FOUNDATIONS OF THE KINETIC DESCRIPTION OF PLASMAS: FROM LINEAR TO NONLINEAR THEORY

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Abstract

Plasma physics for nuclear fusion deals with a many particle system in sophisticated geometry and under the influence of boundary conditions. Thus one will never reach a detailed and complete theoretical understanding of such a complicated system. However, statistical physics offers a description for mean values and probability distributions which can be used. In this contribution the basic principles of equilibrium thermodynamics and non-equilibrium kinetics will be reviewed for plasmas in idealized situations, leaving the more realistic but complicated calculations in the presence of tokamak geometry and walls to subsequent presentations. In the introductory part, the various plasma states are catalogued. Problems of equilibrium statistical mechanics are briefly touched in Sec. I. The main part of this paper begins in Sec. II, where the kinetic theory starting from BBGKY hierarchy is elucidated with the Vlasov, Landau-Fokker-Planck, and Balescu-Lenard equations as the main outcomes. Landau damping and the linear dispersion function are discussed as applications. Then a kinetic theory of waves, resulting in a wave-kinetic equation for quasiparticles, is discussed. The paper is concluded by a short summary.

I. INTRODUCTION

A plasma consists of many particles and thus a statistical description is adequate. When aiming for the latter, several points have to be clarified before getting started. First, we have to specify the species, e.g. electrons and protons. Of course, in general more species, e.g. neutrals, take part in the dynamics. When we consider a fully ionized plasma, we can estimate the region of validity by the Saha[1] equation

$$\frac{n_i n_e}{n_n} = \frac{2g_1}{g_0}\left(\frac{m_e}{2\pi}\right)^{3/2}\hbar^{-3}(k_B T_e)^{3/2} e^{-E_i/(k_B T_e)}$$
$$\approx 2.4\cdot 10^{15}\,(T_e[K])^{3/2}\, e^{-E_i/(k_B T_e)}\quad [cm^{-3}]\,. \qquad (1)$$

This formula follows from equilibrium statistical mechanics. The meaning of the various symbols is as follows: n_e, n_i, n_n, particle number densities of electrons, singly ionized ions, and neutrals, respectively; g_ν, statistical weight factors; k_B, Boltzmann constant $k_B = 1.3807\cdot 10^{-16} erg/K$; T_e, electron temperature; m_e, electron mass; E_i, ionisation energy. From (1) we conclude that for sufficiently high temperatures ($k_B T_e \geq E_i$) the system will be in the fully ionized state. Next, we have to decide whether a relativistic and/or quantum-mechanical consideration is appropriate.[2-7] Again the (electron) temperature is one of the main parameters. Relativistic effects become important when (as an estimate of order of magnitude) $v_{te}^2/c^2 \geq 0.05$, where $v_{te} = (k_B T_e/m_e)^{1/2}$ is the electron thermal velocity and c is the speed of light. Quantum effects become significant when the thermal de-Broglie wavelength $\lambda_B = \hbar/(m_e v_{te})$ of electrons exceeds the mean particle distance $\lambda_n \approx n_e^{-1/3}$. If the latter condition is not satisfied we call the plasma classical. In a classical non-relativistic fully ionized plasma the interactions are governed by the electrostatic Coulomb potential. If on the average the interaction energy is small compared to the kinetic energy, the system is close to the ideal gas, and the plasma is called ideal. Using the electron Debye length $\lambda_{De} = (k_B T_e/4\pi n_e e^2)^{1/2}$ for a classical plasma, the ideality condition means $n_e \lambda_{De}^3 \gg 1$. In a quantum plasma, on the other hand, a characteristic value for the kinetic energy of an electron is $\hbar^2/(2m_e\lambda_n^2)$. If the latter is much larger than e^2/λ_n we have an ideal quantum plasma. Note that a classical ideal plasma requires (for fixed temperature T_e) small densities n_e, whereas in the quantum case we require for the ideal situation large densities.

The Debye length plays a crucial role within plasma physics. The plasma does not show only the irregular (chaotic) motion of individual particles, but is also able to exhibit collective behavior[8] such as waves and nonlinear coherent structures. A good parameter to characterize the collective appearence is the number of particles in

the Debye sphere,

$$\Lambda = \frac{4\pi}{3} n \lambda_D^3 \,, \tag{2}$$

where λ_D can be either the electron Debye length λ_{De} or (following from an analogous definition) the ion Debye lenth λ_{Di}. Very often $\lambda_D^{-2} = \lambda_{De}^{-2} + \lambda_{Di}^{-2}$ is used as a definition for the socalled total Debye length λ_D. In order to have a feeling of the characteristic quantities we can use

$$\lambda_{De,i}[cm] \approx 7.43 \times 10^2 \left(T_{e,i}[eV]/n_{e,i}[cm^{-3}]\right)^{1/2} . \tag{3}$$

Let us now make a few preliminary remarks about collective behavior. First, comparing the mean potential energy (for a classical non-relativistic plasma) $\sim n^{1/3} e^2$ with the mean kinetic energy $\sim k_B T$ we obvious get the result that for $\Lambda \gg 1$ the contribution of the kinetic energy dominates on the average.

Secondly, solving the Poisson equation

$$\nabla^2 \varphi \approx -4\pi e \delta(\vec{r}) + 4\pi e [n_{e0} \exp(e\varphi/k_B T_e) - n_{e0}] \tag{4}$$

for the potential produced by an (quite immobile) ion at position $\vec{r} = 0$ in the presence of (rather mobile) Boltzmann distributed electrons, we find in the linear limit $[\exp(e\varphi/k_B T_e) \approx 1 + e\varphi/k_B T_e]$ the Debye potential

$$\varphi(\vec{r}) = \frac{e}{r} \exp(-r/\lambda_{De}) \tag{5}$$

as a solution of (4). In other words, the Coulomb potential of an ion is screened by the surrounding electrons such that over distances larger than λ_{De} the potential (approximately) disappears. Thus, a plasma of macroscopic dimesion $L \gg \lambda_{De}$ is quasi-neutral. But it is also clear that this shielding mechanism can only work when the mean particle distance λ_n is smaller than λ_D, i.e. $\Lambda \gg 1$. This means that the whole scenario is consistent. From the thermal velocity v_{te} and the Debye length λ_{De} we can construct a characteristic frequency $\omega_{pe} = v_{te}/\lambda_{De}$ (and in a similar way the ion plasma frequency ω_{pi}). The total plasma frequency ω_p is often defined through $\omega_p^2 = \omega_{pe}^2 + \omega_{pi}^2$. To get a feeling for the orders of magnitudes we write

$$\omega_{pe}[rad/s] \approx 5.64 \times 10^4 (n_e[cm^{-3}])^{1/2} . \tag{6}$$

To compare this (collective) frequency with the frequency for individual processes, we have to define a collision frequency, i.e. the inverse mean time for (90°) deflections due to inter-particle collisions. Of course, there is a problem involved here since (without screening) the (Coulomb) potential is a long-range potential. For an estimate we shall use the Debye length as the effective interaction distance. For $\Lambda \gg 1$ we have many particles in the Debye sphere. They rarely suffer strongly deflections and more often are only slightly deflected by weak collisions. The question is which process is more effective, and some simple considerations suggest that the frequent small angle scattering processes sum up to a larger effect compared with the not so frequent strong collisions. Anyway, we can estimate the collision frequency ν_c via

$$\frac{\nu_c}{\omega_{pe}} \approx \frac{\ln \Lambda}{\Lambda} \approx \frac{1}{\Lambda} \ll 1 \tag{7}$$

for a classical non-relativistic ideal plasma.

