Z. Physik B 32, 425-430 (1979)

Zeitschrift für Physik B

Adiabatic Behavior of Sine-Gordon Solitons in the Presence of Perturbations*

K.H. Spatschek
Fachbereich Physik, Universität Essen – Gesamthochschule, Essen, F.R. Germany

Received August 21, Revised Version November 6, 1978

The behavior of sine-Gordon solitons in the presence of weak perturbations is considered. The procedure is based on the exact inverse scattering solution of the unperturbed sine-Gordon equation. Accounting for perturbations such as those arising from impurities, external forces as well damping and spatially inhomogeneous frequencies the corresponding perturbed operator equation can be solved by the Green's function technique if one expands the Green's operator in terms of a set of biorthogonal eigenfunctions. Ordinary linear differential equations prescribing the time evolution of the scattering data are obtained. Instead of solving the inverse scattering problem completely the adiabatic assumption is then used anticipating the result that solitons maintain their integrity to a high degree. Explicit solutions for the one-soliton dynamics are presented.

I. Introduction

The sine-Gordon equation has many applications in nonlinear physics [1]. Solitary wave solutions of the sine-Gordon equation are known for a long time. Recently, Ablowitz, Kaup, Newell, and Segur [2] have solved the initial value problem by using the inverse scattering technique. Emerging from multi-soliton collisions the solitary wave solutions have the same shapes and velocities with which they entered thus satisfying the requirements for considering them as solitons.

However, many real physical situations call for an extension of the idealised sine-Gordon equation: Impurities, external electric and magnetic fields, spatial inhomogeneities etc. are present and their effect on the motion of the solitons has to be considered. Several authors [3-7] have therefore considered a perturbed sine-Gordon equation and developed a technique to solve the dynamics of perturbed sine-Gordon solitons. However, most of them do not use the powerfull inverse scattering technique and their method is therefore restricted to single solitary-wave solutions [3]. Recently, Karpman and Maslov [8] developed a perturbation theory for the nonlinear Schrödinger equation as well as the Korteweg-de Vries equation based on the inverse scattering solution of those equations. This method has the advantage of being not restricted to a single soliton or non-overlapping soliton solutions.

In this paper we use the general outline of Karpman and Maslov to derive the perturbation theory of the sine-Gordon equation. The paper is organized as follows: In Sect. II, a short review of the inverse scattering solution for the sine-Gordon equation is presented since this forms the basis of the perturbation theory. In Sect. III, the basic equations for the time evolution of the scattering data in the presence of perturbations are derived. The method is based on an expansion in terms of the eigenfunctions of the unperturbed evolution operator (and its Hermitean adjoint). The Green's function allows to evaluate the effect of perturbations. The latter are treated as an inhomogeneity of the nonlinear partial differential equation. We apply this theory to the dynamics of a single soliton in the presence of perturbations, anticipating the adiabatic approximation. The latter ignores radiation effects which will be treated else-

* Work supported by the Deutsche Forschungsgemeinschaft through the Sonderforschungsbereich Nr. 162 "Plasmaphysik Bochum/Jülich"

0340-224X/79/0032/0425/$01.20

where. The basic formulas within the adiabatic approximation are summarized in Sect. IV. Finally, in Sect. V we apply the basic findings to the dynamics of a soliton in the presence of (i) an impurity, (ii) an external driver, and (iii) spatial inhomogeneity. The results are compared with previous investigations.

II. Review of the Inverse Scattering Solution

The sine-Gordon equation

$$u_{TT}-u_{XX}+\sin u=0, \tag{1}$$

can be written in the form

$$u_{xt}=\sin u, \tag{2}$$

by using the transformation

$$x=\tfrac{1}{2}(X+T), \tag{3}$$

$$t=\tfrac{1}{2}(X-T). \tag{4}$$

We note that the inversion procedure for the sine-Gordon equation in light-cone coordinates (x, t) is slightly more general than the inversion procedure for laboratory coordinates (X, T). The related questions of the Goursat and Cauchy problems for the sine-Gordon equation have been recently discussed by Kaup and Newell [9]. We therefore develop the perturbation theory based on the inverse scattering solution for the sine-Gordon equation in light-cone coordinates.

