Coupled localized electron-plasma waves and oscillatory ion-acoustic perturbations

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The equations describing the coupling of high-frequency electrostatic waves with ion fluctuations allow stationary one-dimensional localized solutions which have not been reported previously. It is shown that these solutions follow from scaling laws different from those known for sonic Langmuir solitons, but similar to those used for static ion response. The present treatment generalizes the theory for subsonic Langmuir solitons by consistently including the second harmonic contribution in the low-frequency response. The physical importance of the new solutions is discussed.

I. INTRODUCTION

The problem of stationary propagation of waves and related questions of electric field localization have been intensively investigated in the past. For electron plasma waves in the nonrelativistic limit, the basic nonlinear mechanism involved is the ponderomotive force which induces an ion density perturbation (density cavity) which can trap the electron wave.[1-7] In the relativistic limit, electron-mass variations become important and cause a similar field localization.[8,9] Soliton formation is thus a prevailing nonlinear phenomenon in laser interaction with plasmas.

In this paper, we reconsider the dynamics of Langmuir envelope solitons in the nonrelativistic limit. Many authors[2,6,7] have already considered this problem in detail and found different types of stationary rarefraction solitons. The differences in the various theories lie in the description of the ion response: Rudakov[2] investigated the static case whereas Nishikawa *et al.*[6] and Karpman[7] looked for ion perturbations moving with ion-acoustic speed. Physically, the various results differ in the Mach number and the scaling of the relevant parameters, i.e., maximum field amplitude and density dip. Here, we generalize the result of Rudakov[2] for subsonic Langmuir solitons by re-investigating the interaction of a long-wavelength, slowly modulated ion-acoustic wave with the high-frequency field. The ordering is chosen such that the high-frequency field produces no modification of the carrier ion-acoustic wave; it only modifies the equation for the modulation envelope. The reaction back of this on the high-frequency field equation is retained. We show that in the small amplitude limit the corresponding second harmonic contribution can become important and a new soliton with a weakly oscillating density depression exists.

II. BASIC EQUATIONS AND SCALING

In general, Langmuir solitons can be derived from the following equations for the high-frequency electron plasma wave:

$$\frac{m_e}{m_i}\frac{\partial^2 E}{\partial t^2} - 3\frac{\partial^2 E}{\partial x^2} + (1+\delta n_e)E = 0, \tag{1}$$

and the low-frequency electron and ion response,

$$\frac{\partial \delta n_i}{\partial t} + \frac{\partial}{\partial x}[(1+\delta n_i)v_i] = 0, \tag{2}$$

$$\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x}\frac{v_i^2}{2} = -\frac{\partial \varphi}{\partial x}, \tag{3}$$

$$(1+\delta n_e)\frac{\partial |E|^2}{\partial x} = \frac{\partial \varphi}{\partial x}(1+\delta n_e) - \frac{\partial \delta n_e}{\partial x}, \tag{4}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \delta n_e - \delta n_i. \tag{5}$$

Here, we have nondimensionalized the time t by the inverse ion plasma frequency ω_{pi}^{-1}, the space coordinate x by the electron Debye length λ_e, the ion velocity v_i by the sound velocity $c_s = (T_e/m_i)^{1/2}$, the low-frequency potential φ by T_e/e, and the high-frequency electric field E by $(n_0 T_e/m_e)^{1/2}$, where n_0 is the (constant) average particle density.

Decomposing the electric field E into a slowly varying amplitude and a fast varying phase factor,[1] one obtains, from Eq. (1) in the long-wavelength limit,

$$2i\left(\frac{m_e}{m_i}\right)^{1/2}\frac{\partial E}{\partial t} + 3\frac{\partial^2 E}{\partial x^2} - \delta n_e E = 0. \tag{6}$$

Equations (2)–(6) form the basic set which can be solved for different scalings.

Rudakov[2] found a localized solution which one can derive from Eqs. (2)–(6) by assuming δn_e, δn_i, v_i, and φ being of order ϵ^2, whereas E is of order ϵ. Here, $\epsilon \sim (m_e/m_i)^{1/2}$ is an order parameter. Introducing the stretched coordinates

$$\xi = \epsilon(x - \lambda t), \quad \tau = \epsilon^2 t, \tag{7}$$

and assuming a nonoscillatory low-frequency response, the density depression is given by

$$\delta n_e = -|E|^2/(1-\lambda^2). \tag{8}$$

Since $\lambda \ll 1$, Eq. (8) together with Eq. (6) describe a subsonic Langmuir soliton. The density depression is proportional to the square of the electric field amplitude as prescribed by the scaling given here.

