Sequence space representations for (DFN)-algebras of entire functions modulo closed ideals

By *Reinhold Meise* at Düsseldorf

Let $\mathbb{P}=(p_k)_{k\in N}$ be an increasing sequence of plurisubharmonic functions on C^n, which satisfies some mild technical conditions. Denote by $A_P(C^n)$ the vector space of all entire functions f on C^n which satisfy for suitable $C>0$ and $k\in N$ the estimate $|f(z)|\leqq C\exp(p_k(z))$ for all $z\in C^n$. Under its natural inductive limit topology $A_P(C^n)$ becomes a locally convex algebra. Algebras of this type have been studied for a long time. They arise in complex analysis and also in functional analysis since various convolution algebras of distributions and ultradistributions are isomorphic to algebras of this type by Fourier transform.

Extending previous work of Ehrenpreis [8], [9] and others, Berenstein and Taylor [1], [2], [3] have investigated from the point of view of interpolation theory the structure of $A_P(C^n)/I$, where $\mathbb{P}$ is of the special form $(kp)_{k\in N}$ and where I is a localized ideal generated by a slowly decreasing N-tuple of functions in $A_P(C^n)$. Some of their results have been used by Taylor [30] to show that for $\mathbb{P}=(k|z|^s)_{k\in N}$, $s\geqq 1$, every closed ideal in $A_P(C)$ is complemented.

In the present article their investigations are continued and extended in the discrete case where more emphasis is put on the fact that all the algebras $A_P(C^n)$ are (DFN)-spaces, i.e. strong duals of nuclear Fréchet spaces. We show that the nuclearity of $A_P(C^n)$ together with an application of the Auerbach Lemma and easy arguments from functional analysis can be used to derive the following from results and methods of Berenstein and Taylor [1], [2]: Let $(F_1,\dots,F_N)$ be a slowly decreasing N-tuple of functions in $A_P(C^n)$ with a discrete zero variety and denote by $I_{\mathrm{loc}}(F_1,\dots,F_N)$ the local ideal in $A_P(C^n)$ generated by $F_1,\dots,F_N$ which we assume to be infinite codimensional. Then $A_P(C^n)/I_{\mathrm{loc}}(F_1,\dots,F_N)$ is isomorphic to $\lambda(B)'_b$, the strong dual of $\lambda(B)$, where $\lambda(B)$ is the nuclear Köthe sequence space given by $b_{j,k}:=\exp(p_k(w_j))$ for a suitable sequence $(w_j)_{j\in N}$ in C^n with $\lim\limits_{j\to\infty}|w_j|=\infty$. In particular this implies that for $\mathbb{P}=(kp)_{k\in N}$ the quotient space $A_P(C^n)/I_{\mathrm{loc}}(F_1,\dots,F_N)$ is the strong dual of a power series space of infinite type. From the work of Kelleher and Taylor [13] it follows that under some technical conditions on $\mathcal{P}$ every closed ideal in $A_p(\mathbb{C})$ is of the form $I_{loc}(F_1, F_2)$. In this situation the result stated above gives a representation of $A_p(\mathbb{C})/I$ for every closed ideal I in $A_p(\mathbb{C})$.

The sequence space representation of $A_p(\mathbb{C}^n)/I_{loc}(F_1,\dots,F_N)$ allows to use the structure theory of nuclear Fréchet spaces to investigate whether $I_{loc}(F_1,\dots,F_N)$ is complemented in $A_p(\mathbb{C}^n)$. In fact, we show that the results of Taylor [30] mentioned above can be obtained and even extended to a larger class by an application of the splitting theorem of Vogt [32]. On the other hand, we give examples of algebras $A_p(\mathbb{C})$, $\mathcal{P}=(kp)_{k\in\mathbb{N}}$, where p is a radial weight function satisfying $p(2z)=O(p(z))$, in which every proper infinite codimensional closed ideal is not complemented. It is also shown that for many weight systems $\mathcal{P}=(p_k)_{k\in\mathbb{N}}$ for which p_k is radial and which satisfy

$$p_k(2z)=O(p_k(z)) \quad \text{and} \quad \lim_{|z|\to\infty}\frac{p_k(z)}{p_{k+1}(z)}=0 \quad \text{for all} \quad k\in\mathbb{N},$$

every proper closed infinite codimensional ideal I in $A_p(\mathbb{C})$ is not complemented. This is essentially a consequence of the observation that every continuous linear map from $A_p(\mathbb{C})/I$ into $A_p(\mathbb{C})$ is compact, which is derived from the characterization given by Vogt [34]. In some cases it is, however, more convenient to prove this using the remark that $(A_p(\mathbb{C})/I)'_b$ has the property $(\overline{DN})$, introduced by Vogt [34], a linear topological invariant which is rather restrictive. There are even examples of weight systems $\mathcal{P}$ for which $A_p(\mathbb{C})'_b$ has $(\overline{DN})$, and there are other examples $\mathcal{P}$ for which $A_p(\mathbb{C})'_b$ has property $(\bar{\Omega})$ which is also a rather restrictive linear topological invariant. In fact these are the first "natural" examples of nuclear Fréchet spaces having $(\overline{DN})$ resp. $(\bar{\Omega})$.

As a further application of the sequence space representation we obtain results on the structure of the translation invariant subspaces of certain weighted Fréchet spaces of entire functions. We only mention the following two particular cases: Let $A(\mathbb{C})$ denote the Fréchet space of all entire functions on $\mathbb{C}$ and for $s>1$ denote by E_0^s the Fréchet space

$$E_0^s := \left\{ f\in A(\mathbb{C}) \;\middle|\; \sup_{z\in\mathbb{C}} |f(z)| \exp\left(-\frac{1}{k}|z|^s\right) < \infty \quad \text{for all} \quad k\in\mathbb{N} \right\}.$$

Then every closed linear infinite dimensional translation invariant subspace W of $A(\mathbb{C})$ or E_0^s, $s>1$, is isomorphic to a power series space of infinite type and is complemented. Moreover, as we have shown in [19], W has a basis consisting of exponential polynomials. This extends classical results of Schwartz [28], Gelfond [10], Ehrenpreis [8] and Dickson [5] who proved that W has a finite dimensional decomposition. On the other hand, let $\sigma=(s_k)_{k\in\mathbb{N}}$ be a strictly decreasing sequence in $]1,\infty[$ and put

$$A(\sigma) := \{ f\in A(\mathbb{C}) \mid \sup_{z\in\mathbb{C}} |f(z)| \exp(-|z|^{s_k}) < \infty \quad \text{for all} \quad k\in\mathbb{N} \}.$$

Then every closed linear infinite dimensional translation invariant subspace W of $A(\sigma)$ has a regular Schauder basis but is not complemented.

The article is divided in five sections. In the first section we present the tools which are needed from functional analysis. In the second one we introduce the weight systems P and the algebras $A_P(\mathbb{C}^n)$ and we present several examples. The sequence space representation of $A_P(\mathbb{C}^n)/I_{loc}(F_1,\ldots,F_N)$ is proved in section three, and in section four we study the complementation of closed ideals in $A_P(\mathbb{C}^n)$. The results on the structure of translation invariant subspaces are presented in section five.

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1. Nuclear sequence spaces

In this section we introduce the tools from functional analysis which will be needed in the following sections. In particular, we show that certain nuclear Fréchet spaces with a finite dimensional decomposition already have a Schauder basis. This is based on standard techniques and an application of the Auerbach Lemma. We begin by introducing some notation.

1.1 Definition. (a) Let $A=(a_{j,k})_{(j,k)\in N^2}$ be a matrix of non-negative numbers $a_{j,k}$. A is called a Köthe matrix, if

(1) $a_{j,k}\leqq a_{j,k+1}$ for all $j,k\in N$,

(2) $a_{j,1}>0$ for all $j\in N$.

A is called regular or of type (d_0) (see Dubinsky [6], p. 22) if

(3) $\dfrac{a_{j+1,k}}{a_{j+1,k+1}}\leqq\dfrac{a_{j,k}}{a_{j,k+1}}$ for all $j\in N$ and all $k\in N$.

(b) Let A be a Köthe matrix and let $E=(E_j,\|\ \|_j)_{j\in N}$ be a sequence of Banach spaces. For $1\leqq p<\infty$ we define

$$\lambda^p(A,E):=\left\{x\in\prod_{j\in N}E_j\,|\,\pi_{k,p}(x):=\left(\sum_{j=1}^{\infty}(\|x_j\|_j a_{j,k})^p\right)^{\frac{1}{p}}<\infty \text{ for all } k\in N\right\}$$

and for $p=\infty$ we put

$$\lambda^\infty(A,E):=\{x\in\prod_{j\in N}E_j\,|\,\pi_{k,\infty}(x):=\sup_{j\in N}\|x_j\|_j a_{j,k}<\infty \text{ for all } k\in N\}.$$

These spaces of vector-valued sequences are Fréchet spaces under their natural locally convex topology, induced by the norms $(\pi_{k,p})_{k\in N}$. If $E_j=(\mathbb{C},|\cdot|)$ for all $j\in N$, then we write $\lambda^p(A)$ instead of $\lambda^p(A,E)$. Instead of $\lambda^1(A)$ we sometimes write $\lambda(A)$.

(c) Under the assumptions of (b) we put for $1 \leqq p \leqq \infty$

$$k^p(A, E) := \underset{n\to\infty}{\operatorname{ind}}\, k_n^p(A, E),$$

where

$$k_n^p(A, E) := \left\{ x \in \prod_{j \in N} E_j \mid |||x|||_{n,p} := \left(\sum_{j=1}^{\infty} (\|x_j\|_j a_{j,n}^{-1})^p \right)^{\frac{1}{p}} < \infty \right\}$$

for $1 \leqq p < \infty$ and

$$k_n^\infty(A, E) := \{ x \in \prod_{j \in N} E_j \mid |||x|||_{n,\infty} := \sup_{j \in N} \|x_j\|_j a_{j,n}^{-1} < \infty \}.$$

Again we write $k^p(A)$ instead of $k^p(A, E)$ if $E = (C, |\cdot|)_{j \in N}$.

Examples of regular Köthe matrices which are particularly interesting are the following: Let α be an increasing, unbounded sequence of positive real numbers (called exponent sequence) and put $a_{j,k} := e^{k\alpha_j}$ or $b_{j,k} = e^{-\frac{\alpha_j}{k}}$. Then it is easy to see that the matrices $A = (a_{j,k})_{j,k}$ and $B = (b_{j,k})_{j,k}$ are regular. The corresponding spaces $\lambda^p(A)$ (resp. $\lambda^p(B)$) are usually denoted by $\Lambda_\infty^p(\alpha)$ (resp. $\Lambda_1^p(\alpha)$) and are called power series spaces of infinite (resp. finite) type. We recall that the space $C^\infty(S^1)$ of all C^∞-functions on the unit circle S^1 is isomorphic to $\Lambda_\infty^1(\log(n+1))$ and that $\Lambda_\infty^1(\sqrt[k]{n})$ is isomorphic to the space $A(C^k)$ of all entire functions on C^k endowed with the compact-open topology.

1.2 Lemma. *Let $\lambda^1(A)$ be a Schwartz space and let $E = (E_j, \|\ \|_j)_{j \in N}$ denote a sequence of finite dimensional normed spaces. Let $E' := (E_j', \|\ \|_j')_{j \in N}$ denote the sequence of the strong duals E_j' of E_j. Then we have:*

(a) *For $1 \leqq p < \infty$ the strong dual $\lambda^p(A, E)_b'$ of $\lambda^p(A, E)$ is isomorphic to $k^q(A, E')$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

(b) *$k^q(A, E')$ is identical with the locally convex space*

$$\left\{ y \in \prod_{j \in N} E_j' \mid \pi_{a,q}(y) := \left(\sum_{j=1}^{\infty} (\|y_j\|_j' a_j)^q \right)^{\frac{1}{q}} < \infty \text{ for all } a \in \lambda^\infty(A),\ a \geqq 0 \right\}$$

for $1 < q < \infty$ and

$$\{ y \in \prod_{j \in N} E_j' \mid \pi_{a,\infty}(y) := \sup_{j \in N} \|y_j\|_j' a_j < \infty \text{ for all } a \in \lambda^\infty(A),\ a \geqq 0 \}$$

for $q = \infty$, the topology of which is the one induced by the semi-norms $\{\pi_{a,q} \mid a \in \lambda^\infty(A), a \geqq 0\}$.

Proof. (a) It is easy to see that the map $\Phi: k^q(A, E') \to \lambda^p(A, E)_b'$, defined by

$$\langle \Phi(y), x \rangle := \sum_{j=1}^{\infty} \langle y_j, x_j \rangle_j,$$

is linear, continuous and bijective. Since $\lambda^1(A)$ is Schwartz, we have: For every $k \in N$ there exists $m \in N$ such that $\lim_{j\to\infty} \frac{a_{j,k}}{a_{j,m}} = 0$.

From this it follows by well-known arguments that $\lambda^p(A, E)$ is a Fréchet-Schwartz space. Hence $\lambda^p(A, E)_b'$ and $k^q(A, E')$ are (LF)-spaces, whence Φ is an isomorphism by the open mapping theorem for (LF)-spaces.

(b) This follows from the observation that a set M in $\lambda^p(A, E)$ is bounded if and only if there exists $a \in \lambda^\infty(A)$, $a > 0$ with

$$M \subset \left\{ x \in \prod_{j \in N} E_j \mid \sum_{j=1}^{\infty} (\|x_j\|_j a_j^{-1})^p \leqq 1 \right\} \quad \text{for} \quad 1 \leqq p < \infty,$$

and

$$M \subset \{ x \in \prod_{j \in N} E_j \mid \sup_{j \in N} \|x_j\|_j a_j^{-1} \leqq 1 \} \quad \text{for} \quad p = \infty.$$

This can be proved in the same way as Bierstedt, Meise and Summers [1], 2. 5.

1. 3 Proposition. *Let A be a Köthe matrix and let $E = (E_j, \| \ \|_j)_{j \in N}$ be a sequence of finite dimensional normed spaces $E_j \neq \{0\}$. Then the following are equivalent:*

(1) *$\lambda^1(A, E)$ is nuclear.*

(2) *$k^\infty(A, E')$ is nuclear, where $E' := (E_j', \| \ \|_j')_{j \in N}$.*

(3) *For every $k \in N$ there exists $l \in N$ with $\sum_{j=1}^{\infty} (\dim E_j) \frac{a_{j,k}}{a_{j,l}} < \infty$.*

Proof. The equivalence of (1) and (2) follows from Lemma 1. 2(a) since a Fréchet space is nuclear iff its strong dual is nuclear.

To prove that (1) implies (3) we note that for each $k \in N$ the completion of the normed space $(\lambda^1(A, E), \pi_{k,1})$ is the Banach space

$$Y_k = \left\{ y \in \prod_{j \in N} E_j \;\middle|\; \|y\|_k := \sum_{j=1}^{\infty} \|y_j\|_j a_{j,k} < \infty \right\}.$$

Obviously Y_k is isomorphic to $l^1(E)$ by a diagonal transformation. For $l > k$ let $\pi_{l,k}: Y_l \to Y_k$ denote the canonical (inclusion) map. If we identify Y_k and Y_l with $l^1(E)$ then $\pi_{l,k}$ corresponds to the diagonal map $T_{l,k}: l^1(E) \to l^1(E)$ defined by

$$T_{l,k}\big((x_j)_{j \in N}\big) = (T_{l,k}^j x_j)_{j \in N} = \left(\frac{a_{j,k}}{a_{j,l}} x_j \right)_{j \in N}.$$

Since $\lambda^1(A)$ is nuclear by hypothesis, for every $k \in N$ exists $l > k$ such that $\pi_{l,k}$ and consequently $T_{l,k}$ is a nuclear map (see, e.g. Pietsch [26], 4. 1. 2). Hence there exist sequences $(b^n)_{n \in N}$ in $l^1(E)_b' = l^\infty(E')$ and $(y^n)_{n \in N}$ in $l^1(E)$ with $\sum_{n=1}^{\infty} \|b^n\| \, \|y^n\| < \infty$ such that

$$Tx = \sum_{n=1}^{\infty} \langle x, b^n \rangle \, y^n \quad \text{for all} \quad x \in l^1(E).$$

Now for every $n \in N$ we have $b^n = (b_j^n)_{j \in N}$ with $b_j^n \in E_j'$ and $\|b^n\| = \sup_{j \in N} \|b_j^n\|_j'$ and $y^n = (y_j^n)_{j \in N}$ with $y_j^n \in E_j$ and $\|y^n\| = \sum_{j=1}^{\infty} \|y_j^n\|_j$. Since

$$\langle x, b^n \rangle = \sum_{j=1}^{\infty} \langle x_j, b_j^n \rangle \quad \text{for all} \quad n \in N,$$

we get

$$T_j x_j = \sum_{n=1}^{\infty} \langle x_j, b_j^n \rangle \, y_j^n \quad \text{for all} \quad j \in N$$

and hence

$$(\dim E_j) \frac{a_{j,k}}{a_{j,l}} = \operatorname{trace} T_j = \sum_{n=1}^{\infty} \langle y_j^n, b_j^n \rangle.$$

This implies

$$\sum_{j=1}^{\infty} (\dim E_j) \frac{a_{j,k}}{a_{j,l}} \leqq \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \|b_j^n\|_j' \, \|y_j^n\|_j \leqq \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \|b^n\| \, \|y_j^n\|_j$$

$$= \sum_{n=1}^{\infty} \|b^n\| \, \|y^n\| < \infty.$$

Hence (1) implies (3).

