THE SPECTRUM AS ENVELOPE OF HOLOMORPHY OF A DOMAIN OVER AN ARBITRARY PRODUCT OF COMPLEX LINES

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§0. Introduction.

For a domain X over $\mathbf{C}^n$ the following conditions are equivalent ([2], p.283):

(1) X is a domain of holomorphy i.e. X coincides with its envelope of holomorphy.

(2) X is holomorphically convex i.e. for any compact subset K of X, the holomorphically convex hull $\widehat{\mathcal{H}}(K,X)$ is compact.

(3) X is Stein.

(4) X is the domain of existence of a holomorphic function.

(5) X is convex with respect to the plurisubharmonic functions.

(6) $-\log d_X$ is plurisubharmonic.

(7) For any sequence $(x_n)_{n\in \mathbf{N}}$ in X with $d_X(x_n) \to 0$, there exists a holomorphic function f on X such that $f(x_n) \to \infty$.

(8) X coincides with Spec $\mathcal{H}(X)$, the space of all nonzero complex algebra homomorphisms on the algebra $\mathcal{H}(X)$ of all holomorphic functions on X.

The equivalence of (1) – (7) holds for a domain X over the product $\mathbf{C}^\Lambda$ of card(Λ) copies of $\mathbf{C}$ if the boundary distance d_X is defined in a suitable manner generalizing the usual boundary distance in finite dimensions ([1]). Besides other equivalent conditions which are analogous to conditions in the finite dimensional case there exists one which is very useful to reduce infinite dimensional problems to finite dimensions:

(9) There exists a finite subset ϕ of Λ and a Stein domain X_ϕ over $\mathbf{C}^\phi$ such that X is isomorphic (as domain over $\mathbf{C}^\Lambda$) to $X_\phi \times \mathbf{C}^{\Lambda-\phi}$.

The proof of the equivalences is based on [3] and [5]. In [3] HIRSCHOWITZ showed the equivalence of (1), (2), (4) and (9) for a domain X in $\mathbf{C}^\mathbf{N}$. In [5] MATOS generalized these results for domains over $\mathbf{C}^\mathbf{N}$; he replaced, however, (2) and (9) by more complicated technical conditions.

Up to now it is not known whether (8) is equivalent to the other conditions in case Λ is infinite. In what follows we shall prove a modification: X is a domain of holomorphy if and only if X coincides with the space of all nonzero $\mathcal{b}$-continuous algebra homomorphisms $\mathcal{H}(X) \to \mathbf{C}$ where $\mathcal{b}$ denotes the bornological topology associated with the compact-open topology on $\mathcal{H}(X)$. As we shall show in section 1 $\mathcal{b}$ has some further nice properties. Moreover the envelope of holomorphy $\mathcal{E}(X)$ is homeomorphic to Spec($\mathcal{H}(X)$,$\mathcal{b}$) endowed with the weak topology, and this involves that $\mathcal{E}$ is a functor in the category of domains and holomorphic maps.

§ 1. The Space of the Holomorphic Functions on a Domain over C^Λ.

We show that the space $\mathcal{H}(X)$ of the holomorphic functions on a domain X over C^Λ is the union of subspaces $\mathcal{H}^\phi(X)$, $\phi \subset \Lambda$ finite, and each $\mathcal{H}^\phi(X)$ is isomorphic to the space $\mathcal{H}^\phi(X)$ of the holomorphic functions on a finite dimensional domain ${}^\phi X$ over C^ϕ. ${}^\phi X$ is obtained as solution of a universal problem. $X \mapsto {}^\phi X$ defines a functor $\mathcal{J}^\phi$, and there is a natural equivalence between the functors $\mathcal{H}^\phi$ and $\mathcal{H} \circ \mathcal{J}^\phi$. If $\mathcal{H}^\phi(X)$ and $\mathcal{H}({}^\phi X)$ are endowed with the compact-open topology c, they are homeomorphic and $\mathcal{H}_c \circ \mathcal{J}^\phi$ and $\mathcal{H}_c^\phi$ are equivalent. The locally convex inductive topology $\mathcal{L}$ on $\mathcal{H}(X)$ with respect to the subspaces $\mathcal{H}_c^\phi(X)$ is the bornological topology associated with the compact-open topology on $\mathcal{H}(X)$. $\mathcal{L}$ is Montel and every extension pair is normal with respect to $\mathcal{L}$.

Notations: Throughout all sections Λ_0 is a fixed nonempty set. Λ, Λ', Λ_1, Λ_2 will always be subsets of Λ_0. C^Λ will denote the set of all mappings $\Lambda \to C$ endowed with the topology of pointwise convergence. If $\theta \subset \Lambda$, C^θ will be considered as subspace of C^Λ, and π_θ^Λ or π_θ will denote the projection of C^Λ onto C^θ. $F(\Lambda) := \{\phi \subset \Lambda : \phi \text{ is finite}\}$; $\mathscr{O} \in F(\Lambda)$. (Where $:=$ indicates a defining relation).

(1.1) Definition: $p: X \to C^\Lambda$ is called a <u>domain</u> (over C^Λ) iff X is a connected nonempty Hausdorff space and p is locally a homeomorphism.

Let $\Lambda_1 \subset \Lambda_2$. A <u>morphism</u> from a domain $p_1: X_1 \to C^{\Lambda_1}$ to a domain $p_2: X_2 \to C^{\Lambda_2}$ is a continuous mapping $\omega: X_1 \to X_2$ such that the following diagram commutes:

$$\begin{array}{ccc} X_1 & \xrightarrow{\ \omega\ } & X_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ C^{\Lambda_1} & \xrightarrow[\pi^{\Lambda_1}_{\Lambda_2}]{} & C^{\Lambda_2} \end{array}$$

The domains over C^ϑ with $\theta \subset \Lambda$ and the morphisms between them form a category $\mathcal{G}(\Lambda_0)$.

(1.2) Lemma: Every morphism is locally a projection, hence open. If Λ_2 is finite and ω is surjective, then ω is semiproper i.e. for every compact set $K_2 \subset X_2$ there exists a compact set $K_1 \subset X_1$ with $\omega(K_1) = K_2$.

The proof is found in [1].

A domain $p: X \to C^\Lambda$ will always be considered as an analytic manifold with the complex structure induced by p. Then every morphism is a holomorphic map. The C-algebra of the holomorphic functions on X will be denoted - "par abus de langage" - by $\mathcal{H}(X)$.

Let $p: X \to C^\Lambda$ be a domain.