The inverse scattering solution [10] of (2) as well as for the more familiar Eq. (1) in laboratory coordinates has been first presented by Ablowitz et al. [2]. The main finding of these authors is that (2) can be written in the form

$$iL_t=[A, L], \tag{5}$$

where the operators L, A are given by

$$L=i\begin{pmatrix} \dfrac{\partial}{\partial x} & u_x/2 \\ u_x/2 & -\dfrac{\partial}{\partial x} \end{pmatrix}, \tag{6}$$

and

$$A=-\tfrac{1}{4}\begin{pmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{pmatrix} L^{-1}. \tag{7}$$

Considering the eigenvalue problem [11]

$$Lv=\varphi v, \tag{8}$$

the eigenvalue φ is time-independent since v evolves in time according to

$$iv_t=Av. \tag{9}$$

Introducing Jost solutions [11-13] for the scattering problem with the following asymptotic behavior

$$g\to\begin{pmatrix}1\\0\end{pmatrix} e^{-i\varphi x} \quad \text{as } x\to-\infty, \tag{10}$$

$$f\to\begin{pmatrix}0\\1\end{pmatrix} e^{i\varphi x} \quad \text{as } x\to+\infty, \tag{11}$$

for $\operatorname{Im}\varphi\geqq 0$, it can be shown that, if f and g exist, they are linearly independent except when the eigenvalue vanishes. In addition, if $f=(f_1, f_2)$ is a solution of the system at $\varphi=\xi+i\eta$ then $\bar{f}=(f_2^*, -f_1^*)$ satisfies the system at $\varphi^*=\xi-i\eta$. The pair of solutions f and $\bar{f}$ forms a complete system of solutions, and therefore scattering data a, b can be defined through

$$g(\xi)=a(\xi)\bar{f}(\xi)+b(\xi)f(\xi). \tag{12}$$

The coefficient $a(\xi)$ can be analytically continued into the upper half-plane and, in particular, the zeros φ_j $(j=1, \ldots, N)$ of $a(\varphi)$ in the upper half-plane correspond to the discrete eigenvalues. For these,

$$g(\varphi_j)=c_j f(\varphi_j), \tag{13}$$

holds. The time-dependence of the scattering data a, b, c follows as [2]

$$\begin{aligned} &a(\varphi)=a_0(\varphi),\\ &b(\varphi)=b_0(\varphi)\exp(-it/2\varphi),\\ &c_j(\varphi_j)=c_{j0}\exp(-it/2\varphi_j). \end{aligned} \tag{14}$$

The "potential" can be recovered by solving the Gel'fand-Levitan-Marchenko equation

$$K(x, y)+F(x+y)+\int_x^{\infty} K(x, z)\,F(y+z)\,dz=0, \tag{15}$$

where

$$K(x, z)=\begin{pmatrix} K_2^* & K_1 \\ -K_1^* & K_2 \end{pmatrix}, \quad F(z)=\begin{pmatrix} 0 & -H^*(z) \\ H(z) & 0 \end{pmatrix},$$

with

$$H(z)=-\sum_j \frac{ic_j}{a'(\varphi_j)} e^{i\varphi_j z}+\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{b(k)}{a(k)} e^{ikz}.$$

The solution is

$$u(x)=-2iK_1^*(x, x). \tag{16}$$

III. Perturbation Theory

Using the general method of Karpman and Maslov [8] we write the perturbed sine-Gordon equation

$$u_{xt}=\sin u-\varepsilon r[u], \tag{17}$$

in operator form

$$iL_t+[L,A]=i\varepsilon R[u],$$

where the L and A operators are the same as in the preceding section for $\varepsilon=0$ and R is given by

$$R=\begin{pmatrix} 0 & -\frac{i}{2}r[u] \\ -\frac{i}{2}r[u] & 0 \end{pmatrix}. \tag{18}$$

The eigenvalue problem

$$Lv=\varphi v, \tag{19}$$

yields after differentiation

$$(L-\varphi)(v_t+iAv)=-\varepsilon Rv+\varphi_t v. \tag{20}$$

From here it follows already that the time-dependence of the eigenvalue φ is given by

$$\varphi_t=\varepsilon\int w^* Rv\,dx/\int w^* v\,dx, \tag{21}$$

where we will choose [see Eq. (42)] $v=f=(f_1,f_2)$ and $w=\bar{f}=(f_2^*,f_1^*)$.
Next we consider the time evolution of the eigenfunctions. Rewriting (20) in the form

$$(L-\varphi)F=-4\pi P, \tag{22}$$

we solve this inhomogeneous equation by the Green's function technique [14], defining G through