If (3) holds then for each $j \in N$ put $n_j := \dim E_j$ and choose an Auerbach basis (see Jarchow [13], p. 291) $\{y_m^j \mid 1 \leqq m \leqq n_j\}$ of $(E_j, \| \ \|_j)$. Let $\{b_m^j \mid 1 \leqq m \leqq n_j\} \subset E_j'$ denote the corresponding coefficient functionals.

Next let $k \in N$ be given and choose $l \in N$ such that (3) holds. Then define $Y_m^j \in l^1(E)$ by $Y_m^j := \left(\frac{a_{j,k}}{a_{j,l}} \delta_{j,n} y_m^j \right)_{n \in N}$ and $B_m^j \in l^1(E)_b' = l^\infty(E')$ by $B_m^j := (\delta_{j,n}, b_m^j)_{n \in N}$ for $j \in N$ and $1 \leqq m \leqq n_j$. Then we have, in the notation introduced above,

$$T_{l,k} x = \sum_{j=1}^{\infty} \sum_{m=1}^{n_j} \langle x, B_m^j \rangle \, Y_m^j$$

and

$$\sum_{j=1}^{\infty} \sum_{m=1}^{n_j} \|B_m^j\| \, \|Y_m^j\| = \sum_{j=1}^{\infty} n_j \frac{a_{j,k}}{a_{j,l}} = \sum_{j=1}^{\infty} (\dim E_j) \frac{a_{j,k}}{a_{j,l}} < \infty.$$

Hence $T_{l,k}$ and consequently $\pi_{l,k}$ is a nuclear map. Consequently $\lambda^1(A, E)$ is nuclear.

Remark. a) Under the hypotheses of Proposition 1. 3 condition 1. 3 (3) is also equivalent to

(1') $\lambda^0(A, E) := \{x \in \lambda^\infty(A, E) \mid \lim_{j \to \infty} \|x_j\|_j \, a_{j,k} = 0 \text{ for all } k \in N\}$ is nuclear.

This follows by modifying the proof of Proposition 1. 3 in an obvious way.

b) Independently of the present work Dubinsky and Holmström [7] have introduced Fréchet spaces with a locally round finite dimensional decomposition. It turns out that this is the class of all spaces $\lambda^1(A, \mathbb{E})$, where $\mathbb{E}=(E_j, |\ |_j)_{j\in N}$ is a sequence of finite dimensional Hilbert spaces. There is some overlapp between their article and this section 1. In fact, [7], Proposition 2. 2, led to an improvement of a previous version of Proposition 1. 3 and the following Proposition 1. 4.

1. 4 Proposition. *Let A be a Köthe matrix, let $\mathbb{E}=(E_j, \|\ \|_j)_{j\in N}$ be a sequence of finite dimensional normed spaces $E_j \neq \{0\}$ and let $\lambda^1(A, \mathbb{E})$ be nuclear. Then the following holds*:

(a) *There exists a Hilbert norm $|\ |_j$ on E_j for each $j\in N$ so that for $\tilde{\mathbb{E}}:=(E_j, |\ |_j)_{j\in N}$ we have $\lambda^1(A, \mathbb{E})=\lambda^2(A, \tilde{\mathbb{E}})$ as l.c. spaces.*

(b) *$\lambda^1(A, \mathbb{E})\cong\lambda^1(B)$ where the matrix B is defined by $b_{l,k}=a_{j,k}$ for*

$$\sum_{m=1}^{j-1} \dim E_m < l \leqq \sum_{m=1}^{j} \dim E_m.$$

Proof. (a) It is a well-known consequence of the Auerbach Lemma (see e.g. Jarchow [13], p. 291) that for each $j\in N$ there is an inner product $(\cdot\,|\,\cdot)_j$ on E_j such that for the corresponding norm $|\ |_j$ the following estimates hold:

$$(*)\qquad (\dim E_j)^{-\frac{1}{2}}\|\ \|_j \leqq |\ |_j \leqq (\dim E_j)^{\frac{1}{2}}\|\ \|_j.$$

We put $\tilde{\mathbb{E}}:=(E_j, |\ |_j)_{j\in N}$ and show that the Fréchet spaces $\lambda^1(A, \mathbb{E})$ and $\lambda^2(A, \tilde{\mathbb{E}})$ are identical.

To do this let $k\in N$ be given. Since $\lambda^1(A, \mathbb{E})$ is nuclear, it follows from Proposition 1. 3 that there exists l with $C_{k,l}:=\sup_{j\in N}(\dim E_j)\frac{a_{j,k}}{a_{j,l}}<\infty$ and that for a suitable $m>l$ we have $\sum_{j=1}^{\infty}\frac{a_{j,l}}{a_{j,m}}<\infty$. Hence we have for each $x=(x_j)_{j\in N}$ in $\lambda^1(A, \mathbb{E})$:

$$\begin{aligned}(\pi_{k,2}(x))^2 &= \sum_{j=1}^{\infty}(|x_j|_j a_{j,k})^2 \leqq \sum_{j=1}^{\infty}(\dim E_j)(\|x_j\|_j a_{j,k})^2\\ &\leqq \sum_{j=1}^{\infty}\left((\dim E_j)\frac{a_{j,k}}{a_{j,l}}\right)^2\left(\frac{a_{j,l}}{a_{j,m}}\right)^2(\|x_j\|_j a_{j,m})^2\\ &\leqq C_{k,l}^2\left(\sum_{j=1}^{\infty}\left(\frac{a_{j,l}}{a_{j,m}}\right)^2\right)(\pi_{m,1}(x))^2.\end{aligned}$$

For all $x=(x_j)_{j\in N}$ in $\lambda^2(A, \tilde{\mathbb{E}})$ we have

$$\begin{aligned}\pi_{k,1}(x) &= \sum_{j=1}^{\infty}\|x_j\|_j a_{j,k} \leqq \sum_{j=1}^{\infty}(\dim E_j)^{\frac{1}{2}}|x_j|\, a_{j,k}\\ &\leqq \sum_{j=1}^{\infty}(\dim E_j)\frac{a_{j,k}}{a_{j,l}}\cdot\frac{a_{j,l}}{a_{j,m}}|x_j|_j a_{j,m}\\ &\leqq C_{k,l}\left(\sum_{j=1}^{\infty}\left(\frac{a_{j,l}}{a_{j,m}}\right)^2\right)^{\frac{1}{2}}\pi_{m,2}(x).\end{aligned}$$

This shows $\lambda^1(A, E)=\lambda^2(A, \hat{E})$ and also the equivalence of the norm systems $(\pi_{k,1})_{k\in N}$ and $(\pi_{k,2})_{k\in N}$.

(b) To show that $\lambda^1(A, E)$ is isomorphic to $\lambda^1(B)$ we choose an orthonormal basis $\{e_{j,n} \mid 1\leqq n\leqq n_j\}$ of the Hilbert space $(E_j, |\ |_j)$, for each $j\in N$. If $x=(x_j)_{j\in N}$ is in $\lambda^1(A, E)=\lambda^2(A, \hat{E})$, then we have

$$x_j=\sum_{n=1}^{n_j}(x_j\mid e_{j,n})_j\, e_{j,n} \quad\text{and}\quad |x_j|_j^2=\sum_{n=1}^{n_j}|(x_j\mid e_{j,n})_j|^2.$$

This implies that

$$(**)\qquad \sum_{j=1}^{\infty}\sum_{n=1}^{n_j}|(x_j\mid e_{j,n})|^2\, a_{j,k}^2=\sum_{j=1}^{\infty}(|x_j|_j\, a_{j,k})^2=(\pi_{k,2}(x))^2$$

for all $k\in N$. Hence the linear map $F\colon \lambda^2(A, \hat{E})\to\lambda^2(B)$, defined by

$$F(x)=(\xi_i)_{i\in N},\ \xi_i:=(x_j\mid e_{j,l})_j \quad\text{for}\quad i=l+\sum_{m=1}^{j-1}n_m=l+\sum_{m=1}^{j-1}\dim E_m,$$

is continuous. Using $(**)$ it is easy to check that F is a surjective isomorphism. Hence we get by the nuclearity of $\lambda^1(A, E)$

$$\lambda^1(A, E)=\lambda^2(A, \hat{E})\cong\lambda^2(B)=\lambda^1(B).$$

The following elementary lemma will be rather useful in section four.

1.5 Lemma. *Let $F=(F_j, |\ |_j)_{j\in N}$ be a sequence of Hilbert spaces and let A be a Köthe-matrix. For $k\in N$ denote by $\pi_k\colon\lambda^2(A, F)\to\lambda^2(A, F)$ the map*

$$\pi_k((x_j)_{j\in N}):=(\delta_{j,k}x_k)_{j\in N}.$$

Let $T\colon\lambda^2(A, F)\to\lambda^2(A, F)$ be continuous and linear with $T\circ\pi_k=\pi_k\circ T$ for all $k\in N$. Then $\ker T$ *is complemented in $\lambda^2(A, F)$.*

Proof. It is easily checked that there exist $T_j\in L(F_j)$ for $j\in N$ such that

$$T((x_j)_{j\in N})=(T_j x_j)_{j\in N}.$$

This implies that $x=(x_j)_{j\in N}$ belongs to $\ker T$ iff $x_j\in\ker T_j$ for all $j\in N$. Since F_j is a Hilbert space there exists an orthogonal projection P_j on F_j with $P_j(F_j)=\ker T_j$. It is easy to see that $P\colon\lambda^2(A, F)\to\lambda^2(A, F)$, defined by $P((x_j)_{j\in N})=(P_j x_j)_{j\in N}$ is a continuous projection on $\lambda^2(A, F)$ with $P(\lambda^2(A, F))=\ker T$.

2. Weighted (DFN)-algebras of entire functions

In this section we introduce the weighted (DFN)-algebras $A_p(C^n)$ of entire functions on C^n. This class contains the algebras $A_p(C^n)$ which have been investigated by Berenstein and Taylor [2], [3]. Using their notation we introduce weight functions and weight systems and present some examples.

2.1 Definition. A function $p: C^n \to [0, \infty[$ is called a weight function on C^n if it has the following properties:

(1) p is continuous and plurisubharmonic.

(2) $\log(1+|z|^2) = O(p(z))$.

(3) There exists $C \geqq 1$ such that for all $w \in C^n$

$$\sup_{|z-w| \leqq 1} p(z) \leqq C \inf_{|z-w| \leqq 1} p(z) + C.$$

A weight function will be called radial, if $p(z) = p(|z|)$ for all $z \in C^n$, where

$$|z| := \left(\sum_{j=1}^{n} |z_j|^2 \right)^{\frac{1}{2}}.$$

2.2 Definition. A sequence $P = (p_k)_{k \in N}$ of weight functions on C^n is called a weight system if it has the following properties:

(1) For every $k \in N$ there is $M \geqq 0$ with $p_k \leqq p_{k+1} + M$.

(2) For every $k \in N$ there exist $m \in N$ and $L \geqq 0$ with

$$2p_k(z) \leqq p_m(z) + L \quad \text{for all} \quad z \in C^n.$$

A weight system $P = (p_k)_{k \in N}$ is called radial, if p_k is radial for all $k \in N$.

For an open set Ω in C^n let $A(\Omega)$ denote the algebra of all holomorphic functions on Ω. If P is a given weight system on C^n then we define the subalgebra $A_P(C^n)$ of $A(C^n)$ in the following way:

2.3 Definition. (a) For a weight function p on C^n we put

$$H_p^\infty(C^n) := \{ f \in A(C^n) \mid \|f\|_{p,\infty} := \sup_{z \in C^n} |f(z)| \, e^{-p(z)} < \infty \},$$

$$H_p^2(C^n) := \{ f \in A(C^n) \mid \|f\|_{p,2} := \left(\int_{C^n} (|f(z)| \, e^{-p(z)})^2 \, dm_{2n}(z) \right)^{\frac{1}{2}} < \infty \},$$

where m_{2n} denotes the Lebesgue measure on $C^n = R^{2n}$.

(b) For a weight system P on C^n we define

$$A_P(C^n) := \bigcup_{k \in N} H_{p_k}^\infty(C^n)$$

and endow this vector space with its natural inductive limit topology. Following the notation of Berenstein and Taylor [2], [3] we write $A_p(C^n)$ instead of $A_P(C^n)$ if $P = (kp)_{k \in N}$.

By standard arguments one proves:

2.4 Proposition. *For every weight system P on C^n we have:*

(a) *$A_P(C^n)$ is a locally convex algebra with unit under pointwise multiplication.*

(b) *$A_P(C^n)$ is a (DFN)-space, i.e. the strong dual of a nuclear Fréchet space. In particular, $A_P(C^n)$ is nuclear, complete and reflexive.*

(c) *$A_P(C^n) = \bigcup_{k \in N} H_{p_k}^2(C^n)$ holds topologically.*

Next we give some examples of weight functions and weight systems which will be used later on.

2.5 Examples. (1) Let $\varphi:[0,\infty[\to[0,\infty[$ be continuous, convex, increasing with $\lim_{t\to\infty}\varphi(t)=\infty$ and assume that there exists $D\geqq 1$ with $\varphi(2t)\leqq D\varphi(t)+D$ for all $t\in[0,\infty[$. Then it follows easily from Hörmander [12], Theorem 1.6.7, that $\varphi\circ p$ is a weight function on $\mathbb{C}^n$ whenever p is a weight function on $\mathbb{C}^n$.

(2) Let $\varphi:[0,\infty[\to[0,\infty[$ be continuous with $\lim_{t\to\infty}\varphi(t)=\infty$. Assume that $t\mapsto\varphi(e^t)$ is convex and increasing and there exists $D\geqq 1$ with $\varphi(2t)\leqq D\varphi(t)+D$ for all $t\in[0,\infty[$. Then $p:z\mapsto\varphi(|z|^r)$ is a radial weight function on $\mathbb{C}^n$ for every $r>0$.

Most of the following examples can be obtained from (1) or (2):

(3) $p(z)=|z|^r$, $r>0$.

(4) $p(z)=\log(1+|z|^2)+|\mathrm{Im}\,z|$.

(5) $p(z)=(\log(1+|z|^2))^s$, $s\geqq 1$.

(6) $p(z)=|\mathrm{Re}(z)|^r+|\mathrm{Im}\,z|^s$, $r,s\geqq 1$.

(7) $p(z)=|z|^r+|\mathrm{Im}\,z|^s$, $r>0$, $s\geqq 1$.

2.6 Associated functions. (1) Let $M=(M_j)_{j\in\mathbb{N}_0}$ be a sequence of positive real numbers which for the whole article is assumed to satisfy $M_0=1$ and $\lim_{j\to\infty}M_j^{\frac{1}{j}}=\infty$. Then the associated function $p_M:\mathbb{C}^n\to[0,\infty[$ is defined (see Komatsu [15], § 3) by

$$p_M(z):=\begin{cases}\sup_{j\in\mathbb{N}_0}\log\dfrac{|z|^j}{M_j} & \text{for } z\neq 0,\\ 0 & \text{for } z=0.\end{cases}$$

It is easy to see that p_M is plurisubharmonic and continuous and that $t\mapsto p_M(e^t)$ is increasing and convex on $[0,\infty[$.

It is useful (see Komatsu [15]) to consider the following conditions on M:

(M1) $M_j^2\leqq M_{j-1}M_{j+1}$ for all $j\in\mathbb{N}$.