Definition: Let $U \subset X$ and f and g be mappings defined on U. We say that **f depends on g** iff f is constant on the connected components of the fibers of g. If h_1 and h_2 are germs in $x \in X$, we say that h_1 depends on h_2 iff h_1 has a representative which depends on a representative of h_2.

Lemma: Let f_x be the germ of a holomorphic function at $x \in X$. There exists $\phi \in F(\Lambda)$ such that f_x depends on $(\pi_\phi \circ p)_x$. Moreover, there exists a smallest set ϕ with this property. It will be denoted by dep(f_x). (cf. [1], [4], [7])

Proof: x has a neighbourhood U such that $p|U \to B \times \mathbf{C}^{\Lambda-\phi}$, with $B \subset \mathbf{C}^\phi$ open, is topological and f_x has a bounded representative g: $U \to \mathbf{C}$. By Liouville's theorem, $g\circ(p|U)^{-1}$ depends on $\pi_\phi|p(U)$. Obviously, the intersection of all ϕ such that f_x depends on $(\pi_\phi \circ p)_x$ has this property, too. q.e.d.

It is easy to verify the following lemma (cf. [1], [3], [4]).

Lemma: Let $f \in \mathcal{H}(X)$. Then the mapping $X \to F(\Lambda)$, $x \mapsto$ dep(f_x) is constant.

Its value will be denoted by **dep(f)**.

(1.3) Definition of $\mathcal{H}^\theta$: Let p: $X \to \mathbf{C}^\Lambda$ be a domain. For every $\theta \subset \Lambda_0$ $\mathcal{H}^\theta(X) := \{f \in \mathcal{H}(X)$: dep(f) $\subset \theta\}$. $\mathcal{H}^\theta(X)$ is a **C**-algebra.

A morphism ω from the domain p: $X \to \mathbf{C}^\Lambda$ to the domain q: $Y \to \mathbf{C}^{\Lambda'}$ induces a homomorphism $\omega^*: \mathcal{H}^\theta(Y) \to \mathcal{H}^\theta(X), f \mapsto f\circ\omega$ for every $\theta \subset \Lambda_0$.

Hence the $\mathcal{H}^\theta$ are contravariant functors from $\mathcal{G}(\Lambda_0)$ to the category of **C**-algebras.

Since $\mathcal{H}(X) = \cup\{\mathcal{H}^\phi(X): \phi \in F(\Lambda)\}$, $\mathcal{H}(X)$ can be identified with $\lim\limits_{\substack{\longrightarrow \\ \phi \in F(\Lambda)}} \mathcal{H}^\phi(X)$.

(1.4) Definition of $^\theta X$ and $\mathcal{S}^\theta$: Let p: $X \to \mathbf{C}^\Lambda$ be a domain. Let $\theta \subset \Lambda_0$. Consider the following equivalence relation on X:

Two points x and y are identified iff $f(x) = f(y)$ for all $f \in \mathcal{H}^\theta(X)$.

Let $^\theta X$ be the quotient space (endowed with the quotient topology) and $^\theta\sigma_p: X \to {^\theta X}$ the canonical projection.

p factors through $^\theta\sigma_p$ to a map $^\theta p$: $^\theta X \to \mathbf{C}^{\theta\cap\Lambda}$. It can be shown ([1]) that $^\theta p$: $^\theta X \to \mathbf{C}^{\theta\cap\Lambda}$ is a domain which has the following universal property:

For any holomorphically separable domain q: $Y \to \mathbf{C}^\Delta$ with $\Delta \subset \theta \cap \Lambda$ and any morphism $\omega: X \to Y$, there exists a unique morphism $\tilde{\omega}: {^\theta X} \to Y$ such that $\omega = \tilde{\omega}\circ{^\theta\sigma_p}$. (cf. the universal property of a complex base in [9].) This universal property involves that $X \to {^\theta X}$ defines a reflector $\mathcal{S}^\theta$ in $\mathcal{G}(\Lambda_0)$ (for a morphism φ from p to q $\mathcal{S}^\theta(\varphi) := \widetilde{(^\theta\sigma_q \circ \varphi)}$).

Notation: Let p: $X \to \mathbf{C}^\Lambda$ be a domain and $\theta \subset \Lambda$. If we consider $\mathcal{H}^\theta(X)$ as topological vectorspace endowed with a topology τ we shall write $\mathcal{H}^\theta_\tau(X)$. c will denote the compact-open topology. Obviously $\mathcal{H}^\theta_c$ is a contravariant functor from $\mathcal{U}(\Lambda_0)$ to the category of locally convex spaces.

(1.5) Proposition: Let p: $X \to C^\Lambda$ be a domain and $\phi \in F(\Lambda)$. Then $^\phi\sigma_p^*$: $\mathcal{H}(^\phi X) \to \mathcal{H}^\phi(X)$ is a topological isomorphism.

The family ($^\phi\sigma_q^*$: q domain in $\mathcal{U}(\Lambda_0)$) is a natural equivalence of $\mathcal{H}_c^\phi$ and $\mathcal{H}_c \circ \mathcal{T}^\phi$.

Proof: $^\phi\sigma_p^*$ is injective since $^\phi\sigma_p$ is surjective. The surjectivity of $^\phi\sigma_p^*$ follows from the construction of $^\phi X$ and $^\phi\sigma_p$. Clearly $^\phi\sigma_p^*$ is continuous, and its inverse is continuous because $^\phi\sigma_p$ is semiproper by (1.2) q.e.d.

(1.6) Corollary: $\mathcal{H}_c^\phi(X)$ is a Fréchet-Montel space.

Let p: $X \to C^\Lambda$ be a domain. We denote by ℓ the locally convex inductive topology on $\mathcal{H}(X)$ with respect to the subspaces $\mathcal{H}_c^\phi(X)$ with $\phi \in F(\Lambda)$. Obviously $\mathcal{H}_\ell(X)$ is the strict inductive limit of the $\mathcal{H}_c^\phi(X)$.

The adjoint map of a morphism of domains ω: $X \to Y$ induces a continuous linear map $\mathcal{H}^\phi(Y) \to \mathcal{H}^\phi(X)$ for every $\phi \in F(\Lambda)$, hence ω^*: $\mathcal{H}_\ell(Y) \to \mathcal{H}_\ell(X)$ is continuous, too. Thus $\mathcal{H}_\ell$ is a contravariant functor from $\mathcal{U}(\Lambda_0)$ to the category of locally convex spaces.

Obviously, ℓ is finer than c. It can be shown that they are equal if and only if Λ is finite.

