RECENT DEVELOPMENTS IN SOLITON THEORY WITH APPLICATIONS TO PLASMAS

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ABSTRACT: After a short general discussion of integrable systems possesing soliton solutions, the importance of the soliton concept in plasma physics is reviewed. Specific problems, such as existence of finite amplitude localized waves and their multi-dimensional stability are also considered. The paper is concluded by a short examination of related problems of multi-dimensional convective motion.

I. INTRODUCTION

The observation of a solitary wave on the surface of water by Scott-Russell[1] is now understood to be the first known study of nonlinear waves. Since then several hundreds of papers appeared which used the soliton concept in explaining physical phenomena in many branches of physics[2].

The interesting properties of a soliton can be most easily understood by discussing a simple example, the Korteweg-de Vries (KdV) equation[3],

$$\partial_t \varphi + \frac{1}{6}\partial_x^3 \varphi + \frac{3}{2}\varphi\,\partial_x \varphi = 0\,, \tag{I.1}$$

which in an approximate way determines, e. g., the normalized surface elongation φ of a nonlinear wave in shallow water. Besides the time derivative, dispersive and nonlinear terms appear. The dispersive term reveals the linear dispersion[4]

$$\begin{aligned} \omega &= g^{1/2} k^{1/2} \tanh^{1/2}(kh) \\ &\approx g^{1/2} h^{1/2} k \left(1 - \tfrac{1}{6} h^2 k^2\right), \\ &\quad \text{for } kh \ll 1\,, \end{aligned} \tag{I.2}$$

for shallow water waves in systems of depth h, under the action of a gravitational field (g). The nonlinear term has its origin in the convective derivative appearing in the equation of motion. (Note that in Eq. (I.1), and in the following, we use non-dimensional quantities.) Looking for stationary localized solutions of Eq. (I.1) in the moving frame, one finds

$$\varphi_s = \varphi_0 \,\mathrm{sech}^2\left[\frac{3^{1/2}}{2}\varphi_0^{1/2}\left(x - \frac{\varphi_0}{2} t\right)\right]. \tag{I.3}$$

Several features are interesting: (i) For this solution all physically relevant quantities are localized in space; this is the reason for calling it a solitary wave. (ii) Amplitude, width, and velocity are correlated; this is typical for solutions of the KdV equation. (iii) When applied to the observations of Scott-Russell, this solution explains in a surprisingly simple and accurate way the reported data on velocity, height, etc.

However, the balance of dispersive and nonlinear effects in producing a stationary solution is not the only interesting aspect of the KdV equation. First, numerical solutions[5] of the initial value problem (I.1) showed that (I.3) is a quite fundamental solution which always appeared besides oscillatory contributions. (This statement is a little bit oversimplified since also n-soliton solutions can appear; we do not go into the details here.) Secondly, and most interesting, when numerical solutions were produced for colliding solitary waves (head on as well as overtaking collisions), essentially the solitary waves came out of the interaction process with unchanged forms; only a phase shift occured. That was the reason for calling solitary waves of stable form solitons. Nowadays, the distinction between solitary waves (which might be unstable during collisions) and solitons is not so strictly in use and therefore we shall not insist on the difference in terminology here.

Theoretically, all these features of the KdV-equation are now well-understood. Contributions by Lax[6] and others lead to a break-through in the year 1967 when Gardner et al[7] were able to solve the initial value problem (I,1) by the socalled inverse scattering method (ISM). Using the initial distribution $\varphi(x,o)$ as a potential for an ordinary Schrödinger scattering problem, the scattering data at $t = o$ are well-defined. By linear ordinary differential equations the scattering data at time t follow, from which the "potential" $\varphi(x,t)$ can be reconstructed. Lateron, many authors, expecially Zakharov and Shabat[8] as well as Ablowitz et al[9] were able to generalize the scattering problem in order to cover many more one-dimensional nonlinear equations[10] which are now solvable by the ISM. It should also be mentioned, that recently[11] the ISM has been further generalized to solve some two-dimensional nonlinear equations. Much progress is expected in this area during the next years. We cannot summarize the details of the important and extremely interesting recent developments here because of space limitations.

Before concentrating on applications in plasma physics we briefly mention applications in other fields in order to demonstrate the very similar aspects appearing everywhere in nonlinear physics.

In <u>fluid theory</u>, the reductive perturbation method was developed and various nonlinear model equations could be derived. When reconsidering for example the water wave problem, we have now a clear picture[12] not only for shallow water waves but also for systems with arbitrary depth. In addition, not only one-dimensional situations have been treated. As two limits, besides the more general Davey-Stewartson-equations[13], we mention the cubic nonlinear Schrödinger equation for deep water[14],

$$i\partial_t\phi - \partial_x^2\phi + \nabla_\perp^2\phi - |\phi|^2\phi = 0 \quad , \qquad \text{(I.4)}$$

and the Kadomtsev-Petviashvili equation for shallow water[15]

$$\partial_t\varphi + \frac{1}{6}\,\partial_x^3\varphi + \frac{3}{2}\varphi\partial_x\varphi + \frac{1}{2}\int^x \nabla_\perp^2\varphi\,dx' = 0. \qquad \text{(I.5)}$$

The, e.g, one-dimensional, solutions of these two limiting equations (I.4,5) are physically quite different. The soliton solution (I.3) of Eq. (I.5) is a solitary pulse of the surface elongation, whereas the stationary solution of Eq. (I.4),

$$\phi_s = 2^{1/2}\phi_0\,\mathrm{sech}(\phi_0 x)\exp(i\phi_0^2 t) \qquad \text{(I.6)}$$

is a localized solution for the envelope of a wave. A localized envelope can be produced by the modulational instability[16]. Note also that the amplitude-width-velocity-relations are different from (I.3), and a nonlinear frequency shift (ϕ_0^2) appears.

