NONLINEARITIES AND TRANSPORT: SOME CONCLUSIONS FROM SIMPLE MODELS

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Abstract: In this contribution some simple models are reviewed in order to point at anomalous transport effects, both analytically and numerically. The first part is on particle transport; it is inspired by the great success of (neutral) fluid theory to understand, e.g., the nonlinear development of the Bénard instability. When treating nonlinear drift waves, either in the fully ionized or in the weakly ionized cases, nonlinear convective transport can become important. It is shown that nonlinear spatially coherent modes do play an important role in the particle transport beyond a critical magnetic field. In the second part we investigate the role of strongly nonlinear entities, like cavitons, in the particle heating process due to waves. The collapse and nonlinear mode conversion are identified as basic processes for the source terms in transport equations. Finally, the nonlinear wave propagation itself is considered, and some recent results on transport of excitations are presented. It is shown that nonlinear wave propagation can overcome, besides dispersive broadening, other inhibition mechanisms of transmission encountered in linear wave propagation theory. The conclusion is that for such important processes as particle and wave energy transport, as well as particle heating, new nonlinear mechanisms exist which have to be investigated in the future in more detail.

I. REMARKS CONCERNING LINEAR TRANSPORT THEORY

The linear transport theory is very well developed, although due to geometrical complications quite sophisticated theories are needed. But even in some relatively simple collisional plasmas, many transport results are anomalous, and during the last decades very many works were devoted to the understanding of such phenomena. Here we do not want to improve existing *linear* transport theories by incorporating into the already existing algebra (in a more or less rigorous manner) nonlinear contributions, e.g., in an iterative procedure. Our aim is another one: We want to demonstrate that new entities like vortices, cavitons, and solitons can take part in the nonlinear dynamics with severe consequences for transport beyond the linear limit. Let us exemplify this view by several historical remarks.

In a collisional plasma, the particle and momentum balances of species + (ions) or - (electrons) can (very often) be approximated by

$$\partial_t n_\pm + \nabla \cdot (n_\pm \vec{u}_\pm) = 0 \,, \tag{1}$$

$$m_\pm n_\pm \left(\partial_t \vec{u}_\pm + \vec{u}_\pm \cdot \nabla \vec{u}_\pm\right) = \pm e n_\pm \vec{E} - k_B T_\pm \nabla n_\pm - \tau_\pm^{-1} n_\pm m_\pm \vec{u}_\pm \,, \tag{2}$$

where m is the mass, n the particle number density, $\vec{u}$ the velocity, $\vec{E}$ the (external) electric field, T the temperature, and τ the mean collision time (e.g. with neutrals $\tau^{-1} \approx n_0 \sigma_n v_{th}$, where n_0 is the neutral density, v_{th} the thermal velocity, and σ_n the cross section, typically $\sim 5 \times 10^{-15} cm^2$ and weakly dependent on temperature). If the nonlinear convective term $(\vec{u} \cdot \nabla \vec{u})$ as well as the inertia term $(\partial_t \vec{u})$ are neglected, one can simply determine the velocities $\vec{u}_\pm$ from Eq. (2) by setting the l.h.s. equal to zero,

$$\vec{u}_\pm \approx \pm \mu_\pm \vec{E} - D_\pm \nabla \ln n_\pm \,, \tag{3}$$

where $\mu_\pm = (e\tau_\pm / m_\pm)$ and $D_\pm = \mu_\pm k_B T_\pm / e$. Note that from here we can obtain already one transport coefficient: $D_\pm$, the species diffusion coefficient. Two remarks are appropriate at this stage. First, during the diffusion process, a polarization field will appear as a result of charge separation. When the dimension of the system is much larger than the Debye length of the plasma, electrons and ions do not diffuse independently. The diffusion becomes ambipolar due the appearance of the polarization field, i.e. $\vec{\Gamma}_- \equiv n_- \vec{u}_- \equiv n_+ \vec{u}_+ \approx \vec{\Gamma}$. We can easily calculate this by considering in Eq. (3) no external fields and interpreting $\vec{E}$ only as the internal polarization field. Eliminating $\vec{E}$ from Eq. (3) we obtain

$$\vec{\Gamma} \approx -D_a \nabla n \quad , \quad D_a = \frac{D_+ \mu_- + D_- \mu_+}{\mu_- + \mu_+} \approx D_- \frac{\mu_+}{\mu_-} \tag{4}$$

(for $T_e \gg T_i$ and $n \approx n_- \approx n_+$). Secondly, another transport coefficient, the electric conductivity, also follows immediately. When introducing the electric current density $\vec{j} = en_+ \vec{u}_+ - en_- \vec{u}_- \approx -en_- \vec{u}_-$ we find $\vec{j} \sim \sigma \vec{E}$ with $\sigma \approx e\mu_- n_-$. The conductivity of a weakly ionized gas is mostly determined by the degree of ionization n_-/n_0. If ionization is stronger, $\tau^{-1} \approx n_0 v_{th} \sigma_n + n_\pm v_{th} \sigma_{Cb}$ consists of both, collisions with neutrals and Coulomb collisions with other charged particles. The scale of the Coulomb cross section πr_{Cb}^2 is roughly determined through $e^2 / r_{Cb} \approx \frac{3}{2} k_B T$. Due to the long range nature of the Coulomb forces, πr_{Cb}^2 has to be multiplied by the so-called Coulomb logarithm $\ln \Lambda$. The Coulomb collisions become dominant at higher degrees of ionization. Then $\tau^{-1} \sim n_\pm$ and the conductivity is only very weakly dependent on charged particles density (through $\ln \Lambda$), $\sigma \approx 9(k_B T_-)^2 / 4\pi e^2 m_- v_{th-} \ln \Lambda$. Spitzer and his co-workers[1] have refined this formula by a numerical factor.

