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Bounded Holomorphic Embeddings of the Unit Disk into Banach Spaces

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It is shown that, in contrast to $\mathbb{C}^n$, infinite dimensional complex Banach spaces E can possess bounded complex closed submanifolds of positive dimension. If E contains c_0 or L^1/H^1_0 then the unit disk D can be embedded into E as a bounded complex closed submanifold. If, however, E has the analytic Radon-Nikodym property then no bounded embedding exists.

0. INTRODUCTION

It is a well-known consequence of the maximum principle that every bounded analytic subset of $\mathbb{C}^n$ is finite. DOUADY showed [3] that every metrizable compact topological space is homeomorphic to an analytic subset of a suitable Banach space. His examples, however, are highly pathological; they do not possess any regular point. Therefore the question arises whether a Banach space can contain bounded complex closed submanifolds X of positive dimension. Note that the maximum principle does not apply because X need not be compact.

In this note we investigate as a first step the special case where X is isomorphic to the unit disk D. This amounts to asking when there exists a holomorphic embedding of D with bounded image. Recall that a holomorphic embedding of D into a Banach space E is a holomorphic map $f: D \to E$ such that $f(D)$ is a closed complex submanifold and $f|D \to f(D)$ is biholomorphic. D can be embedded into $\mathbb{C}^2$ [6,10] but there is no bounded holomorphic embedding $f:D \to \mathbb{C}^n$. We show that the existence or nonexistence of bounded holomorphic embeddings $f:D \to E$ depends on E. For either case we determine a class of Banach spaces E.

1. NONEXISTENCE OF BOUNDED EMBEDDINGS

A holomorphic map $f:D \to E$ is an embedding if and only if it is regular, injective, and proper. The key to our nonexistence results is a characterization of properness whose simple proof will be omitted.

LEMMA. *A continuous map* $f:D \to E$ *is proper if and only if for every sequence* $(z_n)_n$ *in* D *converging to a boundary point* $w \in \partial D$ *the image sequence* $(f(z_n))_n$ *does not converge.*

The lemma suggests to use vector-valued H^∞-theory. We denote by $H^\infty(D,E)$ the space of bounded holomorphic mappings $f:D \to E$. If every $f \in H^\infty(D,E)$ admits almost everywhere radial boundary values (in the topology of E) then we say that E has the *analytic Radon-Nikodym property* (aRNP). It is classical that $E = \mathbb{C}$ has the aRNP. But there are spaces without the aRNP like e.g. c_o, the disk algebra, $H^\infty(D,\mathbb{C})$, their strong duals, and L^1/H^1_o (see [1,2,5] for details). The aRNP devolves on subspaces. The above lemma implies immediately

PROPOSITION. *If* E *has the* aRNP *then there exists no proper map* $f \in H^\infty(D,E)$.

COROLLARY. *If* E *has the* aRNP *then there exists no bounded holomorphic embedding of* D *into* E.

This statement gets interesting by the fact that the aRNP is equivalent to a weakened form of the usual Radon-Nikodym property (RNP) [5]: E possesses the aRNP if and only if every E-valued measure μ which is defined on the Borel sets of ∂D and has bounded variation and whose Fourier coefficients $\hat{\mu}(n) := \int_0^{2\pi} e^{-int} d\mu(t)$ vanish for $n < 0$

has a L^1-density with respect to the Haar measure on ∂D.

Examples show that the aRNP is strictly weaker than the RNP e.g. $L^1[0,1]$ has the aRNP but not the RNP [1].

COROLLARY. *If* E *has the* RNP *then there exists no bounded holomorphic embedding of* D *into* E .

A list of spaces which have the RNP can be found in [2, p.218]. Among them there are the reflexive spaces, the separable duals, and the scalar-valued Hardy spaces $H^p(D,\mathbb{C})$ with $1 \leq p < \infty$.

DOUADY's examples of non-discrete compact analytic sets live in the strong duals of the Banach spaces Lip(K) of Lipschitz continuous complex functions on compact metric spaces K. W.HENSGEN pointed out to me that for $I := [0,1]$ the space Lip(I)' possesses the aRNP but not the RNP: Since every $f \in \mathrm{Lip}(I)$ is differentiable a.e. with derivative $f' \in L^\infty(I)$ the map $\mathrm{Lip}(I) \to \mathbb{C} \times L^\infty(I)$, $f \mapsto (f(0), f')$ is an isomorphism, and Lip(I)' is isomorphic to $\mathbb{C} \times L^\infty(I)'$. The latter space does not have the RNP because $L^1(I) \subset L^\infty(I)'$ does not have it. By theorem 1 in [1] it has, however, the aRNP since it is a Banach lattice which is weakly sequentially complete and therefore does not contain c_0.

Consequently, Lip(I)' does not contain a bounded copy of the unit disk although it possesses a non-discrete compact analytic subset, namely the spectrum of the Banach algebra Lip(I) which is homeomorphic to I [3].

Finally, let us remark that in [7] it is shown that every bounded strictly convex domain in $\mathbb{C}^n$ can be embedded into the open unit ball of ℓ_2. Of course, this result does not contradict the above corollary because the embedding constructed in [7] is not proper when regarded as a mapping from D into the whole space ℓ_2; its image is not closed in ℓ_2.