By virture of (1.6), ℓ is bornological and barrelled. Hence ℓ is finer than the bornological topology c_0 associated with c. In order to show that they are equal we need the following lemma.

(1.7) Lemma: Let p: $X \to C^\Lambda$ be a domain. If $\mathcal{B}$ is a pointwise bounded subset of $\mathcal{H}(X)$, there is a $\phi \in F(\Lambda)$ such that $\mathcal{B} \subset \mathcal{H}^\phi(X)$.

Proof: Suppose the assertion is not true. Then there exists a sequence $(f_n)_{n \in N}$ in $\mathcal{B}$ such that no finite subset of Λ contains all dep(f_n). We shall construct a subsequence which is not bounded in a point of X. $n_1 := 1$. If n_k is defined, there is $n_{k+1} \in N$ so that $n_i < n_{k+1}$ for all $i \in \{1,\ldots,k\}$ and $\mathrm{dep}(f_{n_{k+1}}) \subset \bigcup_{i=1}^{k} \mathrm{dep}(f_{n_i})$. $\phi_k := \mathrm{dep}(f_{n_k})$ for all $k \in N$.

Let $a \in X$. There exist a neighbourhood U of a and a 0–neighbourhood $V = B(0,r)^{\phi'} \times C^{\Lambda - \phi'}$ in C^Λ so that $p|U \to p(a) + V$ is a homeomorphism.

For all $k \in N$ $g_k := f_{n_k} \circ (p|U)^{-1} \in \mathcal{H}(p(a)+V)$.

There is $k_0 \in N$ so that for all $k > k_0$ $(\phi_k - \bigcup_{i=1}^{k_0} \phi_i) \cap \phi' = \emptyset$. $\xi_0 := \pi_{\phi' \cup \phi_{k_0}}(p(a))$.

Suppose ξ_j is defined for $j \in \{0,\ldots,\ell-1\}$.

Since $g_{k_0+1} \Big| \sum_{j=0}^{\ell-1} \xi_j + C^{\phi_{k_0+\ell} - \bigcup_{i=0}^{k_0+\ell-1} \phi_i}$ is an entire function which cannot be constant because

$$\emptyset \neq \phi_{k_0+1} - \bigcup_{i=0}^{k+\ell-1} \phi_i \subset \mathrm{dep}(f_{n_{k_0+\ell}}),$$

there is $\xi_\ell \in C^{\phi_{k_0+\ell} - \bigcup_{i=0}^{k_0+\ell-1} \phi_i}$ with $|g_{k_0+\ell}(\sum_{j=0}^{\ell} \xi_j)| > 1$. Consequently $|g_{k_0+q}(\sum_{j=0}^{\infty} \xi_j)| \to \infty$ if $q \to \infty$, whence a contradiction. (Clearly $\sum_{j=0}^{\infty} \xi_j$ is well defined.) q.e.d.

(1.8) Proposition: $\mathcal{L}$ **is the bornological topology associated with the compact-open topology.**

Proof: We need only show that $\mathcal{L}$ is coarser than c_0.

First we show that

id: $\mathcal{H}_c(X) \to \mathcal{H}_{\mathcal{L}}(X)$ is sequentially continuous. Let $(f_n)_{n\in\mathbf{N}}$ be a convergent sequence in $\mathcal{H}_c(X)$. By (1.7) there is a $\phi \in F(\Lambda)$ so that $(f_n)_{n\in\mathbf{N}}$ converges in $\mathcal{H}_c^{\phi}(X)$. Since $\mathcal{H}_c^{\phi}(X) \hookrightarrow \mathcal{H}_{\mathcal{L}}(X)$ is continuous, $(f_n)_{n\in\mathbf{N}}$ converges in $\mathcal{H}_{\mathcal{L}}(X)$.

Since id: $\mathcal{H}_{c_0}(X) \to \mathcal{H}_c(X)$ is continuous, $\mathcal{H}_{c_0}(X) \to \mathcal{H}_c(X) \to \mathcal{H}_{\mathcal{L}}(X)$ is sequentially continuous, hence continuous (because c_0 is bornological). This means that $\mathcal{L}$ is coarser than c_0. q.e.d.

(1.9) Proposition: $\mathcal{H}_{\mathcal{L}}(X)$ **is a Montel space.**

Proof: Since $\mathcal{H}_{\mathcal{L}}(X)$ is barrelled it suffices to show that $\mathcal{H}_{\mathcal{L}}(X)$ is semi-Montel.

Let $\mathcal{B}$ be a bounded and closed set in $\mathcal{H}_{\mathcal{L}}(X)$. Then $\mathcal{B}$ is pointwise bounded and, by (1.7), there exists a $\phi \in F(\Lambda)$ such that $\mathcal{B} \subset \mathcal{H}^{\phi}(X)$. $\mathcal{B}$ is bounded and closed in $\mathcal{H}_c^{\phi}(X)$ since $\mathcal{L}$ induces on $\mathcal{H}^{\phi}(X)$ the compact-open topology. By virture of (1.6), $\mathcal{B}$ is compact. q.e.d.

It follows in the same way that $\mathcal{H}_c(X)$ is semi-Montel.

(1.10) Proposition: Let $p: X \to C^{\Lambda}$ **and** $q: Y \to C^{\Lambda}$ **be domains and let** $\omega: X \to Y$ **be a morphism. Then** $\omega^*: \mathcal{H}(Y) \to \mathcal{H}(X)$ **is an algebraic isomorphism if and only if** $\omega^*: \mathcal{H}_{\mathcal{L}}(Y) \to \mathcal{H}_{\mathcal{L}}(X)$ **is a topological isomorphism.**

Proof: Suppose $\omega^*: \mathcal{H}(Y) \to \mathcal{H}(X)$ is an algebraic isomorphism. Then, for each $\phi \in F(\Lambda)$, $\omega^*: \mathcal{H}^{\phi}(Y) \to \mathcal{H}^{\phi}(X)$ is an isomorphism. $\omega^*: \mathcal{H}_c^{\phi}(Y) \to \mathcal{H}_c^{\phi}(X)$ is continuous and $\mathcal{H}_c^{\phi}(Y)$ and $\mathcal{H}_c^{\phi}(X)$ are Fréchet spaces, hence, by the open mapping theorem, $\omega^*: \mathcal{H}_c^{\phi}(Y) \to \mathcal{H}_c^{\phi}(X)$ is a homeomorphism. Consequently, $\omega^*: \mathcal{H}_{\mathcal{L}}(Y) \to \mathcal{H}_{\mathcal{L}}(X)$ is a topological isomorphism. The converse implication is trivial. q.e.d.