When considering the energy equation in the simple form

$$\partial_t \left(\frac{3}{2} n_\pm k_B T_\pm \right) + \nabla \cdot \vec{F}_\pm = \pm n_\pm e \vec{E} \cdot \vec{u}_\pm - \frac{3}{2} n_\pm k_B T_\pm \tilde{\nu}_\pm - q_\pm I \,, \tag{5}$$

we have introduced $\tilde{\nu}_\pm \equiv \tilde{\tau}_\pm^{-1} = \delta / \tau_\pm$ as the effective frequency of energy losses. Furthermore, $q_\pm I$ (consisting of the ionization potential I and the resultant creation rate $q_\pm$) describes the energy spent to create new electrons during ionization. The flux $\vec{F}_\pm$ is composed of the hydrodynamic flux of enthalpy and the heat conduction flux; $\vec{F}_\pm \equiv \frac{5}{2} n_\pm k_B T_\pm \vec{u}_\pm - \lambda_\pm \nabla T_\pm$. The coefficients $\lambda_\pm = \frac{5}{2} k_B n_\pm D_\pm$ are called thermal conductivities; they are the other transport coefficients being of main interest.

Here we should stop now the quite heuristic argumentation. Non-equilibrium thermodynamics tells us some more general relations the transport processes have to obey. Thermodynamics forces (pressure and temperature gradients, electric fields, and so on) cause thermodynamic fluxes (particle, electric, or heat currents, and so on). Their linear interrelations are governed by the Onsager relations. When calculating the transport coefficients we have to specify the system and use some sophisticated transport theory. For plasmas, excellent presentations of the latter are given, e.g., by Braginskii[2] or Balescu[3]. In general, the procedure is the following. Starting from a (microscopic) kinetic description (Fokker-Planck, Landau, or Balescu-Lenard forms), macroscopic equations follow from averaging; moments appear which are, e.g., the particle number density, the mean velocity, the pressure tensor, the heat flux, and so on. The moment equations are not closed *ab initio*; we face a hierarchy problem. To calculate some higher moments, one looks for some approximate solution of the kinetic equation, e.g. in form of a finite Hermitean polynomial expansion. A quite elaborate analysis is necessary in order to calculate up to a certain order the transport coefficients, and still the convergence of the procedure has to be checked by numerical evaluations. The reader is referred to, e.g., the excellent monographs of Balescu for further details. However, it is well-known that some of the linear transport coefficients calculated in this manner do not reveal the experimental results. Most prominent examples are the electron heat conductivity and the perpendicular (to an external magnetic field $\vec{B}$) particle diffusion coefficient.

Another remark is appropriate at this stage. In the transport equations source terms appear. As one example - which we we shall discuss in more detail in this contribution - we mention particle heating due to wave-particle interaction. The latter obviously is another candidate for nonlinear considerations.

II. WHY TO INTRODUCE NONLINEARITIES?

We start this section by a remark on diffusion in magnetized plasmas. If we introduce on the r.h.s. of Eq. (2) the Lorentz force density $\pm en_\pm \vec{u}_\pm \times \vec{B}$, some simple manipulations lead to the factor $[1+(\tau_\pm \Omega_\pm)^2]^{-1}$ in front of the r.h.s. of Eq. (3). Here, the gyrofrequencies $\Omega_\pm = \pm eB/m_\pm$ have been introduced. Clearly the factor will lead to a diffusion coefficient $D \sim 1/B^2$ for strong magnetic fields. But already Bohm suggested $D \sim 1/B$ [the famous Bohm diffusion coefficient $D = k_B T/16eB$ is in its quantitative form still not understood!] in order to explain the unexpected weak confinement. Meanwhile many suggestions and scalings are reported which still await a rigorous analytical theory.

Experimental observations show that most plasmas are not quiescent and enhanced fluctuations appear. Within plasma physics very early the necessity appeared to include nonlinearities; effective collision frequencies were defined in a more or less *ad hoc* manner. One of the most successful attempts was the so-called weak turbulence theory[4]. In our present context we need only a few conclusions from this development.

