinite. Because of 2.3, we can choose $\varepsilon > 0$, $C > 0$ and $B > 0$ such that

(i) each component S of

$$S(f_1,f_2;\varepsilon,C) := \{z \in \mathbb{C} \mid |f_1(z)|^2 + |f_2(z)|^2 < (\varepsilon \exp(-C|z|))^2\}$$

is bounded and satisfies $\operatorname{diam} S \leq B \sup_{z\in S}|z|$ and,

(ii) such that for each component S of $S(f_1,f_2;\varepsilon,C)$,

$$\sup_{z\in S}|z| \leq B(\inf_{z\in S}|z|) + B.$$

As we shall see later, this is the appropriate extension of the slowly decreasing condition of Berenstein and Taylor [1], p. 130.

(2) Labeling the components of $S(f_1,f_2;\varepsilon,C)$

By our assumption, $V(I)$ is an infinite discrete subset of $\mathbb{C}$ contained in $S(f_1,f_2;\varepsilon,C)$. Hence (i) of (1) implies that $S(f_1,f_2;\varepsilon,C)$ has infinitely many components S with $S \cap V(I) \neq \emptyset$. We label these components by natural numbers in such a way that the sequence α, defined by $\alpha_j := \sup_{z\in S_j} |z|$, is non-decreasing.

(3) The Banach spaces $(E_j, \| \ \|_j)$

Let $A^\infty(S_j)$ denote the space of all bounded holomorphic functions on S_j, endowed with the norm $\| \ \| : f \mapsto \sup_{z\in S_j} |f(z)|$. Put $E_j := \prod_{a\in S_j\cap V(I)} \mathcal{O}_a/I_a$ and define $\rho_j : A^\infty(S_j) \to E_j$ by $\rho_j(g) := ([g]_a + I_a)_{a\in S_j\cap V(I)}$. It is easy to see that ρ_j is surjective. Hence we can endow E_j with the corresponding quotient norm, i.e. with the norm

$$\| \ \|_j : \varphi \mapsto \inf\{\|g\|_{A^\infty(S_j)} \mid \rho_j(g) = \varphi\}.$$

(4) The spaces $k(\gamma,\mathbb{F})$

Let $\mathbb{F} = (F_j, \| \ \|_j)_{j\in\mathbb{N}}$ be a sequence of Banach spaces and let γ denote an increasing unbounded sequence of non-negative real numbers. Then we define

$$k(\gamma,\mathbb{F}) := \{x \in \prod_{j=1}^{\infty} F_j \mid \text{there exists } A > 0 \text{ such that } \sup_{j\in\mathbb{N}} \|x_j\|_j e^{-A\gamma_j} < \infty\}$$

and endow $k(\gamma,\mathbb{F})$ with its natural inductive limit topology.
If $\dim F_j < \infty$ for all $j \in \mathbb{N}$, then it is easy to check that $k(\gamma,\mathbb{F})$ is a (DFS)-space, i.e. the strong dual of a Fréchet-Schwartz space.

(5) The semi-local interpolation theorem

Let g denote a holomorphic function on $S(f_1,f_2;\varepsilon,C)$ such that, for some $B > 0$,

$\sup\{|g(z)|e^{-B|z|} \mid z \in S(f_1,f_2;\varepsilon,C)\} < \infty$. Then there exists $G \in \mathrm{Exp}(\mathbb{C})$ with $[g]_a - [G]_a \in I_a$ for all $a \in V(I)$.
For a proof of this result we refer to Berenstein and Taylor [1], p. 120.

(6) The map $\rho : \mathrm{Exp}(\mathbb{C}) \to k(\alpha,\mathbf{E})$

Let $\mathbf{E} = (E_j,\| \ \|_j)_{j\in\mathbb{N}}$ denote the sequence of finite dimensional Banach spaces introduced in (3), and let α denote the sequence introduced in (2). If, for some positive numbers A and D, $f \in \mathrm{Exp}(\mathbb{C})$ satisfies the estimate $|f(z)| \le Ae^{D|z|}$ for all $z \in \mathbb{C}$, then, by the definition of α,

$$\|f|S_j\|_{A^\infty(S_j)} \le Ae^{D\alpha_j} \text{ for all } j \in \mathbb{N}\text{, whence}$$