§2. A Special Construction of the Envelope of Holomorphy.

We need the following generalization of the intersection of domains (cf. [7] 2.3, [8] p.85).

(2.1) Proposition: Let $p: X \to C^{\Lambda}$ **be a domain and let** $(p_\phi: X_\phi \to C^{\phi})_{\phi\in F(\Lambda)}$ **be a family of domains. Suppose given a family** $(\varphi_\phi)_{\phi\in F(\Lambda)}$ **of morphisms** $\varphi_\phi: X \to X_\phi$. **Then there exists a domain** $q: Y \to C^{\Lambda}$, **a morphism** $\psi: X \to Y$ **and morphisms** $\psi_\phi: Y \to X_\phi$ **such that:**

(a) $\varphi_\phi = \psi_\phi \circ \psi$ **for all** $\phi \in F(\Lambda)$

(b) ψ **is maximal in the following sense:**

If q': $Y' \to C^{\Lambda'}$ **is a domain and** ψ': $X \to Y'$ **is a morphism such that each** φ_ϕ **factors through** ψ'**, then there is a unique morphism** γ: $Y' \to Y$ **with** $\psi = \gamma \circ \psi'$ **and** $\psi'_\phi = \psi_\phi \circ \gamma$ **for all** $\phi \in F(\Lambda)$.

Definition: (q,ψ) or simply q or Y is called the intersection of $(\varphi_\phi)_{\phi \in F(\Lambda)}$ or of $(p_\phi)_{\phi \in F(\Lambda)}$ if no confusion can arise. q: $Y \to C^\Lambda$ is unique up to isomorphisms.

Sketch of the proof:

Let $Z \subset \prod_{\phi \in F(\Lambda)} X_\phi$ be the set of all $(x_\phi : \phi \in F(\Lambda))$ with the following properties:

(1) $\pi^{\phi}_{\phi \cap \phi'} \circ p_\phi(x_\phi) = \pi^{\phi'}_{\phi \cap \phi'} \circ p_{\phi'}(x_{\phi'})$ for all $\phi, \phi' \in F(\Lambda)$

(2) there is an open polydisc U with center 0 in C^Λ and there are neighbourhoods U_ϕ of x_ϕ in X_ϕ such that $p_\phi | U_\phi \to p_\phi(x_\phi) + \pi_\phi(U)$ is topological for all $\phi \in F(\Lambda)$.

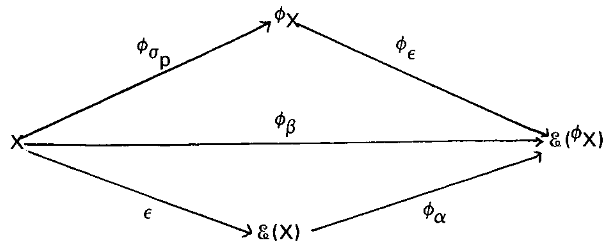
We define a topology on Z in the following way: $W \subset Z$ is called a neighbourhood of $(x_\phi : \phi \in F(\Lambda)) \in Z$ iff there are U and U_ϕ satisfying (2) such that $Z \cap \prod_{\phi \in F(\Lambda)} U_\phi \subset W$. This topology on Z is finer than the trace of the product topology. ψ: $X \to Z$, $x \mapsto (\varphi_\phi(x) : \phi \in F(\Lambda))$ is continuous. Let Y be the connected component which contains $\psi(X)$. The mapping q: $Y \to C^\Lambda$, $(x_\phi : \phi \in F(\Lambda)) \mapsto (p_{\{j\}}(x_{\{j\}}) : j \in \Lambda)$ is well-defined according to (1) and locally a homeomorphism. Hence q: $Y \to C^\Lambda$ is a domain and ψ: $X \to Y$ is a morphism which satisfies (a) if we define ψ_ϕ: $Y \to X_\phi$, $(x_{\phi'} : \phi' \in F(\Lambda)) \mapsto x_\phi$. Condition (b) is easily verified like in [8], [9].

We shall use the following terminology:

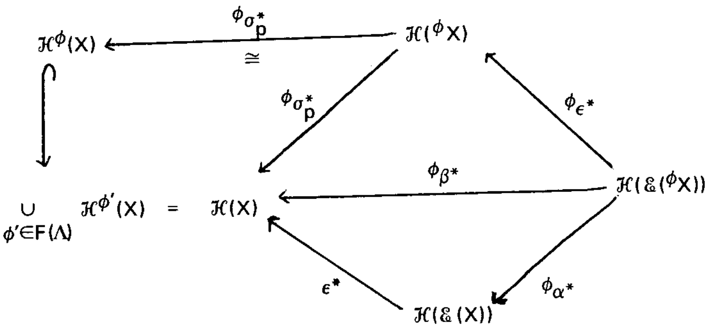
(2.2) Definition: Let p: $X \to C^\Lambda$ and q: $Y \to C^\Lambda$ be domains and ω: $X \to Y$ a morphism.
(q,ω) is called a $\mathcal{H}$(X)–extension of p iff ω^*: $\mathcal{H}(Y) \to \mathcal{H}(X)$ is bijective. (Notice that ω^* is always injective because of the identity theorem.)
The $\mathcal{H}$(X)–extension (q,ω) (or simply Y) is called the **envelope of holomorphy** of p iff, for any $\mathcal{H}$(X)–extension (q': $Y' \to C^\Lambda$, ω'), there is a morphism γ: $Y' \to Y$ such that $\omega = \gamma \circ \omega'$. On these conditions, q: $Y \to C^\Lambda$ is unique up to isomorphisms of domains.

(2.3) Construction of the envelope of holomorphy of a domain p: $X \to C^\Lambda$. For every $\phi \in F(\Lambda)$, the envelope of holomorphy of p: $X \to C^\Lambda$ exists (see e.g. [2], [6], [9]). We denote it by $({}^\phi e$: $\mathcal{E}({}^\phi X) \to C^\phi$, ${}^\phi\epsilon)$ or briefly by $\mathcal{E}({}^\phi X)$.
Set ${}^\phi\beta := {}^\phi\epsilon \circ {}^\phi\sigma_p$ for all $\phi \in F(\Lambda)$.
Let (e: $\mathcal{E}(X) \to C^\Lambda, \epsilon$) be the intersection of $({}^\phi\beta : \phi \in F(\Lambda))$.