Nishikawa *et al.*[6] were the first to generalize this result for ion perturbations moving with the ion-acoustic speed. The ion response became of first order in the pump amplitude and hence nonlinear. The appropriate

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scaling, known from the derivation of the Korteweg–de Vries equation for ion-acoustic waves, is δn_e, δn_i, φ, v_i, $E \sim O(\epsilon)$ and the stretched variables are

$$\xi = \epsilon^{1/2}(x - t), \quad \tau = \epsilon^{3/2} t. \tag{9}$$

Then, the density depression is determined by

$$2\frac{\partial \delta n_e}{\partial \tau} + \frac{\partial}{\partial \xi}(\delta n_e)^2 + \frac{\partial^3}{\partial \xi^3}\delta n_e + \frac{\partial}{\partial \xi}|E|^2 = 0, \tag{10}$$

which has to be solved in connection with Eq. (6). The solutions show a node of the electric field at the density minimum.

Another sonic soliton has been found by Karpman.[7] One can derive it from Eqs. (2)–(6) by using the scaling δn_e, δn_i, v_i, $\varphi \sim O(\epsilon)$; $E \sim O(\epsilon^{3/4})$ and by introducing the stretched coordinates

$$\xi = \epsilon^{1/2}(x - t), \quad \tau = \epsilon t. \tag{11}$$

Then, under the additional assumption $(\partial/\partial\xi)\delta n_e - (\partial/\partial\xi)v_i \sim O(\epsilon^{1/2})$ for sonic solitons, the density response is determined by

$$\frac{\partial \delta n_e}{\partial \tau} = -\frac{1}{2}\frac{\partial |E|^2}{\partial \xi}, \tag{12}$$

which has to be used in connection with Eq. (6). The soliton solutions are similar in shape to those found by Rudakov.[2]

In this paper we show by scaling the appropriate equations (2)–(6) that in addition to the known Langmuir solitons a new subsonic Langmuir soliton exists. The difference from previous treatments results from a more detailed description of the subsonic ion response. We demonstrate that within the scaling of the subsonic Langmuir soliton $[\delta n_e \sim O(\epsilon^2)]$ the second harmonic contribution from the low-frequency response should be included. One finally gets two coupled nonlinear Schrödinger equations for the electron wave amplitude and the (oscillatory) ion response. This description generalizes the previous[2] treatment of static Langmuir solitons. In addition to that purpose it might be important with respect to practical applications since it could explain the appearance of oscillations seen in localized wave structures.[10]

III. SECOND HARMONIC CONTRIBUTION

We solve Eqs. (2)–(6) by introducing the stretched coordinates[11]

$$\xi = \epsilon(x - \lambda t), \quad \tau = \epsilon^2 t, \tag{13}$$

and expanding $\mathrm{U} \equiv (\delta n_e, \delta n_i, \partial\varphi/\partial x, v_i, E)$ in the form

$$\mathrm{U} = \sum_{\alpha=1}^{\infty} \epsilon^{\alpha} \sum_{l=-\infty}^{\infty} \mathrm{U}_l^{(\alpha)}(\xi, \tau)\exp[il(kx - \omega t)]. \tag{14}$$

By choosing this ansatz we allow for oscillatory solutions for the electric field amplitude E. The consistent calculation shows that besides the first-order zeroth harmonic part $E_0^{(1)}$ a second-order first harmonic (sideband) $E_1^{(2)}$ appears.

The main procedure may be summarized as follows: First, we consider the (oscillatory) ion response. Up to order ϵ^2 we do not get any change compared with the results known for the auto-modulation of ion oscillation modes,[12] i.e., we recover the linear dispersion relation

$$\omega^2 = k^2/(1 + k^2), \tag{15}$$

and the compatibility condition

$$\lambda = (1 + k^2)^{-3/2}. \tag{16}$$

Furthermore, within this ordering, $\delta n_{e0}^{(1)} = \delta n_{i0}^{(1)} \equiv \delta n_0^{(1)}$ shows no ξ dependence and thus does not cause a modulation of the plasma wave. [One can set $\delta n_0^{(1)}$ to zero provided $\lambda \neq 1$.]