$$\sup_{j\in\mathbb{N}} \|\rho_j(f|S_j)\|_j\, e^{-D\alpha_j} \le A.$$

At this point the map ρ defined at the beginning can be considered as a map of $\mathrm{Exp}(\mathbb{C})$ into $k(\alpha,\mathbf{E})$, given by $\rho(f) = (\rho_j(f|S_j))_{j\in\mathbb{N}}$. Moreover, the above estimates show that ρ is continuous. Since $\mathrm{Exp}(\mathbb{C})$ and $k(\alpha,\mathbf{E})$ are (DFS)-spaces, the open mapping theorem for (LF)-spaces applies, and ρ is an open map iff it is surjective.
To prove the surjectivity of ρ, let $x = (x_j)_{j\in\mathbb{N}} \in k(\alpha,\mathbf{E})$ be given. Then there exist $A,D > 0$ with $\sup_{j\in\mathbb{N}} \|x_j\|_j e^{-D\alpha_j} \le A$. By the definition of $\| \ \|_j$ and by (ii) of (1), this implies the existence of $g_j \in A^\infty(S_j)$ such that $\rho_j(g_j) = x_j$ and

$$\|g_j\|_{A^\infty(S_j)} \le 2Ae^{D\alpha_j} \le 2Ae^{DB}e^{D|z|} \text{ for all } z \in S_j.$$

Hence the function $g \in A(S(f_1,f_2;\varepsilon,C))$, defined by $g|S_j = g_j$ for $j \in \mathbb{N}$ and $g|S = 0$ for the components S of $S(f_1,f_2;\varepsilon,C)$ with $S \cap V(I) = \emptyset$, satisfies the hypotheses of the semi-local interpolation theorem (5) and, by (5), there is $G \in \mathrm{Exp}(\mathbb{C})$ with $\rho(G) = x$.

Thus, we have already sketched the proof of the following result:

<u>2.6 THEOREM.</u> *Let* I *be a non-zero closed ideal of* $\mathrm{Exp}(\mathbb{C})$ *with infinite codimension. Then* $\mathrm{Exp}(\mathbb{C})/I$ *is isomorphic to* $k(\alpha,\mathbf{E})$.

In order to derive more information from Theorem 2.6, Berenstein and Taylor [1] used Newton interpolation to introduce equivalent norms $|||\ |||_j$ on the spaces E_j. In this way, they obtained a representation of $\mathrm{Exp}(\mathbb{C})/I$ as a space of scalar sequences. But the norms $|||\ |||_j$ are computed by divided differences and, using this representation, it is difficult to discover special structural properties. That $\mathrm{Exp}(\mathbb{C})/I$ really has a very special structure follows from a remark of the author [15] which will be described in the next section.

3. SOLUTION OF THE PROBLEM

If we want to derive the solution of the problem posed in 1.3 from the results presented in section 2, we need some more preparations.

3.1 POWER SERIES SPACES OF INFINITE TYPE

Let γ be an increasing unbounded sequence of non-negative real numbers, and let $\mathbb{F} = (F_j, \| \ \|_j)_{j\in\mathbb{N}}$ be a sequence of Banach spaces. For $1 \leq p < \infty$, we define the spaces $\Lambda^p_\infty(\gamma,\mathbb{F})$ by

$$\Lambda^p_\infty(\gamma,\mathbb{F}) := \{x \in \prod_{j=1}^{\infty} F_j \mid \pi_{r,p}(x) = (\sum_{j=1}^{\infty} (\|x_j\|_j e^{r\gamma_j})^p)^{1/p} < \infty \text{ for all } r \geq 0\}.$$

Obviously, $\Lambda^p_\infty(\gamma,\mathbb{F})$ is a Fréchet space under the canonical norm system $(\pi_{r,p})_{r\geq 0}$. If $(F_j,\| \ \|_j) = (\mathbb{C},|\cdot|)$ for all $j \in \mathbb{N}$, then we write $\Lambda^p_\infty(\gamma)$ instead of $\Lambda^p_\infty(\gamma,\mathbb{F})$. $\Lambda^p_\infty(\gamma)$ is called a power series space of infinite type. We remark that, by the Grothendieck-Pietsch criterion (see Pietsch [17],6.1), $\Lambda^p_\infty(\gamma)$ is nuclear if and only if $\sup_{j\in\mathbb{N}} \frac{\log(j+1)}{\gamma_j} < \infty$. If $\Lambda^p_\infty(\gamma)$ is nuclear, then $\Lambda^p_\infty(\gamma) = \Lambda^q_\infty(\gamma)$ for all $p,q \in [1,\infty)$.

By the work of Dubinsky [5], Vogt [22],[23] and Vogt and Wagner [24],[25], the (stable) nuclear power series spaces of infinite type are a class of Fréchet spaces the structural properties of which are very well understood. We will make use of this fact later on.

