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Sequence space representations for zero-solutions of convolution equations on ultradifferentiable functions of Roumieu type

by

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Abstract. Let $\mathscr{E}_{\{\omega\}}(R)$ denote the space of all ω-ultradifferentiable functions of Roumieu type on R and let T_μ be a convolution operator on $\mathscr{E}_{\{\omega\}}(R)$ which admits a fundamental solution in $\mathscr{D}_{\{\omega\}}(R)'$. We prove that the space $\ker T_\mu$ of all zero-solutions of T_μ has an absolute basis of exponential solutions, hence it is isomorphic to a Köthe sequence space $\lambda(P(\mu))$ if it is infinite-dimensional. The Köthe matrix $P(\mu)$ is computed explicitly in terms of ω and the zeros of the Fourier–Laplace transform of μ. This result is a consequence of a sequence space representation for quotients of certain weighted (LF)-algebras of entire functions modulo slowly decreasing localized ideals.

Classes of non-quasianalytic functions, like the Gevrey classes, were used by Roumieu [20] to extend the notion of a distribution. Then Chou [7] studied convolution equations in these classes, using ideas of Ehrenpreis [9] and Fourier analysis. Recently Braun, Meise and Taylor [5] combined the approaches of Roumieu [20] and Beurling–Björck [2], [4] to introduce classes $\mathscr{E}_{\{\omega\}}(R^N)$ of non-quasianalytic functions which are particularly adapted to the application of Fourier analysis.

In the present paper we show that for each $\mu \in \mathscr{E}_{\{\omega\}}(R)'$ which admits a fundamental solution $\ker T_\mu$, the space of zero-solutions of the convolution operator

$$T_\mu\colon \mathscr{E}_{\{\omega\}}(R) \to \mathscr{E}_{\{\omega\}}(R), \quad T_\mu(f)\colon x \mapsto \langle \mu_y, f(x-y)\rangle,$$

has an absolute Schauder basis consisting of exponential solutions. Moreover, we show that for $\dim_C \ker T_\mu = \infty$ we have a linear topological isomorphism between $\ker T_\mu$ and the sequence space $\lambda(\alpha, \beta)$ which is defined in the following way:

$$\lambda(\alpha, \beta) = \{x \in C^N \mid \pi_{k,y}(x) := \sum_{j=1}^{\infty} |x_j| y_j e^{k\alpha_j} < \infty \quad \text{for each } k \in N \text{ and each } y \in A_\beta\},$$

with

$$A_\beta := \{y \in \mathbf{R}_+^N \mid \lim_{j\to\infty} y_j \exp(-\beta_j/m) = 0 \text{ for each } m \in N\},$$

where $\alpha = (|\mathrm{Im}\, a_j|)_{j\in N}$, $\beta = (\omega(a_j))_{j\in N}$ for an enumeration $(a_j)_{j\in N}$ of the zeros of the Fourier–Laplace transform $\hat{\mu}$ of μ, counted with multiplicities. Note that this representation applies in particular to the Gevrey classes $\mathcal{E}_{\{\omega_s\}}(\mathbf{R}) = \mathcal{E}^{\{(p!)^s\}}(\mathbf{R})$, $s > 1$.

This result is in fact a special case of a theorem on sequence space representations of quotients of weighted nuclear (LF)-algebras $A_{q,r}$ of entire functions modulo closed ideals generated by finitely many slowly decreasing functions. Its proof is based on ideas and results of Berenstein and Taylor [1], Meise [14], Meise and Taylor [16], and Taylor [23]. To get the representation of $\ker T_\mu$ stated above, we use a result of Braun, Meise and Vogt [6] to show that $\mu \in \mathcal{E}_{\{\omega\}}(\mathbf{R})'$ admits a fundamental solution in $\mathcal{D}_{\{\omega\}}(\mathbf{R})'$ if and only if $\hat{\mu}$ is slowly decreasing in $A_{|\mathrm{Im}\, z|,\omega}$.

The sequence space representation for $\ker T_\mu$ derived in the present paper is used in Braun, Meise and Vogt [6] to characterize the surjectivity of convolution operators T_μ: $\mathcal{E}^{\{(p!)^s\}}(\mathbf{R}) \to \mathcal{E}^{\{(p!)^s\}}(\mathbf{R})$ on the Gevrey classes $\mathcal{E}^{\{(p!)^s\}}(\mathbf{R})$, $s > 1$.

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1. Weight functions, weighted algebras and some sequence spaces. In this preliminary section we introduce the basic notation and we state some elementary results which will be needed in the sequel.

1.1. DEFINITION. A function p: $\mathbf{C} \to [0, \infty[$ is called a *weight function* if it has the following properties:

(1) p is continuous and subharmonic.
(2) $\log(1+|z|^2) = o(p(z))$ as $|z|$ tends to ∞.
(3) There exists $d > 0$ such that for all $z \in \mathbf{C}$

$$\sup_{|w|\le 1} p(z+w) \le d\left(1 + \inf_{|w|\le 1} p(z+w)\right).$$

A weight function p is called *radial* if $p(z) = p(|z|)$ for all $z \in \mathbf{C}$.

For a nonempty open set $\Omega \subset \mathbf{C}$ we denote by $A(\Omega)$ the algebra of all holomorphic functions on Ω.

1.2. DEFINITION. Let r be a radial weight function and let q: $\mathbf{R} \to [0, \infty[$ be a convex even function which strictly increases on $[0, \infty[$ and satisfies

$$\lim_{t\to\infty} r(t)/q(t) = 0, \quad \limsup_{t\to\infty} q(t+1)/q(t) < \infty.$$

Then we define

$$A_{q,r} := \{f \in A(\mathbf{C}) \mid \text{there exists } k \in N \text{ such that for each } m \in N$$
$$\|f\|_{k,m} := \sup_{z\in\mathbf{C}} |f(z)| \exp(-kq(\mathrm{Im}\, z) - r(z)/m) < \infty\}.$$

Obviously we have

$$A_{q,r} = \operatorname*{ind}_{k\to} \operatorname*{proj}_{\leftarrow m} A(k, m), \quad \text{where} \quad A(k, m) := \{f \in A(\mathbf{C}) \mid \|f\|_{k,m} < \infty\}.$$

We endow $A_{q,r}$ with this natural (LF)-space topology.

Using standard arguments, one can show the following:

1.3. PROPOSITION. *For r and q as in 1.2, the following holds:*

(a) *$A_{q,r}$ is a nuclear (LF)-algebra with continuous multiplication.*
(b) *$A_{q,r} = \mathrm{ind}_{k\to} \mathrm{proj}_{\leftarrow m} W(k, m)$, where*

$$W_{k,m} := \{f \in A(\mathbf{C}) \mid |f|_{k,m}^2 := \int_{\mathbf{C}} [|f(z)| \exp(-kq(\mathrm{Im}\, z) - r(z)/m)]^2 \, d\lambda(z) < \infty\}$$

and where λ denotes the Lebesgue measure on $\mathbf{C} = \mathbf{R}^2$.

Next we introduce some sequence spaces which we shall use in the subsequent sections.