It is quite accepted that drift wave fluctuations are candidates for effective particle-wave collisions, and thus we may estimate the nonlinear diffusive flux of electrons perpendicular to an external magnetic field by

$$\Gamma_- \approx \langle \delta n_- v_E \rangle \equiv \frac{1}{L_y} \int_{-L_y/2}^{L_y/2} \delta n_- v_E dy \approx \sum_k i k_y \varphi_k^* \delta n_{-k} \frac{n_{-0} k_B T_-}{e B \rho_s} + c.c. \,. \tag{6}$$

Here, the density gradient is in the x-direction, δn_- is the electron density fluctuation, we average over the y-direction, and $\vec{v}_E$ is the $\vec{E}\times\vec{B}$ velocity in the fluctuating electrostatic field [before Fourier transformation $\vec{v}_E = \vec{E}\times\vec{B}/B^2$, and in the last expression on the r.h.s. of Eq. (6) we have normalized the lengths by $\rho_s = v_{th-}/|\Omega_i|$, the density perturbations by n_{-0}, and the potential φ by T_-/e]. The next step is to determine the relation between δn_{-k} and φ_k. If we have a pure Boltzmann distribution $\delta n_{-k}\sim\varphi_k$, the total flux disappears [note the complex conjugate c.c. in Eq. (6)]; thus a phase shift between δn_{-k} and φ_k is essential. Hydrodynamical or gyrokinetic models determine such phase shifts, e.g. in the (non-dimensional) form $\delta n_{-k}\sim\varphi_k+\delta\left(k_\perp^2/k_\parallel^2\right)d_t\varphi_k$ [see Sec. III]. For drift waves we approximate $d_t\to -i\omega\approx ik_y\kappa_n/(1+k_\perp^2)$, with $\kappa_n = d\ln n_{-0}/dx$; $k_\parallel$ being an effective parallel (to $\vec{B}$) wavelength. Thus the expression (6) leads to

$$\Gamma_-\approx-\frac{k_BT_-}{eB}\delta\sum_k\frac{k_\perp^2}{k_\parallel^2}\frac{k_\perp^2}{1+k_\perp^2}|\varphi_k|^2\frac{dn_{-0}}{dx}\,. \tag{7}$$

At the first glance this looks like a great break-through: The coefficient in front of the thermodynamic force dn_{-0}/dx should be identified with the electron diffusion coefficient D_-. Unfortunately, this coefficient depends on the spectrum $|\varphi_k|^2$ for which we generally do not know neither the total energy content nor the spectral distribution over $\vec{k}$-space. Very often, by assumption, an ansatz for $|\varphi_k|^2$ is made and scalings are discussed, but from a rigorous theoretical point of view such a procedure is not very convincing.

In a more systematic procedure we have to calculate $|\varphi_k|^2$ self-consistently. The well-developed methods are "weak turbulence theory" (WTT) or the "direct interaction approximation" (DIA). [Note that a rigorous strong turbulence description is still missing!] The result is

$$\begin{aligned}\partial_t|\varphi_{\vec{k}}|^2-2\,\mathrm{Im}(\omega_{\vec{k}})|\varphi_{\vec{k}}|^2 &= \pi\int d^2k_1d^2k_2\delta\left(\vec{k}-\vec{k}_1-\vec{k}_2\right)\delta\left(\mathrm{Re}[\omega_{\vec{k}}-\omega_{\vec{k}_1}-\omega_{\vec{k}_2}]\right)\\ \times\Big\{\left(\Lambda^{\vec{k}}_{\vec{k}_1,\vec{k}_2}\right)^2|\varphi_{\vec{k}_1}|^2|\varphi_{\vec{k}_2}|^2 &+ \Lambda^{\vec{k}}_{\vec{k}_1,\vec{k}_2}\Lambda^{\vec{k}_1}_{\vec{k}_2,\vec{k}}|\varphi_{\vec{k}_2}|^2|\varphi_{\vec{k}}|^2+\Lambda^{\vec{k}}_{\vec{k}_2,\vec{k}_1}\Lambda^{\vec{k}_2}_{-\vec{k}_1,\vec{k}}|\varphi_{\vec{k}_1}|^2|\varphi_{\vec{k}}|^2\Big\}\,.\end{aligned} \tag{8}$$

Here $\Lambda^{\vec{m}}_{\vec{k},\vec{l}}$ are coupling coefficients which follow from the exact modelling of the drift waves. When solving Eq. (8) for an unstable drift wave situation, the dual cascade process is also revealed in the weak turbulence description. We get a condensation at small k, and in general a more detailed description, taking into account higher nonlinearities becomes necessary. But the appearance of large spatially coherent structures hints at another process which might be dominant in the nonlinear state: convective transport caused by nonlinear structures like vortices.

This actually is not an idea invented by plasma physicists since we have a very prominent praradigm in form of the Rayleigh-Bénard-problem. There one considers a fluid heated from below. Two plates are separated by a distance H, the temperature of the lower one is $T_0+\Delta T$. When the fluid is not moving ($\vec{u}=0$) the stationary (index s) state is $T_s(z)=T_0+\Delta T-(z/H)\Delta T$, $\rho_s(z)=\rho_0(1-\alpha[T_s(z)-T_0])$, $\nabla p_s(z)=-\rho_s(z)g\hat{z}$, where ρ_s is the equilibrium density of the fluid, p_s is the pressure in local equilibrium, and g is the gravitational acceleration. The equations determining $\vec{u}$ and temperature T are similar to those presented in forms of Eqs. (2) and (5), together with the incompressibility condition

$\nabla \cdot \vec{u} = 0$; α is the thermal expansion coefficient. When looking for the stability of the stationary state, one introduces the deviation $\theta(x,y,z,t) := T(x,y,z,t) - T_s(z)$. Within the Boussinesq approximation the equations for $\vec{u}$ and θ are

$$\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} = \alpha \theta g \hat{z} - \frac{1}{\rho_0} \nabla \delta p + \nu \nabla^2 \vec{u} , \tag{9}$$

$$\partial_t \theta + \vec{u} \cdot \nabla \theta = \lambda \nabla^2 \theta - u_z \Delta T / H . \tag{10}$$