3.2 CONSTRUCTION OF A SCHAUDER BASIS

Let $W \neq A(\mathbb{C})$ denote a closed linear subspace of $A(\mathbb{C})$ which is translation invariant and infinite dimensional. By 1.5 and 1.2, $I := F(W^\perp)$ is a closed ideal in $\mathrm{Exp}(\mathbb{C})$ of infinite codimension. By 2.2, $I = I_{loc}(f_1,f_2)$ for appropriate $f_1,f_2 \in \mathrm{Exp}(\mathbb{C}), f_1 \neq 0$. Hence all the hypotheses of 2.5 are satisfied. Using the notation introduced in 2.5, we define now, for all $a \in V(I)$ and $0 \leq k < m_a$, elements $e_{k,a} \in W$, $t_{k,a} \in I^\perp$ and $y_{k,a} \in k(\alpha,\mathbf{E})$ in the following way:

(1) $e_{k,a} : z \mapsto \frac{1}{k!} z^k e^{az}$, $z \in \mathbb{C}$

(2) $t_{k,a} : f \mapsto \frac{1}{k!} f^{(k)}(a)$, $f \in \mathrm{Exp}(\mathbb{C})$

(3) $y_{k,a}$ has germ 0 at every $b \in V(I)$, $b \neq a$, and at a it is the germ $[(z-a)^k]_a$.

Clearly, $t_{k,a}$ belongs to $I^\perp$. Since $F : A(\mathbb{C})' \to \mathrm{Exp}(\mathbb{C})$ is an isomorphism and $I = F(W^\perp)$, we have ${}^tF(I^\perp) = W^{\perp\perp} = W$. Because of

$$< {}^tF(t_{k,a}),\mu > = < t_{k,a}, F(\mu) > = \frac{1}{k!} \sum_{n=0}^{\infty} \frac{1}{n!} \mu_{n+k} a^n = < e_{k,a},\mu >$$

for all $\mu \in A(\mathbb{C})'$, we get

(4) $$^{t}F(t_{k,a}) = e_{k,a},$$

and hence $e_{k,a} \in W$. Obviously $y_{k,a} \in E_j$ if $a \in V(I) \cap S_j$ and if we identify E_j with its canonical image in $k(\alpha,\mathbb{E})$.
Next, identifying $\mathrm{Exp}(\mathbb{C})/I$ with $k(\alpha,\mathbb{E})$ (by the map given by Theorem 2.6) as well as $(\mathrm{Exp}(\mathbb{C})/I)'$ with $I^{\perp}$, it is immediate that

(5) $$t_{l,b}(y_{k,a}) = \delta_{l,k}\delta_{a,b}.$$

In [15], it is shown that one can find a Hilbert norm $|\,|_j$ on E_j, $j \in \mathbb{N}$, such that, with $\tilde{\mathbb{E}} := (E_j, |\,|_j)_{j\in\mathbb{N}}$, the locally convex space $k(\alpha,\mathbb{E})$ is identical with

$$k^2(\alpha,\tilde{\mathbb{E}}) = \{x \in \prod_{j=1}^{\infty} E_j \mid \text{there exists } A > 0 : (\sum(|x_j|_j e^{-A\alpha_j})^2)^{1/2} < \infty\}$$

under its natural inductive limit topology. Putting

$$F_j := \mathrm{span}\{t_{k,a} \mid 0 \le k < m_a,\ a \in V(I) \cap S_j\} \subset I^{\perp},$$

it follows from (5), and from the remark that $\{y_{k,a} \mid 0 \le k < m_a,\ a \in V(I) \cap S_j\}$ is a basis of E_j, that F_j can be interpreted as the dual of the Hilbert space $(E_j, |\,|_j)$. If we denote by $|\,|_j$ the dual norm of $(E_j, |\,|_j)$, then $(F_j, |\,|_j)$ is a Hilbert space, too. Now let $\mathbb{F} := (F_j, |\,|_j)_{j\in\mathbb{N}}$; we remark that, by Theorem 2.6, the map

$$\Phi : \Lambda^2_{\infty}(\alpha,\mathbb{F}) \to \mathrm{Exp}(\mathbb{C})'_b,$$

defined by

$$\Phi((\xi_j)_{j\in\mathbb{N}})[f] := \sum_{j=1}^{\infty} < \xi_j, f > ,\ f \in \mathrm{Exp}(\mathbb{C}),$$

