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LOCAL ANALYTIC GEOMETRY IN BANACH SPACES

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ABSTRACT

After motivating analytic geometry in infinite dimensional spaces we give a survey on the local theory of SF-analytic sets and holomorphic semi-Fredholm maps. Moreover the notion of minimal embedding codimension is introduced. It allows to derive quantitative results although the dimension may be infinite.

1. MOTIVATION

Analytic geometry in $\mathbb{C}^n$ deals with the geometrical properties of the sets of solutions of analytic equations defined in an open subset of $\mathbb{C}^n$. Since many interesting equations in analysis like differential and integral equations are defined in infinite dimensional spaces and are analytic or even polynomial it seems reasonable to develop a concept of analytic geometry in infinite dimensional topological vector spaces. Although in applications mostly real solutions are considered one hopes that as in finite dimensions the complex analytic case yields a simpler and more complete theory. But without further restrictions this is not true: every compact metric space occurs as the set of solutions of a complex quadratic polynomial equation in a suitable Banach space [2]. Hence in order to obtain sets of solutions with nice geometric properties additional conditions have to be imposed on the regularity of the equation.

We compile some sufficient conditions in the case of Banach spaces.

Let E and F be complex Banach spaces and Ω a domain in E. We call a map $f : \Omega \to F$ *holomorphic* if it is complex Frechet differentiable or equivalently if it is complex analytic. $Df(x)$ denotes the differential at x. We say that a linear operator $T : E \to F$ is *splitting* if its kernel and its image are complemented subspaces of

E and F respectively; T will be called a *semi-Fredholm operator* if it is splitting and continuous and if its kernel or its cokernel (or both) are finite dimensional. Obviously every Fredholm operator is semi-Fredholm. The *index* of T is $ind\,T := dim\,Ker\,T - codim\,Im\,T$.

Let X be a subset of Ω. Then we define:

1.1. X is *analytic* in Ω iff X is closed and satisfies the following property:

(A) For each $x \in X$ there exists a neighbourhood U, a complex Banach space H, and a holomorphic map $h : U \to H$ such that $X \cap U = h^{-1}(0)$.

1.2. X is a *complex submanifold* of Ω iff it is analytic and moreover the maps h in (A) can be chosen to have surjective and splitting differentials.

Because of the implicit function theorem we obtain the usual notion of a complex submanifold as a closed subset which is locally in appropriate biholomorphic coordinates an open piece of a complemented linear subspace.

1.3. X is *finitely defined* iff it is analytic and moreover the Banach spaces H in (A) can be chosen finite dimensional.

The finitely defined sets have been investigated intensively by Ramis [6] and later on by Mazet in locally convex spaces [5]. Because these sets have finite codimension induction on the codimension can be used to show that they have nice local geometric properties (see the next section). An example for a finitely defined set is the set of non-surjective linear Fredholm operators in $\mathcal{L}(E,F)$.

1.4. X is *finite dimensional analytic* iff it is analytic and moreover the maps h in (A) can be chosen to have splitting differentials with finite dimensional kernels.

Again by the implicit function theorem it is easy to see that an analytic set X is finite dimensional iff it is locally contained in a finite dimensional complex submanifold of some open set in Ω (Notice that in general a finite dimensional analytic set X is not even locally contained in a linear subspace). Therefore all local results about analytic subsets of $\mathbb{C}^n$ hold also for finite dimensional analytic sets in Banach spaces.

Because an analytic subset in a finite dimensional manifold is always finitely defined we can state that

1.5. *X is finite dimensional analytic iff it is analytic and moreover the maps h in (A) can be chosen such that their differentials are Fredholm operators.*

If we call a holomorphic map a *Fredholm map* or *F-map* (*semi-Fredholm map* or *SF-map*) iff all differentials are Fredholm (semi-Fredholm) operators then the finite dimensional analytic sets are precisely the sets which are locally the fiber of a holomorphic Fredholm map. The importance of nonlinear Fredholm maps is well known, e.g. elliptic differential operators with Dirichlet boundary conditions or maps of the form identity-compact map are Fredholm, and in many cases they are polynomial hence analytic.

Very often an equation $\Phi_y(x) = 0$ depends on a parameter y and one would like to know how the set of solutions changes when the parameter varies. For example the parameter can be the right side of a differential equation $f(x) = y$; hence it is natural that y varies in an infinite dimensional space. Consider y as an additional variable and put $\Phi(x,y) := \Phi_y(x)$. Then the shape of $\Phi^{-1}(0)$ determines how the sets $Z_y := \{x : \Phi_y(x) = 0\}$ depend on y. If Φ_y is a holomorphic Fredholm map and depends holomorphically on y then Φ is a holomorphic semi-Fredholm map. Thus the local bifurcation theory of holomorphic F-maps is related to the local theory of the so-called SF-analytic sets.

1.6. X is *SF-analytic* iff it is analytic and moreover the maps h in (A) can be chosen such that their differentials are SF-operators.

With the aid of the implicit function theorem it can be shown that an analytic set is SF-analytic iff it is locally contained in a complex submanifold where it is finitely defined [1]. Therefore the SF-analytic sets have essentially the same nice local properties as the finitely defined sets. Some of them will be presented in the next section.

The above definitions are also meaningful when Ω is a complex manifold because all occurring notions are local and invariant under biholomorphic maps.

2. LOCAL PROPERTIES OF SF-ANALYTIC SETS

At first we recall some fundamental properties of arbitrary analytic sets [6].

Let X be an analytic set of a domain Ω in the Banach space E. The *codimension* of X in $x \in X$ with respect to Ω is defined as

$\Omega - codim_x X := sup\{n \in \mathbb{N} \cup \{0,\infty\}$: there exists an affine complex subspace H of E with dimension n such that x is isolated in $X \cap H\}$.

Ramis showed that this definition is invariant under biholomorphic maps [6, p. 70, 74], hence it generalizes via charts to complex Banach manifolds Ω.

If no confusion can arise we write simply $codim_x X$. Suppose $x \in X$, H is an affine subspace of E, $dim\, H \leqq codim_x X$, and x is isolated in $X \cap H$. Then for every $n \in \mathbb{N}$ with $dim\, H \leqq n \leqq codim_x X$ there exists an affine subspace G such that $H \subset G$, $dim\, G = n$, and x is isolated in $X \cap G$ [6, II. 3.1.1]. The set of these G's is open in the Grassmannian [6, p. 89], therefore

$$codim_x X \cap Y = min\{codim_x X,\ codim_x Y\}$$

for X and Y analytic and $x \in X \cap Y$. The function $x \mapsto codim_x X$ is upper semicontinuous [6, II. 3.3.1].