2. EXISTENCE OF BOUNDED EMBEDDINGS

PROPOSITION. *There is a bounded holomorphic embedding* f *of* D *into* c_o *(the space of null sequences).*

Proof. Let $f:D \to c_o$ be defined by $f(z):=(z^k)_{k\in N}$. Then f is holomorphic, regular, and injective, and f(D) is contained in the open unit ball of c_o. If $(z_n)_n$ is any sequence in D converging to a boundary point $w \in \partial D$ then $(|z_n^k|)_n$ converges to 1 for every k. Hence $(f(z_n))_n$ does not converge. Therefore f is proper.

COROLLARY. *If* E *contains an isometric copy of* c_o *then there exists a holomorphic embedding* $g:D \to E$ *such that* g(D) *is contained in the unit sphere of* E.

Proof. Let f be the embedding in the above proposition and $j:c_o \to E$ the inclusion. Then the composed map

$$D \xrightarrow[(1,f)]{} \mathbb{C}\times c_o \simeq c_o \xrightarrow[j]{} E$$

has the desired properties.

The disk algebra and $H^\infty(D,\mathbb{C})$ contain c_o [9], hence D can be embedded as a bounded submanifold. For $1 \le p < \infty$ there is no bounded embedding of D into H^p because H^p has the RNP.

A Banach space with the aRNP cannot contain c_o. The converse implication does not hold. The standard counter-example is the space L^1/H_o^1 where $L^1:=L^1(\partial D)$ and $H_o^1:=\{f\in H^1: f(0)=0\}$ is identified with the space of the corresponding boundary values. The usual method of proving that L^1/H_o^1 does not have the aRNP yields also a bounded embedding of D into L^1/H_o^1.

PROPOSITION. *There exists a holomorphic embedding* f *of* D *into* L^1/H_o^1 *such that* f(D) *lies in the open unit ball.*

Proof. For $z\in D$ let $F(z)\in L^1(\partial D)$ be the function $F(z)(t):= = e^{it}(e^{it}-z)^{-1}$, and define $f(z) := F(z)+H_o^1$.
Then $f:D \to L^1/H_o^1$ is holomorphic and regular, and f(D) lies

in the unit ball of L^1/H^1_o (see [1]).

f is proper: Let $(z_n)_n$ be a sequence in D converging to a boundary point $w \in \partial D$. One can extract a subsequence $(x_k)_k$ such that $1-|x_{k+1}| \leq \frac{1}{2}(1-|x_k|)$ for every k. By theorem 9.2 in [4] $(x_k)_k$ is uniformly separated and therefore by CARLESON's interpolation theorem one can find a function $h \in H^\infty(D,\mathbb{C})$ such that $h(x_k) = (-1)^k$ for every k. Hence $\langle f(z_n),h\rangle = \int_0^{2\pi} F(z_n)(t)h(t)dt = 2\pi h(z_n)$ does not converge; and since $(L^1/H^1_o)' = H^\infty$ this shows that $(f(z_n))_n$ is not even weakly convergent.

f is injective: Suppose that $f(z) = f(w)$ for some $z,w \in D$. The identity $h(t):=t$ represents an element of $(L^1/H^1_o)' = H^\infty$ and $2\pi h(z) = \langle f(z),h\rangle = \langle f(w),h\rangle = 2\pi h(w)$, hence $z=w$.

L^1/H^1_o can be considered as a closed subspace of $(H^\infty)'$ by identifying $\psi \in L^1$ with the functional $h \mapsto \int_0^{2\pi} h\psi$. Thus the above embedding f yields a bounded embedding $g:D \to (H^\infty)'$. Up to a factor this g is the canonical map from D into the spectrum of H which assigns to a point $z \in D$ the evaluation homomorphism $\hat{z}:h \mapsto h(z)$.

3. FINAL REMARKS AND OPEN PROBLEMS

(1) Characterize all Banach spaces which admit no bounded holomorphic embedding of D !

The aRNP means that every $f \in H^\infty(D,E)$ has boundary values on almost every radius. For the nonexistence of bounded embeddings it is sufficient that every injective and regular $f \in H^\infty(D,E)$ admits a limit on at least one (not necessarily radial) boundary sequence. Hence it is not unlikely that the nonexistence of bounded embeddings is strictly weaker than the aRNP. For Banach lattices, however, both properties are equivalent.

PROPOSITION. *If E is a Banach lattice then the following properties are equivalent:*

(i) There exists no bounded holomorphic embedding $D \to E$.

(ii) E *has the* aRNP.

(iii) No subspace of E *is isomorphic to* c_o.

(iv) E *is weakly sequentially complete.*

Proof. The equivalence of (iii) and (iv) is well-known (see e.g. [8]). The equivalence of (ii) and (iii) is proved in [1]. In section 1 we showed that (ii) implies (i), and in section 2 that (i) implies (iii).

(2) We return to the original question:

Which Banach spaces contain bounded closed complex submanifolds of positive dimension?

In section 1 we gave a class of Banach spaces which do not contain a bounded copy of the unit disk. Nevertheless, in these spaces - even in the Hilbert spaces - there might exist other bounded closed complex submanifolds of positive dimension.

(3) Conversely one can ask:

Which (reduced) complex spaces X can be realized as bounded closed analytic subspaces of suitable Banach spaces?

It is clear that X must be a Stein space which is holomorphically complete with respect to its bounded entire functions. The results in section 2 imply that every complex space which admits an embedding into a polydisk admits also a bounded embedding in any Banach space E which contains c_o. Is that also true for $E = L^1/H^1_o$?

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