In <u>biophysics</u>, among different approaches to the problem of energy and charge transport on the molecular level, solitons are of increasing interest[17]. The solitons can propagate in molecular systems over comparatively large distances without changing their form and allow us to explain the high efficiency of energy and charge transfer. For example, Davydov and coworkers[18] have worked out a theory for proton motion along one-dimensional chains, called Bernal-Fowler filaments. Let us simplify the discussion by considering a quasi-one-dimensional ordered chain of water molecules formed by hydrogen bonds and resembling the structure of ice. An important property of the hydrogen bond is that the proton potential energy curve in the hydrogen bonds has the form of a well with two minima corresponding to equilibrium positions of a proton. Usually in such a system there are defects which can propagate and there are essentially enough reorientations such that always enough protons can start their motion along the chain. This picture leads, in the continuum limit, to the following mathematical model[19] for the position u of the hydrogen atom between two oxygen atoms which might deviate from their equilibrium positions by ρ,

$$\partial_t^2 u - c_0^2 \partial_x^2 u - \omega_0^2 \left(1 - \frac{u^2}{u_0^2}\right) u + 2 \frac{\chi}{m} \rho u = 0 \,, \tag{I.7}$$

$$\partial_t^2 \rho - v_0^2 \partial_x^2 \rho + \Omega_0^2 \rho + \frac{\chi}{M} (u^2 - u_0^2) = 0 \,. \tag{I.8}$$

Here, u_0 is the equilibrium position of the hydrogen atom and the constants c_0, v_0, ω_0, Ω_0, m, M, and χ can be found in the literature[19]. Eqs. (I.7,8) are two coupled nonlinear Klein-Gordon-equations which posses stationary solitary wave solutions in the form of kinks for u and single humps for ρ. The latter are known numerically and recently, their stability has been proved analytically.[19]

In condensed matter physics the sine-Gordon equation is of importance. The study of soliton dynamics in connection with, e.g., large Josephson tunnel junctions[20] has shown that the latter could support the resonant propagation of a soliton (fluxon) trapped in the junction. The soliton being a 2π-jump in the phase difference (ϕ) across the insulating barrier which separates the two superconductors and has been observed in the current voltage characteristic of the junction[20]. Numerical studies have revealed that configurations with bunched solitons play an important role in explaining the dynamics of the motion, the current voltage characteristic, power output, etc. Restricted to narrow junctions and thin enough oxide layers to permit quantum-mechanical tunneling, the basic Josephson equations can be combined with Maxwell equations to yield, in general, a perturbed sine-Gordon equation. Applying some further approximations for the energy stored within one London penetration depth of the superconducting metal as well as the externally applied bias-current, one arrives (in dimensionless form) at the sine-Gordon equation

$$\partial_x^2 \phi - \partial_t^2 \phi = \sin \phi \,. \tag{I.9}$$

A stationary solution of Eq. (I.9) is the kink

$$\phi_s = 4 \tan^{-1} \left[\exp\left(\pm \frac{x \pm vt}{\sqrt{1 - v^2}} \right) \right] . \tag{I.10}$$

The last model mentioned in this context originates from nonlinear optics. The propagation of coherent optical pulses in a two-level (or many level) system is described by the Maxwell equations together with the Schrödinger equation for the medium (e.g., two-level atoms, where the restriction to two levels follows from some resonance condition). The theory results in the socalled Maxwell-Bloch equations[21],

$$(\partial_t + c\partial_x) \mathcal{E} e^{i\varphi} = -4\pi n_0 \omega_0 p_0 \langle v_1 v_2^* e^{-i\varphi} \rangle e^{i\varphi} , \tag{I.11}$$

$$\partial_t v_1 + \tfrac{1}{2} i \Delta\omega\, v_1 = \frac{p_0 \mathcal{E}}{2\hbar} e^{i\varphi} v_2 \,, \tag{I.12}$$

$$\partial_t v_2 - \tfrac{1}{2} i \Delta\omega\, v_2 = -\frac{p_0 \mathcal{E}}{2\hbar} e^{-i\varphi} v_1 \,. \tag{I.13}$$

The first equation describes the electromagnetic wave with slowly varying envelope $\mathcal{E}$ and phase φ, where on the r. h. s. the excitation and de-excitation within the two-level system is covered by the time derivative of the polarization. ω_0 is the frequency, and $\Delta\omega$ is the frequency mismatch between the wave and the level difference. n_0 is the number of atoms, p_0 the dipole moment, and the average is over the mismatch $\Delta\omega$ originating from Doppler shifts due to thermal motion. ν_1 and ν_2 are measures of the relative populations of levels 1 and 2, respectively. If the thermal motion of the atoms is neglected, and introducing the abbreviation

$$\sigma(x,t) = \int_{-\infty}^{t} \frac{p}{\hbar} \mathcal{E}(x,t')\,dt' , \tag{I.14}$$

one arrives at the sine-Gordon equation in light-cone coordinates,

$$\partial_\xi \partial_\tau \sigma = \pm \sin\sigma , \tag{I.15}$$

where the ± sign characterizes an amplifying or attenuating medium, respectively. Note that (I.15) can be transformed into the form (I.9), and thus the soliton solution (I.10) is here also relevant. It describes the stable propagation of an optical pulse through the medium.