A point $x \in X$ is *regular* iff X is near x a complex submanifold, otherwise x is called *singular*. The set X^* of regular points is not always dense in X, but every point x where $codim_x X < \infty$ is a cluster point of X^* and for every cluster point x the following equation holds

$$codim_x X = \liminf_{\substack{y \to x \\ y \in X^*}} codim_y X.$$

Because a finitely defined analytic set X has everywhere finite codimension X^* is dense in X. The closures of the components of X^* are again analytic and form a locally finite decomposition of X into irreducible components. Also the germ of a finitely defined analytic set can be decomposed into finitely many irreducible germs of finitely defined sets with the usual uniqueness [6, p. 60]. A fundamental result in [6] is

2.1. The local parametrization of finitely defined analytic sets. *Let X be a finitely defined analytic subset of a domain Ω in E. Suppose $0 \in X$ and X is irreducible in 0. Let $E = E_1 \times E_2$ be a topological decomposition such that 0 is isolated in $X \cap (\{0\} \times E_2)$ and $\dim E_2 = \operatorname{codim}_0 X < \infty$. Then each neighbourhood of 0 contains the product of two balls $B_1 \subset E_1$ and $B_2 \subset E_2$ centered at 0 such that the canonical projection $\pi : X \cap (B_1 \times B_2) \to B_1$ is an analytically ramified covering map with finitely many sheets in the following sense:*

(a) *π is finite i.e. $\pi^{-1}(K)$ is compact and nonempty for every compact nonempty $K \subset B_1$ and $m := \sup \{\operatorname{card} \pi^{-1}(x) : x \in B_1\}$ is finite.*

(b) *The bifurcation set $S := \{x \in B_1 : \operatorname{card} \pi^{-1}(x) < m\}$ is a finitely defined nowhere dense analytic subset of B_1. $X \cap ((B_1 - S) \times B_2)$ is complex submanifold of $(B_1 - S) \times B_2$ and is dense in $X \cap (B_1 \times B_2)$. $\pi | X \cap ((B_1 - S) \times B_2) \to B_1 - S$ is a locally biholomorphic unramified covering map with m sheets.* [6, II.2.3.7, II.2.2.4, II.2.2.12].

Since SF-analytic sets are locally finitely defined subsets of submanifolds they enjoy the above mentioned properties. In particular we obtain the following consequences.

2.2. COROLLARY. *Let X be an SF-analytic subset of a Banach manifold Ω.*

(a) *X is locally connected by complex arcs i.e. for each $x \in X$ there are arbitrarily small neighbourhoods U of x such that for each $y \in U$ there exists a holomorphic map γ from the open unit disk D into Ω with $x, y \in \gamma(D) \subset X$.*

(b) *If X is irreducible then every non constant holomorphic function on X is open.*

(c) *If X is irreducible then the maximum principle holds i.e. a holomorphic function on X is constant if its modulus attains a local maximum.*

(d) *If X is compact and Ω is holomorphically separable then*

X is finite.

PROOF. (a) Choose for each irreducible component of X at x a local parametrization as in 2.1. Moreover choose a complex line L through $\pi(x)$ and $\pi(y)$. Then $\pi^{-1}(L)$ is a onedimensional analytic set. The uniformisation of its normalization is isomorpic to D.

(b) By (a) there is through every $x \in X$ a complex curve γ such that $f \circ \gamma$ is not constant and hence open.

(c) follows from (b) and (d) from (c).

The next proposition will serve to define the minimal embedding codimension of an SF-analytic set in a point. This notion will allow to prove and to use in the following sections codimension formulas which correspond to the dimension formulas in finite dimensional complex analysis. Let X be analytic in a domain Ω of E. For $x \in X$ let X_x be the germ and I_x the ideal of germs of holomorphic functions vanishing on X_x. $T_xX := TX_x := \{u \in E : u \in \mathrm{Ker}\, Dh(x) \text{ for every } h \in I_x\}$ is called the *tangent space* of X in x.

2.3. LEMMA. *If X is finitely defined then $E - \mathrm{codim}\, T_xX \leq \Omega - \mathrm{codim}_x X$ for every $x \in X$. In particular T_xX is complemented.*

PROOF. Put $p := \mathrm{codim}_x X$ and $x = 0$. Then $\dim H \cap X > 0$ for every $(p + 1)$-dimensional linear subspace H of E and therefore $\{0\} \neq T_x(H \cap X) \subset H \cap T_xX$. Hence $\mathrm{codim}\, T_xX \leq p$.

2.4. PROPOSITION. *Let X_x be the germ of an SF-analytic set at $x \in \Omega$. Denote $S(X_x)$ the set of all germs of complex submanifolds at x in which X_x is contained and finitely defined. $S(X_x)$ is nonempty and partially ordered by the inclusion. Moreover*

(a) *Each germ in $S(X_x)$ contains a minimal germ.*

(b) *$S_x \in S(X_x)$ is minimal iff $TS_x = TX_x$.*

(c) *Given two minimal germs M_x and N_x in $S(X_x)$ there exists a biholomorphic mapping germ $\Phi_x : M_x \to N_x$ which induces the identity on X_x.*

PROOF. Obviously $S(X_x)$ is nonempty. To prove (a) and (b) let M be a complex submanifold of a neighbourhood of x in E which contains a finitely defined representative X of X_x. Then $T_xX \subset T_xM$ and $T_xM - codim\ T_xX \leq M - codim_x X$.

Suppose $T_xX \neq T_xM$. Then there exists $u \in T_xM$, $u \neq 0$, and $f_x \in I_x$ with $Df_x(x)u \neq 0$. We may assume that f_x has a representative f with nonvanishing derivative on M. Then $S := f^{-1}(0)$ is a submanifold of M which contains X, and $T_xS - codim\ T_xX = (T_xM - codim\ T_xX) - 1$. After finitely many steps we arrive at $T_xX = T_xS$. This S must be minimal since $T_xX \subset T_xM$ for every submanifold M with $S_x \subset M_x$.