Then the following diagram commutes:

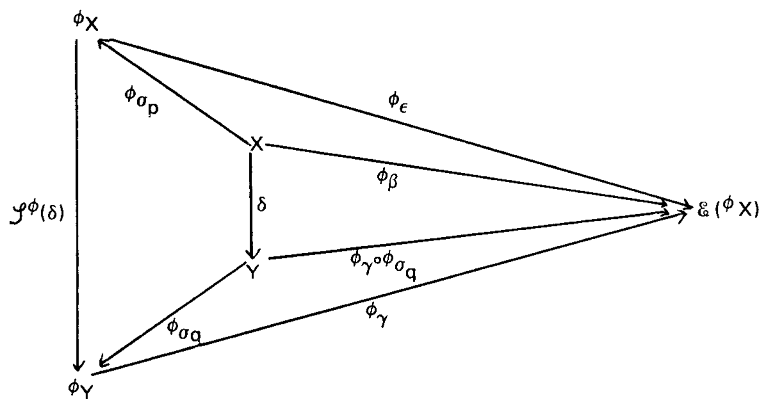


Hence the following one commutes too:



Obviously ϵ^* is surjective, hence (e, ϵ) is a $\mathcal{H}(X)$–extension of p.

Let (q: $Y \to C^{\Lambda}$, δ) be another $\mathcal{H}(X)$–extension. Then δ^*: $\mathcal{H}^{\phi}(Y) \to \mathcal{H}^{\phi}(X)$ is an isomorphism for every $\phi \in F(\Lambda)$. Hence (${}^{\phi}p$, $\mathcal{Y}^{\phi}(\delta)$) is a $\mathcal{H}({}^{\phi}X)$–extension for every $\phi \in F(\Lambda)$. Therefore there are morphisms ${}^{\phi}\gamma$: ${}^{\phi}Y \to \mathcal{E}({}^{\phi}X)$ such that the following diagram commutes:



Because of the intersection property, there exists a morphism γ: $Y \to \mathcal{E}(X)$ such that $\epsilon = \gamma \circ \delta$. q.e.d.

A more detailed consideration yields that the association $[p\colon X \to \mathbf{C}^\Lambda] \mapsto [\mathcal{E}(p) := e\colon \mathcal{E}(X) \to \mathbf{C}^\Lambda]$ is functorial and defines a reflector $\mathcal{E}$ in $\mathcal{G}(\Lambda_0)$.

§ 3. The Spectrum as Envelope of Holomorphy.

If A is a **C**-algebra, Spec A denotes the set of all nonzero algebra homomorphisms $A \to \mathbf{C}$. Let τ be a topology on A. Then Spec(A,τ) will be the set of all τ-continuous nonzero algebra homomorphisms $A \to \mathbf{C}$. The Gelfand map is denoted by $\widehat{\ }\colon A \to \mathbf{C}^{\mathrm{Spec}(A,\tau)}$, $f \to \hat{f}$ ($\hat{f}$ is defined by $\hat{f}(h) = h(f)$ for all $h \in \mathrm{Spec}(A,\tau)$). If Spec$(A,\tau)$ is considered as a topological space then it will always be looked upon as being endowed with the weak topology i.e. the coarsest topology such that all $\hat{f}$ are continuous.

For a domain $q\colon Y \to \mathbf{C}^n$, $\mathcal{E}(Y)$, Spec $\mathcal{H}(Y)$ and Spec$(\mathcal{H}(Y), c)$ are homeomorphic (see e.g. [2]). By virtue of (1.5) this implies that, for a domain $p\colon X \to \mathbf{C}^\Lambda$, $\mathcal{E}({}^\phi X)$, Spec $\mathcal{H}({}^\phi X)$, Spec $\mathcal{H}^\phi(X)$, Spec$(\mathcal{H}({}^\phi X),c)$ and Spec$(\mathcal{H}^\phi(X),c)$ are homeomorphic for every $\phi \in F(\Lambda)$. This suggests the following proposition.

(3.1) Proposition: **Let $p\colon X \to \mathbf{C}^\Lambda$ be a domain. There exists an injective mapping $S\colon \mathcal{E}(X) \to \mathrm{Spec}(\mathcal{H}(X), \mathcal{b})$ which assigns to $(h_\phi\colon \phi \in F(\Lambda)) \in \mathcal{E}(X)$ the homomorphism $h \in \mathrm{Spec}\,\mathcal{H}(X)$ satisfying**

$$(*) \qquad h(f) = h_\phi(({}^\phi\sigma_p)^{-1} f) \quad \text{for} \quad f \in \mathcal{H}^\phi(X).$$

Proof: It suffices to show that $(*)$ defines a mapping $S\colon \mathcal{E}(X) \to \mathrm{Spec}\mathcal{H}(X)$. For all $x \in X$, $({}^\phi\alpha\circ\epsilon(x)\colon \phi \in F(\Lambda)) \in \epsilon(X)$ and ${}^\phi\alpha\circ\epsilon(x) = {}^\phi\epsilon\circ{}^\phi\sigma_p(x)$, hence ${}^\phi\alpha\circ\epsilon(x)$ is the evaluation of $\mathcal{H}({}^\phi X)$ in the point ${}^\phi\sigma_p(x)$ and can be identified (by means of ${}^\phi\sigma_p^{**}$) with the evaluation of $\mathcal{H}^\phi(X)$ in x. Therefore ${}^{\phi'}\alpha\circ\epsilon(x) \mid \mathcal{H}^{\phi''}(X) = {}^{\phi''}\alpha\circ\epsilon(x)$ $(**)$ for all ϕ', $\phi'' \in F(\Lambda)$ with $\phi'' \subset \phi'$. Since, for all $f \in \mathcal{H}({}^{\phi''}X)$, $\hat{f}\colon \mathcal{E}({}^{\phi''}X) \to \mathbf{C}$, $h_{\phi''} \mapsto h_{\phi''}(f)$ is holomorphic ([2], p.49), $\hat{f}\circ{}^{\phi'}\alpha - \hat{f}\circ{}^{\phi''}\alpha$ is a holomorphic function on $\mathcal{E}(X)$. By virtue of $(**)$ and the identity theorem, $\hat{f}\circ{}^{\phi'}\alpha - \hat{f}\circ{}^{\phi''}\alpha = 0$. Consequently $h_{\phi'}|\mathcal{H}^{\phi''}(X) = h_{\phi''}$ for all $(h_\phi\colon \phi \in F(\Lambda)) \in \mathcal{E}(X)$ and all $\phi', \phi'' \in F(\Lambda)$ with $\phi'' \subset \phi'$. Thus, $(*)$ defines a map $S\colon \mathcal{E}(X) \to \mathrm{Spec}\,\mathcal{H}(X)$ which is obviously injective. q.e.d.