gives an isomorphism between $\Lambda_{\infty}(\alpha,\mathbf{F})$ and $I^{\perp}$.
As we have explained in [15], one can now get a basis in $\Lambda^2_{\infty}(\alpha,\mathbb{F})$ in the following way: Choose an orthonormal basis $(h_{k,j})_{k=1}^{n_j}$ in $(F_j, |\,|_j)$ ($n_j := \dim F_j = \dim E_j$) for each $j \in \mathbb{N}$ and identify $h_{k,j}$ with its canonical image in $\Lambda^2_{\infty}(\alpha,\mathbf{F})$. Then $((h_{k,j})_{k=1}^{n_j})_{j\in\mathbb{N}}$ is an absolute basis in $\Lambda^2_{\infty}(\alpha,\mathbb{F})$. If we denote by β the sequence which is obtained by repeating each number α_j n_j-times and if we write the elements of $\Lambda^2_{\infty}(\beta)$ as $((\xi_{k,j})_{k=1}^{n_j})_{j\in\mathbb{N}}$, then the map $A : \Lambda^2_{\infty}(\beta) \to \Lambda^2_{\infty}(\alpha,\mathbb{F})$, defined by

$$A(((\xi_{k,j})_{k=1}^{n_j})_{j\in\mathbb{N}}) := \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \xi_{k,j} h_{k,j},$$

is an isomorphism.

Of course, we can assume that the orthonormal basis $(h_{k,j})_{k=1}^{n_j}$ of $(F_j, |\,|_j)$ is obtained from the basis $\{t_{k,a} \mid 0 \le k < m_a,\ a \in V(I) \cap S_j\}$ by Gram-Schmidt orthonormalization. Since ^{t}F gives an isomorphism from $I^{\perp}$ to W and since (4) holds, we have sketched the complete proof of the following theorem answering question 1.3(b):

3.3 THEOREM. *Let* W *be a proper closed linear subspace of* A($\mathbb{C}$) *which is translation invariant and infinite dimensional. Then* W *has a Schauder basis consisting of exponential polynomials with respect to which* W *is isomorphic to a nuclear power series space of infinite type.*

In the special case $W = \ker T_\mu, \mu \neq 0$, the considerations in 3.2 give the following result which, up to a certain extent, answers question 1.3(a) and which improves previous representation theorems of Dickson [4], Gelfond [9] and Ehrenpreis [8] (see also Berenstein and Taylor [1], Thm. 9).

3.4 THEOREM. *Let* T_μ *be a non-zero convolution operator on* A($\mathbb{C}$) *for which* $\ker T_\mu$ *is infinite dimensional. Then there exist a partition* $(V_j)_{j\in\mathbb{N}}$ *of* $V(F(\mu))$, *linear combinations* $f_{k,j}$, $1 \le k \le n_j := \sum_{a\in V_j} (m_a - 1)$, *of the functions* $\{z^l e^{az} \mid 0 \le l < m_a, a \in V_j\}$ *for each* $j \in \mathbb{N}$, *and an exponent sequence* α *such that the following holds:*

For every family $\xi = ((\xi_{k,j})_{k=1}^{n_j})_{j\in\mathbb{N}}$ *of complex numbers which satisfies* $\sum_{j=1}^{\infty} (\sum_{k=1}^{n_j} |\xi_{k,j}|) e^{r\alpha_j} < \infty$ *for all* $r > 0$, *the series*

$$(*) \qquad \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \xi_{k,j} f_{k,j}$$

converges normally to an element of $\ker T_\mu$, *and every* $f \in \ker T_\mu$ *has a unique representation of this type. In particular,* $\ker T_\mu$ *is isomorphic to a nuclear power series space of infinite type.*

To show that the structural properties derived so far have further implications, we now indicate how they can be used, together with the splitting theorem of Vogt [22] and an observation of [15], to give a new proof of the following result which, by 1.5, is equivalent to Taylor [21], thm. 5.1.

3.5 THEOREM. *Every closed linear translation invariant subspace* W *of* A($\mathbb{C}$) *is complemented. In particular, every non-zero convolution operator on* A($\mathbb{C}$) *has a complemented kernel.*

Sketch of the proof. If $W = \ker T_\mu$, where μ is a non-zero convolution operator, then W is complemented whenever $\dim W < \infty$. But if $\dim W = \infty$, then 2.4b) shows that we have the following exact sequence of Fréchet spaces

$$(*) \qquad 0 \longrightarrow W \xrightarrow{i} A(\mathbb{C}) \xrightarrow{T_\mu} A(\mathbb{C}) \longrightarrow 0.$$