To prove (c) choose a topological decomposition $E = TX_x \oplus H$ and representatives M and N of M_x and N_x. Locally they are the graphs of mappings $TX_x \to H$. Let π be the canonical projection $E \to TX_x$. Then $\Phi := (\pi|N)^{-1} \circ (\pi|M)$ is biholomorphic at x and $\Phi_x \mid X_x = id$.

Let $M(X_x)$ be the set of minimal germs in $S(X_x)$. Because of 2.4. $M(X_x) = \{M_x : M_X$ is the germ of a complex submanifold at x with $X_x \subset M_x$ and $TX_x = TM_x\}$ and

$$emcodim_x X := emcodim\ X_x := M_x - codim\ X_x$$

is independent of $M_x \in M(X_x)$ and will be called the *minimal embedding codimension* of X in x.

This notion should not be confused with the embedding codimension $t_c\ codim_x X$ in [11]. In general they do not coincide. The above considerations show that an SF-analytic set X is near a point $x \in X$ always the zero set of a holomorphic SF-map f with $Ker\ Df(x) = T_xX$ and $codim\ Im Df(x) < \infty$.

2.5. (Local parametrization of SF-analytic sets). *Let X be SF-analytic in a domain Ω in E. Suppose $0 \in X$ and X is irreducible in 0. Choose a topological decomposition $E = T_oX \times H$ and let $p: E \to T_oX$ be the canonical projection. Then $p(X_o)$ is a finitely defined analytic germ in T_oX and $codim\ p(X_o) = emcodim\ X_o$.*

Let $T_oX = E_1 \times G$ be a topological decomposition such that $dim\ G = emcodim\ X_o$ and 0 is isolated in $p(X_o) \cap (\{0\} \times G)$. Put $E_2 := H \times G$.

Then every neighbourhood of x contains the product of two open balls $B_1 \subset E_1$ and $B_2 \subset E_2$ centered at 0 such that the canonical *projection* $\pi : X \cap (B_1 \times B_2) \to B_1$ *is an analytically ramified covering map with finitely many sheets in the sense of 2.1.*

PROOF. Choose $M_o \in M(X_o)$. Then p induces a biholomorphic mapping germ $p_o : M_o \to T_oX = TM_o$. Therefore $p(X_o)$ is finitely defined in T_oX and $codim\, p(X_o) = emcodim\, X_o$. Now apply 2.1 to a representative of $p(X_o)$ and the decompostion $T_oX = E_1 \times G$.

From 2.5 the local bifurcation theorem in [1] can be derived.

We close this section with some remarks on the intersection of SF-analytic sets. A closed subset X of a Banach manifold is SF-analytic iff it is locally the intersection of a complex submanifold M and an analytic set Y where M is finite dimensional or Y is finitely defined.

In general the intersection of two SF-analytic sets X and Y is not SF-analytic, simply because the intersection of complemented linear subspace is not always complemented.

2.6. LEMMA. *Let* X *and* Y *be SF-analytic subsets of a Banach manifold* Ω.

(a) *If* Y *is finite dimensional or finitely defined then* $X \cap Y$ *is SF-analytic.*

(b) *If* Y *is finitely defined and* $X \cap Y$ *is finite dimensional then* X *is finite dimensional.*

PROOF. (a) is obvious. To prove (b) let $x \in X \cap Y$ and $M_x \in M(X_x)$. Then $(X \cap Y)_x$ is finitely defined in M_x, hence its codimension is finite. By 2.3

$$dim\, TM_x = dim\, T(X \cap Y)_x + codim\, T(X \cap Y)_x$$

$$\leqq dim\, T(X \cap Y)_x + codim (X \cap Y)_x < \infty .$$

Therefore X must be finite dimensional in x.

3. HOLOMORPHIC SF-MAPS

Let Ω be a domain in the complex Banach space E. Suppose $0 \in \Omega$ and assume that $f : \Omega \to F$ is a holomorphic map into another Banach space F with splitting differential $Df(0)$ i.e. there are topological decompositions $E = Ker\,Df(0) \oplus M$ and $F = Im\,Df(0) \oplus J$. Then one can find a zero neighbourhood U in E such that

$$\Phi(x) := (\pi_{ImDf(0)} \circ f(x),\ \pi_{KerDf(0)}(x))$$

maps U biholomorphically onto a zero neighbourhood V in $Im\,Df(0) \times Ker\,Df(0)$. Putting $h : V \to J$, $h(y,z) := f \circ \phi^{-1}(y,z) - y$ one obtains

$$f \circ \Phi^{-1}(y,z) = y + h(y,z) \quad \text{for every} \quad (y,z) \in V.$$

Setting $\psi := (id_V, h) \circ \Phi$ we can reformulate the local representation of f in the following way: There are neighbourhoods U' in E and W' in $F \times Ker\,Df(0)$ such that $\psi : U' \to W'$ is a holomorphic embedding, $\psi(0)$ is an isolated point of $\psi(U) \cap J$, and $f(x) = \pi_F \circ \pi(x)$ for every $x \in U'$. (cf. [9]).

This local representation holds in particular for holomorphic SF-maps. If the differential in a point x is (semi-)Fredholm then automatically all differentials in a neighbourhood are (semi-) Fredholm as well. This follows from 3.1.

3.1. LEMMA. *The set $SF(E,F)$ of semi-Fredholm operators is open in $\mathcal{L}(E,F)$.*

PROOF. Let $T \in SF(E,F)$ and $E = Ker\,T \oplus L$, $F = Im\,T \oplus J$ be topological decompositions. $I := Im\,T$. Since $\pi_I \circ T|L$ is isomorphic there is a neighbourhood U of T in $\mathcal{L}(E,F)$ such that $\pi_I \circ S|L$ is isomorphic for every $S \in U$. We want to show $U \subset SF(E,F)$.

Let $S \in U$. Then $\pi_I : S(L) \to I$ is isomorphic, hence $S(L)$ is closed.

FIRST CASE: $codim\,I < \infty$. Then $Im\,S$ has finite codimension and is therefore complemented. Choose a linear subspace $M \supset L$ such that $E = M \oplus Ker\,S$ is an algebraic decomposition. Then $S|M \to Im\,S$ is bijective and induces an isomorphism between M/L and the space $Im\,S / S(L)$ which is finite dimensional because $codim\,S(L) = codim\,I < \infty$. Consequently M is closed and $Ker\,S$ is complemented.