The main effect appears in the equations of third order in ϵ. For the zeroth harmonic ($l = 0$) we obtain $\delta n_{e0}^{(2)} = \delta n_{i0}^{(2)} \equiv \delta n_0^{(2)}$, with

$$\delta n_0^{(2)} = -\frac{1}{1-\lambda^2}\left\{\left(\frac{\omega}{k}\right)^2\left[1 + \left(\frac{\omega}{k}\right)^2\right]|\delta n_1^{(1)}|^2 + |E_0^{(1)}|^2\right\}, \tag{17}$$

and

$$v_{i0}^{(2)} = -\frac{1}{1-\lambda^2}\left\{\left(\frac{\omega}{k}\right)\left[2 + \left(\frac{\omega}{k}\right)^4 - \left(\frac{\omega}{k}\right)^6\right]|\delta n_1^{(1)}|^2 + |E_0^{(1)}|^2\right\}, \tag{18}$$

$$\left(\frac{\partial\varphi}{\partial x}\right)_0^{(2)} = 0. \tag{19}$$

It should be mentioned that for the present ordering the resonance occurring at $\lambda = 1$ in Eqs. (17) and (18) has to be avoided, i.e., we cannot describe sonic envelope solitons. In the latter case, ion nonlinearities have to be included.[6]

Thus, we recover the previous result of Rudakov[2] for $\delta n_1^{(1)} = 0$, i.e., by neglecting the first harmonic part of the density reaction. In that case, the density depression is given by

$$\delta n_0^{(2)} = (\lambda^2 - 1)^{-1}|E|^2, \tag{20}$$

in agreement with Eq. (8). The complete density depression can only be determined from a closed equation governing $\delta n_1^{(1)}$, which is obtained for $l = 1$ within the order ϵ^3,

$$i\frac{\partial}{\partial\tau}\delta n_1^{(1)} + p\frac{\partial^2}{\partial\xi^2}\delta n_1^{(1)} + q|\delta n_1^{(1)}|^2\delta n_1^{(1)} + r|E_0^{(1)}|^2\delta n_1^{(1)} = 0. \tag{21}$$

Here, $p = -3\omega^5/2k^4$, $r = k/(\lambda^2 - 1) + 1/6\omega(1 + k^2)$, and q is given by $q \approx 1/3k$ within the long-wavelength approximation. For the reason of simplicity, in the following we discuss only that limit.

Next, we consistently order Eq. (6) for the high-frequency electric field envelope E. To order ϵ we find $E_l^{(1)} = 0$ for $l \geq 1$; in the next order (ϵ^2) we obtain $E_1^{(2)} \approx -\delta n_1^{(1)}E_0^{(1)}/3k^2$, $E_l^{(2)} = 0$ for $l \geq 2$, whereas the zeroth harmonic contribution of cubic order in ϵ yields

$$2i\left(\frac{m_e}{m_i}\right)^{1/2}\frac{\partial}{\partial\tau}E_0^{(1)} + 3\frac{\partial^2}{\partial\xi^2}E_0^{(1)} - \delta n_0^{(2)}E_0^{(1)} + \frac{2}{3k^2}|\delta n_1^{(1)}|^2 E_0^{(1)} = 0. \tag{22}$$

After simplifying the notation by introducing the abbreviations $a_1=E_0^{(1)}$, $a_2=2\delta n_1^{(1)}$, $\zeta=\xi/3k$, $T_1=6k^2(m_e/m_i)^{1/2}$, $T_2=-6k$, and $k=(1-\lambda^2)^{1/2}/\sqrt{3}$ we obtain from Eqs. (17), (21), and (22)

$$iT_1\frac{\partial a_1}{\partial\tau}+\frac{\partial^2 a_1}{\partial\zeta^2}=-(|a_2|^2+|a_1|^2)a_1, \tag{23a}$$

$$iT_2\frac{\partial a_2}{\partial\tau}+\frac{\partial^2 a_2}{\partial\zeta^2}=-\left(|a_1|^2-\frac{|a_2|^2}{2}\right)a_2, \tag{23b}$$

i.e., two coupled nonlinear Schrödinger equations.

We look for (stationary) solutions in the form

$$a_1=X(\zeta)\exp[i(\alpha/T_1)\tau+i\theta_1(\zeta)], \tag{24}$$

$$a_2=Y(\zeta)\exp[i(\beta/T_2)\tau+i\theta_2(\zeta)]. \tag{25}$$

Inserting Eqs. (24) and (25) into Eqs. (22) and (23), we obtain

$$X^2\dot\theta_1=M,\quad Y^2\dot\theta_2=N, \tag{26}$$

$$\ddot X=-\partial V/\partial X,\quad \ddot Y=-\partial V/\partial Y, \tag{27}$$

where

$$V(X,Y)=\frac{1}{2}\left(\frac{M^2}{X^2}+\frac{N^2}{Y^2}\right)-\frac{1}{2}(\alpha X^2+\beta Y^2)+\frac{1}{4}\left(X^4+2X^2Y^2-\frac{1}{2}Y^4\right). \tag{28}$$

One can easily get the integral

$$\tfrac{1}{2}(\dot X^2+\dot Y^2)+V(X,Y)=C. \tag{29}$$

A particularly simple nonlinear solution can be found for $\alpha=\beta$, $M=A^2N$, and $X=AY$. Introducing $R=zY^2$, where $z=(A^2+1)/(2A^4+4A^2-1)$, Eq. (29) yields

$$\dot R^2=-(R^3+b_2R^2+b_1R+b_0), \tag{30}$$

where $b_2=-4\alpha$, $b_1=-4C/(A^4+2A^2-\frac{1}{2})$, $b_0=4z^2N^2$.