The equation of state is determined by the incompressibility $\nabla \cdot \vec{u} = 0$. The famous result of the Rayleigh-Bénard instability considerations is that besides the usual heat conduction state (with thermal conductivity λ) nonlinearities can trigger a convective transport which clearly overcomes the predictions from linear transport theory. The critical parameter is the dimensionless Rayleigh number R_a. If the latter overcomes a critical value R_c,

$$R_a \equiv \frac{\alpha g H^3 \Delta T}{\lambda \nu} > R_c , \tag{11}$$

roles appear which are able to transport heat through convection. The value of R_c depends on the boundary conditions for the perturbations. The various cases are called rigid-rigid, rigid-free, free-rigid or free-free according to whether the upper and lower boundaries respectively are rigid or free. For $R_a \geq R_c$ instability arises at a certain critical horizontal wavenumber. When $R_a \geq R_c$ the wavenumber of the fastest-growing mode can be determined, and according to linear theory this mode should be the dominant one. For more details to determine the horizontal pattern see the summary in the excellent book of Drazin and Reid[5]. Especially hexagonal cells are of great interest, and in Fig. 1 we show the streamlines for such a hexagonal cell. Beyond the onset of instability the problem

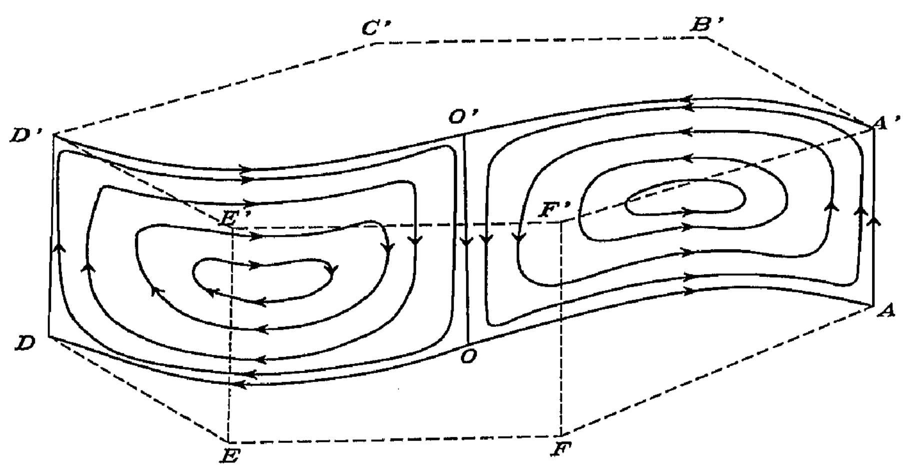

Fig. 1. *Streamlines of the hexagonal cell in Rayleigh-Bénard convection*[5].

becomes fully nonlinear, and several scenarios of nonlinear dynamics have been introduced to mimic the more complicated behavior. The simplest is due to Lorenz[6], who approximated the temperature field θ and the streamfunction ψ $[u_x = -\partial\psi/\partial z \, , \; u_z = \partial\psi/\partial x]$ for free-free boundary conditions $[\theta(0) = \theta(H) = \psi(0) = \psi(H) = \nabla^2\psi(0) = \nabla^2\psi(H) = 0]$ by the modes

$$\psi(x,z,t) = \frac{\sqrt{2}(1+a^2)\lambda}{a} X(t) \sin\left(\frac{\pi a x}{H}\right) \sin\left(\frac{\pi z}{H}\right) , \tag{12}$$

$$\theta(x,z,t) = \frac{\Delta T R_c}{R_a \pi}\left[\sqrt{2}Y(t)\cos\left(\frac{\pi a x}{H}\right)\sin\left(\frac{\pi z}{H}\right) - Z(t)\sin\left(\frac{2\pi z}{H}\right)\right] . \qquad (13)$$

The so-called Lorenz model follows by introducing this ansatz into the original dynamical equations and truncating at the three-modes level:

$$\frac{dX}{dT} = \sigma(Y - X) , \; \frac{dY}{dT} = -XZ + rX - Y , \; \frac{dZ}{dT} = XY - bZ . \qquad (14)$$

Here $T := t\pi^2(1+a^2)\lambda/H^2$, $\sigma := \mu/\lambda$, $r = R_a/R_c$, and $b = 4/(1+a^2)$.

The model (14) shows such interesting features as period-doubling transition to chaos and intermittency. However, one should emphasize that the quantitative agreements with the full theory and/or experiments are not good. But the qualitatively correct suggestion of nonlinear dynamics with spatially coherent structures is fascinating and will be taken up in the following plasma physics models.

Besides particle, momentum, and energy transport (coefficients) the source terms of the transport equations have also to be investigated in a modern manner based on recent developments in nonlinear wave theory. Two of the most successful areas stimulated by plasma physics are soliton physics and chaos. And both have important implications on wave propagation and wave absorption. Thus, source terms in transport equations have to be reconsidered in this situation.