SECOND CASE: $dim\,Ker\,T < \infty$. Then $Ker\,S$ is also finite dimensional and has a topological complement M such that $M = L \oplus N$ with $dim\,N < \infty$.

Because $S(L)$ is complemented and $S(N)$ is finite dimensional $Im\,S = S(M) = S(L) \oplus S(N)$ is also complemented.

3.2. LEMMA. *Let* $f : \Omega \to F$ *be a holomorphic SF-map. If* $X := f^{-1}(0)$ *is finite dimensional then* $Ker\,Df(x)$ *is finite dimensional for every* $x \in f^{-1}(0)$.

PROOF. Assume $dim\,Ker\,Df(x) = \infty$. Then $codim\,Df(x) < \infty$. Choosing near x biholomorphic coordinates Φ as above one can consider X near x as a finitely defined analytic subset of $Ker\,Df(x)$, hence X has finite codimension in $Ker\,Df(x)$. This contradicts the finite dimensionality of X.

3.3. DEFINITION AND PROPOSITION. *Let* X *and* Y *be SF-analytic subsets of domains in Banach spaces* E *and* F. *A mapping* $f : X \to Y$ *is called holomorphic in* $x \in X$ *if for* $M_x \in M(X_x)$ *and* $N_y \in M(Y_y)$ *with* $y = f(x)$ *the germ* f_x *has a holomorphic extension* $\hat{f}_x : M_x \to N_x$. *If it is holomorphic in* x *then the differential*

$$Df(x) := D\hat{f}_x(x) : T_xX \to T_yY$$

is a well-defined continuous linear map. f *is called (semi-)Fredholm in* x *(SF or F for short) if* $Df(x)$ *is a (semi-)Fredholm operator.*

PROOF. Because of 2.4.(c) the existence of a holomorphic extension $\hat{f}_x$ is independent of the particular choice of M_x and N_y. In order to show that $Df(x)$ does not depend on the choice of the extension $\hat{f}_x$ let $\tilde{f}_x : \tilde{M}_x \to \tilde{N}_y$ be another one, $\tilde{M}_x \in M(X_x)$, $\tilde{N}_x \in M(Y_y)$.

Extend $\hat{f}_x$ and $\tilde{f}_x$ to holomorphic germs $\hat{g}_x$ and $\tilde{g}_x$ in E and put $h_x := \hat{g}_x - \tilde{g}_x$. For every $\mu \in F'$ the germ $\mu \circ h_x$ vanishes on X_x and therefore $D(\mu \circ h_x)(x) = \mu \circ Dh_x(X)$ vanishes on T_xX. Hence $Dh_x(x)$ vanishes on T_xX and $D\hat{f}_x(x) = D\tilde{f}_x(x)$ on T_xX.

3.4. LEMMA. *Let* $f : X \to Y$ *be a holomorphic map between SF-analytic sets. If* f *is (semi-)Fredholm in* $x \in X$ *then* f *is (semi-)Fredholm in a neighbourhood of* x.

Notice that the assertion does not follow immediately from 3.1 because the tangent spaces can change with the base point.

PROOF. Let $y := f(x)$. Choose representatives M of $M_x \in M(X_x)$ and N of $N_x \in M(Y_y)$ which contain X and Y locally as finitely defined subsets. Let $\hat{f} : M \to N$ be a holomorphic extension of f. Then $D\hat{f}(x)$ is (semi-)Fredholm and because of 3.1 $D\hat{f}(z)$ is (semi-)Fredholm for every z near x.

Observe that $Df(z) = p \circ D\hat{f}(z) \circ j$ where $j : T_z X \to T_z M$ is the inclusion and $p : T_{f(z)}N \to T_{f(z)}Y$ is a projection. Since j and p are Fredholm operators $Df(z)$ is (semi-)Fredholm.

3.5. COROLLARY. *Suppose X is SF-analytic and irreducible in $x \in X$. Then there exist a neighbourhood U of x in X, a domain V in a Banach space and a finite surjective holomorphic Fredholm map $\Phi : U \to V$ such that $D\Phi(x)$ is surjective and $\dim \operatorname{Ker} D\Phi(x) = \operatorname{emcodim}_x X$.*

PROOF. Choose in 2.5 $U := X \cap (B_1 \times B_2)$, $V := B_1$, and $\Phi := \pi$. Then $D\Phi(x)$ is the canonical projection $T_x X = E_1 \times G \to E_1$, hence it is surjective and $\dim \operatorname{Ker} D\Phi(x) = \dim G = \operatorname{emcodim}_x X$. By 3.4 U and V can be made smaller such that all differentials of Φ are Fredholm.

The permanence properties of holomorphic SF-maps are not very good. For example the composite of two SF-maps is not always SF (if, however, one factor is even Fredholm then the composite map is SF). Moreover the restriction of a holomorphic SF-map $f : \Omega \to Y$ to an SF-analytic subset X of the domain Ω in E is not always SF either. Counter-examples are easily constructed with linear maps.

3.6. PROPOSITION. *Let $f : X \to Y$ be a holomorphic SF-map between SF-analytic subsets of domains Ω and Ξ in Banach spaces. Then the fibers of f are again SF-analytic in Ω.*

PROOF. Let $A := f^{-1}(y)$ and $x \in A$. Choose $M_x \in M(X_x)$ and a holomorphic SF-extension g_x of f_x to M_x. The fiber $g_x^{-1}(y)$ is SF-analytic in M_x. Since X_x is finitely defined in M_x Lemma 2.6(a) implies that $A_x = X_x \cap g_x^{-1}(y)$ is SF-analytic in M_x and hence in Ω_x.

Let us call a closed subset A of an SF-analytic subset X of a Banach manifold Ω *SF-analytic in X* if for every $a \in A$ there exists a neighbourhood U of a in X and a holomorphic SF-map $f : U \to F$ into a Banach space F such that $A \cap U = f^{-1}(0)$.

Since the restriction of a linear SF-operator to a subspace of finite codimension is again SF it is easy to see that a closed set $A \subset X$ is SF-analytic in X if and only if for every $a \in A$ there is a neighbourhood U in Ω and a submanifold M of U such that $X \cap U$ is finitely defined in M and $A \cap U$ is SF-analytic in M.

Obviously 3.6 implies the following transitivity result.

