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Solitons and Vortices in Plasmas

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Abstract

The various types of nonlinear wave solutions in plasmas are reviewed. First, the generation, propagation, and stability of solitary waves are demonstrated for some simple examples. Then, relevant two- and three-dimensional models for Langmuir solitons are proposed and investigated. The collapse as an effective dissipation mechanism in plasmas is discussed in detail. Finally, simple models of two-dimensional vortex motion and their consequences are considered.

1. Introduction

1.1. Nonlinear Physics

The nonlinear nature of physics — with perhaps basic quantum mechanics as the only exception [1] — is very challenging for most of the physicists. Nonlinear phenomena are very fundamental in many areas of natural sciences. Today, big computers make it possible to attack problems which previously were considered as untractable. The use of computers and the numerical results stimulated many new analytical ideas in nonlinear physics and very effective mathematical tools have been developed meanwhile. For solitons, we shall present many impressible examples for that development.

Historically, several milestones can be identified: First, the detection of a solitary water wave by SCOTT-RUSSELL [2] in 1844. This was the first reported observation of a quite stable localized nonlinear wave. Scott-Russell started his report on solitary waves in shallow water with the words: "This is a most beautiful and extraordinary phenomenon: the first day I saw it was the happiest of my life." Lateron, he continues: "It is now known as the solitary wave of translation." That was — looking back — the day of birth of a solitary wave. It stimulated for many decades the research on nonlinear wave propagation. By a solitary wave we now mean a nonlinear localized wave; to day, the word soliton is often used — without further restriction — when solitary waves are meant.

After the year 1844 many researchers tried to work out the theoretical basis for the nonlinear wave phenomena. Nonlinear wave models were proposed but the mathematical understanding of the wave dynamics remained difficult. The numerical experiments of FERMI, PASTA, and ULAM [3] in the year 1965 induced a change. The authors originally wanted to prove thermalization because of nonlinear coupling in a mass-spring-system. In a one-dimensional system they first excited a long-wavelength mode. The numerical results, however, showed no tendency towards thermalization. Although lower modes have been excited, the system returned to its (nearly) original state of excitation after a finite time.

Related to this work are the results of the numerical as well as analytical investigations of Zabusky and Kruskal [4] as well as Gardner, Greene, Kruskal and Miura [5]. Zabusky and Kruskal introduced the notation soliton for nonlinearly stable localized waves. They solved the initial value problem of the Korteweg-deVries equation numerically and found that solitons are created out of initial destributions of quite arbitrary shape. The break-through reported in the work of Gardner et al. [5] was the analytic solution of the initial value problem of the Korteweg-deVries equation. This idea was lateron generalized and initiated a huge amount of work in the soliton area. Before discussing these findings, and their conclusions for plasmas, in greater details, we have to mention another aspect of nonlinear physics.

Nonlinear (ordinary) differential equations show another fundamental phenomenon, which is investigated in these days all over the world. The phenomenon is described as deterministic chaos. We should mention Grossmann and Thomae [6] or Feigenbaum [7] who invented a quite general route to chaos besides many others. In this context it is important to note that deterministic nonlinear equations can have chaotic solutions.

Deterministic chaos and *solitons* can be considered today as the two structural elements of nonlinear physics [8]. Although the words "deterministic" and "chaotic" seem to contradict each other we have learned during the past years that solutions of deterministic equations can become unpredictable for certain ranges of the control parameter. The best-known model in this respect is the Lorenz model. It consists of a set of three coupled nonlinear ordinary differential equations. Another well-known example is the Duffing equation. Characteristic for the driven and damped nonlinear oscillatory systems is the period doubling at certain fixed values of the control parameter. Because of nonlinear interactions, frequency doubling (higher harmonics) might be more expected, but in damped systems the appearence of subharmonics with 1/2, 1/4, etc. of the driving frequency is the more surprising but characteristic behavior [9, 10].

One could think that chaos and solitons have nothing to do with each other since chaos is mostly discussed for ordinary differential equations (and discrete mappings) whereas solitons appear as solutions of nonlinear partial differential equations. However, this is not true. We know already from instabilities (e.g., the Benard instability) that ordinary differential equations and mappings are considered as reduced models for the more complex phenomena in partial differential equations. In addition, recent numerical results of Moon et al. [11] show that driven and damped nonlinear Schrödinger systems posses both structural elements: solitons and chaos. Depending on a control parameter (driving force) such a Schrödinger system is either chaotic (with bursts) or non-chaotic with soliton solutions. Solitons are generated from an unstable situation and they are the counterparts of laminar solutions (limit cycles) in ordinary differential equations. Details will be given below. Let us now discuss the principal results for a typical example. A plane wave solution can become modulationally unstable with respect to long-wavelength perturbations. The modulational instability causes envelope bunching and the generation of localized waves finally results in solitons. In the meanwhile, there are many partial differential equations known which have stable, coherent, pulse-like, localized solutions, whose nonlinear interaction is particle-like. The particle-like behavior is very surprising since nonlinear differential equations do not obey a superposition principle.

Another fact is very important: solitary waves, if they exist for a certain system, are being formed out of any initial distribution. Let us consider as another example a plasma with a sound-like initial pulse. The latter decays in the course of time into completely predictable ion-acoustic solitary waves. Solitary waves can therefore be considered as the elementary excitations of a nonlinear system.

The appearence of solitons is a universal feature in practically all areas of physics and even in situations which are sometimes not thought of to be physical. The universal nature of soliton physics becomes also transparent when one finds that physically quite different and unconnected phenomena are described by the same or similar nonlinear equations, e.g., a Korteweg-de Vries (KdV), or a cubic nonlinear Schrödinger (NLS), or a sine-Gordon (SG) equation.

1.2. Some General Properties of Solitons

Before considering some special properties of solitons in plasmas, we shall summarize some of the general aspects. In order to be able to pinpoint also quantitative pecularities we choose the celibrated KdV equation as the example for demonstration. The KdV equation describes the propagation in one-dimensional, weakly dispersive systems, i.e., for surface waves in shallow water, long waves in anharmonic lattices, magnetohydrodynamic waves, and ion-acoustic waves in plasmas; it is written in the form

$$u_t + auu_x + u_{xxx} = 0. \tag{1}$$

This nonlinear partial differential equation determines the evolution in time t and (one-dimensional) space x of a wave with normalized amplitude u. It contains a nonlinear term auu_x and a dispersive term u_{xxx}. In this case, only the combination of the latter two allows the specific soliton features: the dispersive term alone would cause broadening whereas the nonlinear term alone would always result in steepening and wave breaking.

The one-soliton solution of (1) is given by

$$u = (3v/a) \operatorname{sech}^2 [0.5v^{1/2}(x - vt) + b], \tag{2}$$

where $v > 0$ and b are arbitrary parameters. It is evident that amplitude, width, and velocity are related. The bigger the amplitude, the faster the soliton and the smaller its width. The significance of the solution (2) follows from the universal property that every (one-dimensional) initial distribution will evolve in time into (at least) one soliton (plus radiation). This universal behavior is analogous in importance to, e.g., the period-doubling route to deterministic chaos. Because of the property that solitons self-organize these nonlinear waves are fundamental in physics. In addition, we have already implicitly assumed that solitary waves are stable entities in the sense that they are stable even against strong nonlinear interactions. This can be shown for the interaction of two solitons through overtaking collisions as well as for head-on collisions. Due to the lack of a superposition in the nonlinear regime, the reappearence of the two unchanged forms of the solitons is only asymptotically true ($x, t \to \pm\infty$). But practically this is not a severe restriction because of the exponentially decaying slopes of each soliton.

The stability property just mentioned is very essential. And indeed, the rigorous definition of a *soliton* demands the complete stability of a pulse-like localized wave even as a result of nonlinear interaction. This has to be seen in contrast to the less stringent definition of a *solitary wave*. The latter should also be localized (in its physically relevant quantities like energy, momentum density etc.) but stability during collisions is not required. Sometimes it is simply not known; thus a solitary wave may be a soliton. A soliton, indeed, is always a solitary wave. However, in these days many people do not insist on the difference in notation. One reason is that sometimes we know from computer solutions that a localized wave has soliton character (during interaction) but a rigorous mathematical proof is still missing. Another reason may be that the word soliton is much more attractive than the more specialized term solitary wave originally introduced by Scott Russell [2]. In the following we do not insist in an unambiguous terminology.

The last discussion leads us to the question: What is known *analytically* about solitons? And indeed, the soliton-bearing equations also analytically show a quite universal behavior. The discovery of that general aspect originated back to ABLOWITZ et al. [12] as well as GARDNER et al. [5]. It is known as the *inverse scattering transform* (IST) which strongly resembles the (linear) Fourier transformation for solving linear partial differential equations. When using the (linear) Fourier transformation method we solve the initial value problem by integration. First (i), the spectrum $u(k, 0)$ of the initial wave amplitude $u(x, 0)$ is calculated. Then (ii), by means of the (linear) dispersion relation, the spectrum $u(k, t)$ at time t is derived. Finally (iii), by inverse Fourier transformation the wave amplitude $u(x, t)$ at time t is obtained. This is the well-known procedure to handle linear partial differential equations. ABLOWITZ et al. [12] showed that the inverse scattering transform is a (generalized) Fourier analysis for nonlinear problems. However, this procedure is mathematically much more complicated than the (linear) Fourier transform method. In the case of the KdV equation it is still relatively simple: Starting with one initial wave profile $u(x, 0)$ we have first (i) to solve a Schrödinger scattering problem with a potential identical to the initial profile. The scattering data (reflection and transmission coefficients as well as the discrete eigenvalues) can be calculated by standard procedures. In the next step (ii), the time development of the scattering data can be obtained from ordinary differential equations. Finally (iii), scattering data at time t allow to reconstruct the potential at time t, which is the solution $u(x, t)$. All three steps are in general highly non-trivial but do not involve a new mathematical technique.