Kinetic theory, or more general equilibrium and non-equilibrium statistical mechanics, means to provide a more profound formalism to be used as a tool for calculating the above mentioned individual and collective processes in detail, and some more as, e.g., transport coefficients. In doing that it is necessary to decide which system to deal with, and once the decision took place, to have a physical intuition being necessary for guidance through awful algebraic manipulations. In the following we outline the principles of kinetic theory for a simple plasma in a simple (not tokamak) geometry, in order to give a taste of the principle procedure. More advanced and applied cases will be presented in subsequent lectures by other authors.

II. STATISTICAL MECHANICS IN THERMAL EQUILIBRIUM

For large systems with a huge number of particles we are generally interested in mean values and not in the detailed microscopic information which would be impossible to handle.[9] In the classical description we define the average value of a quantity F, which depends on all coordinates $\vec{q}_i$ and generalized momenta $\vec{p}_i$ of particles $i = 1, \ldots, N$, as

$$\langle F \rangle = \int d^{3N}q \, d^{3N}p \, F \, \rho, \tag{8}$$

where ρ is the weight function which in general depends on coordinates, momenta, and time. The $6N$-dimensional space is called Γ-space. In Γ-space, a system (Ergode) describes a curve. The density ρ in Γ-space follows from the Liouville equation

$$\frac{\partial \rho}{\partial t} = \{H, \rho\}; \tag{9}$$

its quantummechanical analogue is the von-Neuman equation for the density operator. But let us consider in the following only classical non-relativistic plasmas with $\Lambda \gg 1$. In (9), $\{\ldots,\ldots\}$ is the Poisson bracket.

The Liouville equation is an extremly complicated partial differential equation whose solution is impossible to obtain in the general case. The assumption of thermal equilibrium simplifies the situation considerably, since then

we assume $\partial\rho/\partial t \equiv \dot{\rho} = 0$, and the remaining equation $\{H,\rho\} = 0$ has, e.g., the canonical distibution

$$\rho = \frac{1}{h^{3N}N!Z}e^{-\beta H} \tag{10}$$

as a solution. Here, $\theta = k_B T = 1/\beta$, and Z is the partition function

$$\begin{aligned} Z &= \frac{1}{h^{3N}N!}\int\dots\int d^{3N}p\, d^{3N}q\, e^{-\beta H} \qquad (11)\\ &\hat{=} \frac{1}{h^{6N}(N!)^2}\int\dots\int d^{3N}p\, d^{3N}P d^{3N}q\, d^{3N}Q\, e^{-\beta H}, \end{aligned}$$

where the latter expression is an obvious generalization for $2N$ particles (e.g. N electrons and N protons). For the canonical distribution the link to standard (equilibrium) thermodynamics is straightforward via the free energy

$$F = -\theta \ln Z \tag{12}$$

once the Hamiltonian is specified, e.g.

$$\begin{aligned} H &= \frac{1}{2}\sum_{j=1}^{N}\left[\frac{p_j^2}{m_e}+\frac{P_j^2}{m_i}\right]+\frac{1}{2}\sum_{\substack{j,k\\ j\neq k}}^{N}\left[\frac{e^2}{|\vec{Q}_j-\vec{Q}_k|}\right.\\ &\left.+\frac{e^2}{|\vec{q}_j-\vec{q}_k|}\right]-\sum_{j,k}^{N}\frac{e^2}{|\vec{Q}_j-\vec{q}_k|}. \end{aligned} \tag{13}$$

Where is the problem? There is no principal difficulty since we can formulate the free energy F (as an integral) and then have access to the well-developed thermodynamic formalism which allows to calculate all the thermodynamic (equilibrium) quantities in integral form. However, there appears a huge technical problem when we want to evaluate the integrals explicitly. The reason is the (already simplified) interaction energy which in each term depends on two coordinates and forbids factorization of the multiple integrals. Let us demonstrate this for the factor

$$Q'' := \int d^{3N}r \exp\left[-\beta\sum_{i,j}\phi_{ij}\right], \tag{14}$$

where

$$\phi_{ij} = \frac{e^2}{|\vec{r}_i-\vec{r}_j|} \tag{15}$$

designates the Coulomb interaction between like particles. [Note that – as in the case of an ideal gas – the integrals over the momenta can be exactly evaluated since the kinetic energy is additive.] Statistical mechanics suggests to write

$$Q'' = \int d^{3N}r \prod_{i,j}(1+f_{ij}) \tag{16}$$

in terms of Mayer functions $f_{ij} = \exp(-\beta\phi_{ij}) - 1$, and the immense problem is to calculate the cluster integrals

$$b_l = \frac{1}{l!V}\int\dots\int d^3r_1\dots d^3r_l \sum_{l\geq i>j\geq 1}\prod f_{ij}. \tag{17}$$

Of course, it is not possible to evaluate all these integrals, and very sophisticated diagram techniques (and transformations to irreducible cluster integrals, etc.) have been developed to obtain physically relevant results. We cannot report on this immense literature here (see, e.g., Ref. 3). Let us make only two remarks.

First, when determining the equation of state, one obtains

$$\frac{pV}{Nk_BT_e} \approx 1 - \frac{1}{24\pi}\frac{1}{n_e\lambda_{De}^3}, \tag{18}$$

which shows that the corrections to the ideal gas approximation are of order Λ^{-1} for $\Lambda \gg 1$. This is expected since Λ^{-1} measures the ratio of average potential and kinetic energies.

Secondly, the statistical formulation also allows to calculate probability functions for dependent variables as, e.g., the electric microfield $\vec{E}(\vec{r}) = \sum_{j=1}^{N}\vec{E}_j(\vec{r},\vec{r}_j)$ through

$$W(\vec{E}) = \int\dots\int \delta\left[\vec{E}-\vec{E}(\vec{r})\right]\rho\, d^{3N}r\, d^{3N}p. \tag{19}$$

Again, a huge literature exists on this important topic, since the results are very important for interpretation of diagnostic measurements via line shapes. The simplest result, coming out of the very crude Holtsmark approximation, reads

$$W(\beta) = \frac{2}{\pi\beta}\int_0^\infty x\sin x\, e^{-(x/\beta)^{3/2}}\, dx, \tag{20}$$

where the magnitude E of the electric field has been normalized by the mean field E_0, i.e. $\beta = E/E_0$, and $W(E) = 4\pi E^2 W(\vec{E})$ has been used.