3.7. COROLLARY. *Let* X *be an SF-analytic subset of a Banach manifold* Ω *and* $A \subset X$ *be SF-analytic in* X. *Then* A *is also SF-analytic in* Ω.

A holomorphic map $f : X \to Y$ between SF-analytic sets will be called an *embedding* if $f(X)$ is SF-analytic in Y and $f|X \to f(X)$ is a biholomorphic SF-map. f is called an *immersion* in $x \in X$ if $Df(x)$ is an injective SF-operator. As in finite dimensions an immersion is a local embedding.

3.8. LEMMA. *A holomorphic map* $f : X \to Y$ *between SF-analytic sets is an immersion in* $x \in X$ *iff there are neighbourhoods* U *of* x *and* V *of* $f(x)$ *such that* $f|U \to V$ *is an embedding.*

PROOF. Suppose f is an immersion in x. Set $y := f(x)$. Choose $M_x \in M(X_x)$, $N_y \in M(Y_y)$ and a holomorphic extension g_x of f_x. Then g_x is an immersion in x and the analogous result for manifolds implies that for appropriate representatives $g : M \to N$ is an embedding. Since $X \cap M$ is finitely defined in M, $g(X \cap M)$ is finitely defined in the submanifold $g(M)$ of N, hence SF-analytic in N. It follows that $f(X \cap M)$ is SF-analytic in $Y \cap N$.

4. MAPPING THEOREMS

An important theorem in finite dimensional complex analysis is Remmert's proper mapping which states that a proper holomorphic mapping maps analytic sets onto analytic sets.

Recall that a map is called *proper* if it is continuous and the preimages of compact sets are compact, and it is called *finite* if it is proper and has finite fibers. Two infinite dimensional versions of the mapping theorem are known. The first one is proved in [6,9]:

Let Ω *and* Ξ *be Banach manifolds and* $f : \Omega \to \Xi$ *a holomorphic Fredholm map. If* $X \subset \Omega$ *is a finitely defined analytic set and* $f|X$ *is proper then* $f(X)$ *is a finitely defined analytic subset of* Ξ.

The second one is found in [9, 10, 4, 5] (in different generalizations):

Let X be locally finite dimensional complex space, Ξ a Banach manifold and $f : X \to \Xi$ a proper holomorphic map. Then $f(X)$ is a finite dimensional analytic subset of Ξ.

We shall derive a mapping theorem for finite SF-maps. At first a local version.

4.1. THEOREM. *Let X be an SF-analytic subset of a domain Ω in the Banach space E, and $f : X \to F$ a holomorphic map into a Banach space F. Suppose $x \in X$ is isolated in the fiber $f^{-1}(f(x))$ and $Df(x)$ is semi-Fredholm.*

Then there are arbitrarily small open neighbourhoods U of x and V of $f(x)$ such that $f(U) \subset V$ and

(a) *$f|U \to V$ is finite.*

(b) *$f(U)$ is analytic in V. If $\operatorname{ind} Df(x) > -\infty$ then $f(U)$ is finitely defined and*

$$\operatorname{codim}_{f(x)} f(U) = \operatorname{emcodim}_x X - \operatorname{ind} Df(x).$$

(c) *$f|U \to f(U)$ is open in x i.e. f maps neighbourhoods of x onto neighbourhoods of $f(x)$.*

PROOF. We may assume that X lies in Ω with minimal codimension and that g is a holomorphic SF-extension of f to Ω. Because of 2.6(b) $g^{-1}(f(x))$ is finite dimensional and 3.2 implies $\dim \operatorname{Ker} D_g(x) < \infty$. Assume $x = 0$ and $g(x) = 0$. Choose a local representation of g in terms of ψ, U', and W' as in the beginning of section 3. Since 0 is isolated in $f^{-1}(0) \cap X$ there is a ball B in $\operatorname{Ker} Dg(0)$ with center 0 such that $\psi(X)$ and $\{0\} \times \partial B \subset F \times \operatorname{Ker} Dg(0)$ do not meet. $\psi(X)$ is closed in W' and $\{0\} \times \partial B$ is compact because $\operatorname{Ker} Dg(0)$ is finitedimensional. Therefore there exists a zero neighbourhood V in F and positive reals $r < s$ such that $\psi(X)$ and $V \times (B(0,s) - \overline{B}(0,r))$ are disjoint. Hence the projection $\pi_F|\psi(X) \cap (V \times B(0,s)) \to V$ is finite. Putting $U = U' \cap g^{-1}(V) \cap X$ and observing $\pi_F \circ \psi|U = f|U$ we obtain that $f|U \to V$ is finite. This prove (a).

(b) and (c) follow from a theorem on the projection of analytic sets applied to $\pi_F | V \times B(0,s) \to V$ (see prop. II. 3.7 and III. 2.2.1 in [6]).

This result corresponds to a well-known theorem of finite dimensional complex analysis (see e.g. [3, Th. 3.2(b), p. 133]). The usual dimension formula

$$dim_{f(x)} f(U) = dim\, X$$

can be transformed into the codimension formula in 4.1. If E and F are finite dimensional and X is embedded into E with minimal codimension in x then

$$ind\, Df(x) = dim\, E - dim\, F$$

and

$$\begin{aligned} codim_{f(x)} f(U) &= dim\, F - dim_{f(x)} f(U) \\ &= dim\, E - ind\, Df(x) - dim_{f(x)} f(U) \\ &= codim_x X - ind\, Df(x). \end{aligned}$$

4.2. THEOREM. *Let $f : X \to Y$ be a finite holomorphic SF-map between SF-analytic sets in Banach manifolds. Then $f(X)$ is analytic. If $ind\, f > -\infty$ then $f(X)$ is finitely defined and*

$$Y - codim_y f(X) = min\, \{ emcodim_x X - emcodim_y Y - ind\, Df(x) \ : x \in f^{-1}(y) \} \ .$$

PROOF. $f(X)$ is closed since f is proper. Let $y \in f(X)$ and $f^{-1}(y) = \{x_1, \ldots, x_n\}$. Because f is proper there are neighbourhoods V of y and U_j of x_j such that the U_j are pairwise disjoint and $f^{-1}(V) = \cup \, \{U_j : j = 1, \ldots, n\}$. Moreover we may assume that V lies in a domain Ξ of a Banach space with minimal codimension in y. According to 4.1 we can make Ξ and U_j smaller such that each $f(U_j)$ is analytic in Ξ, and if $ind\, Df(x_j) > -\infty$ then

$$\Xi - codim_y f(U_j) = emcodim_{x_j} X - ind\, Df\,(x_j).$$

Hence $f(X) \cap V$ is analytic and the above formula holds.