It is interesting to note that every discrete eigenvalue of the Schrödinger problem corresponds to a soliton and the discrete spectrum is time-independent. Furthermore, a (well-behaved) one-dimensional Schrödinger potential problem always has a discrete eigenvalue; this proves the appearence of solitons out of any initially localized wave profile. The IST is a very fascinating method to solve nonlinear partial differential equations. It has already been generalized considerably. A very difficult task is to solve higher dimensional problems [13] for finite systems; here much is left to do in the future.

2. Specific Properties of Solitons in Unmagnetized Plasma

In an unmagnetized plasma, basically two electrostatic normal modes occur: the ion-acoustic and the Langmuir oscillations. The *nonlinear ion acoustic wave* obeys, in the one-dimensional, small-amplitude limit, a KdV equation. Obviously, then all the conclusions discussed in connection with the KdV equation should be true for ion-acoustic waves. And indeed, in a series of experiments, IKEZI et al. [14] were able to verify experimentally the predicted amplitude-width-relation and the form stability during collisions. In addition, a finite amplitude theory shows that the existence region of ion-acoustic solitons can be discussed in the same way as the motion of a classical particle in a potential well (Sagdeev potential).

All this supports that the ion-acoustic soliton is one of the best-understood classical solitons. But the soliton concept is by no means complete. When applying the soliton theory to the real world we immediately face a problem. One has to take into account the second and third space dimension, and little is known for these physically relevant but mathematically complicated cases. From the KADOMTSEV-PETVIASHVILI equation [15] one can conclude that, at least, one-dimensional stability is preserved when two-dimensional distortions are allowed for. Experimentally, the existence of stable multi-dimensional ion-acoustic solitons seems to be guaranteed; their complete theory is still missing. Only in the case of the ZAKHAROV-KUZNETSOV equation [16] a stability proof exists.

A completely different picture is true for *Langmuir waves*; this basically new scenario will now be explained in more detail. We shall see that qualitatively new aspects of dissipation and turbulence will occur. In these days it seems to be common sense [17] that Langmuir turbulence does not consist of an ensemble of weakly interacting plasmons. Langmuir turbulence is not weak, but also completely different to turbulence of incompressible fluids [18]. The well-known model of weak turbulence leads to a Langmuir condensate, i.e., the energy is concentrated in the small wavenumber regime. Then, a modulational instability can take place and regions of lowered plasma density, so called cavitons, are created. Cavitons can trap the high-frequency Langmuir oscillations and the local energy density can become very high.

So far the scenario fits very well into the soliton concept. Via the modulational instability one-dimensional solitons are formed. The latter consist of density cavities (cavitons) with trapped radiation. The modulational instability is a quite universal mechanism for soliton generation: if initially the threshold for modulational instability is not yet reached the weak turbulence mechanisms will lead to a condensate and finally modulational instability always can occur. It is important to note that a Langmuir soliton is not just a localized nonlinear Langmuir oscillation; it is always accompanied by a density variation.

The Langmuir solitons (cavitons) are the structural elements of Langmuir turbulence. For their dynamical behavior two basically different scenarios exist. In the first one, proposed and developed by Zakharov, all cavitons collapse very rapidly. During the collapse (phase one) they do not loose energy. The collapse is a multi-dimensional nonlinear instability of Langmuir solitons. At the final stage of the collapse, when the diameter of the caviton is of the order of a few Debye lengths, the energy of the Langmuir oscillations is transfered to the particles and the density depression with the trapped Langmuir oscillations burns out. This results in a very effective dissipation mechanism at small wave numbers k. It has to be seen in comparison to Landau damping which only occurs at large k ($k\lambda_D \gtrsim 0.5$, where λ_D is the electron Debye length).

The other model proposed for strong Langmuir turbulence assumes long-living quasistationary cavitons to exist. Then a statistical theory of turbulence, based on cavitons, can be developed. Thus, the stability of Langmuir solitons (cavitons) becomes the central topic of a turbulence theory in plasmas. The two scenarios mentioned above will obviously lead to different conclusions regarding the macroscopic properties of Langmuir turbulence.

Langmuir solitons can be described by a set of two coupled equations which have been first derived by ZAKHAROV [19]:

$$\nabla^2 \left[i\partial_t \Psi + \frac{3}{2}\, \omega_p \lambda_D^2 \nabla^2 \Psi \right] = \frac{\omega_p}{2n_0}\, \nabla \cdot [\delta n \nabla \Psi], \tag{3}$$

$$\partial_t^2 \delta n - c_s^2 \nabla^2 \delta n = \frac{1}{16\pi m_i}\, \nabla^2 |\nabla \Psi|^2. \tag{4}$$

In these equations, ω_p is the electron plasma frequency, n_0 is the (constant) equilibrium density, δn the (quasi-neutral) change of the particle density because of the radiation pressure, and Ψ is the envelope of the electrostatic potential. The electric field is then written in the form

$$\boldsymbol{E} = \frac{1}{2}\, [\nabla \Psi \exp(-i\omega_p t) + \text{c.c.}]. \tag{5}$$

The ion-acoustic velocity is $c_s = (T_e/m_i)^{1/2}$, where T_e is the electron temperature and m_i is the ion mass. The two equations (3) and (4) represent a good model for small field amplitudes ($|\nabla \Psi|^2/nT_e \ll 1$) and negligible damping ($k\lambda_D \ll 1$). The frequency of a

Langmuir wave can be written as

$$\omega_k \approx \omega_p \left[1 + \frac{3}{2} k^2 \lambda_D^2 + \frac{\delta n}{2n_0}\right], \tag{6}$$

where the first two terms result from the linear dispersion relation. The last term originates from the coupling of the density with the high-frequency oscillations. The frequency of the Langmuir wave is, via the plasma frequency, density-dependent; the density has also the (small) contribution δn which represents the reaction due to the radiation pressure (Miller force; ponderomotion force) of the high-frequency oscillations. The most general formula for the radiation pressure in plasmas was presented by KARPMAN [20]. In the present case of an unmagnetized plasma with $\omega_k \approx \omega_p$ one obtains

$$\boldsymbol{F} = -\nabla \frac{|\nabla \Psi|^2}{16\pi}. \tag{7}$$

A better understanding of the averaged equations (3) and (4) can be gained in the following way. From Maxwell's equations for the electric field $\boldsymbol{E}$ we obtain

$$\frac{1}{c^2} \partial_t^2 \boldsymbol{E} + \nabla \times \nabla \times \boldsymbol{E} + \frac{4\pi e}{c^2} [n_0 + \delta n_e] \partial_t \boldsymbol{v}_e = 0. \tag{8}$$

Here, the electron density n is the sum of three terms: an averaged density n_0, a low-frequency disturbance δn_e, and a high-frequency contribution δn_h. Higher harmonics $(2\omega_p, \ldots)$ are neglected. The high-frequency electron velocity $\boldsymbol{v}_e$ follows from the linearized momentum balance,

$$\partial_t \boldsymbol{v}_e + 3v_{te}^2 \nabla \frac{\delta n_h}{n_0} = -\frac{e}{m_e} \boldsymbol{E}, \tag{9}$$

where v_{te} is the electron thermal velocity. Using Eq. (9) in Eq. (8), one gets, within the WKB approximation with $\boldsymbol{E} \approx 1/2\boldsymbol{E} \exp(-i\omega_p t) + \text{c.c.}$,

$$i\varepsilon\, \partial_t \boldsymbol{E} + \nabla^2 \boldsymbol{E} - Q \nabla \times \nabla \times \boldsymbol{E} - \delta n_e \boldsymbol{E} = 0. \tag{10}$$

This equation has been written in dimensionless form [times are measured $\sqrt{3}/\omega_{pi}$, lengths in $\sqrt{3}\,\lambda_D$, potential in T_e/e, density in n_0, and the electric field is measured in $(16\pi n_0 T_e)^{1/2}$, whereas the velocities are in units of $c_s = (T_e/m_i)^{1/2}$]. The parameter ε is $2(m_e/3m_i)^{1/2}$ and the parameter Q is equal to $(m_e c^2/3T_e)^{-1} \gg 1$. To determine the low-frequency density response δn_e we use a Boltzmann distribution of the electrons. The Boltzmann distribution has to be modified by the ponderomotive potential. Thus we have

$$\nabla |\boldsymbol{E}|^2 = \nabla \Phi - \nabla \ln (1 + \delta n_e). \tag{11}$$

The ambipolar potential Φ is coupled via the Poisson equation

$$\frac{1}{3} \nabla^2 \Phi = \delta n_e - \delta n_i \tag{12}$$

to the changes in the ion density. The latter follows from the particle and momentum balance of the ions, i.e.,

$$\partial_t \delta n_i + \nabla \cdot (n_i \boldsymbol{v}_i) = 0, \tag{13}$$

$$\partial_t \boldsymbol{v}_i + \boldsymbol{v}_i \cdot \nabla \boldsymbol{v}_i = -\nabla \Phi. \tag{14}$$

Using the scaling $\varepsilon t \to t$, $\varepsilon \nabla \to \nabla$, $\delta n_e \approx \delta n_i \approx \delta n$, Φ, $v_i \sim \varepsilon^2$, $E \sim \varepsilon$, we obtain the Zakharov equations

$$i\partial_t \boldsymbol{E} + \nabla^2 \boldsymbol{E} - Q\nabla \times \nabla \times \boldsymbol{E} - \delta n \boldsymbol{E} = 0, \tag{15}$$

$$\partial_t^2 \delta n - \nabla^2 \delta n = \nabla^2 |\boldsymbol{E}|^2. \tag{16}$$

Since in non-relativistic plasmas Q is a large parameter, it is reasonable to rewrite Eq. (15) in the form

$$\nabla \times \nabla \times \boldsymbol{E} = \frac{1}{Q} (i\partial_t \boldsymbol{E} + \nabla^2 \boldsymbol{E} - \delta n \boldsymbol{E}). \tag{17}$$

Expanding $\boldsymbol{E}$ in terms of powers of Q^{-1},

$$\boldsymbol{E} = \nabla \Psi + \frac{1}{Q} \boldsymbol{E}_1 + \cdots, \tag{18}$$

we obtain to lowest order

$$\nabla \times \nabla \times \boldsymbol{E}_1 = i\partial_t \nabla \Psi + \nabla^2 \nabla \Psi - \delta n \nabla \Psi. \tag{19}$$

The solvability condition for $\boldsymbol{E}_1$ demands the divergence of the right-hand side of Eq. (19) to vanish. Then we have in dimensionless form

$$\nabla \cdot [i\, \partial_t \nabla \Psi + \nabla^2 \nabla \Psi - \delta n \nabla \Psi] = 0, \tag{20}$$

together with

$$\partial_t^2 \delta n - \nabla^2 \delta n = \nabla^2 |\nabla \Psi|^2. \tag{21}$$

The system of equations (20) and (21) is, when written in dimensional form, identical to the equations (3) and (4).