(M2) There exist $A,H\geqq 1$ with $M_n\leqq AH^n\min_{0\leqq j\leqq n}M_jM_{n-j}$ for all $n\in\mathbb{N}$.

If M satisfies (M1) and (M2) then, by Komatsu [15], 3.6, there exists $D>1$ such that for all $z\in\mathbb{C}^n$

$$2p_M(z)\leqq p_M(Dz)+D.$$

If M satisfies (M2) and if $N:=(M_j\exp(-jg(j)))_{j\in\mathbb{N}_0}$, where $g:\mathbb{N}_0\to[0,\infty[$ is increasing, then it is easily checked that N satisfies (M2).

(2) Assume that $M=(M_j)_{j\in\mathbb{N}_0}$ satisfies (M1). Petzsche [25] has shown that the following conditions (i) and (ii) are equivalent:

(i) there exists $C>0$ with $p_M(2z)\leqq C(p_M(z)+1)$ for all $z\in\mathbb{C}^n$,

(ii) there exists $k\in\mathbb{N}$ with $\liminf_{j\to\infty}\left(\dfrac{M_{jk}}{M_j^k}\right)^{\frac{1}{j}}>1$.

If M satisfies also (M2), then (i) is equivalent to

(iii) there exists $k \in N$ with $\liminf_{j \to \infty} \frac{m_{jk}}{m_j} > 1$, where $m_j := \frac{M_j}{M_{j-1}}$.

It is easy to check that (i) implies that p_M is a weight function on C^n. Moreover, in this case p_M satisfies all the conditions stated in 2. 5. (2).

(3) For $s > 0$ consider the so-called Gevrey sequence $M := ((p!)^s)_{p \in N_0}$. Then M satisfies (M1) and (M2) and it is well-known that for all $z \in C^n$

$$p_M(z) \leqq s|z|^{\frac{1}{s}} \quad \text{and} \quad |z|^{\frac{1}{s}} \leqq \frac{2}{s} p_M(z) + 2\log 2.$$

(4) For $s > 1$ and $a > 0$ consider $M := (e^{ap^s})_{p \in N_0}$. Then M satisfies (M1) but not (M2). Some computation shows that there exists $D > 0$ such that for all $z \in C^n$

$$p_M(z) \leqq 3\frac{s-1}{s}\left(\frac{1}{sa}\right)^{\frac{1}{s-1}} (\log(1+|z|^2))^{\frac{s}{s-1}} + D$$

and

$$\frac{s-1}{3s}\left(\frac{1}{sa}\right)^{\frac{1}{s-1}} (\log(1+|z|^2))^{\frac{s}{s-1}} \leqq p_M(z) + D.$$

Next we describe some methods how to generate weight systems out of a given (weight) function.

2. 7 Example. (1) Let $(\varphi_k)_{k \in N}$ be an increasing sequence of functions each satisfying 2. 5 (1). Assume furthermore that $(\varphi_k)_{k \in N}$ satisfies the condition 2. 2 (1) and (2) if p_k is replaced by φ_k and C^n is replaced by $[0, \infty[$. Then $P := (\varphi_k \circ p)_{k \in N}$ is a weight system on C^n whenever p is a weight function on C^n.

Obvious examples of such sequences $(\varphi_k)_{k \in N}$ are $\varphi_k : x \mapsto kx$ and $\varphi_k : x \mapsto x^{r_k}$, where $(r_k)_{k \in N}$ is a strictly increasing sequence in $[1, \infty[$.

(2) Let φ be as in 2. 5 (2) and let $(r_k)_{k \in N}$ be a strictly increasing sequence in $]0, \infty[$. Then $P := (\varphi(|z|^{r_k}))_{k \in N}$ is a weight system on C^n, provided that one of the following conditions is satisfied:

(α) $\lim_{k \to \infty} r_k = \infty$ and there exists $r > 1$ with $\liminf_{t \to \infty} \frac{\varphi(t^r)}{\varphi(t)} > 2$,

(β) $\liminf_{t \to \infty} \frac{\varphi(t^r)}{\varphi(t)} > 2$ for each $r > 1$.

It is easy to see that (β) is equivalent to $\lim_{t\to\infty}\frac{\varphi(t^r)}{\varphi(t)}=\infty$ for each $r>1$.

(3) Let $M=(M_j)_{j\in N_0}$ satisfy (M1) and 2. 6 (2) (ii). Then

$$\liminf_{t\to\infty}\frac{p_M(t^3)}{p_M(t)}>2$$

holds. To see this, choose $t_0>0$ so that for all $t\geqq t_0$ we have

$$p_M(t)=j(t)\log t-\log M_{j(t)}$$

and $M_{j(t)}\geqq 1$. This implies

$$3p_M(t)=3\big(j(t)\log t-\log M_{j(t)}\big)\leqq j(t)\log t^3-\log M_{j(t)}\leqq p_M(t^3)$$

for all $t\geqq t_0$.

(4) If $M=(M_j)_{j\in N_0}$ satisfies (M1) and (M2) then it follows from the characterization of condition (M2), stated in 2. 6. 1, that

$$\liminf_{t\to\infty}\frac{p_M(t^r)}{p_M(t)}>2 \quad \text{for each} \quad r>1.$$

In section four we shall need information on the structure of $A_p(C^n)'_b$. This information can be obtained easily if a sequence space representation of $A_p(C^n)$ is known. For radial weight functions on C such representations are obtained by estimating the Taylor coefficients of the functions in $A_p(C)$. Here we give only some relevant examples; for a systematic study of $A_p(C)$ using the Young conjugate function we refer to Meise and Taylor [22], where further interesting examples are given.

2. 8 Proposition. *Let $P=(p_k)_{k\in N}$ be a radial weight system on C which satisfies $\log(1+|z|^2)=o(p_1(z))$. Then*

$$A_p(C)\cong k^2(B)=k^\infty(B),$$

where $B:=(b_{j,k})$, $b_{j,k}=\left(2\pi\int_0^\infty r^{2j+1}\exp(-2p_k(r))\,dr\right)^{-\frac{1}{2}}$, $j\in N_0$, $k\in N$.

Proof. Put $l_k^2:=\left\{x\in C^{N_0}\,\middle|\,\sum_{j=0}^\infty |x_j|^2 b_{j,k}^{-2}<\infty\right\}$. Then it is easy to check that the map $T_k: H_{p_k}^2\to l_k^2$, $T_k(f):=\left(\frac{f^{(j)}(0)}{j!}\right)_{j\in N_0}$, is an isometrical isomorphism for each $k\in N$. Obviously we have $T_{k+1}\mid H_{p_k}=T_k$. Hence we get from 2. 4 that

$$A_p(C)=\bigcup_k H_{p_k}^2\cong k^2(B)=k^\infty(B).$$

2. 9 Corollary. *Let p be a radial weight function on C^n which satisfies:*

$$(*)\qquad \begin{cases}\text{There exist } A\geqq 1 \text{ and } B\geqq 0 \text{ such that for all } z\in C\\ p(2z)\leqq Ap(z)+B \text{ and } 2p(z)\leqq p(Az)+B.\end{cases}$$

Then $A_p(C^n)\cong \Lambda_\infty(k^{\frac{1}{n}})'_b$.

Proof. Let us first assume $n=1$. Then (*) implies $A_p(\mathbb{C})=A_{\mathbb{P}}(\mathbb{C})$, where $\mathbb{P}=(p_k)$ with $p_k: z\mapsto p(kz)$. Since

$$\int_0^\infty r^{2j+1}e^{-2p(kr)}dr=\frac{1}{k^{2j+2}}\int_0^\infty s^{2j+1}e^{-2p(s)}ds$$

we get from Proposition 2. 8 that by a diagonal transformation $A_p(\mathbb{C})\cong\Lambda_\infty(k)'_b$. If $n>1$ then (*) implies $A_p(\mathbb{C}^n)=A_q(\mathbb{C}^n)$, where $q: z\mapsto\sum_{j=1}^n p(z_j e_j)$ ($(e_j)_{j=1}^n$ the canonical basis vectors of $\mathbb{C}^n$). Hence it follows from 2. 4 that

$$A_p(\mathbb{C}^n)=A_q(\mathbb{C}^n)\cong A_p(\mathbb{C})\hat{\otimes}\cdots\hat{\otimes}A_p(\mathbb{C})\cong\Lambda_\infty(k^{\frac{1}{n}})'_b.$$

2. 10 Example. Let $M=(M_j)_{j\in\mathbb{N}_0}$ be a sequence satisfying the conditions (M1), (M2) and 2. 6 (2) (ii). Then it follows from 2. 6 (1) and (2) that p_M satisfies condition (*) of Corollary 2. 9. Hence we have

$$A_{p_M}(\mathbb{C}^n)\cong\Lambda_\infty((\sqrt[n]{k})_{k\in\mathbb{N}})'_b.$$

This example can also be derived from Komatsu [15], 4. 5, which we will now use to obtain further examples.

2. 11 Proposition. *For $k\in\mathbb{N}$ let $M^k=(M_j^k)_{j\in\mathbb{N}_0}$ satisfy* (M1) *and* 2. 6 (2) (ii) *and assume that $M_j^k\geqq M_j^{k+1}$ for all $j\in\mathbb{N}_0$ and all $k\in\mathbb{N}$. Then the following holds:*

(a) *If $\mathbb{P}=(p_{M^k})_{k\in\mathbb{N}}$ is a weight system, then $A_{\mathbb{P}}(\mathbb{C})\cong\lambda(B)'_b$, where $b_{j,k}=(M_j^k)^{-1}$, $j\in\mathbb{N}_0$, $k\in\mathbb{N}$.*

(b) *If M^k satisfies* (M2) *for each $k\in\mathbb{N}$ and if $\left(\frac{M_j^{k+1}}{M_j^k}\right)_{j\in\mathbb{N}_0}$ is in $\Lambda_\infty(j)$ for all $k\in\mathbb{N}$, then $\mathbb{P}=(p_{M^k})_{k\in\mathbb{N}}$ is a weight system on $\mathbb{C}$ which satisfies*

$$\lim_{|z|\to\infty}\frac{p_{M^k}(z)}{p_{M^{k+1}}(z)}=0 \quad \textit{for all} \quad k\in\mathbb{N}.$$

Proof. (a) If $f\in H_{p_k}^\infty(\mathbb{C})$ for some $k\in\mathbb{N}$, then the proof of Komatsu [15], 4. 5, shows that for each $j\in\mathbb{N}_0$ we have

$$\left|\frac{f^{(j)}(0)}{j!}\right|\leqq\|f\|_{\infty,k}(M_j^k)^{-1}.$$

If $(a_j)_{j\in\mathbb{N}_0}$ satisfies $\sup_{j\in\mathbb{N}_0}|a_j|\,M_j^k=:\|a\|_k<\infty$ for some $k\in\mathbb{N}$, then Komatsu [15], 4. 5, shows that the function $f: z\mapsto\sum_{j=0}^\infty a_j z^j$ can be estimated by

$$\begin{aligned}|f(z)|&\leqq 2\|a\|_k\exp(p_{M^k}(2z))\leqq 2\|a\|_k\exp(C_k p_{M^k}(z)+C_k)\\&\leqq 2\|a\|_k\exp(C'_k)\exp(p_{M^l}(z)),\end{aligned}$$

since $\mathbb{P}$ is a weight system and since M^k satisfies 2. 6 (2) (i). This proves that

$$A_{\mathbb{P}}(\mathbb{C})\cong\lambda(B)'_b.$$

(b) Since M^{k+1} satisfies (M2) it follows from 2.6 (1) that for some $D>1$ and all $n\in N$ we have

$$2^n p_{M^{k+1}}(z) \leqq p(D^n z) + (2^n - 1)D \quad \text{for all} \quad z \in C,$$

and hence

$$p_{M^{k+1}}\left(\frac{z}{D^n}\right) \leqq \frac{1}{2^n} p_{M^{k+1}}(z) + D.$$

Since $\left(\frac{M_j^{k+1}}{M_j^k}\right)_{j\in N_0}$ is in $\Lambda_\infty(n)$ it follows from this and Komatsu [15], 3.1.10 that

$$\lim_{|z|\to\infty} \frac{p_{M^k}(z)}{p_{M^{k+1}}(z)} = 0.$$

This implies that 2.2 (2) is satisfied. Hence P is a weight system.

2.12 Corollary. *Let* $M=(M_j)_{j\in N_0}$ *be a sequence satisfying* (M1) *and* 2.6 (2) (ii), *and let* $(r_k)_{k\in N}$ *be a strictly increasing sequence in* $[1,\infty[$.

(a) *If* $P := \left(\frac{1}{r_k} p_M(|z|^{r_k})\right)_{k\in N}$ *is a weight system, then*

$$A_P(C)_b' \cong \Lambda_1\left((\log M_j)_{j\in N_0}\right).$$

(b) *If* M *satisfies* (M2) *and if* P *is defined as in* (a) *then*

$$\tilde{P} := \left(p_M(|z|^{r_k})\right)_{k\in N}$$

is a weight system and $A_P(C) = A_{\tilde{P}}(C)$.

Proof. (a) Put $M^k := (M_j^{\frac{1}{r_k}})_{j\in N_0}$ and remark that

$$p_M(|z|^{r_k}) = \sup_{j\in N_0} \log \frac{|z|^{r_k j}}{M_j} = r_k \sup_{j\in N_0} \log \frac{|z|^j}{M_j^{\frac{1}{r_k}}} = r_k p_{M^k}(z).$$

Hence it follows from 2.11 (a) that $A_P(C)_b' \cong \lambda(B)$, where

$$b_{j,k} = (M_j^k)^{-1} = \exp\left(-\frac{1}{r_k}\log M_j\right), \quad j\in N_0,\ k\in N,$$

which shows $A_P(C)_b' \cong \Lambda_1\left((\log M_j)_{j\in N_0}\right)$.

(b) Since $0 < \frac{1}{r_k} \leqq 1$ for all $k\in N$ we have

$$\frac{1}{r_k} p_M(|z|^{r_k}) \leqq p_M(|z|^{r_k}) \quad \text{for all} \quad z\in C.$$

Since M satisfies (M2) it follows from 2.6 (1) that for some $D>1$ we have

$$r_{k+1} p_M(|z|^{r_k}) \leqq p_M(D|z|^{r_k}) + D \leqq p_M(|z|^{r_{k+1}}) + D'$$

for all $z\in C$. This implies $A_P(C) = A_{\tilde{P}}(C)$.

2.13 Examples. (1) Let $(r_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence in $]0, \infty[$ and put $\mathbb{P}:=(|z|^{r_k})_{k\in\mathbb{N}}$. Then $A_\mathbb{P}(\mathbb{C})\cong\Lambda_1((j\log j)_{j\in\mathbb{N}})'_b$. This follows from Corollary 2.12 and 2.6 (3).

(2) For $r>1$ put $p:z\mapsto(\log(1+|z|^2))^r$. Then $A_p(\mathbb{C})\cong\Lambda_1((j^{\frac{r}{r-1}})_{j\in\mathbb{N}})'_b$. To see this, put $s:=\frac{r}{r-1}$ and define $M^k=\left(\exp\left(\frac{j^s}{k}\right)\right)_{j\in\mathbb{N}_0}$ for $k\in\mathbb{N}$. From 2.6 (4) we get that there is $A>0$ such that for each $k\in\mathbb{N}$ there exists $D>0$ with

$$\frac{A}{3}k^{\frac{1}{s-1}}p(z)-D\leqq p_{M^k}(z)\leqq 3Ak^{\frac{1}{s-1}}p(z)+D.$$

This shows that $\mathbb{P}:=(p_{M^k})_{k\in\mathbb{N}}$ is a weight system and that $A_p(\mathbb{C})=A_\mathbb{P}(\mathbb{C})$. Hence the isomorphism follows from 2.11 (a).

(3) Let $(r_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence in $]1,\infty[$ and put

$$\mathbb{P}:=((\log(1+|z|^2))^{r_k})_{k\in\mathbb{N}}.$$

Then $A_\mathbb{P}(\mathbb{C})=\lambda(B)'_b$, where $b_{j,k}=\exp(-j^{\frac{r_k}{r_k-1}})$, $j\in\mathbb{N}_0$, $k\in\mathbb{N}$. To see this, put

$$M^k=(\exp(j^{\frac{r_k}{r_k-1}}))_{j\in\mathbb{N}_0}$$

and put $\tilde{\mathbb{P}}:=(p_{M^k})_{k\in\mathbb{N}}$. Since $\tilde{\mathbb{P}}$ is a weight system and since $A_\mathbb{P}(\mathbb{C})=A_{\tilde{\mathbb{P}}}(\mathbb{C})$ by 2.6 (4), the isomorphism follows from 2.11 (a).