III. NON-EQUILIBRIUM KINETIC THEORY FOR PARTICLES

Starting from the Liouville equation (9) we want to derive a kinetic equation, i.e. a closed equation for the one-particle distribution function[10–13]

$$\begin{aligned} f^\alpha(\vec{q},\vec{p};t) &\equiv f_1^\alpha(\vec{q}_1,\vec{p}_1;t)\\ &:= N_\alpha\int d^3q_2\dots d^3q_N d^3p_2\dots d^3p_N\,\rho, \end{aligned} \tag{21}$$

where α designates the species of the particle under consideration. This (normalized) function describes the probability of finding a particle of species α at postion $\vec{q}$ with momentum $\vec{p}$ at time t. In a similar way one can define multiple-particle distribution functions, e.g. the two-particle distributions

$$f_2^{\alpha\alpha} = N_\alpha(N_\alpha-1)\int d^3q_3\dots d^3q_N d^3p_3\dots d^3p_N\rho, \tag{22}$$

$$f_2^{\alpha\beta} = N_\alpha N_\beta\int d^3q_3\dots d^3q_N d^3p_3\dots d^3p_N\,\rho. \tag{23}$$

The generalizations are obvious. Also the change from $\vec{p} = m\vec{v}$ to $\vec{v}$ as variable is trivial. The idea is to obtain the kinetic equation for the one-particle distribution function by integrating the Liouville equation over the irrelevant coordinates and momenta. And again the procedure is not trivial since we get a hierarchy of coupled equations. Let us elucidate this important point a little bit more in detail.

Introducing the Liouville operator

$$L = -\sum_{j=1}^{N}\left[\vec{v}_j\cdot\frac{\partial}{\partial\vec{q}_j} - \frac{e_j}{m_j}\frac{\partial\phi}{\partial\vec{q}_j}\cdot\frac{\partial}{\partial\vec{v}_j}\right] + \sum_{j=1}^{N}\sum_{i=1}^{j-1}\frac{1}{m_j}\frac{\partial\phi_{ij}}{\partial\vec{q}_j}\cdot\frac{\partial}{\partial\vec{v}_j} \equiv L^{(1)} + L^{(2)} \quad (24)$$

for particles in an external potential ϕ and with interaction potential

$$\phi_{ij} = \frac{e_i e_j}{|\vec{q}_i - \vec{q}_j|}, \quad (25)$$

we get a one-particle propagator $L^{(1)}$ and an interaction contribution $L^{(2)}$. The latter depends on coordinates of two particles in a non-separable manner. It causes the main problems. Note that the Liouville equation can be written in the form

$$\frac{\partial\rho}{\partial t} = L\rho. \quad (26)$$

Now integrating (26) over all the coordinates and momenta of the other particles, exept $\vec{q}_1$ and $\vec{p}_1$ of the particle under consideration, we obtain after some straightforward manipulations

$$\partial_t f^\alpha(\vec{q}_1,\vec{v}_1;t) = L_1^\alpha f^\alpha(\vec{q}_1,\vec{v}_1;t) + \sum_{\beta=e,i}\int d^3q_2\, d^3v_2\, L_{12}^{\alpha\beta} f^{\alpha\beta}(\vec{q}_1,\vec{v}_1,\vec{q}_2,\vec{v}_2;t) \quad (27)$$

where $\partial_t = \partial/\partial t$. Here and in the following we omit some indices when no confusion is expected. Also, the L-operators follow from (24) in a straightforward manner. Obviously, because of the inter-particle interactions, this equation for f^α contains the two-particle distribution function $f^{\alpha\beta}$. The latter we split into two parts:

$$f^{\alpha\beta}(\vec{q}_1,\vec{v}_1,\vec{q}_2,\vec{v}_2;t) = f^\alpha(\vec{q}_1,\vec{v}_1;t)f^\beta(\vec{q}_2,\vec{v}_2;t) + g^{\alpha\beta}(\vec{q}_1,\vec{v}_1,\vec{q}_2,\vec{v}_2;t), \quad (28)$$

where the first contribution on the r.h.s. is the dominating one in dilute gases, when particles approximately move independently, and the second contribution measures the correlation. In a similar way, we can define the triple correlation function via

$$f^{\alpha\beta\gamma} = f^\alpha f^\beta f^\gamma + f^\alpha g^{\beta\gamma} + f^\beta g^{\alpha\gamma} + f^\gamma g^{\alpha\beta} + g^{\alpha\beta\gamma}. \quad (29)$$

In all following discussions we shall assume $g^{\alpha\beta\gamma} \approx 0$, meaning that close clusters of three particles are very rare. This assumption is consistent with previous considerations since $\Lambda(\sim n^{-1/2}) \gg 1$ is good for dilute systems. By the assumption $g^{\alpha\beta\gamma} \approx 0$ we close the BBGKY (Bogoliubov, Born, Green, Kirkwood, Yvon) hierarchy which expresses the fact that the equation for f_1 contains f_2, the equation for f_2 contains f_3, and so on. But still we have not succeeded in a kinetic equation, since our present state of calculation has produced the following coupled set of equations

$$\partial_t f^\alpha(\vec{q}_1,\vec{v}_1;t) = L_1^\alpha f^\alpha(\vec{q}_1,\vec{v}_1;t) + \sum_{\beta=e,i}\int d^6 2\, L_{12}^{\alpha\beta} f^\alpha(\vec{q}_1,\vec{v}_1;t)f^\beta(\vec{q}_2,\vec{v}_2;t) + \sum_{\beta=e,i}\int d^6 2\, L_{12}^{\alpha\beta} g^{\alpha\beta}(\vec{q}_1,\vec{v}_1,\vec{q}_2,\vec{v}_2;t), \quad (30)$$

$$\partial_t g^{\alpha\beta}(\vec{q}_1,\vec{v}_1,\vec{q}_2,\vec{v}_2;t) = (L_1^\alpha + L_2^\beta)g^{\alpha\beta}(\vec{q}_1,\vec{v}_1,\vec{q}_2,\vec{v}_2;t) + L_{12}^{\alpha\beta} g^{\alpha\beta}(\vec{q}_1,\vec{v}_1,\vec{q}_2,\vec{v}_2;t) + \sum_{\gamma=e,i}\int d^6 3\,\Big[L_{13}^{\alpha\gamma} f^\alpha(\vec{q}_1,\vec{v}_1;t)g^{\beta\gamma}(\vec{q}_2,\vec{v}_2,\vec{q}_3,\vec{v}_3;t) + L_{23}^{\beta\gamma} f^\beta(\vec{q}_2,\vec{v}_2;t)g^{\alpha\gamma}(\vec{q}_1,\vec{v}_1,\vec{q}_3,\vec{v}_3;t) + (L_{13}^{\alpha\gamma} + L_{23}^{\beta\gamma})f^\gamma(\vec{q}_3,\vec{v}_3;t)g^{\alpha\beta}(\vec{q}_1,\vec{v}_1,\vec{q}_2,\vec{v}_2;t)\Big] + L_{12}^{\alpha\beta} f^\alpha(\vec{q}_1,\vec{v}_1;t)f^\beta(\vec{q}_2,\vec{v}_2;t). \quad (31)$$