All these considerations lead to a better understanding of the Zakharov equations (3) and (4), including the knowledge of the limitations of validity. Let us draw some more conclusions using the Zakharov model.

Within the quasi-stationary approximation

$$\partial_t^2 \delta n \ll c_s^2 \nabla^2 \delta n \tag{22}$$

we obtain a balance between kinetic and ponderomotive pressure,

$$\delta n \approx -\frac{|\nabla \Psi|^2}{16\pi T_e}. \tag{23}$$

Within this approximation, the Zakharov equations (3) and (4) simplify to one equation of the Schrödinger type,

$$\nabla^2 \left[i\partial_t \Psi + \frac{3}{2} \omega_p \lambda_D^2 \nabla^2 \Psi \right] = -\frac{\omega_p}{32\pi n_0 T_e} \nabla \cdot [|\nabla \Psi|^2 \nabla \Psi]. \tag{24}$$

This equation is simple enough to demonstrate the modulational instability. We rewrite Eq. (24) in the form

$$\nabla \cdot [i\partial_t \nabla \Psi + p \nabla^2 \nabla \Psi + q\, |\nabla \Psi|^2\, \nabla \Psi] = 0, \tag{25}$$

where $p = (3/2)\, \omega_p \lambda_D^2$ and $q = \omega_p / 32\pi n_0 T_e$.

First, it is obvious, that Eq. (25) has the exact solution

$$\nabla\Psi_0 = A \exp(-i\nu t)\,\hat{x} \tag{26}$$

with

$$\nu = -qA^2. \tag{27}$$

We clearly see that ν is a nonlinear frequency shift. The solution (26) describes a Langmuir wave of constant amplitude in space. If we disturb this wave, introducing the notation

$$\delta\nabla\Psi = [X(x, t) + iY(x, t)] \exp(-i\nu t)\,\hat{x}, \tag{28}$$

we get two coupled equations for X and Y. In linearized form, with $X \sim \cos(Kx - \Omega t)$ and $Y \sim \sin(Kx - \Omega t)$ we obtain the dispersion relation

$$\Omega^2 = pK^2(pK^2 - 2qA^2). \tag{29}$$

Instability ($\Omega^2 < 0$) follows for certain values $pq > 0$. If we recall the definitions for p and q, we see that $pq > 0$ is true in our case. Then Eq. (29) yields

$$0 < K < (2q/p)^{1/2} A \tag{30}$$

for instability. Inserting the coefficients, and going back to the dimensional case, we get the condition for modulational instability,

$$\frac{|\nabla\Psi_0|^2}{24\pi n_0 T_e} > K^2\lambda_D^2. \tag{31}$$

The maximum growth rate is

$$\Gamma_{\max} = \frac{\omega_p}{32\pi n_0 T_e} |\nabla\Psi_0|^2 \tag{32}$$

for the wavenumber

$$K_{\max} = \frac{|\nabla\Psi_0|}{(48\pi n_0 T_e)^{1/2}} \lambda_D^{-1}. \tag{33}$$

Several conclusions can be drawn from this consideration: Langmuir oscillations of constant amplitude are modulationally unstable. The constant amplitude breaks into long-wavelengths (small $K\lambda_D$) packets. These results were obtained within a quasi-stationary approximation, which is valid under the condition (22). If we estimate both sides of the inequality (22) by introducing the maximum growth rate (32) and the K-value (33), then we obtain

$$\frac{|\nabla\Psi_0|^2}{32\pi n_0 T_e} \ll \frac{m_e}{m_i}. \tag{34}$$

Inequality (34) means that the quasi-stationary approximation is appropriate to small energy densities. From the condition (30) for modulational instability we have the consistency relation

$$K^2\lambda_D^2 \ll \frac{m_e}{m_i}. \tag{35}$$

Obviously, the regions of validity (34) and (35) can (and should also) be obtained from direct dimensional arguments. For the cubic nonlinear Schrödinger equation we can introduce a characteristic time τ with $\tau^{-1} \sim \omega_p K^2 \lambda_D^2 \sim \omega_p |\nabla \Psi_0|^2/16\pi n_0 T_e$. If we insert this into the inequality (22), we immediately obtain (34) and (35). Analytically, most is known about the behaviour of nonlinear Langmuir waves in the quasi-static approximation. If we consider the equivalent vectorial form of Eq. (25), i.e.

$$i\partial_t \boldsymbol{E} + p\nabla^2 \boldsymbol{E} + Q\nabla \times \nabla \times \boldsymbol{E} + q\,|E|^2\,\boldsymbol{E} = 0, \tag{36}$$

we get the more convenient type of a nonlinear Schrödinger equation. Of course, its divergence agress with Eq. (25). For one-dimensional electric fields E, this equation is the standard cubic nonlinear Schrödinger equation. Without loss of generality we can take $p = q = 1$. The spatially one-dimensional cubic nonlinear Schrödinger equation can be solved by the inverse scattering transform method as it has been shown by ZAKHAROV and SHABAT [21]. The one-soliton solutions of

$$i\partial_t E + \nabla^2 E + |E|^2\, E = 0 \tag{37}$$

are

$$E = \sqrt{2}\,\eta \operatorname{sech}(\eta x)\, e^{i\eta^2 t} \equiv G(x;\eta)\, e^{i\eta^2 t}. \tag{38}$$

They are one-dimensionally stable. For later generalizations it is meaningful to put the stability criterion into a form which can be used for other situations, too. Accordingly we write [22]

$$\frac{\partial}{\partial \eta^2} \int dx G^2 > 0. \tag{39}$$

One recognizes immediately that the solution (38) fulfills the criterion (39). Physically, the criterion (39) means that in the stable case the plasmon number should increase with the nonlinear frequency shift. However, a transverse (2d) instability occurs. For the latter, a variational principle could be derived [23]. One gets for the growth the rate γ of the transverse instability,

$$\gamma^2 = \sup_{\langle a|G\rangle = 0} \frac{-\langle a\,|H_-|\,a\rangle}{\langle a\,|H_+^{-1}|\,a\rangle}. \tag{40}$$

Here, H_+ and H_- are Schrödinger operators defined through

$$H_+ = -\frac{d^2}{dx^2} + k^2 - G^2 + \eta^2, \tag{41}$$

$$H_- = H_+ - 2G^2, \tag{42}$$

where k is the wavenumber of the transverse perturbation. Evaluations of the principle (40) lead to [24] $\gamma \approx 2k\eta$, for small k. A cutoff $k_c = \sqrt{3}\,\eta$ occurs; the system is stable for $k > k_c$. A typical (maximum) growth rate is of the order ω_{pe}. However, there are some physical objections against this mathematically correct result [25]. The ions cannot follow such a fast modulation because of their huge inertia. And indeed, better models and appropriate evaluations, e.g., based on the Zakharov equations, showed that (i) although the unstable k region is not significantly changed, (ii) the magnitude of the maximum growth rate of the transverse instability is drastically reduced [26]. (A similar argument holds, of course, for the modulational instability. However, there we assume small K from the beginning and therefore we do not contradict the quasi-static approximation.)

Summarizing the results obtained so far we can state the following: One-dimensional solitons are stable under the influence of one-dimensional perturbations. Within the Schrödinger equation model, the inverse scattering method of Zakharov and Shabat proves the creation of solitons out of any well-behaved initial distribution. For more complex models, numerical solutions show that parametric instabilities can result in solitons. One-dimensional solitons are unstable for transverse perturbations. The characteristic length $l_\perp$ for the break-up into transverse filaments is of the order of magnitude

$$l_\perp \approx \lambda_D(16\pi n_0 T_e/|E|^2)^{1/2}. \tag{43}$$

The multi-dimensional formation of localized structures initiates the question whether the final states of a modulational instability are stable multi-dimensional solitons. This problem has been intensively investigated during the past years. A fundamental idea originates from ZAKHAROV [17]. He has presented a qualitative as well as quantitative picture of an instability in higher space dimensions. Within the quasi-static approximation (i.e., the cubic nonlinear Schrödinger equation) one has the constant of motion

$$I_1 = \int |E|^2 \, d^i r, \tag{44}$$

where $i(= 1, 2, 3)$ is the dimension of space. Thus, Eq. (44) leads to the scaling

$$|E|^2 \sim L^{-i}, \tag{45}$$

with a characteristic length L. On the other hand, for the cubic nonlinear Schrödinger equation the dispersive term scales like L^{-2} and therefore for $i \geqq 2$ the dispersion cannot hold a collapsing state. Of course, this qualitative argument was analytically proved. Quite general is the following argument based on a virial theorem [17, 27]. In the simplest case of the scalar cubic nonlinear Schrödinger equation

$$i\partial_t E + \nabla^2 E + |E|^2 E = 0, \tag{46}$$

we have the constant of motion.

$$I_2 = \int d^i r \left(|\nabla E|^2 - \frac{1}{2} |E|^4 \right). \tag{47}$$

The result

$$\int d^i r \, |\boldsymbol{r}|^2 \, |E|^2 \to 0 \tag{48}$$

follows for $I_2 < 0$, e.g., in three space dimensions, from

$$\int\limits_0^\infty r^4 |E|^2 \, dr < 3I_2 t^2 + C_1 t + C_2, \tag{49}$$

where C_1 and C_2 are constants.