Many other models for soliton propagation exist. Besides the fields mentioned already, applications in astrophysics[22], particle physics[23], statistical and mathematical physics[24] are well-known.

II. SOLITONS IN PLASMAS

In an unmagnetized plasma basically two electrostatic normal modes occur: the ion-acoustic and the Langmuir oscillations. In the following we shall study the nonlinear versions of these modes including modifications due to external magnetic fields. In magnetized plasmas many more modes exist; this aspect will be briefly touched on in Sec. IV.

a. The ion-acoustic soliton

For small β (= $4\pi n_e T_e/B^2$) the mode is purely longitudinal and may be described by the fluid equations for ions,

$$\partial_t n + \nabla\cdot(n\vec{v}) = 0 , \tag{II.1}$$

$$\partial_t \vec{v} + \vec{v}\cdot\nabla\vec{v} + \nabla\phi + \Omega\hat{z}\times\vec{v} = 0, \tag{II.2}$$

together with Poisson's equation,

$$\nabla^2\phi = \exp(\phi) - n , \tag{II.3}$$

where a Boltzmann distribution for the electrons has been assumed. Note that lengths are normalized by the electron Debye length, times are measured in inverse ion plasma frequencies, and the potential ϕ is normalized by T_e/e. The parameter $\Omega = \Omega_i/\omega_{pi}$ takes care of an external magnetic field (in z-direction) through the ion gyrofrequency Ω_i.

First, for deriving a small-amplitude, simple model from Eqs. (II. 1-3) one can apply the reductive perturbation technique[25] to arrive for Ω = o at the Kadomtsev-Petviashvili equation[15]

$$\partial_t n + \frac{1}{2}\partial_x^3 n + n\partial_x n + \frac{1}{2}\int^x \nabla_\perp^2 n\, dx' = 0, \tag{II.4}$$

and for $\Omega \sim 1$ at the Zakharov-Kuznetsov equation[26],

$$\partial_t n + \frac{1}{2}\partial_x^3 n + n\partial_x n + \frac{1}{2}(1+\Omega^{-2})\partial_x \nabla_\perp^2 n = 0. \tag{II.5}$$

In the intermediate region, $\Omega \ll 1$, the following model equation is valid,[27]

$$\partial_t n + \frac{1}{2}\partial_x^3 n + n\,\partial_x n + \frac{1}{2}(\partial_x^2+\Omega^2)^{-1}\partial_x \nabla_\perp^2 n = 0. \tag{II.6}$$

Let us first discuss the theoretical predictions following from these equations in one space dimension. Obviously, in this case all the conclusions discussed in connection with the KdV-equation should be true for ion-acoustic waves. And indeed, in a series of experiments, Ikezi and coworkers[28] were able to verify experimentally the predicted amplitude-width relation, form stability during collisions, number of solitons, etc. Discrepancies between the theoretical predictions and the experimental data are only in details and are due to finite ion temperatures and deviations from the electron Boltzmann distribution.

For finite amplitudes, the asymptotically valid (for amplitudes tending to zero) reductive perturbation theory cannot be applied. The stationary (in a moving frame) and one-dimensional solutions of the basic equations (II.1-3) can be obtained from[29]

$$\frac{d^2\phi}{dx^2} = \exp(\phi) - \left(1 - \frac{2\phi}{M^2}\right)^{-1/2}, \qquad \text{(II.7)}$$

where M is the Mach number (velocity of the soliton measured in terms of the ion-acoustic speed). The r. h. s. of Eq. (II.7) can be written in terms of a ϕ-derivative of the socalled Sagdeev potential ($-\,dV/d\phi$), where the potential V is given by

$$V = 1 - \exp(\phi) + M^2\left[1 - \left(1 - \frac{2\phi}{M^2}\right)^{1/2}\right]. \qquad \text{(II.8)}$$

A simple interpretation[30] (x interpreted as "time" and ϕ as "position") shows that the existence region of solitons can be discussed in the same way as the motion of a classical particle in a potential well. The result is that for $1 < M \lesssim 1.6$ finite amplitude ion acoustic solitons can be obtained by simple integration.

The analogy with the motion of a classical particle in a potential well allows us to predict the effect of small dissipation. A classical particle will be decelerated, meaning that after reflection it will not reach its starting position again. The same is true for the potential ϕ, and therefore shock solutions are expected.

The finite amplitude theory for ion-acoustic solitons will be limited in application, since at large potential humps ions will be reflected. This process can be considered also as an effective damping mechanism, since reflected ions carry away energy from the wave. Again shock solutions appear. When estimating the amplitudes of potential humps which are necessary to create asymmetries due to reflection of many ions, we find that symmetric relative density humps with

$$\frac{\delta n}{n_0} > 0.3 \qquad \text{(II.9)}$$

are unlikely to be observed. This is consistent with experimental data.