In finite dimensions

$$emcodim_x X - emcodim_y Y - ind\, Df(x) = dim_y Y - dim_x X$$

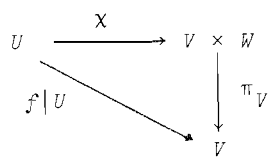
and therefore the above formula is transformed to

$$dim_y f(X) = max\, \{dim_x X : x \in f^{-1}(y)\}.$$

5. LOCAL FACTORIZATION

In this section it is shown that a holomorphic SF-map can be locally factored into a finite map and a projection as it is the case in finite dimensions (see e.g. [3]). As a consequence the fiber dimension is semi-continuous.

5.1. PROPOSITION. *Let $f : X \to Y$ be a holomorphic SF-map between SF-analytic sets and $x \in X$. Then there are arbitrarily small open neighbourhoods U of x, V of $f(x)$, a domain W in a Banach space G, and a finite holomorphic SF-map $\chi : U \to V \times W$ such that the following diagram commutes:*



$\chi(U)$ is analytic in $V \times W$ and $\chi : U \to \chi(U)$ is open in x. If $k := dim\, f^{-1}(f(x)) < \infty$ and $ind\, Df(x) > -\infty$ then $\chi(U)$ is finitely defined and

$$ind\, D\chi(x) = ind\, Df(x) - k,$$

$$codim_{\chi(x)} \chi(U) = emcodim_x X - emcodim_{f(x)} Y - ind\, Df(x) + k.$$

Let us remark that in finite dimensions the above formula corresponds to the well-known dimensions formula

$$dim_{\chi(x)} \chi(U) = dim_x X - k$$

because

$$emcodim_x X - emcodim_{f(x)} Y - ind\, Df(x) = dim_{f(x)} Y - dim_x X.$$

PROOF. According to 3.5 there exist a neighbourhood V of x in $f^{-1}(f(x))$, a domain W in a Banach space G and a finite holomorphic SF-map $\Phi : V \to W$ with $dim\, Ker\, D\Phi(x) < \infty$ and $Im\, D\Phi(x) = G$. Φ can be extended to a holomorphic map $\psi : U \to W$ in a neighbourhood U of x in X. Define $\chi := (f|U, \psi) : U \to Y \times W$. Then the above diagram commutes and

$$Ker\, D\chi(x) = Ker\, Df(x) \cap Ker\, D\psi(x) = Ker\, D\Phi(x)$$

hence

$$dim\, Ker\, D\chi(x) = dim\, Ker\, D\Phi(x) < \infty.$$

To see that $Im\, D\chi(x)$ is complemented apply Lemma 5.2 below to $T_1 := Df(x)$, $T_2 := D\psi(x)$, and $G_o = \{0\}$. Thus $D\chi(x)$ is semi-Fredholm.

If $k = dim_x f^{-1}(f(x)) < \infty$ then $dim\, Ker\, Df(x) < \infty$ by 3.2 and

$$dim\, Ker\, D\chi(x) = dim\, Ker\, D\Phi(x) = dim\, Ker\, Df(x) - dim\, Im\, D\Phi(x),$$

$$codim\, Im\, D\chi(x) = codim\, Im\, Df(x) + codim\, Im\, D\Phi(x) = codim\, Im\, Df(x).$$

Because of $dim\, Im\, D\Phi(x) = dim\, G = k$ (Φ is finite) one obtains

$$ind\, D\chi(x) = ind\, Df(x) - k.$$

Since x is isolated in the fiber $\chi^{-1}(\chi(x))$ Theorem 4.1 can be applied to obtain the other assertion (for possibly smaller U, V, and W).

5.2. LEMMA. *Let E, F, and G be Banach spaces, $T_1 : E \to F$ and $T_2 : E \to G$ be continuous linear maps. Suppose $Im\, T_2 = Im(T_2|Ker\, T_1)$ and $F = Im\, T_1 \oplus F_o$ and $G = Im\, T_2 \oplus G_o$ are topological decompositions. Define $T := (T_1, T_2) : E \to F \times G$. Then $Im\, T = Im\, T_1 \times Im\, T_2$ and*

$$F \times G = Im\, T \oplus (F_o \times G_o)$$

is a topological decomposition.

PROOF. Let $E = E_1 \times Ker\, T_1$. Then

$$Im\, T = T(E_1) + T(Ker\, T_1) = T(E_1) + (\{0\} \times T_2(Ker\, T_1)) = Im\, T_1 \times Im\, T_2 .$$

It is easy to see that the stated decomposition is algebraically correct. Since all factors are closed the decomposition is a topological one.

5.3. COROLLARY (Semi-continuity of the fiber dimension). *Let $f : X \to Y$ be a holomorphic SF-map between SF-analytic sets. Then for every $x \in X$ there is a neighbourhood U such that*

$$dim_z f^{-1}(f(z)) \leq dim_x f^{-1}(f(x)) \quad \text{for every} \quad z \in U$$

i.e. the function $x \mapsto dim_x f^{-1}(f(x))$ is upper semi-continuous.

PROOF. Suppose $k := dim\, f^{-1}(f(x)) < \infty$ (otherwise the inequality is trivial). Choose the local situation as in 5.1. Then $\chi \,|\, f^{-1}(f(x)) \cap U \to \{f(z)\} \times W$ is finite for every $z \in U$ and the inequality follows from 4.1(b) or from the corresponding finite dimensional result.

The rank theorem in [12] has a counterpart for SF-maps with constant fiber dimension.