In addition, self-similar solutions exist which demonstrate the collapse explicitly. They also indicate that the singularity might occur earlier then estimated from the virial theorem. In the spherically symmetric three-dimensional situation, the self-similar solution for E has the form [28]

$$E(r, t) \simeq \frac{1}{\xi} S\left(\frac{r}{\xi}\right), \tag{50}$$

where $\xi := (t_c - t)^{1/2}$. The function S is the solution of an ordinary nonlinear differential equation. In the two-dimensional case, the self-similar solution has a logarithmic dependence [29]

$$E(\boldsymbol{r}, t) = \frac{1}{\xi} R\left(\frac{r}{\xi}\right) e^{-i\theta(r,t)}, \tag{51}$$

where

$$\xi := [[t_c - t]/[-a \log (t_c - t)]]^{1/2}$$

and

$$\theta := r^2/4(t_c - t) - \frac{a}{4} [\log (t_c - t)]^2.$$

The constant a is positive and is determined through the initial values.

There exists no exact proof of the collapse for the Zakharov equations, but numerical solutions constitute a firm basis for the collapse. In addition, we have time-asymptotic self-similar solutions (if the supersonic assumption is adequate; see below).

When we use a rough estimate for the sign of I_2 we find that $I_2 < 0$ in the quasi-static approximation provided

$$\frac{|E|^2}{16\pi n_0 T} > K^2 \lambda_D^2. \tag{52}$$

This agrees quite well with the condition for the modulational instability. One can thus expect that the structures created by a modulational instability can directly collapse.

The sufficient criterion for a Langmuir wave collapse does not exclude at all the possible existence of non-collapsing, stable, multi-dimensional Langmuir solitons. Let us elaborate a little bit more on this point. For the cubic nonlinear Schrödinger equation (or its weakly nonlinear generalizations), the invariant I_2 vanishes for two-dimensional (2d) solitons in two space dimensions. Nevertheless, the collapse argument can still be used in this case to demonstrate instability. The reason is that in the neighbourhood of every 2d Langmuir soliton we can find infinitesimally close states for which the invariant I_2 has the correct sign for a collapse. The situation is completely different for three space dimensions (3d). Here, the collapse argument cannot be used at all for 3d Langmuir solitons since the invariants for existing 3d solitons are positive. A short calculation yields

$$I_2 = \eta^2 \frac{i-2}{4-i} I_1, \tag{53}$$

where i is the space dimension and $I_1 > 0$. Nevertheless, we were able to show by variational principles that 3d Langmuir solitons are unstable in the weakly nonlinear regime [29]. When applying the direct Liapunov method we find that the Hamiltonian is not bounded from below in three space dimensions. Estimates with the help of the Hölder inequality yield

$$\int E^4 \, dx \leqq 2 I_1{}^{3/2} Z_{x,1}^{1/2}, \tag{54}$$

$$\int E^4 \, dx \, dy \leqq \frac{4}{\sqrt{2}} I_1 (Z_{x,2} + Z_{y,2}), \tag{55}$$

$$\int E^4 \, dx \, dy \, dz \leqq \text{const.}\ I_1{}^{1/2} (Z_{x,3} + Z_{y,3} + Z_{z,3})^{3/2}, \tag{56}$$

where $Z_{j,i} = \int |dE/dx_j|^2 \, d^i r$, for the dimension $i = 1, 2, 3$.

From there we obtain for the Hamiltonian in, e.g., one space dimension

$$H \equiv I_2 \geqq Z_{x,1} - 2Z_{x,1}^{1/2} I_1^{3/2} \geqq -I_1^3, \tag{57}$$

i.e., stability of one-dimensional solitons. In three space dimensions, a rough inflationary analysis can be used to support instability arguments. An exact calculation was performed by Laedke and Spatschek [30].

All the previous investigations are valid for small energy densities. They show that Langmuir oscillations are trapped within density depressions. We have used a quasi-static approximation which is valid in the small amplitude limit. However, when the collapse sets in, the local energy density becomes larger and the quasi-static approximation breaks down. In the regime being complementary to (22), i.e.,

$$\frac{|\Delta \Psi|^2}{16\pi n_0 T_e} \gg \frac{m_e}{m_i} \tag{58}$$

a supersonic approximation can be used. Thus, when the estimate (58) is true in the center of the cavity, we investigate the further development with the help of the equations:

$$\nabla^2 \left(i\partial_t \Psi + \frac{3}{2}\, \omega_p \lambda_D^2 \nabla^2 \Psi \right) = \frac{\omega_p}{2n_0}\, \nabla \cdot (\delta n \nabla \Psi), \tag{59}$$

$$\partial_t^2 \delta n = \frac{1}{16\pi m_i}\, \nabla^2 |\nabla \Psi|^2. \tag{60}$$

Qualitative arguments and numerical solutions show that the collapse progresses. Especially time-asymptotic self-similar solutions provide a firm basis for that hypothesis. For example we have

$$\Psi = \frac{1}{(t - t_0)}\, g(\xi) \exp\left[i \int \lambda dt \right], \tag{61}$$

$$\delta n = \frac{V(\xi)}{(t_0 - t)^{2\alpha}}, \tag{62}$$

where

$$\xi = r/(t_0 - t)^\alpha, \qquad \lambda(t) = \lambda_0/(t_0 - t)^{2\alpha},$$

and $\alpha = 2/i$. For self-similar solutions we have to argue that arbitrary initial distributions will finally show the self-similar behavior during the course of time. Numerical solutions will help to give an answer to that question and indeed there are many solutions known which support the self-similar behavior.

The (supersonic) equations (59) and (60) have the constant of motion

$$I_2 = \int \left\{ \frac{3\lambda_D^2}{64\pi} |\nabla^2 \Psi|^2 + \frac{\delta n}{16\pi n_0} |\nabla \Psi|^2 + \frac{n_0}{2} (\nabla \Phi)^2 \right\} d^3r; \tag{63}$$

here Φ has to be calculated from

$$\dot{\Phi} = -\frac{1}{16\pi m_i} |\nabla \Psi|^2. \tag{64}$$

This invariant I_2 indicates for collapsing solutions the scaling

$$\frac{\delta n}{n_0} \sim K^2 \lambda_D^2, \tag{65}$$

$$n_0 (\nabla \Phi)^2 \sim K^2 \lambda_D^2 \frac{|\nabla \Psi|^2}{16\pi}. \tag{66}$$

If we start with a state of the degree of turbulence

$$\frac{W_0}{n_0 T_e} \equiv \frac{|\nabla \Psi_0|^2}{16\pi n_0 T_e} \approx (K_0 \lambda_D)^2 \ll 1, \tag{67}$$

we can estimate the spatial width of the caviton by

$$l_0 \sim \frac{1}{K_0} \approx \lambda_D \left[\frac{n_0 T_e}{W_0}\right]^{1/2}. \tag{68}$$

At later times, during the collapse the spatial width will decrease. The integrals

$$I_1 = \int |\nabla \Psi|^2 \, d^3 r \tag{69}$$

and I_2 remain constant. From I_1 we can conclude

$$W \sim W_0 \left(\frac{K}{K_0}\right)^3 \sim (K \lambda_D)^3 \left(\frac{n_0 T_e}{W_0}\right)^{1/2} n_0 T_e. \tag{70}$$

We can eliminate from here, Eq. (70), and Eq. (65), which describes the scaling of $\delta n/n_0$, the characteristic spatial width at a later time. At the time when $W \approx n_0 T_e$ we obtain a relatively small density depression

$$\frac{\delta n}{n_0} \sim \left(\frac{W_0}{n_0 T_e}\right)^{1/3} \ll 1. \tag{71}$$

These qualitative arguments lead to the conclusion that during the time-development of the collapse, from energy densities $W \ll (m_e/m_i)\, n_0 T_e$ to $W \approx n_0 T_e$, the scaling of the density depression δn changes: initially, we have $\delta n/n_0 \approx -W/n_0 T_e$, but later $\delta n/n_0$ remains small, i.e., $\delta n/n_0 \ll 1$, even when $W \approx n_0 T_e$. Numerical calculations confirm this behavior. In the course of development of the collapse ion inertia effects become important and the above scaling applies.

For high energy densities ($W \approx n_0 T_e$, $\delta n/n_0 < 1$) Landau damping becomes important. For smaller diameters, the effective $K\lambda_D$ values increase and Landau damping can no longer be ignored. The energy of high-frequency waves with $\omega/K \gtrsim v_{te}$ is transfered to the particles and appears as kinetic energy of fast electrons. The cavity burns out. We can estimate the characteristic widths of the cavities (when this effects sets in) by the following way. Particles travelling through the cavity in the time $T = 2\pi/\omega_p$ are in resonance with the oscillations and are highly accelerated. Approximately, if the velocity is less than twice the thermal velocity v_{te} Landau damping becomes important. The estimate $L/T \lesssim 2v_{te}$ yields $L \lesssim 6\lambda_D$ which is in agreement with simulation results. The empty cavity (the density depression) survives for some longer time because of ion inertia effects. Finally, the cavity breaks up into ion-sound. This last period is only possible to observe in numerical calculations.

The whole scenario leads to an effective dissipation mechanism of (initially) small K-values. Via the modulational instability, long-wavelength perturbations evolve into cavities. The latter collapse and burn out. This has to be seen in addition to the usual linear Landau damping for large K-values. The problem of Langmuir condensation is thereby solved.