The problem of existence of multi-dimensional ion-acoustic solitons and their stability will be discussed in Sec. III.b.

b. The Langmuir soliton

The coupled set of wave equation

$$i\epsilon\,\partial_t\vec{E} + \nabla^2\vec{E} - q\,\nabla\times(\nabla\times\vec{E}) - (n_e - 1)\vec{E} = 0, \qquad \text{(II.10)}$$

and plasma response

$$\partial_t n_i + \nabla\cdot(n_i\vec{v}_i) = 0, \qquad \text{(II.11)}$$

$$\partial_t\vec{v}_i + \vec{v}_i\cdot\nabla\vec{v}_i \approx -\nabla\phi, \qquad \text{(II.12)}$$

$$\nabla \ln n_e \approx \nabla\phi - \nabla|E|^2, \qquad \text{(II.13)}$$

$$\frac{1}{3}\nabla^2\phi = n_e - n_i, \qquad \text{(II.14)}$$

is a quite general basis for investigating nonlinear Langmuir waves. This model uses a two-timescale formalism where the timescales are separated by $\epsilon = 2(m_e/3m_i)^{1/2}$, and a hydrodynamic plasma response through the electrons (mass m_e, density n_e) and the ions (mass m_i, density n_i, velocity $\vec{v}_i$); ϕ is the ambipolar potential. For magnetic-field-free situations, the low-frequency field is electrostatic. Furthermore, because of their large mobility, electrons react to the forces via a Boltzmann distribution. For simplicity, we have neglected the ion temperature T_i. For $M < (T_i/T_e)^{1/2}$ the ions are also approximately Boltzmann distributed and the model (II.10-14) breaks down. Note also that all times have been nondimensionalized by $\sqrt{3}\,\lambda_e$. Other units are $(16\pi n_0 T_e)^{1/2}$ for the electric field and T_e/e for the potential, and $q = (m_e c^2/v_{te}) - 1$. Note that the coupling between the hf field and the plasma is due to the ℓf ponderomotive force[31] ($\sim -\nabla|E|^2$). The latter arises because of the radiation pressure.

In the past, many simplified models have been derived, since the time dependent Eqs. (II.10-14) are very difficult to handle. For example, when a static response is assumed, i.e.,

$$n_e - 1 \approx -|E|^2 \qquad \text{(II.15)}$$

in the small amplitude limit, then a cubic nonlinear Schrödinger equation is obtained from Eq. (II.10). Rudakov[32] discussed such a model and presented a single-humped one-dimensional soliton solution for the electric field E. A very important model has been developed by Zakharov[33],

$$\nabla^2(i\partial_t\varphi + \nabla^2\varphi) = \nabla\cdot(n_e\nabla\varphi), \qquad \text{(II.16)}$$

$$(\partial_t^2 - \nabla^2)n_e = \nabla^2|\nabla\varphi|^2, \qquad \text{(II.17)}$$

where φ is the envelope of the hf electrostatic potential. In contrast to (II.15), the plasma response is now dynamic.

Nishikawa et al[34] generalized this result for ion perturbations moving close to ion-acoustic speed. The ion response then becomes of first order in the pump amplitude and hence nonlinear. The plasma response is then described by a KdV-type equation, modified by the ponderomotive force. Other scalings, resulting in different model equations, have been presented by various authors[35].

Let us concentrate in the following on one aspect: the stationary one-dimensional soliton solutions of Eqs. (II.10-14). Using the results of the asymptotic theorys mentioned above, one finds for small amplitudes($\eta \to o$) and $M \to o$

$$E \simeq 2^{1/2}(1-M^2)^{1/2}\eta\,\mathrm{sech}\,\eta x, \qquad \text{(II.18)}$$

$$\phi \simeq -2M^2(1-M^2)^{-1/2}\eta^2\,\mathrm{sech}^2\eta x, \qquad \text{(II.19)}$$

whereas for $\eta \to 0$ and $M \to 1$ one has

$$E \simeq \left[8\eta^4 + 6\eta^2(M^{-2}-1)\right]^{1/2}\mathrm{sech}\,\eta x\,\tanh\eta x, \qquad \text{(II.20)}$$

$$\phi \simeq -6\eta^2\,\mathrm{sech}^2\eta x\,. \qquad \text{(II.21)}$$

In the latter case, only along the path

$$\eta^2 \approx \frac{3}{20}(M^{-2}-1) \qquad \text{(II.22)}$$

the soliton solutions exist. Accepting single humped (in E) solutions for $M = o$ and E-solitons with one node for $M \to 1$ and $\eta \to 0$, we are confronted with the problem what happens for finite M and η. For that, one has to solve the nonlinear eigenvalue problem[36]

$$\partial_x^2 E = \eta^2 E - \left[1 - \exp(\phi - E^2)\right]E\,, \qquad \text{(II.23)}$$

$$\frac{1}{3}\partial_x^2\phi = \exp(\phi - E^2) - \frac{M}{(M^2 - 2\phi)^{1/2}}\,,$$

which follows from Eqs. (II.10-14); η^2 and M play the role of the eigenvalues. We apply boundary conditions for localized solutions: E, $\phi \to o$ for $x \to \pm\infty$. The results of the calculations are rather surprising; one should note however that qualitatively similar features also occur in the small-amplitude sonic model[37]. First, the eigenvalue spectrum is discrete and not continuous as assumed in all previous finite amplitude theories. That means that solitons exist only for specific M-η-relations. Secondly, the soliton solutions for M > o are of the Nishikawa et al[34] type, i.e., E has one (or more than one) node. Thirdly, bands of eigenvalues exist; the bands can be consequently numbered by the number of ϕ-oscillations inbetween two neighbouring maxima and minima of E. The bands with more ϕ-oscillations occur closer to M = o. Fourthly, the distance between the maximum and minimum of E increases with the number of ϕ-oscillations inbetween them. This leads to the following picture for the transition to M = o solitons: As $M \to o$, the minimum of E moves out at infinity and $\phi \to o$.