III. NONLINEAR PARTICLE TRANSPORT DUE TO DRIFT VORTICES

In this section we review some of the latest results from plasma physics showing that indeed also in plasmas a nonlinear particle convective transport can appear. V. Naulin[7] has investigated a model for the dynamics of drift vortices in fully-ionized plasmas; here we concentrate on a similar nonlinear model for nonlinear drift waves in weakly ionized plasmas, being investigated mainly by Th. Eickermann[8]. The starting equations are again Eqs. (1) and (2), enlarged by the Lorentz term in the presence of an external magnetic field $\vec{B}$. Note that (even for constant temperature) these are eight scalar equations which we have to simplify for analytical considerations. Let us assume cold ($T_i = 0$) and magnetized ions, $\Gamma := \Omega_+ \tau_+ \gg 1$, and weakly space-dependent magnetic fields $\vec{B}$, such that we can use the drift approximation. Some lengthy analysis leads to the following coupled equations for $p := \ln(n_-/n_{-0})$ [note quasi-neutrality], $e\varphi/k_B T_- \to \varphi$, and $u = u_{+\|}/c_s$:

$$\partial_t p - \kappa\partial_y\varphi + \delta\nabla_\perp^2(\varphi - p) + \delta^{-1}\nabla_\|^2(\varphi - p) + \eta\partial_y(\varphi - p) + \{\varphi, p\} = 0 , \qquad (15)$$

$$(1+\partial_t)\nabla_\perp^2\varphi + \delta\nabla_\perp^2(\varphi - p) - \nabla_\| u + \delta^{-1}\nabla_\|^2(\varphi - p) - \eta\partial_y p + \{\varphi, \nabla_\perp^2\varphi\} = 0 . \qquad (16)$$

$$(1+\partial_t)u + \{\varphi, u\} = -\nabla_\|\varphi . \qquad (17)$$

In these equations time is normalized by τ_+, and the perpendicular ($\perp$) space coordinates x and y are measured in ρ_s while the parallel ($\|$) space coordinate is measured in $c_s\tau_+$. The parameters $\delta = m_-\tau_+/m_+\tau_-$, $\kappa = d\ln n_0/dx$, $\eta = 2/R$ characterize the ratio of ion to electron mobility, the density gradient, and the curvature of the magnetic field, respectively; $\{a,b\} = (\partial_x a)(\partial_y b) - (\partial_x b)(\partial_y a)$ is the Poisson bracket.

Some remarks are appropriate at this stage. First, the three-fields model (15) - (17) leads for perturbations to a cubic dispersion relation. From this dispersion relation we can determine the critical parameters for instability. [The existence of drift-like instabilities was already discussed by Simon and collaborators.] For example, we can determine a critical magnetic field strength B_c above which the system becomes unstable. Secondly, if we neglect (a) $\delta\nabla_\perp^2(\varphi - p)$ with respect to $\delta^{-1}\nabla_\parallel^2(\varphi - p)$, (b) $\eta \approx 0$, and (c) the parallel ion velocity u, we arrive at

$$\partial_t p - \kappa\partial_y\varphi + \delta^{-1}\nabla_\parallel^2(\varphi - p) + \{\varphi, p\} = 0\,, \tag{18}$$

$$(1 + \partial_t)\,\nabla_\perp^2\varphi + \delta^{-1}\nabla_\parallel^2(\varphi - p) + \{\varphi, \nabla_\perp^2\varphi\} = 0\,, \tag{19}$$

which agree in form with the Hasegawa-Wakatani equations for fully-ionized plasmas [except for a different hyper-ion-viscosity there].

Note that from the second equation of this coupled set we obtain for $\delta \to 0$ [and negligible damping] $p_k \approx \varphi_k + \delta\left(k_\perp^2/k_\parallel^2\right) d_t\varphi_k$, i.e. the deviation from the Boltzmann distribution mentioned already before. If we subtract (18) from (19) and iterate for $\delta \to 0$, we obtain to lowest order the Hasegawa-Mima equation

$$\partial_t\left(1 - \nabla_\perp^2\right)\varphi - \kappa\partial_y\varphi - \{\varphi, \nabla^2\varphi\} - \nabla^2\varphi = 0\,. \tag{20}$$

Thirdly, in the opposite limit (a) $\delta\nabla_\perp^2(\varphi - p) \gg \delta^{-1}\nabla_\parallel^2(\varphi - p)$ and (b) $u \equiv 0$ the system (15) - (17) reduces to

$$\partial_t p - \kappa\partial_y\varphi + \delta\nabla_\perp^2(\varphi - p) + \eta\partial_y(\varphi - p) + \{\varphi, p\} = 0\,, \tag{21}$$

$$(1 + \partial_t)\,\nabla_\perp^2\varphi + \delta\nabla_\perp^2(\varphi - p) - \eta\partial_y p + \{\varphi, \nabla_\perp^2\varphi\} = 0\,. \tag{22}$$