1.4. DEFINITION. Let $\alpha = (\alpha_j)_{j\in N}$ and $\beta = (\beta_j)_{j\in N}$ be sequences of nonnegative real numbers and let $E = (E_j, \|\ \|_j)_{j\in N}$ be a sequence of Banach spaces. For $k \in N$ and $m \in N$ we put

$$\lambda(k, m, E) := \{x = (x_j)_{j\in N} \in \prod_{j\in N} E_j \mid \|x\|_{k,m} := \sum_{j=1}^\infty \|x_j\|_j \exp(k\alpha_j + \beta_j/m) < \infty\},$$

$$K(k, m, E) := \{x = (x_j)_{j\in N} \in \prod_{j\in N} E_j \mid |||x|||_{k,m} := \sup_{j\in N} \|x_j\|_j \exp(-k\alpha_j - \beta_j/m) < \infty\},$$

and we define

$$\lambda(\alpha, \beta, E) := \operatorname*{proj}_{\leftarrow k} \operatorname*{ind}_{m\to} \lambda(k, m, E), \quad K(\alpha, \beta, E) := \operatorname*{ind}_{k\to} \operatorname*{proj}_{\leftarrow m} K(k, m, E).$$

If $E = (\mathbf{C}, |\ |)_{j\in N}$, then we just omit the E in the notation introduced above.

1.5. PROPOSITION. *Let α, β and E be as in 1.4, assume $\dim E_j < \infty$ for all $j \in N$ and put*

$$A_\beta := \{y \in \mathbf{R}_+^N \mid \sup_{j\in N} y_j e^{-\beta_j/m} < \infty \text{ for each } m \in N\}.$$

Then we have

$$\lambda(\alpha, \beta, E) = \{x \in \prod_{j\in N} E_j \mid \pi_{k,y}(x) := \sum_{j=1}^{\infty} \|x_j\|_j y_j e^{k\alpha_j} < \infty \quad \text{for all } k\in N,\ y\in \Lambda_\beta\}$$

and $\{\pi_{k,y} \mid k\in N,\ y\in\Lambda_\beta\}$ *is a fundamental system of seminorms for* $\lambda(\alpha, \beta, E)$.

Proof. We define $E' := (E_j', \|\ \|_j')_{j\in N}$ and we fix $k\in N$. Then we put

$$F_k := \{z\in \prod_{j\in N} E_j' \mid \lim_{j\to\infty} \|z_j\|_j' \exp(-k\alpha_j - \beta_j/m) = 0 \text{ for each } m\in N\}$$

and endow F_k with the Fréchet space topology induced by the norm-system $(\|\ \|_m)_{m\in N}$, where

$$\|z\|_m := \sup_{j\in N} \|z_j\|_j' \exp(-k\alpha_j - \beta_j/m).$$

Then it follows easily as in Bierstedt, Meise and Summers [3], Thm. 3.4, that F_k is a quasi-normable Fréchet space. This implies that $(F_k)_b'$ is bornological. Identifying E_j'' with E_j, this gives

$$(F_k)_b' = \operatorname{ind}_{m\to} \lambda(k, m, E)$$

with the canonical bilinear form $\langle x, z\rangle = \sum_{j=1}^{\infty} \langle x_j, z_j\rangle_j$.

On the other hand, we find as in Bierstedt, Meise and Summers [3], Thm. 2.7, that with the same bilinear form we have

$$(F_k)_b' = \{x\in \prod_{j\in N} E_j \mid \pi_{k,y}(x) < \infty \text{ for each } y\in\Lambda_\beta\}.$$

Hence the result follows from well-known properties of projective limits.

1.6. Proposition. *For* α, β *and* E *as in* 1.4 *assume* $\lim_{j\to\infty}\beta_j = \infty$ *and* $\dim E_j < \infty$ *for all* $j\in N$. *Then:*

(1) $\lambda(\alpha, \beta, E)$ *is a complete Schwartz space.*

(2) $\lambda(\alpha, \beta, E)_b'$ *can be identified with* $K(\alpha, \beta, E')$ *under the canonical bilinear form* $\langle x, y\rangle := \sum_{j=1}^{\infty} \langle x_j, y_j\rangle_j$, *where* $E' = (E_j, \|\ \|_j')_{j\in N}$.

(3) *A subset* M *of* $K(\alpha, \beta, E')$ *is equicontinuous with respect to the identification in* (2) *iff there exists* $k\in N$ *such that for each* $m\in N$

$$\sup_{y\in M}\sup_{j\in N} \|y_j\|_j' \exp(-k\alpha_j - \beta_j/m) < \infty.$$

Proof. (1) It is easy to check that the present hypotheses imply that for each $k\in N$ the space $\operatorname{ind}_{m\to}\lambda(k, m, E)$ is a (DFS)-space. This implies that $\operatorname{ind}_{m\to}\lambda(k, m, E)$ is a complete Schwartz space. Hence $\lambda(\alpha, \beta, E)$ has this property, too.

(2) Fix $k\in N$ and let M be an arbitrary bounded subset of $\operatorname{proj}_{\leftarrow m} K(k, m, E')$. Then we have for each $m\in N$

$$\sup_{y\in M}\sup_{j\in N} \|y_j\|_j' \exp(-k\alpha_j - \beta_j/m) < \infty.$$

This implies that

$$z := (\sup_{y\in M} \|y_j\|_j' \exp(-k\alpha_j))_{j\in N}$$

is in Λ_β. Hence we get for each $y\in M$ and each $x\in\lambda(\alpha, \beta, E)$

$$\Big|\sum_{j=1}^{\infty} \langle x_j, y_j\rangle_j\Big| \leqslant \sum_{j=1}^{\infty} \|x_j\|_j \|y_j\|_j' = \sum_{j=1}^{\infty} \|x_j\|_j e^{k\alpha_j} \|y_j\|_j' e^{-k\alpha_j} \leqslant \pi_{k,z}(x).$$

By Proposition 1.5, this implies that for each $y\in K(\alpha, \beta, E)'$ the functional $\Phi(y)\colon x\mapsto \sum_{j=1}^{\infty}\langle x_j, y_j\rangle_j$ is in $\lambda(\alpha, \beta, E)'$. Moreover, it follows that $\Phi\colon K(\alpha, \beta, E')\to\lambda(\alpha, \beta, E)_b'$ is continuous.

To show that Φ is also surjective, let $T\in\lambda(\alpha, \beta, E)'$ be given. By Proposition 1.5 there exist $k\in N$, $z\in\Lambda_\beta$ and $C > 0$ such that

$$(4) \qquad |T(x)| \leqslant C\pi_{k,z}(x) \quad \text{for all } x\in\lambda(\alpha, \beta, E).$$

Now note that E_j can be identified canonically with a linear subspace of $\lambda(\alpha, \beta, E)$. If y_j denotes the restriction of T to this subspace, then (4) implies

$$(5) \qquad \|y_j\|_j' \leqslant C z_j e^{k\alpha_j} \quad \text{for each } j\in N.$$

From this estimate it is immediate that $y := (y_j)_{j\in N}$ is in $K(\alpha, \beta, E')$. Moreover, it follows easily that $\Phi(y) = T$. Hence Φ is a continuous linear bijection. Now observe that $K(\alpha, \beta, E')$ is an (LF)-space and that $\lambda(\alpha, \beta, E)_b'$ is ultrabornological by (1) and Schwartz [22], p. 43. Hence Φ is a linear topological isomorphism by the open mapping theorem.

(3) If $M\subset\lambda(\alpha, \beta, E)'$ is equicontinuous, then the estimate (4) holds for all $T\in M$, where k, z and C only depend on M. Then (5) implies (3).

1.7. Lemma. *Let* α, β *and* E *be as in* 1.4 *and assume* $1\leqslant n_j := \dim E_j < \infty$ *for all* $j\in E$. *Then the estimate*

(*) *There is* $l\in N$ *such that for each* $m\in N$:

$$\sup_{j\in N} n_j \exp(-l\alpha_j - \beta_j/m) < \infty$$

implies that

$$\lambda(\alpha, \beta, E) \simeq \lambda(\gamma, \delta), \quad K(\alpha, \beta, E) \simeq K(\gamma, \delta),$$

where the sequence γ (resp. δ) is obtained from α (resp. β) by repeating α_j (resp. β_j) n_j times.