By Theorem 3.4, W is a power series space of infinite type. Since $A(\mathbb{C}) \simeq \Lambda_\infty^1(n)$, the sequence (*) splits by Vogt [22], thm. 7.1; hence W is complemented. If W is an arbitrary translation invariant closed linear subspace, then we may assume $W \neq A(\mathbb{C})$ and $\dim W = \infty$. From 2.2b), it follows that there exist non-zero analytic functionals μ and ν with $W = (\ker T_\mu) \cap (\ker T_\nu)$, and consequently $W = \ker(T_\nu | \ker T_\mu)$. By the previous argument, $\ker T_\mu$ is complemented. Hence W is complemented if $\ker (T_\nu | \ker T_\mu)$ is complemented in $\ker T_\mu$. But this follows from the structure of $\ker T_\mu$ as described in 3.2 and an elementary lemma. For details, we refer to [15].

Remark. a) Results on the structure of the closed linear translation invariant subspaces (resp. the kernels of convolution operators) analogous to those given in the theorems 3.3, 3.4 and 3.5 can also be obtained for Fréchet spaces A of entire functions different from $A(\mathbb{C})$. This has been demonstrated in [15] and, more generally, in Meise and Schwerdtfeger [16].
b) A different proof of the fact that every non-zero convolution operator T_μ on $A(\mathbb{C})$ has a complemented kernel was given by Schwerdtfeger [20]. He used results of Gelfond [9] and Dickson [4] to show that $\ker T_\mu$ has property (Ω), which is sufficient for the application of the splitting theorem of Vogt [22].

The answer to question 1.3(a) which we have given in Theorem 3.4 is not yet complete, since it does not exclude that already the exponential monomials $\{z^k e^{az} \mid 0 \leq k < m_a,\ a \in V(F(\mu))\}$ form a Schauder basis of $\ker T_\mu$ for every convolution operator T_μ on $A(\mathbb{C})$. However, as classical results of Leont'ev [12] indicate this is not true in general. To conclude, let us show now how the model of $\ker T_\mu$ obtained so far can be used to derive a simple necessary condition which leads to examples of convolution operators T_μ for which the exponential monomials do not form a Schauder basis of $\ker T_\mu$.

3.6 DEDUCTION OF A NECESSARY CONDITION

Let T_μ be a non-zero convolution operator on $A(\mathbb{C})$ for which $\ker T_\mu$ is infinite dimensional and for which the exponential monomials $\{z^k e^{az} \mid 0 \leq k < m_a, a \in V(F(\mu))\}$ form a Schauder basis of $\ker T_\mu$. We put $W = \ker T_\mu$, $I = F(\mu)\mathrm{Exp}(\mathbb{C})$ and use the notation introduced in 2.5 and 3.2. In 3.2, we have indicated that, by ${}^t F$, $\ker T_\mu$ is isomorphic to $\Lambda^2_\infty(\alpha, \mathbb{F})$. Hence it follows from 3.2(4) that $\{t_{k,a} \mid 0 \leq k < m_a, a \in V(F(\mu))\}$ is a Schauder basis of $\Lambda^2_\infty(\alpha, \mathbf{F})$. Again in 3.2 we have remarked that $k(\alpha, \mathbf{E}) = k^2(\alpha, \tilde{\mathbf{E}})$ and, by the same arguments, we get $\Lambda^2_\infty(\alpha, \mathbf{F}) = \Lambda^1_\infty(\alpha, \mathbf{E}')$, where $\mathbf{E}' = ((E_j, \|\ \|_j)'_b)_{j \in \mathbb{N}}$. It is easy to check that, by $< (x_j)_{j\in\mathbb{N}}, (y_j)_{j\in\mathbb{N}} > := \sum_{j\in\mathbb{N}} < x_j, y_j >_j$, the space $k(\alpha, \mathbf{E}) = k(\alpha, \mathbf{E}'')$ is the dual space of $\Lambda^1_\infty(\alpha, \mathbf{E}')$. Hence 3.2(5) implies that, with respect to this duality, the system $\{y_{k,a}\}$ is the system of coefficient functionals of the basis $\{t_{k,a}\}$.

We claim:

(1) There exists $D>0$ such that $\sup_{j\in\mathbb{N}} \sup_{a\in V_j} \sup_{0\le k<m_a} \|t_{k,a}\|_j' \|y_{k,a}\|_j e^{-D\alpha_j} < \infty$,

where $V_j := V(F(\mu)) \cap S_j$.