In all the results, the ambipolar potential ϕ has a (mainly negative) solitary wave structure. For very large amplitudes, thereby the validity of the present theory will be limited. Deviations from the Boltzmann distribution and the occurrence of reflected electrons by the negative potential could produce an ion rich region behind the negative potential and no symmetric solutions will exist. Such large amplitude effects become important when the potential difference is on the order of the electron temperature T_e/e, and are responsible for weak double layers[38]. Hasegawa and Sato[39] have discussed theoretically the existence of a negative potential solitary wave structure and the consequent formation of a double layer even when the drift velocity is smaller than the electron thermal velocity ($v_{de} < v_{te}$). The peak of the potential can exceed the electron temperature T_e/e. It should be noted that this structure is physically quite different from the socalled strong double layer: a necessary condition for the latter is the existence of a super-thermal electron drift. Another question is whether really a stationary double layer exists. And indeed, the positive (asymmetric) potential will stay only until the ions respond to it. Hence, the weak double layer will last only within the ion inertia time scale, and such a time evolution is clearly seen in the numerical simulation of Sato and Okuda[40]. A weak double layer becomes unstable with respect to the excitation of ion acoustic solitons.

III. <u>THE STABILITY PROBLEM</u>

a. <u>Plane Solitons</u>

So far we have discussed models of ion-acoustic and Langmuir solitons and their stationary one-dimensional solutions. Now we investigate the stability properties of the latter.

Let us start with the ion-acoustic solitons. In the weak amplitude one-dimensional limit everything is clear: the solitons are stable since they are described in by a KdV equation. However, in the finite amplitude case [see Eqs. (II.1-3)] even in one dimension no stability proof exists, although solitons are expected to be stable. Since the multi-dimensional generalizations of the KdV equation are known [see Eqs. (II.4-5)], we can answer the question whether the one-dimensional stability is preserved when two-dimensional distortions are allowed for. It was shown[15,41] that the Kadomtsev-Petviashvili equation (for Ω = O) leads to transversely stable plane soliton solutions. On the other hand, for very strong magnetic fields, plane solitons are unstable[15,42]. Recently, the transition from stable to unstable behavior was computed[43] and for a 1d-solution of the form $n_s = 6\eta^2 \operatorname{sech}^2 \eta x$, a threshold $\Omega_T \approx \eta$ in Ω was found. For $\Omega > \Omega_T$ instability occurs.

We now discuss in more detail the corresponding problem for the Langmuir soliton. Within the static approximation in one dimension everything is clear in the weak amplitude limit:

the solitons are stable since they are described by the integrable cubic nonlinear Schrödinger equation. However, already for $M \to 1$, even in one dimension no stability proof exists if Poisson's equation becomes important[44]. Let us therefore concentrate in the following on the two models: the nonlinear Schrödinger equation and the Zakharov equations (II.16,17). A transverse instability, first discussed by Zakharov and Rubenchik[45], can be derived from complementary variational principles[46]. The latter allow to calculate the growth rate γ in the whole unstable (transverse) wavenumber k-regime. For example, for the scalar, cubic nonlinear Schrödinger equation

$$i\partial_t \Psi + \nabla^2 \Psi + |\Psi|^2 \Psi = 0 , \qquad \text{(III.1)}$$

with 1d soliton solution $\psi_s := \varphi_s \exp(i\eta_s^2 t)$, we have[46]

$$\gamma^2 = \sup_{\langle a|\varphi_s\rangle = 0} \frac{-\langle a|H_-|a\rangle}{\langle a|H_+^{-1}|a\rangle} , \qquad \text{(III.2)}$$

$$\gamma^2 = \inf_{a\in M} \frac{-\langle a|H_-H_+H_-|a\rangle}{\langle a|H_-|a\rangle} , \qquad \text{(III.3)}$$

where $H_+ = -\nabla^2 - \varphi_s^2 + \eta_s^2$, $H_- = H_+ - 2\varphi_s^2$, and M is the set of functions φ for which $\langle \varphi|H_-|\varphi\rangle < 0$. Evaluating (III.2,3) we find instability for $k < \sqrt{3}\,\eta_s$ with a quite large maximum growth rate ($\gamma \sim \omega_{pe}$). However, the latter is significantly reduced when the dynamic ion response is taken into account. In addition, the maximum growth rate is shifted towards smaller k-values.[47]

We now investigate the nonlinear evolution of the transverse instability. The standard method is that of Newell and Whitehead[48], which is widely used in fluid dynamics to investigate the onset of turbulence via successive instabilities. Janssen and Rasmussen[49] used a slightly different perturbation scheme which gives essentially the same results. One can consider k as the critical parameter; for $k < k_c \equiv \sqrt{3}\,\eta_s$ instability sets in. Applying the Newell-Whitehead procedure one finds near the threshold for the amplitude ϕ of the unstable mode[49]

$$\partial_t^2 \phi - \gamma^2 \phi - \beta^2 |\phi|^2 \phi = 0 . \qquad \text{(III.4)}$$

Thus, in contrast to many other instabilities, the transverse instability is not quenched by nonlinear effects. Instead, nonlinearities tend to enhance the growth rate of the linearly unstable mode.