Proof. Obviously we can identify $\lambda(\gamma, \delta)$ with

$$\{\xi = ((\xi_{j,\nu})_{1 \le \nu \le n_j})_{j \in N} \mid \sum_{j=1}^{\infty} \sum_{\nu=1}^{n_j} |\xi_{j,\nu}| y_j e^{k\alpha_j} < \infty \text{ for all } k \in N,\ y \in \Lambda_\beta\}.$$

Next we choose for each $j \in N$ an Auerbach basis $\{e_{j,\nu} \mid 1 \le \nu \le n_j\}$ of E_j with coefficient functionals $\{f_{j,\nu} \mid 1 \le \nu \le n_j\}$ (see e.g. Jarchow [10], p. 291). Using the identification mentioned above, we define

$$A\colon \lambda(\gamma, \delta) \to \lambda(\alpha, \beta, E) \quad \text{by} \quad A(\xi) = \Big(\sum_{\nu=1}^{n_j} \xi_{j,\nu} e_{j,\nu}\Big)_{j \in N}.$$

For each $k \in N$, each $y \in \Lambda_\beta$ and each $\xi \in \lambda(\gamma, \delta)$ we have

$$\sum_{j=1}^{\infty} \Big\| \sum_{\nu=1}^{n_j} \xi_{j,\nu} e_{j,\nu} \Big\| y_j e^{k\alpha_j} \le \sum_{j=1}^{\infty} \sum_{\nu=1}^{n_j} |\xi_{j,\nu}| y_j e^{k\alpha_j},$$

which proves that A is a continuous linear map.

To show that A is a topological isomorphism, fix $x \in \lambda(\alpha, \beta, E)$ and look at the sequence $((\langle f_{j,\nu}, x_j \rangle)_{1 \le \nu \le n_j})_{j \in N}$. From (*) we see that $z := (n_j e^{-l\alpha_j})_{j \in N}$ is in Λ_β. Now let $k \in N$ and $y \in \Lambda_\beta$ be given. Then it is easy to check that $(z_j y_j)_{j \in N}$ is in Λ_β, too. Hence the estimate

$$\sum_{j=1}^{\infty} \sum_{\nu=1}^{n_j} |\langle f_{j,\nu}, x_j \rangle| y_j e^{k\alpha_j} \le \sum_{j=1}^{\infty} n_j \|x_j\|_j y_j e^{k\alpha_j}$$
$$= \sum_{j=1}^{\infty} \|x_j\|_j n_j e^{-l\alpha_j} y_j e^{(l+k)\alpha_j} = \sum_{j=1}^{\infty} \|x_j\|_j z_j y_j e^{(l+k)\alpha_j}$$

shows that the map

$$B\colon \lambda(\alpha, \beta, E) \to \lambda(\gamma, \delta), \quad B(x) := ((\langle f_{j,\nu}, x_j \rangle)_{1 \le \nu \le n_j})_{j \in N},$$

is linear and continuous. It is easy to check that $B \circ A = \mathrm{id}_{\lambda(\gamma,\delta)}$ and $A \circ B = \mathrm{id}_{\lambda(\alpha,\beta,E)}$. Hence we have proved $\lambda(\alpha, \beta, E) \simeq \lambda(\gamma, \delta)$. By similar arguments one can show $K(\alpha, \beta, E) \simeq K(\gamma, \delta)$.

1.8. Remark. Let α and $\tilde{\alpha}$ be sequences of nonnegative real numbers. They are called *equivalent* if there exists $A \ge 1$ with

$$\alpha_j \le A(1 + \tilde{\alpha}_j) \quad \text{and} \quad \tilde{\alpha}_j \le A(1 + \alpha_j) \quad \text{for all } j \in N.$$

If α and $\tilde{\alpha}$ and also β and $\tilde{\beta}$ are equivalent then it follows easily that for each sequence $E = (E_j, \|\ \|_j)$ of Banach spaces we have

$$\lambda(\alpha, \beta, E) = \lambda(\tilde{\alpha}, \tilde{\beta}, E), \quad K(\alpha, \beta, E) = K(\tilde{\alpha}, \tilde{\beta}, E).$$

1.9. Examples. (1) For α and β as in 1.4 assume $\lim_{j\to\infty} \alpha_j/\beta_j = 0$. Then it is easily checked that

$$\lambda(\alpha, \beta) = \{x \in C^N \mid \text{there exists } m \in N\colon \sum_{j=1}^{\infty} |x_j| \exp(\beta_j/m) < \infty\},$$
$$K(\alpha, \beta) = \{y \in C^N \mid \text{for all } m \in N\colon \sup_{j \in N} |y_j| \exp(-\beta_j/m) < \infty\}.$$

Hence $\lambda(\alpha, \beta)$ is a (DF)-space, while $K(\alpha, \beta)$ is a Fréchet space.

(2) For α and β as in 1.4 assume $\liminf_{j\to\infty} \alpha_j/\beta_j > 0$. Then it is easily checked that

$$\lambda(\alpha, \beta) = \{x \in C^N \mid \text{for each } k \in N\colon \sum_{j=1}^{\infty} |x_j| \exp(k\alpha_j) < \infty\},$$
$$K(\alpha, \beta) = \{y \in C^N \mid \text{there exists } k \in N\colon \sup_{j \in N} |y_j| \exp(-k\alpha_j) < \infty\}.$$

Hence $\lambda(\alpha, \beta)$ is a Fréchet space, while $K(\alpha, \beta)$ is a (DF)-space.

1.10. Remark. In general, $\lambda(\alpha, \beta)$ is neither a (DF)- nor an (F)-space. In fact, there are examples of nuclear spaces $\lambda(\alpha, \beta)$ which are neither barrelled nor bornological and for which $\lambda(\alpha, \beta)'_b = K(\alpha, \beta)$ is not sequentially complete. Such examples can be obtained as follows: Choose increasing sequences γ and δ of positive numbers with $\lim_{j\to\infty} (\log j)/\gamma_j = 0$ and $\sup_{j \in N} (\log j)/\delta_j < \infty$, and put

$$\Lambda_1(\gamma) := \{x \in C^N \mid \|x\|_k := \sum_{j=1}^{\infty} |x_j| \exp(-\gamma_j/k) < \infty \text{ for each } k \in N\},$$

$$\Lambda_\infty(\delta) := \{x \in C^N \mid \|x\|_k := \sum_{j=1}^{\infty} |x_j| \exp(k\delta_j) < \infty \text{ for each } k \in N\}.$$

Then $\Lambda_1(\gamma)$ and $\Lambda_\infty(\delta)$ are nuclear Fréchet spaces. It is easy to check that $L_b(\Lambda_1(\gamma), \Lambda_\infty(\delta)) \simeq \Lambda_1(\gamma)'_b \hat{\otimes}_\pi \Lambda_\infty(\delta) = \lambda(\alpha, \beta)$ for suitable sequences α and β. Hence it follows from Krone and Vogt [13], 2.1, by Vogt [24], 4.2, that $\lambda(\alpha, \beta)$ and $K(\alpha, \beta)$ have the properties mentioned above.

For a detailed discussion of the question when $\lambda(\alpha, \beta)$ is barrelled, we refer to Vogt [25] and [26].

2. Sequence space representations of certain quotients of $A_{q,r}$. In this section we fix q and r as in 1.2 and we derive a sequence space representation of $A_{q,r}$ modulo certain closed ideals which are generated by slowly decreasing functions in $A_{q,r}$.