Langmuir condensation is an artifact of the weak turbulence theory. Let us repeat just a few arguments for its occurence. If we define the plasmon number n_k through

$$\langle \Psi_k \Psi_{k'}^* \rangle = \frac{4\pi\omega_p}{k^2} n_k \delta_{k-k'}, \tag{72}$$

then the weak turbulence kinetic description yields the constant of motion

$$N := \int dk n_k = \text{const}. \tag{73}$$

On the other hand, the free energy

$$E := \omega_p \int (k\lambda_D)^2 n_k \, dk \tag{74}$$

decreases, i.e.,

$$\frac{\partial}{\partial t} E \leqq 0. \tag{75}$$

The two results (73) and (75) obviously result in a condensation of the plasmons at $k \to 0$, for increasing time.

If the condition $W/n_0T_e > k^2\lambda_D^2$ is satisfied, a modulational instability can appear and the effective dissipation mechanism (described above) can take place. This situation now resembles the classical picture of hydrodynamic turbulence. Using dimensional arguments, with an inertial region between small k at the source and large k for dissipation, we can assume a constant energy flux in k-space, i.e., $|E_k|^2 k^{i-1}k/\tau \approx$ const., where i is the space dimension. The characteristic coupling time τ can be obtained from the self-similar solutions of the collapse. There we have the combination $(x - x_0)/(t_c - t)^{2/i}$ of variables [compare the Eqs. (61) and (62)] leading to $\tau \sim k^{-i/2}$. We therefore have the spectral energy distribution $|E_k|^2 \sim k^{-3i/2}$ for strong Langmuir turbulence.

The other scenario of strong Langmuir turbulence consists of long-living cavitons which can be considered as the building blocks of a statistical theory. But let us first answer the question how it is possible that long-living stable Langmuir solitons can exist. For the answer to this question we have to go back to the various models, e.g., the equations (3) and (4). As has been mentioned already, the models considered so far are not valid for $W \approx n_0T_e$. If we want to investigate the behavior for larger energy densities we have to repair that defect by taking into account various phenomena: Landau damping, electron nonlinearities, modifications of the dispersion relation, ion nonlinearities, etc. To demonstrate that such changes can completely change the qualitative picture we choose the following example. If we still use the quasi-static approximation in Eqs. (10)–(14), we obtain

$$\frac{\delta n}{n_0} \approx \exp\left(-\frac{|\nabla\Psi|^2}{16\pi n_0 T_e}\right) - 1, \tag{76}$$

and thereby the following equation for the envelope Ψ,

$$\nabla^2\left(i\partial_t\Psi + \frac{3}{2}\,\omega_p\lambda_D^2\nabla^2\Psi\right) - \frac{1}{2}\nabla\cdot\left[\left(\exp\left(-\frac{|\nabla\Psi|^2}{16\pi n_0T_e}\right) - 1\right)\nabla\Psi\right] = 0. \tag{77}$$

For reasons of demonstration we use the equivalent, dimensionless, exponentially nonlinear Schrödinger equation

$$i\partial_t \boldsymbol{E} + \nabla\nabla \cdot \boldsymbol{E} - Q\nabla \times \nabla \times \boldsymbol{E} + [1 - \exp(-|E|^2)]\, \boldsymbol{E} = 0. \tag{78}$$

Let us first consider spatially three-dimensional situations. If we assume spherical symmetry (and disregard electric field configurations with an asymmetry of $\boldsymbol{E}$ at $r = 0$), we can write

$$\boldsymbol{E} = G(r) \exp[i\eta^2 t]\, \hat{r}, \tag{79}$$

where G follows from an ordinary nonlinear differential equation. The stability investigations result for "longitudinal" (i.e., only r-dependent perturbations) in a criterion similar to (39),

$$\frac{\partial}{\partial \eta^2} \int dr r^2 G^2 > 0. \tag{80}$$

In Fig. 1, the plasmon number N is plotted versus the frequency shift η^2. One can conclude that for $\eta^2 \geqq 0.1$ the solitons are stable. Additional calculations show that arbitrary θ- and φ-dependent perturbations do not lead to a stability boundary beyond (80). Therefore, the criterion (80) demonstrates the existence of finite amplitude stable three-dimensional envelope solitons. These results can be generalized for the more complex system of equations (10)–(14), with criterion (80) as the general result. When investigating spatially two-dimensional situations, a significant difference occurs. Let us take for example the two-dimensional scalar exponentially nonlinear Schrödinger equation. Assuming cylindrical symmetry for the stationary solitons, one finds that the threshold in amplitude vanishes and solitons of small amplitudes are stable with respect to longitudinal perturbations. This is a mathematically as well as physically quite remarkable result.

The existence of stable envelope solitons under realistic conditions, such as Landau damping, external sources, higher dispersion effects, ion nonlinearities etc. is very difficult to prove. Because of the very fundamental problems, most cases could not be in-

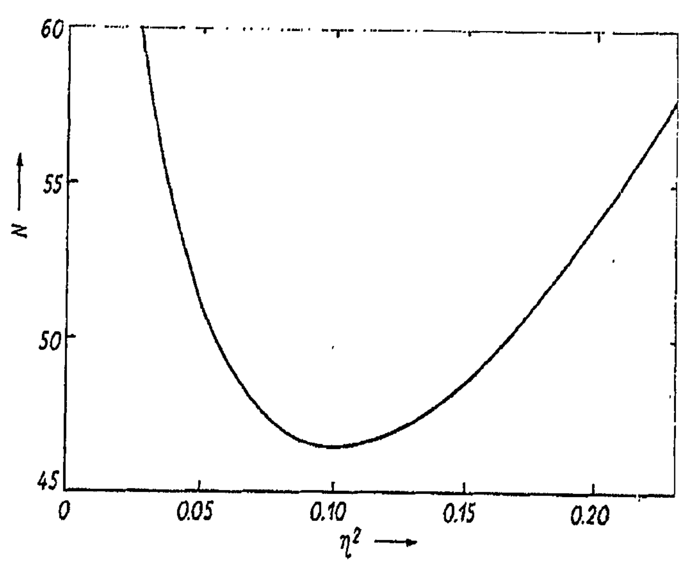

Fig. 1. Plasmon number N as a function of the nonlinear frequency shift η^2 for the nonlinear Schrödinger equation

vestigated analytically so far. But it seems true that existing stable (non-collapsing) three-dimensional Langmuir solitons have to exceed a threshold in amplitude. This is a very basic argument against their natural appearence via the modulational instability.

To make the things absolutely clear: Stable multi-dimensional solitons can exist but if they are thought to be created by a modulational instability, the resulting cavities will collapse before exceeding the amplitude threshold. This is so since the modulational instability sets in for small amplitudes, i.e., $W \ll n_0 T_e$, and then the balance of hydrodynamic and ponderomotive pressure leads to collapsing solutions. When the local energy density increases, i.e., $W/n_0 T_e > m_e/m_i$, ion inertia effects may play a dominating role and the problem becomes essentially non-stationary. The ion inertia causes a continuous decrease of the caviton diameter (central collapse) and does not allow new stationary states [18]. The whole scenario changes, however, if the solitons can be generated in a different way. Then, an ensemble of partially collapsing and partially stable cavitons seems to be appropriate.

In this connection the question of dimensionality should be re-considered. The situation is now clear for one and three space dimensions: One-dimensional Langmuir solitons are one-dimensionally stable and transversely unstable; however, strictly one-dimensional states are quite unrealistic. Three-dimensional Langmuir solitons are only stable if their amplitudes are above a certain threshold. That is the reason why it is unlikely to expect their generation via a modulational instability. Two-dimensional Langmuir solitons are longitudinally stable for arbitrary small amplitudes. Therefore they can be reached via a modulational instability. However, two-dimensional solitons are transversely (and three-dimensionally) unstable. This has some consequences for two-dimensional simulations. There may be no central collapse but a dangereous angular instability. This raises the question how much energy may be transfered to the particles.

Before concluding this section let us compare with numerical calculations and real experiments. Numerically, two different procedures have been used to get some general insight into the dynamical behavior. Firstly, the Zakharov equations (3) and (4) have been solved numerically when Landau damping is included. The disadvantage of this procedure is that other effects (e.g., higher nonlinearities) are not included. In addition, Landau damping could only be included within a crude model. Nevertheless, part of the numerics shows a quite remarkable agreement with the theoretical predictions [32]. Collapsing cavities were observed in a parametrically pumped system. Very rapidly, they are detuned in phase from the driver (phase decoupling and wavenumber changes because of density variations are very pronounced). The results show approximately the predicted self-similar behavior. Fast electrons are generated. However, very often artificial mass ratios m_e/m_i are used to shorten the computational times. This simplification can be dangereous since the ion inertial region, which is important for the collapse, might be underestimated. Another important point is that for the collapse the spatial grid size should not be to large.