This is not unexpected since a collapse[33,50] should occur. (Note, however, that many collapse arguments are not applicable when periodic boundary conditions are imposed in the transverse direction.) We can not go into the details of all the collapse arguments but discuss only qualitatively the multidimensional behavior of the solutions of the cubic nonlinear Schrödinger equation. From the invariant

$$I_1 = \int |\Psi|^2 d^\nu x , \qquad \text{(III.5)}$$

where $\nu (= 1,2,3)$ is the dimension, we find the scaling $\Psi^2 \sim L^{-\nu}$, whereas the dispersive term in Eq. (III.1) scales like L^{-2}. Thus for $\nu \geq 2$, dispersion cannot hold a nonlinear collapsing state. Of course, these rough arguments have to be verified by theoretical considerations[33,50]. The same results can be derived for the Zakharov equations (II.16,17), as long as the ions move with subsonic velocities. In the static limit, when time derivatives can be neglected in the ion equation, the collapse is mathematically proved in general. However, for the full Zakharov equations, variational estimates were obtained and self-similar solutions were found.

At the final stage of the collapse, electron nonlinearities could become essential and wave particle interaction has to be taken into account. Recently, in a very interesting particle simulation[51] the two-dimensional collapse of Langmuir waves was studied. Cavity compression for an initial distribution satisfying the sufficient criteria for collapse was really found. The cavity shrinks and after some limiting size is reached ($\approx 6\lambda_e$) Landau damping becomes important. Then the energy of the oscillations captured in the collapsing cavity is almost totally transmitted to electrons. The absorption is a fast process and the wave energy is mainly transferred to fast electrons, as can be seen from the tail in the velocity distribution function. When the same computations were performed for different ion masses, the collapse time turned out to practically independent on the mass ratio; this means that the collapse is subsonic. Thus Zakharov's idea of collapse plays an important role in plasma turbulence, since it presents a very effective mechanism of energy absorption for long wavelength ($k\lambda_e \ll 1$) via modulational instability and collapse.

b. Multi-dimensional Solitons

The preceding discussion for plane waves showed that most plasma solitons are transversely unstable. This initiated the search for stable solitons in three dimensions. One is first tempted to look for two- and three-dimensional solutions of the cubic nonlinear Schrödinger equation or the Zakharov equations. However, the collapse argument and some additional calculations[52] quickly show that these solutions are also unstable. (The 3d solution is even longitudinally unstable.) There exists a simple way out of this dilemma: if we improve the cubic Schrödinger model we find that the instability arguments do not apply anymore. The models discussed so far resulted from small amplitude theories. If we generalize them to finite amplitudes, we obtain the exponential nonlinear Schrödinger equation (for standing solitons) or a nonlinear Schrödinger equation coupled to the full hydrodynamic equations for the ions (in the general case). In the latter situation, the ℓf-part of the electron density is calculated by a Boltzmann distribution including the ponderomotive force and ambipolar potential. For reason of demonstration we choose here the first model; all the conclusions will also hold for the more complicated (and more realistic) second model.

The stationary spherically symmetric solutions of the exponential nonlinear Schrödinger equation

$$i\partial_t \vec{E} + \nabla\nabla\cdot\vec{E} - q\,\nabla\times(\nabla\times\vec{E}) + \left[1-\exp(-\vec{E}\cdot\vec{E}^*)\right]\vec{E} = 0 \qquad \text{(III.6)}$$

can be written in the form

$$\vec{E} = G(r)\exp(i\eta^2 t)\,\hat{r}\,. \qquad \text{(III.7)}$$

The localized functions G(r) are known[53] numerically; we now comment on their stability and give the final result of a corresponding calculation[54].

A necessary and sufficient condition for stability of a spherically symmetric (three-dimensional) envelope soliton (III.7) (with respect to longitudinal and transversal electrostatic as well as electromagnetic perturbations) is

$$\partial_{\eta^2} N > 0\,, \qquad \text{(III.8)}$$

where N is the plasmon number

$$N = \int_0^\infty dr\, r^2 G^2\,. \qquad \text{(III.9)}$$

For the proof of the sufficient part a Liapunov functional has been constructed; the necessary part follows from variational principles. One can show analytically that the derivative of N with respect to η^2 changes sign when η^2 is increased from small values (where $\partial_{\eta^2} N < 0$) to larger η^2 values. Numerical calculations clearly show that a critical value $\eta_c^2 \approx 0.10125$ exists such that for $\eta^2 > \eta_c^2$ solitons are stable. The existence of such a threshold can be interpreted in the same way as in particle physics: below η_c^2 a lower energy state than the soliton exists such that a decay of a soliton can occur. From the cubic nonlinear Schrödinger equation it is also clear that for $\eta^2 \to 0$ spherical solitons are unstable.

We should briefly mention why the collapse argument does not apply. The reason is that the invariant I_2 is not negative for a spherical soliton, which, according to Refs. 33 and 55 is necessary to find the collapse. Thus one should not expect that type of a collapse here.

The amplitudes ($\sim \eta$) for stable three-dimensional envelope solitons can be still small enough not to violate some of the basic assumptions underlying their physical model, such as, e. g., neglection of Landau damping. At the critical value η_c, the radius of the soliton is larger than ten electron Debye lengths; for much larger η^2-values, one has to take into account various dissipation mechanisms. There exist, however, some arguments[53] that for a spherically symmetric case the actual damping rate will be significantly reduced as compared with the one-dimensional case.