5.4. FACTORIZATION LEMMA. *Let $f : X \to Y$ be a holomorphic SF-map between SF-analytic sets. Suppose that for every z in a neighbourhood of $x \in X$ the dimension of the fiber $f^{-1}(f(z))$ in z is the same finite number k. Then there are arbitrarily small open neighbourhoods U of x and V of $f(x)$, a domain W in $\mathbb{C}^k$, an analytic subset V' of V and a finite surjective holomorphic map $\chi' : U \to V' \times W$ such that the following diagram commutes*

$$\begin{array}{ccc} U & \xrightarrow{\ \chi\ } & V' \times W \\ {\scriptstyle f|U}\downarrow & & \downarrow{\scriptstyle \pi_{V'}} \\ V & \hookleftarrow & V' \end{array}$$

If $ind\, Df(x) > -\infty$ then V' is finitely defined and

$$codim_{f(z)} f(U) = emcodim_x X - emcodim_{f(z)} Y - ind\, D f(z) + k.$$

If X and Y are manifolds then the above formula can be written in a symmetrical form

$$\operatorname{codim} \operatorname{Im} Df(z) - \operatorname{codim}_{f(z)} f(U) = \dim \operatorname{Ker} Df(x) - \dim_z f^{-1}(f(z)).$$

Again notice that for finite dimensional X and Y this codimension formula is equivalent to

$$\dim_{f(z)} f(U) = \dim_z X - \dim_z f^{-1}(f(z)).$$

PROOF. Choose the local situation as in 5.1. Put $\chi(x) = (f(x),0)$. Then $A := \chi(U) \cap (V \times \{0\})$ is analytic in $V \times \{0\}$ and $V' := \pi_V(A)$ is analytic in V.

For each $y \in f(U)$ the map $\chi|f^{-1}(y) \cap U \to \{y\} \times W$ is finite and surjective since W is a domain with the dimension of $f^{-1}(y)$. Hence $\chi(U) = V' \times W$. Define $\chi' := \chi|U \to V' \times W$.

If $\operatorname{ind} Df(x) > -\infty$ then $\chi(U)$ is finitely defined in $V \times W$ and hence V' is so in V. The codimension formula follows from 5.1 and from

$$V - \operatorname{codim}_{f(z)} V' = (V \times W) - \operatorname{codim}_{\chi(z)} V' \times W = \operatorname{codim}_{\chi(z)} \chi(U).$$

If X and Y are manifolds then the embedding codimensions vanish and the stated formula follows from

$$\operatorname{ind} Df(z) = \dim \operatorname{Ker} Df(z) - \operatorname{codim} \operatorname{Im} Df(z).$$

6. A CRITERION FOR OPENESS

Immediately from 5.1 there follows a criterion for openess of holomorphic F-maps.

6.1. PROPOSITION. *Let $f : X \to Y$ be a holomorphic F-map between SF-analytic sets. If*

$$\operatorname{emcodim}_{f(x)} Y - \operatorname{emcodim}_x X = \dim_x f^{-1}(f(x)) - \operatorname{ind} Df(x)$$

then f is open in x.

For manifolds the converse implication holds also.

6.2. THEOREM. *Let $f : X \to Y$ be a holomorphic F-map between manifolds. Then f is open if and only if*

$$\dim_x f^{-1}(f(x)) = \operatorname{ind} Df(x) \quad \text{for} \quad \text{every} \quad x \in X.$$

(Notice that $\operatorname{ind} Df(x)$ is constant if X is connected).

PROOF. Suppose that f is open and that $x \in X$. We may assume that X and Y are domains in Banach spaces E and F. Define $K := \operatorname{Ker} Df(x)$ and $I := \operatorname{Im} Df(x)$. Choose local coordinates such that $E = I \times K$, $F = I \times J$, and $f(y,z) = (y,h(y,z))$ for $(y,z) \in U \subset I \times K$ where $h : U \to I$ and $x = (0,0)$. We show that $h(y, \cdot)$ is open for each y. Let V be open in K and W be an open neighbourhood of y. Then $f(W \times V)$ is open, hence

$$(\{y\} \times J) \cap f(W \times V) = (\{y\} \times J) \cap f(\{y\} \times V)$$

$$= f(\{y\} \times V) = \{y\} \times h(\{y\} \times V)$$

is open in $\{y\} \times J$.

In particular $g := h(0, \cdot) : X \cap \operatorname{Ker} Df(x) \to J$ is open. The criterion for openess in finite dimensions (see e.g. [3, p. 145]) implies $\dim \operatorname{Ker} Df(x) = \dim J + \dim_0 g^{-1}(g(0))$. Since $g^{-1}(g(0)) = f^{-1}(f(x)) \cap U$ and $\operatorname{ind} Df(x) = \dim \operatorname{Ker} Df(x) - \dim J$ we obtain $\dim_x f^{-1}(f(x)) = \operatorname{ind} Df(x)$.

7. THE SINGULAR SET OF A HOLOMORPHIC FREDHOLM MAP

The singular set $S(f)$ of a holomorphic map $f : X \to Y$ between Banach maniforlds X and Y is the set of all points $x \in X$ in which the differentials $Df(x)$ are not surjective. For Fredholm maps f this set is finitely defined analytic and its codimension can be estimated as in finite dimensions (see e.g. [3, p. 97]).

7.1. LEMMA. *Let A and B be analytic subsets of a Banach manifold Ω. Suppose B is near $x \in A \cap B$ a submanifold. Then*

$$\Omega - \operatorname{codim}_x A \geqq B - \operatorname{codim}_x A \cap B.$$

PROOF. We may assume that Ω is a domain in a Banach space E and B

is near x a complemented linear subspace of E. Then there is a linear subspace C of B with $\dim C = B - \operatorname{codim}_x A \cap B$ such that x is isolated in $A \cap B \cap C$. Therefore $\Omega - \operatorname{codim}_x A \geqq \dim C$.

7.2. LEMMA. *Let $f : X \to Y$ be a holomorphic map between Banach manifolds and Z be an analytic subset of Y. Then*

$$\operatorname{codim}_{f(x)} Z \geqq \operatorname{codim}_x f^{-1}(Z) \quad \text{for every} \quad x \in f^{-1}(Z).$$

PROOF. Define $\Omega := X \times Y$, $B := \operatorname{graph} f$, $A := X \times Z$ and apply 7.1.