2.1. Definition. $F = (F_1, \ldots, F_N)$ in $(A_{q,r})^N$ is called *slowly decreasing* if there exist a weight function s with $s = o(r)$ and positive numbers L, M, ε, C and D such that with $p\colon z \mapsto q(\operatorname{Im} z) + s(z)$ the following holds:

(1) $\sup_{z\in C}|F_j(z)|\exp(-Lp(z))\leqslant M$ for $1\leqslant j\leqslant N$.
(2) For each component S of the set

$$S_p(F,\varepsilon,C):=\{z\in C\,|\,|F(z)|:=\Big(\sum_{j=1}^{N}|F_j(z)|^2\Big)^{1/2}<\varepsilon e^{-Cp(z)}\}$$

we have

$$\sup_{z\in S}p(z)\leqslant D\big(1+\inf_{z\in S}p(z)\big),\quad \sup_{z\in S}r(z)\leqslant D\big(1+\inf_{z\in S}r(z)\big).$$

Remark. Definition 2.1 can be regarded as an extension of the slowly decreasing definition of Berenstein and Taylor [1], p. 130, and of the one which was used implicitly in Meise and Taylor [16]. It looks somewhat artificial and as if it were invented just to make the proofs of this section work. However, in Proposition 3.5 we show that for convolution operators on certain spaces of ultradifferentiable functions, this condition has an interesting characterization.

2.2. Definition. Let $F=(F_1,\ldots,F_N)\in(A_{q,r})^N$ be given.

(a) By $I(F)$ we denote the ideal in $A_{q,r}$ which is algebraically generated by $F_1,\ldots,F_N$.

(b) By $I_{loc}(F)$ we denote the set

$$I_{loc}(F):=\{f\in A_{q,r}\,|\,[f]_a\in I_a(F)\text{ for each }a\in C\},$$

where $[f]_a$ denotes the germ of f at a and where $I_a(F)$ denotes the ideal generated by $[F_1]_a,\ldots,[F_N]_a$ in the algebra $\mathcal{O}_a$ of all holomorphic germs at a.

It is easy to check that $I_{loc}(F)$ is a closed ideal in $A_{q,r}$ with $I(F)\subset I_{loc}(F)$.

2.3. Proposition. *If $F\in A_{q,r}$ is slowly decreasing then $I(F)=I_{loc}(F)$. Hence $I(F)$ is closed.*

Proof. By the preceding remark it suffices to show $I_{loc}(F)\subset I(F)$. If $g\in I_{loc}(F)$ is given, then g/F is in $A(C)$. To show that g/F is even in $A_{q,r}$, choose s, ε, C and D according to Definition 2.1. Since g is in $A_{q,r}$ we have

(1) There exists $k\in N$ such that for each $m\in N$ there exists C_m with $\|g\|_{k,m}\leqslant C_m$.

Since $s=o(r)$ we get for each $m\in N$ and each $z\notin S_p(F,\varepsilon,C)$

$$(2)\quad \left|\frac{g(z)}{F(z)}\right|\leqslant C_m\exp\Big(kq(\operatorname{Im}z)+\frac{1}{m}r(z)\Big)\cdot\frac{1}{\varepsilon}\exp\big(Cq(\operatorname{Im}z)+Cs(z)\big)$$
$$\leqslant C_m'\exp\Big((k+C)\,q(\operatorname{Im}z)+\frac{2}{m}r(z)\Big).$$

By 2.1 and the maximum principle it follows from (2) that for each component S of $S_p(F,\varepsilon,C)$ we have

$$(13)\quad \left|\frac{g(z)}{F(z)}\right|\leqslant C_m'\exp\Big((k+C)\sup_{z\in S}p(z)+\frac{2}{m}\sup_{z\in S}r(z)\Big)$$
$$\leqslant C_m'\exp\Big((k+C)D\big(1+p(z)\big)+\frac{2D}{m}\big(1+r(z)\big)\Big)$$
$$\leqslant C_m''\exp\Big((k+C)Dq(\operatorname{Im}z)+\frac{3D}{m}r(z)\Big).$$

Now (3) and (2) imply $g/F\in A_{q,r}$. Hence $g=(g/F)F$ belongs to $I(F)$.

2.4. Lemma. *For $k\in N$ let $L^2(P_k)$ denote the space*

$$L^2(P_k):=\{f\in L^2_{loc}(C)\,|\,|f|_m^2:=\int_C[|f(z)|\exp(-kq(\operatorname{Im}z)-r(z)/m)]^2\,d\lambda(z)<\infty\text{ for all }m\in N\}$$

which is a Fréchet space if we endow it with the l.c. topology induced by the norms $(|\ |_m)_{m\in N}$. Then for each bounded subset B of $L^2(P_k)$ there exists a bounded set C in $L^2(P_k)$ such that for each $u\in B$ there exists $v\in C$ with $\bar\partial v=u$ in the distributional sense.

Proof. Fix $k\in N$ and define

$$Y_k:=\{f\in L^2(P_k)\,|\,\bar\partial f\in L^2(P_k)\}$$

endowed with the system $(\|\ \|_m)_{m\in N}$ of norms, where

$$\|f\|_m:=|f|_m+|\bar\partial f|_m,\quad f\in Y_k.$$

Then it is easy to check that Y_k is a Fréchet space and that $\bar\partial\colon Y_k\to L^2(P_k)$ is continuous and linear. Moreover, Taylor [23], Thm. 5, implies that the proof of Meise and Taylor [16], 2.1, also applies in the present situation. Hence $\bar\partial\colon Y_k\to L^2(P_k)$ is surjective, so that we have the exact sequence of Fréchet spaces

$$0\to\ker\bar\partial\xrightarrow{j}Y_k\xrightarrow{\bar\partial}L^2(P_k)\to 0,$$

where j denotes the inclusion. The definition of the norms $\|\ \|_m$, $m\in N$, implies that the topologies of Y_k and $L^2(P_k)$ coincide on $\ker\bar\partial$. Therefore it follows from Wloka [27], 1, § 4,2., that $\ker\bar\partial$ is a Fréchet–Schwartz space. This implies that $\ker\bar\partial$ is quasi-normable. Hence the result follows from Merzon [19], Thm. 2 (see also De Wilde [8]) and the fact that the topology of Y_k is stronger than the one which is induced by $L^2(P_k)$.

2.5. LEMMA. *Let $F = (F_1, \ldots, F_N) \in (A_{q,r})^N$ be slowly decreasing and choose s, L, M, ε, C and D according to 2.1. Let $Q \subset A(S_p(F, \varepsilon, C))$ be given and assume that for some $k \in N$ and some sequence $(D_m)_{m\in N}$ of positive numbers we have*

(1) *For each $\tilde{f} \in Q$, each $m \in N$ and all $z \in S_p(F, \varepsilon, C)$*

$$|\tilde{f}(z)| \leqslant D_m \exp(kq(\operatorname{Im} z) + r(z)/m).$$

Then there exist positive numbers ε_1, C_1 with $0 < \varepsilon_1 < \varepsilon$ and $C_1 > C$ as well as $l \in N$ and a sequence $(E_m)_{m\in N}$ of positive numbers such that for each $\tilde{f} \in Q$ there exist $f \in A(C)$ and $\sigma_j \in A(S_p(F, \varepsilon_1, C_1))$, $1 \leqslant j \leqslant N$, satisfying (2) and (3):

(2) $f(z) = \tilde{f}(z) + \sum_{j=1}^{N} \sigma_j(z) F_j(z)$ *for all $z \in S_p(F, \varepsilon_1, C_1)$.*

(3) *For each $m \in N$ and all $z \in C$*

$$|f(z)| \leqslant E_m \exp(lq(\operatorname{Im} z) + r(z)/m).$$

Proof. Since $p\colon z \mapsto q(\operatorname{Im} z) + s(z)$ is a weight function, the arguments used in the proof of Berenstein and Taylor [1], p. 120, imply the existence of positive numbers ε_1, C_1, A and B and of a function $\chi \in C^\infty(C)$ with the following properties:

$$(4)\qquad \begin{aligned} &0 \leqslant \chi \leqslant 1, \quad \operatorname{Supp}(\chi) \subset S_p(F, \varepsilon, C), \quad \chi | S_p(F, \varepsilon_1, C_1) \equiv 1, \\ &\left|\frac{\partial \chi}{\partial \bar{z}}(z)\right| \leqslant A \exp(Bp(z)) \quad \text{for all } z \in C. \end{aligned}$$