Note that we have introduced the symbol $d^6 2$ to indicate integration over position and velocity of particle 2. We have to eliminate $g^{\alpha\beta}$ in order to get one closed equation for f^α. Besides mathematical also physical problems rise if we proceed in a straightforward manner (if there exists any). Suppose we would be able to solve (31) for $g^{\alpha\beta}$. Then the correlation function would depend on the whole time-history (which in complete detail is actually not relevant). Kinetic regime means that the correlation function depends on its variables only functionally through the one-particle distribution functions; formally

$$g^{\alpha\beta}(t) \approx g^{\alpha\beta}[f(t)]. \quad (32)$$

Now let us discuss which approximations can be applied to the system (30) and (31).[14–17] The first approximation is due to Vlasov: We close equation (30) directly by putting $g^{\alpha\beta} = 0$. In the second approximation, due to Landau, we neglect in (31) for small coupling ($g^{\alpha\beta} \ll f^\alpha f^\beta$) all terms which contain contributions

of particle 3. Then (31) simpilifies to

$$\left[\partial_t - L_1^\alpha - L_2^\beta\right] g^{\alpha\beta} = L_{12}^{\alpha\beta}\, f^\alpha(\vec{q}_1,\vec{v}_1;t) f^\beta(\vec{q}_2,\vec{v}_2;t). \quad (33)$$

The third and most advanced ansatz is due to Balescu, Lenard, and Guernsey. Screening contributions of the third particle are taken into account when (31) is approximated by

$$\left[\partial_t - L_1^\alpha - L_2^\beta\right] g^{\alpha\beta} = L_{12}^{\alpha\beta}\, f^\alpha(\vec{q}_1,\vec{v}_1;t) f^\beta(\vec{q}_2,\vec{v}_2;t)$$
$$+ \sum_{\gamma=e,i} \int d^6 3 \left[L_{13}^{\alpha\gamma} f^\alpha(\vec{q}_1,\vec{v}_1;t) g^{\beta\gamma}(\vec{q}_2,\vec{v}_2,\vec{q}_3,\vec{v}_3;t)\right.$$
$$\left. + L_{23}^{\beta\gamma} f^\beta(\vec{q}_2,\vec{v}_2;t) g^{\alpha\gamma}(\vec{q}_1,\vec{v}_1,\vec{q}_3,\vec{v}_3;t)\right]. \quad (34)$$

Note that in all cases we shall arrive at a kinetic equation of the form

$$\partial_t f^\alpha(\vec{q}_1,\vec{v}_1;t) = L_1^\alpha f^\alpha(\vec{q}_1,\vec{v}_1;t)$$
$$+ \sum_{\beta=e,i} \int d^3q_2 d^3v_2 L_{12}^{\alpha\beta}\, f^\alpha(\vec{q}_1,\vec{v}_1;t) f^\beta(\vec{q}_2,\vec{v}_2;t)$$
$$+ K^\alpha\{f^\alpha(t)\}. \quad (35)$$

A. The Vlasov equation

Here, we have $K^\alpha \equiv 0$ and (30) can be rewritten in the following form:

$$\frac{\partial f^\alpha}{\partial t} + \vec{v}_1 \cdot \frac{\partial f^\alpha}{\partial \vec{q}_1} + \frac{e_\alpha}{m_\alpha}\left[\frac{1}{c}\vec{v}_1 \times \vec{B}(\vec{q}_1) + \vec{E}_0\right.$$
$$\left. + \vec{E}(\vec{q}_1;t)\right] \cdot \frac{\partial f^\alpha}{\partial \vec{v}_1} = 0, \quad (36)$$

where in the electrostatic approximation ($\nabla \times \vec{E} = 0$, external $\vec{B}$ and external $\vec{E}_0$) the selfconsistent electric field $\vec{E}$ follows from Poisson's equation

$$\nabla \cdot \vec{E} = 4\pi \sum_{\beta=e,i} e_\beta \int d^3v_1 f^\beta(\vec{q}_1,\vec{v}_1;t). \quad (37)$$

The Vlasov equations takes care of the collective effects and is exactly valid in the limit $\Lambda \to \infty$. Because of the long range nature of the interaction potential, particles move under the action of the electric field produced by the others; strong binary reflections are small ($\to 0$). The Vlasov equation expresses the fact that in the limit $\Lambda \to \infty$ the many small influences of all the other particles can dominate over the rare strong deflections (due to close interactions).

B. The Landau-Fokker-Planck equation

When solving (33) we have to remember our physical implication (32). One can estimate that over times of the order $\tau_c = \max(\omega_{pe}^{-1}, \omega_{pi}^{-1})$, i.e. the mean time for a collision process, the initial correlations disappear. And in kinetic theory we are not interested the relaxation phenomena on an atomic scale. In this brief summary we cannot present the details of the algebra, but summarize the result for the non-vanishing collision integral

$$K^\alpha = \sum_{\beta=e,i} 2\pi e_\alpha^2 e_\beta^2 \ln\Lambda_B \int d^3v_2 \frac{1}{m_\alpha}\frac{\partial}{\partial v_{1\nu}} G_{\nu\mu}(\vec{g}) \quad (38)$$
$$\times \left[\frac{1}{m_\alpha}\frac{\partial}{\partial v_{1\mu}} - \frac{1}{m_\beta}\frac{\partial}{\partial v_{2\mu}}\right] f^\alpha(\vec{q}_1,\vec{v}_1;t) f^\beta(\vec{q}_1,\vec{v}_2;t)$$

where

$$G_{\nu\mu}(\vec{g}) = \frac{g^2\delta_{\nu\mu} - g_\nu g_\mu}{g^3} \quad (39)$$

is the Landau tensor. Several comments are in order. First, we have used $\vec{g} = \vec{v}_2 - \vec{v}_1$ and have introduced $\ln\Lambda_B$ as the average Coulomb logarithm,

$$\ln\Lambda_B = \ln\frac{3\lambda_D k_B(T_e + T_i)}{2e^2}. \quad (40)$$

Secondly during the algebraic manipulations the divergent integral

$$A_{\alpha\beta}(0,\infty) := 2\pi e_\alpha^2 e_\beta^2 \int_0^\infty dk \frac{1}{k} \quad (41)$$

appears. The divergence for $k \to \infty$ originates from small distances in the Coulomb potential when the weak coupling approximation fails anyhow. We have used $k_{\max} \approx 3k_B T_\alpha/e_\alpha^2$, since $e_\alpha^2/3k_B T_\alpha$ is the collision parameter for 90° deflections. Finally, the divergence for $k \to 0$, corresponding to large distances in the Coulomb potential, originates from the fact that the shielding is not taken appropriately. We can either introduce a cut-off at $k_{\min} \approx 1/\lambda_D$ or replace in the evaluation of the corresponding integrals the Coulomb potential by a (static) Debye potential in an ad hoc manner. Balescu and Lenard have carried out the latter idea in a mathematically rigorous manner.