(3.2) Lemma: **Let $p\colon X \to \mathbf{C}^\Lambda$ and $p_\phi\colon X \to \mathbf{C}^\phi$ be domains. Suppose that $\phi \in F(\Lambda)$ and that $\rho\colon X \to X_\phi$ is a morphism such that $\rho \times (\pi_{\Lambda-\phi}\circ p)\colon X \to X_\phi \times \mathbf{C}^{\Lambda-\phi}$ is an isomorphism of p and $p_\phi \times \mathrm{id}_{\mathbf{C}^{\Lambda-\phi}}\colon X_\phi \times \mathbf{C}^{\Lambda-\phi} \to \mathbf{C}^\Lambda$. Let $f \in \mathcal{H}(X)$ and $\phi' := \mathrm{dep}(f) - \phi$.**
Then there are holomorphic functions $f_\mu \in \mathcal{H}(X_\phi)$, $\mu \in \mathbf{N}^{\phi'}$, such that for all $y \in X$

$$f(y) = \sum_{\mu \in \mathbf{N}^{\phi'}} f_\mu\circ\rho(y) \prod_{j\in\phi'} (\pi_j\circ p(y))^{\mu_j}.$$

(For $\phi' = \emptyset$, set $\prod_{j\in\phi'} (\pi_j\circ p(y))^{\mu_j} := 1$ and $\mathbf{N}^{\phi'} := \{0\}$.)

The series converges locally absolutely and uniformly.

Proof: Set $q := p_\phi \times \mathrm{id}_{C^{\Lambda-\phi}}$, $Y := X_\phi \times C^{\Lambda-\phi}$ and $g := f\circ(\rho \times (\pi_{\Lambda-\phi}\circ p))^{-1} \in \mathcal{H}(Y)$. Let $x \in Y$ and U be a neighbourhood of x such that $q|U \to q(U)$ is topological. For $\mu \in N^{(\Lambda)}$ we define $D^\mu g(x) := D^\mu g\circ(q|U)^{-1}(qx)$. Clearly, $D^\mu g\colon y \mapsto D^\mu g(y)$ is a well-defined holomorphic function on Y. Since dep(g) is finite, every point $z \in Y$ has a neighbourhood where g can be expanded into a power series

$$g(y) = \sum_{\mu\in N^{\mathrm{dep}(g)}} D^\mu g(y) \prod_{j\in\mathrm{dep}(g)} (\pi_j(qy - qz))^{\mu_j}$$

and where this series converges uniformly and absolutely. Hence for all $(y,z) \in X_\phi \times C^{\Lambda-\phi}$

$$g(y,z) = \sum_{\mu\in N^{\phi'}} D^\mu g(y,0) \prod_{j\in\phi'} (\pi_j(z))^{\mu_j}$$

and evidently this series converges locally uniformly and absolutely, too. Setting $f_\mu(x) = D^\mu g(x,0)$ for $x \in X_\phi$ and $\mu \in N^{\phi'}$, we obtain the desired formula. q.e.d.

Definition: Let $p\colon X \to C^\Lambda$ be a domain. p is called a **domain of holomorphy** iff the canonical morphism ϵ from p to $\mathcal{E}(p)$ (see (2.3)) is an isomorphism of domains (it is equivalent to say that ϵ is bijective).
$E\colon X \to \mathrm{Spec}(\mathcal{E}(X),\mathfrak{T})$ denotes the map which assigns to x the evaluation homomorphism $E_x\colon \mathcal{H}(X) \to C$, $f \mapsto f(x)$.

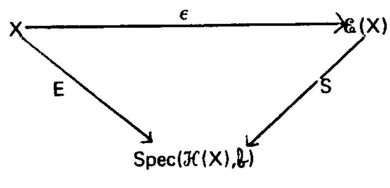
(3.3) Theorem: Let $p\colon X \to C^\Lambda$ be a domain. p is a domain of holomorphy if and only if $E\colon X \to \mathrm{Spec}(\mathcal{H}(X),\mathfrak{T})$ is bijective.

Proof: $\Rightarrow$: By virtue of condition 9 in §0 we can suppose that there is a $\phi \in F(\Lambda)$, a Stein domain $p_\phi\colon X_\phi \to C^\phi$ and a morphism $\rho\colon X \to X_\phi$ such that $\rho \times (\pi_{\Lambda-\phi}\circ p)$ is an isomorphism from p to $p_\phi \times \mathrm{id}_{C^{\Lambda-\phi}}$.
For $h \in \mathrm{Spec}(\mathcal{H}(X),\mathfrak{T})$ define $h \in \mathrm{Spec}\,\mathcal{H}(X_\phi)$ by $h(f) := h(f\circ\rho)$. Since p_ϕ is Stein, there exists $a \in X$ so that, for all $f \in \mathcal{H}(X_\phi)$, $h(f) = f\circ\rho(a)$ ([2]).
$b := (\rho \times \pi_{\Lambda-\phi}\circ p)^{-1}(a, (h(\pi_j\circ p))_{j\in\Lambda-\phi})$.
We show that $h = E_b$. Let $f \in \mathcal{H}(X)$. Using (3.2), we obtain

$$h(f) = \sum_{\mu\in N^{\phi'}} h(f_\mu) \prod_{j\in\phi'} (h(\pi_j\circ p))^{\mu_j}$$

$$= \sum_{\mu\in N^{\phi'}} f_\mu(a) \prod_{j\in\phi'} (h(\pi_j\circ p))^{\mu_j} = f(b).$$

$\Leftarrow$:



The above diagram commutes. Hence S and $\epsilon = S^{-1} \circ E$ are bijective. q.e.d.

Corollary: If p: $X \to \mathbf{C}^\Lambda$ is a domain of holomorphy, E: $X \to \mathrm{Spec}(\mathcal{H}(X),c)$ is bijective, hence $\mathrm{Spec}(\mathcal{H}(X),c) = \mathrm{Spec}(\mathcal{H}(X),\mathfrak{b})$.

Proof: Follows from the proof of (3.3).

(3.4) Lemma: Let p: $X \to \mathbf{C}^\Lambda$ be a domain. There exists $\phi_0 \in F(\Lambda)$ such that, for any $\phi_1 \in F(\Lambda)$ with $\phi_0 \subset \phi_1$, every $(x_\phi\colon \phi \in F(\Lambda)) \in \mathcal{E}(X)$ is determined by the components x_ϕ with $\phi \in \{\phi_1\} \cup F(\Lambda - \phi_1)$.