(4) Let $0<s<\infty$ and $0<q<1$ be given. Some computation shows that for each $k\in\mathbb{N}$ one can find $j_k\in\mathbb{N}$ and $M^k=(M_j^k)_{j\in\mathbb{N}_0}$ which satisfies the hypotheses of Proposition 2.11 (b) such that

$$M_j^k=\exp(sj\ \log j-kj(\log j)^q)\quad\text{for all}\quad j\geqq j_k.$$

Then $\mathbb{P}:=(p_{M^k})_{k\in\mathbb{N}}$ is a weight system on $\mathbb{C}$ with $\lim_{|z|\to\infty}\frac{p_{M^k}(z)}{p_{M^{k+1}}(z)}=0$ for each $k\in\mathbb{N}$, by Proposition 2.11. Moreover, we have $A_\mathbb{P}(\mathbb{C})\cong\lambda(B)'_b$, where $b_{j,k}=(M_j^k)^{-1}$ for $j\in\mathbb{N}_0$, $k\in\mathbb{N}$. Hence we get by a diagonal transformation that

$$A_\mathbb{P}(\mathbb{C})\cong\Lambda_\infty((j(\log(j+1))^q)_{j\in\mathbb{N}})'_b.$$

(5) Let $0<s<\infty$ be given. Some computation shows that for each $k\in\mathbb{N}$ one can find $j_k\in\mathbb{N}$ and $M^k=(M_j^k)_{j\in\mathbb{N}_0}$ which satisfies the hypotheses of Proposition 2.11 (b) such that

$$M_j^k=\exp(sj(\log j-(\log\log j)^k))\quad\text{for all}\quad j\geqq j_k.$$

Then $\mathbb{P}:=(p_{M^k})_{k\in\mathbb{N}}$ is a weight system on $\mathbb{C}$ with $\lim_{|z|\to\infty}\frac{p_{M^k}(z)}{p_{M^{k+1}}(z)}=0$ for each $k\in\mathbb{N}$, by Proposition 2.11. Moreover, we have $A_\mathbb{P}(\mathbb{C})\cong\lambda(B)'_b$, where $b_{j,k}=(M_j^k)^{-1}$ for $j\in\mathbb{N}_0$, $k\in\mathbb{N}$. Hence we get by a diagonal transformation that $A_\mathbb{P}(\mathbb{C})\cong\lambda(B)'_b$, where

$$b_{j,k}=\exp(sj(\log\log(e+j))^k).$$

Later we shall use that $A_{\tilde{P}}(C)=A_P(C)$, where $\tilde{P}:=\left(|z|^{\frac{1}{s}}\exp\left((\log\log|z|^{\frac{1}{s}})^k\right)\right)_{k\in N}$. This is a consequence of the following remark which is proved by some computation:

For $a>0$ and $s>0$ let $M=(M_j)_{j\in N_0}$ be a sequence satisfying (M1) and (M2) such that for some $j_0\in N$ we have $M_j=\exp\left(js(\log j-(\log\log j)^a)\right)$ for all $j\geqq j_0$. Then there exists $B=B(s,a)$ such that for all $z\in C$

$$\frac{s}{4e}|z|^{\frac{1}{s}}\exp\left((\log\log(|z|^{\frac{1}{s}}+e))^a\right)-B\leqq p_M(z)$$

$$\leqq s|z|^{\frac{1}{s}}\exp\left((\log\log(|z|^{\frac{1}{s}}+e))^a\right)+B.$$

3. Weighted algebras modulo localized ideals

In this section we use ideas and results of Berenstein and Taylor [1], [2] to derive a sequence space representation of $A_P(C^n)/I$ for certain closed ideals I in $A_P(C^n)$. Since we also use the nuclearity of $A_P(C^n)$ we can apply the main results of section 1 to get a rather precise information on the structure of $A_P(C^n)/I$.

As a slight extension of a notation introduced by Berenstein and Taylor [1], [2], we define:

3. 1 Definition. Let P be a weight system on C^n and let

$$F=(F_1,\ldots,F_N)\in\left(A_P(C^n)\right)^N$$

be given.

a) F is called slowly decreasing if

$$V(F):=\{z\in C^n \mid F_j(z)=0 \quad \text{for} \quad 1\leqq j\leqq N\}$$

is discrete in C^n (which implies $N\geqq n$) and if there are $\varepsilon>0$ and $m\in N$ with

(i) each component of the set $S_m(F,\varepsilon)$ is bounded, where

$$S_m(F,\varepsilon):=\left\{z\in C^n \;\middle|\; \left(\sum_{j=1}^N |F_j(z)|^2\right)^{\frac{1}{2}}<\varepsilon\exp\left(-p_m(z)\right)\right\},$$

(ii) for every $k\geqq m$ there are D_k and D_k' such that for each component S of $S_m(F,\varepsilon)$ we have

$$\sup_{z\in S}p_k(z)\leqq D_k\inf_{z\in S}p_k(z)+D_k'.$$

b) F is called slowly decreasing in the weak sense if (i) and (ii) are required to hold only for those components S of $S_m(F,\varepsilon)$ for which $S\cap V(F)\neq\emptyset$.

c) F is called regularly slowly decreasing in the weak sense if $V(F)$ is discrete, if there exist $\varepsilon>0$ and $m\in\mathbb{N}$ so that the corresponding conditions (i) and (ii) above are satisfied and if the following conditions (iii) and (iv) hold:

(iii) $V(F)$ is an infinite set,

(iv) there exist an enumeration $(S_j)_{j\in\mathbb{N}}$ of the components S of $S_m(F,\varepsilon)$ with $S\cap V(F)\neq\emptyset$ and $z_j\in S_j$ for each $j\in\mathbb{N}$ such that for the matrix $A=(a_{j,k})$ defined by $a_{j,k}=\exp(p_k(z_j))$, $j\in\mathbb{N}$, $k\in\mathbb{N}$, $k\geqq m$, there exists $j_0\in\mathbb{N}$ such that the submatrix $\tilde{A}:=(a_{j,k})_{j\geqq j_0,k\geqq m}$ is regular in the sense of definition 1.1 (a).

3.2 Remark. a) If $\mathbb{P}=(p_k)_{k\in\mathbb{N}}$ is a radial weight system on $\mathbb{C}$ with

$$p_k(2z)=O\big(p_k(z)\big)$$

for every $k\in\mathbb{N}$, then it follows from an application of the minimum modulus theorem (see Levin [17], p. 20) as in Berenstein and Taylor [1], Proposition 4, that every $f\in A_p(\mathbb{C})$, $f\neq 0$, is slowly decreasing. Hence every $F=(F_1,\dots,F_N)\in\big(A_p(\mathbb{C})\big)^N$ with $F\neq 0$ is slowly decreasing.

b) For examples of $F=(F_1,\dots,F_N)\in\big((A_p(\mathbb{C}^n)\big)^N$ which are slowly decreasing, we refer to Berenstein and Taylor [2].

c) If p is a non-radial weight function, then there might exist $f\in A_p(\mathbb{C})$ which are slowly decreasing in the weak sense, but not slowly decreasing. To see this look at the following example which is essentially due to B. A. Taylor: Let the weight function p on $\mathbb{C}$ be defined by $p: z\mapsto\max(0,\operatorname{Re}(z))+\sqrt{|z|}$ and define the entire function $f: z\mapsto e^z\prod_{j=1}^{\infty}\left(1-\frac{z}{j^2}\right)$. Then f is in $A_p(\mathbb{C})$, but f is not slowly decreasing since for every $\varepsilon>0$ and every $m\in\mathbb{N}$ there is $a<0$ with $S_m(f,\varepsilon)\supset\{x\in\mathbb{R}\mid x\leqq a\}$. However, a careful application of the minimum modulus theorem shows that f is slowly decreasing in the weak sense.

3.3 Lemma. *Let $\mathbb{P}=(p_k)_{k\in\mathbb{N}}$ be a weight system on $\mathbb{C}^n$, let*

$$F=(F_1,\dots,F_N)\in\big(A_p(\mathbb{C}^n)\big)^N$$

be slowly decreasing in the weak sense and let $V(F)$ be infinite. Then F is regularly slowly decreasing in the weak sense provided that $\mathbb{P}$ is of one of the following types:

(1) *$\mathbb{P}=(\varphi_k\circ p)_{k\in\mathbb{N}}$, where $(\varphi_k)_{k\in\mathbb{N}}$ and p are as in 2.7 (1) and where there exists $t_0\geqq 0$ such that $\varphi_{k+1}-\varphi_k$ is increasing on $[t_0,\infty[$ for each $k\in\mathbb{N}$.*

(2) *$\mathbb{P}=\big(\varphi(|z|^{r_k})\big)_{k\in\mathbb{N}}$ is as in 2.7 (2) (α) and there exist $s>1$ and $t_0\geqq 0$ such that $t\mapsto\varphi(t^s)-\varphi(t)$ is increasing on $[t_0,\infty[$.*

(3) *$\mathbb{P}=\big(\varphi(|z|^{r_k})\big)_{k\in\mathbb{N}}$ is as in 2.7 (2) (β) and there exists $t_0\geqq 0$ such that for every $s>1$ the function $t\mapsto\varphi(t^s)-\varphi(t)$ is increasing on $[t_0,\infty[$.*

Proof. By hypothesis there exist $\varepsilon>0$ and $m\in\mathbb{N}$ such that there are infinitely many components S of $S_m(F,\varepsilon)$ with $S\cap V(F)\neq\emptyset$ and such that the corresponding conditions (i) and (ii) of 3.1 a) are satisfied.

Let us assume that P is of type (1). Since p is bounded on each component S of $S_m(F,\varepsilon)$ with $S \cap V(F) \neq \emptyset$, since $\lim_{|z|\to\infty} p(z) = \infty$ and since $V(F)$ is discrete, we can enumerate these conponents by $(S_j)_{j\in N}$ in such a way that $(\sup_{z\in S_j} p(z))_{j\in N}$ is increasing, and we can choose $z_j \in S_j$ such that $(p(z_j))_{j\in N}$ is increasing. Then there exists j_0 with $p(z_j) \geqq t_0$ for all $j \geqq j_0$. Hence (1) implies that for all $j \geqq j_0$ and all $k \geqq m$

$$\varphi_{k+1}(p(z_{j+1})) - \varphi_k(p(z_{j+1})) \geqq \varphi_{k+1}(p(z_j)) - \varphi_k(p(z_j)).$$

By the definition of p_k this implies

$$\frac{a_{j+1,k}}{a_{j+1,k+1}} \leqq \frac{a_{j,k}}{a_{j,k+1}} \quad \text{for all} \quad j \geqq j_0,\ k \geqq m,$$

which completes the proof in case (1).

If P is of type (2), we may assume that $r_k = s^k$ for all $k \in N$. Then we enumerate the countably many components S of $S_m(F,\varepsilon)$ with $S \cap V(F) \neq \emptyset$ by $(S_j)_{j\in N}$ in such a way that $(\sup_{z\in S_j} |z|)_{j\in N}$ is increasing, and we choose $z_j \in S_j$ such that $(|z_j|)_{j\in N}$ is increasing. Next we choose $j_0 \in N$ with $|z_{j_0}| \geqq t_0$. Then we have for all $j \geqq j_0$ and all $k \in N$

$$\varphi(|z_{j+1}|^{s^{k+1}}) - \varphi(|z_{j+1}|^{s^k}) \geqq \varphi(|z_j|^{s^{k+1}}) - \varphi(|z_j|^{s^k}),$$

which completes the proof as in the previous case.

If P is of type (3), then an easy inspection shows that the same choices as in case (2) work.

3. 4 Definition. Let P be a weight system on C^n.

a) For an arbitrary ideal I in $A_p(C^n)$ we define

$$I_{\mathrm{loc}} := \{f \in A_p(C^n) \mid \text{for every } a \in C^n : [f]_a \in I_a\},$$

where $[f]_a$ denotes the germ of f at a and I_a denotes the ideal generated by $\{[f]_a \mid f \in I\}$ in the ring $\mathcal{O}_a$ of all germs of holomorphic functions at the point a. I_{loc} is called the local ideal generated by I or just the localization of I. I is called localized if $I = I_{\mathrm{loc}}$.

b) For $F = (F_1, \ldots, F_N) \in (A_p(C^n))^N$ we denote by $I(F)$ the ideal in $A_p(C^n)$ generated by the functions $F_1, \ldots, F_N$. The localization of $I(F)$ is denoted by $I_{\mathrm{loc}}(F)$.

It has been remarked by Schwartz [28] that for

$$p: z \mapsto |z| \text{ and } p: z \mapsto |\mathrm{Im}\, z| + \log(1 + |z|^2)$$

every closed ideal in $A_p(C)$ is localized (so-called spectral synthesis property). Gurevich [11] gave an example of a closed ideal in $A_p(C^2)$ $(p: z \mapsto |z|)$ which is not localized. Kelleher and Taylor [14] have used the L^2-techniques of Hörmander to study the localization of ideals in $A_p(C^n)$. It turns out that some of their essential results also hold for algebras $A_p(C)$. In the following remark we collect some results which will be used later on.

3.5 Remark. a) From the closure of modules theorem it follows that I_{loc} is a closed ideal in $A_P(\mathbb{C}^n)$.

b) An analysis of the proofs of Kelleher and Taylor [14] shows that for every $F=(F_1,\ldots,F_N)\in(A_P(\mathbb{C}^n))^N$ which is slowly decreasing we have $\overline{I(F)}=I_{loc}(F)$.

c) By Berenstein and Taylor [2], Theorem 4.2 we have $I(F)=I_{loc}(F)$, whenever $F=(F_1,\ldots,F_n)\in(A_p(\mathbb{C}^n))^n$ is slowly decreasing.

d) If $F\in A_P(\mathbb{C})$ is slowly decreasing, then $I(F)=F\cdot A_P(\mathbb{C})=I_{loc}(F)$. This follows in the same way as Proposition 3 of Berenstein and Taylor [1].

e) For $n=1$ every proper ideal in $\mathcal{O}_a$ is of the form $[z-a]_a^{m_a}\,\mathcal{O}_a$. This implies that every proper localized ideal I in $A_P(\mathbb{C})$ is of the form

$$I=\{f\in A_P(\mathbb{C})\mid f^{(j)}(a)=0 \text{ for } 0\leqq j<m_a \text{ and all } a\in V(I)\},$$

where $V(I):=\{a\in\mathbb{C}\mid f(a)=0 \text{ for all } f\in I\}$ and

$$m_a:=\max\{m\in\mathbb{N}\mid f^{(j)}(a)=0 \text{ for } 0\leqq j<m \text{ and all } f\in I\}.$$

f) If I is a proper localized ideal in $A_P(\mathbb{C})$, then there exist $F_1,F_2\in A_P(\mathbb{C})$ with $I=I_{loc}(F_1,F_2)$. This can be shown by the "jiggling of zeros" argument indicated in Berenstein and Taylor [1], p. 120.

g) Let $\mathbb{P}=(p_k)_{k\in\mathbb{N}}$ be a radial weight system on $\mathbb{C}$ and assume that

$$p_k(2z)=O(p_k(z))$$

for every $k\in\mathbb{N}$. Then every closed ideal in $A_P(\mathbb{C})$ is localized. In particular $I=I_{loc}(F_1,F_2)$ for every proper closed ideal in $A_P(\mathbb{C})$. This can be proved by a slight modification of the arguments in Kelleher and Taylor [14].

Next we want to determine the locally convex structure of $A_P(\mathbb{C}^n)/I_{loc}(F)$.