C. The Balescu-Lenard equation

Using the form (34) a mathematically rather sophisticated and physically profound procedure leads to the collision operator

$$K = \frac{1}{m_e^2}\partial_{\vec{v}} \cdot \int d^3v' Q(\vec{v},\vec{v}') \cdot (\partial_{\vec{v}} - \partial_{\vec{v}'}) f(\vec{v}) f(\vec{v}'), \quad (42)$$

where

$$Q = 8\pi^4 \int d^3k \frac{\vec{k}\vec{k}\,\varphi^2(k)}{|\epsilon(\vec{k},\vec{k}\cdot\vec{v})|^2} \delta[\vec{k}\cdot(\vec{v} - \vec{v}')], \quad (43)$$

and $\varphi(k)$ is the Fourier transform of the potential. When comparing with the Landau-Fokker-Planck collision term we clearly see that they agree for $\epsilon \approx 1 + 1/k^2\lambda_{De}^2$. Here

$$\epsilon = 1 - \sum_j \frac{\omega_{pj}^2}{k^2} \vec{k} \cdot \int \frac{\partial f_j/\partial \vec{v}}{\vec{k}\cdot\vec{v} - \omega} d^3v \qquad (44)$$

is the dispersion function, and the limit mentioned above corresponds to the static limit. The latter is equivalent to the Debye shielding. The Balescu-Lenard equation is more precise than the static limit: it takes care of the dynamical shielding of particles.

We conclude this section by a short remark explaining why the kinetic equations with weak collision terms are called "of the Fokker-Planck type". The reason is that these equations can be written in the form

$$\frac{df(\vec{v}_1)}{dt} = -\partial_{\vec{v}_1} \cdot [\vec{A} f(\vec{v}_1)] + \frac{1}{2}\partial_{\vec{v}_1}\partial_{\vec{v}_1} : [\mathcal{B}\, f(\vec{v}_1)], \qquad (45)$$

where $\vec{A}$ and $\vec{\mathcal{B}}$ correspond to $\langle \Delta\vec{v} \rangle$ and $\langle \Delta\vec{v}\Delta\vec{v} \rangle$ in a usual Fokker-Planck type derivation. Finally, although the collision terms may be small they are important for (linear) transport.

III. WAVES AND COLLISIONLESS DAMPING IN THE VLASOV APPROACH

The collective interaction described by the Vlasov equation produces extremly important phenomena in plasmas: waves, instabilities and Landau damping[18]. Let us comment on these phenomena here by sketching the derivation of a dispersion relation and its solutions (in the electrostatic limit). Linearizing the Vlasov equation by assuming

$$f(\vec{q}, \vec{v}; t) = f_0(\vec{v}) + f_1(\vec{q}, \vec{v}; t), \qquad (46)$$

where f_0 is the equilibrium distribution function which may depend on constants of motion, we obtain for the perturbation

$$\partial_t f_1 + \vec{v}\cdot\nabla f_1 - \frac{e}{m_e}\vec{E}_1 \cdot \partial_{\vec{v}} f_0 = 0, \qquad (47)$$

$$\nabla \cdot \vec{E}_1 = -4\pi e \int d^3v f_1. \qquad (48)$$

This is written for the electrons when we can consider the ions as a smeared-out background. Now a principal point has to be made. If we would solve the linear equations (47) and (48) by Fourier transformation we would get the wrong (i.e. a misleading) answer. The reason is that the Fourier transform contains a lot of false modes which do not appear in a correct solution of the initial value problem. The latter is exactly what we have: We want to find out how an initial perturbation evolves, that is to say, whether it is amplified, damped, or disperses away. After Laplace transformation

$$F(p, \vec{v}) = \int_0^\infty f_1(\vec{v}; t) e^{-pt} dt, \qquad (49)$$

$$f_1(\vec{v}; t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(p, \vec{v}) e^{pt} dp, \qquad (50)$$

we obtain from (47) and (48)

$$(p + ikv_x)F(p, \vec{v}) - \frac{e}{m_e}E_{1x}(p)\frac{\partial f_0}{\partial v_x} = f_1(\vec{v}; 0), \qquad (51)$$

$$ikE_{1x}(p) = -4\pi e \int F(p, \vec{v}) d^3v, \qquad (52)$$

where we have introduced the abbreviation $G(u) = \int dv_y\, dv_x\, f_1(u = v_x, v_y, v_z; 0)$ and a similar symbol g originating from f_0.

Combing both we obtain

$$E_{1x}(p) = -\frac{4\pi e}{ik} \frac{\int_{-\infty}^{+\infty} \frac{G(u)}{p+iku} du}{1 - \frac{\omega_{pe}^2}{k}\int \frac{dg}{du}\frac{du}{ku - ip}} \qquad (53)$$

which after backtransformation looks like

$$E_{1x}(t) = \frac{1}{2\pi i}\int_{\tilde{\sigma}-i\infty}^{\tilde{\sigma}+i\infty} E_{1x}(p)\, e^{pt}\, dp. \qquad (54)$$

In evaluating this expression we have to know the contour for integration which is described by the Laplace transform in the initial value problem. In a nutshell, $\tilde{\sigma}$ has to be larger than $\Re p_\nu$ where p_ν designates the singularities of $E_{1x}(p)$ as given by (53). Then the solution (54) can easiest be obtained by functional analysis methods. For $t \to \infty$ we obtain

$$E_{1x}(t) \approx \mathrm{Res}(p)\exp(pt). \qquad (55)$$

Note that the p used here is the pole with the largest real part. For its evaluation we have to discuss the dispersion relation

$$\epsilon(k, \omega = ip) := 1 - \frac{\omega_{pe}^2}{k^2}\int \frac{dg}{du}\frac{du}{u - \frac{\omega}{k}} = 0. \qquad (56)$$

This has to be evaluated along the socalled Landau contour. In the complex u-plane, the path has to be defined in such a way that all poles ip/k lie above the u-contour.

For a Maxwellian equilibrium distribution and low damping ($\gamma/\omega_r \ll 1$) we can obtain from (56) the famous Landau damping rate

$$\gamma \approx -\omega_{pe}\left(\frac{\pi}{8}\right)^{1/2}(k\lambda_{De})^{-3}\exp\left[-\frac{1}{2}(k\lambda_{De})^{-2}-\frac{3}{2}\right]. \tag{57}$$

The physical understanding of this Landau damping due to phase mixing actually is not so difficult, but many presentations in books are misleading. During the summer school these ideas have been worked out in more detail.