7.3. PROPOSITION. *Let E and F be Banach spaces. The set F_o of nonsurjective linear Fredholm operators is a finitely defined analytic subset of the set $\mathsf{F}(E,F)$ of all linear Fredholm operators. Moreover*

$$\operatorname{codim}_T \mathsf{F}_o \leqq \operatorname{ind} T + 1 \quad \text{for every} \quad T \in \mathsf{F}_o \quad \text{with} \quad \operatorname{ind} T \geqq 0.$$

PROOF. In [1] it is proved that F_o is a finitely defined analytic subset of $\mathsf{F}(E,F)$. More precisely it is shown that in a neighbourhood U of $T \in \mathsf{F}_o$ there exists a holomorphic map $\Psi : U \to \mathcal{L}(K,J)$ such that $K = \operatorname{Ker} T$, J is a complement of $\operatorname{Im} T$ and F_o is the preimage $\psi^{-1}(\mathcal{L}_o)$ of the set $\mathcal{L}_o$ of nonsurjective operators in $\mathcal{L}(K,J)$. Notice that $\mathcal{L}(K,J)$ is a finite dimensional vector space. Now suppose $\operatorname{ind} T \geqq 0$. Then $\dim K \geqq \dim J$ and the nonsurjective linear operators from K to J are exactly those linear operators the rank of which is strictly smaller than $\dim J$. According to [3, p.98] they form an irreducible analytic subset of $\mathcal{L}(K,J)$ with the codimension $\dim K - \dim J + 1 = \operatorname{ind} T + 1$. With 7.2 one obtains

$$\operatorname{codim}_T \mathsf{F}_o = \operatorname{codim}_T \psi^{-1}(\mathcal{L}_o) \leqq \operatorname{codim} \mathcal{L}_o = \operatorname{ind} T + 1.$$

7.4. THEOREM. *Let $f : X \to Y$ be a holomorphic Fredholm map between Banach manifolds with $\operatorname{ind} Df(x) \geqq 0$ for every $x \in X$. Then $S(f)$ is a finitely defined analytic subset of X and $\operatorname{codim}_x S(f) \leqq \operatorname{ind} Df(x) + 1$ for every $x \in S(f)$.*

PROOF. We may assume that X and Y are domains in Banach spaces E and F. Then $Df : X \to \mathsf{F}(E,F)$ is holomorphic and $S(f) = (Df)^{-1}(\mathsf{F}_o)$. Now apply 7.2 and 7.3.

7.5. COROLLARY. *Let X and Y be connected Banach manifolds and $f : X \to Y$ a finite holomorphic Fredholm map with nonnegative index. Then $S(f)$, $C := f(S(f))$, and $f^{-1}(C)$ are analytic subsets with codimension one and $f|X - f^{-1}(C) \to Y - C$ is a covering map.*

PROOF. $ind\, Df(x)$ is constant and because of 4.1 it vanishes, and $f(X)$ is open. Therefore f is surjective. According to the theorem of Sard-Smale [12] the set of critical values C is meager in Y, hence $C \neq Y$, $f^{-1}(C) \neq X$, and $S(f) \neq X$. By 7.4 $codim\, S(f) = 1$ and by 4.2 $C = f(S(f))$ is analytic and onecodimensional. $f|X - f^{-1}(C)$ is locally biholomorphic because the differentials are isomorphic since their index is zero. The map is also finite, hence it is a covering map.

8. GRAPH THEOREMS

Graph theorems characterize the regularity of a map by geometrical properties of its graph, for example continuity by closedness. Recall the following characterizations of differentiability.

1. A map $f : X \to Y$ between complex (or real C^r-) Banach manifolds is holomorphic (or real C^r- differentiable) if and only if its graph Γ is a complex (or real C^r-) submanifold of $X \times Y$ and if for every $(x,y) \in \Gamma$ the tangent space of Γ in (x,y) is a topological complement of $\{0\} \times T_yY$ in $T_xX \times T_yY$.

2. If X and Y are (locally finite dimensional) reduced complex spaces and X is normal then Remmert proved that a map $f : X \to Y$ is holomorphic if and only if its graph Γ is analytic in $X \times Y$ and if $dim_{(x,y)}\Gamma = dim_x X$ for every $(x,Y) \in \Gamma$ [8].

In [1] it is shown that in infinite dimensions the analyticity of the graph is too weak to guarantee that the map is holomorphic. There exists a map from the open unit disk into a Banach space which is a homeomorphism onto its image and has an analytic graph but is not holomorphic. Holomorphy can, however, be characterized by the SF-analyticity of the graph.

8.3. THEOREM. *Let $f : \Omega \to F$ be a map from the domain Ω in a Banach space E into the Banach space F and let Γ be its graph. Then the following properties are equivalent:*

(i) *f is holomorphic.*

(ii) *Γ is an SF-analytic subset of $\Omega \times F$ and moreover $\pi_E(T_p\Gamma) = E$; and $\mathrm{emcodim}_p\Gamma = \dim(T_p\Gamma \cap (\{0\} \times F))$ for every $p \in \Gamma$.*

(iii) *For every $p \in \Omega \times F$ there are a neighbourhood U and a holomorphic map $\Phi : U \to H$ in a Banach space H such that $\Gamma \cap U = \Phi^{-1}(0)$ and $D_2\Phi(p)$ is a Fredholm operator with index 0.*

PROOF. The equivalence of (i) and (iii) is proved in [1]. (i) implies (ii) because of 8.1. Thus it remains to show that (ii) implies (i).

Let $p = (x,y) \in \Gamma$. Then there are a neighbourhood U of p, a complex submanifold M of U and a topological decomposition $F = F_1 \oplus F_2$ such that $\Gamma \cap U \subset M$, $\dim F_1 = \mathrm{emcodim}_p\Gamma = M-\mathrm{codim}_p\Gamma$, $T_pM = E \times F_1$, and p is isolated in $\Gamma \cap (\{x\} \times F_1)$. Making U smaller we can achieve that the projection $E \times F \to E \times F_1$ induces a biholomorphic map $h : M \to V$ onto a domain V in $E \times F_1$. The set $A := h(\Gamma)$ is a finitely defined analytic subset of V. Making again V smaller we may assume that A is the finite union of finitely defined analytic sets which are irreducible in p. Because of 2.1 we can find one of them, say A_j, such that the projection $E \times F_1 \to E$ induces an analytically ramified covering map from A_j onto a neighhourhood of x. Since Γ is the graph of a map this covering has only one sheet and furthermore A_j must be the only irreducible component of A. Because A is a submanifold outside of the bifurcation set it is the graph of a locally bounded map $g : W \to F_1$ from a neighbourhood W of x in E into F_1 which is holomorphic outside of proper analytic subset of W. The Riemann removable singularity theorem [6, p. 24] implies that g is holomorphic everywhere. Hence $f|W = h^{-1} \circ g$ is holomorphic.

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