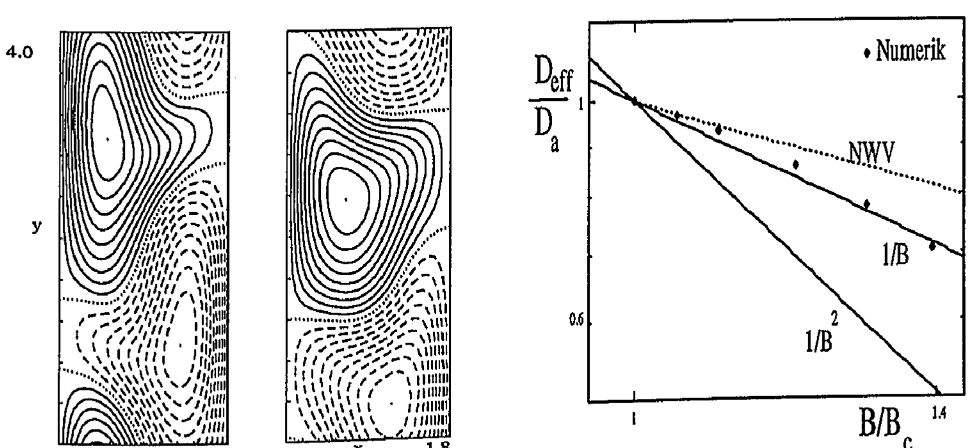

Fig. 2. *(a) Contour plots of φ and p at time t as determined from Eqs. (21) and (22) for $B = 1.8B_c$. (b) Effective diffusion coefficient D_{eff} as a function of B. The dots mark results from numerical experiments, NWV is the analytical result based on the Newell-Whitehead procedure[8].*

Now the interesting physical conclusions. Within the weakly unstable region, one mode dominates the nonlinear behavior and its saturation can be calculated via a Newell-Whitehead procedure. As usual, the corresponding mode amplitude is determined by a generalized Ginzburg-Landau equation, or coupled Ginzburg-Landau equations. The latter can be analyzed by meanwhile standard methods. The phase shift between p_k and φ_k causes an anomalous transport which in the unstable region ($B > B_c$) leads to an effective diffusion coefficient $D_{eff} \sim B^{-1}$, while for $B < B_c$ the ambipolar diffusion is $D_a \sim B^{-2}$. These analytical predictions have been confirmed by Th. Eickermann in his PhD-thesis. Fig. 2.a shows a typical contour plot of φ and p for $B = 1.8B_c$; these results have been obtained from a numerical simulation. In Fig. 2.b the B-dependence of the diffusion coefficient is shown. A cross-over from the B^{-2}-dependence to the B^{-1}-dependence is obvious. In addition, it was found that via quasi-periodicity the system can go to a time-chaotic state. We recognize here a behaviour similar to that mentioned during thermal convection. By analytical methods, simple models could be constructed to mimic that dynamical evolution. The signature of chaos is a back-change to $D_{eff} \sim B^{-2}$. It appears in Fig. 2.b for larger values of B/B_c. More details can be found in a forthcoming publication[9].

IV. ANOMALOUS ABSORPTION AND WAVE COLLAPSE

We now turn to another important process in plasmas which may become anomalous due to nonlinear effects: wave propagation and wave absorption. A classical problem in plasma physics is wave propagation in inhomogeneous plasmas up to a classical turning point. Let us consider the geometry for p-polarization as shown in Fig. 3.a. When the electric field

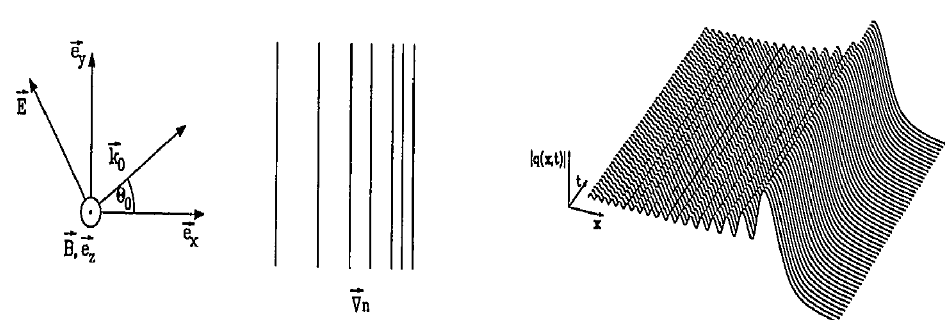

Fig. 3. *(a) Sketch of the geometry for p-polarized wave incidence onto an inhomogeneous plasma. (b) Resonantly excited electrostatic electric field (magnitude of amplitude) at the critical density*[10].

vector lies in the plane built by the wave vector $\vec{k}_0$ and the density gradient $|dn_0/dx|\hat{x}$, the turning point x_r depends on the angle of incidence,

$$\omega_{pe}(x_r) = \omega_0 \cos(\theta_0) , \tag{23}$$

where ω_0 is the frequency of the wave and ω_{pe} is the density- and thereby space-dependent electron plasma frequency. However, at the critical density, defined by $\omega_{pe} = \omega_0$ which gives the critical layer at x_c with $\omega_{pe}(x_c) \approx \omega_0$, an electrostatic component can be driven resonantly; see Fig. 3.b. The driving amplitude is a tunneling B-component of the electromagnetic wave. Several authors have investigated the linear problem, and recently also the nonlinear situation could be solved.