Let $\mathbb{P}$ be a given weight system on $\mathbb{C}^n$ and let $F=(F_1,\ldots,F_N)\in(A_P(\mathbb{C}^n))^N$ be slowly decreasing in the weak sense. Without loss of generality we may assume that $p_k\leqq p_{k+1}$ for all $k\in\mathbb{N}$, that $F_1,\ldots,F_N\in H^\infty_{p_1}(\mathbb{C}^n)$ and that the slowly decreasing condition in the weak sense is satisfied for $m=1$ and an appropriate $\varepsilon>0$.

a) If $S_1(F;\varepsilon)$ has infinitely many components S with $S\cap V(F)\neq\emptyset$, then we can choose an enumeration $(S_j)_{j\in\mathbb{N}}$ of these components and we can choose $z_j\in S_j$ for $j\in\mathbb{N}$. Next we define the matrix $A=(a_{j,k})_{j,k\in\mathbb{N}}$ by

$$a_{j,k}:=\exp(p_k(z_j)). \tag{1}$$

Since F is slowly decreasing, for every $k\in\mathbb{N}$ there exist D_k and D'_k such that for all $j\in\mathbb{N}$

$$\begin{aligned}&\sup_{z\in S_j}p_k(z)\leqq D_k p_k(z_j)+D'_k,\\ &p_k(z_j)\leqq D_k\inf_{z\in S_j}p_k(z)+D'_k.\end{aligned}\tag{2}$$

In the case that F is regularly slowly decreasing in the weak sense, then S_j and z_j can be chosen as in 3.1 c) and we can assume (after a suitable modification of the p_k on some compact zero neighbourhood, which does not change $A_P(\mathbb{C})$) that A is a regular matrix.

Next put $I := I_{\mathrm{loc}}(F)$ and $V := V(F)$ and define for $j \in N$

(3) $$E_j := \prod_{a \in S_j \cap V} \mathcal{O}_a / I_a.$$

We remark that I_a is closed in $\mathcal{O}_a$ for the locally convex topology of simple convergence on $\mathcal{O}_a$ (see Hörmander [12] 6. 3. 5). Hence there exists a locally convex Hausdorff topology τ_j on E_j.

Let $H^\infty(S_j)$ denote the Banach space of all bounded holomorphic functions on S_j. Then the map

(4) $$\rho_j : H^\infty(S_j) \to E_j, \quad \rho_j(f) := ([f]_a + I_a)_{a \in S_j \cap V},$$

is linear and continuous with respect to the topology τ_j on E_j. Since S_j is bounded, it follows from Cartan's theorem B that ρ_j is surjective. Hence we get a norm on E_j by letting

(5) $$\| \ \|_j : E_j \to R, \ \|\varphi\|_j := \inf\{\|g\|_{H^\infty(S_j)} \mid g \in H^\infty(S_j), \rho_j(g) = \varphi\}.$$

Remark that $(E_j, \| \ \|_j)$ is a Banach space.

Now let E denote the sequence $(E_j, \| \ \|_j)_{j \in N}$ of Banach spaces defined by (3) and (5). We want to show that for every $f \in A_p(C^n)$ we have $(\rho_j(f|S_j))_{j \in N} \in k^\infty(A, E)$. To see this, let $f \in H^\infty_{p_k}(C^n)$ be given. Then we have by 2. 2 (2)

$$\begin{aligned}\|\rho_j(f|S_j)\|_j &\leq \|f|S_j\|_{H^\infty(S_j)} \leq \|f\|_{p_k,\infty} \exp\big(\sup_{z \in S_j} p_k(z)\big) \\ &\leq \|f\|_{p_k,\infty} \exp\big(D_k p_k(z_j) + D_k'\big) \\ &\leq \|f\|_{p_k,\infty} \exp\big(p_l(z_j) + D_k''\big) = \|f\|_{p_k,\infty}\, a_{j,l} \exp(D_k'').\end{aligned}$$

Moreover, the estimates show that the linear map

(6) $$\rho : A_p(C^n) \to k^\infty(A, E), \quad \rho(f) := \big(\rho_j(f|S_j)\big)_{j \in N}$$

is continuous.

Next we use the semi-local to global extension theorem of Berenstein and Taylor [2], 2. 2, to show that ρ is surjective. To do this, let

$$\mu = (\mu_j)_{j \in N} \in k^\infty(A, E)$$

be given. Then there exists $k \in N$ with

$$\sup_{j \in N} \|\mu_j\|_j\, a_{j,k}^{-1} = |||\mu|||_k < \infty.$$

Because of (5) we can choose $\lambda_j \in H^\infty(S_j)$ with $\rho_j(\lambda_j) = \mu_j$ and

(7) $$\|\lambda_j\|_{H^\infty(S_j)} \leq 2\, |||\mu|||_k\, a_{j,k} \quad \text{for all} \quad j \in N.$$

Then we define $\lambda : S_1(F; \varepsilon) \to C$ by

$$\lambda(z) := \begin{cases} \lambda_j(z) & \text{if} \quad z \in S_j,\ j \in N, \\ 0 & \text{if} \quad z \in S_1(F; \varepsilon) \setminus \bigcup_{j \in N} S_j. \end{cases}$$

It is obvious that λ is holomorphic on $S_1(F; \varepsilon)$. For each $z \in S_j$, (2) implies the estimate

$$|\lambda_j(z)| \leq 2\, |||\mu|||_k\, e^{p_k(z_j)} \leq 2\, |||\mu|||_k \exp\big(D_k p_k(z) + D_k'\big).$$

Since $S_k(F;\varepsilon)\subset S_1(F;\varepsilon)$ we get from the semi-local to global extension theorem of Berenstein and Taylor [2], 2. 2, the existence of $f\in A_{p_k}(\mathbb{C}^n)$ with $\rho(f)=(\mu_j)_{j\in\mathbb{N}}$. By 2. 2 (2) we have $A_{p_k}(\mathbb{C}^n)\subset A_p(\mathbb{C}^n)$. Hence ρ is surjective. By the open mapping theorem for (LF)-spaces (6) implies that ρ is open. Since $\ker\rho=I_{\mathrm{loc}}(F)$, we have shown

$$(8)\qquad A_P(\mathbb{C}^n)/I_{\mathrm{loc}}(F)\cong k^\infty(A,E).$$

Now remark that $A_P(\mathbb{C}^n)$ is nuclear by 2. 4. Hence $A_p(\mathbb{C}^n)/I_{\mathrm{loc}}(F)$ and consequently $k^\infty(A,E)$ is nuclear. This implies that $\dim E_j$ is finite for every $j\in\mathbb{N}$. Hence we can apply Proposition 1. 4 to obtain that

$$(9)\qquad A_P(\mathbb{C}^n)/I_{\mathrm{loc}}(F)\cong\lambda^1(B)'_b,$$

where B is obtained from A by repeating the j-th row $(\dim E_j)$-times.

b) If there are only finitely many components S of $S_1(F,\varepsilon)$ with $S\cap V(F)\neq\emptyset$, then the arguments used in part a) show that $A_P(\mathbb{C}^n)/I_{\mathrm{loc}}(F)\cong\prod_{j\in M}E_j$, where M is a finite set. Hence $I_{\mathrm{loc}}(F)$ is of finite codimension in $A_p(\mathbb{C}^n)$.

All together we have proved:

3. 7 Theorem. *Let $\mathbb{P}$ be a weight system on $\mathbb{C}^n$ and let*

$$F=(F_1,\dots,F_N)\in(A_P(\mathbb{C}^n))^N$$

be slowly decreasing in the weak sense. Then $A_P(\mathbb{C}^n)/I_{\mathrm{loc}}(F)$ is either finite dimensional or the strong dual of a nuclear Fréchet space $\lambda^1(B)$, where $b_{j,k}=\exp(p_k(w_j))$, $j,k\in\mathbb{N}$, and where $(w_j)_{j\in\mathbb{N}}$ is an appropriate sequence in $\mathbb{C}^n$ with $\lim_{j\to\infty}|w_j|=\infty$.

If F is regularly slowly decreasing in the weak sense, then

$$A_P(\mathbb{C}^n)/I_{\mathrm{loc}}(F)\ .$$

has a regular basis, i.e. B is a regular matrix.

3. 8 Corollary. *Let p be a weight function on $\mathbb{C}^n$ and let*

$$F=(F_1,\dots,F_N)\in(A_p(\mathbb{C}^n))^N$$

be slowly decreasing in the weak sense. Then $A_p(\mathbb{C}^n)/I_{\mathrm{loc}}(F)$ is either finite dimensional or isomorphic to the strong dual of a nuclear power series space of infinite type.

Proof. By hypothesis there exist $m\in\mathbb{N}$ and $\varepsilon>0$ such that 3. 1 (a) (i) and (ii) are satisfied for all components S of $S_m(F,\varepsilon)$ with $S\cap V(F)\neq\emptyset$. If there is an infinite number of such components, then it follows from Lemma 3. 3 (1) that F is regularly slowly decreasing in the weak sense. Hence the result follows from Theorem 3. 7 since the matrix B is given by $b_{j,k}=e^{kp(w_j)}$, where $(p(w_j))_{j\in\mathbb{N}}$ is an increasing and unbounded sequence.

Remark. a) Theorem 3. 7 and Corollary 3. 8 extend Theorem 7 of Berenstein and Taylor [1] to the n-dimensional discrete case and give a more precise formulation of Berenstein and Taylor [2], Theorem 4. 7. The present proof is inspired by their proof in the one dimensional case. However, it is different even for $n=1$, since it does not use explicit extension formulas, but only estimates which every extension has to satisfy. Moreover, the application of the Auerbach Lemma as in section 1 gives more information on the structure of $A_P(\mathbb{C}^n)/I_{\mathrm{loc}}(F)$ than Berenstein and Taylor [1], Theorem 8.

b) Assume that under the hypotheses of Corollary 3. 8 we have

$$\dim (A_p(\mathbb{C}^n)/I_{\text{loc}}(F)) = \infty.$$

Then it follows from Proposition 1. 3 (3) that (in the notation of the proof of Theorem 3. 7) there exist $C > 0$ and $d > 0$ with

$$\dim E_j \leqq C \exp(dp(z_j)) \quad \text{for all} \quad j \in \mathbb{N}.$$

For $n=1$ and one slowly decreasing function $F \in A_p(\mathbb{C})$ this estimate has been derived in Berenstein and Taylor [1], Lemma 4. (f) by function theoretic arguments.

4. On the complementation of closed ideals in $A_p(\mathbb{C}^n)$

The information on the structure of $A_p(\mathbb{C}^n)/I_{\text{loc}}(F)$ obtained in the previous section is now used to decide whether $I_{\text{loc}}(F)$ is complemented in $A_p(\mathbb{C}^n)$. This is done by means of certain linear topological invariants which were introduced and investigated by Vogt [31], [32], [34], Vogt and Wagner [36] and Wagner [37]. In particular it turns out that the complementation results of Taylor [30], 5. 1, can be obtained and improved by an application of the splitting theorem of Vogt [32]. We begin by recalling the definition of the invariants which we shall use later on.

4. 1 Definition. Let E be a metrizable locally convex space and let $(\| \ \|_k)_{k \in \mathbb{N}}$ be an (increasing) fundamental system of semi-norms on E generating the locally convex structure of E. For $k \in \mathbb{N}$ define $\| \ \|_k^* : E' \to [0, \infty]$ by $\|y\|_k^* = \sup \{|y(x)| \mid \|x\|_k \leqq 1\}$. Then we say:

(a) E has property (DN) if there exists $m \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ there exist $n \in \mathbb{N}$ and $C > 0$ with $\| \ \|_k^2 \leqq C \| \ \|_m \| \ \|_n$.

(b) E has property $(\overline{DN})$ if there exists $m \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for every $d > 0$ there exists $C > 0$ with

$$\| \ \|_k^{1+d} \leqq C \| \ \|_m^d \| \ \|_n.$$

(c) E has property (Ω) if for every $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ there exist $d > 0$ and $C > 0$ with $\| \ \|_q^{*1+d} \leqq C \| \ \|_k^* \| \ \|_p^{*d}$.

(d) E has property $(\bar{\Omega})$ if there exists $d > 0$ such that for every $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ there exists $C > 0$ with

$$\| \ \|_q^{*1+d} \leqq C \| \ \|_k^* \| \ \|_p^{*d}.$$

(e) E has property $(\bar{\bar{\Omega}})$ if for every $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ and every $d > 0$ there exists $C > 0$ with $\| \ \|_q^{*1+d} \leqq C \| \ \|_k^* \| \ \|_p^{*d}$.

4. 2 Remark. a) It is easy to check that the properties (DN) and $(\overline{DN})$ are linear topological invariants which are inherited by topological linear subspaces. By Vogt [31], 1. 7, a nuclear metrizable locally convex space E has (DN) iff E is isomorphic to a subspace of s. By Vogt [31], 2. 4, a power series space $\Lambda_R(\alpha)$ has (DN) iff $R = +\infty$.

b) It is easy to check that the properties (Ω), $(\bar{\Omega})$ and $(\bar{\bar{\Omega}})$ are linear topological invariants which are inherited by quotient spaces. By Vogt and Wagner [36], 1. 8, a nuclear Fréchet space E has (Ω) iff E is a quotient space of s. By Vogt [33], 2. 8 and 7. 3, a strongly nuclear Fréchet space E has $(\bar{\Omega})$ iff E is a quotient of a nuclear power series space of finite type. By Vogt [34], 4. 2, a Fréchet space E has $(\bar{\bar{\Omega}})$ iff every continuous linear map $T: E \to \Lambda_1(\alpha)$ is bounded for some (all) power series space $\Lambda_1(\alpha)$ with $\sup_{n\in N} \frac{\alpha_{n+1}}{\alpha_n} < \infty$. For other characterizations of Fréchet spaces satisfying $(\bar{\bar{\Omega}})$ see Vogt [35], Theorem 4. 2, and Meise and Vogt [23], Theorem 3. 3.

c) By Vogt and Wagner [36], 2. 8, a nuclear Fréchet space $\lambda(A)$ has (Ω) and (DN) iff $\lambda(A)$ is isomorphic to a power series space of infinite type. Since $(\overline{DN})$ implies (DN) and since every space $\Lambda_\infty(\alpha)$ fails $(\overline{DN})$, this shows that every nuclear Fréchet space E with a Schauder basis having (Ω) and $(\overline{DN})$ is finite dimensional.

d) From Vogt [33], 1. 6, it follows that a nuclear Fréchet space with $(\bar{\Omega})$ and (DN) is finite dimensional.

4. 3 Proposition. *Let p be a weight function on $\mathbb{C}^n$, let*

$$F=(F_1,\ldots,F_N) \in (A_p(\mathbb{C}^n))^N$$

be slowly decreasing in the weak sense, and assume that $A_p(\mathbb{C}^n)_b'$ has (DN). Then we have:

(a) *$I_{\mathrm{loc}}(F)$ is complemented in $A_p(\mathbb{C}^n)$ if and only if $I_{\mathrm{loc}}(F)_b'$ has (DN).*

(b) *If $I_{\mathrm{loc}}(F)$ equals $I(F)$, then $I_{\mathrm{loc}}(F)$ is complemented in $A_p(\mathbb{C}^n)$.*

Proof. (a) Obviously $I_{\mathrm{loc}}(F)$ is complemented in $A_p(\mathbb{C}^n)$ iff the exact sequence

$$(1) \qquad 0 \to I_{\mathrm{loc}}(F) \xrightarrow{j} A_p(\mathbb{C}^n) \xrightarrow{q} A_p(\mathbb{C}^n)/I_{\mathrm{loc}}(F) \to 0$$

splits. By Proposition 2. 4 all the spaces in the sequence are (DFN)-spaces. Hence the dual sequence

$$(2) \qquad 0 \to (A_p(\mathbb{C}^n)/I_{\mathrm{loc}}(F))_b' \to A_p(\mathbb{C}^n)_b' \to I_{\mathrm{loc}}(F)_b' \to 0$$

is an exact sequence of nuclear Fréchet spaces. By Corollary 3. 8, $(A_p(\mathbb{C}^n)/I_{\mathrm{loc}}(F))_b'$ is either finite dimensional or a power series space of infinite type and consequently has (Ω). Hence the splitting theorem of Vogt [32], Theorem 2. 2, shows that (1) splits if $I_{\mathrm{loc}}(F)_b'$ has (DN). On the other hand, if (2) splits, then $I_{\mathrm{loc}}(F)_b'$ is isomorphic to a subspace of $A_p(\mathbb{C}^n)_b'$, and consequently $I_{\mathrm{loc}}(F)_b'$ has (DN) since $A_p(\mathbb{C}^n)_b'$ has (DN) by hypothesis.

b) Define $M_F: (A_p(\mathbb{C}^n))^N \to I(F)$ by $M_F(g_1,\ldots,g_N) := \sum_{j=1}^N g_j F_j$. By hypothesis $I(F) = I_{\mathrm{loc}}(F)$ is closed and hence a (DFN)-space. Since M_F is surjective, it follows from the open mapping theorem for (LF)-spaces that $I(F) \cong (A_p(\mathbb{C}^n))^N/\ker M_F$ and hence $I(F)_b' \cong (\ker M_F)^\perp$ which is a topological linear subspace of $(A_p(\mathbb{C}^n)_b')^N$. Since $A_p(\mathbb{C}^n)_b'$ has (DN) by hypothesis, it follows that $I(F)_b' = I_{\mathrm{loc}}(F)_b'$ has (DN). Hence $I_{\mathrm{loc}}(F)$ is complemented in $A_p(\mathbb{C}^n)$ by (a).