Let us here mention another point.[19] Usually, for a Maxwellian the integral in (56) is traced back to the socalled Z-function (or G-function)

$$G(\zeta) \equiv Z(-\zeta) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{e^{-p^2}}{\zeta - p}\,dp, \tag{58}$$

and the asymptotic behaviors of Z are well-known. Just to present one example. For an electron-ion plasma the dispersion relation looks like

$$k^2 = \frac{\omega_{pe}^2}{2v_{te}^2}Z'(\zeta_e) + \frac{\omega_{pi}^2}{2v_{ti}^2}Z'(\zeta_i), \tag{59}$$

and for $\omega \geq \omega_{pe}$ and $\omega/k \gg v_{te}$ the solution is

$$\omega^2 \approx \omega_{pe}^2\left(1 + 3k^2\lambda_{De}^2\right), \tag{60}$$

i.e. electron plasma waves. Other examples will be found in the various contributions to this volume.

So far we have presented a general outline for developing the kinetic theory of particles. We succeeded only in elucidating some steps for simple geometries. The whole area of drift-kinetic and gyro-kinetic descriptions, being most valid for tokamaks,[20–23] has not been touched (because of space limitations) and is left to subsequent presentations.

VI. KINETIC DESCRIPTION OF WAVES IN WEAK AND STRONG TURBULENCE

Now we want to emphasize another point. Once waves are excited in a turbulent medium (and of course in principle one first has to specify what we mean by turbulence), one may consider them as quasi-particles. Let us now discuss how one could develop a statistical description of random interacting waves.[24–27]

Let us start from a Fourier-transformed one-field equation being valid up to second order in amplitudes. We formally write

$$\partial_t\phi_{\vec{k}}(t) + i\omega_{\vec{k}}\phi_{\vec{k}}(t) = \frac{1}{2}\sum_{\vec{k}'+\vec{k}''=\vec{k}}\Lambda^{\vec{k}}_{\vec{k}',\vec{k}''}\phi_{\vec{k}'}(t)\phi_{\vec{k}''}(t)\,. \tag{61}$$

Such an equation can be obtained for example by iterating a kinetic equation. The field $\phi_{\vec{k}}$ is in general stochastic (or has at least a stochastic component) and therefore in the turbulent case a statistical description is adequate. In what follows we use the compact notation of the strong turbulence theory which is also known under the name "direct interaction approximation (DIA)". Let us group together the variables $\{\vec{k}_i, t_i\} \to \{i\}$ and introduce the operators

$$L(1\,2) := \left(\partial_{t_1} + i\omega_{\vec{k}_1}\right)\delta(t_1 - t_2)\,\delta_{\vec{k}_1,\vec{k}_2}\,, \tag{62}$$

$$N(1\,2\,3) := \Lambda^{\vec{k}_1}_{\vec{k}_2,\vec{k}_3}\,\delta(t_1 - t_2)\,\delta(t_1 - t_3)\,\delta_{\vec{k}_1,\vec{k}_2+\vec{k}_3}\,. \tag{63}$$

With the convention of summation (integration) over repeated indices, Eq. (61) can be written as

$$L(1\,2)\phi(2) + \frac{1}{2}N(1\,2\,3)\phi(2)\phi(3) = 0\,. \tag{64}$$

In statistical mechanics, the hierarchy problem related with Eq. (64) is well-known. When deriving from Eq. (64) an equation for the double-correlation function

$$\langle 1\,2\rangle := \langle\phi(1)\phi(2)\rangle \equiv \langle\phi_{\vec{k}_1}(t_1)\,\phi_{\vec{k}_2}(t_2)\rangle\,, \tag{65}$$

the latter will be determined by the triple-correlation function,

$$L(1\,2)\langle 2\,4\rangle + \frac{1}{2}N(1\,2\,3)\langle 2\,3\,4\rangle = 0\,, \tag{66}$$

and so on,

$$L(1\,2)\langle 2\,4\,5\rangle + \frac{1}{2}N(1\,2\,3)\langle 2\,3\,4\,5\rangle = 0\,. \tag{67}$$

When the triple-correlation function is set to zero we obtain the linear theory. Since the latter is not of interest here, a higher truncation level is needed. For centered moments [$\langle i\rangle \equiv 0$] a quasi-Gaussian approximation is known in weak-turbulence theory,

$$\langle 1\,2\,3\,4\rangle = \langle 1\,2\rangle\langle 3\,4\rangle + \langle 1\,3\rangle\langle 2\,4\rangle + \langle 1\,4\rangle\langle 2\,3\rangle\,. \tag{68}$$

Here we shall also use (68) for truncation at the "level four". Inserting the expression (68) into Eq. (66) we obtain an equation for the triple-correlation $\langle 2\,4\,5\rangle$ which we have to solve:

$$L(1\,2)\langle 2\,4\,5\rangle + N(1\,2\,3)\langle 2\,4\rangle\langle 3\,5\rangle = 0\,. \tag{69}$$

Note that $N(1\,2\,3)\langle 2\,3\rangle\langle 4\,5\rangle = 0$ [(see Eq. (64) after averaging and for centered moments], and the symmetry

$$N(1\,2\,3) = N(1\,3\,2) \tag{70}$$

has been used. Now, an additional assumption of the DIA comes into play: When solving for $\langle 2\,4\,5\rangle$ we use a *renormalized* resolvent g instead of the linear resolvent L^{-1}.

Furthermore, we symmetrisize with respect to "particles" 1, 2, and 3:

$$\begin{aligned}\langle 1\,2\,3\rangle = &- g(1\,6)N(6\,4\,5)\langle 2\,4\rangle\langle 3\,5\rangle \\ &- g(2\,6)N(6\,4\,5)\langle 1\,4\rangle\langle 3\,5\rangle \\ &- g(3\,6)N(6\,4\,5)\langle 2\,4\rangle\langle 1\,5\rangle. \end{aligned} \quad (71)$$

These are the main assumptions of the DIA; the definition of g will be given below, after some outlook on the general strategy.