$$[L-\varphi]G=-4\pi I. \tag{23}$$

The Green's operator G can be expressed in terms of biorthogonal eigenfunctions of L and L^*. Since the eigenvalue problem is not self-adjoint, one has to be carefull in assuming the completeness of the eigenfunctions. However, one of the important by-products of the basic investigation of Ablowitz et al. [12] is, indeed, the completeness relation (with respect to L_2) for this nonself-adjoint operator [for details see Appendix 6 of Ref. 12].
Using the abbreviation

$$(h(\lambda),t(\mu))\equiv\int_{-\infty}^{+\infty} h^*(x,\lambda)\,t(x,\mu)\,dx, \tag{24}$$

we obtain the orthogonality and normalization conditions

$$\begin{aligned}&(\bar{f}(\lambda_1),f(\lambda_2))\\&=(\bar{\bar{f}}(\lambda_1),\bar{f}(\lambda_2))=0,\quad \text{for } \lambda_1\neq\lambda_2,\end{aligned} \tag{25}$$

$$(\bar{\bar{f}}(\mu),g(\lambda))=(\bar{\bar{g}}(\mu),f(\lambda))=0, \tag{26}$$

$$\begin{aligned}&(\bar{\bar{f}}(\mu),f(\lambda))=(\bar{\bar{g}}(\mu),g(\lambda))\\&=2\pi|a(\lambda)|^2\,\delta(\mu-\lambda).\end{aligned} \tag{27}$$

The "solution" of (20) can now be written in the form [8]

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+iA\right)f(x,\varphi,t)\\&=i\varepsilon\left[\sum_j\frac{c_j\alpha(\varphi_j,\varphi)}{a_j'(\varphi_j-\varphi)}f(x,\varphi_j,t)\right.\\&\left.+\sum_j\frac{c_j^*\bar{\alpha}(\varphi_j,\varphi)}{a_j'^*(\varphi_j^*-\varphi)}\bar{f}(x,\varphi_j,t)\right]\\&-\frac{\varepsilon}{2\pi}\int_{-\infty}^{+\infty}\frac{\bar{\alpha}(\mu,\varphi)f(x,\mu)+\bar{\beta}(\mu,\varphi)g(x,\mu)}{|a(\mu)|^2(\mu-\varphi-i0)}d\mu\\&+d_1 f(x,\varphi,t)+d_2\,g(x,\varphi,t),\end{aligned} \tag{28}$$

where

$$\begin{aligned}&\alpha(\mu,\varphi)=(\bar{f}(\mu),Rf(\varphi)),\\&\bar{\alpha}(\mu,\varphi)=(\bar{\bar{f}}(\mu),Rf(\varphi)),\\&\beta(\mu,\varphi)=(\bar{g}(\mu),Rf(\varphi)),\\&\bar{\beta}(\mu,\varphi)=(\bar{\bar{g}}(\mu),Rf(\varphi)).\end{aligned} \tag{29}$$

First, considering the discrete eigenvalues, one determines the constants d_1, d_2 by taking the limit $x\to-\infty$. Inserting this value into the asymptotic equation for $x\to\infty$, and using (13), one obtains

$$\begin{aligned}&\frac{dc_r}{dt}=-\frac{i}{2\varphi_r}c_r\\&+\frac{i\varepsilon}{a'}\left[\frac{d\alpha(\varphi,\varphi_r)}{d\varphi}c_r^2-\frac{d\beta(\varphi,\varphi_r)}{d\varphi}c_r\right]_{\varphi=\varphi_r}.\end{aligned} \tag{30}$$

A similar calculation for the continuous eigenvalues yields

$$\frac{d\alpha(\varphi)}{dt}=i\varepsilon(a\bar{\alpha}(\varphi,\varphi)+b\alpha(\varphi,\varphi)), \tag{31}$$

$$\frac{db(\varphi)}{dt}=-\frac{ib}{2\varphi}+i\varepsilon[a\alpha^*(\varphi,\varphi)-b\bar{\alpha}^*(\varphi,\varphi)]. \tag{32}$$

Equations (30), (31), and (32) together with Eqs. (15), (16) are the basis for obtaining solutions of the perturbed sine-Gordon equation.
So far the procedure is quite general. The evolution equations for the eigenvalues and scattering coeffi-

cients are exact and constitute a basis for the perturbation theory. However, from a practical point of view, in most applications the evaluation will be limited to the discrete part of the spectrum, but it should be mentioned that recently for the Korteweg-de Vries and nonlinear Schrödinger equations progress has been made in calculating the so-called tail formation too [15].