In principle better, but of course more time-consuming, are particle in cell simulations. Typical [33] two-dimensional computations have $64\lambda_D \times 64\lambda_D$ grids. Then the minimal wavenumber is

$$k_{\min}\lambda_D = \frac{2\pi\lambda_D}{L} \approx 10^{-1}, \tag{81}$$

and the typical length L is 64 Debye lengths large. A minimal particle number is required to reduce the thermal fluctuations. A typical value is (at least) 20 particles per Debye zone. That results in a total number of 10^5 particles in the system. Nowadays, this number can be handled with modern computers. Let us review some of the interesting simulation results of ANISIMOV et al. [33]. These authors start with initial distributions which satisfy the collapse criterion $I_2 < 0$. Furthermore, the electron and ion distributions are prescribed separately at $t = 0$. In the electron distribution we have to include the high-frequency oscillations whereas the (otherwise quasi-neutral) ion modulations (described by an ion-acoustic distribution with modifications due to the ponderomotive force) describe the radiation trapping. Thus, one uses

$$n_e = n_0 + \delta n + \delta n_e, \tag{82}$$

$$n_i = n_0 + \delta n, \tag{83}$$

$$\delta n_e = \varepsilon_1 \left(\cos \frac{\pi x}{L} + \cos \frac{\pi y}{L}\right), \tag{84}$$

$$\delta n = \varepsilon_2 \left(\cos \frac{2\pi x}{L} + \cos \frac{2\pi y}{L}\right). \tag{85}$$

For the reasons mentioned above

$$\nabla^2 \Psi = 4\pi e \delta n_e \tag{86}$$

and

$$\frac{\delta n}{n_0} = -\frac{|\nabla \Psi|^2}{16\pi n_0 T_e}. \tag{87}$$

The last two relations determine ε_2. The condition $I_2 < 0$ leads to the threshold condition $\varepsilon_1 > 0.01$. The simulation results confirm in principle most of the theoretical predictions. On the time-scale of an inverse ion plasma frequency the maximum amplitude of the electric field grows. Localized cavitons with decreasing widths are generated. At $t \sim \omega_{pi}^{-1}$ the caviton burns out. The amplitude of the electric field decreases and the field energy is transfered to the particles via Landau damping. Real experiments for soliton generation, propagation, and collapse have been performed by two groups: NEZLIN et al. [34] at the Kurchatov institute and WONG et al. [35] at UCLA. Three principle regions were observed: (i) Observations of stable one-dimensional Langmuir solitons, if the transverse instability is quenched by small lateral dimensions ($2\pi/L > k_c$). (This result stimulated the conclusion that some strongly turbulent systems might be equivalent to one-dimensional configurations as far as the soliton concept is concerned.) (ii) Non-reproducable situations with many spikes in the electric field intensity. The spikes disappear and are created again (in the characteristic collapse time) and support in their dynamical behavior the basic influence of ion inertia during the dynamical phase. (iii) Reproducable generations of cavitons with subsequent collapses. It seems to the author that these various observations reveal the different stages at the route to chaos in a complex nonlinear dynamical system.

Summarizing, we can conclude that theory, numerics, and experiments support the idea of the collapse as the essential mechanism of strong Langmuir turbulence. However, one should remember that very often in the numerics, as well as in experiments, special initial conditions are prepared and it is by no means clear whether the collapse is a dissipation mechanism at the hundred percent level. It cannot be excluded that in strong Langmuir turbulence also localized fields with quite stable properties might exist; they will contribute to the various phenomena and may be describable by a statistical theory.

3. Vortices in Magnetized Plasmas

Compared to a plasma without an external magnetic field, in a magnetized plasma a huge number of eigenmodes exists. Of course, for each of them one could develop a nonlinear theory which would be of practical interest. However, from the principal point of view, many of the calculations would be quite similar and are therefore not of utmost interest since they do not lead to qualitatively new results. There is however one phenomenon which definitely deserves a more detailed consideration. Because of the $\boldsymbol{E} \times \boldsymbol{B}$ drift, in a magnetized plasma vortices with finite angular momentum can exist. The pioneering work in this direction originates back from Okuda and Dawson [36], as well as Hasegawa and Mima [37].

Generally, in fluids vortices are well-known but because of their quite complicated dynamics we do not have a complete theory for vortex existence and dynamics. The simplest example is the two-dimensional Navier-Stokes model

$$[\partial_t + \boldsymbol{v} \cdot \nabla] \boldsymbol{v} = -\nabla p. \tag{88}$$

When incompressibility is also assumed,

$$\nabla \cdot \boldsymbol{v} = 0. \tag{89}$$

Because of two-dimensionality and incompressibility, the velocity $\boldsymbol{v}$ can be derived from a velocity potential Ψ,

$$\boldsymbol{v} = \nabla \Psi \times \hat{z}. \tag{90}$$

Taking the curl of Eq. (88) we immediately obtain

$$\partial_t \nabla^2 \Psi - \nabla \Psi \times \hat{z} \cdot \nabla \nabla^2 \Psi = 0. \tag{91}$$

It is obvious that in a magnetized plasma because of the dominating $\boldsymbol{E} \times \boldsymbol{B}$ drift [see Eq. (90)] an equation similar to Eq. (90) can be derived for the electrostatic potential. A detailed derivation will be presented below. The corresponding nonlinear model equation is known in plasma physics as "convective cell equation". For a deeper physical understanding of its nature we first have to discuss some basic principles.

In the 1973 paper of Okuda and Dawson [36] it was shown that collective effects can cause a Bohm-like scaling of the diffusion coefficient. Within a linear theory, it is now straightforward to derive a new mode, the so called Okuda-Dawson mode, which is responsible for that behavior. Allowing for finite parallel wavenumbers k_z, we have

$$\omega^2 \approx \frac{k_z^2}{k^2} (1 + k^2). \tag{92}$$

Following Okuda and Dawson one can calculate the thermal energy density associated with that mode by applying the fluctuation-dissipation-theorem. Within a test-particle model, the diffusion coefficient D then scales for $\Omega_i \gg \omega_{pi}$ like

$$D \sim \frac{1}{B}, \tag{93}$$

which is the Bohm scaling. The numerical factor in the Bohm formula, i.e.,

$$D \approx \frac{1}{16} \frac{cT}{eB} \tag{94}$$

cannot be reproduced by these arguments. That raises the question, how the order of magnitude of the diffusion coefficient can be understood at all.

To answer this question, let us briefly review the results of the classical driftwave turbulence theory. The usually accepted driftwave model assumes that for sufficiently small Larmor radii the $\boldsymbol{E} \times \boldsymbol{B}$ convection of the particle density and the vorticity

$$W = c\nabla^2\Phi/B \tag{95}$$

are the dominating mechanisms. Introducing the 2d operator

$$L = -\nabla^2 + L_{\text{ah}}, \tag{96}$$

with an anti-hermitean part L_{ah} due to electron dissipation, we have

$$[1 + L]\,\partial_t\Phi + \hat{z}\cdot\nabla\Phi\times\nabla L\Phi + v_d\partial_y\Phi + \gamma_i\Phi = 0. \tag{97}$$

Here, γ_i designates the ion-dissipation because of ion-ion collisions, ion-neutral collisions, ion Landau damping, etc.; v_d is the drift velocity. Further details can be found in Refs. [38] and [39]; in addition we shall derive the nonlinear terms later by simple arguments. Many authors have produced numerical results for the model (97). A typical procedure starts with a finite number (e.g., 128^2) of Fourier components in the two-dimensional space. The thereby truncated discrete model shows many interesting and important aspects. Let us here concentrate on one of them. If we detune the instability (universal mode) the short wavelength fluctuations disappear very rapidly because of the ion-ion collisional viscosity. However one finds large scale structures, as can be seen in the numerical simulations of McWilliams [40] and Horton [39]. In the results of Horton, clearly monopolar as well as dipolar vortices appear. Without stressing to much attention on the similarity to the generation mechanism of structures in Langmuir turbulence, it is fascinating to see here also spatially localized structures appearing. To work this out a little bit more, let us investigate some simple but concrete models.

Petviashvili [41], as well as Hasegawa and Mima [37] have shown that a strong similarity exists between nonlinear driftwaves in plasmas and Rossby waves in the atmosphere. In the simplest case we can distinguish between the case with a strong temperature gradient, described by the Petviashvili equation

$$\partial_t(1 - \nabla^2)\,\Phi + v_d\partial_y\Phi - \frac{1}{2}\,v_{dT}\partial_y\Phi^2 = 0, \tag{98}$$

and the case with a strong density inhomogeneity, described by the Hasegawa-Mima equation

$$\partial_t(1 - \nabla^2)\,\Phi + (\hat{z}\times\nabla\Phi)\cdot\nabla\nabla^2\Phi - v_d\partial_y\Phi = 0. \tag{99}$$

It is important to note that in a dissipationless situation Eq. (97) agrees with Eq. (99).

A slightly more general model, although without temperature inhomogeneities, has been derived by Balescu et al. [42]. This model consists of three coupled equations [43] of the form

$$\partial_t n + \hat{z}\times\nabla\Phi\cdot\nabla n + \partial_z v = 0, \tag{100}$$

$$\partial_t v + \hat{z}\times\nabla\Phi\cdot\nabla v = \delta\,\frac{m_i}{m_e}\,\partial_z(\Phi - n), \tag{101}$$

$$\partial_t\nabla^2\Phi = (\nabla\Phi\times\hat{z})\cdot\nabla\nabla^2\Phi - \partial_z v. \tag{102}$$

The first equation (100) is the continuity equation for the electron density n (in units n_0), where v is the parallel (to the external magnetic field) electron velocity [in $c_s = (T_e/m_i)^{1/2}$], and the drift motion in the perpendicular direction. In the latter direction the $\boldsymbol{E} \times \boldsymbol{B}$ drift dominates. The second equation (101) is the ion momentum balance, parallel to $\hat{z} = \boldsymbol{B}_0/B_0$; the parameter δ depends on the scaling as discussed below. If $\delta = 1$, as will be the case for the units used below, on the right-hand side of Eq. (101) the large ion-to-electron mass ratio appears. The third equation (102) expresses, in the quasi-neutral situation, $\nabla \cdot \boldsymbol{j} = 0$, i.e. the charge continuity. In that equation the ion polarization drift dominates. In writing Eqs. (100)–(102) we have assumed $\Omega_i \ll \omega_{pi}$, and the transversal lengths, as well as the z-coordinates, are measured in units of $\varrho_s = c_s/\Omega_i$. Furthermore, the electrostatic potential Φ is measured in units of T_e/e. It is interesting to note that for the different units: v in v_{te} and z in v_{te}/Ω_i whereas the other units are unchanged, the system of equations (100)–(102) is invariant except for the change $\delta = m_e/m_i$. Then the order of magnitude factor on the right-hand side of Eq. (101) disappears. In the strictly two-dimensional case ($\partial_z = 0$) the model (100)–(102) reduces to the convective cell equation:

$$\partial_t \nabla^2 \Phi = \nabla \Phi \times \hat{z} \cdot \nabla \nabla^2 \Phi, \tag{103}$$

which is identical to the 2d Navier-Stokes equation (91). In that case $n \approx (m_e/m_i) \nabla^2 \Phi \approx 0$; this is typical for convective cells. In the case $\partial_z \neq 0$ we get from Eq. (101), for $\delta = 1$, the driftwave scaling $\Phi = n$ and therefore from Eqs. (100) and (102) the (homogeneous) Hasegawa-Mima equation

$$\partial_t (1 - \nabla^2) \Phi = \hat{z} \times \nabla \Phi \cdot \nabla \nabla^2 \Phi. \tag{104}$$

Its inhomogeneous generalization (change $\Phi \to -\Phi$) has been presented already in the form (99). [Of course, the system of equations (100)–(102) can be generalized to the inhomogeneous situation, without any difficulty. An additional term $v_d \partial_y \Phi$ has to be added to the left-hand side of Eq. (100).] Except the two scalings mentioned above a third one is known which corresponds to the Dawson-Okuda mode. We then have

$$n \approx \nabla^2 \Phi. \tag{105}$$

In this case, the model (100)–(102) can be reduced to two coupled equations which have been discussed by BALESCU et al. [42].