The following corollary is an immediate consequence of Berenstein and Taylor [2], Theorem 4. 2 and Proposition 4. 3 above.

4. 4 Corollary. *If $A_p(\mathbb{C}^n)'_b$ has (DN) and if $F=(F_1,\ldots,F_n)\in(A_p(\mathbb{C}^n))^n$ is slowly decreasing, then $I(F)=I_{\mathrm{loc}}(F)$ is complemented in $A_p(\mathbb{C}^n)$.*

Remark. From 2. 8 and 2. 9 we know quite a number of weight functions p on $\mathbb{C}^n$ for which $A_p(\mathbb{C}^n)'_b$ has (DN) since it is a power series space of infinite type. However, Example 2. 13 (2) shows that there are radial weight functions p on $\mathbb{C}$ satisfying $p(2z)=O(p(z))$ for which $A_p(\mathbb{C})'_b$ does not have (DN). For a characterization of the radial weight functions p on $\mathbb{C}$ for which $A_p(\mathbb{C})'_b$ has (DN), we refer to Meise and Taylor [23].

4. 5 Lemma. *Let $\mathbb{P}$ be a weight system on $\mathbb{C}^n$, let $F=(F_1,\ldots,F_N)\in A_{\mathbb{P}}(\mathbb{C}^n)^N$ be slowly decreasing and assume that $I_{\mathrm{loc}}(F)$ is complemented in $A_{\mathbb{P}}(\mathbb{C}^n)$. Then $I_{\mathrm{loc}}(F_1,\ldots,F_N,G)$ is complemented in $A_{\mathbb{P}}(\mathbb{C}^n)$ for every $G\in A_{\mathbb{P}}(\mathbb{C}^n)$.*

Proof. Put $I_{\mathrm{loc}}(F,G):=I_{\mathrm{loc}}(F_1,\ldots,F_N,G)$. By the arguments used in the proof of Proposition 4. 3 it follows that $I_{\mathrm{loc}}(F,G)$ is complemented in $A_{\mathbb{P}}(\mathbb{C}^n)$ iff $(A_{\mathbb{P}}(\mathbb{C}^n)/I_{\mathrm{loc}}(F,G))'_b$ is complemented in $A_{\mathbb{P}}(\mathbb{C}^n)'_b$. Identifying $(A_{\mathbb{P}}(\mathbb{C}^n)/I_{\mathrm{loc}}(F,G))'_b$ canonically with $I_{\mathrm{loc}}(F,G)^\perp$, it suffices to show that $I_{\mathrm{loc}}(F,G)^\perp$ is complemented in $A_{\mathbb{P}}(\mathbb{C}^n)'_b$.

To prove this, we first remark that by 3. 5 b) we have

(1) $$I_{\mathrm{loc}}(F)=\overline{I(F)} \quad\text{and}\quad I_{\mathrm{loc}}(F,G)=\overline{I(F,G)}.$$

This implies

(2) $$I_{\mathrm{loc}}(F)^\perp=\overline{I(F)}^\perp=I(F)^\perp=(\operatorname{im}M_F)^\perp=\ker{}^tM_F,$$

where $M_F:(A_{\mathbb{P}}(\mathbb{C}^n))^N\to A_{\mathbb{P}}(\mathbb{C}^n)$ is defined by $M_F(g_1,\ldots,g_N):=\sum_{j=1}^N g_jF_j$. We define $M_G:A_{\mathbb{P}}(\mathbb{C}^n)\to A_{\mathbb{P}}(\mathbb{C}^n)$ by $M_G(f):=f\cdot G$ and note that the following identity is an easy consequence of (1)

(3) $$(\ker{}^tM_G)\cap(\ker{}^tM_F)=I_{\mathrm{loc}}(F,G)^\perp.$$

Furthermore we remark that $\ker{}^tM_F$ is an invariant subspace of ${}^tM_G$ since for every $y\in\ker{}^tM_F$ and every $f=\sum_{j=1}^N g_jF_j\in I(F)$ we have

$$\begin{aligned}\langle {}^tM_G(y),f\rangle&=\left\langle y,\sum_{j=1}^N Gg_jF_j\right\rangle=\langle y,M_F(g_1G,\ldots,g_NG)\rangle\\&=\langle {}^tM_F(y),(g_1G,\ldots,g_NG)\rangle=0.\end{aligned}$$

Since the hypothesis implies that $I_{\mathrm{loc}}(F)^\perp=\ker{}^tM_F$ is complemented in $A_{\mathbb{P}}(\mathbb{C}^n)'_b$ and since $(\ker{}^tM_G)\cap(\ker{}^tM_F)=\ker({}^tM_G\mid\ker{}^tM_F)$, it follows from (3) that $I_{\mathrm{loc}}(F,G)$ is complemented in $A_{\mathbb{P}}(\mathbb{C}^n)$ if the restriction of ${}^tM_G$ to $\ker{}^tM_F=I_{\mathrm{loc}}(F)^\perp$ has a complemented kernel.

To prove this we use the notation and information from the proof of Theorem 3.7. There it has been shown that $\rho : A_p(C^n) \to k^\infty(A, E)$ is a surjective topological homomorphism with $\ker \rho = I_{\mathrm{loc}}(F)$. Hence ${}^t\rho : k^\infty(A, E)'_b \to (A_p(C^n))'_b$ is an injective topological homomorphism with $\operatorname{im} {}^t\rho = I_{\mathrm{loc}}(F)^\perp = \ker {}^tM_F$. We remark that by 1.4 we can find Hilbert norms $|\ |_j$ on $F_j := (E_j, \|\ \|_j)'_b$ such that

$$\lambda^2(A, F) = \lambda^1(A, E') = k^\infty(A, E)'_b.$$

Moreover, we remark that $M_{\varrho_j}(G) : E_j \to E_j$, $M_{\varrho_j}(G)\,[\rho_j(f)] = \rho_j(Gf)$, is a continuous linear map on E_j for every $j \in N$. Then we have for every $y = (y_j)_{j\in N} \in \lambda^2(A, F)$ and each $f \in A_p(C^n)$

$$\begin{aligned} \langle {}^tM_G \circ {}^t\rho(y), f\rangle &= \langle y, \rho(Gf)\rangle = \sum_{j=1}^\infty \langle y_j, \rho_j(G\cdot f)\rangle \\ &= \sum_{j=1}^\infty \langle y_j, M_{\rho_j}(G)\,[\rho_j(f)]\rangle \\ &= \sum_{j=1}^\infty \langle {}^tM_{\rho_j}(G)\,[y_j], \rho_j(f)\rangle \\ &= \langle z, \rho(f)\rangle = \langle {}^t\rho(z), f\rangle, \end{aligned}$$

where $z = ({}^tM_{\rho_j}(G)\,[y_j])_{j\in N}$. This implies

$$({}^t\rho)^{-1} \circ {}^tM_G \circ {}^t\rho : y \mapsto ({}^tM_{\rho_j}(G)\,[y_j])_{j\in N}.$$

Hence it follows from Lemma 1.5 that the kernel of $({}^t\rho)^{-1} \circ {}^tM_G \circ {}^t\rho$ is complemented in $\lambda^2(A, F)$, and consequently ${}^tM_G \mid \ker {}^tM_F$ has a complemented kernel.

4.6 Proposition. *Let p be a weight function on C^n, let*

$$F = (F_1, \ldots, F_n) \in (A_p(C^n))^n$$

be slowly decreasing and assume that $A_p(C^n)'_b$ has (DN). Then

$$I_{\mathrm{loc}}(F_1, \ldots, F_n, G_1, \ldots, G_M) = \overline{I(F_1, \ldots, F_n, G_1, \ldots, G_M)}$$

is complemented in $A_p(C^n)$ for every $G_1, \ldots, G_M \in A_p(C^n)$.

Proof. This follows by induction on M from Corollary 4.4 and Lemma 4.5.

4.7 Theorem. *Let p be a radial weight function on C with $p(2z) = O(p(z))$ and assume that $A_p(C^n)'_b$ has (DN). Then every closed ideal I in $A_p(C)$ is complemented.*

Proof. Let I be a proper closed ideal in $A_p(C)$. By 3.5 g) $I = I_{\mathrm{loc}}(F_1, F_2)$ where we may assume $F_1 \neq 0$. By 3.2 a) F_1 is slowly decreasing in $A_p(C)$. Hence the result follows from Proposition 4.6.

Remark. Theorem 4.7 extends the results of Taylor [30], Theorem 5.1 and Remark 5.1, with a different proof (see also Schwerdtfeger [29], Theorem 1.1.21).

Remark. If p is a non-radial weight function, then in general $A_p(\mathbb{C})'_b$ does not have (DN). For example, for q as in 2. 5 (4) we have $A_q(\mathbb{C}^n)'_b \cong \mathscr{E}(\mathbb{R})$. Since $\mathscr{E}(\mathbb{R})$ does not have a continuous norm, it cannot have (DN). In this situation Taylor [30], Theorem 5. 2, shows that there are infinite codimensional proper closed ideals in $A_q(\mathbb{C})$ which are complemented, and that there are such ideals which are not complemented. In Meise and Vogt [24] the slowly decreasing $f \in A_q(\mathbb{C})$ are characterized for which $I_{\mathrm{loc}}(f)$ is complemented in $A_q(\mathbb{C})$.

Next we want to derive conditions which imply that every proper closed infinite codimensional ideal I in $A_P(\mathbb{C})$ is not complemented. A basic observation for such a result is contained in the following lemma, which shows that $A_P(\mathbb{C})/I_{\mathrm{loc}}(F)$ always belongs to a rather small class of (DFN)-spaces.

4. 8 Lemma. *Let $P=(p_k)_{k\in\mathbb{N}}$ be a weight system on $\mathbb{C}^n$ and let*

$$F=(F_1,\dots,F_N)\in\left(A_P(\mathbb{C}^n)\right)^N$$

be slowly decreasing in the weak sense. Then we have

(a) $\left(A_P(\mathbb{C}^n)/I_{\mathrm{loc}}(F)\right)'_b$ *has (DN).*

(b) *If for every $k\in\mathbb{N}$ there exists $n\in\mathbb{N}$ with* $\lim\limits_{|z|\to\infty}\dfrac{p_k(z)}{p_n(z)}=0$, *then*

$$\left(A_P(\mathbb{C}^n)/I_{\mathrm{loc}}(F)\right)'_b$$

has $(\overline{DN})$.

Proof. If $A_P(\mathbb{C}^n)/I_{\mathrm{loc}}(F)$ is finite dimensional, then (a) and (b) hold trivially. If it is infinite dimensional, then $\left(A_P(\mathbb{C}^n)/I_{\mathrm{loc}}(F)\right)'_b\cong\lambda(B)$ by Theorem 3. 7, where

$$b_{j,k}=\exp\left(p_k(w_j)\right),\quad j,k\in\mathbb{N}$$

for some sequence $(w_j)_{j\in\mathbb{N}}$ in $\mathbb{C}^n$ with $\lim\limits_{j\to\infty}|w_j|=\infty$.

To prove (a) let $k\in\mathbb{N}$ be given. By 2. 2 (2) there exist $m\in\mathbb{N}$ and $L>0$ with $2p_k\leqq p_m+L$. Since $p_1\geqq 0$, this implies

$$b_{j,k}^2\leqq\exp(L)\,b_{j,1}b_{j,m}\quad\text{for all}\quad j\in\mathbb{N}.$$

By Vogt [31], 2. 3, this proves that $\lambda(B)$ has (DN).

To prove (b) let $k\in\mathbb{N}$ be given and choose $n\in\mathbb{N}$ with $\lim\limits_{|z|\to\infty}\dfrac{p_k(z)}{p_n(z)}=0$. If now $d>0$ is given, then we can find $j_0\in\mathbb{N}$ so that for all $j\geqq j_0$

$$p_k(w_j)\leqq\frac{1}{1+d}p_n(w_j).\,.$$

This implies that there exists $C\geqq 1$ with

$$b_{j,k}^{1+d}\leqq C b_{j,1}^d b_{j,n}\quad\text{for all}\quad j\in\mathbb{N}.$$

From this it follows easily that $\lambda(B)$ has $(\overline{DN})$.

4.9 Proposition. *Let $\mathbb{P}=(p_k)_{k\in\mathbb{N}}$ be a weight system on $\mathbb{C}^n$, let*

$$F=(F_1,\ldots,F_N)\in(A_p(\mathbb{C}^n))^N$$

be slowly decreasing in the weak sense and assume that $A_p(\mathbb{C}^n)/I_{loc}(F)$ is infinite dimensional. Then $I_{loc}(F)$ is not complemented in $A_p(\mathbb{C}^n)$ if one of the following conditions is satisfied:

(a) *$A_p(\mathbb{C}^n)'_b$ has $(\bar{\Omega})$.*

(b) *$A_p(\mathbb{C}^n)'_b$ has (Ω) and for every $k\in\mathbb{N}$ there exists $m\in\mathbb{N}$ with*

$$\lim_{|z|\to\infty}\frac{p_k(z)}{p_m(z)}=0.$$

Proof. Let us assume that $I_{loc}(F)$ is complemented in $A_p(\mathbb{C}^n)$. Then it follows that $I_{loc}(F)^\perp\cong(A_p(\mathbb{C}^n)/I_{loc}(F))'_b$ is complemented in $A_p(\mathbb{C}^n)'_b$. By Theorem 3.7 we have $I_{loc}(F)^\perp\cong\lambda(B)$.

If condition (a) is satisfied, then $I_{loc}(F)^\perp$ as a quotient of $A_p(\mathbb{C}^n)'_b$ has $(\bar{\Omega})$. By Lemma 4.8 a) $I_{loc}(F)^\perp$ has (DN). Hence $\dim I_{loc}(F)^\perp<\infty$ by 4.2 d), which contradicts $I_{loc}(F)^\perp\cong\lambda(B)$.

If condition (b) is satisfied, then $I_{loc}(F)^\perp$ has (Ω). By Lemma 4.8 b) $I_{loc}(F)^\perp$ has $(\overline{DN})$. Hence $\dim I_{loc}(F)^\perp<\infty$ by 4.2 c), which is again a contradiction. Thus, $I_{loc}(F)$ is not complemented in $A_p(\mathbb{C}^n)$.

4.10 Corollary. *Let $\mathbb{P}$ be a radial weight system on $\mathbb{C}$ with*

$$p_k(2z)=O(p_k(z))\quad\text{for all}\quad k\in\mathbb{N}.$$

Assume that condition 4.9 (a) or 4.9 (b) is satisfied. Then every proper closed infinite codimensional ideal in $A_p(\mathbb{C})$ is not complemented.

Proof. Let I be an arbitrary proper closed ideal in $A_p(\mathbb{C})$ which has infinite codimension. By 3.5 g) we have $I=I_{loc}(F_1,F_2)$. Since I has infinite codimension, $V(F_1,F_2)$ is an infinite set. Hence we may assume that $F_1\neq 0$ and that F_1 has infinitely many zeros. By 3.2 a) F_1 is slowly decreasing in $A_p(\mathbb{C})$, and consequently (F_1,F_2) is slowly decreasing. By Proposition 4.9, I is not complemented in $A_p(\mathbb{C})$.

4.11 Corollary. *Let the weight system $\mathbb{P}=(\varphi(|z|^{r_k}))_{k\in\mathbb{N}}$ be as in 2.7 (2) and assume that for each $k\in\mathbb{N}$ there exists $m\in\mathbb{N}$ with $\lim\limits_{x\to\infty}\dfrac{\varphi(x^{r_k})}{\varphi(x^{r_m})}=0$. Then every proper closed infinite codimensional ideal in $A_p(\mathbb{C})$ is not complemented.*

Proof. From Meise and Taylor [21], 1. 10, it follows that $A_p(\mathbb{C})_b' = \lambda(A)$, where $A = (a_{j,k})_{j,k \in N}$ is given by $a_{j,k} = \exp\left(-\psi\left(\frac{j}{r_k}\right)\right)$ for some increasing and convex function ψ on $[0, \infty[$. To prove that $\lambda(A)$ has (Ω) let $p \in N$ be given and choose $q = p+1$ and $d > 0$ with $\frac{1}{r_q} \leq \frac{d}{1+d} \cdot \frac{1}{r_p}$. Then the properties of ψ imply that for every $j \in N$ we have

$$\psi\left(\frac{j}{r_q}\right) \leq \psi\left(\frac{1}{1+d}\left(\frac{dj}{r_p} + \frac{j}{r_k}\right)\right) \leq \frac{d}{1+d}\psi\left(\frac{j}{r_p}\right) + \frac{1}{1+d}\psi\left(\frac{j}{r_k}\right),$$

which implies $a_{j,k} a_{j,p}^d \leq a_{j,q}^{1+d}$ for all $j \in N$. By Vogt and Wagner [36], 2. 3, this proves that $\lambda(A)$ has (Ω). Hence the result follows from Corollary 4. 10.