IV. Adiabatic Approximation

Instead of following the general outline given in the last section we now restrict our investigation to the dynamics of a single soliton within the adiabatic approximation. In general, an initial pulse decays—even in the absence of perturbations—into a sequence of solitons and a tail. We now assume that a small perturbation has a neglegible effect on the soliton formation, i.e. that we are below a certain threshold [16]. Starting with different solitons a small perturbation leads to a slow change of the soliton parameters, a weak deformation of their shapes and the formation of a group of small amplitude waves (tail). Within the adiabatic approximation we consider only the first effect and leave the more difficult problem of tail formation to future investigations.

The single soliton (kink) solution of (2) is

$$u=4\tan^{-1}[e^{2\eta(x-\xi)}], \tag{33}$$

and by the adiabatic approximation we mean that the form of the soliton will be preserved although the parameters η and ξ as well as the eigenvalue φ might become t-dependent. We note that the solution (33) belongs to a purely imaginary eigenvalue $\varphi=i\eta$. The (unperturbed) scattering problem (8) now reads

$$\frac{\partial v_1}{\partial x}+i\varphi v_1=-2\eta\,\mathrm{sech}[2\eta(x-\xi)]\,v_2, \tag{34}$$

$$\frac{\partial v_2}{\partial x}-i\varphi v_2=2\eta\,\mathrm{sech}[2\eta(x-\xi)]\,v_1. \tag{35}$$

Applying the boundary conditions (10) and (11) we find from (34, 35)

$$f=\frac{e^{i\varphi x}}{i\varphi-\eta}\begin{pmatrix}-\eta\,\mathrm{sech}[2\eta(x-\xi)]\\ i\varphi-\eta\tanh[2\eta(x-\xi)]\end{pmatrix}, \tag{36}$$

and

$$g=\frac{e^{-i\varphi x}}{i\varphi-\eta}\begin{pmatrix}i\varphi+\eta\tanh[2\eta(x-\xi)]\\ -\eta\,\mathrm{sech}[2\eta(x-\xi)]\end{pmatrix}. \tag{37}$$

One immediately sees from the solutions (36, 37) that the scattering parameter $c(t)$ is related to $\eta(t)$ and $\xi(t)$ by

$$c=e^{2\eta\xi}. \tag{38}$$

Furthermore, from (21) we find

$$\frac{d\varphi}{dt}=i\frac{d\eta}{dt}=\int f^*Rf\,dx/\int f^*f\,dx. \tag{39}$$

Using

$$Rf=-i\frac{e^{-\eta x}}{4}r\begin{pmatrix}1+\tanh 2\eta(x-\xi)\\ \mathrm{sech}\,2\eta(x-\xi)\end{pmatrix}, \tag{40}$$

and

$$\int f^*f\,dx=(2\eta\varphi)^{-1}, \tag{41}$$

one gets

$$\frac{d\eta}{dt}=-\frac{\varepsilon}{4}\int_{-\infty}^{+\infty} r\,\mathrm{sech}\,z\,dz. \tag{42}$$

On the other hand, inserting the ansatz (38) into (30) yields after some algebra

$$\frac{d\xi}{dt}=-\frac{1}{4\eta^2}-\frac{\varepsilon}{8\eta^2}\cdot\int_{-\infty}^{+\infty}(z+2\eta\xi)r\,\mathrm{sech}\,z\,dz-\frac{\xi}{\eta}\frac{d\eta}{dt}. \tag{43}$$

Together with expression (42), (43) can be written as

$$\frac{d\xi}{dt}=-\frac{1}{4\eta^2}-\frac{\varepsilon}{8\eta^2}\int_{-\infty}^{+\infty} rz\,\mathrm{sech}\,z\,dz. \tag{44}$$

Equations (42) and (44) are the main results for the soliton dynamics within the adiabatic approximation.

At this stage we would like to mention that the results obtained here could have been also derived from the conservation laws.