In order to prove this claim, we first remark that, because of the nuclearity of $\Lambda_\infty^2(\alpha,\mathbb{F}) = \Lambda_\infty^1(\alpha,\mathbf{E}')$ and the basis theorem of Dynin and Mityagin [6], $\{t_{k,a}\}$ is an absolute basis. Arguing by contradiction, we assume that (1) does not hold. Then, for every $n\in\mathbb{N}$ there exist $j(n), a(n) \in V_{j(n)}$ and $0\le k(n) < m_{a(n)}$ with

$$\|t_{k(n),a(n)}\|_{j(n)}' \quad \|y_{k(n),a(n)}\|_{j(n)} \ge \exp(2n\alpha_{j(n)}).$$

Without loss of generality we can assume that $(j(n))_{n\in\mathbb{N}}$ is strictly increasing. Next, for each $n\in\mathbb{N}$, we choose $x_{j(n)} \in E_{j(n)}$ with $\|x_{j(n)}\|_{j(n)} = 1$ and $\|t_{k(n),a(n)}\|_{j(n)}' = t_{k(n),a(n)}(x_{j(n)})$. Then we define $y \in \prod_{j\in\mathbb{N}} E_j$ by $y_{j(n)} = x_{j(n)}\exp(-n\alpha_{j(n)})$ for all $n\in\mathbb{N}$ and $y_j = 0$ for all $j\in\mathbb{N}\setminus \bigcup_{n\in\mathbb{N}}\{j(n)\}$. It is easy to check that $y\in\Lambda_\infty^1(\alpha,\mathbf{E}')$. Since we have

$$|t_{k(n),a(n)}(y)|\ \pi_{o,1}(y_{k(n),a(n)}) = t_{k(n),a(n)}(y_{j(n)})\|y_{k(n),a(n)}\|_{j(n)}$$

$$= \|t_{k(n),a(n)}\|_{j(n)}'\|y_{k(n),a(n)}\|_{j(n)}\exp(-n\alpha_{j(n)}) \ge \exp(n\alpha_{j(n)})$$

for each $n\in\mathbb{N}$, the system $\{t_{k,a}\}$ cannot be an absolute basis.

Now let us assume that, in addition to the hypotheses made so far, we also have the following:

(2) There exists $E>0$ such that $\sup_{j\in\mathbb{N}} (\operatorname{diam} S_j)^{m_j} e^{-E\alpha_j} < \infty$, where $m_j := \max_{a\in V_j} (m_a - 1)$.

We note that, for the functions $\varphi_{k,a} : z \mapsto (z-a)^k$ $(0\le k<m_a, a\in V_j)$, we have

$$1 = t_{k,a}(\rho_j(\varphi_{k,a})) \le \|t_{k,a}\|_j' \|\rho_j(\varphi_{k,a})\|_j \le \|t_{k,a}\|_j'\|\varphi_{k,a}\|_{A^\infty(S_j)}$$

$$\le \|t_{k,a}\|_j' \max(1,(\operatorname{diam} S_j)^k).$$

Hence (2) implies the existence of $E>0$ with

$$\inf_{j\in\mathbb{N}} \inf_{a\in V_j} \inf_{0\le k<m_a} \|t_{k,a}\|_j' e^{E\alpha_j} > 0;$$

together with (1), this gives:

(3) There exists $F>0$ such that $\sup_{j\in\mathbb{N}} \sup_{a\in V_j} \sup_{0\le k<m_a} \|y_{k,a}\|_j e^{-F\alpha_j} < \infty$.

Hence $B := \{y_{k,a} \mid 0 \le k < m_a, a \in V(F(\mu))\}$ is a bounded subset of $k(\alpha,\mathbf{E})$. Since $\rho : \mathrm{Exp}(\mathbb{C}) \to k(\alpha,\mathbf{E})$ is a surjective topological homomorphism by 2.5(6), there exists a bounded set M in $\mathrm{Exp}(\mathbb{C})$ with $\rho(M) = B$ and consequently:

(4) There exists $F > 0$ such that for each $a \in V(F(\mu))$ and each $0 \le k < m_a$ one can find $f_{k,a} \in \mathrm{Exp}(\mathbb{C})$ with $\rho(f_{k,a}) = y_{k,a}$ and

$$\sup_{a\in V(F(\mu))} \sup_{0\le k<m_a} \sup_{z\in\mathbb{C}} |f_{k,a}(z)| e^{-F|z|} < \infty.$$

Now we have proved the following proposition:

3.7 PROPOSITION. *Let* T_μ *denote a non-zero convolution operator on* $A(\mathbb{C})$ *for which* $\ker T_\mu$ *is infinite dimensional and which has the following property:*

$$* \begin{cases} \text{There exist positive numbers } \varepsilon, C \text{ and } E \text{ such that, for every component } S \text{ of} \\ \{z \in \mathbb{C} \mid |F(\mu)[z]| < \varepsilon e^{C|z|}\} \text{ with } S \cap V(F(\mu)) \neq \emptyset, \text{ we have} \\ (\operatorname{diam} S)^{m_S} \exp(-E \sup_{z\in S} |z|) \le E, \text{ where } m_S := \max\{m_a - 1 \mid a \in S \cap V(F(\mu))\}. \end{cases}$$

Then, if the exponential monomials $\{z^k e^{az} \mid 0 \le k < m_a, a \in V(F(\mu))\}$ *form a Schauder basis of* $\ker T_\mu$*, assertion 3.6(4) holds.*

Remark. a) From the estimates noted in 2.5(1), it follows easily that the hypothesis (*) in 3.7 is satisfied whenever $\sup\{m_a \mid a \in V(F(\mu))\} < \infty$.
b) It is possible to show that 3.6(4) is equivalent to the fact that the multiplicity variety of the ideal $F(\mu)\mathrm{Exp}(\mathbb{C})$ is an interpolating variety in the notation of Berenstein and Taylor [1]. In view of this, it follows from Berenstein and Taylor [1], thm. 4, and some additional considerations that, under the hypotheses of 3.7, the following assertions are equivalent:

(i) The exponential monomials form a Schauder basis of $\ker T_\mu$;
(ii) 3.6(4) holds;
(iii) there exists $A > 0$ with $\inf_{a\in V(F(\mu))} \left| \frac{F(\mu)^{(m_a)}(a)}{m_a!} \right| e^{A|a|} > 0$.

The necessary condition given in 3.7 looks rather complicated; however, it is easy to derive the following simple explicit condition from it.

3.8 COROLLARY. *Let* T_μ *denote a convolution operator on* $A(\mathbb{C})$ *which satisfies the hypotheses of Proposition 3.7. If the exponential monomials* $\{z^k e^{az} \mid 0 \le k < m_a, a \in V(F(\mu))\}$ *form a Schauder basis of* $\ker T_\mu$*, then there exists* $F > 0$ *such that* $\inf_{a\in V(F(\mu))} \operatorname{dist}(a, V(F(\mu)) \setminus \{a\}) e^{F|a|} > 0$.

PROOF. From 3.7 we get a positive number F such that, for each $a \in V(F(\mu))$, there exists $f_a \in \mathrm{Exp}(\mathbb{C})$ with the properties $\sup_{a\in V(F(\mu))} \sup_{z\in\mathbb{C}} |f_a(z)| e^{-F|z|} \le A < \infty$, $f_a(a) = 1$

and $f_a(b) = 0$ for all $b \in V(F(\mu)) \setminus \{a\}$. Now fix $a \in V(F(\mu))$ and choose $b \in V(F(\mu))$ with $|b-a| = \mathrm{dist}(a, V(F(\mu)) \setminus \{a\})$. Without loss of generality, we can assume $|b-a| < 1$. Letting $\mathbb{D}$ denote the unit disk, we define $g : \mathbb{D} \to \mathbb{C}$ by $g(w) := f_b(a+w)$. Then we have $g(0) = f_b(a) = 0$ and $\|g\|_{A^\infty(\mathbb{D})} \leq A \sup_{|w| \leq 1} e^{F|a+w|} = Ae^F e^{F|a|}$. Now the Schwarz lemma implies $|g(w)| \leq |w| Ae^F e^{F|a|}$, and hence

$$1 = f_b(b) = g(b-a) \leq |b-a| Ae^F e^{F|a|},$$

whence the desired condition.

Finally we show how Corollary 3.8 can be used to construct examples of convolution operators T_μ for which the exponential monomials do not form a Schauder basis. This is done by jiggling the zeros of certain functions (see Berenstein and Taylor [1], p. 120).