Now fix $\tilde{f} \in Q$ and note that $(\partial/\partial\bar{z})(\chi\tilde{f}) = \tilde{f}\,\partial\chi/\partial\bar{z}$ vanishes on $S_p(F, \varepsilon_1, C_1)$. Hence the functions

$$v_j\colon z \mapsto -\bar{F}_j(z)|F(z)|^{-2}\frac{\partial(\chi\tilde{f})}{\partial\bar{z}}(z), \quad 1 \leqslant j \leqslant N,$$

are in $C^\infty(C)$ and satisfy $\operatorname{Supp}(v_j) \subset S_p(F, \varepsilon, C) \setminus S_p(F, \varepsilon_1, C_1)$. Because of (1), (4) and 2.1, the following estimates hold for $1 \leqslant j \leqslant N$, each $m \in N$ and all $z \in C$:

$$(5)\qquad \begin{aligned} |v_j(z)| &\leqslant MAD_m \varepsilon^{-2} \exp((L + 2C + B)p(z) + kq(\operatorname{Im} z) + r(z)/m) \\ &\leqslant D'_m \exp(\varkappa q(\operatorname{Im} z) + 2r(z)/m), \end{aligned}$$

where $\varkappa := k + L + 2C + B$ and where D'_m depends on ε, M, A, m, D_m, s and r, but not on the particular $\tilde{f} \in Q$. Since r satisfies 1.1(2) there exists a sequence $(D''_m)_{m\in N}$ of positive numbers depending on D'_m, r and $\varkappa$, but not on $\tilde{f}$, with

$$(6)\qquad \int_C [|v_j(z)| \exp(-\varkappa q(\operatorname{Im} z) - r(z)/m)]^2 \, d\lambda(z) \leqslant D''_m$$

for each $m \in N$. Hence the ellipticity of the $\bar{\partial}$-equation and Lemma 2.4 imply the existence of a sequence $(D'''_m)_{m\in N}$ of positive numbers such that for $1 \leqslant j \leqslant N$ there exist $u_j \in C^\infty(C)$ with $\partial u_j/\partial\bar{z} = v_j$ and

$$(7)\qquad \int_C [|u_j(z)| \exp(-\varkappa q(\operatorname{Im} z) - r(z)/m)]^2 \, d\lambda(z) \leqslant D'''_m$$

for each $m \in N$.

If we define

$$(8)\qquad f := \chi\tilde{f} + \sum_{j=1}^{N} u_j F_j$$

then it follows easily that $\partial f/\partial\bar{z} \equiv 0$, i.e. $f \in A(C)$. Next put

$$\sigma_j := u_j | S_p(F, \varepsilon_1, C_1) \quad \text{for } 1 \leqslant j \leqslant N$$

and note that

$$\frac{\partial\sigma_j}{\partial\bar{z}} = v_j | S_p(F, \varepsilon_1, C_1) \equiv 0, \quad \text{i.e.} \quad \sigma_j \in A(S_p(F, \varepsilon_1, C_1)).$$

Hence (2) follows from (8). To see that (3) holds, note that because of $\varkappa \geqslant k$ it follows from (1) and (7) that there exists a bounded set B in $\operatorname{proj}_{\leftarrow m} W(\varkappa, m)$ with $f \in B$ for each $\tilde{f} \in Q$, which implies (3) by standard arguments.

2.6. THEOREM. *Let $F = (F_1, \ldots, F_N) \in (A_{q,r})^N$ be slowly decreasing and assume that $V(F) := \{z \in C \mid F_j(z) = 0 \text{ for } 1 \leqslant j \leqslant N\}$ is an infinite set. Then $A_{q,r}/I_{\mathrm{loc}}(F)$ is linear topologically isomorphic to $K(\gamma, \delta)$, where the sequences γ and δ are obtained in the following way: If $(a_j)_{j\in N}$ is an enumeration of the points in $V(F)$, each point counted with the multiplicity of the common zero of $(F_1, \ldots, F_N)$, then*

$$\gamma = (q(\operatorname{Im} a_j))_{j\in N}, \quad \delta = (r(a_j))_{j\in N}.$$

Proof. Choose s, L, M, ε, C and D according to 2.1 and define the weight function p by $p(z) = q(\operatorname{Im} z) + s(z)$. Then label the components S of $S_p(F, \varepsilon, C)$ with $S \cap V(F) \neq \emptyset$ in such a way that the sequence

$$\beta := (\sup_{z\in S_j} r(z))_{j\in N}$$

is increasing and define

$$\alpha := (\sup_{z\in S_j} q(\operatorname{Im} z))_{j\in N}.$$

Note that 1.1(2) implies $\lim_{j\to\infty} \beta_j = \infty$.

Next fix $j \in N$ and denote by $A'(S_j)$ the Banach space of all bounded holomorphic functions on S_j endowed with the supremum norm. Put

$$E_j := \prod_{a \in S_j \cap V(F)} \mathcal{O}_a/I_a(F)$$

and note that the map

$$\varrho_j\colon A^\infty(S_j)\to E_j,\quad \varrho_j(g):=([g]_a+I_a(F))_{a\in S_j\cap V(F)},$$

is a surjective linear map which has a closed kernel. Therefore we can define $\|\ \|_j$ on E_j as the quotient norm induced by ϱ_j, i.e.

(1) $$\|\mu\|_j:=\inf\{\|g\|_{A^\infty(S_j)}\mid g\in A^\infty(S_j),\ \varrho_j(g)=\mu\}.$$

Now let E denote the sequence $(E_j,\|\ \|_j)_{j\in N}$ of finite-dimensional normed spaces. To show that for each $f\in A_{q,r}$ the sequence $(\varrho_j(f|S_j))_{j\in N}$ belongs to $K(\alpha,\beta,E)$, we fix $f\in A_{q,r}$. Then

(2) There exists $k\in N$ such that for each $m\in N$ there exists C_m with $\|f\|_{k,m}\leqslant C_m$.

Hence the definition of α and β and (1) imply by (2)

(3) There exists $k\in N$ such that for each $m\in N$ there exists $C_m>0$ such that for each $j\in N$ we have

$$\|\varrho_j(f|S_j)\|_j\leqslant \|f|S_j\|_{A^\infty(S_j)}\leqslant C_m\exp(k\alpha_j+\beta_j/m).$$

This shows that the linear map

(4) $$\varrho\colon A_{q,r}\to K(\alpha,\beta,E),\quad \varrho(f):=(\varrho_j(f|S_j))_{j\in N},$$

is continuous.