It has been shown that the normalized amplitude q of the electrostatic field $\vec{E} \sim q\exp(i\omega_{pe}t)\hat{x}$ obeys the equation

$$i\partial_t q + \partial_x^2 q + p|q|^2 q - xq = 1 \tag{24}$$

within the quasi-static approximation. Here the electric field is in units of $\sin\theta_0 B_0/\delta$, $\delta := \left(\sqrt{3}\lambda_{De}/L\right)^{1/3}$, and $p = (\sin\theta_0 B_0/\delta)^2$. Note that because of this normalization the driver (due to B_0) has the value one. Equation (24) is a driven and (convectively) damped cubic nonlinear Schrödinger equation in the one-dimensional geometry. The coefficient p determines the degree of nonlinearity responsible for "anomalous" effects. For small p the situation is as expected from linear theory. A standing (electrostatic) wave pattern appears, which for $p = 0.3$ is shown in Fig. 3.b. Of course it is not exactly of the Airy-function-form, but only slightly modified due to nonlinearities. Collisional and/or Landau damping effects will cause a net transfer of wave energy to particles which has been calculated in standard literature. Now the new effect: when p is larger than a critical value $p_{cr} \approx 0.55$, the situation changes drastically. The big electrostatic hump at the critical density (see Fig. 3.b) moves away from the critical layer, down the density gradient, in form of solitary waves. The latter are continuously generated at the critical layer, but convectively moving away from the resonance region. The motion is accelerated down the density gradient. The numerical results and an analytical theory have been published recently where more details can be found[10]. The problem turns out to be of fundamental importance in basic research since it is one of the first examples of a systematic nonlinear dynamics with solitary waves. Here we want to mention only the implications for wave absorption, i.e. the source term in the (particle) energy balance equation. Since the wave pattern is moving away from the linearly expected absorption region, we shall find in the nonlinear regime a less peaked absorption profile. However, there is another competing, essentially nonlinear, effect which we have to take into account. The model equation (24) is one-dimensional in space. However there is no doubt that transverse effects are present. That means that in Eq. (24) we have to replace $\partial_x^2 \to \nabla^2$. Now another (nonlinear) time-scale enters the picture. Let us exemplify it for the undriven and homogeneous case

$$i\partial_t q + \nabla^2 q + |q|^2 q = 0\,, \tag{25}$$

i.e. the standard cubic nonlinear Schrödinger equation. (Note that now the parameter p becomes irrelevant since we can rescale the variable q.) We compare the nonlinear with the dispersive terms. Since Eq. (25) conserves $\int |q|^2 dxdydz$ (for the three-dimensional case) we can estimate for a balanced solution $|q|^2 \sim L^{-3}$ and $\nabla^2 \sim L^{-2}$. Thus for such perturbations which will shrink the solution ($L \to 0$) the nonlinearity dominates and the solution becomes unstable. This phenomenon is known as collapse[11], and it can lead to a very effective dissipation mechanism. When the interaction with the particles is taken into account, resonances with the particle velocities can occur. Let us give a simple (and oversimplified) picture for such a process. We know that wave particle interaction becomes significant when $k\lambda_D \sim \mathcal{O}(1)$. Estimating $k \sim L^{-1}$, we can transfer via the collapse wave energy from small k to large $k \sim \lambda_D^{-1}$ where the strong dissipation sets in; Fig. 4 reviews schematically the observed phenomena. When monitoring in a particle simulation

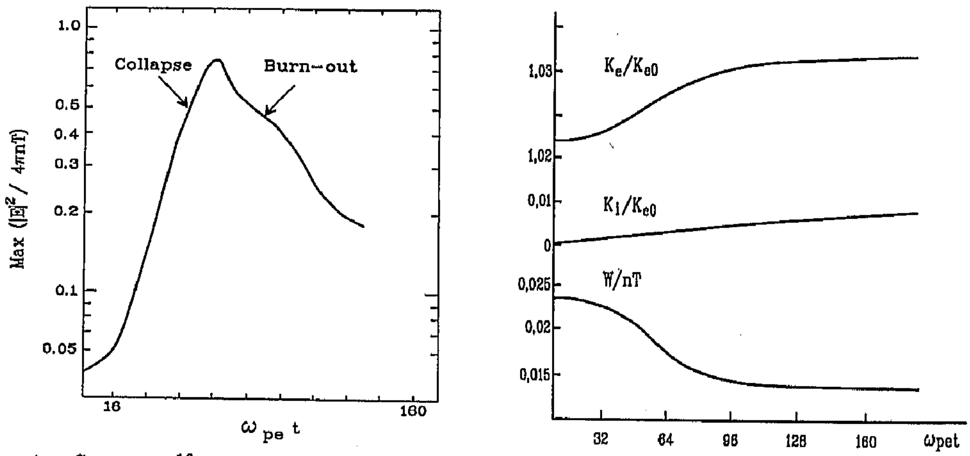

Fig. 4. *Summary*[12] *of the basic collapse processes: (a) Increase of the maximum wave amplitude during the collapse phase, and subsequent burn-out due to wave-particle interaction. (b) change of electron, ion, and wave energy density due to heating by a collapsing caviton.*

"experiment" the wave energy density one can recognize the effectiveness of such a nonlinear (anomalous) process; see Fig. 4.b.

In concluding this section we remind the reader that heating of plasmas by waves occurs besides in laboratory plasmas also in astrophysical plasmas, where many of the phenomena are still not yet very well understood. The nonlinear process mentioned above could contribute to a better understanding of the observed anomalous behavior.