In all cases stationary vortex solutions exist. The monopolar and dipolar vortices can be constructed in the same way as in the atmospheric calculations. There is no need at this stage to present the explicit analytical expressions which include Bessel functions. Let us only mention one important relation. The velocity region for the nonlinear solutions is complementary to the regime of linear phase velocites. If the latter are in the region $0 < v_{ph} < v_d$, then the velocities u of the vortices are $u < 0$ or $u > v_d$. Two principal questions arise at this stage: Are the vortices stable entities and how are vortices generated from arbitrary initial distributions?

The first question of stability was answered in the past by LAEDKE and SPATSCHEK [44, 45]: Monopolar vortices, in the case of strong temperature gradients, are linearly and nonlinearly stable. The linear stability of Hasegawa-Mima dipolar vortices was proved for $u > v_d$. The stability of convective cells could only be shown linearly; the nonlinear regime is in progress.

At this stage it is helpful to explain the principal procedure for a nonlinear stability proof. Let us choose Eq. (103) for demonstration. That equation has as constants of motion the energy

$$\int (\nabla \Phi)^2 \, d^2 r = \text{const.}, \tag{106}$$

the generalized enstrophy

$$\int F(\nabla^2\Phi)\, d^2r = \text{const.}, \tag{107}$$

where F is an arbitrary but well-behaved and integrable function, as well as the z-component of the angular momentum

$$\int \nabla^2\Phi r^2\, dr = \text{const.} \tag{108}$$

Other invariants, especially CASIMIRS [46], are not known at the moment. One tries to build out of the constants of motion a nonlinear functional L in such a way that the first variation vanishes for the stationary solution Φ_s under consideration, i.e.,

$$\delta L|_{\Phi=\Phi_S} = 0. \tag{109}$$

If the functional itself vanishes at the stationary point $\Phi = \Phi_s$, then a stability condition is

$$\delta^2 L|_{\Phi=\Phi_S} > 0. \tag{110}$$

The last condition has to be supplemented by an estimate of the so called convexity conditions. Of course, the sufficient criterion (110) for stability is generally not necessary since there is some arbitraryness in the choice of L. Therefore more effort is generally needed to find some sufficient criteria for instability.

As has been mentioned already, there exist complete nonlinear stability proofs only for some cases in unbounded regions. The reason is that only a few cases allow an exact estimate of the form (110) which otherwise is to complex. For linear stability, the eigenvalue equation can be investigated directly. However, it is by no means clear that linear stability implies nonlinear stability. The actual state of knowledge is the following:

The linear stability proofs for monopolar and dipolar vortices are presumably of different value in the nonlinear regime. It is expected that the linear stability of the dipolar vortices survives in the nonlinear region. On the other hand it is quite evident that the monopolar convective cells cannot be nonlinearly stable. The reason is that solutions with the same vorticity tend to unite, which has been demonstrated numerically and experimentally [47]. These results show that at the end of various interactions two big vortices of opposite polarity survive.

Let us now turn to the second question of the *generation of vortices*. Different routes are possible; the best understood is the following [48]. It is similar in its physical picture to the soliton generation in Langmuir turbulence and applies to the case of a strong temperature gradient. Then the basic model is described by the Petviashvili equation (98) and one-dimensional localized solutions are transversely unstable. The procedures of NEWELL and WHITEHEAD [49] or SCHLÜTER et al. [50] allow to calculate the nonlinear evolution of the unstable modes close to the onset of the instability. It turns out that a stabilization with subsequent (vortex-like) structures is possible in two space dimensions. After this first bifurcation we expect localized 2d vortices to exist. The latter are nonlinearly stable. This scenario shows some similarity to the route to turbulence for Langmuir waves, — with the basic difference that no collapse occurs.

In the case of strong density inhomogeneity the drift instability may be a generation mechanism for drift vortices of different polarity. However, a complete and satisfactory theory for drift vortex excitation does not yet exist. Several proposals for convective cell excitation can be found in the literature. In the scenario of LIU et al. [51] drift waves are the pump waves in a parametric excitation of convection cells. The strength of excitation can be estimated in the following way. The "universal instability" has its maximum growth for $k_0\varrho_s \approx 1$; the growth rate is

$$\gamma_{\max} \approx \sqrt{\pi} \frac{cT_e}{eB_0} [\varrho_s L_n]^{-1}, \tag{111}$$

where L_n is the characteristic inhomogeneity length. The parametric decay was investigated by SHUKLA et al. [52], with the result for the decay growth rate

$$\gamma_{\text{decay}} \approx D_{\text{Bohm}} |\boldsymbol{k} \times \boldsymbol{k}_0 \cdot \hat{z}| \frac{e\Phi_0}{T_e} \left(\frac{T_e}{T_i}\right)^{1/2} \left(\frac{k_0^2 - k^2}{(\boldsymbol{k} - \boldsymbol{k}_0)^2}\right)^{1/2}. \tag{112}$$

Let us assume that the gain of driftwave energy due to the "universal instability" is balanced by the loss via the parametric decay instability, i.e.,

$$\partial_t \sum_k E_{k,\text{drift}} \approx \partial_t \sum_k E_{k,cc}. \tag{113}$$

Then, together with the assumption of saturation by turbulent diffusion, the saturation spectrum of the convective cells follows.

Another possibility for generation of convective cells follows from a linear coupling of driftwaves with convective cells [53]. If we investigate the linear tems of the model equations (100)–(103), including spatial inhomogeneity ($\varkappa := -\partial_x \ln n_0$), we have

$$\partial_t n - \varkappa \partial_y \Phi + \partial_z v = 0, \tag{114}$$

$$\partial_t \nabla^2 \Phi + \partial_z v = 0, \tag{115}$$

$$\partial_t v - \partial_z (\Phi - n) = 0. \tag{116}$$

After Fourier transformation we obtain the linear dispersion relation

$$\omega^3 - \frac{k_z^2}{k_\perp^2} (1 + k_\perp^2) \omega + k_z^2 \frac{k_y |\varkappa|}{k_\perp^2} = 0. \tag{117}$$

Its solution is depicted in Fig. 2. (The ion-acoustic mode is not present since we have suppressed the ion motion along the magnetic field. If we include the latter, nothing is changed in the following arguments.)

We recognize that for small wave numbers k_z (parallel to the external magnetic field) a linear coupling occurs with a bifurcation point at k_c. For $k_z < k_c$ instability sets in. The phase velocities in the unstable region (as well as in most parts of the stable regimes) are larger than the thermal velocity of the electrons. This and a separate kinetic description justifies the relation (117). We have performed a bifurcation analysis in the vicinity of the onset of the instability. As a result we can obtain saturated, vortex-like structures.

Summarizing the results we can conclude as follows: Vortices have been observed in real experiments and numerical simulations. Theoretically, routes have been found for the generation of vortices in plasmas. Analytical investigations show that stable vortices can exist in plasmas. Let us therefore discuss some consequences.

For a better understanding of the driftwave turbulence we discuss in the following a statistical model. Such a choice is adequate since we have found that drift vortices do not collapse and therefore (dipolar) drift vortices can be considered as long-living stable structures. We shall see that a model based on non-interacting drift vortices is qualitatively different in its results compared with the outcomes of the weak turbulence theory.

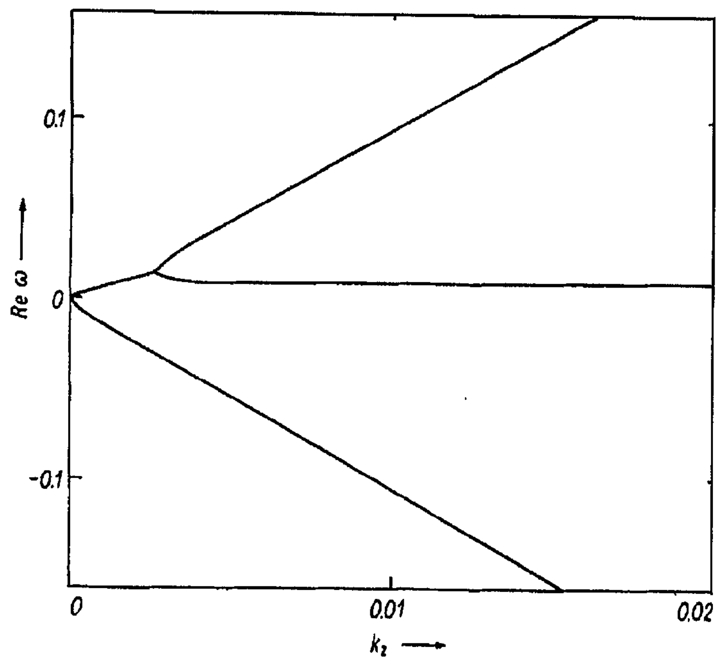

Fig. 2. Solution of the dispersion relation (117), depending on k_z, for the parameter values $k_y\,|\varkappa| = 0.01$ and $k_\perp = 0.1$