For ion-acoustic solitons, the situation is slightly different[41,52]: The two-dimensional solutions of the Kadomtsev-Petviashvili equation (II.5) are stable, whereas in three dimensions no stable soliton solutions of Eq. (II.5) exist. (The question of stable finite-amplitude solitons in three dimensions is still open.) The two-dimensional (cylindrical) soliton solutions of the Zakharov-Kuznetsov equation (II.6) are unstable (with respect to z-perturbations) whereas spherical solutions of Eq. (II.6) are stable.

Thus for both, ion acoustic and Langmuir oscillations, stable three-dimensional soliton solutions exist. It is also clearly expected that similar statements can be made for other types of solitary waves in plasmas.

IV. VORTEX SOLITONS

Finally we mention some soliton models in plasmas which are essentially two-dimensional and no one-dimensional counterparts exist. These are plasma vortices originating from low-frequency modes in magnetized plasmas. Three modes are typical: the zero frequency electrostatic convective cell[56] and the magnetostatic modes[57] in homogeneous plasmas as well as the finite frequency drift mode[53] in inhomogeneous plasmas. These modes are believed to be very important in explaining anomalous (particle and heat) transport in magnetized plasmas[56-59].

The convective cell mode is a zero-frequency, electrostatic mode involving only particle motion and perturbations perpendicular to the external magnetic field. When the damping rate is small, the electric field of this mode appears to be almost d. c. in time. The dispersion relation is

$$\omega = -i\mu_i k^2/(1+\Omega_i^2/\omega_{pi}^2)\,, \qquad \text{(IV.1)}$$

where the ion viscosity is related through $\mu_i = \eta_1^i/n_0 m_i$ to Braginskii's resistivity $\eta_1^i = 0.3\, n_0 T_i \nu_i/\Omega_i^2$. The ion gyrofrequency is Ω_i and ω_{pi} is the ion plasma frequency. The convective cell mode has a similar electric field polarization as the extra-ordinary mode. Although electrostatic, the convective cells are almost incompressible. In fact, one has

$$\left|\frac{\delta n_i}{n_0}\right| \approx \frac{\Omega_e^2}{\omega_{pe}^2}\left|\frac{\delta n_e}{n_0}\right| \approx \lambda_e^2 \nabla^2 \frac{e\phi}{T_e} \ll 1 . \quad \text{(IV.2)}$$

Calculating the damping of the convective cell one finds that the convective cell, if excited, can have a very long life time. The quasi-steady-state cells, expecially the large ones, can therefore cause macroscopic plasma convection and large scale transport because of their capability of mixing various regions of the plasma.

If a density gradient is present, one must add to the right hand side of (IV.2) the term

$$\frac{1+\Omega_i^2/\omega_{pi}^2}{\mu_i} \vec{v}_D \cdot \nabla \nabla^{-2} \frac{e\phi}{T_e} \quad , \quad \text{(IV.3)}$$

where $\vec{v}_D = (c\, T_e/e\, B\,)\hat{z} \times \nabla \ell n\, n_0$ is the diamagnetic drift velocity originating from the non-uniform spatial distribution of the gyro-orbits. The additional term (IV.3) can become important if $(1 + \Omega_i^2/\omega_{pi}^2)(\Omega_i/\nu_i) \gtrsim k^2\lambda_i^2 k_\perp L_n$, where $L_n = |\nabla \ell n\, n_0|^{-1}$ is the density scale-length. In this case, the density fluctuation can no longer be neglected, and one expects the particle motion along the magnetic field to become significant because of flux conservation.

The electron drift wave has exactly the same ion motion as the convective cells. However, since perturbations along the external magnetic field are no longer precluded, electrons can move along the field lines to maintain equilibrium. For $|\partial_x| \gg |\partial_z| > (m_e/m_i)^{1/2}|\partial_x|$ one obtains from the z-component of the electron momentum equation Boltzmann distributed electrons. The dispersion relation for the electron driftwave is

$$\omega = \frac{\vec{k}\cdot\vec{v}_D - i\, k_\perp^4 \mu_i \rho_s^2}{1 + k^2\lambda_e^2 + k_\perp^2 \rho_s^2} \quad , \quad \text{(IV.4)}$$

where $\rho_s = (T_e/m_i)^{1/2}/\Omega_i$ is the ion gyroradius defined in terms of the electron pressure. (When electron collisions are included in the z-component of the momentum equation, the socalled drift-dissipative instability can appear.) The ions in the driftwaves also execute convective cell motion, thus the expressions for $\vec{v}_{i\perp}$ are the same in both cases. The parallel motion of the electrons can short out any large-scale electric fields (associated with large vortices) in a bounded system. Thus, one does not expect that the convective motion induced by drift waves leads to large diffusion similar to that of the convective cells.