V. NONLINEAR WAVE PROPAGATION IN RANDOM SYSTEMS

For wave propagation, nonlinearities have the tendency to balance the dispersive broadening. In some situations this could be an overbalance with resulting collapse as we discussed above. However, in general, nonlinearities help to avoid strong linear dispersive broadening. The fascinating aspect of this well-known effect is that the balance does not only occur for a stationary solution, but also survives strong perturbations and even interactions in many cases. The prominent representative of these highly nonlinear phenomena is the *soliton.*

For example, one-dimensional plasma oscillations can be described by the cubic nonlinear Schrödinger equation (25) with $\nabla^2 \equiv \partial_x^2$. That equation is completely integrable and can be considered as a paradigm for solitons. The soliton theory has been confirmed by several experiments; and nonlinear effects in pulse propagation are well accepted.

Here we would like to introduce another effect of nonlinearity on wave energy transport which is not so well known but to our opinion equally important. The assumption of a quiescent medium (plasma) for wave propagation reflects a strong idealization which will not be true in reality. Therefore the wave propagation in random media has to be considered in detail. Let us come back to our paradigm (25) which we now write (in one space dimension) with a random potential U in the form

$$i\partial_t\varphi_n + D\left(\varphi_{n+1} - 2\varphi_n + \varphi_{n-1}\right) + |\varphi_n|^2\varphi_n + U_n\varphi_n = 0 \, . \tag{26}$$

Here, we have discretized the cubic nonlinear Schrödinger equation (index n replaces the continuous variable x in the previous cases). The reason is that at discrete positions $n = 1, 2, ..., N$ we place irregularities being caused by external or internal sources. Of course, this model is for plasma physics applications still at its infancy, but let us nevertheless anticipate it for the moment since some very interesting ideas can be made plausible from here.

In a famous paper, Anderson[13] showed that randomness causes localization in the *linear* Schrödinger equation with a random potential U_n. For our application this means that the transmission coefficient t_N at $n = N$ falls off exponentially with length Na of the system,

$$t_N \sim \exp\left(-Na/L_{loc}\right) \, , \tag{27}$$

where L_{loc} is the localization length. For $t_N = j_N/j_0$, where $j_n := i\sqrt{D}\left(\varphi_n^*\varphi_{n-1} - \varphi_n\varphi_{n-1}^*\right)$, we calculate the ratio of the at $n = 0$ incoming to the at $n = N$ transmitted current densities j. An incident wave is scattered back from irregularities and fluctuations, and that results in the diminishing of the transmission. In long and random systems this process is quite effective; however, because of space limitations we cannot work out here the quantitative predictions with plasma physics applications.

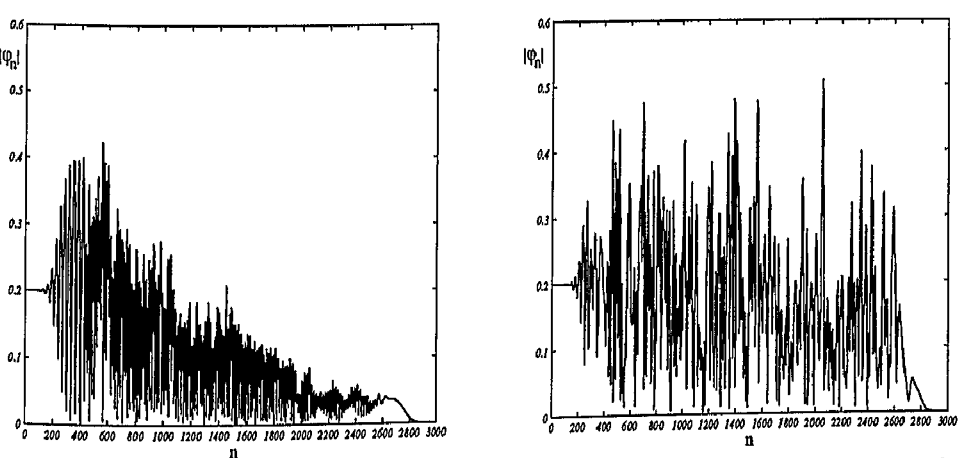

Fig. 5. *Two typical solutions*[14] *of Eq. (26). A wave is incident on a random system from left. (a) Result when the nonlinear term is neglected; the transmission is drastically reduced. (b) With nonlinearity most of the wave energy is transmitted to the right. The transporting modes are solitons. Note that in both cases the waves are artificially damped at the very right end because of numerical reasons.*

Coherent background fluctuations can further increase the "backscattering" due to stimulated "parametric" processes. However, nonlinear wave packets can easier overcome the "negative" effects from irregularities and fluctuations. Let us demonstrate this by the results of a numerical simulation[14], which may apply for plasma waves. We compare the predictions from linear theory with those of the full nonlinear model (26). Typical runs are shown in Fig. 5. When the linear system is investigated for various lengths, formula (27) is confirmed to a good degree of accuracy. The full nonlinear system does not show Anderson localization. Nonlinear pulses can overcome the disturbances, and they travel to the right approximately without changes in shape. We can interpret these fundamental results by a simple picture which is borrowed from transport in nonphysical systems: If we follow the motion of a single waggon under the influence of fluctuating restoring forces, one event might be strong enough to stop the forward motion. But if we couple several waggons to a train, the latter may easier propagate in a random environment because of its big inertia. The waggon corresponds to a single linear Fourier mode, and the train has its counterpart in the soliton which is a nonlinear composition of several Fourier modes.

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