4. 12 Examples. We give some examples of weight systems P on $\mathbb{C}$ which satisfy the hypotheses of Corollary 4. 10. Hence every proper closed infinite codimensional ideal in $A_p(\mathbb{C})$ is not complemented.

(1) $P = \left(k(\log(1+|z|^2))^r\right)_{k \in N}$, $r > 1$. $A_p(\mathbb{C})_b'$ has $(\bar{\Omega})$ by 2. 13 (2).

(2) $P = (|z|^{r_k})_{k \in N}$, where $(r_k)_{k \in N}$ is a strictly increasing sequence in $]0, \infty[$. $A_p(\mathbb{C})_b'$ has $(\bar{\Omega})$ by 2. 13 (1).

(3) $P = \left(p_M(|z|^{r_k})\right)_{k \in N}$, where $(M_j)_{j \in N_0}$ satisfies (M1), (M2) and 2. 6 (2) (ii) and where $(r_k)_{k \in N}$ is a strictly increasing sequence in $[1, \infty[$. $A_p(\mathbb{C})_b'$ has $(\bar{\Omega})$ by Corollary 2. 12.

(4) $P = (p_{M^k})_{k \in N}$, where $(M^k)_{k \in N}$ is defined as in 2. 13 (4). Then condition 4. 9 (b) is satisfied by 2. 13 (4).

(5) $P = \left((\log(1+|z|^2))^{r_k}\right)_{k \in N}$, where $(r_k)_{k \in N}$ is a strictly increasing sequence in $]1, \infty[$. The following considerations show that $A_p(\mathbb{C})_b'$ has $(\bar{\Omega})$ and hence (Ω).

By 2. 13 (3) we have $A_p(\mathbb{C})_b' \cong \lambda(A)$, where $A = \left(\exp(-j^{s_k})\right)_{j,k \in N}$ with $s_k = \frac{r_k}{r_k - 1}$. Hence it follows from Vogt [34], 4. 3, that $\lambda(A)$ and consequently $A_p(\mathbb{C})_b'$ has $(\bar{\Omega})$ if we show:

$$(1) \quad \begin{cases} \text{for every } p \in N, \text{ every } k \in N \text{ and every } d > 0 \text{ there exists} \\ C > 0 \text{ with } a_{j,k} a_{j,p}^d \leq C a_{j,p+1}^{1+d} \text{ for all } j \in N. \end{cases}$$

Since $(s_l)_{l \in N}$ is strictly decreasing in $]1, \infty[$, we have for all but finitely many $j \in N$:

$$\frac{1+d}{d} j^{s_{p+1} - s_p} \leq 1.$$

This implies the existence of $C \geq 1$ such that for all $j \in N$

$$(1+d) j^{s_{p+1}} \leq d j^{s_p} + j^{s_k} + \log C.$$

Obviously, this implies (1).

Remark. a) Example 4.12(1) shows that even for radial weight functions p satisfying $p(2z)=O(p(z))$ the complementation of ideals can change very drastically, if $A_p(\mathbb{C})'_b$ does not have (DN).

b) For a nuclear Fréchet space E condition (Ω) is considered as rather weak in the sense, that "almost all" nuclear Fréchet space which occur in connection with problems in analysis, have property (Ω). Hence Corollary 4.10 might indicate that for most of the weight systems $P=(p_k)_{k\in N}$ satisfying $\lim_{|z|\to\infty}\frac{p_k(z)}{p_{k+1}(z)}=0$ for all $k\in N$, every proper closed infinite codimensional ideal in $A_p(\mathbb{C})'_b$ is not complemented. However, the following example shows that there exist algebras $A_p(\mathbb{C})$ for which $A_p(\mathbb{C})'_b$ does not have (Ω). Consequently, Corollary 4.10 does not apply to these algebras.

4.13 Example. For $k\in N$ put $p_k: z\mapsto |z|\exp((\log\log(e+|z|))^k)$ and let $P:=(p_k)_{k\in N}$. By 2.13(5) we have $A_{\tilde P}(\mathbb{C})=A_P(\mathbb{C})$, where $\tilde P=(p_{M^k})_{k\in N}$ is the weight system defined in 2.13(5). Moreover, we have $A_{\tilde P}(\mathbb{C})'_b\cong\lambda(B)$, where

$$B=\left(\exp\left(j\left(\log\log(j+e)\right)^k\right)\right)_{j,k\in N}.$$

From this it follows by standard arguments that $A_{\tilde P}(\mathbb{C})'_b$ has property $(\overline{DN})$. Hence 4.2c) implies that $A_{\tilde P}(\mathbb{C})'_b$ does not have (Ω).

Without proof we remark that the following can be shown: The pair $(\lambda(B),\lambda(B))$ satisfies condition (S_1^*) of Vogt [35] and hence $\operatorname{Ext}^1(\lambda(B),\lambda(B))=0$. By Vogt [35], 1.6′, 1.7′ and 1.8, this implies that for each quotient space E of $\lambda(B)$, each subspace G of $\lambda(B)$ and each Fréchet space F each exact sequence

$$0\to E\to F\to G\to 0$$

splits. Hence the splitting theorem which we have used in the proof of Proposition 4.3 also holds for $\lambda(B)$.

4.14 Lemma. *Let $P=(p_k)_{k\in N}$ be a radial weight system on $\mathbb{C}$ with*

$$p_k(2z)=O(p_k(z))\quad \textit{for all}\quad k\in N.$$

Define the matrix $A=(a_{j,k})_{j,k\in N}$ by $a_{j,k}:=\exp(p_k(e^j))$, $j,k\in N$, and assume that every continuous linear map from $A_P(\mathbb{C})'_b$ into $\lambda(A)$ is compact. Then every proper closed infinite codimensional ideal I in $A_P(\mathbb{C})$ is not complemented in $A_P(\mathbb{C})$.

Proof. As in the proof of 4.10 we have $I=I_{\mathrm{loc}}(F_1,F_2)$. An application of the minimum modules theorem implies (by the argument used in the proof of Kelleher and Taylor [14], Proposition 5.2) that there exist $\varepsilon>0$, $m\in N$, $n_0\in N$ and a sequence $(r_n)_{n\in N}$ with $e^n<r_n<e^{n+1}$ for all $n\geqq n_0$ such that

$$S_m(\varepsilon,F_1,F_2)\cap\bigcup_{n\geqq n_0}\{z\in\mathbb{C}\mid |z|=r_n\}=\emptyset.$$

This shows that, up to finitely many exceptions, for each component S of $S_m(\varepsilon, F_1, F_2)$ there exists $n \in N$ with $S \subset R_n := \{z \in \mathbb{C} \mid r_n < |z| < r_{n+1}\}$. And without loss of generality we may assume that $n_0 = 1$ and that this holds for all components S of $S_m(\varepsilon, F_1, F_2)$. By Theorem 3.7 we can choose a sequence $(w_j)_{j\in N}$ in $S_m(\varepsilon, F_1, F_2)$ with $(A_P(\mathbb{C})/I)_b' \cong \lambda(B)$ where $b_{j,k} = \exp(p_k(w_j))$. Of course, we may assume that $(|w_j|)_{j\in N}$ is increasing. Define $\tilde{b}_{j,k} := \exp(p_k(e^n))$ if $r_n < |w_j| < r_{n+1}$. Then the properties of P imply that $\lambda(B) = \lambda(\tilde{B})$. Let

$$M := \{n \in N \mid \text{there exists } j \in N \text{ with } r_n < |w_j| < r_{n+1}\}$$

and let $(n_q)_{q\in N}$ be the increasing arrangement of M. If we define $C = (c_{q,k})_{q,k\in N}$, $c_{q,k} := \exp(p_k(\exp(n_q)))$, then $\lambda(C)$ is a complemented subspace of $\lambda(\tilde{B}) = \lambda(B)$ as well as of $\lambda(A)$.

Now assume that I is a complemented subspace of $A_P(\mathbb{C})$. Then there exists a continuous linear surjective map T from $A_P(\mathbb{C})_b'$ onto $(A_P(\mathbb{C})/I)_b' \cong \lambda(B)$. Hence there exists a continuous linear surjective map S from $A_P(\mathbb{C})_b'$ onto $\lambda(C)$. By the preceding considerations this implies that not every continuous linear map from $A_P(\mathbb{C})_b'$ into $\lambda(A)$ is compact, in contradiction to the hypothesis. Since I was arbitrary, this completes the proof.

4.15 Proposition. *Let $P = (p_k)_{k\in N}$ be a radial weight system with*

$$p_k(2z) = O(p_k(z))$$

for all $k \in N$ and let $A_P(\mathbb{C})_b' \cong \lambda(B)$, where $b_{j,k} = \exp(-\mu_{j,k})$. If the following condition is satisfied:

$$(*)\quad \begin{cases} \text{for every } (K(N))_{N\in N} \in N^N \text{ there exists } k \in N \text{ such that for all } n \in N \\ \text{there exists } M \in N \text{ and } C \geqq 0 \text{ such that for all } v, j \in N \\ p_n(e^v) + \mu_{j,k} \leqq \max_{1\leqq N\leqq M} (p_N(e^v) + \mu_{j,K(N)}) + C, \end{cases}$$

then every proper closed infinite codimensional ideal in $A_P(\mathbb{C})$ is not complemented.

Proof. Define the matrix $A = (a_{j,k})_{j,k\in N}$ by $a_{j,k} := \exp(p_k(e^j))$. Then $(*)$ implies by Vogt [34], 1.5, that every continuous linear map from $A_P(\mathbb{C})_b'$ into $\lambda(A)$ is compact. Hence the result follows from Lemma 4.14.

Condition 4.15 $(*)$ looks somewhat complicated. However, it can be used to decide what happens in case of the Example 4.13.

4.16 Example. For $r > 0$ put $P = (|z|^r \exp((\max(1, \log\log|z|^r))^{r_k}))_{k\in N}$, where $(r_k)_{k\in N}$ is a strictly increasing sequence in $]0, \infty[$. Then every proper closed infinite codimensional ideal in $A_P(\mathbb{C})$ is not complemented.

To show this, we argue in a more general situation which can be used also for other examples. We assume the following:

(1) $p_k(e^j) = \varphi_k(j)$ where $\varphi_k(x) = \varphi(x)\, e^{g_k(x)}$ for $x \geqq x_k$.

(2) $A_P(\mathbb{C})_b' \cong \lambda(B)$ with $b_{j,k} = \exp(-\mu_{j,k})$, where $\mu_{j,k} = \varphi_k^*(j)$ for $\varphi_k^*(x) = x\psi(x) - x f_k(x)$ for $x \geqq x_k$.

(3) For every $k \in N$ there exists ξ_k such that $\varphi_{k+1} - \varphi_k$ is strictly increasing on $[\xi_k, \infty[$.

(4) For every $k\in N$ we have

$$\lim_{x\to\infty}\frac{g_k(x)}{g_{k+1}(x)}=0,\quad \lim_{x\to\infty}\frac{f_k(x)}{f_{k+1}(x)}=0,\quad \lim_{x\to\infty}\frac{\exp\circ g_k(x)}{\varphi(x)}=0,\quad \lim_{x\to\infty} g_k(x)=\lim_{x\to\infty} f_k(x)=\infty,$$

and the functions g_k, f_k and φ are positive and continuous.

To show that 4.15 (∗) holds, let $(K(N))_{N\in N}$ be given. Without restriction we can assume that $(K(N))_{N\in N}$ is strictly increasing. Choose $k\doteq K(1)+1$ and let $n\in N$ be given. Then choose $M>n+1$ and $\xi\in[0,\infty[$ by (3) such that $\varphi_M-\varphi_n$ and $\varphi_n-\varphi_1$ are strictly increasing on $[\xi,\infty[$. Next fix $s\geqq s_0$, where s_0 will be determined by the following considerations, and define $T(s)$ (resp. $\tau(s)$) as the solution of the following equation (T) (resp. (τ)):

$$\text{(T)}\qquad \varphi_k^*(s)-\varphi_{K(M)}^*(s)=\varphi_M(t)-\varphi_n(t),$$

$$(\tau)\qquad \varphi_{K(1)}^*(s)-\varphi_k^*(s)=\varphi_n(t)-\varphi_1(t).$$

Assume for a moment that we can show the following:

(5) There exists $s_0\in[0,\infty[$ with $T(s)\leqq\tau(s)$ for all $s\geqq s_0$.

Then we have for all $s\geqq s_0$:

$$\text{(T')}\qquad \varphi_k^*(s)-\varphi_{K(M)}^*(s)\leqq\varphi_M(t)-\varphi_n(t)\quad\text{for all}\quad t\geqq T(s),$$

$$(\tau')\qquad \varphi_{K(1)}^*(s)-\varphi_k^*(s)\geqq\varphi_n(t)-\varphi_1(t)\quad\text{for all}\quad t\in[\xi_0,\tau(s)],$$

where $\xi_0\geqq\xi$ is chosen appropriately. Hence

$$\varphi_n(t)+\varphi_k^*(s)\leqq\max\left(\varphi_1(t)+\varphi_{K(1)}^*(s),\ \varphi_M(t)+\varphi_{K(M)}^*(s)\right)$$

for all $s\geqq s_0$ and all $t\geqq\xi_0$. By (1) and (2) this implies the existence of j_0 and ν_0 such that

$$p_n(e^\nu)+\mu_{j,k}\leqq\max_{1\leqq N\leqq M}\left(p_N(e^\nu)+\mu_{j,K(N)}\right)\quad\text{for all}\quad j\geqq j_0,\ \nu\geqq\nu_0.$$

This implies that we can find $C\geqq 0$ such that 4.15 (∗) holds. Hence every proper closed infinite codimensional ideal in $A_p(\mathbb{C})$ is not complemented by Proposition 4.15 if we can show that (5) holds.

To prove that (5) holds it suffices to show that for all s which are sufficiently large we have

$$(6)\qquad \varphi_n(T(s))-\varphi_1(T(s))\leqq\varphi_{K(1)}^*(s)-\varphi_k^*(s).$$

To show this, we note that by (1), (2) and (T) $T(s)$ satisfies the identity

$$(7)\qquad s\left(f_{K(M)}(s)-f_k(s)\right)=\varphi(T(s))\left(\exp\left(g_M(T(s))\right)-\exp\left(g_n(T(s))\right)\right).$$

We eliminate $\varphi(T(s))$ from (7) and remark that $\lim_{s\to\infty}T(s)=\infty$. Hence $K(M)>k$ implies by (4) for large s

$$\begin{aligned}(8)\quad \varphi_n(T(s))-\varphi_1(T(s))&\leqq\varphi_n(T(s))=\varphi(T(s))\exp\left(g_n(T(s))\right)\\ &=s\left(f_{K(M)}(s)-f_k(s)\right)\exp\left(g_n(T(s))\right)\left(\exp\left(g_M(T(s))\right)-\exp\left(g_n(T(s))\right)\right)^{-1}\\ &\leqq s f_{K(M)}(s)\,2\exp\left(g_n(T(s))-g_M(T(s))\right).\end{aligned}$$

From $k>K(1)$ and (4) we get for s large enough

$$\text{(9)} \qquad \varphi^*_{K(1)}(s)-\varphi^*_k(s)=s\big(f_k(s)-f_{K(1)}(s)\big)\geqq\frac{s}{2}f_k(s).$$

Since $\lim_{s\to\infty} T(s)=\infty$ we get from $M>n$ and (4) for large s

$$\text{(10)} \qquad \exp\left(\frac{1}{2}g_M(T(s))\right)\leqq\exp\big(g_M(T(s))-g_n(T(s))\big).$$

Then (9), (10) and (8) show that (6) is implied by the inequality

$$\text{(11)} \qquad \frac{f_{K(M)}(s)}{f_k(s)}\leqq\frac{1}{4}\exp\left(\frac{1}{2}g_M(T(s))\right).$$

To prove that (11) holds for all large s, one has to estimate $T(s)$ from below. This we shall do only for the functions which are given by the example. By 2.13 (5) we have in this example

$$\varphi_k(x)=\exp\big(rx+(\log rx)^{r_k}\big), \quad \text{i.e.} \quad \varphi(x)=e^{rx},\ g_k(x)=(\log rx)^{r_k}$$

and

$$\varphi^*_k(x)=\frac{x}{r}\log x-\frac{x}{r}(\log\log x)^{r_k}, \quad \text{i.e.} \quad f_k(x)=\frac{1}{r}(\log\log x)^{r_k}.$$

Then we get from (7)

$$\exp\big(rT(s)\big)=s\big(f_{K(M)}(s)-f_k(s)\big)\big(\exp\big(g_M(T(s))\big)-\exp\big(g_n(T(s))\big)\big)^{-1}$$

and hence

$$\text{(12)} \qquad rT(s)=\log s+\log\big(f_{K(M)}(s)-f_k(s)\big)-\log\big(\exp\big(g_M(T(s))\big)-\exp\big(g_n(T(s))\big)\big).$$

This implies

$$\text{(13)} \qquad rT(s)\leqq\log s+\log f_{K(M)}(s)\leqq 2\log s$$

and hence

$$\text{(14)} \qquad rT(s)\geqq\log s-g_M(T(s))\geqq\log s-g_M(2\log s)$$

for all large s.