Inserting (71) into Eq. (66) leads to

$$\begin{aligned}\{L(1\,2) &- \frac{1}{2}N(1\,6\,3)g(6\,7)N(7\,5\,2)\langle 3\,5\rangle \\ &- \frac{1}{2}N(1\,6\,3)g(3\,7)N(7\,5\,2)\langle 6\,5\rangle\}\,\langle 2\,4\rangle \\ &= \frac{1}{2}N(1\,2\,3)g(4\,7)N(7\,5\,6)\langle 3\,5\rangle\langle 2\,6\rangle. \end{aligned} \quad (72)$$

This is a closed equation for the second moment (within the DIA). When looking at Eq. (72) from the physical point of view it is natural to define the terms in the curly brackets on the left-hand-side of Eq. (72) as the renormalized propagator, i.e.

$$\begin{aligned}L(1\,2) \to g^{-1}(1\,2) &:= L(1\,2) \\ &- \frac{1}{2}N(1\,6\,3)g(6\,7)N(7\,5\,2)\langle 3\,5\rangle \\ &- \frac{1}{2}N(1\,6\,3)g(3\,7)N(7\,5\,2)\langle 6\,5\rangle \\ &\equiv L(1\,2) + \Sigma(1\,2)\,. \end{aligned} \quad (73)$$

Thus, we write Eq. (72) in the form

$$g^{-1}(1\,2)\langle 2\,4\rangle = g(4\,5)F(5\,1)\,, \quad (74)$$

where

$$F(5\,1) := \frac{1}{2}N(1\,2\,3)N(5\,6\,7)\langle 3\,6\rangle\langle 2\,7\rangle \quad (75)$$

is the so-called random noise source. Obviously the resolvent g follows from

$$[L(1\,2) + \Sigma(1\,2)]\,g(2\,3) = 1\,, \quad (76)$$

which is a complicated integral equation.

Let us now distinguish between two time-scales,

$$\tau = t_1 - t_2\,,\; T = \frac{1}{2}(t_1 + t_2)\,, \quad (77)$$

for the correlation function and assume that the dependence on T can be treated in an adiabatic manner. I.e., if we Fourier-transform (with respect to τ) a product, we use for

$$C(t_1, t_2) := \int dt_3 A(t_1, t_3)\,B(t_3, t_2) \quad (78)$$

$$\begin{aligned} C(\tau, T) &= \int d\omega C_\omega(T) e^{-i\omega\tau} \\ &= \int dt_3 \left[A\left(\tau_{13} = t_1 - t_3, T + \frac{1}{2}\tau_{32}\right)\right] \\ &\times \left[B\left(\tau_{32} = t_3 - t_2, T - \frac{1}{2}\tau_{13}\right)\right] \\ &= \int dt_3 \{A(\tau_{13}, T)\,B(\tau_{32}, T) \\ &+ \frac{1}{2}\tau_{32}B(\tau_{32}, T)\frac{\partial A(\tau_{13}, T)}{\partial T} \\ &- \frac{1}{2}\tau_{13}A(\tau_{13}, T)\frac{\partial B(\tau_{32}, T)}{\partial T}\} \\ &= 2\pi \int d\omega \{A_\omega(T)B_\omega(T) \\ &+ \frac{i}{2}\frac{\partial A_\omega(T)}{\partial \omega}\frac{\partial B_\omega(T)}{\partial T} \\ &- \frac{i}{2}\frac{\partial A_\omega(T)}{\partial T}\frac{\partial B_\omega(T)}{\partial \omega}\Big\} e^{-i\omega\tau}\,. \end{aligned} \quad (79)$$

Thus we can write

$$\frac{1}{2\pi}C_\omega \approx A_\omega B_\omega + \frac{i}{2}\frac{\partial A_\omega}{\partial \omega}\frac{\partial B_\omega}{\partial T} - \frac{i}{2}\frac{\partial A_\omega}{\partial T}\frac{\partial B_\omega}{\partial \omega}\,, \quad (80)$$

where we have dropped the additional argument T.

Using this rule (and approximation) when Fourier-transforming Eq. (72), we can write

$$\begin{aligned}\Sigma(1\,2) = -\frac{1}{2}\sum_{\vec{k}_4 + \vec{k}_5 = \vec{k}_1 = \vec{k}_2} &\Lambda^{\vec{k}_1}_{\vec{k}_4, \vec{k}_5}\Lambda^{\vec{k}_4}_{-\vec{k}_5, \vec{k}_2} g_{\vec{k}_4}(t_1 - t_2) \\ &\times \langle\phi^2\rangle_{\vec{k}_5}(t_1 - t_2) + \{\vec{k}_4 \leftrightarrow \vec{k}_5\}\,. \end{aligned} \quad (81)$$

Here, we have defined for homogeneous turbulence

$$g(1\,2) := g_{\vec{k}_1}(t_1 - t_2)\,\delta_{\vec{k}_1, \vec{k}_2}, \quad (82)$$

$$\langle 1\,2\rangle := \langle\phi^2\rangle_{\vec{k}_1}(t_1 - t_2)\,\delta_{\vec{k}_1, -\vec{k}_2}; \quad (83)$$

again the T-dependence is suppressed. The corresponding Fourier-transforms are

$$I_{\vec{k},\omega} = \frac{1}{2\pi}\int d\tau\; e^{i\omega\tau}\langle\phi^2\rangle_{\vec{k}}(\tau), \quad (84)$$

$$g_{\vec{k},\omega} = \frac{1}{2\pi}\int d\tau\; e^{i\omega\tau} g_{\vec{k}}(\tau). \quad (85)$$

By the Faltungstheorem we obtain from (81)

$$\Sigma_{\vec{k},\omega} = -\sum_{\vec{k}'}\Lambda^{\vec{k}}_{\vec{k}-\vec{k}',\vec{k}'}\Lambda^{\vec{k}-\vec{k}'}_{-\vec{k}',\vec{k}}\int d\omega' g_{\vec{k}-\vec{k}',\omega-\omega'} I_{k',\omega'}. \quad (86)$$

We are now in a position to write for the left-hand-side of Eq. (74)

$$\begin{aligned}&\frac{1}{2\pi}\int d\tau\; e^{i\omega\tau}[L(1\,2) + \Sigma(1\,2)]\,\langle 2\,3\rangle \\ &\approx 2\pi\left[\frac{1}{2}\frac{\partial}{\partial T}I_{\vec{k}_1,\omega} + \left(-i\omega + i\omega_{\vec{k}_1} + \Sigma_{\vec{k}_1,\omega}\right) I_{\vec{k}_1,\omega}\right]\delta_{\vec{k}_1,-\vec{k}_3}. \end{aligned} \quad (87)$$

We have neglected the ω-derivative of $\Sigma_{\vec{k}_1,\omega}$ as a higher-order effect in the propagator. In the same way, one obtains for the right-hand-side of Eq. (74)

$$\frac{1}{2\pi}\int d\tau\, e^{i\omega\tau} g(3\,4) F(4\,1) \approx 2\pi g_{-\vec{k}_1,-\omega} F_{\vec{k}_1,-\omega}\delta_{\vec{k}_3,-\vec{k}_1}, \tag{88}$$

where

$$\begin{aligned} F_\omega &= \frac{1}{2}\sum_{\vec{k}'}\int d\omega' \left(\Lambda^{\vec{k}_1}_{\vec{k}_1-\vec{k}',\vec{k}'}\right)^2 I_{\vec{k}',\omega'} I_{\vec{k}_1-\vec{k}',\omega-\omega'}\delta_{\vec{k}_4,-\vec{k}_1} \\ &\equiv F_{\vec{k}_1,\omega}\delta_{\vec{k}_4,-\vec{k}_1}. \end{aligned} \tag{89}$$