The solution of Eq. (30) has been discussed by Inoue[13] in connection with the nonlinear coupling of polarized plasma waves. For $b_1\geq 0$ and $b_2<0$, or $b_1<0$, and $r_1^3+r_2^2\leq 0$, where $r_1=\frac{1}{3}b_1-\frac{1}{9}b_2^2$ and $r_2=\frac{1}{6}(b_1b_2-3b_0)-\frac{1}{27}b_2^3$, the solution is

$$R(\zeta)=R_2+(R_1-R_2)\,\mathrm{cn}^2(\zeta/g). \tag{31}$$

Here, $R_1\geq R_2\geq R_3$ are the three real roots of $R^3+b_2R^2+b_1R+b_0=0$, and $g=2(R_1-R_3)^{-1/2}$. In the limiting case $R_2=R_3=0$, which occurs for $N=C=0$, Eq. (31) represents the solitary wave solution

$$R(\zeta)=R_1\,\mathrm{sech}^2(\zeta/g). \tag{32}$$

Thus, we have found a stationary solution of the coupled equations (23a) and (23b). Calculating from that the *total* field envelope

$$E\approx E_0^{(1)}\{1-\delta n_1^{(1)}\exp[i(kx-\omega t)]/3k^2\},$$

and the density depression

$$\delta n\approx\delta n_1^{(1)}\exp[i(kx-\omega t)]+\delta n_0^{(2)},$$

we find a localization of the field envelope E, which is slowly modulated by the low-frequency response. The density reaction consists of a first order localized density oscillation and a second order density dip. The absolute height of the envelope $\delta n_1^{(1)}$ is here simply proportional to that of the envelope $E_0^{(1)}$, whereas the nonoscillatory density dip $\delta n_0^{(2)}$ is (roughly) proportional to $|E_0^{(1)}|^2$.

IV. DISCUSSION

We have found a nonlinear solution which shows a localization of the zeroth harmonic electric field amplitude similar to that reported by Rudakov[2] and Karpman[7]; but, in addition to the (nonoscillatory) part $E_0^{(1)}$, a localized first harmonic contribution $E_1^{(2)}\exp[i(kx-\omega t)]$ being of second order in ϵ exists. The total low-frequency density response consists of an oscillatory part, $\delta n_1^{(1)}\exp[i(kx-\omega t)]+\mathrm{c.c.}$, and a nonoscillatory part $\delta n_0^{(2)}$. Thus, this theory predicts the appearance of slow oscillations of a depression in density produced by a large amplitude Langmuir wave. The wavenumber k is related to the Mach number λ through Eq. (16), and thereby the frequency ω is also a function of the Mach number. The density depression $\delta n_0^{(2)}$ arises due to ponderomotive force effects as well as the second harmonic contribution. Within the present scaling the magnitude of the latter is of the same order as the ponderomotive force contribution and therefore, this theory gives a more correct relation between the zeroth harmonic density depression and the maximum of the electric field amplitude than found previously.[2]

However, within the present scaling we get an upper limit for the maximum amplitude $E_0^{(1)}$ in terms of the modulation wavenumber k. In the small k limit, we obtain from Eq. (21)

$$|\delta n_1^{(1)}|^2\sim|E_0^{(1)}|^2/k^2, \tag{33}$$

from Eq. (17)

$$|\delta n_0^{(2)}|\sim|E_0^{(1)}|^2(2k^{-2}+1)/3k^2, \tag{34}$$

and therefore, the balancing of nonlinearity and dispersion in Eq. (23), yields

$$K^2\sim|E_0^{(1)}|^2/k^2, \tag{35}$$

where $K\sim\partial/\partial\xi$. Thus, our analysis restricts the amplitude to

$$|E_0^{(1)}|^2\sim K^2k^2\ll k^4.$$

Finally, we want to mention that the solution of the nonlinear Schrödinger equation (23b) is related to the oscillatory solution of the Korteweg–de Vries equation (10) in the small-wavenumber region,[12] and thus, following the idea of Nishikawa *et al.*,[6] oscillatory sonic Langmuir envelope solitons can also be expected.

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