Now we can show that (14) implies (11) in the special case:

$$\frac{1}{4}\exp\left(\frac{1}{2}g_M(T(s))\right)\geqq\frac{1}{4}\exp\left(\frac{1}{2}\big(\log(\log s-(\log(2r\log s))^{r_M})\big)^{r_M}\right)$$

$$\geqq\frac{1}{4}\exp\left(\frac{1}{2}\left(\log\left(\frac{1}{2}\log s\right)\right)^{r_M}\right)\geqq\exp\big((r_{K(M)}-r_k)\log\log\log s\big)$$

$$=\frac{f_{K(M)}(s)}{f_k(s)} \quad \text{for all} \quad s\geqq s_0.$$

Remark. We want to remark that also the Corollaries 4.10 and 4.11 can essentially be derived from Proposition 4.15. However, the proof is more involved than those which we have given above.

Remark. If I is a closed ideal in $A_p(C^n)$ then $A_p(C^n)/I$ can also be interpreted as a space of functions $A_p(V_m(I))$ on the multiplicity variety $V_m(I)$ (see Berenstein and Taylor [1], [2] for details). We want to mention that the results of this section on the (non-)complemention of I in $A_p(C^n)$ of course are equivalent to the (non-)existence of continuous linear extension operators

$$E: A_p(V_m(I)) \to A_p(C^n).$$

5. Translation invariant subspaces of some weighted Fréchet spaces of entire functions

Since the work of Ritt [27] on differential equations of infinite order on $A(C)$ and since the work of Schwartz [28] it has been an interesting question to determine the structure of the kernel of a convolution operator on $A(C)$ or more generally, of the translation invariant subspaces of $A(C)$ (see Dickson [5], Ehrenpreis [8], Gelfond [10], Leont'ev [16]). In this section we indicate that the main results of section 3 and section 4 also give a new answer to this question for various weighted Fréchet spaces of entire functions. We concentrate on some classical examples and refer to Meise, Schwerdtfeger and Taylor [20] for a systematic study.

We begin by introducing the Fréchet spaces which we will work with.

5. 1 Definition. Let $Q=(q_k)_{k\in N}$ be a decreasing sequence of weight functions on C. We put

$$A_Q^0(C) := \{f \in A(C) \mid \sup_{z\in C} |f(z)| \exp(-q_k(z)) < \infty \quad \text{for all} \quad k \in N\}$$

and endow $A_Q^0(C)$ with its natural Fréchet space topology. We are interested in the following two particular cases:

(1) $$Q = \left(\frac{1}{k} p_N\right)_{k\in N},$$

where $N=(N_j)_{j\in N_0}$ satisfies (M2) and the hypotheses of 2. 6 (2). Moreover, we assume that there exists a sequence $M=(M_j)_{j\in N_0}$ satisfying the same conditions as N such that for some $C>1$ and $S>1$ we have

$$(CS^j)^{-1} \leqq \frac{M_j N_j}{j!} \leqq CS^j \quad \text{for all} \quad j \in N_0.$$

Then we put $P_Q := (kp_M)_{k\in N}$. By abuse of notation also $A(C)$ is considered as a space of this type. In this case we put $P_Q := (k|z|)_{k\in N}$.

(2) $$Q = (p_{N^k})_{k\in N},$$

where $N^k = (N_j^k)_{j\in N_0}$ satisfies 2. 6 (2), $N_j^k \leqq N_j^{k+1}$ for all $j\in N_0$ and

$$\left(\frac{N_j^k}{N_j^{k+1}}\right)_{j\in N_0} \in \Lambda_\infty(j) \quad \text{for all} \quad k \in N.$$

Moreover, we assume that for each $k \in N$ there exists $M^k = (M_j^k)_{j \in N_0}$ satisfying $M_j^k \geqq M_j^{k+1}$ for all $j \in N_0$, 2. 6 (2) and (M2) such that for each $k \in N$ there exist $C > 1$ and $S > 1$ with

$$(CS^j)^{-1} \leqq \frac{M_j^k N_j^k}{j!} \leqq CS^j \quad \text{for all} \quad j \in N_0.$$

Then we put $P_Q := (p_{M^k})_{k \in N}$.

5. 2 Example. (1) For $s > 1$ put

$$N(s) := \left((j!)^{\frac{1}{s}}\right)_{j \in N_0} \quad \text{and} \quad M(s) := \left((j!)^{\frac{1}{r}}\right)_{j \in N_0},$$

where $\frac{1}{r} + \frac{1}{s} = 1$. Then all the conditions of 5. 1 (1) are satisfied for

$$Q(s) := \left(\frac{1}{k} p_{N(s)}\right)_{k \in N}.$$

By 2. 6 (3) we have

$$A^0_{Q(s)}(C) = \left\{ f \in A(C) \;\middle|\; \sup_{z \in C} |f(z)| \exp\left(-\frac{1}{k}|z|^s\right) < \infty \quad \text{for all} \quad k \in N \right\}.$$

We remark that $A^0_{Q(s)}(C)$ is identical with the space E_0^s of Martineau [18]. Obviously $A(C)$ can be regarded as E_0^∞.

(2) For $0 < \sigma < 1$ and $t \in R$ put

$$N = \left((j!)^\sigma (\log(j+A))^t\right)_{j \in N_0} \quad \text{and} \quad M := \left((j!)^{1-\sigma} (\log(j+A))^{-t}\right)_{j \in N_0},$$

where $A = A(\sigma, t)$ is large enough. Then all the conditions of 5. 1 (1) are satisfied for $Q := \left(\frac{1}{k} p_N\right)_{k \in N}$.

(3) Let $(s_k)_{k \in N}$ be a strictly decreasing sequence in $]1, \infty[$. For $k \in N$ put

$$N^k := \left((j!)^{\frac{1}{s_k}}\right)_{j \in N_0} \quad \text{and} \quad M^k := \left((j!)^{\frac{s_k - 1}{s_k}}\right)_{j \in N_0}.$$

Then $Q := (p_{N^k})_{k \in N}$ satisfies all the conditions of 5. 1 (2), and 2. 6 (3) implies

$$A^0_Q(C) = \{ f \in A(C) \mid \sup_{z \in C} |f(z)| \, exp(-|z|^{s_k}) < \infty \quad \text{for all} \quad k \in N \}.$$

Then we remark that $A_{P_Q}(C) = A_{\tilde{P}}(C)$, where $\tilde{P} = \left(|z|^{\frac{s_k}{s_k - 1}}\right)_{k \in N}$.

(4) For $k \in N$ choose $N^k = (N_j^k)_{j \in N_0}$ and $M^k = (M_j^k)_{j \in N_0}$ and $j_k \in N$ such that

$$N_j^k = \exp\left(j\left(\frac{1}{2}\log(j) + k\sqrt{\log j}\right)\right) \text{ and } M_j^k = \exp\left(j\left(\frac{1}{2}\log(j) - k\sqrt{\log j}\right)\right)$$

for all $j \geqq j_k$ and that all the conditions of 5. 1 (2) are satisfied if we put $Q = (p_{N^k})_{k \in N}$.

The following proposition is an easy consequence of Komatsu [15], 4. 5.

5. 3 Proposition. *Let Q and P_Q be as in 5. 1 (1) or 5. 1 (2). Then the Fourier-Borel transform $\mathscr{F} : A_Q^0(\mathbb{C})_b' \to A_{P_Q}(\mathbb{C})$, defined by*

$$\mathscr{F}(T)[\zeta] := \langle T_z, \exp(z\zeta)\rangle,$$

is a linear topological isomorphism. Moreover, we have

$$\mathscr{F}(T)[\zeta] = \sum_{j=0}^{\infty} \frac{T(z^j)}{j!} \zeta^j.$$

5. 4 Definition. A linear subspace W of $A_Q^0(\mathbb{C})$ is called translation invariant if for every $f \in W$ and every $a \in \mathbb{C}$ the function $z \mapsto f(a+z)$ belongs to W.

5. 5 Proposition. *A closed linear subspace W of $A_Q^0(\mathbb{C})$ is translation invariant if and only if $\mathscr{F}(W^\perp)$ is a closed ideal in $A_{P_Q}(\mathbb{C})$.*

Proof. It is easy to check that $D : f \mapsto f'$ defines a continuous linear endomorphism of $A_Q^0(\mathbb{C})$. Hence we have for every $T \in A_Q^0(\mathbb{C})'$

$$(1) \qquad \mathscr{F}({}^tD(T))[\zeta] = \left\langle T_z, \frac{d}{dz}\exp(z\zeta)\right\rangle = \zeta\mathscr{F}(T)[\zeta] \quad \text{for all} \quad \zeta \in \mathbb{C}.$$

Now, if W is a closed translation invariant subspace of $A_Q^0(\mathbb{C})$, then $f \in W$ implies $f' \in W$ since $f' = \lim_{h \to 0} \frac{f(\cdot + h) - f(\cdot)}{h}$ in the topology of $A_Q^0(\mathbb{C})$. Hence (1) implies $\zeta \cdot \mathscr{F}(W^\perp) \subset \mathscr{F}(W^\perp)$. Since the polynomials are dense in $A_{P_Q}(\mathbb{C})$ and since $\mathscr{F}(W^\perp)$ is closed, this implies that $\mathscr{F}(W^\perp)$ is an ideal in $A_{P_Q}(\mathbb{C})$.

On the other hand, if $\mathscr{F}(W^\perp)$ is an ideal in $A_{P_Q}(\mathbb{C})$, then (1) together with $W^{\perp\perp} = W$ implies that $f' \in W$ for every $f \in W$. Hence $f \in W$ implies $f^{(n)} \in W$ for every $n \in \mathbb{N}_0$. Now remark that in the topology of $A_Q^0(\mathbb{C})$ we have for every $a \in \mathbb{C}$

$$f(\cdot + a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\cdot)}{n!} a^n.$$

Hence W is translation invariant.

5. 6 Theorem. *Let Q be as in 5. 1 (1). Then every proper closed infinite dimensional translation invariant linear subspace W of $A_Q^0(\mathbb{C})$ is complemented and is isomorphic to a power series space of infinite type.*

Proof. From Proposition 5. 3 and classical duality theory it follows that

$$W = W^{\perp\perp} \cong (A_{P_Q}(\mathbb{C})/\mathscr{F}(W^\perp))_b'.$$

By Proposition 5. 5, $\mathscr{F}(W^\perp)$ is a proper closed ideal in $A_{P_Q}(\mathbb{C})$. Since $A_{P_Q}(\mathbb{C}) = A_{P_M}(\mathbb{C})$, where M satisfies the hypotheses of Example 2. 10, it follows from 2. 10 and Theorem 4. 7 that $\mathscr{F}(W^\perp)$ is complemented in $A_{P_Q}(\mathbb{C})$ and hence W is complemented in $A_Q^0(\mathbb{C})$. By 3. 5 g) and 3. 2 a) it follows from Corollary 3. 8 that W is isomorphic to a power series space of infinite type.

The following corollary is an immediate consequence of Theorem 5. 6 and Example 5. 2 (1).

5.7 Corollary. *For $s>1$ put*

$$E_0^s := \left\{ f \in A(\mathbb{C}) \;\middle|\; \sup_{z\in\mathbb{C}} |f(z)| \exp\left(-\frac{1}{k}|z|^s\right) < \infty \quad \text{for all} \quad k \in \mathbb{N} \right\}.$$

Then every proper closed infinite dimensional translation invariant linear subspace W of $A(\mathbb{C})$ resp. E_0^s, $s>1$, is complemented and isomorphic to a power series space of infinite type.

5.8 Theorem. *Let Q be as in 5.1 (2) and assume that $A_Q^0(\mathbb{C})$ has (Ω). Then every proper closed infinite dimensional translation invariant linear subspace W of $A_Q^0(\mathbb{C})$ has a Schauder basis, but is not complemented in $A_Q^0(\mathbb{C})$.*

Proof. As in the proof of Theorem 5.6 we have

$$W \cong (A_{P_Q}(\mathbb{C})/\mathcal{F}(W^\perp))_b'.$$

By Proposition 5.5, $\mathcal{F}(W^\perp)$ is a proper closed infinite codimensional ideal in $A_{P_Q}(\mathbb{C})$. From 5.1 (2) it follows that $(M^k)_{k\in N}$ satisfies the hypotheses of Proposition 2.11 (b). By 3.5 g) we have $\mathcal{F}(W^\perp) = I_{\text{loc}}(F_1, F_2)$ is slowly decreasing in $A_{P_Q}(\mathbb{C})$. Hence $W \cong \lambda(B)$ by Theorem 3.7. By Proposition 2.11 (b) it follows from Lemma 4.8 that W has $(\overline{DN})$. Since $A_Q^0(\mathbb{C}) \cong A._{Q}(\mathbb{C})_b'$ has (Ω) by hypothesis, it follows from Corollary 4.10 that $\mathcal{F}(W^\perp)$ is not complemented in $A_{P_Q}(\mathbb{C})$. Hence W is not complemented in $A_Q^0(\mathbb{C})$.

5.9 Corollary. *Let $\sigma := (s_k)_{k\in N}$ be a strictly decreasing sequence in $]1, \infty[$ and put*

$$A(\sigma) := \{ f \in A(\mathbb{C}) \mid \sup_{z\in\mathbb{C}} |f(z)| \exp(-|z|^{s_k}) < \infty \quad \text{for all} \quad k \in \mathbb{N} \}.$$

Then every proper closed infinite dimensional translation invariant linear subspace W of $A(\sigma)$ has a regular Schauder basis, but is not complemented in $A(\sigma)$.

Proof. By Example 5.2 (3) we have $A(\sigma) = A_{Q(\sigma)}^0(\mathbb{C})$, where $Q(\sigma) = (p_{N^k})_{k\in N}$ with $N^k := ((j!)^{\frac{1}{s_k}})_{j\in N_0}$, and we have $A_{P_Q}(\mathbb{C}) = A_{\tilde{P}}(\mathbb{C})$ where $\tilde{P} = (|z|^{\frac{s_k}{s_k-1}})_{k\in N}$. Hence Proposition 5.3 and the proof of Corollary 4.11 show that $A_{Q(\sigma)}^0(\mathbb{C})$ has (Ω). Moreover, it follows from Lemma 3.3, 3.5 g) and Theorem 3.7 that $W \cong (A_{\tilde{P}}(\mathbb{C})/\mathcal{F}(W^\perp))_b'$ has a regular basis. From Theorem 5.8 it follows that W is not complemented.

5.10 Example. Let $Q = (p_{N^k})_{k\in N}$ be defined as in Example 5.4 (4). Then $A_Q^0(\mathbb{C})$ has (Ω) since it follows from Example 2.13 (4) and Proposition 5.3 that $A_Q^0(\mathbb{C})$ is isomorphic to a power series space of infinite type. Hence it follows from Theorem 5.8 that every proper closed infinite dimensional translation invariant linear subspace of $A_Q^0(\mathbb{C})$ is not complemented.

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Mathematisches Institut der Universität, Universitätsstr. 1, D-4000 Düsseldorf

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