Combing all the results found so far, we can formulate the wave-kinetic equation within the DIA:

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial T} I_{\vec{k}_1,\omega} + \left(-i\omega + i\omega_{\vec{k}_1} + \Sigma_{\vec{k}_1,\omega}\right) I_{\vec{k}_1,\omega} \\ &= \frac{1}{2} g_{-\vec{k}_1,-\omega}\sum_{\vec{k}'}\int d\omega' \left(\Lambda^{\vec{k}_1}_{\vec{k}_1-\vec{k}',\vec{k}'}\right)^2 I_{\vec{k}',\omega'} I_{\vec{k}_1-\vec{k}',\omega-\omega'}. \end{aligned} \tag{90}$$

One point we have to remember: ω is the Fourier-variable conjugate to τ, i.e. the fast variable, and thus it is natural to assume ω to be real. On the slow time-scale T the spectral density $I_{\vec{k}_1,\omega}$ varies due to the imaginary parts of $\omega_{\vec{k}_1}$ and $\Sigma_{\vec{k}_1,\omega}$.

For weak-turbulence we assume

$$I_{\vec{k}_1,\omega} \approx \mid \phi \mid^2_{\vec{k}_1} \delta\left(\omega - \Re(\omega_{\vec{k}_1})\right), \tag{91}$$

$$g_{-\vec{k}_1,-\omega} \approx \frac{1}{i\omega - i\Re(\omega_{\vec{k}_1}) + \delta} \approx \pi\delta\left(\omega - \Re(\omega_{\vec{k}_1})\right) \tag{92}$$

to obtain from Eq. (91)

$$\begin{aligned} &\partial_T \mid \phi \mid^2_{\vec{k}} - 2\Im(\omega_{\vec{k}}) \mid \phi \mid^2_{\vec{k}} \\ &= \pi\int d^2k_1 d^2k_2 \delta(\vec{k}-\vec{k}_1-\vec{k}_2)\delta\left[\Re(\omega_{\vec{k}} - \omega_{\vec{k}_1} - \omega_{\vec{k}_2})\right] \\ &\times \left(\Lambda^{\vec{k}}_{\vec{k}_1,\vec{k}_2}\right)^2 \mid \phi \mid^2_{\vec{k}_1} \mid \phi \mid^2_{\vec{k}_2} \\ &+ \pi\int d^2k_1 d^2k_2 \delta(\vec{k}-\vec{k}_1-\vec{k}_2)\delta\left[\Re(\omega_{\vec{k}} - \omega_{\vec{k}_1} - \omega_{\vec{k}_2})\right] \\ &\times \{\Lambda^{\vec{k}}_{\vec{k}_1,\vec{k}_2}\Lambda^{\vec{k}_1}_{-\vec{k}_2,\vec{k}} \mid \phi \mid^2_{\vec{k}_2} \mid \phi \mid^2_{\vec{k}} \\ &+ \Lambda^{\vec{k}}_{\vec{k}_2,\vec{k}_1}\Lambda^{\vec{k}_2}_{-\vec{k}_1,\vec{k}} \mid \phi \mid^2_{\vec{k}_1} \mid \phi \mid^2_{\vec{k}}\}. \end{aligned} \tag{93}$$

This is the standard weak-turbulence wave-kinetic equation. Actually, the self-damping, described by $\Sigma_{\vec{k},\omega}$, should cause – by assumption – the sign-convention as usually assumed in standard weak-turbulence theory. As being used in Eq. (92), the "damping term δ" is responsible for the physically reasonable decay of correlations. Within the weak turbulence theory this is an assumption. Within a Hamiltonian description of DIA it appears self-consistently.

We do not introduce here the adiabatic density of waves which would allow an elegant Hamiltonian formulation. For our purposes it is sufficient to use the $\mid \phi \mid^2_{\vec{k}}$ or for example in the case of drift waves $C_{\vec{k}} := (1+k^2) \mid \phi \mid^2_{\vec{k}}$. With the latter the more symmetric form

$$\begin{aligned} &\partial_t C_{\vec{k}} - 2\Im(\omega_{\vec{k}}) C_{\vec{k}} \\ &= \pi\int d^2k_1 d^2k_2 \delta(\vec{k}-\vec{k}_1-\vec{k}_2)\delta(\omega_{\vec{k}} - \omega_{\vec{k}_1} - \omega_{\vec{k}_2}) \\ &\times (1+k^2)\{a^{\vec{k}}_{\vec{k}_1,\vec{k}_2} C_{\vec{k}_1} C_{\vec{k}_2} + b^{\vec{k}}_{\vec{k}_1,\vec{k}_2} C_{\vec{k}_2} C_{\vec{k}} + b^{\vec{k}}_{\vec{k}_2,\vec{k}_1} C_{\vec{k}_1} C_{\vec{k}}\} \end{aligned} \tag{94}$$

is obtained where

$$a^{\vec{k}}_{\vec{k}_1,\vec{k}_2} := (1+k_1^2)^{-1}(1+k_2^2)^{-1} \mid \Lambda^{\vec{k}}_{\vec{k}_1,\vec{k}_2} \mid^2, \tag{95}$$

$$b^{\vec{k}}_{\vec{k}_1,\vec{k}_2} := (1+k_1^2)^{-1}(1+k_2^2)^{-1}\Lambda^{\vec{k}}_{\vec{k}_1,\vec{k}_2}\Lambda^{\vec{k}_1}_{-\vec{k}_2,\vec{k}} \tag{96}$$

satisfy the relation

$$a^{\vec{k}}_{\vec{k}_1,\vec{k}_2} = -(b^{\vec{k}}_{\vec{k}_1,\vec{k}_2} + b^{\vec{k}}_{\vec{k}_2,\vec{k}_1}). \tag{97}$$

V. SUMMARY

In this contribution we have outlined the basic principles for a statistical description of plasmas. The equations developed here work very well in simple geometries where such effects as particle trapping, particle drifts, and plasma wall interactions can be neglected. However, for tokamaks exactly the latter may be dominant, and therefore by similar arguments as presented above the relevant statistical description has to be developed. It is known as neoclassical, drift-kinetic, and gyro-kinetic descriptions, being valid in different parameter regions. The following list of references is not complete at all. It contains some references (from a personal point of view) which might be useful for a first penetration into this field. After that, of course, one should closely follow the actual literature which appears in the journals. The types of plasmas are classified in many textbooks, the literature on equilibrium statistical mechanics is huge, and for the kinetic theory of particles the monographs by Balescu are highly recommended. Wave kinetics is usually less emphasized in textbooks. Many relevant hints can be found in the Handbook of Plasmaphysics.[28] Finally, for actual calculations the "plasma formulary" is highly recommended.[29]

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