To prove that ϱ is surjective, let $x=(x_j)_{j\in N}\in K(\alpha,\beta,E)$ be given. Then there exists $k\in N$ such that for each $m\in N$ there exists $D_m>0$ with

$$|||x|||_{k,m}=\sup_{j\in N}\|x_j\|_j\exp(-k\alpha_j-\beta_j/m)\leqslant D_m.$$

By (1) there is for each $j\in N$ an $f_j\in A^\infty(S_j)$ with $\varrho_j(f_j)=x_j$ and

(5) $$\|f_j\|_{A^\infty(S_j)}\leqslant 2\|x_j\|_j.$$

Next we define $\tilde f\colon S_p(F,\varepsilon,C)\to C$ by

$$\tilde f(z):=\begin{cases} f_j(z) & \text{if } z\in S_j,\\ 0 & \text{if } z\in S_p(F,\varepsilon,C)\setminus\bigcup_{j\in N}S_j.\end{cases}$$

Then $\tilde f$ is in $A(S_p(F,\varepsilon,C))$ and from (5) and 2.1 we get for each $m\in N$, each $j\in N$ and each $z\in S_j$

(6) $$\begin{aligned}|\tilde f(z)|=|f_j(z)|&\leqslant 2\|x_j\|_j\leqslant 2|||x|||_{k,m}\exp(k\alpha_j+\beta_j/m)\\ &\leqslant 2D_m\exp\big(kD(1+p(z))+D(1+r(z))/m\big)\\ &\leqslant 2D_mL\exp\big(kDq(\operatorname{Im}z)+2Dr(z)/m\big),\end{aligned}$$

where L depends only on K, D, s and r.

This shows that $\tilde f$ satisfies an estimate of type 2.5(1). Therefore, 2.5 implies the existence of $f\in A_{q,r}$ satisfying 2.5(2), (3). Obviously, 2.5(2) implies $\varrho(f)=(x_j)_{j\in N}$. Hence we have shown that the continuous linear map $\varrho\colon A_{q,r}\to K(\alpha,\beta,E)$ is surjective. By the open mapping theorem for (LF)-spaces, ϱ is an open map. Since $\ker\varrho=I_{\mathrm{loc}}(F)$, this proves

(7) $$A_{q,r}/I_{\mathrm{loc}}(F)\simeq K(\alpha,\beta,E).$$

To obtain the desired sequence space representation from this, note that by 2.1, $F=(F_1,\dots,F_N)$ is slowly decreasing in A_P in the sense of [14], 3.1, for $P=(kp)_{k\in N}$. Hence Remark (b) of Cor. 3.8 of [14] implies

(8) There exists $l\in N$ with $\sup_{j\in N}(\dim E_j)\exp(-l\sup_{z\in S_j}p(z))<\infty$.

Next note that $s=o(r)$ implies that for each $m\in N$ there exists $D_m>0$ such that for all $j\in N$

$$\sup_{z\in S_j}p(z)\leqslant\sup_{z\in S_j}q(\operatorname{Im}z)+\sup_{z\in S_j}s(z)\leqslant\alpha_j+\beta_j/m+D_m.$$

Because of (8) this implies

(9) There exists $l\in N$ such that for each $m\in N$

$$\sup_{j\in N}(\dim E_j)\exp(-l\alpha_j-\beta_j/m)<\infty.$$

By Lemma 1.7, (9) implies $K(\alpha,\beta,E)\simeq K(\tilde\gamma,\tilde\delta)$, where $\tilde\gamma$ (resp. $\tilde\delta$) is obtained from α (resp. β) by repeating α_j (resp. β_j) $\dim E_j$ times. Next note that 2.1(2) implies that (for a suitable enumeration) $\tilde\gamma$ and γ (resp. $\tilde\delta$ and δ) are equivalent in the sense of 1.8, which implies $K(\tilde\gamma,\tilde\delta)=K(\gamma,\delta)$. Hence the result follows from (7).

2.7. Remark. In Theorem 2.6 we can identify $K(\gamma,\delta)$ with $\lambda(\gamma,\delta)'_b$ by Proposition 1.6(2). If we do this, then a subset G of $\lambda(\gamma,\delta)'_b$ is equicontinuous if and only if there exist $k\in N$ and a bounded set M in $A(k):=\operatorname{proj}_{\leftarrow m}A(k,m)\subset A_{q,r}$ with $G=\varrho(M)$.

To see this, note that each set G of this form is certainly equicontinuous in $\lambda(\gamma,\delta)'_b$ by 1.6(3). To show the converse, let $G\subset\lambda(\gamma,\delta)'_b$ be equicontinuous and identify $K(\gamma,\delta)$ with $K(\alpha,\beta,E)$ as in the proof of 2.6. Then an easy inspection of the proof of 2.6 shows that the functions $f_x\in A_{q,r}$ with $\varrho(f_x)=x$ for $x\in G$ are in fact contained in a set M of the required form.

3. Kernels of convolution operators. In this section we use the results of the preceding one to derive sequence space representations for the kernels of convolution operators on ultradifferentiable functions of Roumieu type.

3.1. DEFINITION. Let ω be a radial weight function on C with $\omega|\{z \in C \mid |z| \leqslant 1\} \equiv 0$ which also satisfies:

(α) There exists $K \geqslant 1$ such that for all $z \in C$

$$\omega(2z) \leqslant K(1+\omega(z)).$$

(β) $\int_{-\infty}^{\infty} \frac{\omega(t)}{1+t^2} dt < \infty$.

Note that by the remark following 1.3 in Meise, Taylor and Vogt [17], we have $\lim_{t\to\infty} \omega(t)/t = 0$.

3.2. *Notation.* For ω as in 3.1, the function φ: $[0, \infty[\to [0, \infty[$, $\varphi(t) = \omega(e^t)$, is convex and satisfies $\lim_{t\to\infty} t/\varphi(t) = 0$. Therefore we can define its Young conjugate φ^*: $[0, \infty[\to [0, \infty[$ by

$$\varphi^*(y) := \sup\{xy - \varphi(x) \mid x \geqslant 0\}.$$

From Braun, Meise and Taylor [5] we recall:

3.3. DEFINITION. For ω as in 3.1 define φ and φ^* as in 3.2.

(a) For an open interval I in R we define

$\mathscr{E}_{\{\omega\}}(I) := \{f \in C^\infty(I) \mid$ for each $K \subset I$ compact there exists $m \in N$ with

$$\|f\|_{K,m} := \sup_{x\in K} \sup_{j\in N_0} |f^{(j)}(x)| \exp(-\varphi^*(mj)/m) < \infty\}$$

and we endow $\mathscr{E}_{\{\omega\}}(I)$ with the l.c. topology which is given by taking the projective limit over $K \Subset I$ of the inductive limit over $m \in N$.

(b) For a compact interval $[a, b]$ in R we put

$$\mathscr{D}_{\{\omega\}}[a, b] := \{f \in \mathscr{E}_{\{\omega\}}(R) \mid \mathrm{Supp}(f) \subset [a, b]\}$$

and endow $\mathscr{D}_{\{\omega\}}[a, b]$ with the induced topology. Then we define

$$\mathscr{D}_{\{\omega\}}(R) := \operatorname{ind}_{n\to} \mathscr{D}_{\{\omega\}}[-n, n].$$

3.4. *Convolution operators on* $\mathscr{E}_{\{\omega\}}(R)$. In Braun, Meise and Taylor [5], it was shown that for each $\mu \in \mathscr{E}_{\{\omega\}}(R)'$ the map T_μ: $\mathscr{E}_{\{\omega\}}(R) \to \mathscr{E}_{\{\omega\}}(R)$, $T_\mu(f) := \mu * f$, where

$$\mu * f(x) := \langle \mu_y, f(x-y) \rangle,$$

is continuous and linear. These maps are called *convolution operators.* It was also shown in [5] that for $\mu \in \mathscr{E}_{\{\omega\}}(R)'$ and $\nu \in \mathscr{D}_{\{\omega\}}(R)'$, $\mu * \nu \in \mathscr{D}_{\{\omega\}}(R)'$ can be defined by

$$\langle \mu * \nu, f \rangle := \langle \nu, \check{\mu} * f \rangle, \quad f \in \mathscr{D}_{\{\omega\}}(R),$$

where $\langle \check{\mu}, f \rangle = \langle \mu, \check{f} \rangle$ and $\check{f}$: $x \mapsto f(-x)$.