In contrast to the convective cells, the magnetostatic mode is purely electromagnetic. Furthermore, the particle motion is mainly along the external field. Although the magnetostatic mode was first explicitly identified by considering electron fluid motion, it is actually a magnetohydrodynamic mode derivable from MHD equations including finite conductivity and electron inertia. Its dispersion relation is

$$\omega = -\, i\, \frac{\frac{1}{2}\nu_e + \mu_\perp^e k_\perp^2}{1 + k_\perp^2 c^2/\omega_{pe}^2} \quad . \quad \text{(IV.5)}$$

Like the convective cell mode, the magnetostatic mode is purely damped. However, here the electric field vector is polarized like the ordinary mode. There is no linear perpendicular particle dynamics directly associated with the magnetostatic mode; this is because of $\vec{E}_\perp = 0$. However, any particle is now free to move perpendicular to B_0 along the perturbed magnetic field. Thus, the perpendicular motion of the electrons participating in the magnetostatic mode is of second order. Hence, the diffusion due to the magnetostatic mode is usually unimportant. However, since it is the high energy electrons which contribute to the magnetostatic mode diffusion, one expects enhanced heat loss in the presence of this mode.

Having some real applications in mind, we here concentrate on the nonlinear ℓf quasi-twodimensional electrostatic description in an inhomogeneous plasma with weak magnetic shear[58]. In that case, perturbations along the external magnetic field are not precluded and electrons can move along the magnetic field lines to maintain equilibrium. (Therefore the description is called quasi-twodimensional - in a plane perpendicular to the external magnetic field.)

Let us assume that the shear is not very strong so that the ion motion can be considered as two-dimensional (in the plane perpendicular to the magnetic field). Furthermore, assuming $T_e >> T_i$, the ions can be treated within the cold fluid description. Then the ion momentum and density equations combine,

$$\frac{d}{dt}\left(\frac{\nabla^2\phi}{B_0\Omega_i} - \frac{n}{n_0} - \ln n_0\right) = \mu_i \frac{\nabla^4\phi}{B_0^2\Omega_i^2}, \qquad \text{(IV.6)}$$

where the convective derivative is to lowest order

$$\frac{d}{dt} \approx \frac{\partial}{\partial t} - \frac{\nabla\phi\times\hat{z}}{B_0}\cdot\nabla \quad . \qquad \text{(IV.7)}$$

If one takes the limit of small resistivity with a finite magnetic shear, the z-component of the electron momentum equation yields

$$\frac{n}{n_0} \approx \frac{e\phi}{T_e} \quad . \qquad \text{(IV.8)}$$

Substituting the Boltzmann distribution into Eq. (IV.6) one obtains a nonlinear equation for ϕ, the socalled Hasegawa-Mima equation[58]

$$\partial_t(\nabla^2\phi - \phi) - [(\nabla\phi\times\hat{z})\cdot\nabla](\nabla^2\phi - \ln n_0) = \bar{\mu}\nabla^4\phi, \qquad \text{(IV.9)}$$

where the electrostatic potential ϕ is normalized by T_e/e, the length unit is the ion Larmor radius $\rho_s \equiv c_s/\Omega_i$, the time is measured in Ω_i^{-1}, and $\bar{\mu} \equiv \mu_i/\rho_s^2\Omega_i$ is a normalized viscosity[61]. The Hasegawa-Mima equation is analogous to the two-dimensional incompressible, inviscid, neutral, and homogeneous fluid dynamics when the adiabatic shielding of the electrons and the drift velocity (due to inhomogeneity) are neglected.

For $\bar{\mu} \approx 0$, the Hasegawa-Mima equation conserves the energy

$$E = \int d^2r\, [(\nabla\phi)^2 + \phi^2] \qquad \text{(IV.10)}$$

and the (potential) enstrophy.

$$\mathcal{U} = \int d^2r\, [(\nabla\phi)^2 + (\nabla^2\phi)^2] \, . \qquad \text{(IV.11)}$$

When a large number of modes are present the dynamics is strongly mixing in the ϕ_k phase space. We are not interested in these aspects here, but look for a steady solution $\phi = \phi(x-ut,y)$ in a frame moving with a constant velocity u in the x-direction. Such a solution has been found[62] for the geophysical version of Eq. (IV.9) and is often called modon. One divides the $\xi(\equiv x-ut)$, y plane into two parts by a circle of radius a with its center at the point $\xi = y = 0$. Assuming that ϕ satisfies in each region

$$\nabla^2\phi = f\phi + gy, \qquad \text{(IV.12)}$$

where f and g are step functions, one finds from Eq. (IV.9) the explicit solution for ϕ. It is required that across the circle $r \equiv (\xi^2+y^2)^{1/2} = a$, the quantities ϕ and $\nabla\phi$ are continuous and that at infinity ϕ vanishes.

Numerical solutions[63] of the dynamical evolution of this solution show that drift wave vortices behave like solitons in two-dimensional interaction. In fact, after both, head-on and overtaking collisions, these vortices recover asymptotically their initial shapes at the end of the interaction. An analytical proof of this behavior is still missing; however preliminary results[64] of linear stability calculations (for small perturbations) seem to confirm the stable behavior.

V. CONCLUDING REMARKS

In this short review on recent developments in soliton theory with application to plasmas not all the important aspects of soliton theory could be mentioned. Also the reference list is by no means complete. To demonstrate this lack of completeness we just mention two examples: soliton formation because of relativistic mass variation[65] and perturbed soliton systems.[66] The latter area is extremely important since in reality various effects (e. g., damping, inhomogeneity, boundary conditions, etc.) will perturb ideal model systems. Then one is confronted with the following questions: Is the soliton concept still important? Do solitons behave like "particles" even when external forces, radiation due to acceleration, etc. are taken into account? Partially, some answers already exist but much has still to be done in the future.

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