Proof: Since $\mathcal{E}(X)$ is a domain of holomorphy, we know by virtue of (3.2) and condition 9 in section 0 that there exists $\phi_0 \in F(\Lambda)$ such that every $f \in \mathcal{H}(X)$ can be written

$$f(x) = \sum_{\mu \in \mathbf{N}^{\mathrm{dep}(f)-\phi_0}} f_\mu \circ \rho(x) \prod_{j \in \mathrm{dep}(f)-\phi_0} (\pi_j \circ p(x))^{\mu_j} \quad \text{for all } x \in \mathcal{E}(X).$$

Let $\phi_1 \in F(\Lambda)$, $\phi_0 \subset \phi_1$. For all $x \in \mathcal{E}(X)$ and $\nu \in \mathbf{N}^{\mathrm{dep}(f)-\phi_1}$.

$$F_\nu(x) := \sum_{\substack{\mu \in \mathbf{N}^{\mathrm{dep}(f)-\phi_0} \\ \mu|\mathrm{dep}(f)-\phi_1 = \nu}} f_\mu \circ \rho(x) \prod_{j \in \phi_1 - \phi_0} (\pi_j \circ p(x))^{\mu_j}.$$

Clearly, $F_\nu \in \mathcal{H}^{\phi_1}(\mathcal{E}(X)) = \mathcal{H}^{\phi_1}(X)$ and, for all $x \in \mathcal{E}(X)$,

$$f(x) = \sum_{\nu \in \mathbf{N}^{\mathrm{dep}(f)-\phi_1}} F_\nu(x) \prod_{k \in \mathrm{dep}(f)-\phi_1} (\pi_j \circ p(x))^{\nu_k}.$$

If $x = (x_\phi\colon \phi \in F(\Lambda)) \in \mathcal{E}(X)$ then, writing S_x instead of $S(x)$,

$$S_x(f) = \sum_{\nu \in \mathbf{N}^{\mathrm{dep}(f)-\phi_1}} S_x(F_\nu) \cdot S_x\Big(\prod_{k \in \mathrm{dep}(f)-\phi_1} (\pi_j \circ p)^{\nu_k}\Big)$$

$$= \sum_{\nu \in \mathbf{N}^{\mathrm{dep}(f)-\phi_1}} x_{\phi_1}(F_\nu) \cdot x_{\mathrm{dep}(f)-\phi_1}\Big(\prod_{k \in \mathrm{dep}(f)-\phi_1} (\pi_j \circ p)^{\nu_k}\Big)$$

with identifying $\mathcal{E}({}^\phi X)$, $\mathrm{Spec}\,\mathcal{H}^\phi(X)$ and $\mathrm{Spec}\,\mathcal{H}^\phi(\mathcal{E}(X))$. Since S is injective, the assertion follows. q.e.d.

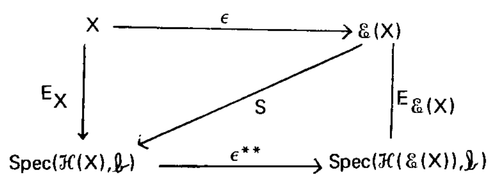
(3.5) Lemma: Let p: $X \to \mathbf{C}^\Lambda$ be a domain. The topology on $\mathcal{E}(X)$ coincides with the trace of the product topology on $\prod_{\phi \in F(\Lambda)} \mathcal{E}({}^\phi X)$.

Proof: Obviously, the topology of $\mathcal{E}(X)$ is finer than the product topology. We show that it is coarser, too. Let $x = (x_\phi: \phi \in F(\Lambda)) \in \mathcal{E}(X)$ and U and U_ϕ like in (2.1)(2). There is $\phi' \in F(\Lambda)$ such that $\pi_{\Lambda-\phi'}(U) = C^{\Lambda-\phi'}$. Choose ϕ_0 like in (3.4). $\phi_1 := \phi_0 \cup \phi'$. For all $\phi \in F(\Lambda-\phi_1)$, $p_\phi|U \to p_\phi(x_\phi) + C^\phi$ is topological, hence ([2], p.44) $U_\phi = \mathcal{E}(^\phi X)$. Therefore (3.4) involves that

$$[\prod_{\phi \in F(\Lambda)} U_\phi] \cap \mathcal{E}(X) = [U_{\phi_1} \times \prod_{\substack{\phi \in F(\Lambda)\\ \phi \neq \phi_1}} \mathcal{E}(^\phi X)] \cap \mathcal{E}(X). \qquad \text{q.e.d.}$$

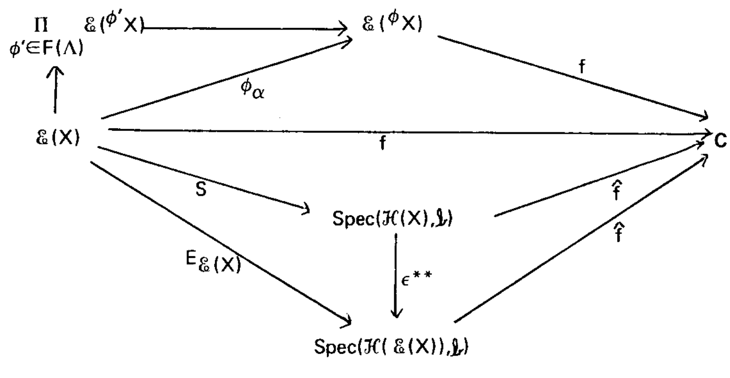
(3.6) Theorem: Let $p: X \to C^\Lambda$ be a domain. Then $S: \mathcal{E}(X) \to \mathrm{Spec}(\mathcal{H}(X), \mathfrak{L})$ is a homeomorphism.

Proof: The following diagram commutes:



The double adjoint ϵ^{**} of ϵ is bijective by virture of (1.10), hence topological. $E_{\mathcal{E}(X)}$ is bijective by (3.3). Consequently S is bijective.

For $f \in \mathcal{H}^\phi(X)$, $\phi \in F(\Lambda)$, the extensions of f to $\mathcal{E}(^\phi X)$ and $\mathcal{E}(X)$ will be denoted by f, too. The following diagram commutes:



Since, for all $f \in \mathcal{H}(X)$, $\hat{f} \circ S = f$ is continuous, S is continuous. By (3.5), the topology of $\mathcal{E}(X)$ is the projective topology with respect to $(^\phi\alpha: \phi \in F(\Lambda))$. Because the $\mathcal{E}(^\phi X)$ are endowed with the projective topology with respect to $\mathcal{H}(^\phi X) \cong \mathcal{H}^\phi(X)$ and because, for all $\phi \in F(\Lambda)$ and all $f \in \mathcal{H}^\phi(X)$, $f \circ {}^\phi\alpha \circ S^{-1} = \hat{f}$ is continuous, S^{-1} is continuous. q.e.d.