Then $(\mathscr{E}_{\{\omega\}}(R)'_b, *)$ is a l.c. algebra with continuous multiplication. If q: $R \to [0, \infty[$ is defined by $q(t) := |t|$, then the Fourier–Laplace transform $\mathscr{F}$: $(\mathscr{E}_{\{\omega\}}(R)'_b, *) \to A_{q,\omega}$ defined by

$$\mathscr{F}(\mu) = \hat{\mu}: z \mapsto \langle \mu_x, e^{-ixz} \rangle$$

is a topological algebra isomorphism by Braun, Meise and Taylor [5]. Moreover, we have

(1) $$T_\mu^t = \mathscr{F}^{-1} \circ M_{\hat{\mu}} \circ \mathscr{F},$$

where $M_{\hat{\mu}}$: $A_{q,\omega} \to A_{q,\omega}$ denotes the multiplication operator induced by $\mathscr{F}(\check{\mu})$.

By Braun, Meise and Taylor [5], $\mathscr{E}_{\{\omega\}}(R)$ is a complete nuclear space. Hence $\ker T_\mu$ has this property, too. By Schwartz [22], p. 43, this implies that $(\ker T_\mu)'_b$ is ultrabornological. Since the restriction map R: $\mathscr{E}_{\{\omega\}}(R)'_b \to (\ker T_\mu)'_b$ is continuous, linear and surjective by the Hahn–Banach theorem and since $\mathscr{E}_{\{\omega\}}(R)'_b \simeq A_{q,\omega}$ is an (LF)-space by 1.3, the open mapping theorem implies

(2) $$(\ker T_\mu)'_b \simeq \mathscr{E}_{\{\omega\}}(R)'_b/\ker R = \mathscr{E}_{\{\omega\}}(R)'_b/(\ker T_\mu)^\perp.$$

Since $\mathscr{E}_{\{\omega\}}(R)$ is semireflexive, $(\ker T_\mu)^\perp$ equals the $\mathscr{E}_{\{\omega\}}(R)'_b$-closure of $\operatorname{im} T_\mu^t$. Hence (1) and (2) imply

(3) $$(\ker T_\mu)'_b \simeq A_{q,\omega}/\overline{I(\mathscr{F}(\check{\mu}))},$$

where the isomorphism is induced by the map $\Phi := R \circ \mathscr{F}^{-1}$. Note that the Hahn–Banach theorem implies

(4) A subset G of $(\ker T_\mu)'$ is equicontinuous if and only if there exist $k \in N$ and a bounded set M in $\operatorname{proj}_{\leftarrow m} A(k, m) \subset A_{q,\omega}$ with $G = \Phi(M)$.

3.5. PROPOSITION. *For ω as in 3.1 and $\mu \in \mathscr{E}_{\{\omega\}}(R)'$ the following conditions are equivalent:*

(1) *$\hat{\mu}$ is slowly decreasing in $A_{q,\omega}$, where $q(t) = |t|$.*
(2) *μ admits a fundamental solution $E \in \mathscr{D}_{\{\omega\}}(R)'$, i.e. $\mu * E = \delta$.*

Proof. By Braun, Meise and Vogt [6], 2.4, μ admits a fundamental solution $E \in \mathscr{D}_{\{\omega\}}(R)'$ if and only if there exist a radial weight function s satisfying 3.1(α)–(β) and $n \in N$ with $s = o(\omega)$ and

$$\sup_{z\in C} |\hat{\mu}(z)| \exp(-n|\operatorname{Im} z| - ns(z)) < \infty,$$

such that $\hat{\mu}$ is slowly decreasing in the algebra A_p for $p(z) = |\operatorname{Im} z| + s(z)$, where A_p is defined as

$$A_p := \{f \in A(C) \mid \text{there is } k \in N \text{ with } \sup_{z\in C} |f(z)| \exp(-kp(z)) < \infty\}.$$

Because of this characterization it is obvious that (1) implies (2). To show that (2) implies (1), one uses property 3.1(α) for ω and the diameter estimates for the components S of $S_p(F, \varepsilon, C)$ which have been derived in the proof of Meise, Taylor and Vogt [17], 2.3.

For $\mu \in \mathscr{E}_{\{\omega\}}(R)'$ and $a \in C$ with $\hat{\mu}^{(j)}(a) = 0$ for $0 \leqslant j < m$ we define $f_j\colon R \to C$ by $f_j(x) := x^j e^{ixa}$, $0 \leqslant j < m$. By Braun, Meise and Taylor [5], we have $f_j \in \mathscr{E}_{\{\omega\}}(R)$. Moreover, it follows easily from the definition of T_μ that $f_j \in \ker T_\mu$ for $0 \leqslant j < m$. Linear combinations of such zero-solutions of T_μ are called *exponential solutions of* T_μ.

3.6. THEOREM. *For ω as in 3.1 and $\mu \in \mathscr{E}_{\{\omega\}}(R)'$ assume that μ admits a fundamental solution in $\mathscr{D}_{\{\omega\}}(R)'$ and $\dim\ker T_\mu = \infty$. Then $\ker T_\mu$ has an absolute basis consisting of exponential solutions and $\ker T_\mu$ is topologically isomorphic to $\lambda(\alpha, \beta)$ for $\alpha = (|\operatorname{Im} a_j|)_{j\in N}$ and $\beta = (\omega(a_j))_{j\in N}$, where $(a_j)_{j\in N}$ is an enumeration of the zeros of $\hat{\mu}$, counted with multiplicities.*

Proof. By Proposition 3.5, $\hat{\mu}$ is slowly decreasing in $A_{q,\omega}$ for $q(t) = |t|$. Obviously, this also holds for $\mathscr{F}(\check{\mu})\colon z \mapsto \hat{\mu}(-z)$. This implies $I(\mathscr{F}(\check{\mu})) = \overline{I(\mathscr{F}(\check{\mu}))} = I_{loc}(\mathscr{F}(\check{\mu}))$ by 2.3. Therefore, Theorem 2.6 and 3.4(3) show

$$(\ker T_\mu)'_b \simeq A_{p,\omega}/\overline{I(\mathscr{F}(\check{\mu}))} \simeq K(\alpha, \beta) = \lambda(\alpha, \beta)'_b.$$

If we identify $(\ker T_\mu)'_b$ with $\lambda(\alpha, \beta)'_b$ by this isomorphism, then it follows from 3.4(4) and 2.7 that both spaces have the same equicontinuous sets. Since they are both semireflexive (because of $\lim_{j\to\infty}\beta_j = \infty$), this implies $\ker T_\mu \simeq \lambda(\alpha, \beta)$. As in Meise, Schwerdtfeger and Taylor [15], we can write out this isomorphism more explicitly. Then it follows that the images of the canonical basis vectors in $\lambda(\alpha, \beta)$ are in fact exponential solutions.

3.7. Remark. Under the hypotheses of Theorem 3.6 we also have

$$\ker T_\mu \simeq \lambda(\alpha, \tilde{\beta}),$$

where $\alpha = (|\operatorname{Im} a_j|)_{j\in N}$, $\tilde{\beta} := (\omega(\operatorname{Re} a_j))_{j\in N}$ and $(a_j)_{j\in N}$ is as in 3.6.

To see this, note that there exists $C > 0$ with $\omega(t) \leqslant |t| + C$ for all $t \in R$. This implies for each $z \in C$

$$\omega(z) \leqslant \omega(|\operatorname{Re} z| + |\operatorname{Im} z|) \leqslant \omega(2\max(|\operatorname{Re} z|, |\operatorname{Im} z|))$$
$$\leqslant K(1 + \omega(\operatorname{Re} z) + \omega(\operatorname{Im} z)) \leqslant K\omega(\operatorname{Re} z) + K|\operatorname{Im} z| + K(1 + C).$$

Since $\omega(\operatorname{Re} z) \leqslant \omega(z)$ for all $z \in C$, this implies $\lambda(\alpha, \beta) = \lambda(\alpha, \tilde{\beta})$.