Since for the unperturbed soliton the quantities

$$I_1=\int\tfrac{1}{2}u^2\,dx, \tag{45}$$

and

$$I_2=\int[2u_{ttx}u_x+4u_{ttx}u_{tx}+u_t^2u_{tx}u_x]\,dx, \tag{46}$$

are conserved, one can derive from the corresponding "conservation laws" of the perturbed Eq. (17), e.g.,

$$\frac{\partial}{\partial t}I_1=-\varepsilon\int ru_x\,dx, \tag{47}$$

differential equations for the two parameters ξ and η. For example, Eq. (47) yields after some algebra the result (42). In a similar way one can obtain (44); however, it is easier to operate on the perturbed sine-Gordon equation by $\int dz\,z\,\mathrm{sech}\,z$ in order to verify

Eq. (44). Since there exists an infinite number of "conservation laws" the adiabatic approximation is not restricted to a two-parameter ansatz. However, using a finite number of conservations laws always lets the question unanswered whether the rest of the totally infinite set of conserved quantities will yield to contradictions or not. Because it is the purpose of this paper to investigate the dynamics of the translation mode by a new perturbation analysis we do not got into the details of such a multi-parameter approach. In addition, the full treatment of the inverse scattering problem as stated at the end of Sect. III would be more promising than starting from the conservation laws.

We conclude this section by outlining the generalizations to more difficult situations than the one-soliton case. In general, N-soliton solutions are known and the corresponding eigenfunctions can be constructed by the procedure given in Ref. [12]. Thus the dynamics of N-solitons in the presence of perturbations reduces to an integration within the adiabatic approximation. For example, the eigenfunctions of the two-soliton solution, which in the case of paired complex eigenvalues corresponds to soliton states which oscillate in time (breather), are explicitely known in the literature [5]. Taking these instead of (36) and (37), the coefficients α, β, $\bar{\alpha}$, $\bar{\beta}$, [Eq. (29)] can be found by integration and, in principle, the time-dependence of the eigenvalues and the soliton parameters are known. Since the formulas are quite lengthy and the algebra is quite involved we renounce the presentation and discussion of this example.

V. Applications

We now report several results for the dynamics of the translation mode by using different expressions for r in Eqs. (42) and (44).

Setting

$$\varepsilon r = \alpha\delta(x+t-x_0-t_0), \tag{48}$$

corresponding to a fixed impurity in real X, T-space, we find

$$\frac{d\eta}{dt} = -\frac{\alpha\eta}{2}\,\mathrm{sech}\,[2\eta(x_0+t_0-t-\xi)], \tag{49}$$

and

$$\frac{d\xi}{dt} = -\frac{1}{4\eta^2} - \frac{\alpha}{2}(x_0+t_0-t-\xi) \cdot \mathrm{sech}\,[2\eta(x_0+t_0-t-\xi)]. \tag{50}$$

The equations show that the soliton travels with constant velocity until it "sees" the impurity. Then it will be accelerrated (decelerated). The acceleration

$$\frac{d}{dt}\frac{d\xi}{dt} \approx \frac{\alpha}{2}\,\mathrm{sech}\,[2\eta(x_0+t_0-t-\xi)], \tag{51}$$

induced the conjecture that solitons behave like Newtonion particles [3].

Next, we consider a constant external driving field by letting

$$\varepsilon r = \chi = \mathrm{const}. \tag{52}$$

We immediately obtain

$$\eta = -\frac{\pi}{4}\chi t + \eta_0, \tag{53}$$

$$\xi = \frac{1}{\pi\chi\left(\eta_0 - \frac{\pi}{4}\chi t\right)}, \tag{54}$$

i.e. the velocity is growing or decreasing depending on the direction of the external field.

Introducing an additional damping by considering

$$\varepsilon r = -\frac{\Gamma}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t}\right) + \chi, \tag{55}$$

one finds

$$\frac{d\eta}{dt} = -\frac{\pi}{4}\chi + \Gamma\eta\left[1 - \frac{1}{4\eta^2}\right], \tag{56}$$

and

$$\frac{d\xi}{dt} = -\frac{1}{4\eta^2}. \tag{57}$$

There exists a "terminal velocity"

$$\left.\frac{d\xi}{dt}\right|_\infty \equiv -\frac{1}{4\eta_\infty^2}, \tag{58}$$

where

$$\eta_\infty = \frac{\pi\chi}{8\Gamma} \pm \sqrt{\frac{1}{4} + \frac{\pi^2\chi^2}{64\Gamma^2}}. \tag{59}$$

On the other hand, in the purely damped case

$$\eta^2 = \tfrac{1}{4}(4\eta_0^2 - 1)e^{2(t-t_0)\Gamma} + \tfrac{1}{4}, \tag{60}$$

i.e., the velocity is exponentially damped.