The weak turbulence theory makes a clear prediction for the autocorrelation coefficient of the potential fluctuations. A simplified argumentation is as follows. We have two constants of motion,

$$E = \sum_k [|\Phi_k|^2 + k^2\,|\Phi_k|^2], \tag{118}$$

$$U = \sum_k [k^2\,|\Phi_k|^2 + k^4\,|\Phi_k|^2], \tag{119}$$

the energy and the enstrophy. Numerical computations show that we can use an argument known from Kolmogorov in fluid turbulence. Here a dual cascade process can be assumed: energy E flows through a inertial region to small k (relative to the pump wave-number k_s) and enstrophy U flows through another inertial region to large k values ($k > k_s$). Let us investigate first the region $k < k_s$. Since driftwave turbulence is three-dimensional, we can conclude from Eq. (118)

$$E \sim \int |\Phi_k|^2\,k^2\,dk \equiv \int W_k dk, \tag{120}$$

and therefore a constant flow in k-space yields

$$\frac{W_k k}{\tau} \sim \frac{|\Phi_k|^2\,k^2 k}{\tau} \sim |\Phi_k|^3\,k^7 = \text{const.} \tag{121}$$

Here we took

$$\frac{1}{\tau} \sim \frac{k^4\Phi_k}{1 + k^2} \approx k^4\Phi_k \tag{122}$$

from the Hasegawa-Mima equation [see also Eq. (97)]. From the right hand side of Eq. (121) we get $|\Phi_k|^2 \sim k^{-14/3}$ for driftwaves with $k < k_s$. In the region $k > k_s$ we can use $U_k \approx k^4 |\Phi_k|^2$ and arguments similar to (121) and (122) yield $|\Phi_k|^2 \sim k^{-6}$. This agrees with the results of some experiments [54].

For convective cells the arguments are changed insofar as

$$E = \sum_k k^2 |\Phi_k|^2 \tag{123}$$

and

$$U = \sum_k k^4 |\Phi_k|^2. \tag{124}$$

In addition, convective cells are two-dimensional, i.e.,

$$E \sim \int dkkk^2 |\Phi_k|^2, \qquad U \sim \int dkkk^4 |\Phi_k|^2, \tag{125}$$

and

$$\frac{1}{\tau} \approx k^2 |\Phi_k|, \tag{126}$$

[compare with the 2d Navier-Stokes equation (91)]. Then, it follows

$$\frac{W_k k}{\tau} \sim k^3 |\Phi_k|^2 kk^2 |\Phi_k| \sim |\Phi_k|^3 k^6 = \text{const.} \quad \text{for} \quad k < k_s, \tag{127}$$

and

$$\frac{U_k k}{\tau} \sim k^5 |\Phi_k|^2 kk^2 |\Phi_k| \sim |\Phi_k|^3 k^8 = \text{const.} \quad \text{for} \quad k > k_s. \tag{128}$$

Characteristic for the weak turbulence results are (for fixed k) that (i) the maximum of the excitation is at the linear drift frequency and (ii) the spectra have a small width. The last phenomenon is quite understandable since weak means that only a weak coupling of the linear modes occurs. Qualitatively, the Fourier transform of the autocorrelation function $S(k, \omega)$ is shown in Fig. 3. The function $S(k, \omega)$ is defined through

$$\langle \Phi(x + \zeta, t + \tau) \; \Phi(x, t) \rangle = \frac{1}{(2\pi)^3} \int dk \int d\omega S(k, w) \, e^{ik\zeta - i\omega\tau}. \tag{129}$$

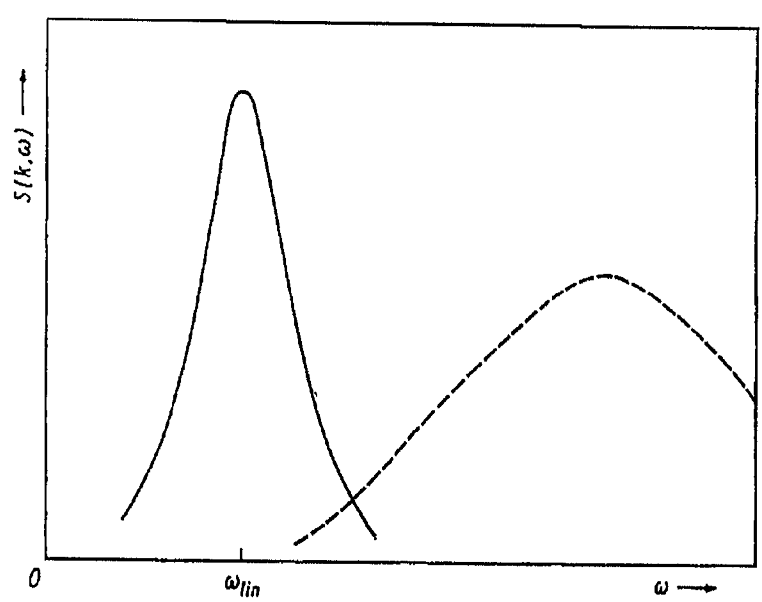

Fig. 3. Qualitative behavior of the dynamic form factor $S(k, \omega)$, for fixed k, vs ω. The solid line depicts the result of the weak turbulence theory whereas the broken line indicates the functional dependence as a result of a statistical vortex model

In Fig. 3 we have also indicated that experimental results show a maximum at higher values (up to a factor 10); in addition, the width is very often quite large (of the order of magnitude of the characteristic frequency). These two phenomena can be understood as strong indications that a statistical vortex model might be appropriate. In the latter, we assume a system of non-interacting, closely packed vortices. Very important for the following arguments will be that the velocites of the nonlinear structures are complementary in magnitudes to the linear phase velocites of the linear waves.

Following MEISS and HORTON [55] we choose the simplest situation of one-dimensional structures in the case of strong temperature inhomogeneities. Then the solutions of the equation

$$[1 - \varrho_s^2 \nabla_\perp^2]\, \partial_t \Phi + v_d \partial_y \Phi - v_d \Phi \partial_y \Phi = 0 \tag{130}$$

are explicitly known,

$$\Phi_s(y, t; u) = -3 \left(\frac{u}{v_d} - 1\right) \operatorname{sech}^2 \left[\frac{1}{2\varrho_s} \left(1 - \frac{v_d}{u}\right)^{1/2} (y - ut)\right]. \tag{131}$$

Let us ignore all non-soliton contributions. Then we can write

$$\Phi = \sum_{n=1}^{N} \Phi_s(y_n, t; u_n). \tag{132}$$

Furthermore, the solitons should be uniformly distributed in space. Their positions are denoted by y_n. Then the dynamical form factor is

$$S(k, \omega) = \begin{cases} \dfrac{1}{L} \sum \left\langle \left[12\pi k \varrho_s \left(\dfrac{u_n}{v_d}\right) \operatorname{csch} \left(\dfrac{\pi k \varrho_s}{\sqrt{1 - v_d/u}}\right)\right]^2 \delta(\omega - k u_n) \right\rangle, \\ 0 \quad \text{for} \quad 0 < \omega < k v_d \end{cases} \tag{133}$$

The averaging $\langle \cdots \rangle$ is performed with a Gibbs distribution

$$P(E_s) = \frac{2\pi}{Z} \left(\frac{\partial E_s}{\partial J}\right)^{-1} e^{-\beta E_s}. \tag{134}$$

Here, the derivative of the soliton energy E_s with respect to the action variable J can be estimated by

$$\frac{\partial E_s}{\partial J} \sim \dot{\theta} \sim 2\pi \frac{u}{L}. \tag{135}$$

Note that the energy of a soliton is

$$E_s(u) = \frac{12}{5} \left(\frac{U}{v_d}\right)^2 \left(1 - \frac{v_d}{u}\right)^{3/2} \left(6 - \frac{v_d}{u}\right), \tag{136}$$

i.e., we can calculate analytically the distribution function (134) as a function of the new variable u. The final result for $S(k, \omega)$ is

$$S(k, \omega) \approx \begin{cases} \omega^2 f_s(\omega/k), & \text{for} \quad k v_d [1 + (\pi k \varrho_s)^2] < \omega, \\ \omega^2 f_s(\omega/k), & \text{for} \quad \omega < 0, \\ 0, & \text{for} \quad 0 < \omega < v_d, \\ \exp\left[-2\pi k \varrho_s \left\{\dfrac{k v_d}{\omega - k v_d}\right\}^{1/2}\right], & \text{for} \quad k v_d < \omega < k v_d [1 + (\pi k \varrho_s)^2], \end{cases} \tag{137}$$

where

$$f_s(u) = N_s P(u).$$

The distribution depends on two parameters, N_s and β; the latter follow from the total energy and the additional assumption for a close soliton packing. The main result of the whole theory is that for small β the dynamical form factor (at prescribed k) does not peak at the linear frequency ω_0. It becomes largest at considerably higher ω-values. Furthermore, the width is proportional to the square root of the small (relative to the thermal energy) total energy and not to the total energy density itself as it is true for the weak turbulence theory. Both results are supported by experimental results.

Of course, such a one-dimensional statistical soliton model cannot be considered as a final, closed theory. Especially, the effects in higher space dimensions have to be taken into account. Nevertheless, many phenomena let us expect that a statistical vortex model tends into the physically correct direction and conclusions for the anomalous transport can be expected in the future.

4. Concluding Remarks

Solitons (and solitary waves) are solutions of nonlinear model equations in different areas in plasma physics. The behavior of solitons in one space dimension is quite well understood. In two and three space dimensions, quite surprising and important phenomena can occur which lead to qualitatively new results. The best example which shows all these phenomena is the Langmuir soliton. As we have seen, for certain initial distributions a three-dimensional cavity collapses and the collapse is an effective nonlinear dissipation mechanism in plasmas. In systems with an external magnetic field new low-frequency modes occur. In two space dimensions their nonlinear versions are vortices.

Since vortices can be stable, convective transport and vortex dynamics becomes important. In conclusion, nonlinear structures of different types exist in plasmas and they can influence the plasma properties considerably.

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