3.8. COROLLARY. *For ω as in 3.1 and $\mu \in \mathscr{E}_{\{\omega\}}(R)'$ assume that the convolution operator $T_\mu\colon \mathscr{E}_{\{\omega\}}(R) \to \mathscr{E}_{\{\omega\}}(R)$ is surjective, that T_μ admits a continuous*

linear right inverse and that $\dim\ker T_\mu = \infty$. Then $\ker T_\mu$ is isomorphic to $\Lambda_1(\beta)'_b$, where

$$\Lambda_1(\beta) = \{x \in C^N \mid \|x\|_m := \sum_{j=1}^{\infty} |x_j| \exp(-\beta_j/m) < \infty \text{ for all } m \in N\}$$

and $\beta := (\omega(a_j))_{j\in N}$ for $(a_j)_{j\in N}$ as in 3.6.

Proof. Note that Theorem 4.4 of Meise and Vogt [18] extends to the present class $\mathscr{E}_{\{\omega\}}(R)$. Hence T_μ admits a fundamental solution in $\mathscr{D}_{\{\omega\}}(R)'$. By Theorem 3.6, this implies $\ker T_\mu \simeq \lambda(\alpha, \beta)$. Now observe that by Meise and Vogt [18], 4.7 (which also extends to the present class) we have $\lim_{j\to\infty}\alpha_j/\beta_j = 0$. Since $\lambda(\alpha, \beta)$ is nuclear, this and 1.9(1) imply $\ker T_\mu \simeq \lambda(\alpha, \beta) \simeq \Lambda_1(\beta)'_b$.

Remark. Define $\mu \in \mathscr{E}(R)'$ by $\mu := \delta_\pi - \delta_{-\pi}$ and fix ω as in 3.1. Then μ is in $\mathscr{E}_{\{\omega\}}(R)'$ and $\ker T_\mu$ is the space of all 2π-periodic functions in $\mathscr{E}_{\{\omega\}}(R)$. It is easy to show that T_μ admits fundamental solutions E_+ and E_- in $\mathscr{D}(R)' \subset \mathscr{D}_{\{\omega\}}(R)'$ with $\operatorname{Supp} E_+ \subset [a, \infty[$ and $\operatorname{Supp} E_- \subset]-\infty, b]$ for suitable $a, b \in R$. Hence the proof of Meise and Vogt [18], 4.4, (7)$\Rightarrow$(1), shows that T_μ is surjective on $\mathscr{E}_{\{\omega\}}(R)$ and that T_μ admits a continuous linear right inverse. By Corollary 3.8, this implies

$$\ker T_\mu \simeq \Lambda_1((\omega(j))_{j\in Z})'_b \simeq \Lambda_1((\omega(j))_{j\in N})'_b.$$

This shows that Corollary 3.8 extends the results of Petzsche [21], Sect. 3, to the present class $\mathscr{E}_{\{\omega\}}(R)$.

The observation that the results of Petzsche [21], Sect. 3, could be obtained from Komatsu [12], 1.1, by a modification of the arguments of Berenstein and Taylor [1], Sect. 3, and Meise [14], 3.7, was in fact the starting point for the investigations of the present paper.

3.9. EXAMPLE. It is easy to check that the following functions $\omega\colon C \to [0, \infty[$ satisfy all the conditions of 3.1 after a suitable change on a compact disk with center zero:

(1) $\omega(z) = |z|^\alpha, \quad 0 < \alpha < 1,$

(2) $\omega(z) = |z|^\alpha (\log(1+|z|^2))^\beta, \quad 0 < \alpha < 1,\ 0 \leqslant \beta < \infty,$

(3) $\omega(z) = |z|(\log(2+|z|^2))^{-\beta}, \quad \beta > 1,$

(4) $\omega(z) = (\log(1+|z|^2))^\beta, \quad \beta > 1,$

(5) $\omega(z) = \exp((\log(1+|z|^2))^\alpha), \quad 0 < \alpha < 1.$

3.10. EXAMPLE. Let $(M_j)_{j\in N_0}$ be a sequence in $([1, \infty[)^{N_0}$ which satisfies:

(M1) $\quad M_j^2 \leqslant M_{j-1} M_{j+1} \quad$ for all $j \in N$, (M2) there exist $A, H \geqslant 1$ with $M_n \leqslant AH^n \min_{0 \leqslant j \leqslant n} M_j M_{n-j}$ for all $n \in \boldsymbol{N}$,

(M3)′ $\sum_{j=1}^{\infty} M_{j-1}/M_j < \infty$,

(M4) there exists $k \in \boldsymbol{N}$ with $\liminf_{j \to \infty} (M_{jk}/M_j^k)^{1/j} > 1$,

and define $\omega_M \colon \boldsymbol{C} \to [0, \infty[$ by

$$\omega_M(z) := \begin{cases} \sup_{j \in N_0} \log \dfrac{|z|^j}{M_j} M_0 & \text{for } |z| > 1, \\ 0 & \text{for } |z| \leqslant 1. \end{cases}$$

Then it follows from Komatsu [11], Sect. 3, and [14], 2.6(2), that ω_M satisfies all the conditions in 3.1. Using the notation of Komatsu [11], 2.5, we have

$$\mathcal{E}_{\{\omega_M\}}(\boldsymbol{R}) = \mathcal{E}^{\{M_j\}}(\boldsymbol{R}).$$

From this it follows that for $s > 1$ and $\omega_s \colon z \mapsto |z|^{1/s}$, the space $\mathcal{E}_{\{\omega_s\}}(\boldsymbol{R})$ coincides with the Gevrey class $\mathcal{E}^{\{(p!)^s\}}(\boldsymbol{R})$.

3.11. Example. Let $\omega \colon \boldsymbol{R} \to [0, \infty[$ be a continuous even function which satisfies:

(α) $0 = \omega(0) \leqslant \omega(s+t) \leqslant \omega(s) + \omega(t)$ for all $s, t \in \boldsymbol{R}$,

(β) $\int_{-\infty}^{+\infty} \frac{\omega(t)}{1+t^2} dt < \infty$,

(γ)′ $\log(1+|t|) = o(\omega(t))$ for $|t| \to \infty$,

(δ) $\varphi \colon t \to \omega(e^t)$ is convex on $\boldsymbol{R}$.

Then it follows from Björck [4], 1.2.8, that after a suitable change on a compact disk with center zero, the function $\tilde{\omega} \colon z \mapsto \omega(|z|)$ satisfies all the conditions in 3.1. By Braun, Meise and Taylor [5] we have

$$\mathcal{S}_{\{\tilde\omega\}}(\boldsymbol{R}) = \{f \in \mathcal{S}'(\boldsymbol{R}) \mid \text{there exists } \varepsilon > 0\colon \int_{-\infty}^{+\infty} |\hat{f}(t)| \exp(\varepsilon \tilde{\omega}(t))\, dt < \infty\},$$

$$\mathcal{O}_{\{\tilde\omega\}}(\boldsymbol{R}) = \{f \in \mathcal{O}'(\boldsymbol{R}) \mid gf \in \mathcal{S}_{\{\tilde\omega\}}(\boldsymbol{R}) \text{ for each } g \in \mathcal{S}_{\{\tilde\omega\}}(\boldsymbol{R})\}.$$

Hence $\mathcal{O}_{\{\tilde\omega\}}(\boldsymbol{R})$ defined in 3.3 coincides with $\mathcal{E}_{\{\omega\}}(\boldsymbol{R})$ as used in Meise and Vogt [18], Sect. 4.

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