3.9 EXAMPLE. Let $f \in \mathrm{Exp}(\mathbb{C})$ be a function for which $V(f)$ is infinite and for which $m_a = 2$ for all $a \in V(f)$, e.g. $f(z) = \left(\sum_{k=0}^{\infty} \frac{z^k}{(2k)!}\right)^2$. Label the elements of $V(f)$ by $(a_k)_{k \in \mathbb{N}}$ in such a way that $(|a_k|)_{k \in \mathbb{N}}$ is non-decreasing. Next, choose a sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ of complex numbers with the following properties:

(1) $(\varepsilon_k)_{k \in \mathbb{N}} \in \Lambda^1_\infty(|a_k|)$,

(2) $|\varepsilon_k| > 0$ for all $k \in \mathbb{N}$,

(3) $\sum_{k=1}^{\infty} |\varepsilon_k| < 1$,

(4) $V(f) \cap (a_k + \varepsilon_k \overline{\mathbb{D}}) = \{a_k\}$

and put $g : z \mapsto f(z) \prod_{k=1}^{\infty} \frac{z-(a_k+\varepsilon_k^2)}{z-a_k}$ for $z \in \mathbb{C} \setminus V(f)$. Since $m_{a_k} = 2$ for all $k \in \mathbb{N}$, g defines an entire function and, in view of our choice, it is not difficult to show that $g \in \mathrm{Exp}(\mathbb{C})$. Obviously, $V(g) = V(f) \cup \{a_k + \varepsilon_k^2 \mid k \in \mathbb{N}\}$, and every zero of g is simple. Hence it follows from (1) and Corollary 3.8 that the exponential monomials do not form a Schauder basis in $\ker T_\mu$ if we put $\mu := F^{-1}(g)$.

By Example 3.9, it is clear that, as a general answer to question 1.3(a), Theorem 3.4 is optimal.

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elements of Exp($\mathbb{C}$) which we will now recall.

2.1 LOCALIZATION OF IDEALS

For $a \in \mathbb{C}$, we denote by $\mathcal{O}_a$ the ring of germs of holomorphic functions at a. If I is an ideal in Exp($\mathbb{C}$), then I_a denotes the ideal generated by the canonical image of I in $\mathcal{O}_a$. The localization I_{loc} of the ideal I is defined as

$$I_{loc} = \{f \in \mathrm{Exp}(\mathbb{C}) \mid [f]_a \in I_a \text{ for all } a \in \mathbb{C}\},$$

where $[f]_a$ denotes the germ of f at a. It is easy to check that I_{loc} is a closed ideal in Exp($\mathbb{C}$) which contains I.
Since every non-zero ideal in $\mathcal{O}_a$ is of the form $[(z-a)]_a^{m_a} I_a$ for a suitable $m_a \in \mathbb{N}_0$, the non-zero localized ideals $I = I_{loc}$ are completely determined by the set $V(I) := \{a \in \mathbb{C} \mid m_a > 0\}$ and the numbers m_a, $a \in V(I)$. As an example, let us look at the ideal $I(f_1,\dots,f_n)$ generated by $f_1,\dots,f_n$ in Exp($\mathbb{C}$). For its localization $I_{loc}(f_1,\dots,f_n)$, it is easy to see that

$$V(I_{loc}(f_1,\dots,f_n)) = \{a \in \mathbb{C} \mid f_j(a) = 0 \text{ for } 1 \le j \le n\}$$

and that m_a equals the minimum of the order of the zeros of the functions f_j at a.

In the following theorem we state the special property of the ideals in Exp($\mathbb{C}$) which we are going to use. The theorem is due to Schwartz [19] and Ehrenpreis [7]; for a proof, we refer to [7], sect. 6, or to Kelleher and Taylor [11], where rather general extensions of this result are presented.

2.2 THEOREM. *a) Every closed ideal in* Exp($\mathbb{C}$) *is localized.*
b) For every closed ideal I *in* Exp($\mathbb{C}$), *there exist* $f_1, f_2 \in \mathrm{Exp}(\mathbb{C})$ *such that* $I = I_{loc}(f_1,f_2)$.

Besides Theorem 2.2, we shall use the following property of the non-zero functions in Exp($\mathbb{C}$), which can be derived from the minimum modulus theorem (see e.g. Levin [13],I,§8).

2.3 PROPOSITION. *For every* $f \in \mathrm{Exp}(\mathbb{C})$, $f \neq 0$, *there exist* $\varepsilon > 0$, $C > 0$ *and* $(r_n)_{n\in\mathbb{N}}$ *with* $2^n < r_n < 2^{n+1}$ *such that, for all large* $n \in \mathbb{N}$, *we have*

$$\inf_{t\in[0,2\pi]} |f(r_n e^{it})| \ge \varepsilon e^{-Cr_n}.$$

Using some functional analysis, it is easy to conclude, from this property, the following classical result on analytic convolution operators, due to Ehrenpreis [7] and Malgrange [14]; it already gives a partial answer to 1.3(a).

2.4 PROPOSITION. *a) Every principal ideal in* Exp($\mathbb{C}$) *is closed.*
b) Every non-zero convolution operator T_μ *is surjective, and the exponential*