(3.7) Corollary: Let $p: X \to C^{\Lambda}$ be a domain.

p is a domain of holomorphy if and only if $E: X \to \mathrm{Spec}(\mathcal{H}(X),\mathfrak{T}_r)$ is a homeomorphism.

Proof: Clear, because $E = S \circ \epsilon$.

(3.8) Theorem: Let $p: X \to C^{\Lambda_1}$ and $q: Y \to C^{\Lambda_2}$ be domains and let $\varphi: X \to Y$ be a holomorphic map. There exists a unique holomorphic map $\mathcal{E}(\varphi): \mathcal{E}(X) \to \mathcal{E}(Y)$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & Y \\ {\scriptstyle \epsilon_X}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon_Y} \\ \mathcal{E}(X) & \xrightarrow[\ \mathcal{E}(\varphi)\ ]{} & \mathcal{E}(Y) \end{array}$$

Proof: For $j \in \Lambda_2$ set $\phi_j := \mathrm{dep}(\pi_j \circ q \circ \varphi)$. It is clear that, for $\phi'' \in F(\Lambda_2)$ and $\phi' = \bigcup_{j \in \phi''} \phi_j$, the adjoint map φ^* maps continuously $\mathcal{H}^{\phi''}(Y)$ in $\mathcal{H}^{\phi'}(X)$. Consequently φ^{**}: Spec $\mathcal{H}(X) \to$ Spec $\mathcal{H}(Y)$, $h \to h \circ \varphi^*$ maps Spec $(\mathcal{H}(X),\mathfrak{T})$ in $\mathrm{Spec}(\mathcal{H}(Y),\mathfrak{T})$. Because $\hat{f} \circ \varphi^{**} = \widehat{f \circ \varphi}$ for all $f \in \mathcal{H}(Y)$, $\varphi^{**}: \mathrm{Spec}(\mathcal{H}(X),\mathfrak{T}) \to \mathrm{Spec}(\mathcal{H}(Y),\mathfrak{T})$ is continuous. Since by (3.6) S_X: $\mathcal{E}(X) \to \mathrm{Spec}(\mathcal{H}(X),\mathfrak{T})$ and S_Y: $\mathcal{E}(Y) \to \mathrm{Spec}(\mathcal{H}(Y),\mathfrak{T})$ are topological, $\mathcal{E}(\varphi) := S_Y^{-1} \circ \varphi^{**} \circ S_X$ is continuous. Moreover $\mathcal{E}(\varphi)$ makes the above diagram commutative. Thus, all we need to show is that $e_Y \circ \mathcal{E}(\varphi)$ is holomorphic or, equivalently, that $\pi_j \circ e_Y \circ \mathcal{E}(\varphi)$ is holomorphic for all $j \in \Lambda_2$. For every $j \in \Lambda_2$, φ^{**} induces a map

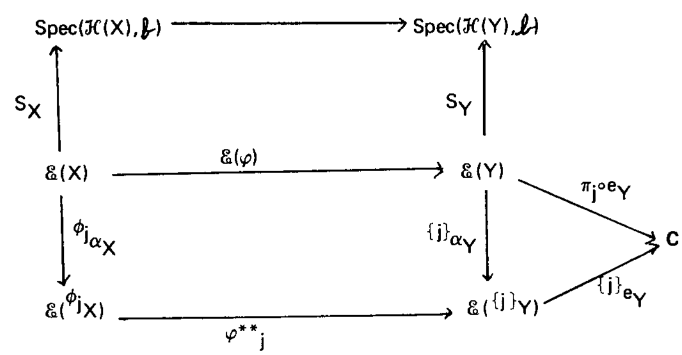
$$\begin{array}{ccc} \varphi^{**}_j : \mathrm{Spec}\,\mathcal{H}^{\phi_j}(X) & \longrightarrow & \mathrm{Spec}\,\mathcal{H}^{\{j\}}(Y) \\ \wr\!\wr & & \wr\!\wr \\ \mathrm{Spec}\,\mathcal{H}({}^{\phi_j}X) & & \mathrm{Spec}\,\mathcal{H}({}^{\{j\}}Y) \\ \wr\!\wr & & \wr\!\wr \\ \mathcal{E}({}^{\phi_j}X) & & \mathcal{E}({}^{\{j\}}Y) \end{array}$$

We show that ${}^{\{j\}}e_Y \circ \varphi^{**}_j$ (considered as map $\mathcal{E}({}^{\phi_j}X) \to C$) is holomorphic. Let $h \in \mathcal{E}({}^{\phi_j}X)$. A basis of neighbourhoods of h is given by the sets of the form $\{ h_z = \sum_{\nu \in N^{\phi_j}} \frac{1}{\nu!} z^{\nu} \cdot h \circ D^{\nu} : z \in B(0,r)^{\phi_j} \}$ where $r > 0$ (cf. [2], p.50)

$${}^{\{j\}}e_Y \circ \varphi^{**}_j(h_z) = \varphi^{**}_j(h_z)(\pi_j^{\Lambda_2} \circ q) = \sum_{\nu \in N_0^{\phi_j}} \frac{1}{\nu!} h \circ D^{\nu} (\pi_j^{\Lambda_2} \circ q \circ \varphi)\, z^{\nu}$$

is obviously a holomorphic function of z. Hence ${}^{\{j\}}e_Y \circ \varphi^{**}_j$ is holomorphic.

The following diagram commutes:



$$\pi_j \circ e_Y \circ \mathcal{E}(\varphi) = {}^{\{j\}}e_Y \circ {}^{\{j\}}\alpha_Y \circ \mathcal{E}(\varphi) = {}^{\{j\}}e_Y \circ \varphi^{**}{}_j \circ {}^{\{j\}}\alpha_X$$

is holomorphic for all $j \in \Lambda_2$. q.e.d.

Corollary: $p \mapsto \mathcal{E}(p)$, $\varphi \mapsto \mathcal{E}(\varphi)$ is a reflector in the category of domains and holomorphic maps.

Corollary: The analytic structure of $\mathcal{E}(X)$ depends only on the analytic structure of X (and not on the choice of p).

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