Finally, we consider soliton motion in an inhomogeneous medium. The simplest conclusion can be drawn from a model which yields a jump in the frequency, i.e., we take

$$\varepsilon r = -\alpha\Theta(x+t)\sin u, \tag{61}$$

where $\Theta(z)$ is the step function ($\Theta(z)=0$ for $z<0$ and $\Theta(z)=1$ for $z>0$). After some algebra we find

$$\frac{d\eta}{dt} = -\frac{\alpha}{4}\operatorname{sech}^2[2\eta(\xi+t)], \tag{62}$$

and

$$\frac{d\xi}{dt} = -\frac{1}{4\eta^2} + \frac{\alpha}{8\eta^2}\{e^{2\eta(\xi+t)}$$

$$+2\eta(\xi+1)\operatorname{sech}[2\eta(\xi+t)]\}\operatorname{sech}[2\eta(\xi+t)]. \tag{63}$$

The dominant contribution arises from $d\eta/dt$. If initially the soliton is moving (for $\alpha>0$) from the high density to the low density η decreases and thus $d\xi/dt$ increases.

All these results are in good agreement with the conclusions by Fogl et al. [3] and Eilenberger [4] based on a completely different approach.

VI. Conclusions

In this paper, we investigated the effect of perturbations on localized solutions of the sine-Gordon equation. We used a perturbation theory for the scattering data, whose time dependences in the case of no perturbations are well-known. The method used here, which was originally developed by Karpman and Maslov [8] for the nonlinear Schrödinger equation and the Korteweg-de Vries equation, has several advantages compared to other procedures [3, 6]: It allows to consider overlapping solitary waves and it is not restricted to shape-conserving solutions.

The physical problem consists of two parts: the dynamics of the translation mode (in order to draw some conclusions about the particle-like behavior of solitons), and the radiation. In this paper we have not attacked the second problem although it is tractable within the general outline given at the end of Sect. III. These investigations are in progress and will be published later. For the dynamical behavior of the translation mode we found a very general and simple description which allows, just by straightforward integration, to calculate the influence of different types of perturbations. The results are in good agreement with those obtained previously, but the procedure proposed here seems to be more powerfull than those used earlier, especially if one wants to consider radiation problems for overlapping solitons. Thus, the applications presented in Sect. V of this paper are promising enough to attack the more difficult radiation problem.

References

1. Scott, A.C., Chu, F.Y.F., McLaughlin, D.W.: IEEE **61**, 1443 (1973)
2. Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H.: Phys. Rev. Lett. **30**, 1262 (1973)
3. Fogl, M.B., Trullinger, S.E., Bishop, A.R., Krumhansl, J.A.: Phys. Rev. B**15**, 1578 (1977)
4. Eilenberger, G.: Z. Physik B**27**, 199 (1977)
5. Kaup, D.J., Newell, A.C.: Proc. R. Soc. Lond. A**361**, 413 (1978)
6. Anderson, D., Bondeson, A., Lisak, M.: A Variational Approach to Perturbed Soliton Equations, Preprint CTH-IEFT/PP-1978-4, Chalmers Univ. of Technology, Gothenburg (1978)
7. Keener, J.P., McLaughlin, D.W.: J. Math. Phys. **18**, 2008 (1977)
8. Karpman, V.I., Maslov, E.M.: Phys. Lett. A**60**, 307 (1977); ibid. **61**, 493 (1977);
   Karpman, V.I., Maslov, E.M.: Zh. Eksp. Teor. Fiz. **73**, 537 (1977); ibid. **75**, in press 1978)
9. Kaup, D.J., Newell, A.C.: SIAM J. Appl. Math. **34**, 37 (1978)
10. Gardner, C.S., Greene, J.M., Kruskal, M.D., Miura, R.M.: Phys. Rev. Lett. **19**, 1095 (1967)
11. Zakharov, V.E., Shabat, A.B.: Sov. Phys.-JETP **34**, 62 (1972)
12. Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H.: Stud. Appl. Math. **53**, 249 (1974)
13. Chadan, K., Sabatier, P.C.: Inverse Problems in Quantum Scattering Theory, Berlin-Heidelberg-New York: Springer 1977
14. Morse, P.M., Feshbach, H.: Methods of Theoretical Physics, Vol. I, p. 884, New York: McGraw-Hill 1953
15. Karpman, V.I.: private communication
16. Chen, H.H., Liu, C.S.: Phys. Rev. Lett. **37**, 693 (1976)

K.H. Spatschek
Fachbereich Physik
Universität Essen – Gesamthochschule
Postfach 6843
D-4300 Essen 1
Federal Republic of Germany